Solutions for exercises in chapter 1

E1.1 *Verify that*

$$S_0 \to \neg S_1 = \langle 2,3,1,4\rangle$$

and

$$(S_0 \to S_1) \to (\neg S_1 \to \neg S_0) = \langle 2,2,3,4,2,1,4,1,3\rangle.$$

$$\begin{aligned} S_0 \to \neg S_1 &= \langle 2\rangle^\frown S_0^\frown \neg S_1 \\ &= \langle 2\rangle^\frown \langle 3\rangle^\frown \langle 1\rangle^\frown S_1 \\ &= \langle 2,3,1,4\rangle; \end{aligned}$$

$$\begin{aligned} (S_0 \to S_1) \to (\neg S_1 \to \neg S_0) &= \langle 2\rangle^\frown (S_0 \to S_1)^\frown (\neg S_1 \to \neg S_0) \\ &= \langle 2\rangle^\frown \langle 2\rangle^\frown S_0^\frown S_1^\frown \langle 2\rangle^\frown \neg S_1^\frown \neg S_0 \\ &= \langle 2,2,3,4,2\rangle^\frown \langle 1\rangle^\frown S_1^\frown \langle 1\rangle^\frown S_0 \\ &= \langle 2,2,3,4,2,1,4,1,3\rangle. \end{aligned}$$

E1.2 *Prove that there is a sentential formula of each positive integer length.*

If m is a positive integer, then

$$\langle \overbrace{1,1,\ldots,1}^{m-1 \text{ times}}, S_0\rangle$$

is a formula of length m, it is

$$\overbrace{\neg\neg\cdots\neg}^{m-1 \text{ times}} S_0.$$

E1.3 *Prove that m is the length of a sentential formula not involving $\neg$ iff m is odd.*

Proof. $\Rightarrow$: We prove by induction on φ that if φ is a sentential formula not involving $\neg$, then the length of φ is odd. This is true of sentential variables, which have length 1. Suppose that it is true of φ and ψ, which have length $2m+1$ and $2n+1$ respectively. Then $\varphi \to \psi$, which is $\langle 1\rangle^\frown \varphi^\frown \psi$, has length $1 + 2m + 1 + 2n + 1 = 2(m + n + 1) + 1$, which is again odd. This finishes the inductive proof.

$\Leftarrow$. We construct formulas without $\neg$ with length any odd integer by induction. $\langle S_0\rangle$ is a formula of length 1. If φ has been constructed of length $2m+1$, then $S_0 \to \varphi$, which is $\langle 1, S_0\rangle^\frown \varphi$, has length $2m+3$. This finishes the inductive construction.

E1.4 *Prove that a truth table for a sentential formula involving n basic formulas has 2^n rows.*

We prove this by induction on n. For $n = 1$, there are two rows. Assume that for n basic formulas there are 2^n rows. Given $n+1$ basic formulas, let φ be one of them. For the others, by the inductive hypothesis there are 2^n rows. For each such row there are two possibilities, 0 or 1, for φ. So for the $n+1$ basic formulas there are $2^n \cdot 2 = 2^{n+1}$ rows.

E1.5 *Use the truth table method to show that the formula*

$$(\varphi \to \psi) \leftrightarrow (\neg\varphi \vee \psi)$$

is a tautology.

φ	ψ	$\varphi \to \psi$	$\neg\varphi$	$\neg\varphi \vee \psi$	$(\varphi \to \psi) \leftrightarrow (\neg\varphi \vee \psi)$
1	1	1	0	1	1
1	0	0	0	0	1
0	1	1	1	1	1
0	0	1	1	1	1

E1.6 *Use the truth table method to show that the formula*

$$[\varphi \vee (\psi \wedge \chi)] \leftrightarrow [(\varphi \vee \psi) \wedge (\varphi \vee \chi)]$$

is a tautology.

Let θ be the indicated formula.

φ	ψ	χ	$\varphi \vee \psi$	$\varphi \vee \chi$	$(\varphi \vee \psi) \wedge (\varphi \vee \chi)$	$\psi \wedge \chi$	$\varphi \vee (\psi \wedge \chi)$	θ
1	1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1	1
1	0	1	1	1	1	0	1	1
1	0	0	1	1	1	0	1	1
0	1	1	1	1	1	1	1	1
0	1	0	1	0	0	0	0	1
0	0	1	0	1	0	0	0	1
0	0	0	0	0	0	0	0	1

E1.7 *Use the truth table method to show that the formula*

$$(\varphi \to \psi) \to (\varphi \to \neg\psi)$$

is not a tautology. It is not necessary to work out the full truth table.

φ	ψ	$\varphi \to \psi$	$\neg\psi$	$\varphi \to \neg\psi$	$(\varphi \to \psi) \to (\varphi \to \neg\psi)$
1	1	1	0	0	0

E1.8 *Determine whether or not the following is a tautology:*

$$S_0 \to (S_1 \to (S_2 \to (S_3 \to S_1))).$$

Suppose that f is an assignment making the indicated formula false; we work towards a contradiction. Thus

(1) $S_0[f] = 1$ and

(2) $(S_1 \to (S_2 \to (S_3 \to S_1)))[f] = 0$.

From (2) we get

(3) $S_1[f] = 1$ and

(4) $(S_2 \to (S_3 \to S_1))[f] = 0$.

From (4) we get

(5) $S_2[f] = 1$ and

(6) $(S_3 \to S_1)[f] = 0$.

From (6) we get $S_1[f] = 0$, contradicting (3).

E1.9 *Determine whether or not the following is a tautology; an informal method is better than a truth table:*

$$(\{[(\varphi \to \psi) \to (\neg\chi \to \neg\theta)] \to \chi\} \to \tau) \to [(\tau \to \varphi) \to (\theta \to \varphi)].$$

Suppose that f is an assignment which makes the given formula false; we want to get a contradiction. Thus we have

(1) $(\{[(\varphi \to \psi) \to (\neg\chi \to \neg\theta)] \to \chi\} \to \tau)[f] = 1$ and

(2) $[(\tau \to \varphi) \to (\theta \to \varphi)][f] = 0$.

By (2) we have

(3) $(\tau \to \varphi)[f] = 1$ and

(4) $(\theta \to \varphi)[f] = 0$.

By (4) we have

(5) $\theta[f] = 1$ and

(6) $\varphi[f] = 0$.

By (3) and (6) we get

(7) $\tau[f] = 0$.

By (1) and (7) we get

(8) $\{[(\varphi \to \psi) \to (\neg\chi \to \neg\theta)] \to \chi\}[f] = 0$.

It follows that

(9) $[(\varphi \to \psi) \to (\neg\chi \to \neg\theta)][f] = 1$ and

(10) $\chi[f] = 0$.

Now by (6) we have

(11) $(\varphi \to \psi)[f] = 1$,

and hence by (9),

(12) $(\neg\chi \to \neg\theta)[f] = 1$.

By (5) we have

(13) $(\neg\theta)[f] = 0$,

and hence by (12),

(14) $(\neg\chi)[f] = 0$.

This contradicts (10).

E1.10 *Determine whether the following statements are logically consistent. If the contract is valid, then Horatio is liable. If Horation is liable, he will go bankrupt. Either Horatio will go bankrupt or the bank will lend him money. However, the bank will definitely not lend him money.*

Let S_0 correspond to "the contract is valid", S_1 to "Horatio is liable", S_2 to "Horatio will go bankrupt", and S_3 to "the bank will lend him money". Then we want to see if there is an assignment of values which makes the following sentence true:

$$(S_0 \to S_1) \wedge (S_1 \to S_2) \wedge (S_2 \vee S_3) \wedge \neg S_3.$$

We can let $f(0) = f(1) = f(2) = 1$ and $f(3) = 0$, and this gives the sentence the value 1.

E1.11 *Write out an actual proof for $\{\psi\} \vdash \neg\psi \to \varphi$. This can be done by following the proof of Lemma 1.9, expanding it using the proof of the deduction theorem.*

Following the proof of Lemma 1.9, the following is a $\{\psi, \neg\psi\}$-proof:

(a)	$\neg\psi$	
(b)	$\neg\psi \to (\neg\varphi \to \neg\psi)$	(1)
(c)	$\neg\varphi \to \neg\psi$	(a), (b), MP
(d)	$(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$	(3)
(e)	$\psi \to \varphi$	(c), (d), MP
(f)	ψ	
(g)	φ	(e), (f), MP

Now applying the proof of the deduction theorem, the following is a $\{\psi\}$-proof:

(a) $[\neg\psi \to [(\neg\psi \to \neg\psi) \to \neg\psi]] \to [[\neg\psi \to (\neg\psi \to \neg\psi)] \to (\neg\psi \to \neg\psi)]$ (2)

(b)	$\neg\psi \to [(\neg\psi \to \neg\psi) \to \neg\psi]$	(1)
(c)	$[\neg\psi \to (\neg\psi \to \neg\psi)] \to (\neg\psi \to \neg\psi)$	(a), (b), MP
(d)	$\neg\psi \to (\neg\psi \to \neg\psi)$	(1)
(e)	$\neg\psi \to \neg\psi$	(c), (d), MP
(f)	$[\neg\psi \to (\neg\varphi \to \neg\psi)] \to [\neg\psi \to [\neg\psi \to (\neg\varphi \to \neg\psi)]]$	(1)
(g)	$\neg\psi \to (\neg\varphi \to \neg\psi)$	(1)
(h)	$\neg\psi \to [\neg\psi \to (\neg\varphi \to \neg\psi)]]$	(f), (g), MP
(i)	$[(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)] \to [\neg\psi \to [(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)]]$	(1)
(j)	$(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$	(3)
(k)	$\neg\psi \to [(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)]$	(i), (j), MP
(l)	$[\neg\psi \to [(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)]] \to [[\neg\psi \to (\neg\varphi \to \neg\psi)]$ $\to [\neg\psi \to (\psi \to \varphi)]]$	(2)
(m)	$[\neg\psi \to (\neg\varphi \to \neg\psi)] \to [\neg\psi \to (\psi \to \varphi)]$	(k), (l), MP
(n)	$\neg\psi \to (\psi \to \varphi)$	(g), (m), MP
(o)	$\psi \to (\neg\psi \to \psi)$	(1)
(p)	ψ	
(q)	$\neg\psi \to \psi$	(o), (p), MP
(r)	$[\neg\psi \to (\psi \to \varphi)] \to [(\neg\psi \to \psi) \to (\neg\psi \to \varphi)]$	(2)
(s)	$(\neg\psi \to \psi) \to (\neg\psi \to \varphi)$	(n), (r), MP
(t)	$\neg\psi \to \varphi$	(q), (s), MP

Solutions for exercises in Chapter 2

E2.1 *Give an exact definition of a language for the structure* $(\omega, <)$.

The quadruple $(\{11\}, \emptyset, \emptyset, rnk)$, where rnk is the function with domain $\{11\}$ such that $rnk(11) = 2$.

E2.2 *Give an exact definition of a language for the set A (no individual constants, function symbols, or relation symbols).*

The quadruple $(\emptyset, \emptyset, \emptyset, \emptyset)$. Note that the last $\emptyset$ is the empty function.

E2.3 *Describe a term construction sequence which shows that* $+ \bullet v_0 v_0 v_1$ *is a term in the language for* $(\mathbb{R}, +, \cdot, 0, 1, <)$.

$\langle v_0, \bullet v_0 v_0, v_1, + \bullet v_0 v_0 v_1\rangle$.

E2.4 *In any first-order language, show that the sequence* $\langle v_0, v_0\rangle$ *is not a term. Hint: use Proposition 2.2.*

Suppose that $\langle v_0, v_0\rangle$ is a term. This contradicts Proposition 2.2(ii).

E2.5 *In the language for* $(\omega, S, 0, +, \cdot)$, *show that the sequence* $\langle +, v_0, v_1, v_2\rangle$ *is not a term. Here* $S(i) = i + 1$ *for any* $i \in \omega$. *Hint: use Proposition 2.2.*

Suppose it is a term. By Proposition 2.2(ii)(c), there are terms σ, τ such that $\langle +, v_0, v_1, v_2\rangle$ is $\langle +\rangle^\frown\sigma^\frown\tau$. Thus $\langle v_0, v_1, v_2\rangle = \sigma^\frown\tau$. So the term v_0 is an initial segment of the term σ. By Proposition 2.2(iii) it follows that $v_0 = \sigma$. Hence $\langle v_1, v_2\rangle = \tau$. This contradicts Proposition 2.2(ii).

E2.6 *Prove Proposition 2.5.*

We show by complete induction on i that $\varphi_i \in \Gamma$ for all $i < m$. So, suppose that $i < m$ and $\varphi_j \in \Gamma$ for all $j < i$. By the definition of formula construction sequence, we have the following cases.

Case 1. φ_i is an atomic formula. Then $\varphi_i \in \Gamma$ by (i).

Case 2. There is a $j < i$ such that φ_i is $\neg\varphi_j$. By the inductive hypothesis, $\varphi_j \in \Gamma$. Hence by (ii), $\varphi_i \in \Gamma$.

Case 2. There are $j, k < i$ such that φ_i is $\varphi_j \to \varphi_k$. By the inductive hypothesis, $\varphi_j \in \Gamma$ and $\varphi_k \in \Gamma$. Hence by (iii), $\varphi_i \in \Gamma$.

Case 4. There exist $j < i$ and $k \in \omega$ such that φ_i is $\forall v_k \varphi_j$. By the inductive hypothesis, $\varphi_j \in \Gamma$. Hence by (iv), $\varphi_i \in \Gamma$.

This completes the inductive proof.

E2.7 *Show how the structure $(\omega, S, 0, +, \cdot)$ can be put in the general framework of structures.*

$(\omega, S, 0, +, \cdot)$ can be considered to be the structure $(\omega, Rel', Fcn', Cn')$ where $Rel' = \emptyset$, Cn' is the function with domain $\{8\}$ such that $Cn'(8) = 0$, and Fcn' is the function with domain $\{6, 7, 9\}$ such that $Fcn'(6) = S$, $Fcn'(7) = +$, and $Fcn'(9) = \cdot$.

E2.8 *Prove that in the language for the structure $(\omega, +)$, a term has length m iff m is odd.*

First we show by induction on terms that every term has odd length. This is true for variables. Suppose that it is true for terms σ and τ. Then also $\sigma + \tau$ has odd length. Hence every term has odd length.

Second we prove by induction on m that for all m, there is a term of length $2m + 1$. A variable has length 1, so our assertion holds for $m = 0$. Assume that there is a term σ of length $2m + 1$. Then $\sigma + v_0$ has length $2m + 3$. This finishes the inductive proof.

E2.9 *Give a formula φ in the language for $(\mathbf{Q}, +, \cdot)$ such that for any $a : \omega \to \mathbb{Q}$, $(\mathbb{Q}, +, \cdot) \models \varphi[a]$ iff $a_0 = 1$.*

Let φ be the formula $\forall v_1[v_0 \cdot v_1 = v_1]$.

E2.10 *Give a formula φ which holds in a structure, under any assignment, iff the structure has at least 3 elements.*

$$\exists v_0 \exists v_1 \exists v_2(\neg(v_0 = v_1) \wedge \neg(v_0 = v_2) \wedge \neg(v_1 = v_2)).$$

E2.11 *Give a formula φ which holds in a structure, under any assignment, iff the structure has exactly 4 elements.*

$$\exists v_0 \exists v_1 \exists v_2 \exists v_3(\neg(v_0 = v_1) \wedge \neg(v_0 = v_2) \wedge \neg(v_0 = v_3) \wedge \neg(v_1 = v_2) \wedge \neg(v_1 = v_3) \wedge \neg(v_2 = v_3)$$
$$\wedge \forall v_4(v_0 = v_4 \vee v_1 = v_4 \vee v_2 = v_4 \vee v_3 = v_4)).$$

E2.12 *Write a formula φ in the language for $(\omega, <)$ such that for any assignment a, $(\omega, <) \models \varphi[a]$ iff $a_0 < a_1$ and there are exactly two integers between a_0 and a_1.*

$$v_0 < v_1 \wedge \exists v_2 \exists v_3[v_0 < v_2 \wedge v_2 < v_3 \wedge v_3 < v_1$$
$$\wedge \forall v_4[v_0 < v_4 \wedge v_4 < v_1 \to v_4 = v_2 \vee v_4 = v_3]].$$

E2.13 *Prove that the formula*

$$v_0 = v_1 \rightarrow (\mathbf{R}v_0v_2 \rightarrow \mathbf{R}v_1v_2)$$

is universally valid, where $\mathbf{R}$ *is a binary relation symbol.*

Let $\overline{A}$ be a structure and $a : \omega \rightarrow A$ an assignment. Suppose that $\overline{A} \models (v_0 = v_1)[a]$. Then $a_0 = a_1$. Also suppose that $\overline{A} \models \mathbf{R}v_0v_2[a]$. Then $(a_0, a_2) \in \mathbf{R}^{\overline{A}}$. Hence $(a_1, a_2) \in \mathbf{R}^{\overline{A}}$. Hence $\overline{A} \models \mathbf{R}v_1v_2[a]$, as desired.

