Appendix E: the logical foundations of set theory

We develop logic rigorously here, ending with a precise statement of the axioms of ZFC. Although the development is rigorous, it takes a lot of practice with logic to be comfortable with the idea that our usual mathematical arguments, including those in the main part of these notes, can be cast in the strict form described here. A development of logic giving extensive practice with these notions is beyond the scope of this appendix. We hope that our exposition will clarify the rigorous foundation of set theory, and will motivate readers to go learn about these matters in a more thorough fashion.

The fundamental idea is to define a formal language: define explicitly what symbols are allowed, how to combine them into meaningful expressions, and how a sequence of expressions constitutes a mathematical proof.

Of course our formal language is specially designed for set theory. It is, however, just one instance of more general languages in the general first-order classification of languages.

The language is built up from certain symbols. These symbols are supposed to all be distinct from one another. We assume as given an intuitive notion of a finite sequence. The meaningful expressions of the language will be certain finite sequences of symbols. We also assume that no symbol is a finite sequence of symbols.

The symbols of our language are as follows.

Sentential symbols: $\neg$ and $\rightarrow$. Intuitively these mean "not" and "implies" respectively.

The universal quantifier $\forall$, meaning intuitively "for all".

Variables: $v_0, v_1, \ldots$. We assume that we have an infinite supply of these.

The equality symbol $=$.

The membership symbol $\in$.

Punctuation symbols (and).

These are all of the symbols of our language.

A *string* is a finite sequence of symbols. Now we define the notion of a formula.

(1) For any variables v_i, v_j, the string $(v_i = v_j)$ is a formula.

(2) For any variables v_i, v_j, the string $(v_i \in v_j)$ is a formula.

(3) If φ is a formula, then so is $(\neg\varphi)$.

(4) If φ and ψ are formulas, then so is $(\varphi \rightarrow \psi)$.

(5) If φ is a formula and v_i is a variable, then the string $(\forall v_i \varphi)$ is a formula.

(6) Formulas can only be formed by finitely many applications of the rules (1)–(5).

Formulas of the types (1) and (2) are called *atomic formulas*.

The use of parentheses is rather strict here, in order to be precise. In practice we omit and add parentheses or change their form, using [,], {, and } instead, in order to increase readability.

We define some additional logical notions in terms of the ones taken as symbols:

(7) $(\varphi \vee \psi)$ abbreviates $((\neg\varphi) \rightarrow \psi)$.

(8) $(\varphi \wedge \psi)$ abbreviates $(\neg(\varphi \rightarrow (\neg\psi)))$.

(9) $(\varphi \leftrightarrow \psi)$ abbreviates $((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi))$.

(10) $(\exists v_i \varphi)$ abbreviates $(\neg(\forall v_i(\neg\psi)))$.

Intuitively these mean, respectively, "or", "and", "iff ", and "there exist". Note that "or" is taken in the non-exclusive sense.

A fundamental fact, which we will not formulate precisely, is that formulas are uniquely readable. For example, if $\varphi, \psi, \varphi', \psi'$ are formulas and $(\varphi \rightarrow \psi)$ is the same formula as $(\varphi' \rightarrow \psi')$, then φ is the formula φ' and ψ is the formula ψ'.

We now describe formulas which we call *logical axioms*. For any formulas φ, ψ, χ the following are axioms.

(A1) $\varphi \rightarrow (\psi \rightarrow \varphi)$.

(A2) $[\varphi \rightarrow (\psi \rightarrow \chi)] \rightarrow [(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi)]$.

(A3) $(\neg\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \varphi)$.

(A4) $\forall v_i(\varphi \rightarrow \psi) \rightarrow (\forall v_i\varphi \rightarrow \forall v_i\psi)$ for any $i = 0, 1, \ldots$.

(A5) $\varphi \rightarrow \forall v_i\varphi$, if v_i does not occur in φ.

(A6) $\exists v_i(v_i = v_j)$ for $i \neq j$.

(A7) $v_i = v_j \rightarrow (\varphi \rightarrow \psi)$, if $i \neq j$, φ is atomic, and ψ is obtained from φ by replacing one occurrence of v_i by v_j.

