**Solutions, exercise set 11**

**Chapter 11,** exercise 6 The left side is the number of elements of

$$\text{(1)} \qquad \mathsf{X}_{i\in I}\left(\bigcup_{j\in J_i}(\kappa_{ij}\times\{j\})\right),$$

and the right side is the number of elements of

$$\text{(2)} \qquad \bigcup_{f\in P}\left(\left(\mathsf{X}_{i\in I}\kappa_{i,f(i)}\right)\times\{f\}\right).$$

Now given $x$ in (1) we define $G(x)$ in (2) as follows. For each $i \in I$ we have $x_i \in \bigcup_{j\in J_i}(\kappa_{ij}\times\{j\})$, and so there is a unique $j \in J_i$ such that $x \in \kappa_{ij}\times\{j\}$; let $f_x(i)$ be this $j$. Thus $f_x \in P$. Now $1^{\rm st}(x_i) \in \kappa_{i,f_x(i)}$ for all $i \in I$, so $\langle 1^{\rm st}(x_i) : i \in I\rangle \in \mathsf{X}_{i\in I}\kappa_{i,f_x(i)}$. We define $G(x) = (\langle 1^{\rm st}(x_i) : i \in I\rangle, f_x)$. Clearly $G(x)$ is in (2).

Suppose that $G(x) = G(y)$. Now $f_x = 2^{\rm nd}(G(x)) = 2^{\rm nd}(G(y) = f_y$. Write $f_x = g$. Then for any $i \in I$,

$$1^{\rm st}(x_i) = (1^{\rm st}(G(x)))_i = (1^{\rm st}(G(y)))_i = 1^{\rm st}(y_i),$$

and $2^{\rm nd}(x_i) = g(i) = 2^{\rm nd}(y_i)$. So $x_i = y_i$. Hence $x = y$. So $G$ is one-one.

To show that $G$ maps onto (2), suppose that $z$ is a member of (2). Choose $h \in P$ such that $z \in (\mathsf{X}_{i\in I}\kappa_{i,h(i)})\times\{h\}$. For each $i \in I$ we let

$$x_i = ((1^{\rm st}(z))_i, h(i)).$$

Then $x$ is in (1). Moreover, clearly $f_x = h$. Then $1^{\rm st}(x_i) = (1^{\rm st}(z))_i$ for each $i$, hence

$$\begin{aligned} G(x) &= (\langle 1^{\rm st}(x_i) : i \in I\rangle), f_x) \\ &= (\langle 1^{\rm st}(x_i) : i \in I\rangle), h) \\ &= (1^{\rm st}(z))), h) \\ &= (1^{\rm st}(z), 2^{\rm nd}(z)) \\ &= z, \end{aligned}$$

as desired.

**Chapter 11,** exercise 8 First suppose that $\alpha < \kappa^+$. Then $|\alpha| \le \alpha$ by the definition of $|\alpha|$, so $|\alpha| < \kappa^+$. Since $\kappa^+$ is the least cardinal greater than $\kappa$, and $|\alpha|$ is a cardinal, it follows that $|\alpha| \le \kappa$.

Now suppose that $|\alpha| \le \kappa$. Thus there is a one-one function from $\alpha$ into $\kappa$. If $\kappa^+ \le \alpha$, then we could also get a one-one function from $\kappa^+$ into $\kappa$, so $\kappa^+ = |\kappa^+| \le |\kappa| = \kappa$, contradiction. So $\alpha < \kappa^+$, as desired.

$\boxed{\textbf{Chapter 11, } \text{exercise 9}}$ Let $\kappa = 2^\lambda$. So $\lambda < \kappa$. Furthermore, $\kappa^\lambda = (2^\lambda)^\lambda = 2^{\lambda\cdot\lambda} = 2^\lambda = \kappa$.

$\boxed{\textbf{Chapter 11, } \text{exercise 11}}$ For each $\xi < \beta$, let $\lambda_\xi = \kappa_{\xi+1}$. Then by 11.11 we have $\sum_{\xi<\beta}\kappa_\xi < \prod_{\xi<\beta}\lambda_\xi$. Hence it suffices to show that $\prod_{\xi<\beta}\lambda_\xi \leq \prod_{\xi<\beta}\kappa_\xi$, and to do this it suffices to find a one-one function mapping $\mathsf{X}_{\xi<\beta}\lambda_\xi$ into $\mathsf{X}_{\xi<\beta}\kappa_\xi$. For each $x \in \mathsf{X}_{\xi<\beta}\lambda_\xi$ define $f(x) \in \mathsf{X}_{\xi<\beta}\kappa_\xi$ by setting, for each $\xi < \beta$,

$$(f(x))_\xi = \begin{cases} x_\eta & \text{if } \xi = \eta + 1, \\ 0 & \text{if } \xi = 0 \text{ or if } \xi \text{ is a limit ordinal.} \end{cases}$$

Then $f$ is one-one, since if $x, y \in \prod_{\xi<\beta}\lambda_\xi$ and $x \neq y$, choose $\xi < \beta$ such that $x_\xi \neq y_\xi$; then $(f(x))_{\xi+1} = x_\xi \neq y_\xi = (f(y))_{\xi+1}$, so $f(x) \neq f(y)$.