E2.14 *Give an example showing that the formula*

$$v_0 = v_1 \rightarrow \forall v_0(v_0 = v_1)$$

is not universally valid.

Consider the structure $\overline{A} \stackrel{\text{def}}{=} (\omega, <)$, and let $a : \omega \rightarrow \omega$ be defined by $a(i) = 0$ for all $i \in \omega$. Then $\overline{A} \models (v_0 = v_1)[a]$. Now $\overline{A} \not\models (v_0 = v_1)[a^0_1]$ since $1 \neq 0$, so $\overline{A} \not\models \forall v_0(v_0 = v_1)[a]$. Therefore $\overline{A} \not\models (v_0 = v_1 \rightarrow \forall v_0(v_0 = v_1))[a]$.

E2.15 *Prove that* $\exists v_0 \forall v_1 \varphi \rightarrow \forall v_1 \exists v_0 \varphi$ *is universally valid.*

Assume that $a : \omega \rightarrow A$ and $\overline{A} \models \exists v_0 \forall v_1 \varphi[a]$. Choose $u \in A$ so that $\overline{A} \models \forall v_1 \varphi[a^0_u]$. In order to show that $\overline{A} \models \forall v_1 \exists v_0 \varphi[a]$, let $w \in A$ be given. Then $\overline{A} \models \varphi^{01}_{uw}$. It follows that $\overline{A} \models \exists v_0 \varphi[u^1_w]$. Hence $\overline{A} \models \forall v_1 \exists v_0 \varphi[a]$, as desired.

Solutions to exercises in Chapter 3

E3.1 *Do the case* $\mathbf{R}\sigma_0 \ldots \sigma_{m-1}$ *for some* m*-ary relation symbol and terms* $\sigma_0, \ldots, \sigma_{m-1}$ *in the proof of Theorem 3.1, (L3).*

We are assuming that v_i does not occur in $\mathbf{R}\sigma_0 \ldots \sigma_{m-1}$; hence it does not occur in any term σ_i.

$$\begin{aligned} \overline{A} \models (\mathbf{R}\sigma_0 \ldots \sigma_{m-1})[a] \quad &\text{iff} \quad \langle \sigma_0^{\overline{A}}(a), \ldots, \sigma_{m-1}^{\overline{R}}(a) \rangle \in \mathbf{R}^{\overline{A}} \\ &\text{iff} \quad \langle \sigma_0^{\overline{A}}(b), \ldots, \sigma_{m-1}^{\overline{R}}(b) \rangle \in \mathbf{R}^{\overline{A}} \\ &\qquad \text{(by Proposition 2.4)} \\ &\text{iff} \quad \overline{A} \models (\mathbf{R}\sigma_0 \ldots \sigma_{m-1})[b]. \end{aligned}$$

E3.2 *Prove that (L6) is universally valid, in the proof of Theorem 3.1.*

Assume that $\overline{A} \models (\sigma = \tau)[a]$ and $\overline{A} \models (\rho = \sigma)[a]$. Then $\sigma^{\overline{A}}(a) = \tau^{\overline{A}}(a)$ and $\rho^{\overline{A}}(a) = \sigma^{\overline{A}}(a)$, so $\rho^{\overline{A}}(a) = \tau^{\overline{A}}(a)$, hence $\overline{A} \models (\rho = \tau)[a]$.

E3.3 *Prove that (L8) is universally valid, in the proof of Theorem 3.1.*

Assume that $\overline{A} \models (\sigma = \tau)[a]$. Then $\sigma^{\overline{A}}(a) = \tau^{\overline{A}}(a)$. Assume that

$$\begin{aligned} &\overline{A} \models (\mathbf{R}\xi_0 \ldots \xi_{i-1}\sigma\xi_{i+1} \ldots \xi_{m-1})[a]; \text{ hence} \\ &\langle \xi_0^{\overline{A}}(a), \ldots, \xi_{i-1}^{\overline{A}}(a), \sigma^{\overline{A}}(a), \xi_{i+1}^{\overline{A}}(a), \ldots, \xi_{m-1}^{\overline{A}}(a) \rangle \in \mathbf{R}^{\overline{A}}; \text{ hence} \\ &\langle \xi_0^{\overline{A}}(a), \ldots, \xi_{i-1}^{\overline{A}}(a), \tau^{\overline{A}}(a), \xi_{i+1}^{\overline{A}}(a), \ldots, \xi_{m-1}^{\overline{A}}(a) \rangle \in \mathbf{R}^{\overline{A}}; \text{ hence} \\ &\overline{A} \models (\mathbf{R}\xi_0 \ldots \xi_{i-1}\tau\xi_{i+1} \ldots \xi_{m-1})[a]; \end{aligned}$$

hence (L8) is universally valid.

E3.4 *Finish the proof of Proposition 3.9.*

If $i = 0$ then φ itself is the desired segment, unique by Proposition 2.6(iii). If $i > 0$ then actually $i > 1$ so that φ_i is within ψ, and the inductive hypothesis applies.

E3.5 *Finish the proof of Proposition 3.11.*

Suppose inductively that φ is $\neg\psi$. Thus φ is $\langle 1\rangle^\frown\psi$. It follows that $i > 0$, so that φ_i appears in ψ; then the inductive hypothesis applies.

Suppose inductively that φ is $\psi \to \chi$. Thus φ is $\langle 2\rangle^\frown\psi^\frown\chi$. It follows that $i > 0$, so that φ_i appears in ψ or χ; then the inductive hypothesis applies.

Finally, suppose that φ is $\forall v_s\psi$ with ψ a formula and $s \in \omega$. Thus φ is $\langle 4, 5(s+1)\rangle^\frown\psi$. Hence $i > 0$. If $i = 1$, then $\langle 5(s+1)\rangle$ is the desired segment, unique by Proposition 2.6(iii). Suppose that $i > 1$. So φ_i is an entry in ψ and hence by the inductive assumption, there is a segment $\langle\varphi_i, \varphi_{i+1}, \ldots \varphi_m\rangle$ which is a term; this is also a segment of φ, and it is unique by Proposition 2.6(iii).

E3.6 *Indicate which occurrences of the variables are bound and which ones free for the following formulas.*

$\exists v_0(v_0 < v_1) \wedge \forall v_1(v_0 = v_1)$.
$v_4 + v_2 = v_0 \wedge \forall v_3(v_0 = v_1)$.
$\exists v_2(v_4 + v_2 = v_0)$.

First formula: the first and second occurrences of v_0 are bound, and the third one is free. The first occurrence of v_1 is free, and the other two are bound.

Second formula: the occurrence of v_3 is bound. All other occurrences of variables are free.

Third formula: the two occurrences of v_2 are bound. The other occurrences of variables are free.

E3.7 *Prove Proposition 3.14.*

Induction on φ. Suppose that φ is $\rho = \xi$. Then by Proposition 3.13, σ occurs in ρ or ξ. Suppose that it occurs in ρ. Let ρ' be obtained from ρ by replacing that occurrence of σ by τ. Then ρ' is a term by Proposition 3.14. Since ψ is $\rho' = \xi$, ψ is a formula. The case in which σ occurs in ξ is similar. Now suppose that φ is $\mathbf{R}\eta_0 \ldots \eta_{m-1}$ with $\mathbf{R}$ an m-ary relation symbol and $\eta_0, \ldots, \eta_{m-1}$ are terms. Then the occurrence of σ is within some η_i. Let η'_i be obtained from η_i by replacing that occurrence by τ. Now ψ is $\mathbf{R}\eta_0 \ldots \eta_{i-1}\eta'_i \ldots \eta_{m-1}$, so ψ is a formula.

Now suppose that the result holds for φ', and φ is $\neg\varphi'$. Then σ occurs in φ', so if ψ' is obtained from φ' by replacing the occurrence of σ by τ, then ψ' is a formula by the inductive assumption. Since ψ is $\neg\psi'$ also ψ is a formula.

Next, suppose that the result holds for φ' and φ'', and φ is $\varphi' \to \varphi''$. Then the occurrence of σ is within φ' or is within φ''. If it is within φ', let ψ' be obtained from φ' by replacing that occurrence of σ by τ. Then ψ' is a formula by the inductive hypothesis. Since ψ is $\psi' \to \varphi''$, also ψ is a formula. If the occurrence is within φ'', let ψ'' be obtained from φ'' by replacing that occurrence of σ by τ. Then ψ'' is a formula by the inductive hypothesis. Since ψ is $\varphi' \to \psi''$, also ψ is a formula.

Finally, suppose that the result holds for φ', and φ is $\forall v_k\varphi'$. If $i = 1$, then σ is v_k, and by hypothesis τ is some variable v_l. Then ψ is $\forall v_l\varphi'$, which is a formula. If $i > 1$, then σ occurs in φ', so if ψ' is obtained from φ' by replacing the occurrence of σ by τ, then ψ' is a formula by the inductive assumption. Since ψ is $\forall v_k\psi'$ also ψ is a formula.

E3.8 *Indicate all free and bound occurrences of terms in the formula* $v_0 = v_1 + v_1 \to \exists v_2(v_0 + v_2 = v_1)$.

v_0 is free in both of its occcurrences.
v_1 is free in all three of its occurrences.
v_2 is bound in both of its occurrences.
$v_1 + v_1$ is free in its occurrence.
$v_0 + v_2$ is bound in its occurrence.

E3.9 *Prove Proposition 3.17*

Induction on φ. If φ is atomic, then ψ is equal to φ, and θ is equal to χ and hence is a formula. Suppose the result is true for φ' and φ is $\neg\varphi'$. If $\psi = \varphi$, again the desired conclusion is clear. Otherwise the occurrence of ψ is within the subformula φ'. If θ' is obtained from φ' by replacing that occurrence by χ, then θ' is a formula by the inductive hypothesis. Since θ is $\neg\theta'$, also θ is a formula.

Now suppose the result is true for φ' and φ'', and φ is $\varphi' \to \varphi''$. If $\psi = \varphi$, again the desired conclusion is clear. Otherwise the occurrence of ψ is within the subformula φ' or is within the subformula φ''. If it is within φ' and θ' is obtained from φ' by replacing that occurrence by χ, then θ' is a formula by the inductive hypothesis. Since θ is $\theta' \to \varphi''$, also θ is a formula. If it is within φ'' and θ'' is obtained from φ'' by replacing that occurrence by χ, then θ'' is a formula by the inductive hypothesis. Since θ is $\varphi' \to \theta''$, also θ is a formula.

Finally, suppose the result is true for φ' and φ is $\forall v_i\varphi'$. If $\psi = \varphi$, again the desired conclusion is clear. Otherwise the occurrence of ψ is within the subformula φ'. If θ' is obtained from φ' by replacing that occurrence by χ, then θ' is a formula by the inductive hypothesis. Since θ is $\forall v_i\theta'$, also θ is a formula.

E3.10 *Show that the condition in Lemma 3.15 that the resulting occurrence of* τ *is free is necessary. Hint: use Theorem 3.2; describe a specific formula of the type in Proposition 3.15, but with* τ *not free, such that the formula is not universally valid.*

Consider the language for (ω, S), and the formula

$$v_0 = v_1 \to (\exists v_1(\mathbf{S}v_0 = v_1) \leftrightarrow \exists v_1(\mathbf{S}v_1 = v_1)).$$

Taking an assignment $a : \omega \to \omega$ with $a_0 = a_1$ makes this sentence false; hence it is not provable, by Theorem 3.2.

E3.11 *Do the case of implication in the proof of Lemma 3.15.*

Suppose inductively that φ is $\chi \to \theta$.

Case 1. The occurrence of σ in φ is within χ. Let χ' be obtained from χ by replacing that occurrence by τ, such that that occurrence is free in ψ, hence free in χ'. By the inductive hypothesis, $\vdash \sigma = \tau \to (\chi \leftrightarrow \chi')$. Since ψ is $\chi' \to \theta$, a tautology gives the desired result.

Case 2. The occurrence of σ in φ is within θ. Let θ' be obtained from θ by replacing that occurrence by τ, such that that occurrence is free in ψ, hence free in θ'. By the inductive hypothesis, $\vdash \sigma = \tau \to (\theta \leftrightarrow \theta')$. Since ψ is $\chi \to \theta'$, a tautology gives the desired result.

E3.12 *Prove that the hypothesis of Theorem 3.25 is necessary.*

Consider the formula

$$\forall v_0 \exists v_1 (v_0 < v_1) \to \exists v_1 (v_1 < v_1).$$

This formula is not universally valid; it fails to hold in $(\omega, <)$, for example.

E3.13 *Prove Proposition 3.29.*

Proof. By definition, $\exists v_i \neg\varphi$ is $\neg\forall v_i \neg\neg\varphi$. Now $\vdash \varphi \leftrightarrow \neg\neg\varphi$ by a tautology. Hence using generalization and (L2) we get $\vdash \forall v_i \varphi \leftrightarrow \forall v_i \neg\neg\varphi$. Hence another tautology yields $\vdash \neg\forall v_i \varphi \leftrightarrow \neg\forall v_i \neg\neg\varphi$, i.e., $\vdash \neg\forall v_i \varphi \leftrightarrow \exists v_i \neg\varphi$. □

E3.14 *Prove Proposition 3.30.*

Proof. $\neg\exists v_i \varphi$ is the formula $\neg\neg\forall v_i \neg\varphi$, so a simple tautology gives the result. □

E3.15 *Prove Proposition 3.31.*

Proof. By Theorem 3.25 we have $\vdash \forall v_i \neg\varphi \to \mathrm{Subf}^{v_i}_{\sigma}(\neg\varphi)$. Since clearly $\mathrm{Subf}^{v_i}_{\sigma}(\neg\varphi)$ is the same as $\neg\mathrm{Subf}^{v_i}_{\sigma}\varphi$, a tautology gives $\vdash \mathrm{Subf}^{v_i}_{\sigma}\varphi \to \exists v_i \varphi$. □

E3.16 *Prove Proposition 3.33.*

By Corollary 3.26 and Corollary 3.32.

E3.17 *Prove Proposition 3.34.*

Proof. $\vdash \neg\varphi \leftrightarrow \forall v_i \neg\varphi$. Now use a tautology.

E3.18 *Prove Proposition 3.41.*

Assume $\vdash \varphi \leftrightarrow \psi$. By a tautology, $\vdash \varphi \to \psi$. Hence by generalization and (L2), $\vdash \forall v_i \varphi \to \forall v_i \psi$. Similarly, $\vdash \forall v_i \psi \to \forall v_i \varphi$. The exercise follows by a tautology.

E3.19 *Prove Proposition 3.42*

Assume $\vdash \varphi \leftrightarrow \psi$. By a tautology, $\vdash \neg\varphi \leftrightarrow \neg\psi$. Hence by exercise E3.18, $\vdash \forall v_i \neg\varphi \leftrightarrow \forall v_i \neg\psi$. Now a tautology gives the desired result.

