7. Comparing the size of infinite sets

April 16, 2015

In this chapter we give the basic facts about comparing the size of infinite sets. We start by a brief introduction and discussion of ordinal numbers; this discussion will be continued in a later chapter. Then the basic notion of cardinal number is defined in terms of ordinals, and we can then begin the real task of this chapter, to show how one computes with infinite cardinal numbers and uses them to count various infinite collections.

For any set A we let $\in_A = \{(a, b) : a, b \in A \text{ and } a \in b\}$. A set A is *transitive* if every member of A is a subset of A. This may sound strange, but actually we know already many transitive sets: all natural numbers are transitive by Proposition 6.4. An *ordinal number*, or simply an *ordinal*, is a transitive set A such that $(A, \in_A)$ is a well-ordering.

Proposition 7.1. *Every natural number is an ordinal.*

Proof. Let m be a natural number. By 6.4, m is transitive. Every member of m is a natural number, by 6.3, and so by 6.9, every nonempty subset of m has a least element under the ordering $<$ of natural numbers. If $i, j \in m$, then by definition $i < j$ iff $i \in j$, and hence $i \in_m j$. Thus $(m, \in_m)$ is a well-ordering. □

Proposition 7.2. *ω is an ordinal.*

Proof. ω is transitive by 6.3. $(\omega, \in_\omega)$ is the same as $(\omega, <)$, and it is a well-ordering by 6.9. □

Generally we use $\alpha, \beta, \gamma, \ldots$ to denote ordinals. If α and β are ordinals, we write $\alpha < \beta$ to mean that $\alpha \in \beta$, and $\alpha \leq \beta$ to mean that $\alpha < \beta$ or $\alpha = \beta$.

Now we give a few elementary facts about ordinals analogous to those about natural numbers. Remember that $<$ means $\in$ for ordinal numbers; but be careful not to assume this for partial orders which are not ordinals!

Proposition 7.3. *For any ordinals α, β, γ, if $\alpha < \beta < \gamma$, then $\alpha < \gamma$.*

Proof. This is true because γ is transitive. □

Proposition 7.4. *Every element of an ordinal is an ordinal.*

Proof. Suppose that α is an ordinal, and $x \in \alpha$. To show that x is transitive, suppose that $y \in z \in x$. Then $z \in x \in \alpha$, so $z \in \alpha$, since α is transitive. Thus $y \in z \in \alpha$, so also $y \in \alpha$ since α is transitive. Now we have $x, y, z \in \alpha$ and $y \in_\alpha z \in_\alpha x$, so $y \in_\alpha x$ since $(\alpha, \in_\alpha)$ is a well-ordering, and hence in particular a linear ordering. Thus $y \in x$. This shows that x is transitive.

For any $y, z \in x$, we have $y \in_x z$ iff $y \in z$, and $y, z \in \alpha$ by the transitivity of α, so $y \in_x z$ iff $y \in_\alpha z$. Clearly then $(x, \in_x)$ is a linear order.

Any nonempty subset M of x is also a nonempty subset of α, since α is transitive. Hence we can choose a $y \in M$ which is the smallest member of M in the ordering $\in_\alpha$. If $z \in M$ and $z \in_x y$, then $z \in y$ and $z, y \in \alpha$, so $z \in_\alpha y$, contradiction. So y is the smallest member of M in the ordering $(x, \in_x)$. So $(x, \in_x)$ is a well-ordering, as desired. □

Proposition 7.5. *For any ordinals α, β, $\alpha \subset \beta$ iff $\alpha \in \beta$.*

Proof. For $\Leftarrow$, if $\alpha \in \beta$, then $\alpha \subseteq \beta$ since β is transitive. Also, $\alpha \notin \alpha$, but $\alpha \in \beta$, so $\alpha \neq \beta$. Thus $\alpha \subset \beta$.

For $\Rightarrow$, assume that $\alpha \subset \beta$. Let x be the smallest member of $\beta \backslash \alpha$ under the well-ordering $\in_\beta$. If $y \in x$, then $y \in \beta$ by the transitivity of β, so $y \in_\beta x$, and hence $y \in \alpha$ by the minimality of x. This shows that $x \subseteq \alpha$. We claim that $x = \alpha$; this of course implies that $\alpha \in \beta$, as desired. In fact, suppose that $z \in \alpha \backslash x$. Then x and z are members of β, so they are comparable under $\in_\beta$, since it is a linear order. But $z \notin x$, so $z = x$ or $x \in z$. However, $z = x$ contradicts the facts that $z \in \alpha$ and $x \notin \alpha$, and $x \in z$ implies that $x \in \alpha$ since α is transitive. So we get a contradiction in any case. Hence such an element z cannot exist, and so $x = \alpha$. □

Proposition 7.6. *If α and β are ordinals, then $\alpha \in \beta$, $\alpha = \beta$, or $\beta \in \alpha$.*

Proof. Suppose that α and β are ordinals, $\alpha \notin \beta$, and $\alpha \neq \beta$; we want to show that $\beta \in \alpha$. If $\alpha \cap \beta = \alpha$, then $\alpha \subseteq \beta$, and since $\alpha \neq \beta$, even $\alpha \subset \beta$; then $\alpha \in \beta$ by 7.5, contradicting a supposition. Thus $\alpha \cap \beta \neq \alpha$. Let x be the least element of $\alpha \backslash (\alpha \cap \beta)$ under the well-ordering $\in_\alpha$. If $y \in x$, then also $y \in \alpha$ since α is transitive, so $y \in_\alpha x$, and then it follows by the minimality of x that $y \in \beta$. This proves that $x \subseteq \beta$. Now x is an ordinal by 7.4, so $x \subset \beta$ would imply that $x \in \beta$ by 7.5, contradiction. So $x = \beta$. Since $x \in \alpha$, this shows that $\beta \in \alpha$. □

Proposition 7.7. *If A is a nonempty set of ordinals, then A has a least element. That is, there is an $\alpha \in A$ such that $\alpha < \beta$ for all $\beta \in A$ such that $\alpha \neq \beta$.*

Proof. Let γ be any element of A. If γ is the least element of A, we are through. Otherwise, there is an $\varepsilon \in A$ with $\varepsilon \neq \gamma$ and $\gamma \not< \varepsilon$. By 7.6, $\varepsilon < \gamma$. Thus $M \stackrel{\text{def}}{=} \{\delta \in \gamma : \delta \in A\}$ is a nonempty subset of γ, so from the fact that $(\gamma, \in_\gamma)$ is a well-ordering we see that M has a least element α under $\in_\gamma$. We claim that α is the least element of A. For, suppose that $\beta \in A$ and $\alpha \neq \beta$. If $\gamma \leq \beta$, then $\alpha < \beta$ since $\alpha < \gamma$ and β is transitive. If $\beta < \gamma$, then $\beta \in M$, and hence $\alpha < \beta$ by the minimality of α. □

This gives us enough properties of ordinals to define and work with the notion of cardinal. A *cardinal number*, or *cardinal* is an ordinal which is not equipotent with any smaller ordinal. In particular, we have a relation $<$ defined on the cardinals, and it satisfies the formal requirements of a well-ordering. We cannot say that it really is a well-ordering, since as will be seen later, the set of all cardinals does not exist; it is too big.

Proposition 7.8. *Every natural number is a cardinal.*

Proof. We observed above that every natural number is an ordinal. If m is a natural number and m is equipotent with $x < m$, then $x \in m$, and so x is a natural number by 6.3; and then 6.25 is contradicted. □

Proposition 7.9. *ω is a cardinal; in fact, it is the smallest infinite cardinal.*

Proof. By 6.31 and 6.33. □

Before defining the size of sets, we prove two results which be useful later.

Proposition 7.10. *If f is a bijection from A to C, g is a bijection from B to D, and $A \cap B = \emptyset = C \cap D$, then $f \cup g$ is a bijection from $A \cup B$ to $C \cup D$.*

Proof. Clearly $f \cup g$ is a relation. If $(x, y), (x, z) \in f \cup g$, then clearly $x \in A$ or $x \in B$. If $x \in A$, then $(x, y), (x, z) \in f$ and so $y = z$. Similarly if $x \in B$. So $f \cup g$ is a function. Similarly, it is a one-one function. Clearly it has domain $A \cup B$ and range $C \cup D$. □

Theorem 7.11. (Cantor, Schröder, Bernstein theorem) *If there are injections of A into B and of B into A, then A and B are equipotent.*

Proof. Suppose that $f : A \to B$ and $g : B \to A$ are injections. For any $X \subseteq A$ let $F(X) = A \backslash g[B \backslash f[X]]$.

(1) If $X, Y \subseteq A$ and $X \subseteq Y$, then $F(X) \subseteq F(Y)$.

In fact, suppose that $X, Y \subseteq A$ and $X \subseteq Y$. Then $f[X] \subseteq f[Y]$, hence $B \backslash f[Y] \subseteq B \backslash f[X]$, hence $g[B \backslash f[Y]] \subseteq g[B \backslash f[X]]$, hence

$$F(X) = A \backslash g[B \backslash f[X]] \subseteq A \backslash g[B \backslash f[Y]] = F(Y).$$

Now let $\mathscr{A} = \{X : X \subseteq A \text{ and } X \subseteq F(X)\}$, and set $X_0 = \bigcup_{X \in \mathscr{A}} X$.

(2) $X_0 \subseteq F(X_0)$.

In fact, if $X \in \mathscr{A}$, then $X \subseteq X_0$, and hence $F(X) \subseteq F(X_0)$ by (1). But also $X \subseteq F(X)$ by the definition of $\mathscr{A}$, so $X \subseteq F(X_0)$. Since this is true for every $X \in \mathscr{A}$, we have $X_0 = \bigcup_{X \in \mathscr{A}} X \subseteq F(X_0)$, as desired in (2).

(3) $X_0 = F(X_0)$.

For, $F(X_0) \subseteq F(F(X_0))$ by (2) and (1), so $F(X_0) \in \mathscr{A}$, and hence $F(X_0) \subseteq \bigcup_{X \in \mathscr{A}} X = X_0$. Together with (2), this gives (3).

Now note that f is a bijection from X_0 onto $f[X_0]$. Moreover, $A \backslash X_0 = A \backslash F(X_0) = g[B \backslash f[X_0]]$, so g^{-1} is a bijection from $A \backslash X_0$ onto $B \backslash f[X_0]$. Hence by 7.10, A is equipotent with B. □

To define the notion of the size of a set, we have to use the axiom of choice. But the version of the axiom of choice introduced in section 3 is inconvenient for this purpose, so we give a variant of that axiom. Later we will show that these two versions are equivalent (and are equivalent to several other widely used variants). Recall that results using the axiom of choice are indicated with a superscript $^{\mathrm{ch}}$.

Axiom 6′ *Axiom of choice, second form.* **For every set A there is a unique cardinal number equipotent with A.**

We denote this cardinal number by $|A|$, and call it the *number of elements* or *cardinality,* or *size* of A. Clearly this just extends the definition of cardinality of finite sets given in chapter 6.

The most basic fact about cardinality is as follows.

Theorem 7.12$^{\text{ch}}$**.** *For any sets X and Y, the following conditions are equivalent:*

(i) $|X| = |Y|$.

(ii) There is a one-one function mapping X onto Y.

Proof. (i)⇒(ii): Let $\kappa = |X| = |Y|$, and let f and g be one-one functions from κ onto X and from κ onto Y respectively. Then $g \circ f^{-1}$ is a one-one function from X onto Y.

