11. More cardinal arithmetic

We now extend the cardinal arithmetic of chapters 7 and 8. We first give some results about sequences of ordinals. These again are theorems about class functions. We say that F is an *ordinal class function* iff F applies to any ordinal number α to produce another ordinal number $F(\alpha)$. An ordinal class function F is *strictly increasing* iff for any ordinals α, β, if $\alpha < \beta$ then $F(\alpha) < F(\beta)$.

Proposition 11.1. *If F is a strictly increasing ordinal class function, then $\alpha \leq F(\alpha)$ for every ordinal α.*

Proof. We prove this by transfinite induction. Suppose that $\xi \leq F(\xi)$ for every ordinal $\xi < \eta$. If $\xi < \eta$, then $\xi \leq F(\xi) < F(\eta)$. Put another way, $\xi \in F(\eta)$ for every $\xi \in \eta$, i.e., $\eta \subseteq F(\eta)$, and hence $\eta \leq F(\eta)$. By the transfinite induction principle, our statement holds for every ordinal. □

An ordinal class function F is *continuous* iff for every limit ordinal α, $F(\alpha) = \bigcup_{\beta<\alpha} F(\beta)$.

Proposition 11.2. *If F is a continuous ordinal class function, and $F(\alpha) < F(\alpha +_o 1)$ for every ordinal α, then F is strictly increasing.*

Proof. Fix an ordinal γ, and suppose that there is an ordinal δ with $\gamma < \delta$ and $F(\delta) \leq F(\gamma)$; we want to get a contradiction. Take the least such δ.

Case 1. $\delta = \theta +_o 1$ for some θ. Thus $\gamma \leq \theta$. If $\gamma = \theta$, then $F(\gamma) < F(\delta)$ by the hypothesis of the proposition, contradicting our supposition. Hence $\gamma < \theta$. Hence $F(\gamma) < F(\theta)$ by the minimality of δ, and $F(\theta) < F(\delta)$ by the assumption of the proposition, so $F(\gamma) < F(\delta)$, contradiction.

Case 2. δ is a limit ordinal. Then there is a $\theta < \delta$ with $\gamma < \theta$, and so by the minimality of δ we have

$$F(\gamma) < F(\theta) \leq \bigcup_{\varepsilon<\delta} F(\varepsilon) = F(\delta),$$

contradiction. □

Proposition 11.3. *Suppose that F is a strictly increasing and continuous ordinal class function, and ξ is a limit ordinal. Then $F(\xi)$ is a limit ordinal too.*

Proof. Suppose that $\gamma < F(\xi)$. Thus $\gamma \in \bigcup_{\eta<\xi} F(\eta)$, so there is a $\eta < \xi$ such that $\gamma < F(\eta)$. Now $F(\eta) < F(\xi)$. Hence $F(\xi)$ is a limit ordinal. □

Proposition 11.4. *Suppose that F and G are strictly increasing and continuous ordinal class functions. Then also $G \circ F$ is strictly increasing and continuous.*

Proof. Clearly $G \circ F$ is strictly increasing. Now suppose that ξ is a limit ordinal. Then $F(\xi)$ is a limit ordinal by 11.3.

Suppose that $\rho < \xi$. Then $F(\rho) < F(\xi)$, so $G(F(\rho)) \leq \bigcup_{\eta<F(\xi)} G(\eta) = G(F(\xi))$. Thus

$$(*) \qquad \bigcup_{\rho<\xi} G(F(\rho)) \leq G(F(\xi)).$$

Now if $\eta < F(\xi)$, then by the continuity of F, $\eta < \bigcup_{\rho<\xi} F(\rho)$, and hence there is a $\rho < \xi$ such that $\eta < F(\rho)$; so $G(\eta) < G(F(\rho))$. So for any $\eta < F(\xi)$ we have $G(\eta) \leq \bigcup_{\rho<\xi} G(F(\rho))$. Hence

$$G(F(\xi)) = \bigcup_{\eta<F(\xi)} G(\eta) \leq \bigcup_{\rho<\xi} G(F(\rho));$$

together with $(*)$ this gives the continuity of $G \circ F$. □

Now we continue the study of cardinal arithmetic, by discussing infinite products of cardinals, and a theorem which goes far towards describing just how exponentiation can be calculated.

By 7.31 we know that for every cardinal κ there is a larger cardinal. We let κ^+ be the least cardinal number greater than κ. We say that λ is a *successor cardinal* iff there is a κ such that $\lambda = \kappa^+$. If λ is not a successor cardinal and is not 0, then we say that λ is a *limit cardinal*.

We can now define the standard sequence of infinite cardinal numbers, by transfinite recursion:

$$\begin{aligned} \aleph_0 &= \omega; \\ \aleph_{\alpha+1} &= \aleph_\alpha^+; \\ \aleph_\beta &= \bigcup_{\alpha<\beta} \aleph_\alpha \quad \text{for } \beta \text{ a limit ordinal.} \end{aligned}$$

For historical reasons, one sometimes writes ω_α in place of $\aleph_\alpha$. Note that $\aleph$ is a continous class function, defined for every ordinal.

Lemma 11.5. *If $\alpha < \beta$, then $\aleph_\alpha < \aleph_\beta$.* □

Proof. By 11.2 and the definition. □

Lemma 11.6. *$\alpha \leq \aleph_\alpha$ for every ordinal α.*

Proof. By 11.5 and 11.1. □

Theorem 11.7. *For every infinite cardinal κ there is an ordinal α such that $\kappa = \aleph_\alpha$.*

Proof. Let κ be any infinite cardinal. Then $\kappa \leq \aleph_\kappa < \aleph_{\kappa+_o 1}$ by 11.5 and 11.6. This shows that there is an ordinal α such that $\kappa < \aleph_\alpha$; choose the least such α. Clearly $\alpha \neq 0$ and α is not a limit ordinal. Say $\alpha = \beta + 1$. Then $\aleph_\beta \leq \kappa < \aleph_{\beta+1}$, so $\kappa = \aleph_\beta$. □

This theorem shows that there are as many cardinals as there are ordinals; namely, both "collections" are too big to be sets, so they are really just properties of sets. (See 9.3.)

We can now say a little more about the continuum hypothesis. Not only is it consistent that it fails, but it is even consistent that $|\mathscr{P}(\omega)| = \aleph_2$, or $|\mathscr{P}(\omega)| = \aleph_{17}$, or $|\mathscr{P}(\omega)| = \aleph_{\omega+1}$; the possibilities have been spelled out in great detail. However, it is not possible to have $\mathscr{P}(\omega)| = \aleph_\omega$; we will establish this later in this chapter.

By chapter 8, the binary operations of addition and multiplication of cardinals are trivial when applied to infinite cardinals; and the infinite sum is also easy to calculate. We now introduce infinite products which, as we shall see, are not so trivial.

The infinite product of a system of cardinals will be denoted by $\prod_{i\in I}\kappa_i$. This is different from the cartesian product of sets introduced in chapter 3. To reduce confusion we shall use the notation $\mathsf{X}_{i\in I}\kappa_i$ for the cartesian product of sets whenever there is a possibility of confusion. Thus

$$\mathsf{X}_{i\in I}X_i=\{f : f \text{ is a function and } \mathrm{dmn}(f)=I \text{ and } f(i)\in X_i \text{ for all } i\in I\}.$$

If $\langle\kappa_i : i\in I\rangle$ is a system of cardinals, we define

$$\prod_{i\in I}\kappa_i=\left|\mathsf{X}_{i\in I}\kappa_i\right|.$$

Some elementary properties of this notion are summarized in the following proposition.

Proposition 11.8. *(i)* $\left|\mathsf{X}_{i\in I}A_i\right|=\prod_{i\in I}|A_i|$.

(ii) If $\kappa_i=0$ *for some* $i\in I$, *then* $\prod_{i\in I}\kappa_i=0$.

(iii) $\prod_{i\in 0}\kappa_i=1$.

(iv) $\prod_{i\in I}\kappa_i=\prod_{i\in I,\kappa_i\neq 1}\kappa_i$.

(v) $\prod_{i\in I}1=1$.

(vi) If $\kappa_i\le\lambda_i$ *for all* $i\in I$, *then* $\prod_{i\in I}\kappa_i\le\prod_{i\in I}\lambda_i$.

(vii) $\prod_{i\in 2}\kappa_i=\kappa_0\cdot\kappa_1$.

(viii) $\prod_{i\in I}\kappa=\kappa^{|I|}$.

(ix) $\kappa^{\sum_{i\in I}\lambda_i}=\prod_{i\in I}\kappa^{\lambda_i}$.

(x) $\left(\prod_{i\in I}\kappa_i\right)^{\lambda}=\prod_{i\in I}\kappa_i^{\lambda}$.

Proof. Some details here are left to the exercises.

(i): According to the definition of product, it suffices to find a one-one function g mapping $\mathsf{X}_{i\in I}A_i$ onto $\mathsf{X}_{i\in I}|A_i|$. For each $i\in I$, let f_i be a one-one function mapping A_i onto $|A_i|$. (We are using the axiom of choice here.) Then for each $x\in\mathsf{X}_{i\in I}A_i$ and each $i\in I$ let $g(x)_i=f_i(x_i)$. It is easily checked that g is as desired.

(ii): Under the hypothesis here, the set $\mathsf{X}_{j\in I}\kappa_j$ is empty, since a member x of it would have to satisfy $x_i\in 0$, and 0 is the empty set. So (ii) follows by definition.

(iii): This is true because the only member of $\mathsf{X}_{i\in 0}\kappa_i$ is the empty set.

(iv): by definition, it suffices to find a one-one function mapping

$$\mathsf{X}_{i\in I}\kappa_i \quad \text{onto} \quad \mathsf{X}_{\substack{i\in I,\\ \kappa_i\neq 1}}\kappa_i.$$

For each $f\in\mathsf{X}_{i\in I}\kappa_i$, let $F(f)=f\restriction\{i\in I : \kappa_i\neq 1\}$. It is easy to see that this works. Note that both products of sets are empty if some κ_i is 0.

(v): this works because there is only one member of $\mathsf{X}_{i\in I}1$, namely the function which assigns 0 to each member of I.

(vi): this is true because, clearly,

$$\mathsf{X}_{i\in I}\kappa_i \subseteq \mathsf{X}_{i\in I}\lambda_i.$$

(vii): here it suffices to define a bijection between $\mathsf{X}_{i\in 2}\kappa_i$ and $\kappa_0 \times \kappa_1$. For each $f \in \mathsf{X}_{i\in 2}\kappa_i$ let $g(f) = (f(0), f(1))$. It is easy to check that g is as desired.

(viii): Since $\mathsf{X}_{i\in I}\kappa = {}^I\kappa$, the conclusion here is clear.

(ix): It suffices to show that the following two sets are equipotent:

$$(1) \qquad {}^{\{(\alpha,i):i\in I \text{ and } \alpha<\lambda_i\}}\kappa$$

and

$$(2) \qquad \mathsf{X}_{i\in I}\left({}^{\lambda_i}\kappa\right).$$

So, take any member f of (1), and take any $i \in I$ and $\alpha < \lambda_i$. Define

$$(F(f))_i(\alpha) = f(\alpha, i);$$

it is straightforward to check that F is a one-one function mapping the set (1) onto the set (2).