E3.20 *Prove that*

$$\vdash \forall v_0 \forall v_1 (v_0 = v_1) \to \forall v_0 (v_0 = v_1 \lor v_0 = v_2).$$

$$\vdash \forall v_0\forall v_1(v_0 = v_1) \to v_0 = v_1; \quad \text{Cor. 3.26 twice, taut.} \tag{1}$$
$$\vdash \forall v_1(v_0 = v_1) \to v_0 = v_2; \quad \text{Thm. 3.25} \tag{2}$$
$$\vdash \forall v_0\forall v_1(v_0 = v_1) \to v_0 = v_2; \quad \text{(2), Cor. 3.26, taut.} \tag{3}$$
$$\vdash \forall v_0\forall v_1(v_0 = v_1) \to v_0 = v_1 \vee v_0 = v_2; \quad \text{(1), (3), taut.} \tag{4}$$
$$\vdash \forall v_0\forall v_0\forall v_1(v_0 = v_1) \to \forall v_0(v_0 = v_1 \vee v_0 = v_2); \quad \text{(4), (L2), taut.} \tag{5}$$
$$\vdash \forall v_0\forall v_1(v_0 = v_1) \to \forall v_0(v_0 = v_1 \vee v_0 = v_2). \quad \text{(5), Prop. 3.27, taut.}$$

E3.21 *Prove that*

$$\vdash \exists v_0(\neg v_0 = v_1 \wedge \neg v_0 = v_2) \to \exists v_0\exists v_1(\neg v_0 = v_1).$$

$$\vdash \neg\forall v_0(v_0 = v_1 \vee v_0 = v_2) \to \neg\forall v_0\forall v_1(v_0 = v_1); \quad \text{E3.20, taut.} \tag{1}$$
$$\vdash \neg\forall v_0(v_0 = v_1 \vee v_0 = v_2) \leftrightarrow \exists v_0\neg(v_0 = v_1 \vee v_0 = v_2); \quad \text{Prop. 3.29} \tag{2}$$
$$\vdash \neg(v_0 = v_1 \vee v_0 = v_2) \leftrightarrow (\neg(v_0 = v_1) \wedge \neg(v_0 = v_2)); \quad \text{taut.} \tag{3}$$
$$\vdash \exists v_0\neg(v_0 = v_1 \vee v_0 = v_2) \leftrightarrow \exists v_0(\neg(v_0 = v_1) \wedge \neg(v_0 = v_2)); \quad \text{(3), Prop. 3.42} \tag{4}$$
$$\vdash \neg\forall v_0(v_0 = v_1 \vee v_0 = v_2) \leftrightarrow \exists v_0(\neg(v_0 = v_1) \wedge \neg(v_0 = v_2)); \quad \text{(2), (4), taut.} \tag{5}$$
$$\vdash \neg\forall v_1(v_0 = v_1) \leftrightarrow \exists v_1\neg(v_0 = v_1); \quad \text{Prop. 3.29} \tag{6}$$
$$\vdash \exists v_0\neg\forall v_1(v_0 = v_1) \leftrightarrow \exists v_0\exists v_1\neg(v_0 = v_1); \quad \text{(6), Prop. 3.42} \tag{7}$$
$$\vdash \neg\forall v_0\forall v_1(v_0 = v_1) \leftrightarrow \exists v_0\neg\forall v_1(v_0 = v_1); \quad \text{Prop. 3.29} \tag{8}$$
$$\vdash \neg\forall v_0\forall v_1(v_0 = v_1) \leftrightarrow \exists v_0\exists v_1\neg(v_0 = v_1) \quad \text{(7), (8), taut.} \tag{9}$$
$$\vdash \exists v_0(\neg v_0 = v_1 \wedge \neg v_0 = v_2) \to \exists v_0\exists v_1(\neg v_0 = v_1). \quad \text{(1), (5), (9), taut.} \qquad \square$$

Solutions to exercises in Chapter 4

E4.1 *Suppose that* $\Gamma \vdash \varphi \to \psi$, $\Gamma \vdash \varphi \to \neg\psi$, *and* $\Gamma \vdash \neg\varphi \to \varphi$. *Prove that* Γ *is inconsistent.*

The formula $(\neg\varphi \to \varphi) \to \varphi$ is a tautology. Hence by Lemma 3.3, $\Gamma \vdash (\neg\varphi \to \varphi) \to \varphi$. Since also $\Gamma \vdash \neg\varphi \to \varphi$, it follows that $\Gamma \vdash \varphi$. Hence $\Gamma \vdash \psi$ and $\Gamma \vdash \neg\psi$. Hence by Lemma 4.1, Γ is inconsistent.

E4.2 *Let* $\mathscr{L}$ *be a language with just one non-logical constant, a binary relation symbol* $\mathbf{R}$. *Let* Γ *consist of all sentences of the form* $\exists v_1\forall v_0[\mathbf{R}v_0v_1 \leftrightarrow \varphi]$ *with* φ *a formula with only* v_0 *free. Show that* Γ *is inconsistent. Hint: take* φ *to be* $\neg\mathbf{R}v_0v_0$.

By Theorem 3.25 we have

$$\Gamma \vdash \forall v_0[\mathbf{R}v_0v_1 \leftrightarrow \neg\mathbf{R}v_0v_0] \to [\mathbf{R}v_1v_1 \leftrightarrow \neg\mathbf{R}v_1v_1]. \tag{1}$$

Now $[\mathbf{R}v_1v_1 \leftrightarrow \neg\mathbf{R}v_1v_1] \to \neg(v_0 = v_0)$ is a tautology, so from (1) we obtain

$$\Gamma \vdash \forall v_0[\mathbf{R}v_0v_1 \leftrightarrow \neg\mathbf{R}v_0v_0] \to \neg(v_0 = v_0);$$

then generalization gives

$$\Gamma \vdash \forall v_1[\forall v_0[\mathbf{R}v_0v_1 \leftrightarrow \neg\mathbf{R}v_0v_0] \to \neg(v_0 = v_0)].$$

Then by Proposition 3.37 we get

$$\Gamma \vdash \exists v_1\forall v_0[\mathbf{R}v_0v_1 \leftrightarrow \neg\mathbf{R}v_0v_0] \to \neg(v_0 = v_0).$$

But the hypothesis here is a member of Γ, so we get $\Gamma \vdash \neg(v_0 = v_0)$. Hence by Lemma 4.1, Γ is inconsistent.

Alternate proof (due to a couple of students). Suppose that Γ is consistent. By the completeness theorem let $\overline{A}$ be a model of Γ. Taking φ to be $\neg\mathbf{R}v_0v_0$, we get $\overline{A} \models \exists v_1\forall v_0[\mathbf{R}v_0v_1 \leftrightarrow \neg\mathbf{R}v_0v_0]$. Let $a : \omega \to A$ be any assignment. Then by Proposition 2.8(iv) there is a $b \in A$ such that $\overline{A} \models \forall v_0[\mathbf{R}v_0v_1 \leftrightarrow \neg\mathbf{R}v_0v_0][a^1_b]$. By the definition of satisfaction of $\forall$, it follows that for any $c \in A$ we have $\overline{A} \models [\mathbf{R}v_0v_1 \leftrightarrow \neg\mathbf{R}v_0v_0][a^{0\,1}_{c\,b}]$. Hence $(c, b) \in \mathbf{R}^{\overline{A}}$ iff $(c, b) \notin \mathbf{R}^{\overline{A}}$, contradiction.

E4.3 *Show that the first-order deduction theorem fails if the condition that φ is a sentence is omitted. Hint: take $\Gamma = \emptyset$, let φ be the formula $v_0 = v_1$, and let ψ be the formula $v_0 = v_2$.*

$$\begin{aligned}
&\{v_0 = v_1\} \vdash v_0 = v_1\\
&\{v_0 = v_1\} \vdash \forall v_1(v_0 = v_1)\\
&\{v_0 = v_1\} \vdash \forall v_1(v_0 = v_1) \to v_0 = v_2 \quad \text{by Theorem 3.25}\\
&\{v_0 = v_1\} \vdash v_0 = v_2.
\end{aligned}$$

On the other hand, let $\overline{A}$ be the structure with universe ω and define $a = \langle 0, 0, 1, 1, \ldots\rangle$. Clearly $\overline{A} \not\models [v_0 = v_1 \to v_0 = v_2][a]$. Hence $\not\vdash v_0 = v_1 \to v_0 = v_2$ by Theorem 3.2.

E4.4 *In the language for $\overline{A} \stackrel{\text{def}}{=} (\omega, S, 0, +, \cdot)$, let τ be the term $v_0 + v_1 \cdot v_2$ and ν the term $v_0 + v_2$. Let a be the sequence $\langle 0, 1, 2, \ldots\rangle$. Let ρ be obtained from τ by replacing the occurrence of v_1 by ν.*

(a) Describe ρ as a sequence of integers.

(b) What is $\rho^{\overline{A}}(a)$?

(c) What is $\nu^{\overline{A}}(a)$?

(d) Describe the sequence $a^1_{\nu^{\overline{A}}(a)}$ as a sequence of integers.

(e) Verify that $\rho^{\overline{A}}(a) = \tau^{\overline{A}}(a^1_{\nu^{\overline{A}}(a)})$ (cf. Lemma 4.4.)

(a) ρ is $v_0 + (v_0 + v_2) \cdot v_2$; as a sequence of integers it is $\langle 7, 5, 9, 7, 5, 15, 15\rangle$.

(b) $\rho^{\overline{A}}(a) = 0 + (0 + 2) \cdot 2 = 4$.

(c) $\nu^{\overline{A}}(a) = 0 + 2 = 2$.

(d) $a^1_{\nu^{\overline{A}}(a)} = \langle 0, 2, 2, 3, \ldots\rangle$.

(e) $\rho^{\overline{A}}(a) = 4$, as above; $\tau^{\overline{A}}(a^1_{\nu^{\overline{A}}(a)}) = 0 + 2 \cdot 2 = 4$.

$\boxed{\text{E4.5}}$ *In the language for* $\overline{A} \stackrel{\text{def}}{=} (\omega, S, 0, +, \cdot)$, *let* φ *be the formula* $\forall v_0(v_0 \cdot v_1 = v_1)$, *let* ν *be the formula* $v_1 + v_1$, *and let* $a = \langle 1, 0, 1, 0, \ldots \rangle$.

(a) *Describe* $\text{Subf}_\nu^{v_1}\varphi$ *as a sequence of integers*

(b) *What is* $\nu^{\overline{A}}(a)$?

(c) *Describe* $a^1_{\nu^{\overline{A}}(a)}$ *as a sequence of integers.*

(d) *Determine whether* $\overline{A} \models \text{Subf}_\nu^{v_1}\varphi[a]$ *or not.*

(e) *Determine whether* $\overline{A} \models \varphi[a^1_{\nu^{\overline{A}}(a)}]$ *or not.*

(a) $\text{Subf}_\nu^{v_1}\varphi$ is $\forall v_0(v_0 \cdot (v_1 + v_1) = v_1 + v_1$; as a sequence of integers it is

$$\langle 4, 5, 3, 9, 5, 7, 10, 10, 7, 10, 10 \rangle.$$

(b) $\nu^{\overline{A}}(a) = (v_1 + v_1)^{\overline{A}}(\langle 1, 0, 1, 0, \ldots \rangle) = 0 + 0 = 0.$

(c) $a^1_{\nu^{\overline{A}}(a)} = \langle 1, 0, 1, 0, \ldots \rangle.$

(d) $\overline{A} \models \text{Subf}_\nu^{v_1}\varphi[a]$ iff $\overline{A} \models [\forall v_0(v_0 \cdot (v_1 + v_1) = v_1 + v_1][\langle 1, 0, 1, 0, \ldots \rangle]$ iff for all $a \in \omega$, $a \cdot (0 + 0) = 0 + 0$; this is true.

(e) $\overline{A} \models \varphi[a^1_{\nu^{\overline{A}}(a)}]$ iff $\overline{A} \models [\forall v_0(v_0 \cdot v_1 = v_1][\langle 1, 0, 1, 0, \ldots \rangle]$ iff for all $a \in \omega$, $a \cdot 0 = 0$; this is true.

$\boxed{\text{E4.6}}$ *Show that the condition in Lemma 4.6 that*

no free occurrence of v_i *in* φ *is within a subformula of the form* $\forall v_k \mu$ *with* v_k *a variable occurring in* ν

is necessary for the conclusion of the lemma.

In the language for $\overline{A} = (\omega, S, 0, +, \cdot)$, let φ be the formula $\exists v_1[Sv_1 = v_0]$, $\nu = v_1$, and $a = \langle 1, 1, \ldots \rangle$. Note that the condition on v_0 fails. Now $\text{Subf}_{v_1}^{v_0}\varphi$ is the formula $\exists v_1[Sv_1 = v_1]$, and there is no $a \in \omega$ such that $Sa = a$, and hence $\overline{A} \not\models \text{Subf}_{v_1}^{v_0}\varphi[a]$. Also, $\nu^{\overline{A}}(a) = v_1^{\overline{A}}(a) = a_1 = 1$, and hence $a^0_{\nu^{\overline{A}}(a)} = \langle 1, 1, \ldots \rangle$. Since $S0 = 1$, it follows that $\overline{A} \models \varphi[a^0_{\nu^{\overline{A}}(a)}]$.

$\boxed{\text{E4.7}}$ *Let* $\overline{A}$ *be an* $\mathscr{L}$*-structure, with* $\mathscr{L}$ *arbitrary. Define* $\Gamma = \{\varphi : \varphi$ *is a sentence and* $\overline{A} \models \varphi[a]$ *for some* $a : \omega \to A\}$. *Prove that* Γ *is complete and consistent.*

Note by Lemma 4.4 that $\overline{A} \models \varphi[a]$ for some $a : \omega \to A$ iff $\overline{A} \models \varphi[a]$ for every $a : \omega \to A$. Let φ be any sentence. Take any $a : \omega \to A$. If $\overline{A} \models \varphi[a]$, then $\varphi \in \Gamma$ and hence $\Gamma \vdash \varphi$. Suppose that $\overline{A} \not\models \varphi[a]$. Then $\overline{A} \models \neg\varphi[a]$, hence $\neg\varphi \in \Gamma$, hence $\Gamma \vdash \neg\varphi$.

This shows that Γ is complete. Suppose that Γ is not consistent. Then $\Gamma \vdash \neg(v_0 = v_0)$ by Lemma 4.1. Then $\Gamma \models \neg(v_0 = v_0)$ by Theorem 3.2. Since $\overline{A}$ is a model of Γ, it is also a model of $\neg(v_0 = v_0)$, contradiction.

$\boxed{\text{E4.8}}$ *Call a set* Γ *strongly complete iff for every formula* φ, $\Gamma \vdash \varphi$ *or* $\Gamma \vdash \neg\varphi$. *Prove that if* Γ *is strongly complete, then* $\Gamma \vdash \forall v_0 \forall v_1(v_0 = v_1)$.

Assume that Γ is strongly complete. Then $\Gamma \vdash v_0 = v_1$ or $\Gamma \vdash \neg(v_0 = v_1)$. If $\Gamma \vdash v_0 = v_1$, then by generalization, $\Gamma \vdash \forall v_0 \forall v_1(v_0 = v_1)$. Suppose that $\Gamma \vdash \neg(v_0 = v_1)$. Then by generalization, $\Gamma \vdash \forall v_0 \neg(v_0 = v_1)$. By Theorem 3.25, $\Gamma \vdash \forall v_0 \neg(v_0 = v_1) \to \neg(v_1 = v_1)$. Hence $\Gamma \vdash \neg(v_1 = v_1)$. But also $\Gamma \vdash v_1 = v_1$ by Proposition 3.4, so Γ is inconsistent by Lemma 4.1, and hence again $\Gamma \vdash \forall v_0 \forall v_1 (v_0 = v_1)$.

E4.9 *Prove that if Γ is rich, then for every term σ with no variables occurring in σ there is an individual constant* $\mathbf{c}$ *such that* $\Gamma \vdash \sigma = \mathbf{c}$.

By richness we have $\Gamma \vdash \exists v_0(v_0 = \sigma) \to \mathbf{c} = \sigma$ for some individual constant $\mathbf{c}$. Then using (L4) it follows that $\Gamma \vdash \mathbf{c} = \sigma$.

E4.10 *Prove that if Γ is rich, then for every sentence φ there is a sentence ψ with no quantifiers in it such that $\Gamma \vdash \varphi \leftrightarrow \psi$.*

We proceed by induction on the number m of symbols $\neg$, $\to$, $\forall$ in φ. (More exactly, by the number of the integers 1,2,4 that occur in the sequence φ.) If $m = 0$, then φ is atomic and we can take $\psi = \varphi$. Assume the result for m and suppose that φ has $m+1$ integers 1,2,4 in it. Then there are three possibilities. First, $\varphi = \neg\varphi'$. Let ψ' be a quantifier-free sentence such that $\Gamma \vdash \varphi' \leftrightarrow \psi'$. Then $\Gamma \vdash \varphi \leftrightarrow \neg\psi'$. Second, $\varphi = (\varphi' \to \varphi'')$. Choose quantifier-free sentences ψ' and ψ'' such that $\Gamma \vdash \varphi' \leftrightarrow \psi'$ and $\Gamma \vdash \varphi'' \leftrightarrow \psi''$. Then $\Gamma \vdash \varphi \leftrightarrow (\psi' \to \psi'')$. Third, $\varphi = \forall v_i \varphi'$. By richness, let c be an individual constant such that $\Gamma \vdash \exists v_i \neg\varphi' \to \mathrm{Subf}^{v_i}_c \neg\varphi'$. Then by Theorem 3.31 we get

(1) $\Gamma \vdash \exists v_i \neg\varphi' \leftrightarrow \mathrm{Subf}^{v_i}_c \neg\varphi'$.

Now $\mathrm{Subf}^{v_i}_c \varphi'$ has only m integers 1,2,4 in it, so by the inductive hypothesis there is a sentence ψ with no quantifiers in it such that $\Gamma \vdash \mathrm{Subf}^{v_i}_c \varphi' \leftrightarrow \psi$ and hence

(2) $\Gamma \vdash \mathrm{Subf}^{v_i}_c \neg\varphi' \leftrightarrow \neg\psi$.

From (1) and (2) and a tautology we get $\Gamma \vdash \neg\exists v_i \neg\varphi' \leftrightarrow \psi$. Then by Proposition 3.31, $\Gamma \vdash \forall v_i \varphi' \leftrightarrow \psi$, finishing the inductive proof.

E4.11 *Describe sentences in a language for ordering which say that $<$ is a linear ordering and there are infinitely many elements. Prove that the resulting set Γ of sentences is not complete.*

Let Γ consist of the following sentences:

$$\neg\exists v_0(v_0 < v_0);$$
$$\forall v_0 \forall v_1 \forall v_2 [v_0 < v_1 \wedge v_1 < v_2 \to v_0 < v_2];$$
$$\forall v_0 \forall v_1 [v_0 < v_1 \vee v_0 = v_1 \vee v_1 < v_0];$$
$$\bigwedge_{i<j<n} \neg(v_i = v_j) \quad \text{for every positive integer } n.$$

The following sentence φ holds in $(\mathbb{Q}, <)$ but not in $(\omega, <)$:

$$\forall v_0 \forall v_1 [v_0 < v_1 \to \exists v_2 (v_0 < v_2 \wedge v_2 < v_1)].$$

Since φ does not hold in $(\omega, <)$, we have $\Gamma \nvdash \varphi$, by Theorem 4.2. But since φ holds in $(\mathbb{Q}, <)$, we also have $\Gamma \nvdash \neg\varphi$ by Theorem 4.2. So Γ is not complete.

