Appendix D: the real numbers

We repeat a definition from Chapter 6. A subset A of $\mathbb{Q}$ is a *Dedekind cut* provided the following conditions hold:

(1) $\mathbb{Q} \neq A \neq \emptyset$;
(2) For all $r, s \in \mathbb{Q}$, if $r < s$ and $s \in A$, then $r \in A$.
(3) A has no largest element.

Let $\mathbb{R}'$ be the set of all Dedekind cuts.

If A and B are Dedekind cuts, then we define

$$A + B = \{x : \text{there are } a \in A \text{ and } b \in B \text{ such that } x = a + b\}.$$

Proposition D1. *If A and B are Dedekind cuts, then so is $A + B$.*

Proof. Since A and B are both nonempty, clearly $A + B$ is nonempty. Now take $r \in \mathbb{Q}\backslash A$ and $s \in \mathbb{Q}\backslash B$. So $t < r$ for all $t \in A$, and $u < s$ for all $u \in B$. Then $a + b < r + s$ for all $a \in A$ and $b \in B$, so that $x < r + s$ for all $x \in A + B$. In particular, $r + s \notin A + B$, by the irreflexivity of $<$. So we have shown that (1) holds for $A + B$.

Now suppose that $r < s \in A + B$. Write $s = a + b$ with $a \in A$ and $b \in B$. Then $r < s = a + b$, so $r - a < b$, and hence $r - a \in B$ by (2) for B. Hence $r = a + (r - a)$ shows that $r \in A + B$. So (2) holds for $A + B$.

Suppose that $x \in A + B$. Write $x = a + b$ with $a \in A$ and $b \in B$. Since a is not the greatest element of A, by (3) choose $a' \in A$ such that $a < a'$. Then $x = a + b < a' + b \in A + B$, proving (3) for $A + B$. □

Proposition D2. *Let A, B, C be Dedekind cuts. Then*
(i) $A + B = B + A$.
(ii) $A + (B + C) = (A + B) + C$.

Proof. (i): obvious. (ii): Suppose that $x \in A + (B + C)$. Then there are $a \in A$ and $y \in (B + C)$ such that $x = a + y$; and there are $b \in B$ and $c \in C$ such that $y = b + c$. So $x = a + b + c$. Now $a + b \in (A + B)$, so $x \in ((A + B) + C)$. This shows that $A + (B + C) \subseteq (A + B) + C$. Since this is generally true for all Dedekind cuts A, B, C, we also have $(A + B) + C = C + (B + A) \subseteq (C + B) + A = A + (B + C)$. □

Now we define, following Chapter 6,

$$Z = \{r \in \mathbb{Q} : r < 0\}.$$

Clearly Z is a Dedekind cut.

Proposition D3. *$A + Z = A$ for every Dedekind cut A.*

Proof. Let $a \in A$. Since A does not have a largest element, choose $b \in A$ such that $a < b$. Then $a - b < 0$, hence $a - b \in Z$, and so $a = b + (a - b)$ shows that $a \in A + Z$.

Conversely, suppose that $x \in A + Z$. Then there exist $a \in A$ and $b \in Z$ such that $x = a + b$. Since $b < 0$, we have $x < a$, and so $x \in A$, as desired. □

It is easy to check that Z is the only element of $\mathbb{R}'$ such that $A + Z = A$ for all A.

Next, for any Dedekind cut A we define

$$-A = \{r \in \mathbb{Q} : \text{there is an } s \in \mathbb{Q} \text{ such that } r < s \text{ and } -s \notin A\}.$$

Proposition D4. *$A + -A = Z$ for any Dedekind cut A.*

Proof. First we show that $-A$ is itself a Dedekind cut. Since $A \neq \mathbb{Q}$, choose $r \in \mathbb{Q}\backslash A$. Then also $r+1 \notin A$. so $-(r+1) < -r$ and $-(-r) = r \notin A$. It follows that $-(r+1) \in -A$. Hence $-A \neq \emptyset$. Next, choose $r \in A$. Then $-r \notin -A$, as otherwise there is an s such that $-r < s$ and $-s \notin A$; but $-s < r$, contradiction. So $-A \neq \mathbb{Q}$. Finally, suppose that $r \in -A$; we want to find a larger element in A. Choose s such that $r < s$ and $-s \notin A$. Take $t \in \mathbb{Q}$ such that $r < t < s$; for example, take $t = (r + s)/2$. Clearly then $t \in -A$, as desired. This checks that $-A$ is a Dedekind cut.

