Solutions, exercise set 9

$\boxed{\textbf{Chapter 9, } \text{exercise } 9}$ *For any set x, let $\rho(x)$ be the least ordinal such that $x \in V_{\alpha +_o 1}$. Show that for any set x, $\rho(x) = \bigcup_{y \in x}(\rho(y) +_o 1)$.*

Let $\beta = \bigcup_{y \in x}(\rho(y) +_o 1)$. Now if $y \in x$, then $y \in V_{\rho(y)+_o 1} \subseteq V_\beta$ by 9.17. Thus $x \subseteq V_\beta$, and so $x \in V_{\beta +_o 1}$. It follows that $\rho(x) \leq \beta$.

Suppose that $\rho(x) < \beta$. Then by the definition of β there is a $y \in x$ such that $\rho(x) < \rho(y) +_o 1$, and hence $\rho(x) \leq \rho(y)$. Now $x \in V_{\rho(x)+_o 1}$, so $x \subseteq V_{\rho(x)}$. So $y \in V_{\rho(x)}$. If $\rho(x) = \gamma +_o 1$ for some γ, then $\gamma +_o 1 < \rho(y) +_o 1$ and $y \in V_{\gamma +_o 1}$, contradicting the definition of $\rho(y)$. If $\rho(x)$ is a limit ordinal, then $y \in V_\gamma$ for some $\gamma < \rho(x)$, hence $y \in V_{\gamma +_o 1}$, and $\gamma +_o 1 < \rho(x) \leq \rho(y) < \rho(y) +_o 1$, again a contradiction.

It follows that $\rho(x) = \beta$.

$\boxed{\textbf{Chapter 9, } \text{exercise } 10}$ *(Continuing 9) Show that if $x \subseteq y$, then $\rho(x) \leq \rho(y)$.*

If $x \subseteq y$, then

$$\rho(x) = \bigcup_{z \in x}(\rho(z) +_o 1) \subseteq \bigcup_{z \in y}(\rho(z) +_o 1) = \rho(y).$$

Hence $\rho(x) \leq \rho(y)$.

$\boxed{\textbf{Chapter 9, } \text{exercise } 11}$ *(Continuing 9) Show that $\rho(\alpha) = \alpha$ for any ordinal α. Hint: use complete induction on α.*

We use complete induction on α. Assume that $\rho(\beta) = \beta$ for all $\beta < \alpha$. Then by exercise 9,

$$\rho(\alpha) = \bigcup_{\beta < \alpha}(\rho(\beta) +_o 1) = \bigcup_{\beta < \alpha}(\beta +_o 1) = \alpha.$$

The last equality is seen as follows. If $\beta < \alpha$, then $\beta +_o 1 \leq \alpha$, so $\leq$ holds. Also, $\beta < \alpha$ implies that $\beta < \beta +_o 1 \leq \bigcup_{\gamma < \alpha}(\gamma +_o 1)$. So $\alpha \subseteq \bigcup_{\gamma < \alpha}(\gamma +_o 1)$, hence $\alpha \leq \bigcup_{\gamma < \alpha}(\gamma +_o 1)$.

$\boxed{\textbf{Chapter 9, } \text{exercise } 15}$ *Show that if $A \subseteq \omega_1$ and A is countable, then $\bigcup A < \omega_1$.*

By 8.28, $\bigcup A$ is an ordinal. If $\omega_1 \leq \bigcup A$, then $\bigcup A$ is uncountable. But this union is a countable union of countable sets, contradiction.