E4.12 *Prove that if a sentence φ holds in every infinite model of a set Γ of sentences, then there is an $m \in \omega$ such that it holds in every model of Γ with at least m elements.*

Suppose that φ holds in every infinite model of a set Γ of sentences, but for every $m \in \omega$ there is a model $\overline{M}$ of Γ with at least m elements such that φ does not hold in $\overline{M}$. Let Δ be the following set:

$$\Gamma \cup \left\{ \bigwedge_{i<j<n} \neg(v_i = v_j) : n \text{ a positive integer} \right\} \cup \{\neg\varphi\}.$$

Our hypothesis implies that every finite subset Δ' of Δ has a model; for if m is the maximum of all n such that the above big conjunction is in Δ', then the hypothesis yields a model of Δ'. By the compactness theorem we get a model $\overline{N}$ of Δ. Thus $\overline{N}$ is an infinite model of Γ in which φ does not hold, contradiction.

E4.13 *Let $\mathscr{L}$ be the language of ordering. Prove that there is no set Γ of sentences whose models are exactly the well-ordering structures.*

Suppose there is such a set. Let us expand the language $\mathscr{L}$ to a new one $\mathscr{L}'$ by adding an infinite sequence $\mathbf{c}_m$, $m \in \omega$, of individual constants. Then consider the following set Θ of sentences: all members of Γ, plus all sentences $\mathbf{c}_{m+1} < \mathbf{c}_m$ for $m \in \omega$. Clearly every finite subset of Θ has a model, so let $\overline{A} = (A, <, a_i)_{i<\omega}$ be a model of Θ itself. (Here a_i is the 0-ary function, i.e., element of A, corresponding to $\mathbf{c}_i$.) Then $a_0 > a_1 > \cdots$; so $\{a_i : i \in \omega\}$ is a nonempty subset of A with no least element, contradiction. □

E4.14 *Suppose that Γ is a set of sentences, and φ is a sentence. Prove that if $\Gamma \models \varphi$, then $\Delta \models \varphi$ for some finite $\Delta \subseteq \Gamma$.*

We prove the contrapositive: Suppose that for every finite subset Δ of Γ, $\Delta \not\models \varphi$. Thus every finite subset of $\Gamma \cup \{\neg\varphi\}$ has a model, so $\Gamma \cup \{\neg\varphi\}$ has a model, proving that $\Gamma \not\models \varphi$.

E4.15 *Suppose that f is a function mapping a set M into a set N. Let $R = \{(a, b) : a, b \in M \text{ and } f(a) = f(b)\}$. Prove that R is an equivalence relation on M.*

If $a \in M$, then $f(a) = f(a)$, so $(a, a) \in R$. Thus R is reflexive on M. Suppose that $(a, b) \in R$. Then $f(a) = f(b)$, so $f(b) = f(a)$ and hence $(b, a) \in R$. Thus R is symmetric. Suppose that $(a, b) \in R$ and $(b, c) \in R$. Then $f(a) = f(b)$ and $f(b) = f(c)$, so $f(a) = f(c)$ and hence $(a, c) \in R$.

E4.16 *Suppose that R is an equivalence relation on a set M. Prove that there is a function f mapping M into some set N such that $R = \{(a, b) : a, b \in M \text{ and } f(a) = f(b)\}$.*

Let N be the collection of all equivalence classes under R. For each $a \in M$ let $f(a) = [a]_R$. Then $(a, b) \in R$ iff $a, b \in M$ and $[a]_R = [b]_R$ iff $a, b \in M$ and $f(a) = f(b)$.

E4.17 *Let Γ be a set of sentences in a first-order language, and let Δ be the collection of all sentences holding in every model of Γ. Prove that $\Delta = \{\varphi : \varphi \text{ is a sentence and } \Gamma \vdash \varphi\}$.*

For $\subseteq$, suppose that $\varphi \in \Delta$. To prove that $\Gamma \vdash \varphi$ we use the compactness theorem, proving that $\Gamma \models \varphi$. Let $\overline{A}$ be any model of Γ. Since $\varphi \in \Delta$, it follows that $\overline{A}$ is a model of Γ, as desired.

For $\supseteq$, suppose that φ is a sentence and $\Gamma \vdash \varphi$. Then by the easy direction of the completeness theorem, $\Gamma \models \varphi$. That is, every model of Γ is a model of φ. Hence $\varphi \in \Delta$.

Solutions, Chapter 6

E6.1 *Prove that if $f : A \to B$ and $\langle C_i : i \in I\rangle$ is a system of subsets of A, then $f\left[\bigcup_{i\in I} C_i\right] = \bigcup_{i\in I} f[C_i]$.*

$$\begin{aligned}
x \in f\left[\bigcup_{i\in I} C_i\right] \quad &\text{iff} \quad x \in \operatorname{rng}\left(f \restriction \bigcup_{i\in I} C_i\right)\\
&\text{iff} \quad \exists y \in \bigcup_{i\in I} C_i[f(y) = x]\\
&\text{iff} \quad \exists i \in I \exists y \in C_i[f(y) = x]\\
&\text{iff} \quad \exists i \in I[x \in \operatorname{rng}(f \restriction C_i)]\\
&\text{iff} \quad \exists i \in I[x \in f[C_i]]\\
&\text{iff} \quad x \in \bigcup_{i\in I} f[C_i].
\end{aligned}$$

E6.2 *Prove that if $f : A \to B$ and $C, D \subseteq A$, then $f[C \cap D] \subseteq f[C] \cap f[D]$. Give an example showing that equality does not hold in general.*

Take any $x \in f[C \cap D]$. Choose $y \in C \cap D$ such that $x = f(y)$. Since $y \in C$, we have $x \in f[C]$. Similarly, $x \in f[D]$. So $x \in f[C] \cap f[D]$. Since x is arbitrary, this shows that $f[C \cap D] \subseteq f[C] \cap f[D]$.

For the required example, let $\operatorname{dmn}(f) = \{a, b\}$ with $a \neq b$ and with $f(a) = a = f(b)$. Let $C = \{a\}$ and $D = \{b\}$. Then $C \cap D = \emptyset$, so $f[C \cap D] = \emptyset$, while $f[C] = \{a\} = f[D]$ and hence $f[C] \cap f[D] = \{a\} \neq \emptyset$. So $f[C \cap D] \neq f[C] \cap f[D]$.

E6.3 *Given $f : A \to B$ and $C, D \subseteq A$, compare $f[C\backslash D]$ and $f[C]\backslash f[D]$: prove the inclusions (if any) which hold, and give counterexamples for the inclusions that fail to hold.*

We claim that $f[C]\backslash f[D] \subseteq f[C\backslash D]$. For, suppose that $x \in f[C]\backslash f[D]$. Choose $c \in C$ such that $x = f(c)$. Since $x \notin f[D]$, we have $c \notin D$. So $c \in C\backslash D$ and hence $x \in f[C\backslash D]$, proving the claim.

The other inclusion does not hold. For, take the same f, C, D as for exercise E6.2. Then $C\backslash D = \{a\}$ and so $f[C\backslash D] \neq \emptyset$. But $f[C] = \{a\} = f[D]$, so $f[C]\backslash f[D] = \emptyset$.

E6.4 *Prove that if $f : A \to B$ and $\langle C_i : i \in I\rangle$ is a system of subsets of B, then $f^{-1}\left[\bigcup_{i\in I} C_i\right] = \bigcup_{i\in I} f^{-1}[C_i]$.*

For any $b \in B$ we have

$$\begin{aligned}
b \in f^{-1}\left[\bigcup_{i\in I} C_i\right] \quad &\text{iff} \quad f(b) \in \bigcup_{i\in I} C_i\\
&\text{iff} \quad \exists i \in I[f(b) \in C_i]
\end{aligned}$$

$$\begin{aligned} &\text{iff} \quad \exists i \in I[b \in f^{-1}[C_i]] \\ &\text{iff} \quad b \in \bigcup_{i\in I} f^{-1}[C_i]. \end{aligned}$$

E6.5 *Prove that if $f : A \to B$ and $\langle C_i : i \in I\rangle$ is a system of subsets of B, then $f^{-1}\left[\bigcap_{i\in I} C_i\right] = \bigcap_{i\in I} f^{-1}[C_i]$.*

For any a,

$$\begin{aligned} a \in f^{-1}\left[\bigcap_{i\in I} C_i\right] \quad &\text{iff} \quad f(a) \in \bigcap_{i\in I} C_i \\ &\text{iff} \quad \forall i \in I[f(a) \in C_i] \\ &\text{iff} \quad \forall i \in I[a \in f^{-1}[C_i]] \\ &\text{iff} \quad a \in \bigcap_{i\in I} f^{-1}[C_i]. \end{aligned}$$

E6.6 *Prove that if $f : A \to B$ and $C, D \subseteq B$, then $f^{-1}[C\backslash D] = f^{-1}[C]\backslash f^{-1}[D]$.*

For any a,

$$\begin{aligned} a \in f^{-1}[C\backslash D] \quad &\text{iff} \quad f(a) \in C\backslash D \\ &\text{iff} \quad f(a) \in C \text{ and } f(a) \notin D \\ &\text{iff} \quad a \in f^{-1}[C] \text{ and } a \notin f^{-1}[D] \\ &\text{iff} \quad a \in f^{-1}[C]\backslash f^{-1}[D]. \end{aligned}$$

E6.7 *Prove that if $f : A \to B$ and $C \subseteq A$, then*

$$\{b \in B : f^{-1}[\{b\}] \subseteq C\} = B\backslash f[A\backslash C].$$

First suppose that b is in the left side; but suppose also, aiming for a contradiction, that $b \in f[A\backslash C]$. Say $b = f(a)$, with $a \in A\backslash C$. Then $a \in f^{-1}[\{b\}]$, so $a \in C$, contradiction.

Second, suppose that b is in the right side. Take any $a \in f^{-1}[\{b\}]$. Then $f(a) = b$, and it follows that $a \in C$, as desired.

E6.8 *For any sets A, B define $A\triangle B = (A\backslash B) \cup (B\backslash A)$; this is called the symmetric difference of A and B. Prove that if A, B, C are given sets, then $A\triangle(B\triangle C) = (A\triangle B)\triangle C$.*

Let $D = A \cup B \cup C$, $A' = D\backslash A$, $B' = D\backslash B$, and $C' = D\backslash C$. Then

$$\begin{aligned} A\triangle B &= (A \cap B') \cup (B \cap A'); \\ (A\triangle B)' &= ((A \cap B') \cup (B \cap A'))' \\ &= (A \cap B')' \cap (B \cap A')' \\ &= (A' \cup B) \cap (B' \cup A) \\ &= (A' \cap B') \cup (A \cap B). \end{aligned}$$

These equations hold for any sets A, B. Now

$$\begin{aligned} A\triangle(B\triangle C) &= (A\cap(B\triangle C)')\cup((B\triangle C)\cap A' \\ &= (A\cap((B'\cap C')\cup(B\cap C)))\cup(((B\cap C')\cup(C\cap B'))\cap A') \\ &= (A\cap B'\cap C')\cup(A\cap B\cap C)\cup(A'\cap B\cap C')\cup(A'\cap B'\cap C). \end{aligned}$$

This holds for any sets A, B, C. Hence

$$\begin{aligned} (A\triangle B)\triangle C &= C\triangle(A\triangle B) \\ &= (C\cap A'\cap B')\cup(C\cap A\cap B)\cup(C'\cap A\cap B')\cup(C'\cap A'\cap B) \\ &= A\triangle(B\triangle C). \end{aligned}$$

E6.9 *For any set A let*

$$Id_A = \{\langle x, x\rangle : x \in A\}.$$

Justify this definition on the basis of the axioms.

$$\mathrm{Id}_A = \{y \in A \times A : \exists x \in A[y = \langle x, x\rangle]\}.$$

E6.10 *Suppose that $f : A \to B$. Prove that f is surjective iff there is a $g : B \to A$ such that $f \circ g = Id_B$. Note: the axiom of choice might be needed.*

$\Leftarrow$: given $b \in B$, we have $b = (f \circ g)(b) = f(g(b))$; so f is surjective.

$\Rightarrow$: Assume that f is surjective. Let

$$\mathscr{A} = \{\{(b, a) : a \in A, f(a) = b\} : b \in B\}.$$

Each member of $\mathscr{A}$ is nonempty; for let $x \in \mathscr{A}$. Choose $b \in B$ such that $x = \{(b, a) : a \in A, f(a) = b\}$. Choose $a \in A$ such that $f(a) = b$. So $(b, a) \in x$.

The members of $\mathscr{A}$ are pairwise disjoint: suppose $x, y \in \mathscr{A}$ with $x \neq y$. Choose b, c so that $x = \{(b, a) : a \in A, f(a) = b\}$ and $y = \{(b, a) : a \in A, f(a) = c\}$. If $u \in x \cap y$, then there exist $a, a' \in A$ such that $u = (b, a)$, $f(a) = b$, and also $(u = (c, a')$, $f(a') = c$. So by Theorem 6.3, $b = c$. But then $x = y$, contradiction.

Now by the axiom of choice, let C have exactly one element in common with each member of $\mathscr{A}$. Then define

$$g = \{(b, a) \in C : a \in A, f(a) = b\}.$$

Now g is a function. For, suppose that $(b, a), (b, a') \in g$. Let $x = \{(b, a'') : a'' \in A, f(a'') = b\}$. Then $(b, a), (b, a') \in C \cap x$, so $(b, a) = (b, a')$. Hence $a = a'$.

Clearly $g \subseteq B \times A$. Next, $\mathrm{dmn}(g) = B$, for suppose that $b \in B$. Choose $x \in C \cap \{(b, a'') : a'' \in A, f(a'') = b\}$; say $x = (b, a)$ with $a \in A$, $f(a) = b$. Then $x \in g$ and so $b \in \mathrm{dmn}(g)$.

Thus $g : B \to A$. Take any $b \in B$, and let $g(b) = a$. So $(b, a) \in g$ and hence $f(a) = b$. So $f \circ g = \mathrm{Id}_B$.

E6.11 *Let A be a nonempty set. Suppose that $f : A \to B$. Prove that f is injective iff there is a $g : B \to A$ such that $g \circ f = Id_A$.*

First suppose that f is injective. Fix $a \in A$, and let

$$g = f^{-1} \cup \{(b, a) : b \in B \backslash \mathrm{rng}(f)\}.$$

Then g is a function. In fact, suppose that $(b, c), (b, d) \in g$. If both are in f^{-1}, then $(c, b(, (d, b) \in f$, so $f(c) = b = f(d)$ and hence $c = d$ since f is injective. If $(b, c) \in f^{-1}$ and $b \in B\backslash\mathrm{rng}(f)$, the $(c, b) \in f$, so $b \in \mathrm{rng}(f)$, contradiction. If $(b, c), (b, d) \notin f^{-1}$, then $c = d = a$.

Clearly then $g : B \to A$. For any $a \in A$ we have $(a, f(a)) \in f$, hence $(f(a), a) \in f^{-1} \subseteq g$, and so $g(f(a)) = a$.

Second, suppose that $g : B \to A$ and $g \circ f = \mathrm{Id}_A$. Suppose that $f(a) = f(a')$. Then $a = (g \circ f)(a) = g(f(a)) = g(f(a')) = (g \circ f)(a') = a'$.

E6.12 *Suppose that $f : A \to B$. Prove that f is a bijection iff there is a $g : B \to A$ such that $f \circ g = Id_B$ and $g \circ f = Id_A$. Prove this without using the axiom of choice.*

$\Rightarrow$: Assume that f is a bijection. By E6.11 there is a $g : B \to A$ such that $g \circ f = \mathrm{Id}_A$. We claim that $f \circ g = \mathrm{Id}_B$. Since f is a bijection, the relation f^{-1} is also a bijection. Now for any $b \in B$,

$$(f \circ g)(b) = f(g(b)) = f(g(f(f^{-1}(b)))) = f((g \circ f)(f^{-1}(b))) = f(f^{-1}(b)) = b.$$

So $f \circ g = \mathrm{Id}_B$, as desired.

$\Leftarrow$: Assume that g is as indicated. Then f is injective, since $f(a) = f(b)$ implies that $a = g(f(a)) = g(f(a')) = a'$. And f is surjective, since for a given $b \in B$ we have $f(g(b)) = b$.