Finally, for (x), it suffices to show that the following two sets are equipotent:

$$(3) \qquad {}^{\lambda}\left(\mathsf{X}_{i\in I}\lambda_i\right)$$

and

$$(4) \qquad \mathsf{X}_{i\in I}\left({}^{\lambda}\kappa_i\right).$$

So, take any f in the set (3), any $i \in I$, and any $\alpha < \lambda$. We define

$$(F(f))_i(\alpha) = (f(\alpha))_i.$$

It is straightforward to check that F is a one-one function mapping the set (3) onto the set (4). □

General commutative, associative, and distributive laws hold also:

Proposition 11.9. (Commutative law) *If $\langle \kappa_i : i \in I\rangle$ is a system of cardinals and $f : I \to I$ is one-one and onto, then*

$$\prod_{i\in I}\kappa_i = \prod_{i\in I}\kappa_{f(i)}.$$ □

Proof. For each $x \in \bigtimes_{i\in I}\kappa_i$ define $F(x) \in \bigtimes_{i\in I}\kappa_{f(i)}$ by setting, for any $i \in I$, $(F(x))(i) = x(f(i))$. Clearly $F : \bigtimes_{i\in I}\kappa_i \to \bigtimes_{i\in I}\kappa_{f(i)}$. To see that F is one-one, suppose that $F(x) = F(y)$. Take any $i \in I$. Then

$$x(i) = x(f(f^{-1}(i))) = (F(x))(f^{-1}(i)) = (F(y))(f^{-1}(i)) = y(f(f^{-1}(i))) = y(i);$$

hence $x = y$. To see that F maps onto $\bigtimes_{i\in I}\kappa_{f(i)}$, take any $z \in \bigtimes_{i\in I}\kappa_{f(i)}$. Let $x(i) = z(f^{-1}(i))$ for all $i \in I$. Thus $x(i) \in \kappa_{f(f^{-1}(i))} = \kappa_i$, for each $i \in I$. So $x \in \bigtimes_{i\in I}\kappa_i$. Since for any $i \in I$ we have $(F(x))(i) = x(f(i)) = z(f^{-1}(f(i))) = z(i)$, it follows that $z = F(x)$. □

Proposition 11.10. (Associative law) *If $\langle \kappa_{ij} : (i,j) \in I \times J\rangle$ is a system of cardinals, then*

$$\prod_{i\in I}\left(\prod_{j\in J}\kappa_{ij}\right) = \prod_{(i,j)\in I\times J}\kappa_{ij}. \qquad \square$$

Proof. Using 11.8(i), we see that the left side of this equation is the number of elements of

$$(*) \qquad \bigtimes_{i\in I}\left(\bigtimes_{j\in J}\kappa_{ij}\right),$$

and the right side is the number of elements of

$$(**) \qquad \bigtimes_{(i,j)\in I\times J}\kappa_{ij}.$$

So it suffices to show that these two sets are equipotent. For any element x of $(*)$, let $(F(x))(i,j) = (x(i))(j)$; clearly then $F(x)$ is in $(**)$. To show that F is one-one, suppose that $F(x) = F(y)$. Fix $i \in I$. Then for any $y \in J$, $(x(i))(j) = (F(x))(i,j) = (F(y))(i,j) = (y(i))(j)$. This being true for all $j \in J$, we must have $x(i) = y(i)$. Since $i \in I$ is arbitrary, it follows that $x = y$.

For onto, suppose that z is a member of $(**)$. Define x in $(*)$ by setting $(x(i))(j) = z(i,j)$ for any $i \in I$ and $j \in J$. Clearly $F(x) = z$. □

We now leave the very elementary portion of cardinal arithmetic and start working towards the fundamental theorem explaining (to a great extent) cardinal exponentiation.

The following theorem gives a fundamental fact about infinite products and sums.

Theorem 11.11. (König) *Suppose that $\langle\kappa_i : i \in I\rangle$ and $\langle\lambda_i : i \in I\rangle$ are systems of cardinals such that $\lambda_i < \kappa_i$ for all $i \in I$. Then*

$$\sum_{i\in I}\lambda_i < \prod_{i\in I}\kappa_i.$$

Proof. The proof is another instance of Cantor's diagonal argument. Suppose that this is not true; thus $\prod_{i\in I}\kappa_i \le \sum_{i\in I}\lambda_i$. It follows that there is a one-one function f mapping $\bigtimes_{i\in I}\kappa_i$ into $\{(\alpha,i) : i \in I, \alpha < \lambda_i\}$. For each $i \in I$ let

$$K_i = \{(f^{-1}(\alpha,i))_i : \alpha < \lambda_i, (\alpha,i) \in \mathrm{rng}(f)\}.$$

Clearly $K_i \subseteq \kappa_i$. Now $|K_i| \le \lambda_i < \kappa_i$, so we can choose $x_i \in \kappa_i \backslash K_i$ (using the axiom of choice). Say $f(x) = (\alpha,i)$. Then $x_i = (f^{-1}(\alpha,i))_i \in K_i$, contradiction. □

Cofinality, and
regular and singular cardinals

Further cardinal arithmetic depends on the notion of cofinality. A subset Γ of an ordinal α is *unbounded* in α provided that for every $\xi < \alpha$ there is a $\eta \in \Gamma$ such that $\xi \leq \eta$. We define the *cofinality* of an ordinal α to be the ordinal

$$\mathrm{cf}(\alpha) \stackrel{\mathrm{def}}{=} \min\{\beta : \text{ there is an unbounded subset } \Gamma \text{ of } \alpha \text{ of order type } \beta\}.$$

Note that this is really only of interest if α is a limit ordinal, since clearly $\mathrm{cf}(0) = 0$ and $\mathrm{cf}(\beta + 1) = 1$.

Proposition 11.12. *If α is a limit ordinal and $\Gamma \subseteq \alpha$, then Γ is unbounded in α iff $\bigcup \Gamma = \alpha$.*

Proof. $\Rightarrow$: Suppose that Γ is unbounded in α. If $\beta \in \bigcup \Gamma$, then $\beta \in \gamma$ for some $\gamma \in \Gamma$, so that $\gamma < \alpha$, and hence $\beta < \alpha$. Thus $\bigcup \Gamma \leq \alpha$. On the other hand, suppose that $\beta < \alpha$. Then also $\beta +_o 1 < \alpha$ since α is a limit ordinal. Choose $\delta \in \Gamma$ such that $\beta +_o 1 \leq \delta$. Then $\beta < \delta \in \Gamma$, so $\beta \in \bigcup \Gamma$. So $\alpha \leq \bigcup \Gamma$.

$\Leftarrow$: Assume that $\bigcup \Gamma = \alpha$, and suppose that $\beta < \alpha$. Choose $\gamma \in \Gamma$ such that $\beta < \gamma$. Then $\gamma < \alpha$, as desired. □

The following easy proposition will be useful.

Proposition 11.13. *For α a limit ordinal, $\mathrm{cf}(\alpha)$ is the least ordinal β such that there is a strictly increasing function $f : \beta \to \alpha$ such that $\mathrm{rng}(f)$ is unbounded in α.*

Proof. Let Γ be an unbounded subset of α of order type $\mathrm{cf}(\alpha)$. Let $g : \mathrm{cf}(\alpha) \to \Gamma$ be the isomorphism of $\mathrm{cf}(\alpha)$ onto Γ. Thus g is strictly increasing, and its range is Γ, which is unbounded in α. So the β mentioned in the proposition is $\leq \mathrm{cf}(\alpha)$. On the other hand, if f is as in the definition of β, then β is the order type of $\mathrm{rng}(f)$, and so $\mathrm{cf}(\alpha) \leq \beta$. □

Lemma 11.14. *If α is a limit ordinal and $\Gamma \subseteq \alpha$ is unbounded in α, then $\mathrm{cf}(\alpha) \leq |\Gamma|$.*

Proof. Let $\gamma = |\Gamma|$. Let g be a one-one function from γ onto Γ. We define a function $f : \gamma \to \alpha + 1$ by recursion, as follows. If $f(\delta)$ has been defined for all $\delta < \beta$, where $\beta < \gamma$, we define $f(\beta)$ as follows.

Case 1. There is a $\xi \in \Gamma$ such that $f(\delta) < \xi$ for all $\delta < \beta$, and also $g(\beta) < \xi$. We let $f(\beta)$ be the least such ξ.

Case 2. There is no such ξ. In this case, let $f(\beta) = \alpha$.

Now for all δ and β, if $\delta < \beta < \gamma$ and $f(\beta) \neq \alpha$, then also $f(\delta) \neq \alpha$, and $f(\delta) < f(\beta)$. In fact, the hypothesis implies that $f(\varepsilon) < f(\beta)$ for all $\varepsilon < \beta$, and $g(\beta) < f(\beta)$. In particular, $f(\delta) < f(\beta) < \alpha$.

Now we consider two possibilities:

(1) There is a $\beta < \gamma$ such that $f(\beta) = \alpha$. Take the least such β. Then $f[\beta] \subseteq \alpha$. We claim that $f[\beta]$ is unbounded in α. For, suppose that $f(\delta) < \eta < \alpha$ for all $\delta < \beta$. Let $\xi = \max(\eta, g(\beta)+1)$. Then $f(\delta) < \xi$ for each $\delta < \beta$, and also $g(\beta) < \xi$. Thus by definition,

$f(\beta) < \alpha$, contradiction. Thus our claim holds. By Proposition 11.13, $\mathrm{cf}(\alpha) \le \beta < \gamma$, as desired.

(2) There is no $\beta < \gamma$ such that $f(\beta) = \alpha$. Then we claim that $\mathrm{rng}(f)$ is unbounded in α. For, take any $\eta < \alpha$. Choose $\xi \in \Gamma$ such that $\eta < \xi$. Say $g(\beta) = \xi$. Now $g(\beta) < f(\beta)$. So the claim holds. By Proposition 11.13, $\mathrm{cf}(\alpha) \le \gamma$, as desired. □

Proposition 11.15. *If α is a limit ordinal, then $\mathrm{cf}(\alpha) = \min\{|\Gamma| : \Gamma$ is an unbounded subset of $\alpha\}$. Hence $\mathrm{cf}(\alpha)$ is a cardinal.*

Proof. $\le$ holds by Lemma 11.14. On the other hand, choose an unbounded subset Γ of α such that $\mathrm{cf}(\alpha)$ is the order type of Γ. Then $|\Gamma| \le \mathrm{cf}(\alpha)$. This proves $\ge$. □

Proposition 11.16. $\mathrm{cf}(\alpha) \le \alpha$ *for any ordinal α.* □

Proposition 11.17. $\mathrm{cf}(\mathrm{cf}(\alpha)) = \mathrm{cf}(\alpha)$ *for any ordinal α.*

Proof. This is obvious for $\alpha = 0$ or α a successor ordinal, so assume that α is a limit ordinal. Then $\le$ holds by 11.16. Now suppose that $\Gamma \subseteq \mathrm{cf}(\alpha)$ is unbounded and has order type $\beta \stackrel{\text{def}}{=} \mathrm{cf}(\mathrm{cf}(\alpha))$. Let $f : \mathrm{cf}(\alpha) \to \alpha$ be strictly increasing such that $\mathrm{rng}(f)$ is unbounded in α (by 11.13); and let $g : \beta \to \mathrm{cf}(\alpha)$ be strictly increasing with $\mathrm{rng}(g)$ unbounded in $\mathrm{cf}(\alpha)$. Then $f \circ g : \beta \to \alpha$ and $\mathrm{rng}(f \circ g)$ is unbounded in α. In fact, given $\beta < \alpha$, choose $\gamma < \mathrm{cf}(\alpha)$ such that $\beta \le f(\gamma)$. Then choose $\delta < \beta$ such that $\gamma \le g(\delta)$. Then $\beta \le f(\gamma) \le f(g(\delta)) = (f \circ g)(\delta)$, as desired. This shows that $\mathrm{cf}\alpha \le \beta$, as desired. □

We call an ordinal α *regular* provided that $\mathrm{cf}\alpha = \alpha$; otherwise α is called *singular*. Usually we are interested in these notions only when α is a cardinal. The main reason for this is the following:

Theorem 11.18. *Every regular ordinal is a cardinal.*

Proof. By 11.15. □

Note that if κ is regular and $\Gamma \subseteq \kappa$ with $|\Gamma| < \kappa$, then $\bigcup \Gamma < \kappa$ by 11.15.

Proposition 11.19. *A cardinal κ is regular iff for every system $\langle \lambda_i : i \in I \rangle$ of cardinals less than κ, with $|I| < \kappa$, one also has $\sum_{i \in I} \lambda_i < \kappa$.*

Proof. $\Rightarrow$: $\sum_{i \in I} \lambda_i \le |I| \cdot \bigcup_{i \in I} \lambda_i < \kappa$. $\Leftarrow$: if $\Gamma \subseteq \kappa$ and $|\Gamma| < \kappa$, then $|\bigcup \Gamma| \le \sum_{\lambda \in \Gamma} |\lambda| < \kappa$, so also $\bigcup \Gamma < \kappa$. Thus κ is regular by 11.15. □

The following theorem gives the single most important fact about regular cardinals:

Theorem 11.20. *For every infinite cardinal κ, the cardinal κ^+ is regular.*

Proof. Suppose that $\Gamma \subseteq \kappa^+$, Γ is unbounded in κ^+, and the order type of Γ is less than κ^+. In particular, $|\Gamma| < \kappa^+$. Hence

$$\kappa^+ = \left| \bigcup_{\gamma \in \Gamma} \gamma \right| \le \sum_{\gamma \in \Gamma} |\gamma| \le \sum_{\gamma \in \Gamma} \kappa = \kappa \cdot \kappa = \kappa,$$

contradiction. The first equality here holds because Γ is unbounded in κ^+ and κ^+ is a limit ordinal. $\square$

This theorem almost tells the full story about when a cardinal is regular. Examples of singular cardinals are $\aleph_{\omega+\omega}$ and $\aleph_{\omega_1}$. But it is conceivable that there are regular cardinals not covered by Theorem 11.20. A cardinal κ is a *limit cardinal* if it is infinite and there is no greatest cardinal less than it. This is equivalent to saying that it does not have the form λ^+. For example, $\aleph_0$ and $\aleph_{\omega_1}$ are limit cardinals. A regular limit cardinal is said to be *weakly inaccessible*. A cardinal κ is said to be *inaccessible* if it is regular and has the property that for any cardinal $\lambda < \kappa$, also $2^\lambda < \kappa$. Clearly every inaccessible cardinal is also weakly inaccessible. Under GCH, the two notions coincide. It is consistent with ZFC that 2^ω is weakly inaccessible; but of course it definitely is not inaccessible. It is consistent with ZFC that there are no uncountable weak inaccessibles at all. But it is reasonable to postulate their existence, and they are useful in some situations. In fact, the subject of *large cardinals* is one of the most studied in contemporary set theory, with many spectacular results.

