Solutions, exercise set 12

$\boxed{\text{Page 120, 5}}$ *Show that if $\alpha, \beta > 1$ then $\alpha + \beta \leq \alpha \cdot \beta$.*

Solution: We fix $\alpha > 1$ and go by induction on β.

$\beta = 2$: $\alpha + \beta \leq \alpha + \alpha = \alpha \cdot 2 = \alpha \cdot \beta$.

Now assume that $\alpha + \beta \leq \alpha \cdot \beta$, with $\beta > 1$. Then

$$\alpha + \beta + 1 \leq \alpha \cdot \beta + 1 < \alpha \cdot \beta + \alpha = \alpha \cdot (\beta + 1),$$

completing the inductive proof.

$\boxed{\text{Page 120, 9 (as corrected in class)}}$ *Show that if α is a limit ordinal, then $\alpha = \omega \cdot \beta$ for some $\beta \neq 0$.*

Solution: Write $\alpha = \omega \cdot \beta + n$ with $n < \omega$. Since α is a limit ordinal, we have $n = 0$. Clearly $\beta \neq 0$.

$\boxed{\text{Page 120, 10 (as corrected in class)}}$ *Show that if $\alpha = \omega \cdot \beta$ for some $\beta \neq 0$, then for every $m \in \omega \backslash 1$ we have $m \cdot \alpha = \alpha$, and $\alpha \neq 0$.*

Solution: Assume that $\alpha = \omega \cdot \beta$ with $\beta \neq 0$. Clearly $\alpha \neq 0$. Now let m be any positive integer. Then

$$m \cdot \alpha = m \cdot \omega \cdot \beta = \left(\bigcup_{n \in \omega} (m \cdot n) \right) \cdot \beta = \omega \cdot \beta = \alpha.$$

$\boxed{\text{Page 120, 11}}$ *Show that if for every $m \in \omega \backslash 1$ we have $m \cdot \alpha = \alpha$, and $\alpha \neq 0$, then α is a limit ordinal.*

Solution: Assume the hypothesis, but suppose that α is a successor ordinal; say $\alpha = \beta + 1$. Then

$$\alpha = 2 \cdot \alpha = 2 \cdot (\beta + 1) = 2 \cdot \beta + 2 \geq \beta + 2 > \beta + 1 = \alpha,$$

contradiction.