E6.13 *For any sets R, S define*

$$R|S = \{(x, z) : \exists y((x, y) \in R \wedge (y, z) \in S)\}.$$

Justify this definition on the basis of the axioms.

$$R|S = \{(x, z) \in \mathrm{dmn}(R) \times \mathrm{rng}(S) : \exists y((x, y) \in R \wedge (y, z) \in S)\}.$$

E6.14 *Suppose that $f, g : A \to A$. Prove that*

$$(A \times A)\backslash[((A \times A)\backslash f)|((A \times A)\backslash g)]$$

is a function.

Suppose that $(x, y), (x, z)$ are in the indicated set, with $y \neq z$. By symmetry say $f(x) \neq y$. Then $(x, y) \in [(A \times A)\backslash f]$, so it follows that $(y, z) \in g$, as otherwise $(x, z) \in [((A \times A)\backslash f)|((A \times A)\backslash g)]$. Hence $(y, y) \notin g$, so $(x, y) \in [((A \times A)\backslash f)|((A \times A)\backslash g)]$, contradiction.

$\boxed{\text{E6.15}}$ *Suppose that $f : A \to B$ is a surjection, $g : A \to C$, and $\forall x, y \in A[f(x) = f(y) \to g(x) = g(y)]$. Prove that there is a function $h : B \to C$ such that $h \circ f = g$. Define h as a set of ordered pairs.*

Let $h = \{(f(a), g(a)) : a \in A\}$. Then h is a function, for suppose that $(x, y), (x, z) \in h$. Choose $a, a' \in A$ so that $x = f(a)$, $y = g(a)$, $x = f(a')$, and $y = g(a')$. Thus $f(a) = f(a')$, so $g(a) = g(a')$, as desired.

Since f is a surjection it is clear that $\text{dmn}(h) = B$. Clearly $\text{rng}(h) \subseteq C$. So $h : B \to C$. If $a \in A$, then $(f(a), b(a)) \in h$, hence $h(f(a)) = g(a)$. This shows that $h \circ f = g$.

$\boxed{\text{E6.16}}$ *The statement*

$\forall A \in \mathscr{A} \forall B \in \mathscr{B}(A \subseteq B)$ *implies that* $\bigcup \mathscr{A} \subseteq \bigcap \mathscr{B}$

is slightly wrong. Fix it, and prove the result.

If $\mathscr{A}$ has a nonempty member and $\mathscr{B}$ is empty, the implication does not hold. Add the hypothesis $\mathscr{B} \neq \emptyset$.

Suppose that $a \in \bigcup A$ and $B \in \mathscr{B}$; we want to show that $a \in B$. Choose $A \in \mathscr{A}$ such that $a \in A$. Since $A \subseteq B$, we have $a \in B$.

$\boxed{\text{E6.17}}$ *Suppose that $\forall A \in \mathscr{A} \exists B \in \mathscr{B}(A \subseteq B)$. Prove that $\bigcup \mathscr{A} \subseteq \bigcup \mathscr{B}$.*

Suppose that $a \in \bigcup \mathscr{A}$; we want to show that $a \in \mathscr{B}$. Choose $A \in \mathscr{A}$ such that $a \in A$. Then choose $B \in \mathscr{B}$ such that $A \subseteq B$. Then $a \in B$. Hence $a \in \bigcup \mathscr{B}$.

$\boxed{\text{E6.18}}$ *The statement*

$\forall A \in \mathscr{A} \exists B \in \mathscr{B}(B \subseteq A)$ *implies that* $\bigcap \mathscr{B} \subseteq \bigcap \mathscr{A}$.

is slightly wrong. Fix it, and prove the result.

If $\mathscr{A}$ is empty and $\bigcap \mathscr{B}$ is nonempty, the statement is false. Fix it by adding the hypothesis that $\mathscr{A}$ is nonempty.

Suppose that $b \in \bigcap \mathscr{B}$ and $A \in \mathscr{A}$; we want to show that $b \in A$. Choose $B \in \mathscr{B}$ such that $B \subseteq A$. Now $b \in B$ since $b \in \bigcap \mathscr{B}$, so $b \in A$.

Solutions, exercises in Chapter 7

$\boxed{\text{E7.1}}$ *Prove that if x is an ordinal, then x is transitive and $(x, \{(y, z) \in x \times x : y \in z\})$ is a well-ordered set.*

By definition, x is transitive. Let $R = \{(y, z) \in x \times x : y \in z\})$. Obviously R is a relation. By definition, $R \subseteq x \times x$. R is irreflexive on x by Theorem 7.5. R is transitive since x is transitive. R is linear on x by Theorem 7.7. The final well-ordering property follows from Theorem 7.13.

$\boxed{\text{E7.2}}$ *Assume that x is transitive and $(x, \{(y, z) \in x \times x : y \in z\})$ is a well-ordered set. Prove that for all $y, z \in x$, either $y = z$ or $y \in z$ or $z \in y$.*

This is obvious.

$\boxed{\text{E7.3}}$ *Assume that x is transitve and for all $y, z \in x$, either $y = z$ or $y \in z$ or $z \in y$. Prove that for all y, if $y \subset x$ and y is transitive, then $y \in x$. Hint: apply the foundation axiom to $x \backslash y$.*

Assume the hypothesis, and suppose that $y \subset x$ and y is transitive. Choose $z \in x \backslash y$ such that $z \cap (x \backslash y) = \emptyset$. If $u \in y$, then $u \in x$ since $y \subset x$. So $u, z \in x$, so by hypothesis we have $u \in z$, $u = z$, or $z \in u$. Now $u \neq z$ since $z \notin y$ and $u \in y$. And $z \notin u$, since $z \in u$ would imply, because y is transitive and $u \in y$, that $z \in y$, which is not true. Hence $u \in z$. This is true for any $u \in y$. So $y \subseteq z$. Clearly also $z \subseteq y$, so $y = z \in x$.

E7.4 *Assume that x is transitive and for all y, if $y \subset x$ and y is transitive, then $y \in x$. Show that x is an ordinal. Hint: let $y = \{z \in x : z \text{ is an ordinal}\}$, and get a contradiction from the assumption that $y \subset x$.*

Assume the hypothesis. Let $y = \{z \in x : z \text{ is an ordinal}\}$. So $y \subseteq x$. Suppose that $y \subset x$. Now y is transitive, for assume that $z \in y$. Thus $z \in x$ and z is an ordinal. Suppose that $w \in z$. Then $w \in x$ since x is transitive, and w is an ordinal since z is an ordinal. So $w \in y$. Thus, indeed, y is transitive. So by assumption $y \in x$. Now y is a transitive set of transitive sets, so y is an ordinal. It follows that $y \in y$, contradiction. This proves that $x = y$. So x is a transitive set of transitive sets, and hence x is an ordinal.

E7.5 *Show that if x is an ordinal, then the following two conditions hold:*
(i) For all $y \in x$, either $y \cup \{y\} = x$ or $y \cup \{y\} \in x$.
(ii) For all $y \subseteq x$, either $\bigcup y = x$ or $\bigcup y \in x$.

Assume that x is an ordinal. Then (i) holds by Proposition 7.10. Now suppose that $y \subseteq x$. If $z \in \bigcup y$, choose $w \in y$ such that $z \in w$. Then also $w \in x$, so $z \in x$ since x is transitive. This shows that $\bigcup y \subseteq x$. By Proposition 7.3, $\bigcup y$ is an ordinal. Hence by Proposition 7.8, $\bigcup y \leq x$.

E7.6 *Assume the two conditions of exercise E7.5. Show that x is an ordinal. Hint: Show that there is an ordinal α not in x. Taking such an ordinal α, show that there is a least $\beta \in \alpha \cup \{\alpha\}$ such that $\beta \notin x$. Work with such a β to show that x is an ordinal.*

By Theorem 7.6 there is an ordinal α not in x. Then by Theorem 7.13 there is a least $\beta \in \alpha \cup \{\alpha\}$ such that $\beta \notin x$. Now we have two possibilities:

Case 1. $\beta = \bigcup \beta$. Now $\beta \subseteq x$, so by (ii) second clause, since $\bigcup \beta = \beta \notin x$ we have $x = \bigcup \beta$, hence x is an ordinal, as desired.

Case 2. $\beta = (\bigcup \beta) + 1$. Thus $\bigcup \beta$ is an ordinal smaller than β, so it is in x. By (i), since $\beta = \bigcup \beta + 1 \notin x$ we have $x = (\bigcup \beta) + 1$, hence x is an ordinal.

Solutions, exercises in Chapter 8

E8.1 *Give an example of* **A**, **R** *such that* **R** *is not well-founded on* **A** *and is not set-like on* **A**.

We take **On** and **R**, where $\mathbf{R} = \{(\alpha, \beta) : \alpha > \beta\}$. As shown after 8.2, **R** is not set-like on **On**. It is also not well-founded on **On**, since ω is a nonempty set of ordinals, but if $m \in \omega$ then $(m +' 1, m) \in \mathbf{R}$, so that ω does not have an **R**-minimal element.

E8.2 *Give an example of* **A**, **R** *such that* **R** *is not well-founded on* **A** *but is set-like on* **A**. *Give one example with* **R** *and* **A** *are proper classes, and one example where they are sets.*

Both are sets: let $\mathbf{A} = \omega$ and $\mathbf{R} = \{(m,n) : m,n \in \omega$ and $m > n$. Then ω does not have an $\mathbf{R}$-minimal element, since for any $m \in \omega$ we have $(m +' 1, m) \in \mathbf{R}$.

Both are proper classes: let $\mathbf{A} = \mathbf{On}$ and let

$$\mathbf{R} = \{(m,n) : m,n \in \omega \text{ and } m > n\} \cup \{(\alpha,\beta) : \alpha < \beta\}.$$

E8.3 *Give an example of* $\mathbf{A}, \mathbf{R}$ *such that* $\mathbf{R}$ *is well-founded on* $\mathbf{A}$ *but is not set-like on* $\mathbf{A}$.

Let $\mathbf{A} = \mathbf{V}$ and $\mathbf{R} = \{(a,\emptyset) : a \in \mathbf{V}, a \neq \emptyset\}$. Thus $\mathrm{pred}_{\mathbf{AR}}(\emptyset) = \mathbf{V}$, so $\mathbf{R}$ is not set-like on $\mathbf{V}$. Now let X be a nonempty set. If $X = \{\emptyset\}$, then $\emptyset \in X$ and $\forall a \in X[(a,\emptyset) \notin \mathbf{R}]$. If $X \neq \{\emptyset\}$, take any $a \in X \backslash \{\emptyset\}$. Then $\forall b \in X[(b,a) \notin \mathbf{R}]$.

E8.4 *Suppose that* $\mathbf{R}$ *is a class relation contained in* $\mathbf{A} \times \mathbf{A}$, $x \in \mathbf{A}$, *and* $v \in \mathrm{pred}_{\mathbf{AR}^*}(x)$. *Prove by induction on* n *that if* $n \in \omega\backslash 1$, f *is a function with domain* $n +' 1$, $\forall i < n[(f(i), f(i +' 1)) \in \mathbf{R}]$ *and* $f(n) = v$, *then* $f(0) \in \mathrm{pred}_{\mathbf{AR}^*}(x)$.

Suppose that $\mathbf{R}$ is a class relation contained in $\mathbf{A} \times \mathbf{A}$, $x \in \mathbf{A}$, and $v \in \mathrm{pred}_{\mathbf{AR}^*}(x)$. We take $n = 1$ in the condition to be proved. So, suppose that f is a function with domain 2 such that $\forall i < 1[(f(i), f(i +' 1)) \in \mathbf{R}]$ and $f(1) = v$. Thus $(f(0), v) \in \mathbf{R}$, so $f(0) \in \mathrm{pred}_{\mathbf{AR}}(v)$. By Lemma 8.3(ii), $f(0) \in \mathrm{pred}_{\mathbf{AR}^*}(x)$.

Now suppose that if $n \in \omega\backslash 1$, f is a function with domain $n +' 1$, $\forall i < n[(f(i), f(i +' 1)) \in \mathbf{R}]$ and $f(n) = v$, then $f(0) \in \mathrm{pred}_{\mathbf{AR}^*}(x)$. Suppose also now that f is a function with domain $n +' 2$, $\forall i < n +' 1[(f(i), f(i +' 1)) \in \mathbf{R}]$ and $f(n +' 1) = v$. Define g with domain $n +' 1$ by setting $g(i) = f(i +' 1)$ for all $i < n +' 1$. Then $\forall i < n[(g(i), g(i +' 1)) = (f(i +' 1), f(i +' 2)) \in \mathbf{R}]$ and $g(n) = f(n +' 1) = v$. Hence by the inductive assumption, $f(1) = g(0) \in \mathrm{pred}_{\mathbf{AR}^*}(x)$. We also have $(f(0), f(1)) \in \mathbf{R}$, so by Lemma 8.3(ii), $f(0) \in \mathrm{pred}_{\mathbf{AR}^*}(x)$.

E8.5 *Suppose that* $\mathbf{R}$ *is a class relation contained in* $\mathbf{A} \times \mathbf{A}$, $(u,v) \in \mathbf{R}^*$, *and* $(v,w) \in \mathbf{R}^*$. *Show that* $(u,w) \in \mathbf{R}^*$.

Assume that $\mathbf{R}$ is a class relation contained in $\mathbf{A} \times \mathbf{A}$, $(u,v) \in \mathbf{R}^*$, and $(v,w) \in \mathbf{R}^*$. Since $(u,v) \in \mathbf{R}^*$, there exist $n \in \omega\backslash 1$ and a function f with domain $n +' 1$ such that $\forall i < n[(f(i), f(i +' 1)) \in \mathbf{R}$, $f(0) = u$, and $f(n) = v$. From exercise E8.4 it follows that $(u,w) \in \mathbf{R}^*$.

E8.6 *Give an example of a proper class* $\mathbf{X}$ *which has a proper class of* $\in$*-minimal elements.*

Let $\mathbf{X} = \{\{\alpha\} : \alpha \geq 2\}$. We claim that all elements of $\mathbf{X}$ are $\in$-minimal. Suppose that $\alpha, \beta \geq 2$ and $\{\alpha\} \in \{\beta\}$. Then $\{\alpha\} = \beta$, Since $\beta \geq 2$ we have $0, 1 \in \beta$, so $0 = \alpha = 1$, contradiction.

E8.7 *Give an example of a proper class relation* $\mathbf{R}$ *contained in* $\mathbf{A} \times \mathbf{A}$ *for some proper class* $\mathbf{A}$, *and a class function* $\mathbf{G}$ *mapping* $\mathbf{A} \times \mathbf{V}$ *into* $\mathbf{V}$ *such that* $\mathbf{R}$ *is set-like on* $\mathbf{A}$ *but not well-founded on* $\mathbf{A}$ *and there is no class function* $\mathbf{F}$ *mapping* $\mathbf{A}$ *into* $\mathbf{V}$ *such that* $\mathbf{F}(a) = \mathbf{G}(a, \mathbf{F}\langle \mathrm{pred}_{\mathbf{AR}}(a))$ *for all* $a \in \mathbf{A}$.

Let $\mathbf{A} = \mathbf{On}$ and

$$\mathbf{R} = \{(m,n) : m,n \in \omega \text{ and } m > n\} \cup \{(\alpha,\beta) : \omega \leq \alpha < \beta\}.$$

Thus $\mathbf{R}$ is a proper class relation contained in $\mathbf{A}\times\mathbf{A}$. Clearly $\mathbf{R}$ is set-like on $\mathbf{On}$ but it is not well-founded on $\mathbf{On}$. Define $\mathbf{G}:\mathbf{On}\times\mathbf{V}\to\mathbf{V}$ by setting

$$\mathbf{G}(\alpha,a)=\begin{cases}\{a(\alpha+'1)\} & \text{if } \alpha\in\omega \text{ and } a \text{ is a function with domain } \{m\in\omega : m>\alpha\},\\ \emptyset & \text{otherwise.}\end{cases}$$

Suppose that $\mathbf{F}:\mathbf{A}\to\mathbf{V}$ is such that $\mathbf{F}(\alpha)=\mathbf{G}(\alpha,\mathbf{F}\restriction \text{pred}_{\mathbf{OnR}}(\alpha))$ for all $\alpha\in\mathbf{On}$. Let $f=\mathbf{F}\restriction\omega$. Choose $b\in\text{rng}(f)$ such that $b\cap\text{rng}(f)=\emptyset$. Say $b=f(m)$ with $m\in\omega$. Now

$$\begin{aligned}f(m)&=\mathbf{F}(m)=\mathbf{G}(m,\mathbf{F}\restriction\text{pred}_{\mathbf{OnR}}(m))=\mathbf{G}(m,\mathbf{F}\restriction\{n:n\in\omega,n>m\})\\&=\{\mathbf{F}(m+'1)\}=\{f(m+'1)\},\end{aligned}$$

so that $f(m+'1)\in f(m)\cap\text{rng}(f)$, contradiction.