The main theorem of cardinal arithmetic

Now we return to the general treatment of cardinal arithmetic.

Theorem 11.21. (König) *If κ is infinite and* $\mathrm{cf}\kappa \leq \lambda$, *then* $\kappa^\lambda > \kappa$.

Proof. Let $\Gamma \subseteq \kappa$ be unbounded and of order type $\mathrm{cf}(\kappa)$. Let $\langle \nu_\xi : \xi < \mathrm{cf}(\kappa)\rangle$ be an isomorphism of $\mathrm{cf}(\kappa)$ onto Γ. Then, using 11.11,

$$\kappa = \left|\bigcup \Gamma\right| = \left|\bigcup_{\xi<\mathrm{cf}(\kappa)} \nu_\xi\right| \leq \sum_{\xi<\mathrm{cf}(\kappa)} |\nu_\xi| < \prod_{\xi<\mathrm{cf}(\kappa)} \kappa = \kappa^{\mathrm{cf}(\kappa)} \leq \kappa^\lambda. \qquad \square$$

Corollary 11.22. *For λ infinite we have* $\mathrm{cf}(2^\lambda) > \lambda$.

Proof. Suppose that $\mathrm{cf}(2^\lambda) \leq \lambda$. Then by 11.21, $(2^\lambda)^\lambda > 2^\lambda$. But $(2^\lambda)^\lambda = 2^{\lambda\cdot\lambda} = 2^\lambda$, contradiction. $\square$

We can now verify a statement made earlier about possibilities for $|\mathscr{P}(\omega)|$. Since $|\mathscr{P}(\omega)| = 2^\omega$, the corollary says that $\mathrm{cf}(2^\omega) > \omega$. So this implies that $|\mathscr{P}(\omega)|$ cannot be $\aleph_\omega$ or $\aleph_{\omega+_o\omega}$. ($\omega +_o \omega$ is the ordinal sum introduced below.) It rules out many other possibilities of this sort.

We now prove a lemma needed for the last major theorem of this subsection, which says how to compute exponents (in a way).

Lemma 11.23. *If κ is a limit cardinal and $\lambda \geq \mathrm{cf}\kappa$, then* $\kappa^\lambda = \left(\bigcup_{\mu<\kappa} \mu^\lambda\right)^{\mathrm{cf}\kappa}$.

Proof. Let $\gamma : \mathrm{cf}\kappa \to \kappa$ be strictly increasing with $\mathrm{rng}(\gamma)$ unbounded in κ. We define $F : {}^\lambda\kappa \to \prod_{\alpha<\mathrm{cf}\kappa} {}^\lambda\gamma_\alpha$ as follows. If $f \in {}^\lambda\kappa$, $\alpha < \mathrm{cf}\kappa$, and $\beta < \lambda$, then

$$((F(f))_\alpha)_\beta = \begin{cases} f(\beta) & \text{if } f(\beta) < \gamma_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Now F is a one-one function. For, if $f, g \in {}^\lambda\kappa$ and $f \neq g$, say $\beta < \lambda$ and $f(\beta) \neq g(\beta)$. Choose $\alpha < \mathrm{cf}\kappa$ such that $f(\beta)$ and $g(\beta)$ are both less than γ_α. Then $((F(f))_\alpha)_\beta = f(\beta) \neq g(\beta) = ((F(g))_\alpha)_\beta$, from which it follows that $F(f) \neq F(g)$. Since F is one-one,

$$\begin{aligned}
\kappa^\lambda = |{}^\lambda\kappa| &\leq \left|\prod_{\alpha<\mathrm{cf}\kappa} {}^\lambda\gamma_\alpha\right| \\
&\leq \left|\prod_{\alpha<\mathrm{cf}\kappa} \left(\bigcup_{\mu<\kappa} {}^\lambda\mu\right)\right| \\
&= \left(\bigcup_{\mu<\kappa} \mu^\lambda\right)^{\mathrm{cf}\kappa} \\
&\leq (\kappa^\lambda)^{\mathrm{cf}\kappa} = \kappa^{\lambda\cdot\mathrm{cf}\kappa} = \kappa^\lambda,
\end{aligned}$$

and the lemma follows. $\square$

The following theorem is not needed for the main result, but it is a classical result about exponentiation.

Theorem 11.24. (Hausdorff) *If κ and λ are infinite cardinals, then $(\kappa^+)^\lambda = \kappa^\lambda \cdot \kappa^+$.*

Proof. If $\kappa^+ \leq \lambda$, then both sides are equal to 2^λ. Suppose that $\lambda < \kappa^+$. Then

$$\begin{aligned}
(\kappa^+)^\lambda = |{}^\lambda(\kappa^+)| &= \left|\bigcup_{\alpha<\kappa^+} {}^\lambda\alpha\right| \\
&\leq \sum_{\alpha<\kappa^+} |\alpha|^\lambda \leq \kappa^\lambda \cdot \kappa^+ \leq (\kappa^+)^\lambda,
\end{aligned}$$

as desired. $\square$

Here is the promised theorem giving computation rules for exponentiation. It essentially reduces the computation of κ^λ to two special cases: 2^λ, and $\kappa^{\mathrm{cf}\kappa}$. Generalizations of the results mentioned about the continuum hypothesis give a pretty good picture of what can happen to 2^λ. The case of $\kappa^{\mathrm{cf}\kappa}$ is more complicated, and there is still work being done on what the possibilities here are. Recent deep work of Shelah on pcf theory has shed some light on this. For example, he showed that $\aleph_\omega^{\aleph_0} \leq 2^{\aleph_0} + \aleph_{\omega_4}$. The role of ω_4 here is still unclear.

Theorem 11.25. (Main theorem of cardinal arithmetic) *Let κ and λ be cardinals with $2 \leq \kappa$ and $\lambda \geq \omega$. Then*

(i) If $\kappa \leq \lambda$, then $\kappa^\lambda = 2^\lambda$.

(ii) If κ is infinite and there is a $\mu < \kappa$ such that $\mu^\lambda \geq \kappa$, then $\kappa^\lambda = \mu^\lambda$.

(iii) Assume that κ is infinite and $\mu^\lambda < \kappa$ for all $\mu < \kappa$. Then $\lambda < \kappa$, and:

(a) if $\mathrm{cf}\kappa > \lambda$, then $\kappa^\lambda = \kappa$;

(b) if $\mathrm{cf}\kappa \leq \lambda$, then $\kappa^\lambda = \kappa^{\mathrm{cf}\kappa}$.

Proof. (i) has already been noted in 8.22. Under the hypothesis of (ii),

$$\kappa^\lambda \le (\mu^\lambda)^\lambda = \mu^\lambda \le \kappa^\lambda,$$

as desired.

Now assume the hypothesis of (iii). In particular, $2^\lambda < \kappa$, so of course $\lambda < \kappa$. Next, assume the hypothesis of (iii)(a): $\text{cf}\kappa > \lambda$. Then

$$\begin{aligned}\kappa^\lambda = |{}^\lambda\kappa| &= \left|\bigcup_{\alpha<\kappa} {}^\lambda\alpha\right| \quad \text{(since } \lambda < \text{cf}\kappa)\\ &\le \sum_{\alpha<\kappa} |\alpha|^\lambda \le \kappa,\end{aligned}$$

giving the desired result.

Finally, assume the hypothesis of (iii)(b): $\text{cf}\kappa \le \lambda$. Since $\lambda < \kappa$, it follows that κ is singular, so in particular it is a limit cardinal. Then, using Lemma 11.23,

$$\kappa^\lambda = \left(\bigcup_{\mu<\kappa} \mu^\lambda\right)^{\text{cf}(\kappa)} \le \kappa^{\text{cf}(\kappa)} \le \kappa^\lambda,$$

finishing the proof. □

Now 11.25 can be used to compute any power κ^λ with λ infinite and κ at least 2, as follows. If $\kappa \le \lambda$, then $\kappa^\lambda = 2^\lambda$ by 11.25(i). If there is a $\mu < \kappa$ such that $\mu^\lambda \ge \kappa$, take the least such μ. Then by 11.25(ii), we have $\kappa^\lambda = \mu^\lambda$. If $\nu < \mu$, then $\nu^\lambda < \mu$, since if $\nu^\lambda \ge \mu$, then $\nu^\lambda \ge \mu^\lambda \ge \kappa$, contradicting the minimality of μ. Thus μ is either $\le \lambda$, or satisfies the conditions in 11.25(iii). Finally, if κ satisfies the conditions in (iii), then (iii) gives the value of κ^λ (at least in terms of $\kappa^{\text{cf}(\kappa)}$).

Under the generalized continuum hypothesis the computation of exponents is very simple:

Corollary 11.26. *Assume GCH, and suppose that κ and λ are cardinals with $2 \le \kappa$ and λ infinite. Then:*

(i) If $\kappa \le \lambda$, then $\kappa^\lambda = \lambda^+$.

(ii) If $\text{cf}(\kappa) \le \lambda < \kappa$, then $\kappa^\lambda = \kappa^+$.

(iii) If $\lambda < \text{cf}(\kappa)$, then $\kappa^\lambda = \kappa$.

Proof. (i) is immediate from 11.25(i). For (ii), assume that $\text{cf}(\kappa) \le \lambda < \kappa$. Then κ is a limit cardinal, and so for each $\mu < \kappa$ we have $\mu^\lambda \le (\max(\mu, \lambda))^+ < \kappa$; hence by 11.25(iii)(b) we have $\kappa^\lambda = \kappa^{\text{cf}(\kappa)} = \kappa^+$, using 11.24. For (iii), assume that $\lambda < \text{cf}(\kappa)$. If there is a $\mu < \kappa$ such that $\mu^\lambda \ge \kappa$, then by 11.25(ii), $\kappa^\lambda = \mu^\lambda \le (\max(\lambda, \mu))^+ \le \kappa$, as desired. If $\mu^\lambda < \kappa$ for all $\mu < \kappa$, then $\kappa^\lambda = \kappa$ by 11.25(iii)(a). □

Exercises, Chapter 11

1. Finish the proof of 11.8(i).
2. Finish the proof of 11.8(iv).
3. Finish the proof of 11.8(vii).
4. Finish the proof of 11.8(ix).
5. Finish the proof of 11.8(x).
6. Prove the following general distributive law:

$$\prod_{i\in I}\sum_{j\in J_i}\kappa_{ij}=\sum_{f\in P}\prod_{i\in I}\kappa_{i,f(i)},$$

where $P = \bigtimes_{i\in I} J_i$.

