Solutions, exercise set 8

$\boxed{\textbf{Chapter 8,}\text{ exercise 14}}$ We prove this by induction on m. For $m = 1$, the function given by $a \mapsto \{(0, a)\}$ is a bijection from A to ${}^1A$. Assume the result for m. Then the function given by $a \mapsto (a \restriction m, a(m))$ is a bijection from ${}^{m+1}A$ to ${}^mA \times A$, and hence

$$|{}^{m+1}A| = |{}^mA \times A| = |{}^mA| \cdot |A| = |A| \cdot |A| = |A|.$$

$\boxed{\textbf{Chapter 8,}\text{ exercise 15}}$ Let B be the collection of all finite sequences of members of A. Note that this includes the empty sequence $\emptyset$. Hence

$$\begin{aligned}
|A| \leq |B| &= \left|\bigcup_{m\in\omega} {}^mA\right| \\
&\leq \sum_{m\in\omega} |{}^mA| \\
&\leq \sum_{m\in\omega} |A| \quad \text{by exercise 14} \\
&= \omega \cdot |A| \\
&= |A|.
\end{aligned}$$

Note that the initial $|A| \leq |B|$ is true by exercise 14 too, and the final inequality is not quite an equality because of the summand ${}^0A$. Anyway, this string of inequalities and equalities shows that $|B| = |A|$.

$\boxed{\textbf{Chapter 8,}\text{ exercise 16}}$ For each finite sequence f of members of A, let $F(f) = \mathrm{rng}(f)$. Clearly F maps the set B of all finite sequences of members of A onto the set C of all finite subsets of A. Hence $|C| \leq |B| = |A|$ by exercise 15. The function which takes each $a \in A$ to $\{a\}$ is clearly an injection of A into C, so also $|A| \leq |C|$. Hence $|C| = |A|$.

$\boxed{\textbf{Chapter 8,}\text{ exercise 18}}$ Since $\kappa \cdot |A| = |A|$, there is a bijection f from $\kappa \times A$ onto A. For each $\alpha < \kappa$ let $X_\alpha = f[\{\alpha\} \times A]$. We claim that $\{X_\alpha : \alpha < \kappa\}$ is the desired partition. Clearly each set X_α is nonempty. If $\alpha, \beta < \kappa$ and $\alpha \neq \beta$, then $(\alpha, a) \neq (\beta, b)$ for all $a, b \in A$, so $(\{\alpha\} \times A) \cap (\{\beta\} \times A) = \emptyset$, and hence $X_\alpha \cap X_\beta = \emptyset$ since f is a bijection. For any $a \in A$, choose $\alpha < \kappa$ and $b \in A$ such that $f(\alpha, b) = a$. Then $a \in X_\alpha$. Thus we have a partition. Clearly each X_α has size $|A|$.