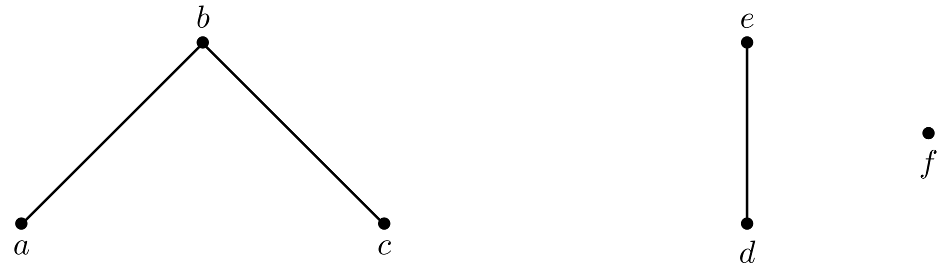
Solutions, exercise set 7

Chapter 8, exercise 1 We can do this with pictures:



$(A, <)$ $(B, \prec)$

We take the function which maps a to d, b to e, and c to d.

Officially, $A = \{a, b, c\}$ with a, b, c distinct objects, $<$ is $\{(a, b), (c, b)\}$, $B = \{d, e, f\}$ with d, e, f distinct objects not in A, and $\prec$ is $\{(d, e)\}$.

Chapter 8, exercise 2 Take the integers $\mathbb{Z}$ with their usual ordering, and let $f(i) = i+1$ for all $i \in \mathbb{Z}$.

Chapter 8, exercise 3 We want to apply 8.7 with A, B, C, D replaced by $\lambda \times \{0\}$, $\mu \times \{1\}$, $\lambda \times \{1\}$, and $\mu \times \{2\}$ respectively.

• $|\lambda \times \{0\}| = |\lambda \times \{1\}$: The mapping $(\alpha, 0) \mapsto (\alpha, 1)$ is clearly a bijection from $\lambda \times \{0\}$ to $\lambda \times \{1\}$.

• $|\mu \times \{1\}| = |\mu \times \{2\}|$: The mapping $(\alpha, 1) \mapsto (\alpha, 2)$ is clearly a bijection from $\mu \times \{1\}$ to $\mu \times \{2\}$.

• $(\lambda \times \{0\}) \cap (\mu \times \{1\}) = \emptyset$: suppose that $x \in (\lambda \times \{0\}) \cap (\mu \times \{1\})$. Then there are $\alpha < \lambda$ and $\beta < \mu$ such that $x = (\alpha, 0) = (\beta, 1)$, so $0 = 1$, contradiction.

• $(\lambda \times \{1\}) \cap (\mu \times \{2\}) = \emptyset$: suppose that $x \in (\lambda \times \{1\}) \cap (\mu \times \{2\})$. Then there are $\alpha < \lambda$ and $\beta < \mu$ such that $x = (\alpha, 1) = (\beta, 2)$, so $1 = 2$, contradiction.

This verifies the hypotheses of 8.7, so our result follows.

Chapter 8, exercise 4 The mapping $\alpha \mapsto (\alpha, 0)$ is clearly a bijection from $\lambda + \mu$ to $(\lambda + \mu) \times \{0\}$, so (2) holds.

Next, suppose that $x \in [(\lambda \times \{1\}) \cup (\mu \times \{2\})] \cap (\kappa \times \{0\})$. Then there is an $\alpha < \kappa$ such that $x = (\alpha, 0)$, and either there is a $\beta < \lambda$ such that $x = (\beta, 1)$, or there is a $\gamma < \mu$ such that $x = (\gamma, 2)$; this leads to the conclusion that $1 = 0$ or $2 = 0$, contradiction. So, this checks (3).

Now we can prove (4):

$$\begin{aligned} \kappa + (\lambda + \mu) &= |(\kappa \times \{0\}) \cup [(\lambda + \mu) \times \{1\}]| \quad \text{by the definition of } +, \\ &= |(\kappa \times \{0\}) \cup (\lambda \times \{1\}) \cup (\mu \times \{2\})| \quad \text{by (2), (3), 8.7.} \end{aligned}$$