(ii)⇒(i). Let h be a one-one function from X onto Y, and let k be a one-one function from $|X|$ onto X. Then $h \circ k$ is a one-one function from $|X|$ onto Y, so $|Y| = |X|$. □

Lemma 7.13. *If κ and λ are cardinals and $f : \kappa \to \lambda$ is one-one, then $\kappa \le \lambda$.*

Proof. Suppose that $\lambda < \kappa$. Then Id_λ is a one-one function from λ into κ. By Theorem 7.11, $\kappa = \lambda$. But $\lambda < \kappa$, contradiction. □

Theorem 7.14$^{\text{ch}}$**.** *If $A \subseteq B$, then $|A| \le |B|$.*

Proof. Let $\kappa = |A|$, $\lambda = |B|$, and let f and g be one-one functions from κ onto A and of λ onto B, respectively. Then $g \circ f^{-1}$ is a one-one function from κ into λ, so $\kappa \le \lambda$ by Lemma 7.13. □

Lemma 7.15. *If κ and λ are cardinals and f is a function mapping λ onto κ, then $\kappa \le \lambda$.*

Proof. For any $\alpha < \kappa$ let $g(\alpha)$ be the least element of λ such that $f(g(\alpha)) = \alpha$. Thus $f \circ g = \mathrm{Id}_\kappa$, so g is a one-one function mapping κ into λ. Hence $\kappa \le \lambda$ by 7.13. □

Corollary 7.16$^{\text{ch}}$**.** *For any sets A and B the following conditions are equivalent:*

(i) $|A| \le |B|$.

(ii) There is a one-one function mapping A into B.

(iii) $A = \emptyset$, or there is a function mapping B onto A.

Proof. (i)⇒(ii): Let f and g be such that f is a one-one function mapping $|A|$ onto A and g is a one-one function mapping $|B|$ onto B. Then $g \circ f^{-1}$ is a one-one function mapping A into B. (Recall that $\le$ means $\subseteq$ for ordinals, hence for cardinals.)

(ii)⇒(iii): by 3.14. (The function g there clearly maps B onto A.)

(iii)⇒(i): This is clear for $A = \emptyset$, so suppose that $A \ne \emptyset$, and there is a function mapping B onto A. Clearly then there is a function mapping $|B|$ onto $|A|$, so $|A| \le |B|$ by 7.15. □

We now have the basic notions of cardinality and size. Next we turn to the comparison of sizes of infinite sets. The situation here is somewhat confusing. It is quite possible to have sets A and B of the same size with $A \subset B$. Perhaps the simplest example is to take $A = \omega\backslash 1$, the set of positive integers, and $B = \omega$. The function f assigning $m + 1$ to each natural number m is clearly a one-one function from B onto A. Hence $|A| = |B|$. Another example is to let B be the set of even natural numbers and $A = \omega$. This time the function assigning $2m$ to each $m \in \omega$ is a one-one function from B onto A, so that again $|A| = |B|$. Here not only is A a proper subset of B, but also $B\backslash A$ is infinite. We explore things of this sort in the rest of this chapter.

We begin our discussion of the size of infinite sets by considering the simplest case. A set A is *denumerable* iff $|A| = \omega$; it is *countable* if $|A| \le \omega$. Thus a set A is countable iff it is either finite, or can be put in one-one correspondence with the set of natural numbers.

Proposition 7.17$^{\mathrm{ch}}$**.** *Any infinite set has a denumerable subset.*

Proof. If A is infinite, then by 7.9, $\omega \le |A|$. By 7.16, there is then a one-one function f mapping ω into A. Thus $\mathrm{rng}(f)$ is a denumerable subset of A. $\square$

Proposition 7.18$^{\mathrm{ch}}$**.** *Any subset of a countable set is countable.*

Proof. If A is countable and $B \subseteq A$, then $|B| \le |A| \le \omega$ by 7.14. $\square$

Proposition 7.19$^{\mathrm{ch}}$**.** *For any set A the following conditions are equivalent:*
(i) A is countable.
(ii) There is a one-one function mapping A into ω.
(iii) $A = \emptyset$, or there is a function mapping ω onto A.

Proof. Since $|\omega| = \omega$, this is immediate from 7.16. $\square$

An *ω-sequence* is a function whose domain is ω.

Proposition 7.20$^{\mathrm{ch}}$**.** *If $\langle a_n : n \in \omega\rangle$ is an ω-sequence, then $\{a_n : n \in \omega\}$ is countable.*

Proof. By definition, $\langle a_n : n \in \omega\rangle$ is just a function, naturally denoted by a, with domain ω, and the indicated set is just $\mathrm{rng}(a)$. Thus its size is $|\mathrm{rng}(a)| \le \omega$ by 7.16. $\square$

For the rest of this chapter, we assume an elementary knowledge of integers, rational numbers, and real numbers; see the end of chapter 6.

Proposition 7.21$^{\mathrm{ch}}$**.** *If $\mathbb{Z}$ is the collection of all integers, then $|\mathbb{Z}| = \omega$.*

Proof. We define $f(2m) = m$ and $f(2m+1) = -m-1$ for every natural number m. Clearly f is the desired bijection. $\square$

If we take 7.21 just to say that there is a bijection from ω to $\mathbb{Z}$, then the axiom of choice is not required. This is true of several other results of this chapter.

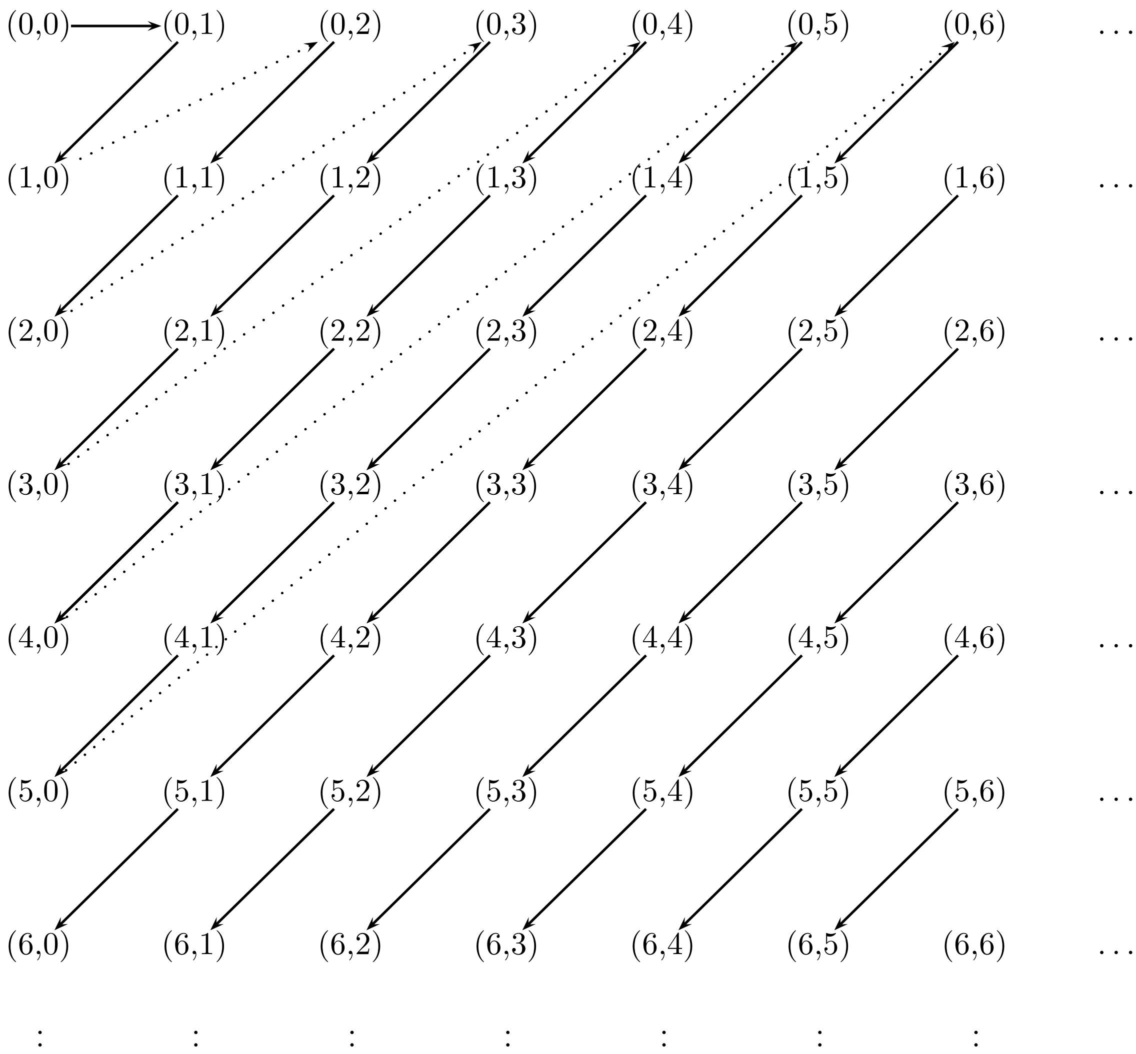
Now we want to prove a similar theorem for the rational numbers. This depends on the following purely set-theoretic fact.

Proposition 7.22$^{\mathrm{ch}}$**.** $|\omega \times \omega| = \omega$.

Proof. We give an illustration of the proof on the next page. The progression of values of the function f is indicated by arrows. Thus $f(0) = (0,0)$, $f(1) = (0,1)$, $f(2) = (1,0)$, $f(3) = (0,2)$, $f(4) = (1,1)$, etc.

Thus we define f by recursion. $f(0) = (0,0)$. If $f(m)$ has been defined, write $f(m) = (a,b)$. Then if $b \ne 0$, write $b = c+1$ and let $f(m+1) = (a+1,c)$. If $b = 0$, let $f(m+1) = (0,a+1)$. We leave the task of formally applying the recursion theorem to an exercise.

We show that f is onto by proving by induction on m that for all natural numbers n, p, if $m = n + p$ then (n, p) is in the range of f. This is clear for $m = 0$. Assume that it is true for m. Now we prove by induction on n that for all n and p, if $n + p = m + 1$ then (n, p) is in the range of f. For $n = 0$ we want to show that $(0, m + 1)$ is in the range of f. Choose $a \in \omega$ such that $f(a) = (m, 0)$. Then $f(a+1) = (0, m+1)$, as desired. Now suppose that $(n, p) \in \text{rng}(f)$, with $n + p = m + 1$. We want to show that for all p, if $n + 1 + p = m$ then $(n + 1, p) \in \text{rng}(f)$. By the inductive assumption we have $(n, p + 1) \in \text{rng}(f)$. Say $f(a) = (n, p + 1)$. Then $f(a + 1) = (n + 1, p)$, as desired. This finishes the inductive proof. Note that this proof is funny, in that the inductive statement "for all n, for all p, if $n + p = m + 1$ then $(n, p) \in \text{rng}(f)$" becomes vacuously true if n is greater than $m + 1$. Formally, though, the staightforward proof given is correct. Anyway, this shows that f maps onto $\omega \times \omega$.



The one-one-ness follows from the following statement:

(1) If $a, b \in \omega$, $a < b$, $f(a) = (m, n)$, and $f(b) = (p, q)$, then either $m + n < p + q$, or else $m + n = p + q$ and $m < p$.