7. Show that for any ordinal α, $|\alpha|$ is the smallest ordinal equipotent with α.
8. Show that for any cardinal κ we have $\kappa^+ = \{\alpha : \alpha \text{ is an ordinal and } |\alpha| \leq \kappa\}$.
9. For every infinite cardinal λ there is a cardinal $\kappa > \lambda$ such that $\kappa^\lambda = \kappa$.
10. For every infinite cardinal λ there is a cardinal $\kappa > \lambda$ such that $\kappa^\lambda > \kappa$.
11. Show that if β is a limit ordinal and $\langle \kappa_\xi : \xi < \beta\rangle$ is a strictly increasing sequence of cardinals, then $\sum_{\xi<\beta}\kappa_\xi < \prod_{\xi<\beta}\kappa_\xi$.
12. Prove that for every $n \in \omega$, and every infinite cardinal κ, $\aleph_n^\kappa = 2^\kappa \cdot \aleph_n$.
13. Prove that $\aleph_\omega^{\aleph_1} = 2^{\aleph_1} \cdot \aleph_\omega^{\aleph_0}$.
14. Prove that $\aleph_\omega^{\aleph_0} = \prod_{n\in\omega}\aleph_n$.
15. Prove that for any infinite cardinal κ, $(\kappa^+)^\kappa = 2^\kappa$.
16. Show that if κ is an infinite cardinal and C is the collection of all cardinals less than κ, then $|C| \leq \kappa$.
17. Show that if κ is an infinite cardinal and C is the collection of all cardinals less than κ, then

$$2^\kappa = \left(\sum_{\nu\in C} 2^\nu\right)^{\mathrm{cf}(\kappa)}.$$

18. Prove that for any limit ordinal τ, $\prod_{\xi<\tau} 2^{\aleph_\xi} = 2^{\aleph_\tau}$.
19. Show that if κ is an infinite cardinal and C is the set of all cardinals $\leq \kappa$, then $\sum_{\mu\in C}\kappa^\mu = 2^\kappa$.
20. Show that if λ is an infinite cardinal, $\kappa = \lambda^+$, and C is the set of all cardinals $< \kappa$, then $\sum_{\mu\in C}\kappa^\mu = 2^\lambda$.
21. Assume that κ is an infinite cardinal, and $2^\lambda < \kappa$ for every cardinal $\lambda < \kappa$. Show that $2^\kappa = \kappa^{\mathrm{cf}(\kappa)}$.

12. Ordinal arithmetic

We introduce addition, multiplication, and exponentiation for ordinal numbers. These are different from the corresponding operations on cardinals. We cannot consider them as extensions of those operations. They do extend the operations on natural numbers, just like the cardinal operations do. But, for example, the commutative law for addition of ordinals can fail. The ordinal operations are not as important as the cardinal operations, but they are frequently of use.

We will use transfinite induction and recursion several times in this chapter, but always in a fairly elementary way. Although we give full proofs here, many of them are for independent reading. The basic properties of the ordinal operations are easy to use, and the proofs are a useful way to learn how transfinite inductions work. Thus in class we will only go over some of these proofs.

Now we turn to the main topic of this chapter, arithmetic of ordinals. We begin with addition. The definition extends the definition of addition of natural numbers. For clarity, we do not use the notation $\alpha +_o 1$, defined earlier to be $\alpha \cup \{\alpha\}$, until after $+_o$ has been officially defined.

Lemma 12.1. *For every ordinal α there is a unique class function F defined for every ordinal and having the following properties.*

(i) $F(0) = \alpha$.

(ii) For any ordinal β, $F(\beta \cup \{\beta\}) = F(\beta) \cup \{F(\beta)\}$.

(iii) For any limit ordinal β, $F(\beta) = \bigcup_{\gamma<\beta} F(\gamma)$.

Proof. We use the class version of transfinite recursion. The class function G is defined for every function k as follows.

$$G(k) = \begin{cases} \alpha & \text{if } k = \emptyset, \\ k(\beta) \cup \{k(\beta)\} & \text{if } \mathrm{dmn}(k) = \beta \cup \{\beta\} \text{ for some ordinal } \beta, \\ \bigcup_{\gamma<\beta} k(\gamma) & \text{if } \mathrm{dmn}(k) = \beta \text{ for some limit ordinal } \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Now by the class version of transfinite recursion, 9.14, there is a class function F, defined for all ordinals, such that $F(\beta) = G(F \restriction \beta)$ for every ordinal β. Then for any ordinal β,

$$\begin{aligned} F(0) &= G(F \restriction 0) = G(\emptyset) = \alpha; \\ F(\beta \cup \{\beta\}) &= G(F \restriction (\beta \cup \{\beta\})) \\ &= (F \restriction (\beta \cup \{\beta\}))(\beta) \cup \{(F \restriction (\beta \cup \{\beta\}))(\beta)\} \\ &= F(\beta) \cup \{F(\beta)\}; \quad \text{and for } \beta \text{ limit,} \\ F(\beta) &= G(F \restriction \beta) \\ &= \bigcup_{\gamma<\beta} (F \restriction \beta)(\gamma) \\ &= \bigcup_{\gamma<\beta} F(\gamma). \end{aligned}$$

This proves that F exists as in the lemma.

For uniqueness, suppose that F' also satisfies the conditions on F, i.e., assume that (1) $F'(0) = \alpha$, (2) for any ordinal β, $F'(\beta \cup \{\beta\}) = F'(\beta) \cup \{F'(\beta)\}$, and (3) for any limit ordinal β, $F'(\beta) = \bigcup_{\gamma<\beta} F'(\gamma)$. We show that $F(\beta) = F'(\beta)$ for every ordinal β by transfinite induction (where the inductive hypotheses should be clear):

$$\begin{aligned}
F(0) &= \alpha = F'(0);\\
F(\beta \cup \{\beta\}) &= F(\beta) \cup \{F(\beta)\}\\
&= F'(\beta) \cup \{F'(\beta)\}\\
&= F(\beta \cup \{\beta\});\\
F(\beta) &= \bigcup_{\gamma<\beta} F(\gamma)\\
&= \bigcup_{\gamma<\beta} F'(\gamma)\\
&= F'(\beta). \qquad \square
\end{aligned}$$

We denote the unique class function proved to exist in 12.1 by Pl_α. Then we define, for any ordinals α, β,,

$$\alpha +_o \beta = \mathrm{Pl}_\alpha(\beta).$$

Proposition 12.2. $\alpha +_o 1 = \alpha \cup \{\alpha\}$ *for any ordinal* α*.*

Here we are taking $\alpha +_o 1$ in the sense of the definition just given. This proposition shows that our earlier definition of $\alpha +_o 1$ is consistent with the new general definition.

Proof.

$$\alpha +_o 1 = \alpha +_o \{0\} = \mathrm{Pl}_\alpha(\{0\}) = \mathrm{Pl}_\alpha(0) \cup \{\mathrm{Pl}_\alpha(0)\} = \alpha \cup \{\alpha\}. \qquad \square$$

Using this proposition we can express the definition of $+_o$ as follows.

Proposition 12.3. *For any ordinals* α, β*,*

$$\begin{aligned}
\alpha +_o 0 &= \alpha;\\
\alpha +_o (\beta +_o 1) &= (\alpha +_o \beta) +_o 1;\\
\alpha +_o \beta &= \bigcup_{\gamma<\beta} (\alpha +_o \gamma) \quad \text{for } \beta \text{ limit.} \qquad \square
\end{aligned}$$

Before we proceed to simple properties of ordinal addition, we want to indicate an equivalent definition which helps the intuition.

Theorem 12.4. *For any ordinals* α, β *let*

$$\alpha \oplus \beta = (\alpha \times \{0\}) \cup (\beta \times \{1\}).$$

We define a relation $\prec$ on $\alpha \oplus \beta$ as follows. For any $x, y \in \alpha \oplus \beta$, $x \prec y$ iff one of the following three conditions holds:

(i) There are $\xi, \eta < \alpha$ such that $x = (\xi, 0)$, $y = (\eta, 0)$, and $\xi < \eta$.

(ii) There are $\xi, \eta < \beta$ such that $x = (\xi, 1)$, $y = (\eta, 1)$, and $\xi < \eta$.

(ii) There are $\xi < \alpha$ and $\eta < \beta$ such that $x = (\xi, 0)$ and $y = (\eta, 1)$.

Then $(\alpha \oplus \beta, \prec)$ is a well-order which is order-isomorphic to $\alpha +_o \beta$.

A simple picture helps to explain the construction in this theorem:

$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{\alpha}\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{\beta}$$

Thus a copy of β is put to the right of a copy of α. The purpose of the definition of $\alpha \oplus \beta$ is to make the copies of α and β disjoint.

Proof. Clearly $\prec$ is a well-order. We show by transfinite induction on β, with α fixed, that $(\alpha \oplus \beta, \prec)$ is order isomorphic to $\alpha +_o \beta$. For $\beta = 0$ we have $\alpha +_o \beta = \alpha +_o 0 = \alpha$, while $\alpha \oplus \beta = \alpha \oplus 0 = \alpha \times \{0\}$. Clearly $\xi \mapsto (\xi, 0)$ defines an order-isomorphism from α onto $(\alpha \times \{0\}, \prec)$. So our result holds for $\beta = 0$. Assume it for β, and suppose that f is an order-isomorphism from $\alpha +_o \beta$ onto $(\alpha \oplus \beta, \prec)$. Now the last element of $\alpha \oplus (\beta +_o 1)$ is $(\beta, 1)$, and the last element of $\alpha +_o (\beta +_o 1)$ is $\alpha +_o \beta$, so the function

$$f \cup \{(\alpha +_o \beta, (\beta, 1))\}$$

is an order-isomorphism from $\alpha +_o (\beta +_o 1)$ onto $\alpha \oplus (\beta +_o 1)$.

Now assume that β is a limit ordinal, and for each $\gamma < \beta$, the ordinal $\alpha +_o \gamma$ is isomorphic to $\alpha \oplus \gamma$. For each such γ let f_γ be the unique isomorphism from $\alpha +_o \gamma$ onto $\alpha \oplus \gamma$. Note that if $\gamma < \delta < \beta$, then $f_\delta \restriction \gamma$ is an isomorphism from $\alpha +_o \gamma$ onto $\alpha \oplus \gamma$; hence $f_\delta \restriction \gamma = f_\gamma$. It follows that

$$\bigcup_{\gamma < \beta} f_\gamma$$

is an isomorphism from $\alpha +_o \beta$ onto $\alpha \oplus \beta$, finishing the inductive proof. □

Theorem 12.5. *For natural numbers m, n, addition in the sense of chapter 6 coincides with the ordinal addition of this chapter.*

Proof. This is obvious by the recursive definitions of the two operations; they are the same for natural numbers. Officially, one can prove that they are the same by ordinary induction. □

Since we will not be concerned much with cardinal addition in this chapter, from now on we omit the subscript in $+_o$. Remember, though, that this is different from cardinal addition. For example, $\omega + \omega = \omega$ in the cardinal sense, but $\omega < \omega + \omega$ in the ordinal sense.

Now we state the most important facts about ordinal addition. An ordinal class function F is *weakly increasing* iff $\alpha < \beta$ implies that $F(\alpha) \leq F(\beta)$, for all ordinals α, β.

Theorem 12.6. *(i) For any ordinal α, the class function F which takes each ordinal β to $\alpha+\beta$ is strictly increasing and continuous.*

(ii) For any ordinal β, the class function F which takes each ordinal α to $\alpha+\beta$ is weakly increasing.

Now suppose that α, β, γ are arbitrary ordinals.

(iii) $\alpha+(\beta+\gamma)=(\alpha+\beta)+\gamma$.

(iv) $\beta \leq \alpha+\beta$.

(v) $0+\alpha=\alpha$.

(vi) $\alpha<\beta$ iff there is a $\delta>0$ such that $\alpha+\delta=\beta$.

Proof. (i): This is true by the definition of $+$ and 11.2, since $\alpha+\beta<\alpha+(\beta+1)$.

(ii): Suppose that $\alpha<\gamma$; we prove by transfinite induction that $\alpha+\beta \leq \gamma+\beta$ for any ordinal β. This is obviously true for $\beta=0$. Assume that $\alpha+\beta \leq \gamma+\beta$. Then

$$\alpha+\beta \leq \gamma+\beta<\gamma+(\beta+1),$$

so $\alpha+(\beta+1) \leq \gamma+(\beta+1)$ by 9.2. Now assume that β is a limit ordinal, and $\alpha+\delta \leq \gamma+\delta$ for every $\delta<\beta$. Then

$$\alpha+\beta=\bigcup_{\delta<\beta}(\alpha+\delta) \leq \bigcup_{\delta<\beta}(\gamma+\delta)=\gamma+\beta,$$

finishing the inductive proof.

