

### Solutions, exercise set 3

**Chapter 3, exercise 2** Prove that if  $f : A \rightarrow B$ , then the following conditions are equivalent:

- (i)  $f$  is one-one.
- (ii)  $X = f^{-1}[f[X]]$  for every  $X \subseteq A$ .

Assume that  $f : A \rightarrow B$ . (i) $\Rightarrow$ (ii): assume that  $f$  is one-one. If  $x \in X$ , then  $f(x) \in f[X]$  and hence  $x \in f^{-1}[f[X]]$ . Now suppose that  $x \in f^{-1}[f[X]]$ . Then by 3.3 we have  $x \in A$  and  $f(x) \in f[X]$ . Hence by 3.2 there is an  $a \in X$  such that  $f(x) = f(a)$ . Since  $f$  is one-one, we get  $x = a$ , so  $x \in X$ . This proves the other inclusion in (ii).

(ii) $\Rightarrow$ (i): Assume (ii), and suppose that  $x, y \in A$  with  $f(x) = f(y)$ . Let  $X = \{y\}$ . Then  $f(x) = f(y) \in f[X]$ , so  $x \in f^{-1}[f[X]] = X = \{y\}$ , so  $x = y$ .

**Chapter 3, exercise 9** Prove that if  $f : A \rightarrow B$  and  $\langle Y_i : i \in I \rangle$  is a system of subsets of  $B$ , then  $f^{-1}[\bigcup_{i \in I} Y_i] = \bigcup_{i \in I} f^{-1}[Y_i]$ .

Assume that  $f : A \rightarrow B$  and  $\langle Y_i : i \in I \rangle$  is a system of subsets of  $B$ . Take any set  $a$ . Then

$$\begin{aligned}
 a \in f^{-1} \left[ \bigcup_{i \in I} Y_i \right] & \text{ iff } a \in A \text{ and } f(a) \in \bigcup_{i \in I} Y_i \\
 & \text{ iff } a \in A \text{ and } f(a) \in Y_i \text{ for some } i \in I \\
 & \text{ iff } a \in f^{-1}[Y_i] \text{ for some } i \in I \\
 & \text{ iff } a \in \bigcup_{i \in I} f^{-1}[Y_i].
 \end{aligned}$$

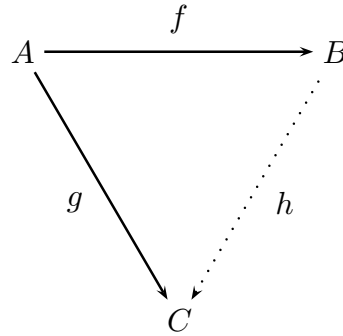
**Chapter 3, exercise 15** Suppose that  $f : A \rightarrow B$ . Show that the following conditions are equivalent:

- (i)  $f$  maps onto  $B$ .
- (ii) For all  $C, g, h$ , if  $g : B \rightarrow C$ ,  $h : B \rightarrow C$ , and  $g \circ f = h \circ f$ , then  $g = h$ .

(i) $\Rightarrow$ (ii): Assume (i) and the hypothesis of (ii). Take any  $b \in B$ . Since  $f$  maps onto  $B$ , choose  $a \in A$  such that  $f(a) = b$ . Then  $g(b) = g(f(a)) = (g \circ f)(a) = (h \circ f)(a) = h(f(a)) = h(b)$ . Since  $b$  is an arbitrary element of  $B$ , it follows that  $g = h$ .

(ii) $\Rightarrow$ (i): Assume (ii), and suppose that  $b \in B$ ; we want to find  $a \in A$  such that  $f(a) = b$ . Suppose that there is no such  $a$ ; we will get a contradiction. Let  $C = \{c, d\}$ , with  $c$  and  $d$  distinct elements, for example  $c = 0$  and  $d = \{0\}$ . Define  $g : B \rightarrow C$  by setting  $g(x) = c$  for all  $x \in B$ , and define  $h : B \rightarrow C$  by setting  $h(x) = c$  for all  $x \in B \setminus \{b\}$  and  $h(b) = d$ . Then for any  $x \in A$  we have  $f(x) \neq b$  by hypothesis, so  $(h \circ f)(x) = h(f(x)) = c = g(f(x)) = (g \circ f)(x)$ . Thus  $h \circ f = g \circ f$ . So  $h = g$ . But obviously  $h \neq g$ , contradiction.

**Chapter 3, exercise 26** Suppose that  $f : A \rightarrow B$ ,  $f$  maps onto  $B$ ,  $g : A \rightarrow C$ , and for all  $x, y \in A$ , if  $f(x) = f(y)$  then  $g(x) = g(y)$ . Show that there is an  $h : B \rightarrow C$  such that the diagram



commutes, i.e.,  $g = h \circ f$ .

We define

$$h = \{(b, c) \in B \times C : \exists a \in A [b = f(a) \text{ and } c = g(a)]\}.$$

First we check that  $h$  is a function. Suppose that  $(b, c), (b, d) \in h$ . Choose, accordingly,  $a, a' \in A$  such that  $b = f(a)$ ,  $c = g(a)$ ,  $b = f(a')$ , and  $d = g(a')$ . Thus  $f(a) = f(a')$ , so by hypothesis  $g(a) = g(a')$ , i.e.,  $c = d$ , as desired.

Since  $f$  maps onto  $B$ , it is clear that the domain of  $h$  is  $B$ , and  $h$  maps into  $C$ . From its definition it follows that  $h(f(a)) = g(a)$  for any  $a \in A$ , as desired.