# Update on Monk [14]

(1) Page 123, Problem 7 has a positive solution. See

Santos, M. J. *Questions on cardinal invariants of Boolean algebras.* Archive Math. logic 62 (2023), 947-963.

(2) Page 125, Problem 8. A partial solution was given in

Kurilic, M. *The minimal size of infinite maximal antichains in direct products of parttial orders.* Order 34 (2017), 235-251.

He showed:
If $\min(\mathfrak{a}(A), \mathfrak{a}(B)) \leq \omega_1$, then $\mathfrak{a}(A \oplus B) = \min(\mathfrak{a}(A), \mathfrak{a}(B))$.
If $A$ is atomic, then $(\mathfrak{a})(A \oplus A) = \mathfrak{a}(A)$.

(3) Page 182, Problem 37. Answered in part by Santos; $A \leq_s B$ implies $tow(B) \leq tow(A)$.

(4) Page 182, Problem 38 answered positively by Santos.

(5) Page 190, Problem 44. Answered in part by Santos: $A \leq_s B$ implies that $\mathfrak{p}(B) \leq \mathfrak{p}(A)$.

(6) Page 190, Problem 45. Answered positively by Santos.

(7) Page 191. Problem 46 has a positive solution: $\mathfrak{p}(A \oplus B) = \min\{\mathfrak{p}(A), \mathfrak{p}(B)\}$.

**Proof.** $\leq$ is clear. For $\geq$, assume that $X \subseteq A \oplus B$, $\prod X = 0$, and $\forall F \in [X]^{<\omega}[\prod F \neq 0]$, with $|X| = \mathfrak{p}(A \times B) < \min\{\mathfrak{p}(A), \mathfrak{p}(B)\}$. For each $x \in X$ write $x = \sum_{i \in m_x}(a_{ix} \cdot b_{ix})$, with each $a_{ix} \in A$ and each $b_{ix} \in B$. For each $F \in [X]^{<\omega}$ let

$$C_F = \left\{ f \in \prod_{x \in X} m_x : \prod_{x \in F} (a_{f(x),x} \cdot b_{f(x)x}) \right\} \neq 0.$$

We claim that each $C_F$ is nonempty. For,

$$0 \neq \prod F = \prod_{x \in F} \sum_{i \in m_x} (a_{ix} \cdot b_{ix}) = \sum_{f \in \prod_{x \in F} m_x} \left( \prod_{x \in F} (a_{f(x)x} \cdot b_{f(x)x}) \right),$$

so there is an $f \in \prod_{x \in F} m_x$ such that $\prod_{x \in F}(a_{f(x)x} \cdot b_{f(x)x}) \neq 0$. Thus any extension of $f$ to a member of $\prod_{x \in X} m_x$ is in $C_f$. So $C_f$ is nonempty.

Now each $C_f$ is closed in $\prod_{xinX} m_x$. For, suppose that $f \in \prod_{x \in X} m_x \backslash C_F$. Thus $\prod_{x \in F}(a_{f(x)x} \cdot b_{f(x))x}) = 0$. Then $f$ is in the open set $\{g \in \prod_{x \in X} m_x : g \restriction F = f\}$, and this set is disjoint from $C_F$.

Now let $f \in \bigcap_{F \in [X]^{<\omega}} C_F$. Then $\forall F \in [X]^{<\omega}(\prod_{x \in F}(a_{f(x)x} \cdot b_{f(x)x})) \neq \emptyset$. Hence $\forall F \in [X]^{<\omega}[\prod_{x \in F} a_{f(x)x} \neq 0]$, so $\prod_{x \in X} a_{f((x)x} \neq 0$. Similarly $\prod_{x \in X} b_{f((x)x} \neq 0$. Hence

$$0 \neq \prod_{x \in X} a_{f((x)x} \cdot \prod_{x \in X} b_{f((x)x} \leq \prod X = 0.$$

contradiction. $\square$

(8) Page 192, Problem 48. Answered negativaly by Santos.

(9) Page 192. Problem 49 has been solved by Malliaris and Shelah, who showed that $\mathfrak{p}(\mathscr{P}(\omega/fin) = tow(\mathscr{P}(\omega)/fin$ in ZFC. See

Malliaris, M.; Shelah, S. *Cofinality spectrum theorems in model theory, set theory, and general topology.* Publication 998 of Shelah.

(10) Page 198, Problem 51. Solved positively by Santos.

(11) Page 238, Problem 71. Answered negatively by Santos.

(12) Page 291, Problem 71. Solved in part by Santos.

(13) Page 310. Problem 90 has been solved by Kunen, who showed assuming that $2^{\aleph_1} = \aleph_2$ that there is an atomic BA $A$ such that $\pi(A) = \aleph_1 < irr(A) = \aleph_2$. See

Kunen, K. *Irredundant sets in atomic Boolean algebras.* http://arxiv.org/abs/1307.3533.

(14) Page 383. Theorem 11.20 is false.

(15) Page 417. Problem 126 was answered. consistently: there is a model with $s_{mm}(\mathscr{P}(\omega)) < \mathfrak{i}(\mathscr{P}(\omega)/fin))$. See

Cancino, J.; Guzman.O.; Miller, A. *Irrdundant generators.*

(16) Page 455. Problem 149 is solved by the following result, which shows that $\mathrm{hd}^{id}_{mm}$ is trivial.

**Proposition.** $\forall A[\mathrm{hd}^{id}_{mm}(A) = \omega]$.

**Proof.** Let $\langle b_n : n \in \omega\rangle$ be a partition in $\overline{A}$. Define

$$I_n = \{a \in A : \forall m < n[a \cdot b_m = 0\}$$

Clearly $I_n \supseteq I_p$ if $n < p$. Suppose that $0 \neq a \in \bigcap_{n\in\omega} I_n$. Choose $n \in \omega$ such that $a \cdot b_n \neq 0$ and choose $c \in A^+$ such that $c \leq a \cdot b_n$. Now $c \in I_{n+1}$, so $c \cdot b_n = 0$, contradiction. Hence $\bigcap_{n\in\omega} I_n = 0$. Hence $\mathrm{hd}^{id}_{mm}(A) = \omega$.

(17) Page 466, Problem 156. Campero-Arena, G.; Cancino, J; Hrusak, M.; Miranda-Perea, F. E. showed that consistently the answer is negative. See

Campero-Arena, G.; Cancino, J; Hrusak, M.; Miranda-Perea, F. E. *Incomparable families and maximal trees.* Fundamenta Mathematica 234 (2016), 73-89.

(18) Page 468, Problem 157. They also showed that $inc_{mm}(\mathscr{P}(\omega)/fin) = 2^\omega$, while it is consistent that $inc^{tree}_{mm}(\mathscr{P}(\omega)/fin) < 2^\omega$.

(19) Page 468, Problem 158. They also showed that the answer is yes.