We prove (1) by induction on b, with a fixed, and we start with $b = a + 1$. If $n \neq 0$, then $m + n = p + q$ and $p = m + 1 > m$, as desired. If $n = 0$, then $p = m + 1$, $q = 0$, and $p + q = m + 1 > m = m + 0$, as desired. So the case $b = a + 1$ is ok. Assume our statement for $b > a$; we prove it for $b + 1$. So suppose that $f(a) = (m, n)$, $f(b) = (r, s)$, and $f(b+1) = (p, q)$. If $s = 0$, then $p + q = r + s + 1 > r + s \geq m + n$, the last inequality by the inductive assumption, as desired. If $s > 0$, then $p > r$. If $m+n = r+s$, then $r > m$ by the inductive assumption, so $p > m$, as desired. If $m + n < r + s$, then $m + n < p + q$, as desired. So (1) holds. $\square$

Proposition 7.23[ch]**.** *The set of all rational numbers is countable.*

Proof. For any $m, n \in \omega$ let

$$f(m,n) = \begin{cases} \frac{m}{2n} & \text{if } m \text{ is even and } n \neq 0, \\ -\frac{m+1}{2n} & \text{if } m \text{ is odd and } n \neq 0, \\ 0 & \text{if } n = 0. \end{cases}$$

To show that f maps onto the set of all rationals, let r be any rational. If $r > 0$, write $r = \frac{u}{v}$ with $u, v > 0$. then $f(2u, v) = \frac{2u}{2v} = \frac{u}{v} = r$. If $r = 0$, then $f(0,0) = 0 = r$. If $r < 0$, write $r = -\frac{u}{v}$ with $u, v > 0$. Then

$$f(2u-1, v) = -\frac{2u}{2v} = -\frac{u}{v} = r.$$

By 7.22, let $g : \omega \to \omega \times \omega$ be a bijection. Then $f \circ g$ maps ω onto the set of rationals. Hence that set is countable by 7.19. $\square$

Proposition 7.24. *If A and B are finite, then so is $A \cup B$.*

Proof. Let A be a fixed finite set. We show by induction on $m \in \omega$ that if B is a finite set with m elements, then $A \cup B$ is finite. This is true for $m = 0$ since if B is a finite set with 0 elements, then $B = \emptyset$, so $A \cup B = A$, which we are assuming is finite. Suppose that our statement is true for m, and now B is a finite set with $m+1$ elements. Let $b \in B$. Then $|B \backslash \{b\}| = m$ by 6.27. Hence $A \cup (B \backslash \{b\})$ is finite by the inductive assumption. Then $A \cup B$ is finite by 6.26, since either $A \cup B = A \cup (B \backslash \{b\})$, or $b \notin A$ and 6.26 applies. This finishes the inductive proof. $\square$

Proposition 7.25[ch]**.** *If A and B are countable, then $A \cup B$ is countable.*

Proof. We may assume that A and B are nonempty. By 7.19 there are functions $f : \omega \to A$, mapping onto A, and $g : \omega \to B$, mapping onto B. Define $h : \omega \times \omega \to A \cup B$ by setting, for any $m, n \in \omega$,

$$h(m,n) = \begin{cases} f(m) & \text{if } n = 0, \\ g(m) & \text{otherwise.} \end{cases}$$

Clearly h maps onto $A \cup B$. By 7.22 let $k : \omega \to \omega \times \omega$ be a bijection. Then $h \circ k$ maps ω onto $A \cup B$, so $A \cup B$ is countable by 7.19. $\square$

Proposition 7.26. *A finite union of finite sets is finite. More formally, if $\langle A_i : i \in I \rangle$ is a system of sets, I is finite, and each A_i is finite, then $\bigcup_{i \in I} A_i$ is finite.*

Proof. We leave this to an exercise. $\square$

We can considerably generalize 7.25:

Theorem 7.27$^{\text{ch}}$. *A countable union of countable sets is countable.*

More symbolically, Theorem 7.27 says that if $\langle A_i : i \in I \rangle$ is a system of sets, each A_i is countable, and I is countable, then also $\bigcup_{i \in I} A_i$ is countable.

Proof. We may assume that I is nonempty, as otherwise the union is empty, and the emptyset is certainly countable. Also, we may assume that each A_i is nonempty, since clearly

$$\bigcup_{i \in I} A_i = \bigcup_{\substack{i \in I \\ A_i \neq \emptyset}} A_i.$$

Let g be a function mapping ω onto I. Define $M = \{(f, i) : i \in I$ and f is a function mapping ω onto $A_i\}$. For each $(f, i) \in M$ let $F(f, i) = i$. Then F maps onto I, since for every $i \in I$ there is a function f mapping ω onto A_i. Let $G : I \to M$ be such that $F \circ G = \mathrm{Id}_I$. Then for each $i \in I$, $G(i)$ has the form (f, i) with f a function mapping ω onto A_i. Now we define, for any $m, n \in \omega$,

$$H(m, n) = (1^{\text{st}}(G(g(m))))(n).$$

Let us decode what this means. Let $m, n \in \omega$. Then $g(m)$ is a member of I. Applying G to it, we get an ordered pair of the form $(f, g(m))$. Then $1^{\text{st}}(G(g(m)))$ picks out that function f, and $H(m, n)$ is defined to be $f(n)$. From this description it follows that $H(m, n) \in A_{g(m)} \subseteq \bigcup_{j \in I} A_j$. We claim that H maps onto that union. For, suppose that $a \in \bigcup_{j \in I} A_j$. Choose $j \in I$ such that $a \in A_j$. Since g maps onto I, choose $m \in \omega$ such that $g(m) = j$. Let $f = 1^{\text{st}}(G(g(m)))$. Then f is a function mapping ω onto $A_{g(m)} = A_j$. So, choose $n \in \omega$ such that $f(n) = a$. Then $H(m, n) = a$, as desired.

Finally, taking a bijection h from ω to $\omega \times \omega$ given by Theorem 7.22, we see that $H \circ h$ maps ω onto the union, and hence the union is countable by 7.19. $\square$

Recall that a finite sequence is just a function with domain a natural number.

Proposition 7.28$^{\text{ch}}$. *The collection of all finite sequences of natural numbers is countable.*

Proof. First we show by induction on m that the set ${}^m\omega$ of all sequences of length m of natural numbers is countable. This is true for $m = 0$ since ${}^0\omega = \{\emptyset\}$ has just one element, and so it is countable. If we know that ${}^m\omega$ is countable, note that ${}^{m+1}\omega$ is equipotent with ${}^m\omega \times \omega$ via the function $f \mapsto (f \restriction m, f(m))$ for any $f \in {}^{m+1}\omega$. By the induction hypothesis, ${}^{m}\omega$ is countable, so 7.22 clearly implies that ${}^{m+1}\omega$ is countable as well.

Thus this proves that each set ${}^{m}\omega$ is countable. Since the collection of all finite sequences is the union of all of these sets ${}^{m}\omega$, our result now follows from 7.27. $\square$

Continuing to look at the size of sets encountered in normal mathematical usage, we next consider the size of the set $\mathbb{R}$ of real numbers. Since $\omega \subseteq \mathbb{R}$, this set is infinite. It is one of the main theorems of elementary set theory that $\mathbb{R}$ is uncountable. We first give the standard proof of this using Cantor's diagonal argument, and then we give generalizations of it whose proofs use that argument in somewhat disguised form.

Theorem 7.29$^{\text{ch}}$**.** *$\mathbb{R}$ is uncountable.*

Proof. Suppose to the contrary that $\mathbb{R}$ is countable. Then the same is true of the interval $[0,1)$. Since even this interval is clearly infinite, our supposition implies that $|[0,1)| = \omega$. Thus there is a bijection f from ω to $[0,1)$. Now we write each number in $[0,1)$ in its decimal expansion, where to assure uniqueness of the expansion we do not allow expansions which end with a string of 9's. To illustrate the diagonal argument coming up, we write the range of f as an infinite array using these expansions:

$$\begin{array}{lcl}
f(0) & = & .a_{00}\,a_{01}\;\; a_{02}\;\; a_{03}\;\; a_{04}\;\; a_{05}\;\; a_{06}\;\; \cdots\;\; \cdots\;\; \cdots \\
f(1) & = & .a_{10}\,a_{11}\;\; a_{12}\;\; a_{13}\;\; a_{14}\;\; a_{15}\;\; a_{16}\;\; \cdots\;\; \cdots\;\; \cdots \\
f(2) & = & .a_{20}\,a_{21}\;\; a_{22}\;\; a_{23}\;\; a_{24}\;\; a_{25}\;\; a_{26}\;\; \cdots\;\; \cdots\;\; \cdots \\
f(3) & = & .a_{30}\,a_{31}\;\; a_{32}\;\; a_{33}\;\; a_{34}\;\; a_{35}\;\; a_{36}\;\; \cdots\;\; \cdots\;\; \cdots \\
f(4) & = & .a_{40}\,a_{41}\;\; a_{42}\;\; a_{43}\;\; a_{44}\;\; a_{45}\;\; a_{46}\;\; \cdots\;\; \cdots\;\; \cdots \\
f(5) & = & .a_{50}\,a_{51}\;\; a_{52}\;\; a_{53}\;\; a_{54}\;\; a_{55}\;\; a_{56}\;\; \cdots\;\; \cdots\;\; \cdots \\
f(6) & = & .a_{60}\,a_{61}\;\; a_{62}\;\; a_{63}\;\; a_{64}\;\; a_{65}\;\; a_{66}\;\; \cdots\;\; \cdots\;\; \cdots \\
. & . & .\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;. \\
. & . & .\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;. \\
. & . & .\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.
\end{array}$$

Here each a_{ij} is one of the digits $0, 1, \ldots, 9$. Now we define another real number. Let

$$\mathrm{x} \quad = \quad .b_0\;\; b_1\;\; b_2\;\; b_3\;\; b_4\;\; b_5\;\; b_6\;\; \cdots\;\; \cdots\;\; \cdots,$$

Where the digits b_i are determined by this rule:

$$b_i = \begin{cases} 4 & \text{if } a_{ii} > 5, \\ 7 & \text{if } a_{ii} \leq 5. \end{cases}$$

Now clearly $x \in [0,1)$, so it is equal to $f(k)$ for some natural number k. But $b_k \neq a_{kk}$, which contradicts the uniqueness of decimal expansions. $\square$

Thus $|\mathbb{R}|$ is a new infinite cardinal number, strictly greater than ω. The *continuum hypothesis* is the statement that $|\mathbb{R}|$ is the smallest uncountable cardinal. Another way of putting this is that it says that there is no set P of real numbers such that $|\omega| < |P| < |\mathbb{R}|$.

This hypothesis is independent of our axioms of set theory: it cannot be derived from our axioms, nor can its negation. We will expand on this point later.

Using the binary expansion of real numbers, it should be clear that $\mathbb{R}$ has the same number of elements as the collection ${}^{\omega}2$ of all infinite sequences of 0's and 1's. We use the fact as motivation for a result that generalizes 7.29, but otherwise this fact will not play a role in what follows.

Proposition 7.30. *For any set A, the sets ${}^{A}2$ and $\mathscr{P}(A)$ are equipotent.*

Proof. For any $X \subseteq A$, the *characteristic function* of X is the function $\chi_X \in {}^{A}2$ defined by setting, for each $a \in A$,

$$\chi_X(a) = \begin{cases} 1 & \text{if } a \in X, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that χ is a bijection from $\mathscr{P}(A)$ to ${}^{A}2$. It is one-one, since if X and Y are different subsets of A, then if we take an element $a \in A$ which is in one of them but not the other, then clearly $\chi_X(a) \neq \chi_Y(a)$, and hence $\chi_X \neq \chi_Y$. And χ maps onto ${}^{A}2$, since if $f \in {}^{A}2$, let $X = \{a \in A : f(a) = 1\}$. Then for any $a \in A$,

$$\begin{aligned} \chi_X(a) &= \begin{cases} 1 & \text{if } a \in X, \\ 0 & \text{if } a \notin X, \end{cases} \\ &= \begin{cases} 1 & \text{if } f(a) = 1, \\ 0 & \text{if } f(a) = 0, \end{cases} \\ &= f(a). \end{aligned}$$

Thus $\chi_X = f$, as desired. □

In view of this proposition and the remarks preceding it, the following theorem is a generalization of Theorem 7.29.