E8.8 *Give an example of a proper class relation* $\mathbf{R}$ *contained in some* $\mathbf{A}\times\mathbf{A}$ *for some proper class* $\mathbf{A}$ *and a class function* $\mathbf{G}$ *mapping* $\mathbf{A}\times\mathbf{V}$ *into* $\mathbf{V}$ *such that* $\mathbf{R}$ *is set-like on* $\mathbf{A}$ *but not well-founded on* $\mathbf{A}$ *but still there is a class function* $\mathbf{F}$ *mapping* $\mathbf{A}$ *into* $\mathbf{V}$ *such that* $\mathbf{F}(a)=\mathbf{G}(a,\mathbf{F}\langle\text{pred}_{\mathbf{AR}}(a))$ *for all* $a\in\mathbf{A}$.

Let $\mathbf{A}$ and $\mathbf{R}$ be as in exercise E8.7, but define $\mathbf{G}(\alpha,a)=\alpha$ for all $\alpha\in\mathbf{On}$ and all $a\in\mathbf{V}$. Then the function $\mathbf{F}:\mathbf{On}\to\mathbf{V}$ such that $\mathbf{F}(\alpha)=\alpha$ for all $\alpha\in\mathbf{On}$ is as desired.

Solutions to exercises in chapter 9

E9.1 *Let* $(A,<)$ *be a well order. Suppose that* $B\subset A$ *and* $\forall b\in B\forall a\in A[a<b\to a\in B]$. *Prove that there is an element* $a\in A$ *such that* $B=\{b\in A:b<a\}$.

Let a be the least element of $A\backslash B$. We claim that a is as desired. For, if $b\in B$, then it cannot happen that $a\le b$, since this would imply that $a\in B$; so $b<a$. And if $b<a$, then $b\in B$ by the minimality of a.

E9.2 *Let* $(A,<)$ *be a well order. Suppose that* $B\subset A$ *and* $\forall b\in B\forall a\in A[a<b\to a\in B]$. *Prove that* $(A,<)$ *is not isomorphic to* $(B,<)$.

Suppose that f is such an isomorphism from $(A,<)$ onto $(B,<)$. By exercise E9.1, let $a\in A$ be such that $B=\{x\in A:x<a\}$. By Proposition 9.11, $a\le f(a)$, contradicting the assumption that f maps into B.

E9.3 *Suppose that* f *is a one-one function mapping an ordinal* α *onto a set* A. *Define a relation* $\prec$ *which is a subset of* $A\times A$ *such that* $(A,<)$ *is a well-order and* f *is an isomorphism of* $(\alpha,<)$ *onto* $(A,\prec)$.

Define $\prec=\{(a,b)\in A\times A : f^{-1}(a)<f^{-1}(b)\}$. We check that $(A,<)$ is a well-order. If $a\in A$ and $a\prec a$, then $f^{-1}(a)<f^{-a}(a)$, contradiction. So $\prec$ is irreflexive. Suppose that $a\prec b\prec c$. Then $f^{-1}(a)<f^{-1}(b)<f^{-1}(c)$, so $f^{-1}(a)<f^{-1}(c)$ and hence $a\prec c$. So $\prec$ is transitive. Now given $a,b\in A$, either $f^{-1}(a)<f^{-1}(b)$ or $f^{-1}(a)=f^{-1}(b)$ or $f^{-1}(b)<f^{-1}(a)$, so $a\prec b$ or $a=b$ or $b\prec a$. Thus $(A,\prec)$ is a linear order. Finally, suppose that $\emptyset\ne X\subseteq A$. Then $\emptyset\ne f^{-1}[X]$, so let ξ be the least element of $f^{-1}[X]$. Then $f(\xi)\in X$. Suppose that $b\in X$. Then $f^{-1}(b)\in f^{-1}[X]$, so $\xi\le f^{-1}(b)$. Hence

$f(\xi) \preceq b$. This shows that $f(\xi)$ is the $\prec$-least element of X. We have shown that $(A, \prec)$ is a well-order.

We are given that f is a bijection from α onto A. If $\xi, \eta \in \alpha$ and $\xi < \eta$, then $f(\xi) \prec f(\eta)$. If $f(\xi) \prec f(\eta)$, then $\xi < \eta$. Thus f is an isomorphism.

E9.4 *Prove that* $1 + m = m + 1$ *for any* $m \in \omega$.

(Ordinary) induction on m. $0 + 1 = 1 = 1 + 0$ using Theorem 9.21(vi). Assume that $1 + m = m + 1$. Then $1 + (m + 1) = (1 + m) + 1 = (m + 1) + 1$.

E9.5 *Prove that* $m + n = n + m$ *for any* $m, n \in \omega$.

With m fixed, induction on n. $0 + m = m = m + 0$ using Theorem 9.21(vi). Assume that $m + n = n + m$. Then $(n + 1) + m = n + (1 + m) = n + (m + 1)$ (by exercise E9.4) $= (n + m) + 1 = (m + n) + 1 = m + (n + 1)$.

E9.6 *Prove that* $\omega \leq \alpha$ *iff* $1 + \alpha = \alpha$.

First note that $1 + \omega = \bigcup_{m \in \omega}(1 + m) = \bigcup_{m \in \omega}(m + 1) = \omega$, using Theorem 9.21(vi).

$\Rightarrow$: Assume that $\omega \leq \alpha$. By Theorem 9.21(vii) let δ be such that $\omega + \delta = \alpha$. Then $1 + \alpha = 1 + (\omega + \delta) = (1 + \omega) + \delta = \omega + \delta = \alpha$.

$\Leftarrow$: It suffices to show that if $m < \omega$ then $1 + m \neq m$. This is true by Theorem 9.21(vi).

E9.7 *For any ordinals* α, β *let*

$$\alpha \oplus \beta = (\alpha \times \{0\}) \cup (\beta \times \{1\}).$$

We define a relation $\prec$ *as follows. For any* $x, y \in \alpha \oplus \beta$, $x \prec y$ *iff one of the following three conditions holds:*

(i) There are $\xi, \eta < \alpha$ *such that* $x = (\xi, 0)$, $y = (\eta, 0)$, *and* $\xi < \eta$.

(ii) There are $\xi, \eta < \beta$ *such that* $x = (\xi, 1)$, $y = (\eta, 1)$, *and* $\xi < \eta$.

(ii) There are $\xi < \alpha$ *and* $\eta < \beta$ *such that* $x = (\xi, 0)$ *and* $y = (\eta, 1)$.

Prove that $(\alpha \oplus \beta, \prec)$ *is a well order which is isomorphic to* $\alpha + \beta$.

Clearly $\prec$ is a well-order. We show by transfinite induction on β, with α fixed, that $(\alpha \oplus \beta, \prec)$ is order isomorphic to $\alpha + \beta$. For $\beta = 0$ we have $\alpha + \beta = \alpha + 0 = \alpha$, while $\alpha \oplus \beta = \alpha \oplus 0 = \alpha \times \{0\}$. Clearly $\xi \mapsto (\xi, 0)$ defines an order-isomorphism from α onto $(\alpha \times \{0\}, \prec)$. So our result holds for $\beta = 0$. Assume it for β, and suppose that f is an order-isomorphism from $\alpha + \beta$ onto $(\alpha \oplus \beta, \prec)$. Now the last element of $\alpha \oplus (\beta + 1)$ is $(\beta, 1)$, and the last element of $\alpha + (\beta + 1)$ is $\alpha + \beta$, so the function

$$f \cup \{(\alpha + \beta, (\beta, 1))\}$$

is an order-isomorphism from $\alpha + (\beta + 1)$ onto $\alpha \oplus (\beta + 1)$.

Now assume that β is a limit ordinal, and for each $\gamma < \beta$, the ordinal $\alpha + \gamma$ is isomorphic to $\alpha \oplus \gamma$. For each such γ let f_γ be the unique isomorphism from $\alpha + \gamma$ onto

$\alpha \oplus \gamma$. Note that if $\gamma < \delta < \beta$, then $f_\delta \restriction \gamma$ is an isomorphism from $\alpha + \gamma$ onto $\alpha \oplus \gamma$; hence $f_\delta \restriction \gamma = f_\gamma$. It follows that

$$\bigcup_{\gamma < \beta} f_\gamma$$

is an isomorphism from $\alpha + \beta$ onto $\alpha \oplus \beta$, finishing the inductive proof.

E9.8 *Given ordinals α, β, we define the following relation $\prec$ on $\alpha \times \beta$:*

$$(\xi, \eta) \prec (\xi', \eta') \quad \textit{iff} \quad ((\xi, \eta) \textit{ and } (\xi', \eta') \textit{ are in } \alpha \times \beta \textit{ and:} \\ \eta < \eta', \textit{ or } (\eta = \eta' \textit{ and } \xi < \xi').$$

We may say that this is the anti-dictionary or anti-lexicographic order.

Show that the set $\alpha \times \beta$ under the anti-lexicographic order is a well order which is isomorphic to $\alpha \cdot \beta$.

We may assume that $\alpha \neq 0$. It is straightforward to check that $\prec$ is a well-order.

Now we define, for any $(\xi, \eta) \in \alpha \times \beta$,

$$f(\xi, \eta) = \alpha \cdot \eta + \xi.$$

We claim that f is the desired order-isomorphism from $\alpha \times \beta$ onto $\alpha \cdot \beta$. If $(\xi, \eta) \in \alpha \times \beta$, then

$$f(\xi, \eta) = \alpha \cdot \eta + \xi < \alpha \cdot \eta + \alpha = \alpha \cdot (\eta + 1) \leq \alpha \cdot \beta.$$

Thus f maps into $\alpha \cdot \beta$.

To show that f is one-one, suppose that $(\xi, \eta), (\xi', \eta') \in \alpha \times \beta$ and $f(\xi, \eta) = f(\xi', \eta')$. Then by Theorem 9.26, $(\xi, \eta) = (\xi', \eta')$. So f is one-one.

To show that f maps onto $\alpha \cdot \beta$, let $\gamma < \alpha \cdot \beta$. Choose ξ and η so that $\gamma = \alpha \cdot \eta + \xi$ with $\xi < \alpha$. Now $\eta < \beta$, as otherwise

$$\gamma = \alpha \cdot \eta + \xi \geq \alpha \cdot \eta \geq \alpha \cdot \beta.$$

It follows that $f(\xi, \eta) = \alpha \cdot \eta + \xi = \gamma$. so f is onto.

Finally, we show that the order is preserved. Suppose that $(\xi, \eta) \prec (\xi', \eta')$. Then one of these cases holds:

Case 1. $\eta < \eta'$. Then

$$f(\xi, \eta) = \alpha \cdot \eta + \xi < \alpha \cdot \eta + \alpha = \alpha \cdot (\eta + 1) \leq \alpha \cdot \eta' \leq \alpha \cdot \eta' + \xi' = f(\xi', \eta'),$$

as desired.

Case 2. $\eta = \eta'$ and $\xi < \xi'$. Then $f(\xi, \eta) < f(\xi', \eta')$.

Now it follows that f is the desired isomorphism.

E9.9 *Suppose that α and β are ordinals, with $\beta \neq 0$. We define*

$${}^{\alpha}\beta^{\mathrm{w}} = \{f \in {}^{\alpha}\beta : \{\xi < \alpha : f(\xi) \neq 0\} \textit{ is finite}\}.$$

For $f, g \in {}^{\alpha}\beta^{\mathrm{w}}$ we write $f \prec g$ iff $f \neq g$ and $f(\xi) < g(\xi)$ for the **greatest** *$\xi < \alpha$ for which $f(\xi) \neq g(\xi)$.*

Prove that $({}^{\alpha}\beta^{\mathrm{w}}, \prec)$ is a well-order which is order-isomorphic to the ordinal exponent β^{α}. (A set X is finite iff there is a bijection from some natural number onto X.)

If $\alpha = 0$, then $\beta^{\alpha} = 1$, and ${}^{\alpha}\beta^{\mathrm{w}}$ also has only one element, the empty function (= the emptyset). So, assume that $\alpha \neq 0$. If $\beta = 1$, then ${}^{\alpha}\beta^{\mathrm{w}}$ has only one member, namely the function with domain α whose value is always 0. This is clearly order-isomorphic to 1, as desired. So, suppose that $\beta > 1$.

Now we define a function f mapping β^{α} into ${}^{\alpha}\beta^{\mathrm{w}}$. Let $f(0)$ be the member of ${}^{\alpha}\beta^{\mathrm{w}}$ which takes only the value 0. Now suppose that $0 < \varepsilon < \beta^{\alpha}$. By Theorem 9.29 write

$$\varepsilon = \beta^{\gamma(0)} \cdot \delta(0) + \beta^{\gamma(1)} \cdot \delta(1) + \cdots + \beta^{\gamma(m-1)} \cdot \delta(m-1),$$

where $\varepsilon \geq \gamma(0) > \gamma(1) > \cdots > \gamma(m-1)$ and $0 < \delta(i) < \beta$ for each $i < m$. Note that $\beta^{\gamma(0)} \leq \varepsilon < \beta^{\alpha}$, so $\gamma(0) < \alpha$. Then we define, for any $\zeta < \alpha$,

$$(f(\varepsilon))(\zeta) = \begin{cases} 0 & \text{if } \zeta \notin \{\gamma(0), \ldots, \gamma(m-1)\}, \\ \delta(i) & \text{if } \zeta = \gamma(i) \text{ with } i < m. \end{cases}$$

Clearly $f(\varepsilon) \in {}^{\alpha}\beta^{\mathrm{w}}$. To see that f maps onto ${}^{\alpha}\beta^{\mathrm{w}}$, suppose that $x \in {}^{\alpha}\beta^{\mathrm{w}}$. If x takes only the value 0, then $f(0) = x$. Suppose that x takes on some nonzero value. Let

$$\{\xi < \alpha : x(\xi) \neq 0\} = \{\gamma(0), \gamma(1), \ldots, \gamma(m-1)\},$$

where $\gamma(0) > \gamma(1) > \cdots > \gamma(m-1)$. Let $\delta(i) = x(\gamma(i))$ for each $i < m$, and let

$$\varepsilon = \beta^{\gamma(0)} \cdot \delta(0) + \beta^{\gamma(1)} \cdot \delta(1) + \cdots + \beta^{\gamma(m-1)} \cdot \delta(m-1).$$

Clearly then $f(\varepsilon) = x$.

Now we complete the proof by showing that for any $\varepsilon, \theta < \beta^{\alpha}$, $\varepsilon < \theta$ iff $f(\varepsilon) < f(\theta)$. This equivalence is clear if one of ε, θ is 0, so suppose that both are nonzero. Write

$$\varepsilon = \beta^{\gamma(0)} \cdot \delta(0) + \beta^{\gamma(1)} \cdot \delta(1) + \cdots + \beta^{\gamma(m-1)} \cdot \delta(m-1),$$

where $\alpha \geq \gamma(0) > \gamma(1) > \cdots > \gamma(m-1)$ and $0 < \delta(i) < \beta$ for each $i < m$, and

$$\theta = \beta^{\gamma'(0)} \cdot \delta'(0) + \beta^{\gamma'(1)} \cdot \delta'(1) + \cdots + \beta^{\gamma'(n-1)} \cdot \delta'(n-1),$$

where $\alpha \geq \gamma'(0) > \gamma'(1) > \cdots > \gamma'(n-1)$ and $0 < \delta'(i) < \beta$ for each $i < n$.

By symmetry we may suppose that $m \leq n$. Note that $N(\beta, m, \gamma, \delta)$, $k(\beta, m, \gamma, \delta) = \varepsilon$, $N(\beta, n, \gamma', \delta')$, and $k(\beta, n, \gamma', \delta') = \theta$. We now consider several possibilities.

Case 1. $\varepsilon = \theta$. Then clearly $f(\varepsilon) = f(\theta)$.

Case 2. $\gamma \subseteq \gamma'$, $\delta \subseteq \delta'$, and $m < n$. Thus $\varepsilon < \theta$. Also, $\gamma'(m)$ is the largest $\xi < \alpha$ such that $(f(\varepsilon))(\xi) \neq (f(\theta))(\xi)$, and $(f(\varepsilon))(\xi) = 0 < \delta'(m) = (f(\theta))(\gamma'(m))$, so $f(\varepsilon) < f(\theta)$.

Case 3. There is an $i < m$ such that $\gamma(j) = \gamma'(j)$ and $\delta(j) = \delta'(j)$ for all $j < i$, while $\gamma(i) \neq \gamma'(i)$. By symmetry, say that $\gamma(i) < \gamma'(i)$. Then we have $\varepsilon < \theta$. Since $\gamma'(i)$ is the largest $\xi < \alpha$ such that $(f(\varepsilon))(\xi) \neq (f(\theta))(\xi)$, and $(f(\varepsilon))(\gamma'(i)) = 0 < \delta'(i) = (f(\theta))(\gamma'(i))$, we also have $f(\varepsilon) < f(\theta)$.

Case 4. There is an $i < m$ such that $\gamma(j) = \gamma'(j)$ and $\delta(j) = \delta'(j)$ for all $j < i$, while $\gamma(i) = \gamma'(i)$ and $\delta(i) \neq \delta'(i)$. By symmetry, say that $\delta(i) < \delta'(i)$. Then we have $\varepsilon < \theta$. Since $\gamma(i)$ is the largest $\xi < \alpha$ such that $(f(\varepsilon))(\xi) \neq (f(\theta))(\xi)$, and $(f(\varepsilon))(\gamma'(i)) = \delta(i) < \delta'(i) = (f(\theta))(\gamma'(i))$, we also have $f(\varepsilon) < f(\theta)$.