Now suppose that $x \in A+-A$. Then there are $a \in A$ and $b \in -A$ such that $x = a+b$. Choose $c \in \mathbb{Q}$ such that $b < c$ and $-c \notin A$. Suppose that $0 \leq x$. Then $x = a + b < a + c$, and so $-c < a + -x \leq a$, and hence $-c \in A$, contradiction. Hence $x < 0$, so that $x \in Z$.

Second suppose that $r \in Z$. Fix $b \notin A$.

(1) There is a positive integer p such that $b + \frac{pr}{2} \in A$.

In fact, to prove (1), also fix $a \in A$. Then $a < b$, as otherwise we would have $b \in A$. Hence there are positive integers s, t such that $b - a = \frac{s}{t}$. Since $\frac{r}{2} < 0$, there are also positive integers u, v such that $\frac{r}{2} = -\frac{u}{v}$. Then $b - a = \frac{s}{t} \leq s \leq su = sv(-\frac{r}{2})$. Hence $b + sv\frac{r}{2} \leq a$, and so $b + sv\frac{r}{2} \in A$, proving (1).

Let p be the smallest positive integer such that $b+p\frac{r}{2} \in A$. Recall that $b \notin A$, so that even if $p = 1$ we can assert that $b + (p-1)\frac{r}{2} \notin A$. Now

$$r = b + pr + (-b + (-p+1)\frac{r}{2} + \frac{r}{2}),$$

and $(-b+(-p+1)\frac{r}{2} + \frac{r}{2}) < (-b+(-p+1)\frac{r}{2}$, and $-(-b+(-p+1)\frac{r}{2}) = b+(p-1)\frac{r}{2} \notin A$. This shows that $r \in A + -A$. □

The element $-A$ is unique: if $A + B = Z$, then $B = -A$. In particular, $-Z = Z$.

Next, we call a Dedekind cut A *positive* iff if has at least one positive member.

Proposition D5. *For any Dedekind cut A, exactly one of the following holds:*

(i) A is positive;

(ii) $A = Z$;

(iii) $-A$ is positive.

Proof. Suppose that A is not positive, and $A \neq Z$. Since A is not positive, all its members are negative or zero; since it has no largest element, $0 \notin A$. Thus $A \subseteq Z$. Since $A \neq Z$, we actually have $A \subset Z$. Choose $r \in Z\backslash A$. Now $r + r < 0 + r = r < 0$, and so $r < \frac{r}{2} < 0$. Hence $0 < -\frac{r}{2} < -r$. So $-\frac{r}{2} \in -A$, since $-(-r) = r \notin A$. This shows that $-A$ is positive.

So we have shown that one of (i)–(iii) holds.

Obviously (i) and (ii) do not simultaneously hold. Suppose that both A and $-A$ are positive. Hence there is a positive element $r \in A$, and a positive element $s \in -A$. By the definition of $-A$, choose t such that $s < t$ and $-t \notin A$. Then $-t < -s < 0 < r$, so $-t \in A$, contradiction. Thus (i) and (iii) do not simultaneously hold. Finally, suppose that $-Z$ is positive. Let r be a positive element of $-Z$. Then by definition there is an s such that $r < s$ and $-s \notin Z$. So $0 \le -s < -r$, contradicting r being positive. $\square$

On the basis of Proposition D5, the following definition makes sense. For any Dedekind cut A,

$$|A| = \begin{cases} A & \text{if } A = Z \text{ or } A \text{ is positive,} \\ -A & \text{if } -A \text{ is positive.} \end{cases}$$

Now we repeat the definition of product from Chapter 6. Let A and B be Dedekind cuts.