Theorem 7.31$^{\text{ch}}$. *For any set A we have $|A| < |\mathscr{P}(A)|$.*

Proof. The function given by $a \mapsto \{a\}$ is a one-one function from A into $\mathscr{P}(A)$, and so $|A| \leq |\mathscr{P}(A)|$. [Saying that $a \mapsto \{a\}$ is giving the value of the function at the argument a; this notation was introduced earlier.] Suppose equality holds. Then there is a one-one function f mapping A onto $\mathscr{P}(A)$. Let $X = \{a \in A : a \notin f(a)\}$. Since f maps onto $\mathscr{P}(A)$, choose $a_0 \in A$ such that $f(a_0) = X$. Then $a_0 \in X$ iff $a_0 \notin X$, contradiction. □

This theorem gives us even bigger sets than $\mathbb{R}$; for example, $\mathscr{P}(\mathbb{R})$. The *generalized continuum hypothesis* is the statement that for any infinite set A, $|\mathscr{P}(A)|$ is the smallest cardinal greater than $|A|$. Obviously the generalized continuum hypothesis implies the continuum hypothesis itself. Whether the generalized continuum hypothesis holds is again independent of our axioms.

Exercises, chapter 7

1. Show that $|\alpha| \leq \alpha$ for every ordinal α.

2. Show that for any set A, $|A|$ is the smallest ordinal which is equipotent with A.

3. Show that for any ordinal α, $|\alpha| = \alpha$ iff α is a cardinal.

4. Prove that if L is a linearly ordered set, then every finite nonempty subset of L has a greatest element.

5. Justify the recursive definition in the proof of 7.22.

6. Prove that an equivalence relation on a countable set has a countable number of equivalence classes.

7. Show that if A is countable, then $\{X \subseteq A : X \text{ is finite}\}$ is countable.

8. Show that the set of all finite sequences of rationals is countable.

9. Prove Proposition 7.26.

10. Show that the collection of all polynomials with rational coefficients is countable.

11. A real number a is *algebraic* iff there is a polynomial $f(x)$ with rational coefficients such that $f(a) = 0$; otherwise a is *transcendental.* Show that the collection of algebraic real numbers is countable. Deduce from this that there are transcendental real numbers.

12. Show that the following sets are all equipotent with each other: $\mathbb{R}$, $[0, 1]$, $(0, 1)$, $[0, \infty)$.

13. Let F be the set of all functions mapping $\mathbb{R}$ into $\mathbb{R}$. Show that $|\mathbb{R}| < |F|$.

14. Show that if $|A| \leq |B|$ then $|\mathscr{P}(A)| \leq |\mathscr{P}(B)|$.

15. Show that if F is a finite subset of an infinite set A, then A and $A \backslash F$ are equipotent.

16. Prove that if $f : A \to B$, B is countable, and $f^{-1}[\{b\}]$ is countable for all $b \in B$, then A is countable.

17. For each $f \in {}^{\omega}2$ let $A_f = \{f \restriction m : m \in \omega\}$. Prove that if f and g are different members of ${}^{\omega}2$ then $A_f \cap A_g$ is finite.

18. Let $m \in \omega$, and suppose that $\mathscr{A}$ is a system of subsets of ω such that any two members of $\mathscr{A}$ intersect in a set of size at most m. Show that $\mathscr{A}$ is countable. (Cf. exercises 17, 20.)

19. Let S be the collection of all finite sequences of 0's and 1's. Prove that there is an uncountable system $\mathscr{A}$ of infinite subsets of S such that any two members of $\mathscr{A}$ have finite intersection.

20. Prove that there is an uncountable system $\mathscr{A}$ of infinite subsets of ω such that any two members of $\mathscr{A}$ have finite intersection.

21. Assume that A is countable. An infinite sequence $a \in {}^{\omega}A$ is *eventually constant* iff there is some $m \in \omega$ such that $a_n = a_m$ for all $n \geq m$. Prove that the collection of all eventually constant infinite sequences of members of A is countable.

8. Simple cardinal arithmetic

In this chapter we define and give the simplest properties of the addition, multplication, and exponentiation operations on cardinal numbers. These generalize the operations on natural numbers.

There is one main theorem about these operations: $\kappa \cdot \kappa = \kappa$ for every infinite cardinal κ. This generalizes the fact from the last chapter that $\omega \times \omega$ is equipotent with ω. In order to prove this main theorem, we need an important result about ordinals, and we begin this chapter with lemmas leading up to it.

First we need to explicitly define the notion of isomorphism between ordered sets that was briefly mentioned in chapter 5. If $(A, <)$ and $(B, \prec)$ are partially ordered sets, then an *isomorphism* from $(A, <)$ to $(B, \prec)$ is a bijection f from A to B such that for any $a_0, a_1 \in A$, $a_0 < a_1$ iff $f(a_0) \prec f(a_1)$. Sometimes we simply say that f is an isomorphism from A to B if the orderings are understood. A function $f : A \to B$ is *strictly increasing* iff for all $a_0, a_1 \in A$, if $a_0 < a_1$ then $f(a_0) \prec f(a_1)$. Thus the converse direction is not assumed, and f is not assumed to be one-one or to map onto B.

Lemma 8.1. *If $(A, <)$ and $(B, \prec)$ are linearly ordered sets and $f : A \to B$ is strictly increasing, then for all $a_0, a_1 \in A$, $a_0 < a_1$ iff $f(a_0) \prec f(a_1)$.*

Proof. The direction $\Rightarrow$ is given by the definition. Now suppose that it is not true that $a_0 < a_1$. Then $a_1 \le a_0$, so $f(a_1) \le f(a_0)$. So $f(a_0) < f(a_1)$ is not true. $\square$

Lemma 8.2. *If $(A, <)$ is a well-ordered set and $f : A \to A$ is strictly increasing, then $x \le f(x)$ for all $x \in A$.*

Proof. Suppose not. Then then set $B \stackrel{\text{def}}{=} \{x \in A : f(x) < x\}$ is nonempty. Let b be the least element of B. Thus $f(b) < b$. Hence by the choice of b, we have $f(b) \le f(f(b))$. Hence by 8.1, $b \le f(b)$, contradiction. $\square$

Lemma 8.3. *If α and β are order-isomorphic ordinals, then $\alpha = \beta$.*

Proof. Suppose that α and β are order-isomorphic and different; say $\alpha < \beta$. Let f be an order isomorphism of β onto α. [Notice that the relation of being isomorphic is symmetric, so we can assume that such an f exists.] Now $\alpha \subseteq \beta$ by 7.5, so f is a strictly increasing function from β into β. Hence by 8.2, $x \le f(x)$ for all $x \in \beta$. In particular, $\alpha \le f(\alpha) < \alpha$, contradiction. $\square$

Lemma 8.4. *If $(A, <)$ is a well-ordered set, then the only isomorphism of $(A, <)$ onto $(A, <)$ is the identity mapping.*

Proof. Clearly the identity mapping Id_A *is* an isomorphism from $(A, <)$ onto $(A, <)$. Now suppose that f is any isomorphism from $(A, <)$ onto $(A, <)$. Then $x \le f(x)$ for all $x \in A$, by 8.2. Also, f^{-1} is an isomorphism from $(A, <)$ onto $(A, <)$, so $x \le f^{-1}(x)$ for all $x \in A$. Since f is strictly increasing, it follows that $f(x) \le f(f^{-1}(x)) = x$ for all $x \in A$. So $f(x) = x$ for all $x \in A$, and thus $f = \mathrm{Id}_A$. $\square$

Lemma 8.5. *If $(A, <)$ and $(B, \prec)$ are isomorphic well-ordered sets, then there is exactly one isomorphism between them.*

Proof. Suppose that f and g are isomorphisms from A to B. Then $g^{-1} \circ f$ is an isomorphism from A to A, so $g^{-1} \circ f$ is the identity on A. It follows that $f = g$. $\square$

One more thing is needed for the indicated result about ordinals: our last axiom. This axiom depends on the intuitive notion of a function, which we put in a set-theoretical framework in Chapter 3. Now we go back to the intuition. A *class function* is such an intuitive function: a rule which assigns to every element of its domain some uniquely determined object. This is not a precise definition (although it could be made precise by developing some logical notions). Some examples of class functions are as follows. For any set x we can consider the identity function which assigns x to x; or the function which assigns $x \cup \{x\}$ to x; or the function which assigns $\mathscr{P}(x)$ to x. These cannot really be considered as functions in the sense of Chapter 3 because they are too big; each of them has domain the collection of all sets, which we saw at the beginning of the notes is too big to be itself considered as a set. There are even class functions which turn out to be small enough to fit in the official definition which are not immediately seen to be small. This applies to the class function which we will use in our theorem about ordinals. The last axiom deals with all kinds of class functions, big and small.

Axiom 9. (*Replacement*) **If F is a class function and A is a set, then the collection of all sets $F(a)$ with $a \in A$ and a in the domain of F is a set.**

To help the intuition, let us apply this axiom to the three examples of class functions above. The first one yields, for any set A, the set A itself; not very exciting. The second one gives for any set A the set $\{a \cup \{a\} : a \in A\}$, and the third one, $\{\mathscr{P}(x) : x \in A\}$. It is not immediately clear how to prove that these exist without using the replacement axiom.

Now we can prove the theorem needed for our treatment of cardinal arithmetic. This theorem says that the ordinals represent all well-ordered sets.

Theorem 8.6. (Ordinal representation theorem) *Every well-ordered set is isomorphic to a unique ordinal.*

Proof. Let $(A, <)$ be a well-ordered set. For each $a \in A$ let $\mathrm{pred}(a, A) = \{x \in A : x < a\}$. Clearly $\mathrm{pred}(a, A)$ is a well-ordered set under the order induced by $<$. That is, under the ordering $x \prec y$ iff $x, y \in \mathrm{pred}(a, A)$ and $x < y$; of course we still denote $\prec$ by $<$ although we are really considering its intersection with $\mathrm{pred}(a, A) \times \mathrm{pred}(a, A)$. Let

$$B = \{a \in A : \mathrm{pred}(a, A) \text{ is isomorphic to an ordinal}\}.$$

(1) If $a \in B$ and $b < a$, then $b \in B$. In fact, if g is an isomorphism from $\mathrm{pred}(a, A)$ onto an ordinal β, then $g \restriction \mathrm{pred}(b, A)$ is an isomorphism from $\mathrm{pred}(b, A)$ onto the ordinal $g(b)$.

For, assume that $a \in B$, $b < a$, and g is as indicated. Let $h \stackrel{\text{def}}{=} g \restriction \mathrm{pred}(b, A)$. Let $g(b) = \alpha$. If $c \in \mathrm{pred}(b, A)$ then $c < b$, and hence $h(c) = g(c) < g(b) = \alpha$, so h maps into α. For any $c, d \in \mathrm{pred}(b, A)$ we have $c < d$ iff $g(c) < g(d)$ iff $h(c) < h(d)$. Given $\delta < \alpha$, we have $\delta < \beta$ by the transitivity of β, so there is a $c \in \mathrm{pred}(a, A)$ such that $g(c) = \delta$. Then $g(c) = \delta < \alpha = g(b)$, so $c < b$; hence $h(c) = \delta$. This shows that h maps onto α, and finishes the proof that h is an order-isomorphism from $\mathrm{pred}(b, A)$ onto α. Thus (1) holds.

If $a \in B$, and $\mathrm{pred}(a, A)$ is isomorphic to ordinals α and β, then clearly α is isomorphic to β since isomorphism satisfies the formal conditions of an equivalence relation (although we do not need to worry about whether there is a *set* equal to this relation). Then $\alpha = \beta$ by 8.3. Thus we can define $F(a)$ to be the unique ordinal isomorphic to $\mathrm{pred}(a, A)$, for each $a \in B$. Note that F is a class function; we do not claim (yet) that it is a set. By the axiom of replacement, let $C = \{F(a) : a \in B\}$, a set. Then $F = \{(a, c) : a \in B, c \in C, F(a) = c\}$, so F really is a set.

(2) C is an ordinal.