E9.10 *Show that for every nonzero ordinal α there are only finitely many ordinals β such that $\alpha = \gamma \cdot \beta$ for some γ.*

Suppose there are infinitely many such β; let $\langle \beta_i : i \in \omega \rangle$ be a one-one enumeration of infinitely many of them. For each $i \in \omega$ let γ_i be such that $\alpha = \gamma_i \cdot \beta_i$. Clearly $\beta_i < \beta_j$ iff $\gamma_j < \gamma_i$. We define $\langle i_j : j \in \omega \rangle$ by recursion. Let i_0 be such that β_{i_0} is the smallest element of $\{\beta_k : k \in \omega\}$. Having defined $i_0, \ldots, i_s$, let i_{s+1} be such that $\beta_{i_{s+1}}$ is the smallest element of

$$\{\beta_k : k \in \omega\} \backslash \{\beta_{i_k} : k \leq s\}$$

Clearly $\beta_{i_0} < \beta_{i_1} < \cdots$, and hence $\gamma_{i_0} > \gamma_{i_1} > \cdots$, contradiction.

E9.11 *Prove that $n^{(\omega^\omega)} = \omega^{(\omega^\omega)}$ for every natural number $n > 1$.*

Note that $n^\omega = \omega$ by an easy argument. Hence

$$\begin{aligned} \omega^{(\omega^\omega)} &= (n^\omega)^{(\omega^\omega)} \\ &= n^{(\omega \cdot (\omega^\omega))} \\ &= n^{(\omega^\omega)}. \quad \text{by Theorem 9.32} \end{aligned}$$

E9.12 *Show that the following conditions are equivalent for any ordinals α, β:*

(i) $\alpha + \beta = \beta + \alpha$.

(ii) There exist an ordinal γ and natural numbers k, l such that $\alpha = \gamma \cdot k$ and $\beta = \gamma \cdot l$.

$\Rightarrow$: Assume that $\alpha + \beta = \beta + \alpha$. The desired conclusion is clear if $\alpha = 0$ or $\beta = 0$, so assume that $\alpha, \beta \neq 0$. Write $\alpha = \omega^\delta \cdot k + \varepsilon$ with $\delta \leq \alpha$, $0 < k \in \omega$, and $\varepsilon < \omega^\delta$, and write $\beta = \omega^\rho \cdot l + \sigma$ with $\rho \leq \beta$, $0 < l \in \omega$, and $\sigma < \omega^\rho$. If $\delta < \rho$, then

$$\alpha + \beta = \beta < \beta + \alpha,$$

contradiction. A similar contradiction is reached if $\rho < \delta$. So $\delta = \rho$. Now

$$\alpha + \beta = \omega^\delta \cdot (k + l) + \sigma = \beta + \alpha = \omega^\delta \cdot (k + l) + \varepsilon,$$

so $\sigma = \varepsilon$. Hence $\alpha = (\omega^\delta + \varepsilon) \cdot k$ and $\beta = (\omega^\delta + \varepsilon) \cdot l$, as desired.

$\Leftarrow$: Obvious.

E9.13 *Suppose that $\alpha < \omega^\gamma$. Show that $\alpha + \beta + \omega^\gamma = \beta + \omega^\gamma$.*

Suppose that α, β, γ are ordinals and $\alpha < \omega^\gamma$. If also $\beta < \omega^\gamma$, then $\alpha+\beta < \omega^\gamma$ by Theorem 9.31, and also by Theorem 9.31 $\alpha + \beta + \omega^\gamma = \omega^\gamma$ and $\beta + \omega^\gamma = \omega^\gamma$.

Now suppose that $\omega^\gamma \leq \beta$. Write $\beta = \omega^\gamma \cdot \delta + \varepsilon$ with $\delta > 0$ and $\varepsilon < \omega^\gamma$.

(1) $\alpha + \omega^\gamma \cdot \varphi = \omega^\gamma \cdot \varphi$ for every positive φ.

We prove (1) by induction on φ. It is true for $\varphi = 1$ by Theorem 9.31. Assume that it holds for φ. Then

$$\alpha + \omega^\gamma \cdot (\varphi + 1) = \alpha + \omega^\gamma \cdot \varphi + \omega^\gamma = \omega^\gamma \cdot \varphi + \omega^\gamma = \omega^\gamma \cdot (\varphi + 1),$$

as desired. Finally, assume that φ is limit and (1) holds for all $\psi < \varphi$. Let $F(\varphi) = \alpha + \varphi$ for all φ, and $G(\varphi) = \omega^\gamma \cdot \varphi$. Both of these are normal functions. Hence

$$\alpha + \omega^\gamma \cdot \varphi = F(G(\varphi)) = \bigcup_{\psi<\varphi} F(G(\psi)) = \bigcup_{\psi<\varphi} (\alpha + \omega^\gamma \cdot \psi) = \bigcup_{\psi<\varphi} (\omega^\gamma \cdot \psi) = \omega^\gamma \cdot \varphi,$$

finishing the inductive proof of (1).

Now by (1) we have

$$\alpha + \beta + \omega^\gamma = \alpha + \omega^\gamma \cdot \delta + \varepsilon + \omega^\gamma = \omega^\gamma \cdot \delta + \varepsilon + \omega^\gamma = \beta + \omega^\gamma.$$

E9.14 *Show that the following conditions are equivalent:*
(i) α is a limit ordinal
(ii) $\alpha = \omega \cdot \beta$ for some $\beta \neq 0$.
(iii) For every $m \in \omega \backslash 1$ we have $m \cdot \alpha = \alpha$, and $\alpha \neq 0$.

(i)$\Rightarrow$(ii): Assume (i). By Theorem 9.26 write $\alpha = \omega \cdot \beta + n$ with $n < \omega$. If $\beta = 0$, then $\alpha = n$, contradiction. If $n \neq 0$, then $\alpha = \omega \cdot \beta + (n-1) + 1$, contradiction.

(ii)$\Rightarrow$(iii): Assume (ii). By Theorem 9.23(iii), $\alpha \neq 0$. Suppose that $m \in \omega \backslash 1$. Then $m \cdot \omega = \bigcup_{n\in\omega}(m \cdot n) = \omega$ by Theorem 9.23(iii), so $m \cdot \alpha = \alpha$.

(iii)$\Rightarrow$(i): Assume (iii), but suppose that $\alpha = \beta + 1$. Then $\alpha = 2 \cdot \alpha = 2 \cdot (\beta + 1) = 2 \cdot \beta + 2 > \alpha$, contradiction.

E9.15 *Show that $(\alpha + \beta) \cdot \gamma \leq \alpha \cdot \gamma + \beta \cdot \gamma$ for any ordinals α, β, γ.*

Assume that $\alpha, \beta.\gamma \neq 0$. Write $\alpha = \omega^\delta \cdot k + \varepsilon$ with $\delta \leq \alpha$, $0 \neq k \in \omega$, $\varepsilon < \omega^\delta$, and $\beta = \omega^\rho \cdot l + \sigma$ with $\rho \leq \beta$, $0 \neq l \in \omega$, $\sigma < \omega^\rho$. Also, write $\gamma = \omega \cdot \xi + m$ with $m \in \omega$. Now we consider some cases.

Case 1. $\delta < \rho$. Then $\alpha + \beta = \beta$, and the desired conclusion follows.

Case 2. $\delta = \rho$. Note that if $m > 0$, then

$$\alpha \cdot m = \omega^\delta \cdot k \cdot m + \varepsilon;$$
$$\beta \cdot m = \omega^\delta \cdot l \cdot m + \sigma;$$
$$(\alpha + \beta) \cdot m = \omega^\delta \cdot (k + l) \cdot m + \sigma.$$

If $\xi = 0$ it is then clear that $(\alpha+\beta)\cdot\gamma = \alpha\cdot\gamma+\beta\cdot\gamma$. Hence assume that $\xi > 0$. Then

$$\begin{aligned}
\alpha\cdot\gamma &= \alpha\cdot\omega\cdot\xi+\alpha\cdot m\\
&= \omega^{\delta+1}\cdot\xi+\alpha\cdot m;\\
\beta\cdot\gamma &= \omega^{\delta+1}\cdot\xi+\beta\cdot m;\\
\alpha\cdot\gamma+\beta\cdot\gamma &= \omega^{\delta+1}\cdot\xi\cdot 2+\beta\cdot m;\\
(\alpha+\beta)\cdot\gamma &= \omega^{\delta+1}\cdot\xi+(\alpha+\beta)\cdot m,
\end{aligned}$$

and the desired conclusion is clear.

Case 3 $\rho < \delta$. Then if $m > 0$ we have

$$\begin{aligned}
\alpha\cdot m &= \omega^{\delta}\cdot k\cdot m+\varepsilon;\\
\beta\cdot m &= \omega^{\rho}\cdot l\cdot m+\sigma;\\
\alpha\cdot m+\beta\cdot m &= \omega^{\delta}\cdot k\cdot m+\varepsilon+\omega^{\rho}\cdot l\cdot m+\sigma;\\
(\alpha+\beta)\cdot m &= \omega^{\delta}\cdot k\cdot m+\varepsilon+\omega^{\rho}\cdot l+\sigma.
\end{aligned}$$

Hence the desired conclusion follows if $\xi = 0$. Assume now that $\xi \neq 0$. Then

$$\begin{aligned}
\alpha\cdot\gamma &= \omega^{\delta+1}\cdot\xi+\alpha\cdot m;\\
\beta\cdot\gamma &= \omega^{\rho+1}\cdot\xi+\beta\cdot m;\\
\alpha\cdot\gamma+\beta\cdot\gamma &= \omega^{\delta+1}\cdot\xi+\alpha\cdot m+\omega^{\rho+1}\cdot\xi+\beta\cdot m;\\
(\alpha+\beta)\cdot\gamma &= \omega^{\delta+1}\cdot\xi+\omega^{\delta}\cdot k\cdot m+\varepsilon+\omega^{\rho}\cdot l+\sigma.
\end{aligned}$$

Again the desired conclusion holds.

Solutions to exercises in chapter 10

E10.1 *Show that any vector space over a field has a basis (possibly infinite).*

Let V be any vector space over F. Let $A = \{X \subseteq V : X \text{ is linearly independent}\}$, partially ordered by $\subseteq$. Then $A \neq \emptyset$, since trivially $\emptyset \in A$. Now suppose that B is a subset of A simply ordered by $\subseteq$. We claim that $\bigcup B \in A$; this will verify the hypothesis of Zorn's lemma. Suppose that $v_1, \ldots, v_n \in \bigcup B$, $a_1, \ldots, a_n \in F$, and $a_1v_1 + \cdots + a_nv_n = 0$; we want to show that all a_i are 0. For each $i = 1, \ldots, n$ choose $X_i \in B$ such that $v_i \in X_i$. Now $\{X_i : i = 1, \ldots, n\}$ has a largest member X_j under $\subseteq$, since B is simply ordered. [Easy proof by induction on n.] Clearly $v_i \in X_j$ for all $i = 1, \ldots, n$. Since X_j is linearly independent, it follows that each $a_i = 0$, as desired.

Now we apply Zorn's lemma to obtain a maximal member Y of A under $\subseteq$. We claim that Y is a basis for V. Since Y is linearly independent, it suffices to show that Y spans V. Suppose that $w \in V$. If $w \in Y$, then obviously w is in the span of Y. Suppose that $w \notin Y$. Then $Y \subset Y \cup \{w\}$ so by the maximality of Y, $Y \cup \{w\}$ is linearly dependent. Hence there is a natural number n, distinct elements $v_1, \ldots, v_n \in Y \cup \{w\}$, and elements $a_1, \ldots, a_n \in F$, not all 0, such that $a_1v_1 + \cdots + a_nv_n = 0$. Since Y is linearly independent, not all v_i are in Y; say that $v_j = w$. Then again because Y is linearly independent, we must have $a_j \neq 0$. So

$$w = \left(-\frac{a_1}{a_j}v_1\right) + \cdots + \left(-\frac{a_{j-1}}{a_j}v_{j-1}\right) + \left(-\frac{a_{j+1}}{a_j}v_{j+1}\right) + \cdots + \left(-\frac{a_n}{a_j}v_n\right),$$

so that w is in the span of Y, as desired.

E10.2 *A subset C of $\mathbb{R}$ is closed iff the following condition holds:*

For every sequence $f \in {}^{\omega}C$, if f converges to a real number x, then $x \in C$.

Here to say that f converges to x means that

$$\forall \varepsilon > 0 \exists M \forall m \geq M[|f_m - x| < \varepsilon].$$

Prove that if $\langle C_m : m \in \omega\rangle$ is a sequence of nonempty closed subsets of $\mathbb{R}$, $\forall m \in \omega \forall x, y \in C_m[|x - y| < 1/(m + 1)]$, and $C_m \supseteq C_n$ for $m < n$, then $\bigcap_{m\in\omega} C_m$ is nonempty. Hint: use the Cauchy convergence criterion.

Let c be a choice function for $\mathscr{P}(\mathbb{R})\backslash\{\emptyset\}$. For each $m \in \omega$ let $f_m = c(C_m)$. We claim that f is a Cauchy sequence, and hence it converges to some point x. For, let $\varepsilon > 0$ be given. Choose $m \in \omega$ such that $\frac{1}{m+1} < \varepsilon$. Then for any $n, p \geq m$ we have $f_n, f_p \in C_m$ and hence by a hypothesis of the exercise, $|f_n - f_p| < \frac{1}{m+1} < \varepsilon$, as desired. Now for any $m \in \omega$ we have $f_n \in C_m$ for all $n \geq m$, and hence $x \in C_m$. Thus $x \in \bigcap_{m\in\omega} C_m$.

E10.3 *Prove that every nontrivial commutative ring with identity has a maximal ideal. Nontrivial means that $0 \neq 1$. Only very elementary definitions and facts are needed here; they can be found in most abstract algebra books. Hint: use Zorn's lemma.*

Let R be a nontrivial commutative ring with identity. Let $\mathscr{A}$ be the collection of all proper ideals, partially ordered under $\subset$. Obviously $\mathscr{A} \neq \emptyset$. Suppose that $\mathscr{B}$ is a nonempty subset of $\mathscr{A}$ simply ordered by $\subset$. Let $I = \bigcup \mathscr{B}$. We claim that I is a proper ideal, so that it is an upper bound for $\mathscr{B}$. In fact, if $a, b \in I$, choose $J, K \in \mathscr{B}$ such that $a \in J$ and $b \in K$. Since $\mathscr{B}$ is simply ordered by $\subset$, by symmetry say $J \subset K$. Then $a, b \in K$, hence $a + b$ and $a - b$ are also in K, and hence they are in I too. Also, if $a \in I$ and $b \in R$, then $a \cdot b \in I$ by an even easier argument. Thus I is an ideal. Clearly $1 \notin I$, so I is proper.

By Zorn's lemma, $\mathscr{A}$ has a maximal element L. Clearly L is a maximal ideal.

E10.4 *A function $g : \mathbb{R} \to \mathbb{R}$ is continuous at $a \in \mathbb{R}$ iff for every sequence $f \in {}^{\omega}\mathbb{R}$ which converges to a, the sequence $g \circ f$ converges to $g(a)$. (See Exercise E10.2.) Show that g is continuous at a iff the following condition holds:*

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x \in \mathbb{R}[|x - a| < \delta \to |g(x) - g(a)| < \varepsilon].$$

Hint: for $\to$, argue by contradiction.

$\to$: Suppose that g is continuous at a but the indicated condition fails. Thus

$$(*) \qquad \exists \varepsilon > 0 \forall \delta > 0 \exists x \in \mathbb{R}[|x - a| < \delta \text{ and } |g(x) - g(a)| \geq \varepsilon].$$

Let c be a choice function for $\mathbb{R}$. For each $m \in \omega$ let

$$f_m = c\left\{x \in \mathbb{R}\left[|x-a| < \frac{1}{m+1} \text{ and } |g(x)-g(a)| \geq \varepsilon\right]\right\}.$$

Then f converges to a. In fact, given $\xi > 0$, choose M such that $\frac{1}{M-1} < \xi$. Then for any $m \geq M$, $|f_m - a| < \frac{1}{m+1} \leq \frac{1}{M-1} < \xi$. Since f converges to a and g is continuous at a, it follows that $g \circ f$ converges to $g(a)$. Hence we can choose N such that $\forall n \geq N[|g(f_m) - g(a)| < \varepsilon]$. But by the definition of f, $|g(f_N) - g(a)| \geq \varepsilon$, contradiction.