$$\begin{aligned} A \cdot B = \{r \in \mathbb{Q} : {}& \text{there are } s \in A \text{ and } t \in B \text{ such that } 0 < s \\ & \text{and } 0 < t \text{ and } r < s \cdot t\} \text{ if } A \text{ and } B \text{ are positive,} \end{aligned} \tag{a}$$

$$A \cdot B = Z \text{ if } A = Z \text{ or } B = Z, \tag{b}$$

$$A \cdot B = -(|A| \cdot |B|) \text{ if } A \neq Z \neq B \text{ and exactly one of } A, B \text{ is positive} \tag{c}$$

$$A \cdot B = (-A) \cdot (-B) \text{ if } -A \text{ and } -B \text{ are both positive.} \tag{d}$$

Proposition D6. *Let A, B, C be Dedekind cuts.*
(i) $A \cdot B = B \cdot A$.
(ii) $(-A) \cdot B = -(A \cdot B) = A \cdot (-B)$.
(iii) $A \cdot (B \cdot C) = (A \cdot B) \cdot C$.
(iv) $A \cdot (B + C) = A \cdot B + A \cdot C$.

Proof. (i): this is clear if both A and B are positive, or if one of them is Z. If both are different from Z and exactly one of them is positive, then $|A|$ and $|B|$ are both positive, and

$$A \cdot B = -(|A| \cdot |B|) = -(|B| \cdot |A|) = B \cdot A.$$

If $-A$ and $-B$ are both positive, then

$$A \cdot B = (-A) \cdot (-B) = (-B) \cdot (-A) = B \cdot A.$$

Thus (i) holds.

(ii): First we prove that $(-A) \cdot B = -(A \cdot B)$. This is true by (b) if one of A, B is Z, since $-Z = Z$. If A and B are positive, then

$$(-A) \cdot B = -(A \cdot B) \quad \text{by (c).}$$

If $-A$ and B are positive, then

$$\begin{aligned} -(A \cdot B) &= -(-((-A) \cdot B)) \quad \text{by (c)} \\ &= (-A) \cdot B. \end{aligned}$$

If A and $-B$ are positive, then

$$\begin{aligned} (-A) \cdot B &= B \cdot (-A) && \text{by (i)} \\ &= -(B \cdot A) && \text{by the previous case} \\ &= -(A \cdot B) && \text{by (i).} \end{aligned}$$

Finally, if $-A$ and $-B$ are positive, then

$$\begin{aligned} (-A) \cdot B &= -((-A) \cdot (-B)) \quad \text{by (c)} \\ &= -(A \cdot B) \quad \text{by (d).} \end{aligned}$$

Thus $(-A) \cdot B = -(A \cdot B)$ in general. The other part of (ii) follows from (i).
(iii):

(1) If A, B, C are all positive, then $A \cdot (B \cdot C) \subseteq (A \cdot B) \cdot C$.

For, assume that A, B, C are all positive. Clearly then $A \cdot B$ and $B \cdot C$ are positive. Now let $x \in A \cdot (B \cdot C)$. Then there exist s, t such that $x < s \cdot t$, $0 < s \in A$, and $0 < t \in B \cdot C$. Since $t \in B \cdot C$, there exist u, v such that $t < u \cdot v$, $0 < u \in B$, and $0 < v \in C$. Choose $s' \in A$ such that $s < s'$. Then $s \cdot u < s' \cdot u$, $0 < s' \in A$, and $0 < u \in B$, so $s \cdot u \in A \cdot B$. Then $x < s \cdot u \cdot v$, $0 < s \cdot u \in A \cdot B$, and $0 < v \in C$, so $x \in (A \cdot B) \cdot C$. This proves (1).

(2) If one of A, B, C is equal to Z, then $A \cdot (B \cdot C) = Z = (A \cdot B) \cdot C$.

This is clear.

(3) If A, B, C are all positive, then $A \cdot (B \cdot C) = (A \cdot B) \cdot C$.

In fact,

$$\begin{aligned} A \cdot (B \cdot C) &\subseteq (A \cdot B) \cdot C && \text{by (1)} \\ &= C \cdot (B \cdot A) && \text{by (i)} \\ &\subseteq (C \cdot B) \cdot A && \text{by (1)} \\ &= A \cdot (B \cdot C) && \text{by (i).} \end{aligned}$$

So (3) holds.