In fact, to see that it is transitive, suppose that $\alpha \in \beta \in C$. Then we can choose $a \in B$ such that $F(a) = \beta$. Let g be an isomorphism of $\mathrm{pred}(a, A)$ onto β. Choose $b < a$ such that $g(b) = \alpha$. Then by (1), $g \restriction \mathrm{pred}(b, a)$ is an isomorphism from $\mathrm{pred}(b, a)$ onto α, so $\alpha \in C$. This proves that C is transitive. Since C is a set of ordinals, it is hence an ordinal by 7.7. So (2) holds.

(3) If $a, b \in B$, then $a < b$ iff $F(a) < F(b)$.

In fact, assume that $a, b \in B$ and $b < a$. Let g be an isomorphism from $\mathrm{pred}(a, A)$ onto $F(a)$. Then by (1) we get $F(b) < F(a)$. Thus F is strictly increasing. So (3) follows from 8.1.

Now we will be through if we show that $A = B$, since then by (2) and (3) F is an isomorphism from A to C. Suppose that $A \neq B$, and let a be the least element of $A \backslash B$. If $b < a$, then $b \in B$, and if $b \in B$, then $b < a$ by (1). ($b \neq a$ by the choice of a, and $a < b$ would contradict (1).) Thus $\mathrm{pred}(a, A) = B$, and F is an isomorphism of B onto an ordinal, so $B \in B$, contradiction. □

Now we turn to the main topic of this chapter: simple cardinal arithmetic. First we define the *sum* of cardinals κ and λ:

$$\kappa + \lambda = |(\kappa \times \{0\}) \cup (\lambda \times \{1\})|.$$

The idea here is that we "disjoint" κ and λ and take the number of elements in the result. Note that κ and $\kappa \times \{0\}$ are equipotent, and so have the same number of elements. In fact, the function given by $\alpha \mapsto (\alpha, 0)$ is clearly a bijection from κ to $\kappa \times \{0\}$. Similar comments apply to λ and $\lambda \times \{1\}$. The sets $\kappa \times \{0\}$ and $\lambda \times \{1\}$ are disjoint since elements of $\kappa \times \{0\}$ have the form $(\alpha, 0)$ for some $\alpha < \kappa$, while elements of $\lambda \times \{1\}$ have the form $(\beta, 1)$ for some $\beta < \lambda$. We cannot have $(\alpha, 0) = (\beta, 1)$, as that would imply that $0 = 1$.

The particular way of "disjointing" κ and λ is not really important, as the following proposition indicates.

Proposition 8.7$^{\mathrm{ch}}$. *If A, B, C, D are sets, $|A| = |C|$, $|B| = |D|$, $A \cap B = \emptyset$, and $C \cap D = \emptyset$, then $|A \cup B| = |C \cup D|$.*

Proof. Assume the hypotheses. Then there is a bijection f from A to C, and a bijection g from B to D. Define $h : A \cup B \to C \cup D$ by setting, for any $x \in A \cup B$,

$$h(x) = \begin{cases} f(x) & \text{if } x \in A, \\ g(x) & \text{if } x \in B. \end{cases}$$

This definition is unambigous because $A \cap B = \emptyset$. Clearly h really does map $A \cup B$ into $C \cup D$. To show that h is one-one, suppose that $x, y \in A \cup B$ and $h(x) = h(y)$. If $x, y \in A$, then $f(x) = h(x) = h(y) = f(y)$, so $x = y$ because f is one-one. If $x, y \in B$, then $g(x) = h(x) = h(y) = g(y)$, so $x = y$ because g is one-one. If $x \in A$ and $y \in B$, then $f(x) = h(x) = h(y) = g(y)$, so this set is in both C and D, contradicting our assumption that $C \cap D = \emptyset$. Similarly if $x \in B$ and $y \in A$. So h is one-one.

h maps onto $C \cup D$: suppose that $z \in C \cup D$. If $z \in C$, choose $x \in A$ such that $f(x) = z$; then $h(x) = f(x) = z$. If $z \in D$, choose $x \in B$ such that $g(x) = z$; then $h(x) = g(x) = z$. □

One of our first tasks with the definition of addition is to prove that this definition extends the definition of addition for natural numbers. Thus we are justified in using the same notation $+$ for the general notion.

Proposition 8.8$^{\text{ch}}$. *If m and n are natural numbers, then addition in the sense of chapter 6 and in the cardinal number sense are the same.*

Proof. For clarity, let the addition in section 6 be denoted by $+'$, and the new cardinal addition by $+$. Fix m. By induction on n we show that $m +' n = m + n$ for every natural number n. For $n = 0$, we have $m +' 0 = m$ by definition, and $m + 0 = m$ since the second set in the definition of $m + 0$ is empty, and so the definition reduces to $m + 0 = |m \times \{0\}|$; we observed above that m is equipotent with $m \times \{0\}$, so $|m \times \{0\}| = m$, as desired. Now assume that $m+'n = m+n$. Now by the definitions involved, $m+'(n+'1) = (m+'n)+'1 = (m +' n) \cup \{m +' n\}$, while

$$\begin{aligned} m + (n +' 1) &= m + (n \cup \{n\}) \\ &= |(m \times \{0\}) \cup ((n \cup \{n\}) \times \{1\})| \\ &= |(m \times \{0\}) \cup (n \times \{1\}) \cup \{(n, 1)\}|. \end{aligned}$$

Now by the inductive hypothesis we have $m +' n = m + n$, so there is a one-one function f mapping $m +' n$ onto $(m \times \{0\}) \cup (n \times \{1\})$. Let $g = f \cup \{(m +' n, (n, 1))\}$. Then g is a one-one function from $(m +' n) +' 1$ onto $(m \times \{0\}) \cup ((n \cup \{n\}) \times \{1\})$ from which it follows that $m +' (n +' 1) = m + (n +' 1)$, completing the inductive proof. □

Proposition 8.9$^{\text{ch}}$. *For any sets A, B, $|A \cup B| \leq |A| + |B|$.*

Proof. Let f be a bijection from A to $|A|$, and let g be a bijection from B to $|B|$. Define $h : A \cup B \to (|A| \times \{0\}) \cup (|B| \times \{1\})$ by setting, for any $x \in A \cup B$,

$$h(x) = \begin{cases} (f(x), 0) & \text{if } x \in A, \\ (g(x), 1) & \text{if } x \notin A \text{ (and hence } x \in B). \end{cases}$$

Clearly $h : A \cup B \to (|A| \times \{0\}) \cup (|B| \times \{1\})$. We claim that it is one-one. For, suppose that $x, y \in A \cup B$ and $h(x) = h(y)$. If $x, y \in A$, then $f(x) = 1^{\text{st}}(h(x)) = 1^{\text{st}}(h(y)) = f(y)$, so $x = y$ since f is one-one. If both x and y are not in A, then $g(x) = 1^{\text{st}}(h(x)) = 1^{\text{st}}(h(y)) = g(y)$, so $x = y$ since g is one-one. If $x \in A$ and $y \notin A$, then $0 = 2^{\text{nd}}(h(a)) = 2^{\text{nd}}(h(y)) = 1$, contradiction. Similarly if $x \notin A$ and $y \in A$. Hence h is one-one.

Hence

$$\begin{aligned}|A \cup B| &\leq |(|A| \times \{0\}) \cup (|B| \times \{1\}) \quad \text{by 7.16}\\ &= |A| + |B| \quad \text{by definition}\end{aligned}$$ □

Proposition 8.10$^{\text{ch}}$**.** *If $\kappa, \lambda, \mu, \nu$ are cardinals, $\kappa \leq \mu$, and $\lambda \leq \nu$, then $\kappa + \lambda \leq \mu + \nu$.*

Proof. By the definition of sum,

(1) $$\kappa + \lambda = |(\kappa \times \{0\}) \cup (\lambda \times \{1\})| \quad \text{and}$$

(2) $$\mu + \nu = |(\mu \times \{0\}) \cup (\nu \times \{1\})|.$$

Now $\kappa \leq \mu$ implies that $\kappa \subseteq \mu$, and hence clearly $\kappa \times \{0\} \subseteq \mu \times \{0\}$. Similarly, $\lambda \times \{1\} \subseteq \nu \times \{1\}$. Thus the set on the right side of (1) is a subset of that on the right side of (2). Hence our desired result follows from 7.14. □

Now we give some elementary properties of addition.

Proposition 8.11$^{\text{ch}}$**.** *Suppose that κ, λ, μ are cardinals. Then*

(i) $\kappa + 0 = \kappa$.

(ii) $\kappa + \lambda = \lambda + \kappa$.

(iii) $\kappa + (\lambda + \mu) = (\kappa + \lambda) + \mu$.

Proof. (i): Since $0 = \emptyset$, we also have $0 \times \{1\} = \emptyset$, so

$$\kappa + 0 = |(\kappa \times \{0\}) \cup (0 \times \{1\})| = |\{\kappa \times \{0\}| = \kappa,$$

using again the fact that $\{\kappa \times \{0\}$ is equipotent with κ, and hence $|\{\kappa \times \{0\}| = \kappa$.

(ii): We have

$$\begin{aligned}\kappa + \lambda =& |(\kappa \times \{0\}) \cup (\lambda \times \{1\})| \quad \text{and}\\ \lambda + \kappa =& |(\lambda \times \{0\}) \cup) \kappa \times \{1\}|,\end{aligned}$$

so it suffices to show that $(\kappa \times \{0\}) \cup (\lambda \times \{1\})$ and $(\lambda \times \{0\}) \cup) \kappa \times \{1\}$ are equipotent. For each $\alpha < \kappa$ let $f(\alpha, 0) = (\alpha, 1)$. Clearly f is a bijection from $\kappa \times \{0\}$ onto $\kappa \times \{1\}$. Similarly, there is a bijection from $\lambda \times \{1\}$ onto $\lambda \times \{0\}$. So our desired result follows from 8.7.

(iii): We have

$$\begin{aligned}\lambda + \mu &= |(\lambda \times \{0\}) \cup (\mu \times \{1\})| \quad \text{by definition}\\ &= |(\lambda \times \{1\}) \cup (\mu \times \{2\})| \quad \text{using 8.7} \qquad (1)\end{aligned}$$

We also clearly have

(2) $$\lambda + \mu = |(\lambda + \mu) \times \{1\}|.$$

Moreover,

$$(\kappa \times \{0\}) \cap [(\lambda \times \{1\}) \cup (\mu \times \{2\})] = \emptyset. \tag{3}$$

It follows that

$$\kappa + (\lambda + \mu) = |(\kappa \times \{0\}) \cup (\lambda \times \{1\}) \cup (\mu \times \{2\})|. \tag{4}$$

similarly,

$$(\kappa + \lambda) + \mu = |(\kappa \times \{0\}) \cup (\lambda \times \{1\}) \cup (\mu \times \{2\})|. \tag{5}$$

So (iii) follows. □

There are other very important properties of addition; for example, $\kappa + \kappa = \kappa$ whenever κ is infinite. This and analogous facts follow from a similar fact about multiplication, so we turn to it now.

For any cardinals κ, λ we define

$$\kappa \cdot \lambda = |\kappa \times \lambda|.$$

Before showing that multiplication, so defined, coincides with our old definition when restricted to natural numbers, it is convenient to establish some simple arithmetical laws.

