Solutions, exercise set 6

$\boxed{\text{Chapter 7, exercise 12}}$ Since $(0,1) \subseteq [0,1] \subseteq [0,\infty) \subseteq \mathbb{R}$, it follows by Theorem 7.14 that $|(0,1)| \le |[0,1]| \le |[0,\infty)| \le |\mathbf{R}|$. Hence by Corollary 7.16 and Theorem 7.11 it suffices to find an injection from $\mathbb{R}$ into $(0,1)$. For each $x \in \mathbf{R}$ let

$$f(x) = \frac{1}{1+e^x}.$$

Since the range of the exponential function is $(0,\infty)$, it is clear that the range of f is contained in $(0,1)$. (Actually it is equal to $(0,1)$.) If $x \neq y$, then $e^x \neq e^y$, hence $1+e^x \neq 1+e^y$, hence $f(x) \neq f(y)$. So f is an injection.

$\boxed{\text{Chapter 7, exercise 13}}$

$$\begin{aligned} |\mathbb{R}| &< |\mathscr{P}(\mathscr{R})| \quad \text{by 7.31,} \\ &= |{}^{\mathbb{R}}2| \quad \text{by 7.30} \\ &\le |{}^{\mathbb{R}}\mathbb{R}|.\text{by 7.14} \end{aligned}$$

$\boxed{\textbf{Chapter 7, }\text{exercise 15}}$ Note by 7.24 that $A\backslash F$ is infinite. Hence by 7.17 there is an injection $f : \omega \to A\backslash F$. Let $m \in \omega$ with $g : F \to m$ a bijection. Now the following sets are pairwise disjoint: $f[m]$, $f[\omega\backslash m]$, F, and $A\backslash(f[\omega] \cup F)$. Now we define a function h which maps $f[\omega]$ one-one onto $f[\omega\backslash m]$, F one-one onto $f[m]$, and pointwise fixes $A\backslash(f[\omega] \cup F)$. Namely,

$$h(a) = \begin{cases} f(m+n) & \text{if } a = f(n) \text{ for some } n \in \omega, \\ f(g(a)) & \text{if } a \in F, \\ a & \text{otherwise.} \end{cases}$$

Clearly $h : A \to A\backslash F$ is an injection. By 7.16, $|A| \le |A\backslash F| \le |A|$, so $|A| = |A\backslash F|$ by 7.11.

$\boxed{\text{Chapter 7, exercise 17}}$ Choose $m \in \omega$ such that $f(m) \neq g(m)$. We claim that $A_f \cap A_g \subseteq \{f \restriction n : n \le m\}$. In fact, suppose that $x \in A_f \cap A_g$. Write $x = f \restriction p$ and $x = g \restriction q$. Then $p = q$ since p is the domain of $f \restriction p = x$ and also q is the domain of $g \restriction q = x$. If $m < p$, then $x(m) = f(m) \neq g(m) = x(m)$, contradiction. Thus $p \le m$, and so $x \in \{f \restriction n : n \le m\}$. There is an obvious surjective map from m onto $\{f \restriction n : n \le m\}$, so this set is finite.