Now we can use (ii) to finish (iii):

$$\begin{aligned} A, B, -C \text{ positive: } A \cdot (B \cdot C) &= A \cdot -(B \cdot -C) \\ &= -(A \cdot (B \cdot -C) \\ &= -((A \cdot B) \cdot -C) \end{aligned}$$

$$
\begin{aligned}
&= (A \cdot B) \cdot C; \\
A, -B, C \text{ positive: } A \cdot (B \cdot C) &= A \cdot -(-B \cdot C) \\
&= -(A \cdot (-B \cdot C)) \\
&= -((A \cdot -B) \cdot C) \\
&= (A \cdot B) \cdot C; \\
A, -B, -C \text{ positive: } A \cdot (B \cdot C) &= A \cdot ((-B) \cdot (-C)) \\
&= (A \cdot -B) \cdot -C \\
&= (A \cdot B) \cdot C; \\
C \text{ positive: } (A \cdot B) \cdot C &= C \cdot (B \cdot A) \\
&= (C \cdot B) \cdot A \\
&= A \cdot (B \cdot C); \\
-A, B, -C \text{ positive: } A \cdot (B \cdot C) &= A \cdot -(B \cdot -C) \\
&= -((-A) \cdot -(B \cdot -C)) \\
&= (-A) \cdot (B \cdot -C) \\
&= ((-A) \cdot B) \cdot -C \\
&= (A \cdot B) \cdot C; \\
-A, -B, -C \text{ positive: } A \cdot (B \cdot C) &= A \cdot ((-B) \cdot (-C)) \\
&= -((-A) \cdot ((-B) \cdot (-C))) \\
&= -(((-A) \cdot (-B)) \cdot -C) \\
&= (A \cdot B) \cdot C.
\end{aligned}
$$

(iv): Clearly

(4) If one of A, B, C is Z, then $A \cdot (B + C) = A \cdot B + A \cdot C$.

(5) If A, B, C are positive, then $A \cdot (B + C) = A \cdot B + A \cdot C$.

For, first suppose that $x \in A \cdot (B + C)$. Then we can choose s, t so that $0 < s \in A$, $0 < t \in B + C$, and $x < s \cdot t$. Since $t \in B + C$, there are $b \in B$ and $c \in C$ such that $t = b + c$. Now choose $b' \in B$ with $b \leq b'$ and $0 < b'$, and choose $c' \in C$ such that $c \leq c'$ and $0 < c'$. Now $x = s \cdot b' + (x - s \cdot b')$, and clearly $s \cdot b' \in A \cdot B$, while

$$x - s \cdot b' < s \cdot (b' + c') - s \cdot b' = s \cdot c',$$

and clearly $s \cdot c' \in A \cdot C$. This proves $\subseteq$ in (5).

Now suppose that $y \in A \cdot B + A \cdot C$. Then we can write $y = u + v$ with $u \in A \cdot B$ and $v \in A \cdot C$. Say $u < s \cdot t$ with $0 < s \in A$ and $0 < t \in B$, and $v < a \cdot c$ with $0 < a \in A$ and $0 < c \in C$. Let s' be the maximum of s and a. Then $y < s' \cdot (t + c)$, $0 < s' \in A$, and $t + c \in B + C$. So $y \in A \cdot (B + C)$. This proves $\supseteq$ in (5).

(6) If $A, B, -C$ are positive, and also $B + C$ is positive, then $A \cdot (B + C) = A \cdot B + A \cdot C$.

For,

$$\begin{aligned} A \cdot B &= A \cdot (B + C + -C) \\ &= A \cdot (B + C) + A \cdot (-C) \quad \text{by (5)} \\ &= A \cdot (B + C) + -(A \cdot C), \quad \text{by (ii)} \end{aligned}$$

and (6) follows.

(7) If $A, B, -C$ are positive, and $B + C$ is negative, then $A \cdot (B + C) = A \cdot B + A \cdot C$.