(iii): Fix α and β; we proceed by induction on γ. The case $\gamma=0$ is obvious. Assume that $\alpha+(\beta+\gamma)=(\alpha+\beta)+\gamma$. Then

$$\begin{aligned}
\alpha+(\beta+(\gamma+1)) &= \alpha+((\beta+\gamma)+1) \\
&= (\alpha+(\beta+\gamma))+1 \\
&= ((\alpha+\beta)+\gamma)+1 \\
&= (\alpha+\beta)+(\gamma+1).
\end{aligned}$$

Finally, suppose that γ is a limit ordinal and we know our result for all $\delta<\gamma$. Let F, G, H be the ordinal class functions such that, for any ordinal δ,

$$\begin{aligned}
F(\delta) &= \alpha+\delta; \\
G(\delta) &= (\alpha+\beta)+\delta; \\
H(\delta) &= \beta+\delta.
\end{aligned}$$

Thus according to (i), all three of these functions are strictly increasing and continuous. Hence, using 11.4,

$$\begin{aligned}
\alpha+(\beta+\gamma) &= F(H(\gamma)) \\
&= \bigcup_{\delta<\gamma} F(H(\delta))
\end{aligned}$$

$$
\begin{aligned}
&= \bigcup_{\delta<\gamma} (\alpha + (\beta + \delta)) \\
&= \bigcup_{\delta<\gamma} ((\alpha + \beta) + \delta \\
&= \bigcup_{\delta<\gamma} G(\delta) \\
&= G(\gamma) \\
&= (\alpha + \beta) + \gamma.
\end{aligned}
\qquad \square
$$

(iv) follows from (i) and 11.1.

We prove (v) by induction on α: $0+0 = 0$. If it is true for α, then $0+(\alpha+1) = (0+\alpha)+1 = \alpha+1$. If it is true for every $\beta < \alpha$, α limit, then $0 + \alpha = \bigcup_{\beta<\alpha}(0 + \beta) = \bigcup_{\beta<\alpha} \beta = \alpha$.

(vi): By (iv) we have $\beta \leq \alpha + \beta$, so there is a γ such that $\beta \leq \alpha + \gamma$; choose the least such γ. Clearly $\beta < \alpha + \gamma$ gives a contradiction, so $\beta = \alpha + \gamma$, as desired. This proves $\Rightarrow$.

The direction $\Leftarrow$ in (vi) is true by (i). $\square$

We give one more useful result about ordinal addition.

Proposition 12.7. *α is infinite iff $1 + \alpha = \alpha$.*

Proof. If α is infinite, then $\omega \leq \alpha$, and hence there is an ordinal β such that $\omega+\beta = \alpha$, by 12.6(vi). Hence

$$
\begin{aligned}
1 + \alpha &= 1 + (\omega + \beta) \\
&= (1 + \omega) + \beta \\
&= \left(\bigcup_{m<\omega} (1 + m) \right) + \beta \\
&= \omega + \beta \\
&= \alpha.
\end{aligned}
\qquad \square
$$

From 12.7 it follows that ordinal addition is not commutative in general. In fact, clearly $\omega = 1 + \omega \neq \omega + 1$.

Now we turn to ordinal multiplication.

Lemma 12.8. *For every ordinal α there is a unique class function F defined for every ordinal and having the following properties.*

(i) $F(0) = 0$.

(ii) For any ordinal β, $F(\beta + 1) = F(\beta) + \alpha$.

(iii) For any limit ordinal β, $F(\beta) = \bigcup_{\gamma<\beta} F(\gamma)$.

Proof. We use the class version of transfinite recursion. The class function G is defined for every function k as follows.

$$
G(k) = \begin{cases}
0 & \text{if } k = \emptyset, \\
k(\beta) + \alpha & \text{if } \mathrm{dmn}(k) = \beta + 1 \text{ for some ordinal } \beta, \\
\bigcup_{\gamma<\beta} k(\gamma) & \text{if } \mathrm{dmn}(k) = \beta \text{ for some limit ordinal } \beta, \\
0 & \text{otherwise.}
\end{cases}
$$

Now by the class version of transfinite recursion, 9.14, there is a class function F, defined for all ordinals, such that $F(\beta) = G(F \restriction \beta)$ for every ordinal β. Then for any ordinal β,

$$\begin{aligned}
F(0) &= G(F \restriction 0) = G(\emptyset) = 0; \\
F(\beta + 1) &= G(F \restriction (\beta + 1)) \\
&= (F \restriction (\beta + 1))(\beta) + \alpha \\
&= F(\beta) + \alpha; \quad \text{and for } \beta \text{ limit,} \\
F(\beta) &= G(F \restriction \beta) \\
&= \bigcup_{\gamma<\beta} (F \restriction \beta)(\gamma) \\
&= \bigcup_{\gamma<\beta} F(\gamma).
\end{aligned}$$

This proves that F exists as in the lemma.

For uniqueness, suppose that F' also satisfies the conditions on F, i.e., assume that (1) $F'(0) = 0$, (2) for any ordinal β, $F'(\beta + 1) = F'(\beta) + \alpha$, and (3) for any limit ordinal β, $F'(\beta) = \bigcup_{\gamma<\beta} F'(\gamma)$. We show that $F(\beta) = F'(\beta)$ for every ordinal β by transfinite induction (where the inductive hypotheses should be clear):

$$\begin{aligned}
F(0) &= 0 = F'(0); \\
F(\beta + 1) &= F(\beta) \cdot \alpha \\
&= F'(\beta) \cdot \alpha \\
&= F'(\beta + 1); \\
F(\beta) &= \bigcup_{\gamma<\beta} F(\gamma) \\
&= \bigcup_{\gamma<\beta} F'(\gamma) \\
&= F'(\beta).
\end{aligned}$$

□

We denote the unique class function proved to exist in 12.8 by Ti_α. Then we define, for any ordinals $\alpha, \beta,$,

$$\alpha \odot \beta = \mathrm{ti}_\alpha(\beta).$$

Proposition 12.9. *For any ordinals α, β,*

$$\begin{aligned}
\alpha \odot 0 &= 0; \\
\alpha \odot (\beta + 1) &= (\alpha \odot \beta) + \alpha; \\
\alpha \odot \beta &= \bigcup_{\gamma<\beta} (\alpha \odot \gamma) \quad \text{for } \beta \text{ limit.}
\end{aligned}$$

□

Proposition 12.10. *Ordinal multiplication coincides with the multiplication of chapter 6 when restricted to natural numbers.*

Proof. Again this is clear from the definitions; a rigorous proof is easily supplied using ordinary induction. □

From now on we use $\cdot$ for ordinal multiplication in place of $\odot$. Note that this is different from cardinal multiplication, though. For example, $\omega \cdot \omega = \omega$ in the cardinal sense, but $\omega \cdot \omega > \omega$ in the ordinal sense.

The most basic properties of ordinal multiplication are given in the following theorem.

Theorem 12.11. *(i) If $\alpha \neq 0$, then $\alpha \cdot \beta < \alpha \cdot (\beta + 1)$;*

(ii) If $\alpha \neq 0$, then the class function assigning to each ordinal β the product $\alpha \cdot \beta$ is strictly increasing and continuous;

(iii) $0 \cdot \alpha = 0$;

(iv) If $\alpha, \beta \neq 0$, then $\alpha \cdot \beta \neq 0$;

(v) $\alpha \cdot (\beta + \gamma) = (\alpha \cdot \beta) + (\alpha \cdot \gamma)$;

(vi) $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$;

(vii) If $\alpha \neq 0$, then $\beta \leq \alpha \cdot \beta$;

(viii) If $\alpha < \beta$ then $\alpha \cdot \gamma \leq \beta \cdot \gamma$;

(ix) $\alpha \cdot 1 = 1 \cdot \alpha = \alpha$.

(x) $\alpha \cdot 2 = \alpha + \alpha$.

Proof. (i): $\alpha \cdot (\beta + 1) = \alpha \cdot \beta + \alpha > \alpha \cdot \beta + 0 = \alpha \cdot \beta$.

(ii): by (i) and 11.2.

(iii): this is an easy induction on α. With obvious assumptions, $0 \cdot 0 = 0$; $0 \cdot (\alpha + 1) = (0 \cdot \alpha) + 0 = 0 + 0 = 0$; $0 \cdot \beta = \bigcup_{\alpha<\beta}(0 \cdot \beta) = 0$.

(iv): by (ii).

For (v) and (vi) we introduce some temporary notation. For any ordinal α, let F_α be the class function such that $F_\alpha(\beta) = \alpha + \beta$ for every ordinal β, and let G_α be the class function such that $G_\alpha(\beta) = \alpha \cdot \beta$ for all β. Then F_α is continuous by definition, as is G_α. Also note that F_α is strictly increasing by 12.6(i), and G_α is strictly increasing when $\alpha \neq 0$, by (ii).

Now we turn to the proof of (v). Fix α and β. By (iii) we may assume that $\alpha \neq 0$; we then proceed by induction on γ. First of all,

$$\alpha \cdot (\beta + 0) = \alpha \cdot \beta = (\alpha \cdot \beta) + 0 = (\alpha \cdot \beta) + (\alpha \cdot 0),$$

so (v) holds for $\gamma = 0$. Now assume that (v) holds for γ. Then

$$\begin{aligned}
\alpha \cdot (\beta + (\gamma + 1)) &= \alpha \cdot ((\beta + \gamma) + 1) \\
&= \alpha \cdot (\beta + \gamma) + \alpha \\
&= (\alpha \cdot \beta) + (\alpha \cdot \gamma) + \alpha \quad \text{(induction hypothesis)} \\
&= (\alpha \cdot \beta) + (\alpha \cdot (\gamma + 1)),
\end{aligned}$$

as desired.

Finally, suppose that δ is a limit ordinal and we know (v) for all $\gamma < \delta$. Then

$$\begin{aligned}
\alpha \cdot (\beta + \delta) &= G_\alpha(F_\beta(\delta)) \\
&= (G_\alpha \circ F_\beta)(\delta) \\
&= \bigcup_{\gamma<\delta} (G_\alpha \circ F_\beta)(\gamma) \quad \text{by 11.4} \\
&= \bigcup_{\gamma<\delta} (\alpha \cdot (\beta + \gamma)) \\
&= \bigcup_{\gamma<\delta} ((\alpha \cdot \beta) + (\alpha \cdot \gamma)) \quad \text{induction hypothesis} \\
&= \bigcup_{\gamma<\delta} F_{\alpha\cdot\beta}(G_\alpha(\gamma)) \\
&= \bigcup_{\gamma<\delta} (F_{\alpha\cdot\beta} \circ G_\alpha)(\gamma) \\
&= (F_{\alpha\cdot\beta} \circ G_\alpha)(\delta) \quad \text{by 11.4} \\
&= (\alpha \cdot \beta) + (\alpha \cdot \delta),
\end{aligned}$$

as desired. This completes the proof of (v).

For (vi), we fix α and β and proceed by induction on γ. Again we may assume that $\alpha \neq 0$ and $\beta \neq 0$. Hence by (iv), also $\alpha \cdot \beta \neq 0$. Now

$$\alpha \cdot (\beta \cdot 0) = \alpha \cdot 0 = 0 = (\alpha \cdot \beta) \cdot 0,$$

so it works for $\gamma = 0$. Now assume that (iv) is true for γ. Then

$$\begin{aligned}
\alpha \cdot (\beta \cdot (\gamma + 1)) &= \alpha \cdot ((\beta \cdot \gamma) + \beta) \\
&= (\alpha \cdot (\beta \cdot \gamma)) + (\alpha \cdot \beta) \quad \text{by (v)} \\
&= ((\alpha \cdot \beta) \cdot \gamma) + (\alpha \cdot \beta) \quad \text{induction hypothesis} \\
&= (\alpha \cdot \beta) \cdot (\gamma + 1),
\end{aligned}$$

as desired.

Finally, suppose that δ is a limit ordinal and (vi) holds for all $\gamma < \delta$. Then

$$\begin{aligned}
\alpha \cdot (\beta \cdot \delta) &= G_\alpha(G_\beta(\delta)) \\
&= (G_\alpha \circ G_\beta)(\delta) \\
&= \bigcup_{\gamma<\delta} (G_\alpha \circ G_\beta)(\gamma) \quad \text{by 11.4} \\
&= \bigcup_{\gamma<\delta} (\alpha \cdot (\beta \cdot \gamma)) \\
&= \bigcup_{\gamma<\delta} ((\alpha \cdot \beta) \cdot \gamma) \quad \text{induction hypothesis}
\end{aligned}$$

$$
\begin{aligned}
&= \bigcup_{\gamma<\delta} G_{\alpha\cdot\beta}(\gamma) \\
&= G_{\alpha\cdot\beta}(\delta) \\
&= (\alpha\cdot\beta)\cdot\delta,
\end{aligned}
$$

as desired.