Proposition 8.12$^{\text{ch}}$. *If A is equipotent with C and B is equipotent with D, then $A \times B$ is equipotent with $C \times D$.*

Proof. Let $f : A \to C$ be one-one and onto, and let $g : B \to D$ be one-one and onto. Define $h(a, b) = (f(a), g(b))$ for any $a \in A$ and $b \in B$. Then $h : A \times B \to C \times D$ is one-one and onto. In fact, to show that h is one-one, suppose that $h(a, b) = h(a', b')$. Then $(f(a), g(b)) = (f(a'), g(b'))$, so $f(a) = f(a')$ and $g(b) = g(b')$. Hence $a = a'$ and $b = b'$ since f and g are one-one. So $(a, b) = (a', b')$, proving that h is one-one. To show that h maps onto $C \times D$, suppose that $(c, d) \in C \times D$. Since f maps onto C, choose $a \in A$ such that $f(a) = c$. Similarly we get $b \in B$ such that $g(b) = d$. So $h(a, b) = (f(a), g(b)) = (c, d)$, showing that h maps onto $C \times D$. □

Proposition 8.13$^{\text{ch}}$. *$A \times (B \times C)$ is equipotent with $(A \times B) \times C$.*

Proof. Left to an exercise. □

Now we give the most common properties of multiplication.

Proposition 8.14$^{\text{ch}}$. *Assume that κ, λ, μ are cardinals.*

(i) $\kappa \cdot \lambda = \lambda \cdot \kappa$;
(ii) $\kappa \cdot (\lambda \cdot \mu) = (\kappa \cdot \lambda) \cdot \mu$;
(iii) $\kappa \cdot (\lambda + \mu) = \kappa \cdot \lambda + \kappa \cdot \mu$;
(iv) $\kappa \cdot 0 = 0$;
(v) $\kappa \cdot 1 = \kappa$;

(vi) $\kappa \cdot 2 = \kappa + \kappa$;
(vii) If $\kappa \leq \mu$ *and* $\lambda \leq \nu$, *then* $\kappa \cdot \lambda \leq \mu \cdot \nu$.

Proof. We leave many details here to the reader; they are too easy even for exercises. For (i), the mapping $(\alpha, \beta) \mapsto (\beta, \alpha)$ is a bijection from $\kappa \times \lambda$ to $\lambda \times \kappa$, and (i) follows.
(ii): Use 8.13.
(iii):

$$\begin{aligned} \kappa \cdot (\lambda + \mu) &= |\kappa \times (\lambda + \mu)| \\ &= |\kappa \times [(\lambda \times \{0\}) \cup (\mu \times \{1\})]| \quad \text{using 8.12} \\ &= |(\kappa \times (\lambda \times \{0\})) \cup (\kappa \times (\mu \times \{1\}))| \\ &= |((\kappa \times \lambda) \times \{0\}) \cup ((\kappa \times \mu) \times \{1\})| \quad \text{by 8.13} \\ &= |((\kappa \cdot \lambda) \times \{0\}) \cup ((\kappa \cdot \mu) \times \{1\})| \quad \text{by 8.12} \\ &= \kappa \cdot \lambda + \kappa \cdot \mu. \end{aligned}$$

(iv): The set $\kappa \times 0 = \kappa \times \emptyset$ is empty, and (iv) follows.
(v): Note that $\kappa \cdot 1 = |\kappa \times 1| = |\kappa \times \{0\}|$; (v) follows.
(vi):

$$\begin{aligned} \kappa \cdot 2 &= |\kappa \cdot 2| \\ &= |\kappa \cdot \{0, 1\}| \\ &= |(\kappa \cdot \{0\}) \cup (\kappa \cdot \{1\})|; \end{aligned}$$

the last two sets here are disjoint and each of size κ, so 8.7 can be used.
(vii): Note that, under the indicated hypotheses, $\kappa \times \lambda \subseteq \mu \times \nu$. □

Now we check that for natural numbers multiplication has the same meaning as before:

Proposition 8.15$^{\mathrm{ch}}$. *Multiplication of natural numbers means the same in the cardinal number sense as in the sense of section 4.*

Proof. Let multiplication in the sense of section 3 be denoted by $\cdot'$, and in the general cardinal number sense by $\cdot$. Fix $m \in \omega$. We prove by induction on n that $m \cdot' n = m \cdot n$ for every $n \in \omega$. It is obvious for $n = 0$, since both sides are then 0. Now assume true for n. Then

$$\begin{aligned} m \cdot' (n +' 1) &= m \cdot' n + m \quad \text{by definition of } \cdot' \text{ and 8.8} \\ &= m \cdot n + m \quad \text{by the inductive hypothesis} \\ &= m \cdot n + m \cdot 1 \\ &= m \cdot (n + 1) \\ &= m \cdot (n +' 1), \end{aligned}$$

as desired. □

The basic theorem about multiplication of infinite cardinals is as follows.

Theorem 8.16$^{\text{ch}}$**.** *$\kappa \cdot \kappa = \kappa$ for every infinite cardinal κ.*

Proof. Suppose not, and let κ be the least infinite cardinal such that $\kappa \cdot \kappa \neq \kappa$. Then $\kappa = \kappa \cdot 1 \leq \kappa \cdot \kappa$, and so $\kappa < \kappa \cdot \kappa$. We now define a relation $\prec$ on $\kappa \times \kappa$. For all $\alpha, \beta, \gamma, \delta \in \kappa$,

$$\begin{aligned}(\alpha, \beta) \prec (\gamma, \delta) \text{ iff } & \max(\alpha, \beta) < \max(\gamma, \delta) \\ \text{or } & \max(\alpha, \beta) = \max(\gamma, \delta) \text{ and } \alpha < \gamma \\ \text{or } & \max(\alpha, \beta) = \max(\gamma, \delta) \text{ and } \alpha = \gamma \text{ and } \beta < \delta.\end{aligned}$$

In words, one pair comes before another iff the maximum of the two terms of the first pair is less than the maximum for the second pair, or else those maximums are the same and the first pair is less, in the dictionary order, than the second. This relation $\prec$ is clearly a simple order of $\kappa \times \kappa$. It is, in fact, a well-order. For suppose that Γ is a non-empty subset of κ. Let γ be the least ordinal such that $\max(\alpha, \beta) = \gamma$ for some $(\alpha, \beta) \in \Gamma$. Then let α be the least ordinal such that $\max(\alpha, \beta) = \gamma$ for some β with $(\alpha, \beta) \in \Gamma$. And finally, let β be the least ordinal such that $(\alpha, \beta) \in \Gamma$. Clearly (α, β) is the least member of Γ in the sense $\prec$.

It follows that $(\kappa \times \kappa, \prec)$ is isomorphic to an ordinal α; let f be the isomorphism. We have $|\alpha| = |\kappa \times \kappa| = \kappa \cdot \kappa > \kappa$ by the remark at the beginning of this proof. So $\kappa < \alpha$. Therefore there exist $\beta, \gamma \in \kappa$ such that $f(\beta, \gamma) = \kappa$. Now

$$f[\{(\delta, \varepsilon) \in \kappa \times \kappa : (\delta, \varepsilon) \prec (\beta, \gamma)\}] = \kappa,$$

so, with $\varphi = \max(\beta, \gamma) + 1$,

$$\begin{aligned}\kappa &= |\{(\delta, \varepsilon) \in \kappa \times \kappa : (\delta, \varepsilon) \prec (\beta, \gamma)\}| \\ &\leq |\varphi \times \varphi| = |\varphi| \cdot |\varphi|.\end{aligned}$$

But $\varphi < \kappa$, so either φ is finite, and $|\varphi| \cdot |\varphi|$ is then also finite, or else φ is infinite, and $|\varphi| \cdot |\varphi| = |\varphi|$ by the minimality of κ. In any case, $|\varphi| \cdot |\varphi| < \kappa$, contradiction. □

With the aid of this theorem we can completely describe how addition and multiplication of cardinals work, when one of them is infinite.

Corollary 8.17$^{\text{ch}}$**.** *Let κ and λ be cardinals.*

(i) If κ is infinite, then $\kappa + \kappa = \kappa$.
(ii) If at least one of κ, λ is infinite, then $\kappa + \lambda = \max(\kappa, \lambda)$.
(iii) $\kappa \cdot 0 = 0$.
(iv) $\kappa \cdot 1 = \kappa$.
(v) If both κ, λ are at least 2 and one of them is infinite, then $\kappa \cdot \lambda = \max(\kappa, \lambda)$.

Proof. We have already proved (iii) and (iv): see 8.14(iv),(v). Next we prove (v). By symmetry, let $\lambda = \max(\kappa, \lambda)$. Then

$$\begin{aligned}\kappa \cdot \lambda &\leq \lambda \cdot \lambda \quad \text{by 8.14(vii)} \\ &= \lambda \quad \text{by 8.16} \\ &= 1 \cdot \lambda \quad \text{by 8.14(v)} \\ &\leq \kappa \cdot \lambda \quad \text{by 8.14(vii).}\end{aligned}$$

Thus (v) holds.

Now we prove (ii); by symmetry say $\lambda = \max(\kappa, \lambda)$.

$$\begin{aligned}\kappa + \lambda &\leq \lambda + \lambda \quad \text{by 8.10}\\ &= \lambda \cdot 2 \quad \text{by 8.14(vi)}\\ &= \lambda \quad \text{by (v)}\\ &= 0 + \lambda\\ &\leq \kappa + \lambda \quad \text{by 8.10.}\end{aligned}$$

Thus (ii) holds. Clearly (i) is a special case of (ii). □

Next we consider exponentiation. For any cardinals κ, λ we define

$$\kappa^\lambda = |{}^\lambda \kappa|.$$

This is a reasonable definition, since in considering the number of functions f from λ to κ one can select each of the λ values of f in κ many ways, all independently of one another. Thus κ^λ should be the result of multiplying κ by itself λ many times. In a later chapter we will introduce infinite products, and this will be formally stated and proved.

Proposition 8.18$^{\text{ch}}$. *For any sets A, B, C, D, if $|A| = |C|$ and $|B| = |D|$, then $|{}^A B| = |{}^C D|$.*

Proof. Let f be a bijection from A to C, and g a bijection from B to D. For each $h \in {}^A B$ define $F(h) = g \circ h \circ f^{-1}$. Since $f^{-1} : C \to A$, $h : A \to B$, and $g : B \to D$, we have $F(h) : C \to D$, and so $F(h) \in {}^C D$. To show that F is one-one, suppose that $h, k \in {}^A B$ and $F(h) = F(k)$. Thus $g \circ h \circ f^{-1} = g \circ k \circ f^{-1}$, so

$$\begin{aligned}g \circ h &= g \circ h \circ \mathrm{Id}_A\\ &= g \circ h \circ f^{-1} \circ f\\ &= g \circ k \circ f^{-1} \circ f\\ &= g \circ k \circ \mathrm{Id}_A\\ &= g \circ k.\end{aligned}$$

Hence $h = \mathrm{Id}_B \circ h = g^{-1} \circ g \circ h = g^{-1} \circ g \circ k = \mathrm{Id}_B \circ k = k$. This proves that F is one-one.

To show that F maps onto ${}^C D$, let $l \in {}^C D$. Define $h = g^{-1} \circ l \circ f$. Since $f : A \to$, $l : C \to D$, and $g^{-1} : D \to B$, we have $h : A \to B$. Hence

$$F(h) = g \circ h \circ f^{-1} = g \circ g^{-1} \circ l \circ f \circ f^{-1} = \mathrm{Id}_D \circ l \circ \mathrm{Id}_C = l. \qquad \square$$

The elementary arithmetic of exponentiation is summarized in the following proposition:

Proposition 8.19$^{\text{ch}}$. *Let κ, λ, μ be cardinals.*

(i) $\kappa^0 = 1$.

(ii) If $\kappa \neq 0$, *then* $0^\kappa = 0$.

(iii) $\kappa^1 = \kappa$.
(iv) $1^\kappa = 1$.
(v) $\kappa^2 = \kappa \cdot \kappa$.
(vi) $\kappa^\lambda \cdot \kappa^\mu = \kappa^{\lambda+\mu}$.
(vii) $(\kappa \cdot \lambda)^\mu = \kappa^\mu \cdot \lambda^\mu$.
(viii) $(\kappa^\lambda)^\mu = \kappa^{\lambda\cdot\mu}$.
(ix) *If* $\kappa \leq \lambda \neq 0$ *and* $\mu \leq \nu$, *then* $\kappa^\mu \leq \lambda^\nu$.