For,

$$\begin{aligned} -(A \cdot (B + C)) &= A \cdot (-(B + C)) \quad \text{by (ii)} \\ &= A \cdot (-B + -C) \\ &= A \cdot (-B) + A \cdot (-C) \quad \text{by (6)} \\ &= -(A \cdot B) + -(A \cdot C), \quad \text{by (ii)} \end{aligned}$$

and (7) follows.

(8) If $A, B, -C$ are positive, and $B + C = Z$, then $A \cdot (B + C) = A \cdot B + A \cdot C$.

For, under these hypotheses, $C = -B$, and so

$$A \cdot (B + C) = A \cdot Z = Z = A \cdot B + -(A \cdot B) = A \cdot B + A \cdot (-B) = A \cdot B + A \cdot C.$$

(9) If $A, -B, C$ are positive, then $A \cdot (B + C) = A \cdot B + A \cdot C$.

This follows from (6)–(8) since $+$ is commutative.

(10) If $A, -B, -C$ are positive, then $A \cdot (B + C) = A \cdot B + A \cdot C$.

For,

$$\begin{aligned} A \cdot (B + C) &= -(A \cdot (-B + -C)) \quad \text{by (ii)} \\ &= -(A \cdot (-B) + A \cdot (-C)) \quad \text{by (5)} \\ &= -(-(A \cdot B) + -(A \cdot C)) \quad \text{by (ii)} \\ &= A \cdot B + A \cdot C. \end{aligned}$$

(11) If A is positive, then $A \cdot (B + C) = A \cdot B + A \cdot C$.

This is true by (6)–(10).

(12) If $-A$ is positive, then $A \cdot (B + C) = A \cdot B + A \cdot C$.

In fact, $(-A) \cdot (B + C) = (-A) \cdot B + (-A) \cdot C$ by (11), and (12) follows, using (ii). $\square$

Now we define

$$I = \{r \in \mathbb{Q} : r < 1\}.$$

Clearly I is a Dedekind cut.

Proposition D7. *$A \cdot I = A$ for any Dedekind cut A.*

Proof. This is clear if $A = Z$. Now suppose that A is positive. Suppose that $r \in A \cdot I$. Then there are $s, t \in \mathbb{Q}$ such that $0 < s \in A$, $0 < t \in I$, and $r < s \cdot t$. Clearly then $r < s$, so $r \in A$ by the definition of Dedekind cut.

Conversely, suppose that $r \in A$. Choose $r', r'' \in A$ such that $r < r' < r''$ and $0 < r'$. Let $s = \frac{r'}{r''}$. Then $0 < s < 1$, so $s \in I$. Since $r < r' = r'' \cdot s$, it follows that $r \in A \cdot I$. Thus we have shown that $A \cdot I = A$ for A positive.

If $-A$ is positive, then $A \cdot I = -((-A) \cdot I) = -(-A) = A$, using D6(ii). $\square$

Proposition D8. *If A is a Dedekind cut and $A \neq Z$, then there is a Dedekind cut B such that $A \cdot B = I$.*

Proof. First suppose that A is positive. Let

$$B = \{r \in \mathbb{Q} : r < 0, \text{ or } 0 \leq r \text{ and } r \cdot s < 1 \text{ for every } s \in A \text{ for which } 0 < s\}.$$

Then $B \neq \emptyset$, since clearly $0 \in B$. Clearly if $r' < r \in B$, then also $r' \in B$. If $0 < s \in A$, then $\frac{1}{s} \notin B$. So B is a Dedekind cut.

We claim that $A \cdot B = I$. Suppose that $r \in A \cdot B$. Choose s, t so that $0 < s \in A$, $0 < t \in B$, and $r < s \cdot t$. Then by the definition of B, $s \cdot t < 1$, so $r < 1$. Hence $r \in I$.

Conversely, suppose that $r \in I$, so that $r < 1$. Choose r', r'', r''' so that $0, r < r' < r'' < r''' < 1$. Let $C = \{s \in \mathbb{Q} : s < r'''\}$. Clearly C is a Dedekind cut.

(1) $(A \cdot C) \subset A$.