(vii): by (ii) and 12.1.

(viii): a straightforward transfinite induction on γ.

(ix): $\alpha\cdot 1 = \alpha\cdot(0+1) = \alpha\cdot 0+\alpha = 0+\alpha = \alpha$. That $1\cdot\alpha = \alpha$ for all α is seen by an easy transfinite induction.

(x): $\alpha\cdot 2 = \alpha\cdot(1+1) = (\alpha\cdot 1)+(\alpha\cdot 1) = \alpha+\alpha$. □

The following is a generalization of the division algorithm for natural numbers. That algorithm is very useful for the arithmetic of natural numbers, and the ordinal version is a basic result for more advanced arithmetic of ordinals.

Proposition 12.12. (Division algorithm) *Suppose that α and β are ordinals, with $\beta \neq 0$. Then there are unique ordinals ξ,η such that $\alpha = \beta\cdot\xi+\eta$ with $\eta<\beta$.*

Proof. First we prove the existence. Note that $\alpha < \alpha+1 \le \beta\cdot(\alpha+1)$ by 12.11(ii) and 11.1. Thus there is an ordinal number ρ such that $\alpha < \beta\cdot\rho$; take the least such ρ. Obviously $\rho\neq 0$. If ρ is a limit ordinal, then because $\beta\cdot\rho = \bigcup_{\sigma<\rho}(\beta\cdot\sigma)$, it follows that there is a $\sigma<\rho$ such that $\alpha<\beta\cdot\sigma$, contradicting the minimality of ρ. Thus ρ is a successor ordinal $\xi+1$. By the definition of ρ we have $\beta\cdot\xi\le\alpha$. Hence by 12.6(vi) there is an ordinal η such that $\beta\cdot\xi+\eta=\alpha$. We claim that $\eta<\beta$. Otherwise, $\alpha = \beta\cdot\xi+\eta \ge \beta\cdot\xi+\beta = \beta\cdot(\xi+1) = \beta\cdot\rho$, contradicting the definition of ρ. This finishes the proof of existence.

For uniqueness, suppose that also $\alpha = \beta\cdot\xi'+\eta'$ with $\eta'<\beta$. Suppose that $\xi\neq\xi'$. By symmetry, say $\xi<\xi'$. Then

$$
\alpha = \beta\cdot\xi+\eta < \beta\cdot\xi+\beta = \beta\cdot(\xi+1) \le \beta\cdot\xi' \le \beta\cdot\xi'+\eta' = \alpha,
$$

contradiction. Hence $\xi=\xi'$. Hence also $\eta=\eta'$ by 12.6(i). □

We now give, similarly to the case of ordinal addition, an equivalent definition of ordinal multiplication which is somewhat more intuitive than the definition above. Given ordinals α,β, we define the following relation $\prec$ on $\alpha\times\beta$:

$$
\begin{aligned}
(\xi,\eta)\prec(\xi',\eta') \quad \text{iff} \quad & ((\xi,\eta) \text{ and } (\xi',\eta') \text{ are in } \alpha\times\beta \text{ and:} \\
& \eta<\eta', \text{ or } (\eta=\eta' \text{ and } \xi<\xi').
\end{aligned}
$$

We may say that this is the *anti-dictionary* or *anti-lexicographic* order.

Proposition 12.13. *For any two ordinals α,β, the set $\alpha\times\beta$ under the anti-lexicographic order is a well-ordering which is isomorphic to $\alpha\cdot\beta$.*

Proof. We may assume that $\alpha\neq 0$.

Clearly $\prec$ is irreflexive. Suppose that $(\xi,\eta)\prec(\xi',\eta')\prec(\xi'',\eta'')$. Then $\eta\leq\eta'\leq\eta''$. If $\eta<\eta'$ or $\eta'<\eta''$, then $\eta<\eta''$, and so $(\xi,\eta)\prec(\xi'',\eta'')$. Otherwise we have $\eta=\eta'=\eta''$, and so $\xi<\xi'<\xi''$, hence $\xi<\xi''$ and so again $(\xi,\eta)\prec(\xi'',\eta'')$. Given two members $(\xi,\eta),(\xi',\eta')$ of $\alpha\times\beta$, if $\eta\neq\eta'$, then one of them precedes the other under $\prec$. If $\eta=\eta'$, then still it is clear that they are equal or else comparable under $\prec$. Thus we have checked that $\prec$ is a linear order. To see that it is a well-order, let X be a nonempty subset of $\alpha\times\beta$. Then $\mathrm{rng}(X)$ is nonempty; let η be its least element. Then $\{\xi:(\xi,\eta)\in X\}$ is nonempty; let ξ be its least element. Clearly (ξ,η) is the least element of X under $\prec$. Thus $\prec$ is a well-order.

Now we define, for any $(\xi,\eta)\in\alpha\times\beta$,

$$f(\xi,\eta)=\alpha\cdot\eta+\xi.$$

We claim that f is the desired order-isomorphism from $\alpha\times\beta$ onto $\alpha\cdot\beta$. If $(\xi,\eta)\in\alpha\times\beta$, then

$$f(\xi,\eta)=\alpha\cdot\eta+\xi<\alpha\cdot\eta+\alpha=\alpha\cdot(\eta+1)\leq\alpha\cdot\beta.$$

Thus f maps into $\alpha\times\beta$.

To show that f is one-one, suppose that (ξ,η) and (ξ',η') are distinct members of $\alpha\times\beta$.

Case 1. $\eta\neq\eta'$. By symmetry, say $\eta<\eta'$. Then

$$\begin{aligned}f(\xi,\eta)&=\alpha\cdot\eta+\xi\\&<\alpha\cdot\eta+\alpha\\&=\alpha\cdot(\eta+1)\\&\leq\alpha\cdot\eta'\\&\leq(\alpha\cdot\eta')+\xi'\\&=f(\xi',\eta').\end{aligned}$$

Thus $f(\xi,\eta)\neq f(\xi',\eta')$ in this case.

Case 2. $\eta=\eta'$ but $\xi\neq\xi'$. Then $f(\xi,\eta)\neq f(\xi',\eta')$ by 12.6(i).

To show that f maps onto $\alpha\cdot\beta$, let $\gamma<\alpha\cdot\beta$. By 12.12, choose ξ and η so that $\gamma=(\alpha\cdot\eta)+\xi$ with $\xi<\alpha$. Now $\eta<\beta$, as otherwise

$$\gamma=(\alpha\cdot\eta)+\xi\geq\alpha\cdot\eta\geq\alpha\cdot\beta.$$

It follows that $f(\xi,\eta)=(\alpha\cdot\eta)+\xi=\gamma$. so f is onto.

Finally, we show that the order is preserved. Suppose that $(\xi,\eta)\prec(\xi',\eta')$. Then one of these cases holds:

Case 1. $\eta<\eta'$. Then

$$f(\xi,\eta)=(\alpha\cdot\eta)+\xi<(\alpha\cdot\eta)+\alpha=(\alpha\cdot(\eta+1)\leq(\alpha\cdot\eta')\leq(\alpha\cdot\eta')+\xi'=f(\xi',\eta'),$$

as desired.

Case 2. $\eta=\eta'$ and $\xi<\xi'$. Then $f(\xi,\eta)<f(\xi',\eta')$ by 12.6(i).

Now it follows from 8.1 that f is the desired isomorphism. $\square$

Our final ordinal operation is exponentiation. Its definition follows the same pattern as addition and multiplication.

Proposition 12.14. *For every ordinal α there is a unique class function F defined for every ordinal and having the following properties.*

(i) $F(0) = 1$.

(ii) For any ordinal β, $F(\beta + 1) = F(\beta) \cdot \alpha$.

(iii) For any limit ordinal β, $F(\beta) = \bigcup_{\gamma<\beta} F(\gamma)$.

Proof. We use the class version of transfinite recursion. The class function G is defined for every function k as follows.

$$G(k) = \begin{cases} 1 & \text{if } k = \emptyset, \\ k(\beta) \cdot \alpha & \text{if } \mathrm{dmn}(k) = \beta + 1 \text{ for some ordinal } \beta, \\ \bigcup_{\gamma<\beta} k(\gamma) & \text{if } \mathrm{dmn}(k) = \beta \text{ for some limit ordinal } \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Now by the class version of transfinite recursion, 9.14, there is a class function F, defined for all ordinals, such that $F(\beta) = G(F \restriction \beta)$ for every ordinal β. Then for any ordinal β,

$$\begin{aligned} F(0) &= G(F \restriction 0) = G(\emptyset) = 1; \\ F(\beta + 1) &= G(F \restriction (\beta + 1)) \\ &= (F \restriction (\beta + 1))(\beta) \cdot \alpha \\ &= F(\beta) \cdot \alpha; \quad \text{and for } \beta \text{ limit}, \\ F(\beta) &= G(F \restriction \beta) \\ &= \bigcup_{\gamma<\beta} (F \restriction \beta)(\gamma) \\ &= \bigcup_{\gamma<\beta} F(\gamma). \end{aligned}$$

This proves that F exists as in the lemma.

For uniqueness, suppose that F' also satisfies the conditions on F, i.e., assume that (1) $F'(0) = 1$, (2) for any ordinal β, $F'(\beta + 1) = F'(\beta) \cdot \alpha$, and (3) for any limit ordinal β, $F'(\beta) = \bigcup_{\gamma<\beta} F'(\gamma)$. We show that $F(\beta) = F'(\beta)$ for every ordinal β by transfinite induction (where the inductive hypotheses should be clear):

$$\begin{aligned} F(0) &= 1 = F'(0); \\ F(\beta + 1) &= F(\beta) \cdot \alpha; \\ &= F'(\beta) \cdot \alpha \\ &= F'(\beta + 1); \\ F(\beta) &= \bigcup_{\gamma<\beta} F(\gamma) \\ &= \bigcup_{\gamma<\beta} F'(\gamma) \\ &= F'(\beta). \end{aligned}$$

$\square$

We denote the unique class function proved to exist in 12.14 by Ex_α. Then we define, for any ordinals α, β,,

$$^o\alpha^\beta = \mathrm{Ex}_\alpha(\beta).$$

This definition can be formulated as follows; this is what will be used below.

Proposition 12.15. *For any ordinals α, β,*

$$\begin{aligned} ^o\alpha^0 &= 1; \\ ^o\alpha^{\beta+1} &= {}^o\alpha^\beta \cdot \alpha; \\ ^o\alpha^\beta &= \bigcup_{\gamma<\beta} {}^o\alpha^\gamma \quad \text{for } \beta \text{ limit.} \end{aligned}$$ □

Again this is an extension of the exponentiation operation on natural numbers, and we leave the proof to the reader:

Proposition 12.16. *If m and n are natural numbers, then $m^n = {}^om^n$.* □

We omit the superscipt ^{o} in ordinal exponentiation from now on. Again note that ordinal exponentiation is different from cardinal exponentiation. For example, $\omega < 2^\omega$ in the cardinal sense, but $\omega = 2^\omega$ in the ordinal sense.

Now we give the simplest properties of exponentiation.

Theorem 12.17. *(i) $0^0 = 1$;*

(ii) $0^{\beta+1} = 0$;

(iii) $0^\beta = 1$ for β a limit ordinal;

(iv) $1^\beta = 1$;

(v) If $\alpha \neq 0$, then $\alpha^\beta \neq 0$;

(vi) If $\alpha > 1$ then $\alpha^\beta < \alpha^{\beta+1}$;

(vii) If $\alpha > 1$, then the operation assigning to each ordinal β the value α^β is strictly increasing and continuous;

(viii) If $\alpha > 1$, then $\beta \leq \alpha^\beta$;

(ix) If $0 < \alpha < \beta$, then $\alpha^\gamma \leq \beta^\gamma$;

(x) For $\alpha \neq 0$, $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$;

(xi) For $\alpha \neq 0$, $(\alpha^\beta)^\gamma = \alpha^{\beta\cdot\gamma}$.

