Solutions, exercise set 3

$\boxed{\textbf{Chapter 3}, \text{exercise 2}}$ *Prove that if $f : A \to B$, then the following conditions are equivalent:*

(i) f is one-one.

(ii) $X = f^{-1}[f[X]]$ for every $X \subseteq A$.

Assume that $f : A \to B$. (i)$\Rightarrow$(ii): assume that f is one-one. If $x \in X$, then $f(x) \in f[X]$ and hence $x \in f^{-1}[f[X]]$. Now suppose that $x \in f^{-1}[f[X]]$. Then by 3.3 we have $x \in A$ and $f(x) \in f[X]$. Hence by 3.2 there is an $a \in X$ such that $f(x) = f(a)$. Since f is one-one, we get $x = a$, so $x \in X$. This proves the other inclusion in (ii).

(ii)$\Rightarrow$(i): Assume (ii), and suppose that $x, y \in A$ with $f(x) = f(y)$. Let $X = \{y\}$. Then $f(x) = f(y) \in f[X]$, so $x \in f^{-1}[f[X]] = X = \{y\}$, so $x = y$.

$\boxed{\textbf{Chapter 3,} \text{ exercise 9}}$ *Prove that if $f : A \to B$ and $\langle Y_i : i \in I\rangle$ is a system of subsets of B, then $f^{-1}[\bigcup_{i\in I} Y_i] = \bigcup_{i\in I} f^{-1}[Y_i]$.*

Assume that $f : A \to B$ and $\langle Y_i : i \in I\rangle$ is a system of subsets of B. Take any set a. Then

$$\begin{aligned}
a \in f^{-1}\left[\bigcup_{i\in I} Y_i\right] \quad &\text{iff} \quad a \in A \text{ and } f(a) \in \bigcup_{i\in I} Y_i \\
&\text{iff} \quad a \in A \text{ and } f(a) \in Y_i \text{ for some } i \in I \\
&\text{iff} \quad a \in f^{-1}[Y_i] \text{ for some } i \in I \\
&\text{iff} \quad a \in \bigcup_{i\in I} f^{-1}[Y_i].
\end{aligned}$$

$\boxed{\textbf{Chapter 3,} \text{ exercise 15}}$ *Suppose that $f : A \to B$. Show that the following conditions are equivalent:*

(i) f maps onto B.

(ii) For all C, g, h, if $g : B \to C$, $h : B \to C$, and $g \circ f = h \circ f$, then $g = h$.

(i)$\Rightarrow$(ii): Assume (i) and the hypothesis of (ii). Take any $b \in B$. Since f maps onto B, choose $a \in A$ such that $f(a) = b$. Then $g(b) = g(f(a)) = (g \circ f)(a) = (h \circ f)(a) = h(f(a)) = h(b)$. Since b is an arbitrary element of B, it follows that $g = h$.

(ii)$\Rightarrow$(i): Assume (ii), and suppose that $b \in B$; we want to find $a \in A$ such that $f(a) = b$. Suppose that there is no such a; we will get a contradiction. Let $C = \{c, d\}$, with c and d distinct elements, for example $c = 0$ and $d = \{0\}$. Define $g : B \to C$ by setting $g(x) = c$ for all $x \in B$, and define $h : B \to C$ by setting $h(x) = c$ for all $x \in B\backslash\{b\}$ and $h(b) = d$. Then for any $x \in A$ we have $f(x) \neq b$ by hypothesis, so $(h \circ f)(x) = h(f(x)) = c = g(f(x)) = (g \circ f)(x)$. Thus $h \circ f = g \circ f$. So $h = g$. But obviously $h \neq g$, contradiction.

$\boxed{\textbf{Chapter 3, } \text{exercise 26}}$ *Suppose that $f : A \to B$, f maps onto B, $g : A \to C$, and for all $x, y \in A$, if $f(x) = f(y)$ then $g(x) = g(y)$. Show that there is an $h : B \to C$ such that the diagram*

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "g"'] & & B \arrow[dl, dotted, "h"] \\ & C & \end{tikzcd}$$

commutes, i.e., $g = h \circ f$.

We define

$$h = \{(b, c) \in B \times C : \exists a \in A[b = f(a) \text{ and } c = g(a)\}.$$

First we check that h is a function. Suppose that $(b, c), (b, d) \in h$. Choose, accordingly, $a, a' \in A$ such that $b = f(a)$, $c = g(a)$, $b = f(a')$, and $d = g(a')$. Thus $f(a) = f(a')$, so by hypothesis $g(a) = g(a')$, i.e., $c = d$, as desired.

Since f maps onto B, it is clear that the domain of h is A, and h maps into C. From its definition it follows that $h(f(a)) = g(a)$ for any $a \in A$, as desired.