In fact, clearly $(A \cdot C) \subseteq A$. Suppose that $A \cdot C = A$. Now

$$A = A \cdot I = (A \cdot C) + (A \cdot (I - C)) = A + (A \cdot (I - C)),$$

so $A \cdot (I - C) = Z$. Choose s, t so that $r''' < s < t < 1$. Then $-s < -r'''$ and $r''' \notin C$, so $-s \in -C$. Hence $0 < t - s \in (I - C)$. So $I - C$ is positive. Since A is also positive, it follows that $A \cdot (I - C)$ is positive, contradiction. Hence (1) holds.

By (1), choose $s \in A \backslash (A \cdot C)$. We may assume that $0 < s$. Thus

(2) For all a, c, if $0 < a \in A$ and $0 < c \in C$, then $a \cdot c \leq s$.

Now let $v = \frac{r'}{s}$. Thus $s \cdot v = r' > r$. Hence we will get $r \in A \cdot B$ as soon as we show that $v \in B$. Suppose that $0 < a \in A$. Now $0 < r'' \in C$, so by (2) we have $a \cdot r'' \leq s$. Hence

$$a \cdot v = a \cdot \frac{r'}{s} < a \cdot \frac{r''}{s} \leq 1,$$

so that $a \cdot v < 1$, as desired.

Thus we have finished the proof in the case that A is positive. If $-A$ is positive, then choose B so that $(-A) \cdot B = I$. Then $(A \cdot (-B)) = (-A) \cdot B = I$, using D7(ii). $\square$

This finishes the purely arithmetic part of the construction of the real numbers. Now we discuss ordering. We define $A < B$ iff $B - A$ is positive. Elementary properties of $<$ are given in the following proposition.

Proposition D9. *Let $A, B, C \in \mathbb{R}'$. Then*

(i) $A \not< A$.

(ii) If $A < B < C$, then $A < C$.

(iii) $A < B$, $A = B$, or $B < A$.

(iv) $A < B$ iff $A + C < B + C$.

(v) $Z < I$.

(vi) If $Z < A$ and $Z < B$, then $Z < A \cdot B$.

(vii) If $Z < C$, then $A < B$ implies that $A \cdot C < B \cdot C$.

(viii) $A < B$ iff $A \subset B$.

Proof. (i): $A - A = Z$, so $A \not< A$ by D5.

(ii) Suppose that $A < B < C$. Thus $B - A$ and $C - B$ are positive. Hence clearly also $C - A = C - B + B - A$ is positive.

(iii): Given A, B, by D5 we have $A - B$ positive, $A - B = Z$, or $-(A - B) = B - A$ is positive. By definition this gives $A < B$, $A = B$, or $B < A$.

(iv): First suppose that $A < B$. Thus $B - A$ is positive. Since $B + C - (A + C) = B - A$, it follows that $A + C < B + C$.

Second, suppose that $A + C < B + C$. Thus $B - A = B + C - (A + C)$ is positive, so $A < B$.

(v): Obviously I is positive.

(vi): Assume that $Z < A$ and $Z < B$. Thus A and B are positive. Clearly then $A \cdot B$ is positive. So $Z < A \cdot B$.

(vii): Assume that $Z < C$ and $A < B$. Then C and $B - A$ are positive, so also $C \cdot (B - A) = C \cdot B - (A \cdot C)$ is positive, and so $A \cdot C < B \cdot C$.

(viii): Suppose that $A < B$. Thus $B - A$ is positive. Choose x so that $0 < x \in B - A$. Then we can write $x = b + a$ with $b \in B$ and $a \in -A$. By the definition of $-A$, choose $s \in \mathbb{Q}$ so that $a < s$ and $-s \notin A$. Then $-s < -a$, so also $-a \notin A$. Also $b + a > 0$, so $b > -a$, and it follows that $b \notin A$. Now if $y \in A$, then $y < b$, as otherwise $b \leq y$ would imply that $b \in A$. But then $y \in B$. So $A \subseteq B$, and since $b \in B \backslash A$, even $A \subset B$.

Conversely, suppose that $A \subset B$. Choose $b \in B \backslash A$. Choose c, d such that $b < c < d \in B$. Now $-c < -b$ and $b \notin A$, so $-c \in -A$. Thus $d - c$ is a positive element of $B - A$, hence $B - A$ is positive and $A < B$. □

The following theorem expresses the essential new property of the reals as opposed to the rationals.