This is a very simple set of axioms. Together with "rules of inference" which we will describe later, they suffice to derive any purely logical fact. There are many notions in logic which can be developed on the basis of what we have so far. For example, tautologies, which are formulas true just on the basis of $\neg$ and $\rightarrow$, can be derived. The important notions of free and bound variables can be defined, and standard facts about them proved.

Before giving our rules of inference, we now formulate all of our set theoretic axioms in our formal language.

(S1) (extensionality) $\forall v_0 \forall v_1[\forall v_2(v_2 \in v_0 \leftrightarrow v_2 \in v_1) \rightarrow v_0 = v_1]$.

(S2) (pairing) $\forall v_0 \forall v_1 \exists v_2 \forall v_3(v_3 \in v_2 \leftrightarrow v_3 = v_0 \vee v_3 = v_1)$.

(S3) (union) $\forall v_0 \exists v_1 \forall v_2[v_2 \in v_1 \leftrightarrow \exists v_3(v_2 \in v_3 \wedge v_3 \in v_0)]$.

(S4) (comprehension) If φ is a formula with all of its variables in the list $v_0, \ldots, v_m$, with $m \geq 1$, then the following formula is an instance of this axiom:

$$\forall v_1 \ldots \forall v_m \exists v_{m+1} \forall v_0(v_0 \in v_{m+1} \leftrightarrow v_0 \in v_m \wedge \varphi).$$

(Note that *formulas* thus are the rigorous expression of *properties* in the text.)

(S5) (power set) $\forall v_0 \exists v_1 \forall v_2(v_2 \in v_1 \leftrightarrow \forall v_3(v_3 \in v_2 \rightarrow v_3 \in v_0))$.

For the axiom of choice, we need several abbreviations. Our abbreviated formulas will involve variables of two sorts: variables v_i with $i < 20$, and variables with higher indices. If φ is an abbreviation, and all the variables of the first sort which appear in φ are in the list $v_0, \ldots, v_m$, then by $\varphi(v_{i_0}, \ldots, v_{i_m})$ we mean the result of simultaneously replacing $v_0, \ldots, v_m$ by $v_{i_0}, \ldots, v_{i_m}$, leaving the variables of the second sort alone.

$Unord$ is $\forall v_{20}(v_{20} \in v_0 \leftrightarrow v_{20} = v_1 \vee v_{20} = v_2)$;

Intuitively, $v_0 = \{v_1, v_2\}$;

Ord is $\exists v_{21} \exists v_{22}[Unord(v_{21}, v_1, v_1) \wedge Unord(v_{22}, v_1, v_2) \wedge Unord(v_0, v_{21}, v_{22})]$;

Intuitively, $v_0 = (v_1, v_2)$;

Rel is $\forall v_{23}[v_{23} \in v_0 \rightarrow \exists v_{24} \exists v_{25} Ord(v_{23}, v_{24}, v_{25})]$;

Intuitively, v_0 is a relation;

Fcn is $Rel \wedge \forall v_{26} \forall v_{27} \forall v_{28} \forall v_{29} \forall v_{30}[v_{26} \in v_0 \wedge v_{27} \in v_0 \wedge Ord(v_{26}, v_{28}, v_{29})$
$\wedge\, Ord(v_{27}, v_{28}, v_{30}) \rightarrow v_{29} = v_{30}]$;

Intuitively, v_0 is a function;

Map is $Fcn \wedge \forall v_{31} \forall v_{32} \forall v_{33}[Ord(v_{31}, v_{32}, v_{33}) \wedge v_{31} \in v_0 \rightarrow v_{32} \in v_1 \wedge v_{33} \in v_2]$
$\wedge\, \forall v_{31}[v_{31} \in v_1 \rightarrow \exists v_{32} \exists v_{33}(Ord(v_{33}, v_{31}, v_{32}) \wedge v_{33} \in v_1)]$;

Intuitively, $v_0 : v_1 \rightarrow v_2$;

$Onto$ is $Map \wedge \forall v_{34}[v_{34} \in v_2 \rightarrow \exists v_{35} \exists v_{36}(Ord(v_{36}, v_{35}, v_{34}) \wedge v_{36} \in v_0)]$;

Intuitively, $v_0 : v_1 \rightarrow v_2$ and v_0 maps onto v_2;

