Solutions, exercise set 10

$\boxed{\textbf{Chapter 10,}\text{ exercise 6}}$ The problem should be formulated more rigorously, as follows; see the definitions in Chapter 5.

Suppose that $(A,<)$ is a partial ordering, $(B,\prec)$ is a simple ordering, $B\subseteq A$, and $\prec$ is a subset of $<$. Then there is a simple ordering $(C,\ll)$ such that $B\subseteq C\subseteq A$, $\prec$ is a subset of $\ll$ and $\ll$ is a subset of $<$, and $(C,\ll)$ is maximal in the sense that if $(D,\sqsubset)$ is a simple ordering such that $C\subseteq D\subseteq A$ and $\ll$ is a subset of $\sqsubset$ and $\sqsubset$ is a subset of $<$, then $C=D$ and $\ll$ is the same as $\sqsubset$.

So, to prove this, let

$$\mathscr{A}=\{(C,\ll):(C,\ll)\text{ is a simple ordering, }B\subseteq C\subseteq A,\\ \prec\text{ is a subset of }\ll\text{ and }\ll\text{ is a subset of }<\}.$$

We partially order $\mathscr{A}$ by saying that

$$(C,\ll)\leq(C',\ll')\quad\text{iff}\quad C\subseteq C'\text{ and }\ll\text{ is a subset of }\ll'.$$

Clearly this *does* partially order $\mathscr{A}$. Now we want to check the hypotheses of Zorn's lemma. First of all, $\mathscr{A}$ is nonempty, since clearly $(B,\prec)\in\mathscr{A}$. Now suppose that $\mathscr{B}$ is a subset of $\mathscr{A}$ simply ordered by $\leq$. We may assume that $\mathscr{B}$ is nonempty, since $(B,\prec)$ is an upper bound for the empty set. Let

$$D=\bigcup_{(C,\ll)\in\mathscr{B}}C\quad\text{and}\quad\sqsubset=\bigcup_{(C,\ll)\in\mathscr{B}}\ll.$$

In order to verify the second hypothesis of Zorn's lemma it suffices to prove that $(D,\sqsubset)\in\mathscr{A}$. Clearly $B\subseteq D\subseteq A$. Now we check that $\sqsubset$ is a simple order.

If $x\sqsubset x$, then $x\ll x$ for some $(C,\ll)\in\mathscr{B}$, contradiction. So $\sqsubset$ is irreflexive.

Suppose that $x\sqsubset y\sqsubset z$. Say $x\ll y\ll' z$ with $(C,\ll),(C',\ll')\in B$. By symmetry, say that $(C,\ll)\leq(C',\ll')$. Then $x\ll' y\ll' z$, so $x\ll' z$ and hence $x\sqsubset y$. So $\sqsubset$ is transitive.

Suppose that $x,y\in D$. Choose $(C,\ll),(C',\ll')\in\mathscr{B}$ such that $x\in C$ and $y\in C'$. By symmetry, say that $(C,\ll)\leq(C',\ll')$. Then $x\ll' y$ or $y\ll' x$, hence $x\sqsubset y$ or $y\sqsubset x$. So $\sqsubset$ is a simple order.

So clearly $(D,\sqsubset)\in\mathscr{A}$ and it is hence the desired upper bound for $\mathscr{B}$.

Thus Zorn's lemma applies, and a maximal element under $\subseteq$ is clearly what is needed for the exercise.

$\boxed{\textbf{Chapter 10,}\text{ exercise 7}}$ By the hint, we consider the partial order $(C,\subset)$, where C is the set of all one-one functions contained in $A\times B$. We apply (1) to $(C,\subset)$ in place of $(A,<)$ and $(\emptyset,\emptyset)$ in place of $(B,\prec)$. We then get $(D,\ll)$ with the following properties: $D\subseteq C$, $\ll$ is a subset of $\subset$, $(D,\ll)$ is a simple order, and if $(E,\sqsubset)$ is such that $D\subseteq E\subseteq C$, $\ll$ is a subset of $\sqsubset$ and $\sqsubset$ is a subset of $\subset$, and $(E,\sqsubset)$ is a simple order, then $D=E$ and $\ll$ is $\sqsubset$.

Let $g = \bigcup_{f \in D} f$. Since D is simply ordered by $\subset$, it is clear that g is a function. Also, clearly $g \subseteq A \times B$. We claim that $\mathrm{dmn}(g) = A$ or $\mathrm{rng}(g) = B$. For, suppose not; choose $a \in A \backslash \mathrm{dmn}(g)$ and $b \in B \backslash \mathrm{rng}(g)$. Clearly then $h \stackrel{\mathrm{def}}{=} g \cup \{(a, b)\}$ is a one-one function. Since $f \subseteq g \subset h$ for each $f \in D$, we have $h \notin D$ and $D \cup \{h\}$ is still simply ordered by $\subset$. This contradicts the maximality of $(D, \ll)$.

Hence the claim holds. If $\mathrm{dmn}(g) = A$, then g is the desired one-one function mapping A into B. If $\mathrm{rng}(g) = B$, then g^{-1} is the desired one-one function mapping B into A.

$\boxed{\textbf{Chapter 10, } \text{exercise 8}}$ Immediate by 3.14.

$\boxed{\textbf{Chapter 10, } \text{exercise 9}}$ We follow the revised hint. We may assume that $A \neq \emptyset$. Hence also $\alpha \neq 0$. To show that $\langle f^{-1}[\{\beta\} : \beta < \alpha\rangle$ is one-one, suppose that $f^{-1}[\{\beta\}] = f^{-1}[\{\gamma\}]$. Choose $a \in A$ such that $f(a) = \beta$. Then $a \in f^{-1}[\{\beta\}] = f^{-1}[\{\gamma\}]$, so $\beta = f(a) = \gamma$. Thus $\langle f^{-1}[\{\beta\} : \beta < \alpha\rangle$ is one-one. This contradicts the choice of α. So there is no function f such that f maps A onto α. Hence by (3) there is a function g mapping α onto A. For each nonempty $X \subseteq A$ let $g(X) = \bigcap\{\beta < \alpha : f(\beta) \in X\}$; note that the set here is nonempty because X is nonempty and f maps onto A. Now let $h(X) = f(g(X))$. Clearly h is the desired choice function.