Theorem D10. *Every nonempty subset of $\mathbb{R}'$ which is bounded above has a least upper bound. That is, if $\emptyset \neq \mathscr{X} \subseteq \mathbb{R}'$, and there is a Dedekind cut D such that $A \leq D$ for all $A \in \mathbb{R}'$, then there is a Dedekind cut B such that the following two conditions hold:*

(i) $A \leq B$ for every $A \in \mathscr{X}$.

(ii) For any Dedekind cut C, if $A \leq C$ for every $A \in \mathscr{X}$, then $B \leq C$.

Proof. Let $B = \bigcup_{A \in \mathscr{X}} A$. Since $\mathscr{X}$ is nonempty, and each Dedekind cut is nonempty, it follows that B is nonempty. To show that B does not consist of all rationals, we use the assumption that $\mathscr{X}$ has an upper bound. Let D be an upper bound for $\mathscr{X}$. Thus $A \leq D$ for all $A \in \mathscr{X}$. By D9(viii), $A \subseteq D$ for all $A \in \mathscr{X}$, and hence $B \subseteq D$. Since $D \neq \mathbb{Q}$, also

$B \neq \mathbb{Q}$. If $x < y \in B$, then $y \in A$ for some $A \in \mathscr{X}$, hence $x \in A$, hence $x \in B$. Thus B is a Dedekind cut.

For any $A \in \mathscr{X}$ we have $A \subseteq B$, and hence $A \leq B$ by D9(viii).

Now suppose that $A \subseteq C$ for all $A \in \mathscr{X}$, where C is a Dedekind cut. Then $B \subseteq C$, hence $B \leq C$ by D9(viii). $\square$

Next we want to embed the rationals into $\mathbb{R}'$. For every rational r we define $f(r) = \{q \in \mathbb{Q} : q < r\}$. Clearly $f(r)$ is a Dedekind cut.

Proposition D11. *(i) f is one-one.*

(ii) $f(r+s) = f(r) + f(s)$ for any $r, s \in \mathbb{Q}$.

(iii) $f(r \cdot s) = f(r) \cdot f(s)$ for any $r, s \in \mathbb{Q}$.

Proof. (i): Suppose that $r, s \in \mathbb{Q}$; say $r < s$. Then $r \in f(s) \backslash f(r)$, so $f(r) \neq f(s)$.

(ii): First suppose that $x \in f(r+s)$. Thus $x < r+s$, so $x - s < r$. Let r' be a rational number such that $x - s < r' < r$. Then $x = r' + (x - r')$, and $x - r' < s$, so $x \in f(r) + f(s)$.

Conversely, suppose that $x \in f(r) + f(s)$. Choose $a \in f(r)$ and $b \in f(s)$ so that $x = a + b$. Then $a < r$ and $b < s$, so $x < r + s$, and so $x \in f(r+s)$.

(iii): Note that $f(0) = Z$; hence (iii) is clear if $r = 0$ or $s = 0$. Suppose that $r, s > 0$. Suppose that $x \in f(r \cdot s)$. So $x < r \cdot s$. Hence $\frac{x}{s} < r$. Choose $r' \in \mathbb{Q}$ such that $\frac{x}{s} < r' < r$ and $0 < r'$. Hence $\frac{x}{r'} < s$. Choose $s' \in \mathbb{Q}$ such that $\frac{x}{r'} < s' < s$ and $0 < s'$. Then $x < r' \cdot s'$, $0 < r' \in f(r)$, and $0 < s' \in f(s)$, so $x \in f(r) \cdot f(s)$.

Conversely, suppose that $x \in f(r) \cdot f(s)$. Then there are $r' \in f(r)$ and $s' \in f(s)$ such that $0 < r'$, $0 < s'$, and $x < r' \cdot s'$. Hence $x < r \cdot s$, so $x \in f(r \cdot s)$, as desired. This finishes the case in which $r, s > 0$.

To continue we need the following little fact:

(1) $-f(r) = \{q \in \mathbb{Q} : q < -r\}$ for any rational number r.