Proof. (i)–(ii) are immediate from the definition. For (iii), suppose it is false, and let β be the smallest limit ordinal such that $0^\beta \neq 1$. Then $0^\beta = \bigcup_{\gamma<\beta} 0^\gamma = 1$, contradiction. (iv): transfinite induction on β; with obvious inductive assumptions,

$$\begin{aligned} 1^0 &= 1; \\ 1^{\beta+1} &= 1^\beta \cdot 1 = 1 \cdot 1 = 1; \\ 1^\beta &= \bigcup_{\gamma<\beta} 1^\gamma = \bigcup_{\gamma<\beta} 1 = 1. \end{aligned}$$

(v): We fix $\alpha > 0$ and proceed by induction on β, with obvious inductive assumptions:

$$\begin{aligned}
\alpha^0 &= 1 \neq 0; \\
\alpha^{\beta+1} &= \alpha^\beta \cdot \alpha \neq 0 \quad \text{by 12.11(iv)} \\
\alpha^\beta &= \bigcup_{\gamma<\beta} \alpha^\beta \neq 0.
\end{aligned}$$

(vi):

$$\begin{aligned}
\alpha^\beta &= \alpha^\beta \cdot 1 \quad \text{by 12.11(ix)} \\
&< \alpha^\beta \cdot \alpha \quad \text{by (v) and 12.11(ii)} \\
&= \alpha^{\beta+1}.
\end{aligned}$$

(vii): by (vi) and 11.2.
(viii): by (vii) and 11.1.
(ix): Fix α and β such that $0 < \alpha < \beta$; we proceed by induction on γ, with obvious assumptions:

$$\begin{aligned}
\alpha^0 &= 1 = \beta^0; \\
\alpha^{\gamma+1} &= \alpha^\gamma \cdot \alpha \leq \beta^\gamma \cdot \alpha \leq \beta^\gamma \cdot \beta = \beta^{\gamma+1}; \\
\alpha^\gamma &= \bigcup_{\delta<\gamma} \alpha^\delta \leq \bigcup_{\delta<\gamma} \beta^\delta = \beta^\gamma.
\end{aligned}$$

(x): The case $\alpha = 1$ holds by (iv). So suppose that $\alpha > 1$. Also fix β; then we go by induction on γ. $\gamma = 0$:

$$\alpha^{\beta+0} = \alpha^\beta = \alpha^\beta \cdot 1 = \alpha^\beta \cdot \alpha^0,$$

The successor case:

$$\alpha^{\beta+\gamma+1} = \alpha^{\beta+\gamma} \cdot \alpha = \alpha^\beta \cdot \alpha^\gamma \cdot \alpha = \alpha^\beta \cdot \alpha^{\gamma+1}.$$

Now assume that we know (x) for all $\delta < \gamma$, where γ is a limit ordinal. We define class functions F, G, H as follows: for any ordinal δ,

$$\begin{aligned}
F(\delta) &= \beta + \delta; \\
G(\delta) &= \alpha^\delta; \\
H(\delta) &= \alpha^\beta \cdot \delta.
\end{aligned}$$

These are strictly increasing and continuous for the following reasons: F by 12.6(i); G by (vii); H by (v) and 12.11(ii). Hence

$$\begin{aligned}
\alpha^{\beta+\gamma} &= G(F(\gamma)) = \bigcup_{\delta<\gamma} G(F(\delta)) = \bigcup_{\delta<\gamma} \alpha^{\beta+\delta} \\
&= \bigcup_{\delta<\gamma} (\alpha^\beta \cdot \alpha^\delta) = \bigcup_{\delta<\gamma} H(G(\delta)) = H(G(\gamma)) = \alpha^\beta \cdot \alpha^\gamma.
\end{aligned}$$

(xi): clear for $\alpha = 1$. Assume that $\alpha > 1$. Also fix β. Then we proceed by induction on γ. $\gamma = 0$:

$$(\alpha^\beta)^0 = 1 = \alpha^0 = \alpha^{\beta\cdot 0}.$$

The successor step:

$$(\alpha^\beta)^{\gamma+1} = (\alpha^\beta)^\gamma \cdot \alpha^\beta = \alpha^{\beta\cdot\gamma} \cdot \alpha^\beta = \alpha^{\beta\cdot\gamma+\beta} = \alpha^{\beta\cdot(\gamma+1)}.$$

The limit step, with γ limit. Let $F(\delta) = \beta \cdot \delta$ and $G(\delta) = \alpha^\delta$ for any ordinal δ. Then

$$(\alpha^\beta)^\gamma = \bigcup_{\delta<\gamma} (\alpha^\beta)^\delta = \bigcup_{\delta<\gamma} G(F(\delta)) = G(F(\gamma)) = \alpha^{\beta\cdot\gamma}. \qquad \square$$

The following is another kind of division algorithm for ordinals.

Theorem 12.18. *Let α and β be ordinals, with $\alpha \neq 0$ and $1 < \beta$. Then there exist unique ordinals $\gamma, \delta, \varepsilon$ such that the following conditions hold:*

(i) $\alpha = \beta^\gamma \cdot \delta + \varepsilon$.

(ii) $\gamma \le \alpha$.

(iii) $0 < \delta < \beta$,

(iv) $\varepsilon < \beta^\gamma$.

Proof. By 12.17(viii) we have $\alpha < \alpha + 1 \le \beta^{\alpha+1}$; so there is an ordinal φ such that $\alpha < \beta^\varphi$. We take the least such φ. Clearly φ is a successor ordinal $\gamma + 1$. So we have $\beta^\gamma \le \alpha < \beta^{\gamma+1}$. Now $\beta^\gamma \neq 0$, since $\beta > 1$ and by 12.17(vii). Hence by the division algorithm 12.12 there are ordinals δ, ε such that $\alpha = \beta^\gamma \cdot \delta + \varepsilon$, with $\varepsilon < \beta^\gamma$. Now $\delta < \beta$; for if $\beta \le \delta$, then

$$\alpha = \beta^\gamma \cdot \delta + \varepsilon \ge \beta^\gamma \cdot \beta = \beta^{\gamma+1} > \alpha,$$

contradiction. We have $\delta \neq 0$, as otherwise $\alpha = \varepsilon < \beta^\gamma$, contradiction.. Also, $\gamma \le \alpha$, since

$$\alpha = \beta^\gamma \cdot \delta + \varepsilon \ge \beta^\gamma \ge \gamma.$$

This proves the existence of $\gamma, \delta, \varepsilon$ as called for in the theorem.

Suppose that $\gamma', \delta', \varepsilon'$ also satisfy the indicated conditions; thus

(1) $\alpha = \beta^{\gamma'} \cdot \delta' + \varepsilon'$,

(2) $\gamma' \le \alpha$,

(3) $0 < \delta' < \beta$,

(4) $\varepsilon' < \beta^{\gamma'}$.

Suppose that $\gamma \neq \gamma'$; by symmetry, say that $\gamma < \gamma'$. Then

$$\alpha = \beta^\gamma \cdot \delta + \varepsilon < \beta^\gamma \cdot \delta + \beta^\gamma = \beta^\gamma \cdot (\delta + 1) \le \beta^\gamma \cdot \beta = \beta^{\gamma+1} \le \beta^{\gamma'} \le \alpha,$$

contradiction. Hence $\gamma = \gamma'$. Now by 12.12 we also have $\delta = \delta'$ and $\varepsilon = \varepsilon'$. $\square$

We can obtain an interesting normal form for ordinals by re-applying 12.18 to the "remainder" ε over and over again. That is the purpose of the following definitions and results.

To abbreviate some long expressions, we let $N(\beta, m, \gamma, \delta)$ stand for the following statement:

> *β is an ordinal > 1, m is a positive integer γ and δ are sequences of ordinals each of length m, and:*
>
> *(1) $\gamma(0) > \gamma(1) > \cdots > \gamma(m-1)$;*
>
> *(2) $0 < \delta(i) < \beta$ for each $i < m$.*

If $N(\beta, m, \gamma, \delta)$, then we define

$$k(\beta, m, \gamma, \delta) = \beta^{\gamma(0)} \cdot \delta(0) + \beta^{\gamma(1)} \cdot \delta(1) + \cdots + \beta^{\gamma(m-1)} \cdot \delta(m-1).$$

Lemma 12.19. *Assume that $N(\beta, \gamma, \delta, m)$ and $N(\beta, \gamma', \delta', n)$. Then*

(i) $k(\beta, \gamma, \delta, m) \geq \gamma(0)$.

(ii) $k(\beta, \gamma, \delta, m) \leq \beta^{\gamma(0)} \cdot (\delta(0) + 1) \leq \beta^{\gamma(0)+1}$.

(iii) If $\gamma(0) \neq \gamma'(0)$, then $k(\beta, \gamma, \delta, m) < k(\beta, \gamma', \delta', n)$ iff $\gamma(0) < \gamma'(0)$.

(iv) If $\gamma(0) = \gamma'(0)$ and $\delta(0) \neq \delta'(0)$, then $k(\beta, \gamma, \delta, m) < k(\beta, \gamma', \delta', n)$ iff $\delta(0) < \delta'(0)$.

(v) If $\gamma(j) = \gamma'(j)$ and $\delta(j) = \delta'(j)$ for all $j < i$, while $\gamma(i) \neq \gamma'(i)$, then $k(\beta, \gamma, \delta, m) < k(\beta, \gamma', \delta', n)$ iff $\gamma(i) < \gamma'(i)$.

(vi) If $\gamma(j) = \gamma'(j)$ and $\delta(j) = \delta'(j)$ for all $j < i$, while $\gamma(i) = \gamma'(i)$ and $\delta(i) \neq \delta'(i)$, then $k(\beta, \gamma, \delta, m) < k(\beta, \gamma', \delta', n)$ iff $\delta(i) < \delta'(i)$.

(vii) If $\gamma \subseteq \gamma'$, $\delta \subseteq \delta'$, and $m < n$, then $k(\beta, \gamma, \delta, m) < k(\beta, \gamma', \delta', n)$

Proof. (i): $k(\beta, \gamma, \delta, m) \geq \beta^{\gamma(0)} \geq \gamma(0)$.

(ii): We prove this by induction on m. It is clear for $m = 1$. Now assume that it holds for $m - 1$, where $m > 1$. Then

$$\begin{aligned}
\beta^{\gamma(0)} \cdot \delta(0) + \beta^{\gamma(1)} \cdot \delta(1) + \cdots + \beta^{\gamma(m-1)} \cdot \delta(m-1) &< \beta^{\gamma(0)} \cdot \delta(0) + \beta^{\gamma(1)+1} \\
&\leq \beta^{\gamma(0)} \cdot \delta(0) + \beta^{\gamma(0)} \\
&= \beta^{\gamma(0)} \cdot (\delta(0) + 1) \\
&\leq \beta^{\gamma(0)} \cdot \beta \\
&= \beta^{\gamma(0)+1},
\end{aligned}$$

finishing the inductive proof.

For (iii), assume the hypothesis, and suppose that $\gamma(0) < \gamma'(0)$. Then

$$\begin{aligned}
k(\beta, \gamma, \delta, m) \leq \beta^{\gamma(0)} \cdot (\delta(0) + 1) &\leq \beta^{\gamma(0)+1} \quad \text{by (ii)} \\
&\leq \beta^{\gamma'(0)} \\
&\leq k(\beta, \gamma', \delta', n) \quad \text{by (i)}.
\end{aligned}$$

By symmetry (iii) now follows.

For (iv), assume the hypothesis, and suppose that $\delta(0) < \delta'(0)$. Then

$$\begin{aligned} k(\beta,\gamma,\delta,m) \le \beta^{\gamma(0)}\cdot(\delta(0)+1) &\le \beta^{\gamma(0)}\cdot(\delta(0)+1) \quad \text{by (ii)} \\ &\le \beta^{\gamma'(0)}\cdot\delta'(0) \\ &\le k(\beta,\gamma',\delta',n) \end{aligned}$$

By symmetry (iv) now follows.

(v) is clear from (iii), by deleting the first i summands of the sums.

(vi) is clear from (iv), by deleting the first i summands of the sums.

(vii) is clear. □

Theorem 12.20. *Let α and β be ordinals, with $\alpha \neq 0$ and $1 < \beta$. Then there exist unique finite sequences $\langle\gamma(i) : i < m\rangle$ and $\langle\delta(i) : i < m\rangle$ of ordinals such that the following conditions hold:*

(i) $\alpha = \beta^{\gamma(0)}\cdot\delta(0) + \beta^{\gamma(1)}\cdot\delta(1) + \cdots + \beta^{\gamma(m-1)}\cdot\delta(m-1)$.

(ii) $\alpha \ge \gamma(0) > \gamma(1) > \cdots > \gamma(m-1)$.

(iii) $0 < \delta(i) < \beta$ *for each* $i < m$.

