Solutions, exercise set 5

$\boxed{\textbf{Chapter 6, } \text{exercise 16}}$ Following the hint, we prove "for all m, for all n, $n < f(m, n)$]" by induction on m. First suppose that $m = 0$. Then for any n, $f(0, n) = n + 1 > n$, as desired. Now assume

$(*)$ for all n, $n < f(m, n)$ (first inductive hypothesis)

We want to prove that for all n, $n < f(m + 1, n)$. We do this by induction on n. First suppose that $n = 0$. Then $f(m + 1, 0) = f(m, 1) > 1$ by the first inductive hypothesis, so $f(m + 1, 0) > 0$. Thus our statement holds for $n = 0$. Now assume

$(**)$ $n < f(m + 1, n)$. (second inductive hypothesis)

Then

$$\begin{aligned} f(m+1, n+1) &= f(m, f(m+1, n)) \\ &> f(m+1, n) \quad \text{(first inductive hypothesis,} \\ &\qquad\qquad\qquad\quad \text{applied with } f(m+1,n) \text{ in place of } n) \\ &> n \quad \text{(second inductive hypothesis)} \end{aligned}$$

Since there are two ">"s here, we have $f(m + 1, n + 1) > n + 1$, finishing the inductive proofs.

$\boxed{\textbf{Chapter 6, } \text{exercise 17}}$ For $m = 0$, we have $f(0, n) = n + 1 < n + 2 = f(0, n + 1)$, so it holds. Now assume that $m \neq 0$, and write $m = p+1$. Then $f(m, n+1) = f(p+1, n+1) = f(p, f(p + 1, n)) > f(p + 1, n) = f(m, n)$, where we used exercise 16.

$\boxed{\textbf{Chapter 6, } \text{exercise 21}}$ First note that $\bigcup y \subseteq y$ for any natural number y, since if $x \in \bigcup y$ then $x \in w \in y$ for some w and so $x \in y$ by 6.4.

Now we prove the implication of the statement by induction on y. The statement is vacuously true for $y = 0$. Assume it for y, that is, assume that if $y \neq 0$ then $\bigcup y$ is a natural number and $y = \bigcup y \cup \{\bigcup y\}$. Then

$$\begin{aligned} \bigcup(y+1) &= \bigcup(y \cup \{y\}) \\ &= \bigcup y \cup y \\ &= y \end{aligned}$$

Hence $y + 1 = y \cup \{y\} = \bigcup(y + 1) \cup \{\bigcup(y + 1)\}$, finishing the inductive proof.

$\boxed{\textbf{Chapter 6, } \text{exercise 23}}$ Note that we can, in general, start induction at any specified natural number m. This is the content of exercise 10, which we assume here.

For $n = 1$, the left side is 1, and the right side is

$$\frac{1}{6} \cdot 1 \cdot 2 \cdot 3 = 1$$

too, so our statement holds for $n = 1$. Now assume it for n. Then

$$\begin{aligned}
1^2 + 2^2 + \cdots + (n+1)^2 &= 1^2 + 2^2 + \cdots + n^2 + (n+1)^2 \\
&= \frac{1}{6}n(n+1)(2n+1) + (n+1)^2 \quad \text{by the inductive assumption} \\
&= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} \\
&= \frac{(n+1)[n(2n+1) + 6(n+1)]}{6} \\
&= \frac{(n+1)[2n^2 + n + 6n + 6]}{6} \\
&= \frac{(n+1)[2n^2 + 7n + 6]}{6} \\
&= \frac{(n+1)(n+2)(2n+3)}{6} \\
&= \frac{1}{6}(n+1)(n+2)(2(n+1)+1),
\end{aligned}$$

which finishes the inductive proof.