In fact, suppose that $q \in -f(r)$. Then there is a rational t such that $q < t$ and $-t \notin f(r)$. thus $-t \not< r$, so $r \leq -t$. Hence $t \leq -r$, so $q < -r$. Conversely, suppose that $q < -r$. Now $r \notin f(r)$, so $q \in -f(r)$. Thus (1) holds.

Now suppose that $r < 0 < s$. Then, using (1),

$$f(r) \cdot f(s) = -((-f(r)) \cdot f(s)) = -(f(-r) \cdot f(s)) = -f((-r) \cdot s) = f(r \cdot s).$$

Similarly if $s < 0 < r$. If $r, s < 0$, then

$$(f(r) \cdot f(s) = (-f(r)) \cdot (-f(s)) = f(-r) \cdot f(-s) = f((-r) \cdot (-s)) = f(r \cdot s). \qquad \square$$

Proposition D12. $\mathbb{Q} \cap \mathbb{R}' = \emptyset$.

Proof. First, $\omega \cap \mathbb{R}' = \emptyset$, since the members of ω are all finite, while the members of $\mathbb{R}'$ are all infinite.

Now suppose that $a \in \mathbb{Z} \cap \mathbb{R}'$. Then $a \notin \omega$ by the preceding paragraph, so $a = [(m, n)]$ for some $m, n \in \omega$. But also $a \in \mathbb{R}'$, so a is a set of rationals. In particular, (m, n) is a rational. Now (m,n) has either one or two elements; the only rationals with only one or two elements are 1 and 2. Since $\emptyset \in 1$ and $\emptyset \in 2$, we get $\emptyset \in (m,n)$, contradiction.

A similar argument shows that $a \in \mathbb{Q} \cap \mathbb{R}'$ leads to a contradiction. $\square$

We can now proceed very much like in previous appendices. We define $\mathbb{R} = (\mathbb{R}' \backslash \mathrm{rng}(f)) \cup \mathbb{Q}$. There is a one-one function $g : \mathbb{R} \to \mathbb{R}'$, defined by $g(A) = A$ if $A \in \mathbb{R}' \backslash \mathrm{rng}(f)$, and $g(A) = f(A)$ for $A \in \mathbb{Q}$. Clearly g is a bijection. Now the operations $+'$ and $\cdot'$ are defined on $\mathbb{R}$ as follows. For any $a, b \in \mathbb{R}$,

$$\begin{aligned} a +' b &= g^{-1}(g(a) + g(b)); \\ a \cdot' b &= g^{-1}(g(a) \cdot g(b)). \end{aligned}$$

moreover, we define $a <' b$ iff $g(a) < g(b)$. With these definitions, g becomes an isomorphism of $\mathbb{R}$ onto $\mathbb{R}'$. Namely, if $a, b \in \mathbb{R}$, then

$$\begin{aligned} g(a +' b) &= g(g^{-1}(g(a) + g(b))) = g(a) + g(b); \\ g(a \cdot' b) &= g(g^{-1}(g(a) \cdot g(b))) = g(a) \cdot g(b); \\ a <' b \quad &\text{iff} \quad g(a) < g(b). \end{aligned}$$

Moreover, the operations $+'$ and $\cdot'$ on $\mathbb{Q}$ coincide with the ones defined in appendix C, since if $a, b \in \mathbb{Q}$, then

$$\begin{aligned} a +' b &= g^{-1}(g(a) + g(b)) = g^{-1}(f(a) + f(b)) = g^{-1}(f(a + b)) = a + b; \\ a \cdot' b &= g^{-1}(g(a) \cdot g(b)) = g^{-1}(f(a) \cdot f(b)) = g^{-1}(f(a \cdot b)) = a \cdot b; \end{aligned}$$

$$\begin{aligned} a <' b \quad &\text{iff} \quad g(a) < g(b) \\ &\text{iff} \quad f(a) < f(b) \\ &\text{iff} \quad a < b. \end{aligned}$$

All of the properties above, like the associative, commutative, and distributive laws, hold for $\mathbb{R}$ since g is an isomorphism. Of course we use $+, \cdot, <$ now rather than $+', \cdot', <'$.