$Comp$ is $Map(v_0, v_2, v_3) \wedge Map(v_1, v_3, v_2) \wedge \forall v_{37}[v_{37} \in v_3 \rightarrow$
$\exists v_{38} \exists v_{39} \exists v_{40}[Ord(v_{39}, v_{37}, v_{38}) \wedge Ord(v_{40}, v_{38}, v_{37})$
$\wedge\, v_{39} \in v_1 \wedge v_{40} \in v_0]]$;

Intuitively, $v_0 : v_2 \rightarrow v_3,\ v_1 : v_3 \rightarrow v_2,$ and $v_0 \circ v_1 = \mathrm{Id}_{v_3}$;

Now we can give the axiom of choice:

(S6) (choice) $\forall v_0 \forall v_1 \forall v_2[Onto \rightarrow \exists v_{41} Comp(v_0, v_{41}, v_1, v_2)]$.

This axiom is quite long if written out in full; some equivalents of the axiom of choice are much shorter.

(S7) (foundation) $\exists v_1(v_1 \in v_0) \rightarrow \exists v_1[v_1 \in v_0 \wedge \neg \exists v_2(v_2 \in v_0 \wedge v_2 \in v_1)]$.

(S8) (infinity)

$$\begin{aligned}&\exists v_0[\exists v_1(v_1 \in v_0 \wedge \forall v_2(\neg(v_2 \in v_1))) \wedge \forall v_1[v_1 \in v_0 \rightarrow \exists v_2(v_2 \in v_0\\ &\quad \wedge \forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_1 \vee v_3 = v_1))]].\end{aligned}$$

(S9) (replacement) If φ is a formula with all of its variables in the list $v_0, \ldots, v_m$, and i is minimum such that v_i does not occur in φ and $i > 1$, then the following is an instance of the replacement axiom:

$$\begin{aligned}&\forall v_0 \ldots \forall v_m[\forall v_0 \forall v_1 \forall v_i[\varphi \wedge \forall v_1(v_1 = v_i \rightarrow \varphi) \rightarrow v_1 = v_i]\\ &\qquad \rightarrow \exists v_{m+1} \forall v_1(v_1 \in v_{m+1} \leftrightarrow \exists v_0(v_0 \in v_m \wedge \varphi))].\end{aligned}$$

The hypothesis here is supposed to say that φ is a "class function" with respect to the argument v_0 and result v_1. The conclusion asserts the existence of a new set v_{m+1} obtained by replacing the elements of v_m according to this class function.

This finishes our list of axioms of ZFC.

Now we define the crucial notion of mathematical proof. A *mathematical proof* is a finite sequence $\varphi_0, \varphi_1, \ldots, \varphi_m$ such that for each $i = 0, \ldots, m$ one of the following holds:

(1) φ_i is a logical axiom.

(2) φ_i is an axiom of ZFC.

(3) There exist $j, k < i$ such that φ_k is the formula $\varphi_j \to \varphi_i$. (Inference rule of modus ponens.)

(4) There exist $j < i$ and a natural number m such that φ_i is the formula $\forall v_m \varphi_j$. (Inference rule of generalization.)

This completes our sketch of the logical foundations of set theory. A thorough developement of set theory on this basis would start by proving some of the purely logical facts mentioned above. For a small illustration of the notions introduced in this appendix, we give a mathematical proof of the formula $v_0 = v_1 \to v_0 = v_1$:

$$\begin{array}{lll}
(1) & (v_0 = v_1 \to [(v_0 = v_1 \to v_0 = v_1) \to v_0 = v_1]) & \\
 & \quad \to ([v_0 = v_1 \to (v_0 = v_1 \to v_0 = v_1)] \to (v_0 = v_1 \to v_0 = v_1)) & \text{(A2)} \\
(2) & v_0 = v_1 \to [(v_0 = v_1 \to v_0 = v_1) \to v_0 = v_1] & \text{(A1)} \\
(3) & [v_0 = v_1 \to (v_0 = v_1 \to v_0 = v_1)] \to (v_0 = v_1 \to v_0 = v_1) & \text{(1),(2),MP} \\
(4) & v_0 = v_1 \to (v_0 = v_1 \to v_0 = v_1) & \text{(A1)} \\
(5) & v_0 = v_1 \to v_0 = v_1 & \text{(3),(4),MP}
\end{array}$$