Proof. (i) is true since there is exactly one function mapping 0 into κ, namely 0 itself.

(ii): If $\kappa \neq 0$, then there are no functions mapping κ into the empty set.

(iii): The mapping $\alpha \mapsto \{(0, \alpha)\}$ is a bijection from κ to ${}^1\kappa$. Note that the members of ${}^1\kappa$ are tiny functions with domain 1, which is $\{0\}$.

(iv): There is only one function mapping κ into 1, namely the function that assigns 0 to each $\alpha < \kappa$.

(v): the function $(\alpha, \beta) \mapsto \{(0, \alpha), (1, \beta)\}$ for $\alpha, \beta < \kappa$ is a bijection from $\kappa \times \kappa$ to ${}^2\kappa$.

(vi): This is more complicated that the preceding facts. First note that

$$\begin{aligned}
\kappa^\lambda \cdot \kappa^\mu &= |(\kappa^\lambda) \times (\kappa^\mu)| \quad \text{by definition}\\
&= |({}^\lambda\kappa) \times ({}^\mu\kappa)| \quad \text{by 8.12, and}\\
\kappa^{\lambda+\mu} &= |{}^{\lambda+\mu}\kappa| \quad \text{by definition}\\
&= |{}^{(\lambda\times\{0\})\cup(\mu\times\{1\})}\kappa| \quad \text{by 8.18.}
\end{aligned}$$

Hence it suffices to show that the two sets

$$\begin{aligned}
A &\stackrel{\text{def}}{=} ({}^\lambda\kappa) \times ({}^\mu\kappa) \quad \text{and}\\
B &\stackrel{\text{def}}{=} {}^{(\lambda\times\{0\})\cup(\mu\times\{1\})}\kappa
\end{aligned}$$

are equipotent. We define a function F with domain $({}^\lambda\kappa) \times ({}^\mu\kappa)$ as follows. If $f \in {}^\lambda\kappa$ and $g \in {}^\mu\kappa$, then $F(f, g)$ is itself a function with domain $(\lambda \times \{0\}) \cup (\mu \times \{1\})$, and for any $x \in (\lambda \times \{0\}) \cup (\mu \times \{1\})$,

$$(F(f,g))(x) = \begin{cases} f(\alpha) & \text{if } x = (\alpha, 0) \text{ for some } \alpha < \lambda,\\ g(\beta) & \text{if } x = (\beta, 1) \text{ for some } \beta < \mu. \end{cases}$$

(Note that we really should write $F((f, g))$ rather than $F(f, g)$.) Thus F clearly maps A into B. To show that F is one-one, suppose that $f, f' \in {}^\lambda\kappa$, $g, g' \in {}^\mu\kappa$, and $F(f, g) = F(f', g')$. Then for any $\alpha < \lambda$ we have $f(\alpha) = (F(f, g))((\alpha, 0)) = (F(f', g'))((\alpha, 0)) = f'(\alpha)$. Hence $f = f'$. Similarly $g = g'$. So $(f, g) = (f', g')$, proving that F is one-one.

To show that F maps onto B, suppose that $y \in B$. Then we define $x \in A$ as follows. Let $x = (u, v)$, where $u \in {}^\lambda\kappa$ and $v \in {}^\mu\kappa$ are defined by setting, for any $\alpha < \lambda$ and $\beta < \mu$, $u(\alpha) = y(\alpha, 0)$ and $v(\beta) = y(\beta, 1)$. Then for any $\alpha < \lambda$ and $\beta < \mu$ we have

$$\begin{aligned}
((F(x))(\alpha, 0) &= ((F(u, v))(\alpha, 0) = u(\alpha) = y(\alpha, 0) \quad \text{and}\\
(F(x))(\beta, 1) &= ((F(u, v))(\beta, 1) = v(\beta) = y(\beta, 1);
\end{aligned}$$

hence $F(x) = y$, as desired.

(vii): We have $(\kappa \cdot \lambda)^\mu = |{}^\mu(\kappa \cdot \lambda)|$ by definition, so by 8.18, $(\kappa \cdot \lambda)^\mu = |{}^\mu(\kappa \times \lambda)|$. Similarly, $\kappa^\mu \cdot \kappa^\mu = |({}^\mu\kappa) \times ({}^\mu\lambda)|$. Thus it suffices to show that ${}^\mu(\kappa \times \lambda)$ and $({}^\mu\kappa) \times ({}^\mu\lambda)$ are equipotent. This can be done as follows. Given a function f mapping μ into $\kappa \times \lambda$, let $F(f) = (g, h)$, where $g : \mu \to \kappa$ and $h \to \lambda$ are defined as follows: for any $\alpha < \mu$, define $g(\alpha) = 1^{\text{st}}(f(\alpha))$ and $h(\alpha) = 2^{\text{nd}}(f(\alpha))$.

To show that F is one-one, suppose that $f, k \in {}^\mu(\kappa \times \lambda)$ and $F(f) = F(k)$. Let $F(f) = (g, h)$. Then for any $\alpha \in \mu$,

$$f(\alpha) = (1^{\text{st}}(f(\alpha)), 2^{\text{nd}}(f(\alpha))) = (g(\alpha), h(\alpha)) = (1^{\text{st}}(k(\alpha)), 2^{\text{nd}}(k(\alpha))) = k(\alpha).$$

So $f = k$.

To show that F maps onto $({}^\mu\kappa) \times ({}^\mu\lambda)$, take any $(g, h) \in ({}^\mu\kappa) \times ({}^\mu\lambda)$. Define $f \in {}^\mu(\kappa \times \lambda)$ by setting, for each $\alpha < \mu$, $f(\alpha) = (g(\alpha), h(\alpha))$. Clearly $F(f) = (g, h)$.

(viii): By the usual arguments, we need to see that there is a one-one correspondence between ${}^\mu({}^\lambda\kappa)$ and ${}^{\lambda\times\mu}\kappa$. Given $f \in {}^\mu({}^\lambda\kappa)$, define $F(f) \in {}^{\lambda\times\mu}\kappa$ by setting, for any $\alpha \in \lambda$ and $\beta \in \mu$,

$$F(f)(\alpha, \beta) = (f(\alpha))(\beta).$$

We leave to an exercise the task of showing that F is the desired bijection.

(ix): Assume the hypotheses. For each $f \in {}^\mu\kappa$ we define $f^+ \in {}^\nu\lambda$ by setting, for any $\alpha \in \nu$,

$$f^+(\alpha) = \begin{cases} f(\alpha) & \text{if } \alpha < \mu, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $^+$ is a one-one function, and (ix) follows. □

Proposition 8.20. *For any natural numbers m, n, m^n in the sense of this chapter coincides with its meaning in the sense of Chapter 6. In particular, $m^n \in \omega$ for all $m, n \in \omega$.*

Proof. This is easily seen for a fixed m by ordinary induction on n. □

We recall the following result (7.28):

Proposition 8.21$^{\text{ch}}$. $|\mathscr{P}A| = 2^{|A|}$. □

Theorem 8.22$^{\text{ch}}$. *If $2 \leq \kappa \leq \lambda \geq \omega$, then $\kappa^\lambda = 2^\lambda$.*

Proof. We have

$$2^\lambda \leq \kappa^\lambda \leq \lambda^\lambda \leq |\mathscr{P}(\lambda \times \lambda)| = |\mathscr{P}(\lambda)| = 2^\lambda,$$

and the theorem follows. □

As the last topic of this chapter we take infinite sums. Infinite products will be treated later. The definition of infinite sums generalizes that of the sum of two cardinals.

Let $\langle \kappa_i : i \in I \rangle$ be a system of cardinals (this just means that κ is a function with domain I whose values are always cardinals). Then we define

$$\sum_{i \in I} \kappa_i = \left| \bigcup_{i \in I} (\kappa_i \times \{i\}) \right|.$$

Proposition 8.23$^{\mathrm{ch}}$**.** *If $\langle \kappa_i : i \in 2 \rangle$ is a system of cardinals (meaning that κ is a function with domain 2 such that both κ_0 and κ_1 are cardinals), then $\sum_{i \in 2} \kappa_i = \kappa_0 + \kappa_1$.* □

Proof. Recall the definition of $\kappa_0 + \kappa_1$:

$$\kappa_0 + \kappa_1 = |(\kappa_0 \times \{0\}) \cup (\kappa_1 \times \{1\})|.$$

On the other hand, our new definition gives

$$\sum_{i \in 2} \kappa_i = \left| \bigcup_{i \in I} (\kappa_i \times \{i\}) \right| = |(\kappa_0 \times \{0\}) \cup (\kappa_1 \times \{1\}|,$$

so the two sets we count with are the same. □

The following is easily proved by induction on $|I|$:

Proposition 8.24. *If $\langle m_i : i \in I \rangle$ is a system of natural numbers, with I finite, then $\sum_{i \in I} m_i$ is a natural number.* □

We mention some important but easy facts concerning the cardinalities of unions:

Proposition 8.25$^{\mathrm{ch}}$**.** *If $\langle A_i : i \in I \rangle$ is a system of pairwise disjoint sets, then $\left| \bigcup_{i \in I} A_i \right| = \sum_{i \in I} |A_i|$.*

Proof. We define a function f with domain $\bigcup_{i \in I} A_i$ as follows. Let $x \in \bigcup_{i \in I} A_i$. Then there is a unique $i \in I$ such that $x \in A_i$. We define $f(x) = (x, i)$. It is easily checked that this defines a one-one function mapping $\bigcup_{i \in I} A_i$ onto $\bigcup_{i \in I} (A_i \times \{i\})$, and so the definition of sum gives the desired result. □

Proposition 8.26$^{\mathrm{ch}}$**.** *If $\langle A_i : i \in I \rangle$ is any system of sets, then $\left| \bigcup_{i \in I} A_i \right| \leq \sum_{i \in I} |A_i|$.*

Proof. We define a function f mapping $\bigcup_{i \in I} (A_i \times \{i\})$ into $\bigcup_{i \in I} A_i$ by setting, for each $i \in I$ and $a \in A_i$, $f(a, i) = a$. We claim that f maps onto the union. (Note in this proof that it is possible that the union is empty; in this case the function f is also empty, and these details are taken care of by the hypthetical nature of the proof.) In fact, suppose that $x \in \bigcup_{i \in I} A_i$. Choose $i \in I$ such that $x \in A_i$. Then $f(x, i) = x$, as desired. □

Finally, we gather together some simple arithmetic of infinite sums:

Proposition 8.27$^{\mathrm{ch}}$**.** *(i)* $\sum_{i \in I} 0 = 0$.

(ii) $\sum_{i \in \emptyset} \kappa_i = 0$.

(iii) $\sum_{i\in I}\kappa_i = \sum_{i\in I,\kappa_i\neq 0}\kappa_i$.
(iv) If $I\subseteq J$*, then* $\sum_{i\in I}\kappa_i \leq \sum_{i\in J}\kappa_i$.
(v) If $\kappa_i\leq\lambda_i$ *for all* $i\in I$*, then* $\sum_{i\in I}\kappa_i\leq\sum_{i\in I}\lambda_i$.
(vi) $\sum_{i\in I}1=|I|$.
(vii) $\sum_{i\in I}\kappa=\kappa\cdot|I|$.