Proof. For the existence, with $\beta > 1$ fixed we proceed by induction on α. Assume that the theorem holds for every $\alpha' < \alpha$ such that $\alpha' \neq 0$, and suppose that $\alpha \neq 0$. By 12.18, let φ, ψ, θ be such that

(1) $\alpha = \beta^\varphi\cdot\psi + \theta$,

(2) $\varphi \le \alpha$,

(3) $0 < \psi < \beta$,

(4) $\theta < \beta^\varphi$.

If $\theta = 0$, then we can take our sequences to be $\langle\gamma(0)\rangle$ and $\langle\delta(0)\rangle$, with $\gamma(0) = \varphi$ and $\delta(0) = \psi$. Now assume that $\theta > 0$. Then

$$\theta < \beta^\varphi \le \beta^\varphi\cdot\psi + \theta = \alpha;$$

so $\theta < \alpha$. Hence by the inductive assumption we can write

$$\theta = \beta^{\gamma(0)}\cdot\delta(0) + \beta^{\gamma(1)}\cdot\delta(1) + \cdots + \beta^{\gamma(m-1)}\cdot\delta(m-1)$$

with

(5) $\theta \ge \gamma(0) > \gamma(1) > \cdots > \gamma(m-1)$.

(6) $0 < \delta(i) < \beta$ for each $i < m$.

Then our desired sequences for α are

$$\langle\varphi, \gamma(0), \gamma(1), \ldots, \gamma(m-1)\rangle \quad \text{and} \quad \langle\psi, \delta(0), \delta(1), \ldots, \delta(m-1)\rangle.$$

To prove this, we just need to show that $\varphi > \gamma(0)$. If $\varphi \leq \gamma(0)$, then

$$\beta^\varphi \leq \beta^{\gamma(0)} \leq \theta,$$

contradiction.

This finishes the existence part of the proof.

For the uniqueness, we use the notation introduced above, and proceed by induction on α. Suppose the uniqueness statement holds for all nonzero $\alpha' < \alpha$, and now we have $N(\beta, \gamma, \delta, m)$, $N(\beta, \gamma', \delta', n)$, and

$$\alpha = k(\beta, \gamma, \delta, m) = k(\beta, \gamma', \delta', n).$$

We suppose that the uniqueness fails. Say $m \leq n$. By 12.19 (vii), there must be an $i < m$ such that $\gamma(i) \neq \gamma'(i)$ or $\delta(i) \neq \delta'(i)$; we take the least such i. Then we have a contradiction by 12.19. $\square$

We finish this chapter by giving an equivalent definition of exponentiation similar to those given above for addition and multiplication.

Theorem 12.21. *Suppose that α and β are ordinals, with $\beta \neq 0$. We define*

$$^\alpha\beta^{\mathrm{w}} = \{f \in {^\alpha\beta} : \{\xi < \alpha : f(\xi) \neq 0\} \textit{ is finite}\}.$$

For $f, g \in {^\alpha\beta^{\mathrm{w}}}$ we write $f \prec g$ iff $f \neq g$ and $f(\xi) < g(\xi)$ for the **greatest** *$\xi < \alpha$ for which $f(\xi) \neq g(\xi)$.*

Then $({^\alpha\beta^{\mathrm{w}}}, \prec)$ is a well-order which is order-isomorphic to the ordinal exponent β^α.

Proof. If $\beta = 1$, then ${^\alpha\beta^{\mathrm{w}}}$ has only one member, namely the function with domain α whose value is always 0. This is clearly order-isomorphic to 1, as desired. So, suppose that $\beta > 1$.

Now we define a function f mapping β^α into ${^\alpha\beta^{\mathrm{w}}}$. Let $f(0)$ be the member of ${^\alpha\beta^{\mathrm{w}}}$ which takes only the value 0. Now suppose that $0 < \varepsilon < \beta^\alpha$. By 12.19 write

$$\varepsilon = \beta^{\gamma(0)} \cdot \delta(0) + \beta^{\gamma(1)} \cdot \delta(1) + \cdots + \beta^{\gamma(m-1)} \cdot \delta(m-1),$$

where $\alpha \geq \gamma(0) > \gamma(1) > \cdots > \gamma(m-1)$ and $0 < \delta(i) < \beta$ for each $i < m$. Note that $\beta^{\gamma(0)} \leq \varepsilon < \beta^\alpha$, so $\gamma(0) < \alpha$. Then we define, for any $\zeta < \alpha$,

$$(f(\varepsilon))(\zeta) = \begin{cases} 0 & \text{if } \zeta \notin \{\gamma(0), \ldots, \gamma(m-1)\}, \\ \delta(i) & \text{if } \zeta = \gamma(i) \text{ with } i < m. \end{cases}$$

Clearly $f(\varepsilon) \in {^\alpha\beta^{\mathrm{w}}}$. To see that f maps onto ${^\alpha\beta^{\mathrm{w}}}$, suppose that $x \in {^\alpha\beta^{\mathrm{w}}}$. If x takes only the value 0, then $f(0) = x$. Suppose that x takes on some nonzero value. Let

$$\{\xi < \alpha : x(\xi) \neq 0\} = \{\gamma(0), \gamma(1), \ldots, \gamma(m-1)\},$$

where $\gamma(0) > \gamma(1) > \cdots > \gamma(m-1)$. Let

$$\varepsilon = \beta^{\gamma(0)} \cdot \delta(0) + \beta^{\gamma(1)} \cdot \delta(1) + \cdots + \beta^{\gamma(m-1)} \cdot \delta(m-1).$$

Clearly then $f(\varepsilon) = x$.

Now we complete the proof by showing that for any $\varepsilon, \theta < \beta^\alpha$, $\varepsilon < \theta$ iff $f(\varepsilon) < f(\theta)$. This equivalence is clear if one of ε, θ is 0, so suppose that both are nonzero. Write

$$\varepsilon = \beta^{\gamma(0)} \cdot \delta(0) + \beta^{\gamma(1)} \cdot \delta(1) + \cdots + \beta^{\gamma(m-1)} \cdot \delta(m-1),$$

where $\alpha \geq \gamma(0) > \gamma(1) > \cdots > \gamma(m-1)$ and $0 < \delta(i) < \beta$ for each $i < m$, and

$$\theta = \beta^{\gamma'(0)} \cdot \delta'(0) + \beta^{\gamma'(1)} \cdot \delta'(1) + \cdots + \beta^{\gamma'(n-1)} \cdot \delta'(n-1),$$

where $\alpha \geq \gamma'(0) > \gamma'(1) > \cdots > \gamma'(n-1)$ and $0 < \delta'(i) < \beta$ for each $i < n$.

By symmetry we may suppose that $m \leq n$. Note that $N(\beta, \gamma, \delta, m)$, $k(\beta, \gamma, \delta, m) = \varepsilon$, $N(\beta, \gamma', \delta', n)$, and $k(\beta, \gamma', \delta', n) = \theta$. We now consider several possibilities.

Case 1. $\varepsilon = \theta$. Then clearly $f(\varepsilon) = f(\theta)$.

Case 2. $\gamma \subseteq \gamma'$, $\delta \subseteq \delta'$, and $m < n$. Thus $\varepsilon < \theta$. Also, $\gamma'(m)$ is the largest $\xi < \alpha$ such that $(f(\varepsilon))(\xi) \neq (f(\theta))(\xi)$, and $(f(\varepsilon))(\xi) = 0 < \delta'(m) = (f(\theta))(\gamma'(m))$, so $f(\varepsilon) < f(\theta)$.

Case 3. There is an $i < m$ such that $\gamma(j) = \gamma'(j)$ and $\delta(j) = \delta'(j)$ for all $j < i$, while $\gamma(i) \neq \gamma'(i)$. By symmetry, say that $\gamma(i) < \gamma'(i)$. Then by 12.19(v) we have $\varepsilon < \theta$. Since $\gamma'(i)$ is the largest $\xi < \alpha$ such that $(f(\varepsilon))(\xi) \neq (f(\theta))(\xi)$, and $(f(\varepsilon))(\gamma'(i)) = 0 < \delta'(i) = (f(\theta))(\gamma'(i))$, we also have $f(\varepsilon) < f(\theta)$.

Case 4. There is an $i < m$ such that $\gamma(j) = \gamma'(j)$ and $\delta(j) = \delta'(j)$ for all $j < i$, while $\gamma(i) = \gamma'(i)$ and $\delta(i) \neq \delta'(i)$. By symmetry, say that $\delta(i) < \delta'(i)$. Then by 12.19(vi) we have $\varepsilon < \theta$. Since $\gamma(i)$ is the largest $\xi < \alpha$ such that $(f(\varepsilon))(\xi) \neq (f(\theta))(\xi)$, and $(f(\varepsilon))(\gamma'(i)) = \delta(i) < \delta'(i) = (f(\theta))(\gamma'(i))$, we also have $f(\varepsilon) < f(\theta)$. $\square$

Exercises, Chapter 12

1. Give examples of ordinals α, β such that $\alpha + \beta \neq \beta + \alpha$.

2. Determine all pairs α, β such that $\alpha + \beta = \beta + \alpha$.

3. Give examples of ordinals α, β such that $\alpha \cdot \beta \neq \beta \cdot \alpha$.

4. Determine all pairs α, β such that $\alpha \cdot \beta = \beta \cdot \alpha$.

5. Show that if $\alpha, \beta > 1$ then $\alpha + \beta \leq \alpha \cdot \beta$.

6. Show that if $\alpha, \beta \neq 0$ then $\alpha \cdot \beta \neq 0$.

7. Give examples of ordinals α, β, γ such that $(\alpha + \beta) \cdot \gamma \neq \alpha \cdot \gamma + \beta \cdot \gamma$.

8. Determine all ordinals α, β, γ such that $(\alpha + \beta) \cdot \gamma = \alpha \cdot \gamma + \beta \cdot \gamma$.

9. Show that if α is a limit ordinal, then $\omega \cdot \beta$ for some $\beta \neq 0$.

10. Show that if $\omega \cdot \beta$ for some $\beta \neq 0$, then for every $m \in \omega \backslash 1$ we have $m \cdot \alpha = \alpha$, and $\alpha \neq 0$.

11. Show that if for every $m \in \omega\backslash 1$ we have $m \cdot \alpha = \alpha$, and $\alpha \neq 0$, then α is a limit ordinal.

12. Give examples of ordinals α, β, γ such that $(\alpha \cdot \beta)^\gamma \neq \alpha^\gamma \cdot \beta^\gamma$.

13. Determine all ordinals α, β, γ such that $(\alpha \cdot \beta)^\gamma = \alpha^\gamma \cdot \beta^\gamma$.

14. If $\alpha, \beta > 1$, then $\alpha \cdot \beta < \alpha^\beta$.

15. $(2 \cdot 2)^\omega \neq 2^\omega \cdot 2^\omega$.

16. If $\alpha < \omega^\beta$, then $\alpha + \omega^\beta = \omega^\beta$.

17. Show that if $\beta + \alpha = \alpha$ for all $\beta < \alpha$, then $\beta + \gamma < \alpha$ for all $\beta, \gamma < \alpha$.

18. Show that if $\beta + \gamma < \alpha$ for all $\beta, \gamma < \alpha$, then $\alpha = 0$, or $\alpha = \omega^\beta$ for some β.

19. Show that if $\alpha = 0$, or $\alpha = \omega^\beta$ for some β, then $\beta + \gamma < \alpha$ for all $\beta, \gamma < \alpha$.

20. Show that if $\beta \cdot \alpha = \alpha$ for every nonzero $\beta < \alpha$, then $\beta \cdot \gamma < \alpha$ for all $\beta, \gamma < \alpha$.

21. Show that if $\beta \cdot \gamma < \alpha$ for all $\beta, \gamma < \alpha$, then $\alpha \in \{0, 1, 2\}$ or $\alpha = \omega^{\omega^\beta}$ for some β.

22. Show that if $\alpha \in \{0, 1, 2\}$ or $\alpha = \omega^{\omega^\beta}$ for some β, then $\beta \cdot \alpha = \alpha$ for every nonzero $\beta < \alpha$.

23. Show that if $\beta^\alpha = \alpha$ for all β with $1 < \beta < \alpha$, then $\alpha = 1$ or $\beta^\gamma < \alpha$ for all $\beta, \gamma < \alpha$.

24. Show that if $\alpha = 1$ or $\beta^\gamma < \alpha$ for all $\beta, \gamma < \alpha$, then $\beta^\alpha = \alpha$ for all β with $1 < \beta < \alpha$.

25. Show that if $\beta^\alpha = \alpha$ for all β with $1 < \beta < \alpha$, then $\beta + \gamma < \alpha$ for all $\beta, \gamma < \alpha$.