Solutions, exercise set 4

$\boxed{\textbf{Chapter 4}\text{, exercise 1}}$ To simplify life, we take all relations relative to the set N of all natural numbers.

(1) Symmetric, transitive, reflexive on N: $R = \{(n, n) : n \in N\}$.

(2) Symmetric, transitive, not reflexive on N: $R = \{(0, 0), (1, 1)\}$.

(3) Symmetric, not transitive, reflexive on N: $R = \{(0, 1), (1, 2), (1, 0), (2, 1)\} \cup \{(m, m) : m \in N\}$.

(4) Symmetric, not transitive, not reflexive on N: $R = \{(0, 1), (1, 2), (1, 0), (2, 1)\}$.

(5) Not symmetric, transitive, reflexive on N: $\{(m, n) : m, n \in N, m \leq n\}$.

(6) Not symmetric, transitive, not reflexive on N: $\{(m, n) : m, n \in N, m < n\}$.

(7) Not symmetric, not transitive, reflexive on N: $R = \{(m, m) : m \in N\} \cup \{(0, 1), (1, 2)\}$.

(8) Not symmetric, not transitive, not reflexive on N: $R = \{(0, 1), (1, 2)\}$.

$\boxed{\textbf{Chapter 4,}\text{ exercise 5}}$ $\Rightarrow$: Assume that $R \subseteq S$, and suppose that $x \in A/R$. So we can write $x = [a]_R$ for some $a \in A$. If $b \in x$, then $(a, b) \in R$, hence $(a, b) \in S$, hence $b \in [a]_S$. Thus $x \subseteq [a]_S$, as desired.

$\Leftarrow$: Assume that every member of A/R is a subset of some member of A/S. Suppose that $(a, b) \in R$. Choose $[c]_S \in A/S$ such that $[a]_R \subseteq [c]_S$. Now $b \in [a]_R$, so $b \in [c]_S$, and hence $(b, c) \in S$. But also $a \in [a]_R$, hence $a \in [c]_S$, so $(a, c) \in S$. Hence $(a, b) \in S$. This shows that $R \subseteq S$.

$\boxed{\textbf{Chapter 5,}\text{ exercise 5}}$ If $a \in A$ and $b \in B$, then we cannot have $(a, b) \ll (a, b)$, since $a \not< a$ (another reason is that $b \not< b$). Suppose that $(a, b) \ll (c, d) \ll (e, f)$. Then $a < c < e$ and $b < d < f$, so $a < e$ and $b < f$, hence $(a, b) \ll (e, f)$. Thus $(A \times B, \ll)$ is a partial order.

$\boxed{\textbf{Chapter 5,}\text{ exercise 6}}$ We can take both A and B to be the natural numbers $0, 1, 2, \ldots$ under the usual order. Note that $(1, 2)$ and $(2, 1)$ are not comparable under the order of exercise 5.