Proof. (i):

$$\sum_{i\in I}0=\left|\bigcup_{i\in I}(0\times\{i\})\right|=\left|\bigcup_{i\in I}(\emptyset)\right|=|\emptyset|=0.$$

(ii):

$$\sum_{i\in\emptyset}\kappa_i=\left|\bigcup_{i\in\emptyset}\kappa_i\right|=|\emptyset|=0.$$

(iii):

$$\begin{aligned}\sum_{i\in I}\kappa_i&=\left|\bigcup_{i\in I}(\kappa_i\times\{i\})\right|=\left|\bigcup_{\substack{i\in I,\\ \kappa_i\neq 0}}(\kappa_i\times\{i\})\cup\bigcup_{\substack{i\in I,\\ \kappa_i=0}}(\kappa_i\times\{i\})\right|\\&=\left|\bigcup_{\substack{i\in I,\\ \kappa_i\neq 0}}(\kappa_i\times\{i\})\cup\emptyset\right|=\left|\bigcup_{\substack{i\in I,\\ \kappa_i\neq 0}}(\kappa_i\times\{i\})\right|=\sum_{\substack{i\in I,\\ \kappa_i\neq 0}}\kappa_i.\end{aligned}$$

(iv):

$$\begin{aligned}\sum_{i\in I}\kappa_i&=\left|\bigcup_{i\in I}(\kappa_i\times\{i\})\right|\leq\left|\bigcup_{i\in J}(\kappa_i\times\{i\})\right|\quad\text{by 7.14}\\&=\sum_{i\in J}\kappa_i.\end{aligned}$$

(v): Recall here that $\kappa_i\leq\lambda_i$ implies that $\kappa_i\subseteq\lambda_i$. Thus

$$\begin{aligned}\sum_{i\in I}\kappa_i&=\left|\bigcup_{i\in I}(\kappa_i\times\{i\})\right|\leq\left|\bigcup_{i\in J}(\lambda_i\times\{i\})\right|\quad\text{by 7.14}\\&=\sum_{i\in J}\lambda_i.\end{aligned}$$

(vi):

$$\sum_{i\in I}1=\left|\bigcup_{i\in I}(1\times\{i\})\right|=|1\times I|=1\cdot|I|=|I|.$$

(vii):

$$\sum_{i\in I}\kappa = \left|\bigcup_{i\in I}(\kappa\times\{i\})\right| = |\kappa\times I| = \kappa\cdot|I|. \qquad \square$$

Now we need another fact about ordinals and cardinals.

Proposition 8.28. *If Γ is a set of ordinals, then $\bigcup\Gamma$ is also an ordinal. For any $\alpha\in\Gamma$ we have $\alpha\leq\bigcup\Gamma$, and if $\alpha\leq\beta$ for all $\alpha\in\Gamma$, then $\bigcup\Gamma\leq\beta$.*

If Γ is a set of cardinals, then $\bigcup\Gamma$ is a cardinal.

Proof. To show that $\bigcup\Gamma$ is transitive, suppose that $x\in y\in\bigcup\Gamma$. Choose $\alpha\in\Gamma$ such that $y\in\alpha$. Since α is transitive, we have $x\in\alpha$, and so $x\in\bigcup\Gamma$.

To show that $\bigcup\Gamma$ is well-ordered under $\in$, suppose that X is a nonempty subset of $\bigcup\Gamma$. Thus each member of X is a member of some member of Γ, and the members of Γ are ordinals; so each member of X is a member of some ordinal, and hence is itself an ordinal by 7.4. Hence X is a nonempty set of ordinals, and hence it has a least element. So we have proved the first part of the proposition.

If $\alpha\in\Gamma$, then $\alpha\subseteq\bigcup\Gamma$, and hence $\alpha\leq\bigcup\Gamma$ by 7.5. Suppose that $\alpha\leq\beta$ for all $\alpha\in\Gamma$. By 7.6 we have $\bigcup\Gamma\leq\beta$ or $\beta<\bigcup\Gamma$. If $\beta<\bigcup\Gamma$, then $\beta\in\bigcup\Gamma$, so $\beta\in\alpha\in\Gamma$ for some α; but $\alpha\leq\beta$, and $\in$ means $<$ for ordinals. So $\bigcup\Gamma\leq\beta$.

This finishes the part of the proposition concerning ordinals.

Now suppose that Γ is a set of cardinals. Suppose that $\bigcup\Gamma$ is not a cardinal. Then it is equipotent with some $\alpha<\bigcup\Gamma$. Say that f is a bijection from $\bigcup\Gamma$ onto α. Now $<$ means $\in$ for ordinals, so $\alpha\in\beta\in\Gamma$ for some β. Hence $\alpha\subseteq\beta\subseteq\bigcup\Gamma$. So $f\restriction\beta$ is a one-one function mapping β into α, so by 7.16, $\beta=|\beta|\leq|\alpha|=\alpha$, contradicting $\alpha<\beta$. $\square$

The first part of 8.28 says that $\bigcup\Gamma$ is an ordinal which is the least upper bound of Γ.

Proposition 8.29$^{\mathrm{ch}}$. *If $\langle A_i : i\in I\rangle$ is any system of sets, then*

$$\left|\bigcup_{i\in I}A_i\right|\leq|I|\cdot\bigcup_{i\in I}|A_i|.$$

Proof. This is an easy computation:

$$\left|\bigcup_{i\in I}A_i\right|\leq\sum_{i\in I}|A_i|\leq\sum_{i\in I}\left(\bigcup_{i\in I}|A_i|\right)=|I|\cdot\bigcup_{i\in I}|A_i|. \qquad \square$$

This theorem has the following very useful corollary, which generalizes the theorem in Chapter 7 which says that a countable union of countable sets is countable.

Corollary 8.30$^{\mathrm{ch}}$. *Let κ be an infinite cardinal. Then a union of at most κ sets, each of size at most κ, has size at most κ.*

Proof. Suppose that $\langle A_i : i \in I\rangle$ is a system of sets with $|I| \le \kappa$ and $|A_i| \le \kappa$ for each $i \in I$. Then

$$\begin{aligned}\left|\bigcup_{i\in I} A_i\right| &\le |I| \cdot \bigcup_{i\in I} |A_i| \quad \text{by 8.30}\\ &\le \kappa \cdot \kappa \quad \text{by the assumptions}\\ &= \kappa \quad \text{by 8.16.}\end{aligned}$$

□

The next theorem reduces infinite sums to the operations of unions (taking the supremum, by 8.28) and binary products.

Theorem 8.31ch**.** *If $\langle \kappa_i : i \in I\rangle$ is a system of nonzero cardinals, and either I is infinite or some κ_i is infinite, then $\sum_{i\in I} \kappa_i = |I| \cdot \bigcup_{i\in I} \kappa_i$.*

Proof. We have $|I| = \sum_{i\in I} 1 \le \sum_{i\in I} \kappa_i$ and, for each $j \in I$ we have $\kappa_j \le \sum_{i\in I} \kappa_i$, so $\bigcup_{i\in I} \kappa_i \le \sum_{i\in I} \kappa_i$. So the hypothesis of the proposition implies that $\sum_{i\in I} \kappa_i$ is infinite, and

$$\begin{aligned}\sum_{i\in I} \kappa_i &= \left|\sum_{i\in I} \kappa_i\right|\\ &\le |I| \cdot \bigcup_{i\in I} \kappa_i \quad \text{by 7.29}\\ &\le \left(\sum_{i\in I} \kappa_i\right) \cdot \left(\sum_{i\in I} \kappa_i\right)\\ &= \sum_{i\in I} \kappa_i,\end{aligned}$$

and the desired conclusion follows. □

As the last theorem of this chapter we calculate the size of the closure of a set under operations; recall 6.35 and 6.36 for this important concept.

Theorem 8.32ch**.** (Closure theorem, II) *Let F be a set of finitary partial operations on a set A, and let $X \subseteq A$. Then*

$$|\mathrm{Cl}_F(X)| \le \max(\omega, |F|, |X|).$$

Proof. Let $\kappa = \max(\omega, |F|, |X|)$. We consider the sets D_i, $i \in \omega$ as in 6.36(ii). We first show that $|D_m| \le \kappa$ for every $m \in \omega$, by induction. First, $|D_0| = |X| \le \kappa$. Now suppose that we have shown that $|D_m| \le \kappa$. For each positive integer i, let

$$M_i = \{(i, f, b) : f \in F,\ \mathrm{dmn}(f) = i,\ b \in \mathrm{dmn}(f) \cap D_m\}$$

and $N = \bigcup_{i\in\omega, i\ne 0} M_i$. Then for each positive integer i,

$$|M_i| \le |F \times {}^i D_m| \le |F| \cdot |D_m|^i \le \kappa \cdot \kappa^i = \kappa,$$

and

$$|N| \leq \sum_{\substack{i\in\omega,\\ i\neq 0}} |M_i| \leq \omega \cdot \kappa = \kappa.$$

Now for each positive integer i and each $(i, f, b) \in N_i$ define $F(i, f, b) = f(b)$. Clearly F maps N onto the set $\{f(b) : f \in F,\ b \in \mathrm{dmn}(f),\ \mathrm{rng}(b) \subseteq D_m\}$, so that set has size at most κ. Hence $|D_{m+1}| \leq \kappa$. This finishes the inductive proof that each D_m has size at most κ. Hence

$$|\mathrm{Cl}_F(X)| = \left|\bigcup_{m\in\omega} D_m\right| \leq \omega \cdot \kappa = \kappa. \qquad \square$$

Exercises, Chapter 8

1. Give an example of partially ordered sets $(A, <), (B, \prec)$ with a strictly increasing function from A to B which is neither one-one nor onto.

2. Give an example of a linearly ordered set A and a strictly increasing function from A onto A which is not the identity.

3. Give details for (1) in the proof of 8.11.

4. Give details for (2), (3), and (4) in the proof of 8.11.

5. Prove (5) in the proof of 8.11.

6. Prove Proposition 8.13.

7. Finish the proof of 8.19(viii).

8. Prove 8.20.

9. Prove 8.24.

10. Finish the proof of 8.25.

11. A *Group* is an ordered pair $(A, \cdot)$ satisfying some simple conditions, where $\cdot$ is a binary operation on A, i.e., a function mapping $A \times A$ into A. Show that if A is infinite, then there are at most $2^{|A|}$ groups with underlying set A. The same upper bound applies to the number of groups of size $|A|$, up to isomorphism; this can be shown by showing that every group of size $|A|$ is isomorphic to one with underlying set A. On the other hand, there does not exist a set of all groups of size $|A|$; here assume the well-known fact that there is at least one group of size $|A|$.

12. If A is an infinite set then there are at most $2^{|A|}$ simple orderings on A; and the same bound applies for the number of simple orderings of size $|A|$, up to isomorphism.

13. Prove that there are exactly 2^ω continuous functions mapping $\mathbb{R}$ into $\mathbb{R}$. Hint: show that any continuous function is determined by what it does to the rationals.

14. Prove that if A is infinite and m is a positive integer, then there are exactly $|A|$ sequences of elements of A of length m.

15. Prove that if A is infinite, then there are exactly $|A|$ finite sequences of elements of A.

16. Show that if A is infinite, then the number of finite subsets of A is $|A|$.

17. Show that for any sets A, B, C,

$$|A \cup B \cup C| + |A \cap B| + |A \cap C| + |B \cap C| = |A| + |B| + |C| + |A \cap B \cap C|.$$

18. Show that if A is an infinite set, κ is a cardinal, and $1 \leq \kappa \leq |A|$, then there is a partition of A into κ sets each of size $|A|$. Hint: use the fact that $\kappa \cdot |A| = |A|$.

19. Show that if A is an infinite set, κ is a cardinal, and $1 \leq \kappa \leq |A|$, then there is a partition of A into $|A|$ sets each of size κ. Hint: use the fact that $\kappa \cdot |A| = |A|$.

20. Suppose that A is an infinite set, and let $a \in A$. Let F be the collection of all functions $f : A \to A$ such that $\{x \in A : f(x) \neq a\}$ is finite. Show that $|F| = |A|$.

21. Show that the set of all permutations of ω (i.e., bijections of ω to ω) has size 2^ω.

22. Show that the set of all equivalence relations on an infinite set A has size $2^{|A|}$.