$\leftarrow$: Assume the indicated condition, and suppose that $f \in {}^{\omega}R$ converges to a. In order to show that $g \circ f$ also converges to a, let $\varepsilon > 0$ be given. By the condition, choose $\delta > 0$ such that $\forall x \in \mathbb{R}[|x-a| < \delta \rightarrow |g(x) - g(a)| < \varepsilon]$. Since f converges to a, choose M such that $\forall m \geq M[|f_m - a| < \delta]$. Then for any $m \geq M$ we have $|g(f_m) - g(a)| < \varepsilon$, as desired.

E10.5 *Show by induction on m, without using the axiom of choice, that if $m \in \omega$ and $\langle A_i : i \in m\rangle$ is a system of nonempty sets, then there is a function f with domain m such that $f(i) \in A_i$ for all $i \in m$.*

For $m = 0$, the system itself is empty, and the desired function f is the empty set.

Now suppose that $\langle A_i : i \in m+1\rangle$ is a system of nonempty sets, and we know our result for a system of m nonempty sets. So, let f be a function with domain m such that $f(i) \in A_i$ for all $i \in m$. Pick $a \in A_m$, and let $g = f \cup \{(m,a)\}$. Clearly g is as desired, completing the inductive proof.

E10.6 *Using AC, prove the following, which is called the Principle of Dependent Choice (which is also weaker than the axiom of choice, but cannot be proved in ZF). If A is a nonempty set, R is a relation, $R \subseteq A \times A$, and for every $a \in A$ there is a $b \in A$ such that aRb, then there is a function $f : \omega \rightarrow A$ such that $f(i)Rf(i+1)$ for all $i \in \omega$.*

Let c be a choice function for nonempty subsets of A. We define $f : \omega \rightarrow A$ by recursion, as follows. Fix $a \in A$. For any $m \in \omega$ let

$$f(m) = \begin{cases} a & \text{if } m = 0, \\ c(\{x \in A : f(n)Rx\}) & \text{if } m = n+1 \text{ and } \{x \in A : f(n)Rx\} \neq 0, \\ a & \text{otherwise.} \end{cases}$$

By induction, $f(i)Rf(i+1)$ for every $i \in \omega$, as desired.

E10.7 *Show that the axiom of choice implies (1), where (1) is*

(1) If $<$ is a partial ordering and $\prec$ is a simple ordering which is a subset of $<$, then there is a maximal (under $\subseteq$) simple ordering $\ll$ such that $\prec$ is a subset of $\ll$, which in turn is a subset of $<$.

Let

$$\mathscr{A} = \{\ll : \ll \text{ is a simple ordering and } \prec \subseteq \ll \subseteq <\}.$$

Note that $\mathscr{A}$ is nonempty, since $\prec \in \mathscr{A}$. We partially order $\mathscr{A}$ by inclusion. To check the hypothesis of Zorn's lemma, suppose that $\mathscr{B}$ is a nonempty subset of $\mathscr{A}$ simply ordered by inclusion. We claim that $\bigcup\mathscr{B} \in \mathscr{A}$; this is clear, by checking all the necessary conditions. For example, $\bigcup\mathscr{B}$ is transitive since if $(a,b),(b,c) \in \bigcup\mathscr{B}$, then there are $R, S \in \mathscr{B}$ with $(a,b) \in R$ and $(b,c) \in S$; by symmetry $R \subseteq S$, hence $(a,b),(b,c) \in S$, hence $(a,c) \in S$, hence $(a,c) \in \bigcup\mathscr{B}$.

So we apply Zorn's lemma to obtain a maximal member $\ll$ of $\mathscr{A}$; this is as desired.

E10.8 *Prove that (1) implies (2). [Given sets A and B, define $f < g$ iff f and g are one-one functions which are subsets of $A \times B$, and $f \subset g$. Apply (1) to $<$ and the empty simple ordering.] Here (2) is*

(2) For any two sets A and B, either there is a one-one function mapping A into B or there is a one-one function mapping B into A.

Following the hint, we get a maximal simple ordering $\prec$ such that $\prec\subseteq<$. Let $f = \bigcup(\prec)$. Since $\prec$ is a simply ordered collection of one-one functions, it is clear that f is a one-one function. It suffices to show that $\mathrm{dmn}(f) = A$ or $\mathrm{rng}(f) = B$. Suppose that this is not true, and choose $a \in A\backslash\mathrm{dmn}(f)$ and $b \in B\backslash\mathrm{rng}(f)$. Let $g = f \cup \{(a,b)\}$. Clearly g is a one-one function contained in $A \times B$. Thus if we define $\prec'$ as an extension of $\prec$ with $g \prec' f$ for all g in the domain of $\prec$, we get a proper extension of $\prec$, contradiction.

E10.9 *Prove that (2) implies (3). [Easy] Here (3) is*

(3) For any two nonempty sets A and B, either there is a function mapping A onto B or there is a function mapping B onto A.

Assume (2), and let A and B be nonempty sets. By (2) and symmetry, say that f is a one-one function mapping A into B. Fix $a \in A$, and define g with domain B by setting, for each $b \in B$,

$$g(b) = \begin{cases} f^{-1}(b) & \text{if } b \in \mathrm{rng}(f), \\ a & \text{otherwise.} \end{cases}$$

Clearly g maps B onto A, as desired.

E10.10 *Show in ZF that for any set A there is an ordinal α such that there is no one-one function mapping α into A. Hint: consider all well-orderings contained in $A \times A$.*

Let X be the set of all well-orderings contained in $A \times A$. Now each $\prec\in X$ is isomorphic to an ordinal $\beta_\prec$. Let $\alpha = \bigcup_{\prec\in X}(\beta_\prec + 1)$. Suppose that f is a one-one function mapping α into A. Let $\prec= \{(f(\xi), f(\eta)) : \xi < \eta\}$. Then $\prec$ is a well-ordering contained in $A \times A$, and so $\beta_X = \alpha$; consequently $\alpha \in \alpha$, contradiction.

E10.11 *Prove that (3) implies the axiom of choice. [Show that any set A can be well-ordered, as follows. Use exercise E10.10 to find an ordinal α which cannot be mapped one-one into $\mathscr{P}(A)$. Show that if $f : A \to \alpha$ maps onto α, then $\langle f^{-1}[\{\beta\}] : \beta < \alpha\rangle$ is a one-one function from α into $\mathscr{P}(A)$.]*

We follow the hint. Suppose that $f : A \to \alpha$ maps onto α. Let $g = \langle f^{-1}[\{\beta\}] : \beta < \alpha\rangle$. Clearly g maps α into $\mathscr{P}(A)$. Suppose that $g(\beta) = g(\gamma)$. Thus $f^{-1}[\{\beta\}] = f^{-1}[\{\gamma\}]$. Choose $a \in A$ such that $f(a) = \beta$; this is possible because f maps onto α. thus $a \in f^{-1}[\{\beta\}] = f^{-1}[\{\gamma\}]$, so $f(a) \in \{\gamma\}$, hence $\beta = f(a) = \gamma$. So g is one-one, contradicting the choice of α.

Now it follows from (3) that there is a function f mapping α onto A. Define $a \prec b$ iff the least element of $f^{-1}[\{a\}]$ is less than the first element of $f^{-1}[\{b\}]$. Clearly $\prec$ is a linear order on A. To show that it is a well-order, let B be a nonempty subset of A. Let β be the least element of $f^{-1}[B]$. Then $f(\beta)$ is clearly the $\prec$-least element of B.

E10.12 *Show that the axiom of choice implies (4). [Use Zorn's lemma.] Here (4) is*

(4) A family $\mathscr{F}$ of subsets of a set A has finite character if for all $X \subseteq A$, $X \in \mathscr{F}$ iff every finite subset of X is in $\mathscr{F}$. Principle (4) says that every family of finite character has a maximal element under $\subseteq$.

Let $\mathscr{F}$, a nonempty family of subsets of A, have finite character. We consider $\mathscr{F}$ as a partially ordered set under inclusion. It is nonempty by assumption. Now suppose that $\mathscr{G}$ is a nonempty subset of $\mathscr{F}$ linearly ordered by inclusion. To show that $\bigcup \mathscr{G} \in \mathscr{F}$, it suffices to show that every finite F subset of it is in $\mathscr{F}$, by the definition of finite character. For each $a \in F$ choose $X_a \in \mathscr{G}$ such that $a \in X_a$. Since $\mathscr{G}$ is linearly ordered by inclusion, choose $a \in F$ such that $X_b \subseteq X_a$ for all $b \in F$. Now $X_a \in \mathscr{F}$ since $\mathscr{G} \subseteq \mathscr{F}$, and F is a finite subset of X_a, so $F \in \mathscr{F}$ by the definition of finite character.

Thus we have verified the hypotheses of Zorn's lemma, and it gives the desired maximal element.

E10.13 *Show that (4) implies (5). Here (5) is*

(5) For any relation R there is a function $f \subseteq R$ such that $dmn\, R = dmn\, f$.

[Given a relation R, let $\mathscr{F}$ consist of all functions contained in R.]

Taking $\mathscr{F}$ as indicated, we verify that $\mathscr{F}$ has finite character. It is obviously nonempty, since $\emptyset \in \mathscr{F}$. Of course, if $f \in \mathscr{F}$, then every finite subset of f is in $\mathscr{F}$. Now suppose that $f \subseteq R$ and every finite subset of f is in $\mathscr{F}$. We just need to show that f is a function. Suppose that $(a, b), (a, c) \in f$. Then $\{(a, b), (a, c)\}$ is a finite subset of f, and so it is in $\mathscr{F}$, which means that it is a function, and so $b = c$. Thus f is a function.

Now by (4), let f be a maximal member of $\mathscr{F}$ under inclusion. So, f is a function included in R. Suppose that $a \in \text{dmn}(R)\backslash\text{dmn}(f)$. Choose b such that $(a, b) \in R$. Then $f \subset f \cup \{(a, b)\} \in \mathscr{F}$, contradiction. Therefore, $\text{dmn}(R) = \text{dmn}(f)$, as desired.

E10.14 *Show that (5) implies the axiom of choice. [Given a family $\langle A_i : i \in I\rangle$ of nonempty sets, let $R = \{(i, x) : i \in I$ and $x \in A_i\}$.]*

We follow the hint. Let f be a function such that $\text{dmn}(f) = \text{dmn}(R)$. Thus $\text{dmn}(f) = I$ and $f(i) \in A_i$ for all $i \in I$.

Solutions to exercises in chapter 12

E12.1 *Define sets A, B with $|A| = |B|$ such that there is a one-one function $f : A \to B$ which is not onto.*

Let $A = B = \omega$ and define $f(m) = m + 1$ for all $m \in \omega$. Then f is not onto, since $0 \notin \text{rng}(f)$. Suppose that $f(m) = f(n)$ and $m \neq n$. Say $m < n$. By Proposition 4.10, $m + 1 \leq n < n + 1 = m + 1$, contradiction.

E12.2 *Define sets A, B with $|A| = |B|$ such that there is an onto function $f : A \to B$ which is not one-one.*

Let $A = B = \omega$. Define $f(m+1) = m$ for any $m \in \omega$ and $f(0) = 0$.

E12.3 *Show that the restriction $\lambda \neq 0$ is necessary in Proposition 12.43(ix).*

Let $\kappa = \lambda = 0$, $\mu = 0$, $\nu = 1$. Then $\kappa^\mu = 0^0 = 1$ by Theorem 12.43(i), and $\kappa^\nu = 0^1 = 0$ by Theorem 12.43(ii).

E12.4 *Let $F : \mathscr{P}(A) \to \mathscr{P}(A)$, and assume that for all $X, Y \subseteq A$, if $X \subseteq Y$, then $F(X) \subseteq F(Y)$. Let $\mathscr{A} = \{X : X \subseteq A$ and $X \subseteq F(X)\}$, and set $X_0 = \bigcup_{X \in \mathscr{A}} X$. Then $X_0 \subseteq F(X_0)$.*

For any $Y \in \mathscr{A}$ we have $Y \subseteq X_0$, and hence $Y \subseteq F(Y) \subseteq F(X_0))$, so $X_0 \subseteq F(X_0)$.

E12.5 *Under the assumptions of exercise E12.4 we actually have $X_0 = F(X_0)$.*

By exercise E12.4, $X_0 \subseteq F(X_0)$, so $F(X_0) \subseteq F(F(X_0))$, hence $F(X_0) \in \mathscr{A}$, hence $F(X_0) \subseteq X_0$; together with exercise E12.4 this proves that $X_0 = F(X_0)$.

E12.6 *Suppose that $f : A \to B$ is one-one and $g : B \to A$ is also one-one. For every $X \subseteq A$ let $F(X) = A \backslash g[B \backslash f[X]]$. Show that for all $X, Y \subseteq A$, if $X \subseteq Y$ then $F(X) \subseteq F(Y)$.*

We have $f[X] \subseteq f[Y]$, hence $B \backslash f[Y] \subseteq B \backslash f[X]$, hence $g[B \backslash f[Y]] \subseteq g[B \backslash f[X]]$, hence $F(X) = A \backslash g[B \backslash f[X]] \subseteq A \backslash g[B \backslash f[Y]] = F(Y)$.

E12.7 *Prove the Cantor-Schröder-Bernstein theorem as follows. Assume that f and g are as in exercise E12.6, and choose F as in that exercise. Let X_0 be as in exercise E12.4. Show that $A \backslash X_0 \subseteq rng(g)$. Then define $h : A \to B$ by setting, for any $a \in A$,*

$$h(a) = \begin{cases} f(a) & \text{if } a \in X_0, \\ g^{-1}(a) & \text{if } a \in A \backslash X_0. \end{cases}$$

Show that h is one-one and maps onto B.

$A \backslash X_0 = A \backslash F(X_0) = g[B \backslash f[X]] \subseteq \text{rng}(g)$. Now note that $h \restriction X_0$ maps X_0 onto $f[X_0]$ and is one-one, and $h \restriction (A \backslash X_0)$ maps $A \backslash X_0$ onto $g^{-1}[A \backslash X_0] = g^{-1}[g[B \backslash f[X_0]]] = B \backslash f[X_0]$ and is one-one. So h is the union of two functions with disjoint domains and disjoint ranges, so h is a one-one function, and it maps A onto B.

E12.8 *Show that if α and β are ordinals, then $|\alpha \dotplus \beta| = |\alpha| + |\beta|$, where $\dotplus$ is ordinal addition and $+$ is cardinal addition.*

This is immediate from Proposition 12.21.

E12.9 *Show that if α and β are ordinals, then $|\alpha \odot \beta| = |\alpha| \cdot |\beta|$, where $\odot$ is ordinal multiplication and $\cdot$ is cardinal multiplication.*

This is immediate from Proposition 12.29.

E12.10 *Show that if α and β are ordinals, $2 \leq \alpha$, and $\omega \leq \beta$, then $|{}^{\cdot}\alpha^\beta| = |\alpha| \cdot |\beta|$. Here the dot to the left of the first exponent indicates that ordinal exponentiation is involved.*

First note that $|\alpha| \leq \alpha \leq {}^{\cdot}\alpha^{\beta}$ and $|\beta| \leq \beta \leq {}^{\cdot}\alpha^{\beta}$, so $|\alpha| \cdot |\beta| \leq |{}^{\cdot}\alpha^{\beta}|$. Hence it suffices to prove the other direction, which we do by induction on β, starting with $\beta = \omega$. First, $\beta = \omega$: If $\alpha < \omega$, then $|{}^{\cdot}\alpha^{\omega}| = \left|\bigcup_{m\in\omega} {}^{\cdot}\alpha^{m}\right| = \omega = |\alpha| \cdot |\omega|$. If $\alpha \geq \omega$, then

$$|{}^{\cdot}\alpha^{\omega}| = |\bigcup_{m\in\omega} {}^{\cdot}\alpha^{m}| \leq \sum_{m\in\omega} |{}^{\cdot}\alpha^{m}| \leq \sum_{m\in\omega} |\alpha| \leq \omega \cdot |\alpha|,$$

as desired.

Now we assume the result for $\beta \geq \omega$. Then

$$|{}^{\cdot}\alpha^{\beta+1}| = |{}^{\cdot}\alpha^{\beta} \odot \alpha| = |{}^{\cdot}\alpha^{\beta}| \cdot |\alpha| = |\alpha| \cdot |\beta| \cdot |\alpha| = |\alpha| \cdot |\beta|.$$

using the inductive hypothesis. Finally, if β is a limit ordinal $> \omega$ and the result is true for all $\gamma < \beta$, then

$$\begin{aligned}
|{}^{\cdot}\alpha^{\beta}| &= \left|\bigcup_{\gamma<\beta} {}^{\cdot}\alpha^{\gamma}\right| = \left|\bigcup_{\omega\leq\gamma<\beta} {}^{\cdot}\alpha^{\gamma}\right| \leq \sum_{\omega\leq\gamma<\beta} |{}^{\cdot}\alpha^{\gamma}| \\
&= \sum_{\omega\leq\gamma<\beta} |\alpha| \cdot |\gamma| \leq \sum_{\omega\leq\gamma<\beta} |\alpha| \cdot |\beta| \leq |\alpha| \cdot |\beta| \cdot |\beta| = |\alpha| \cdot |\beta|.
\end{aligned}$$

E12.11 *Prove that if $|A| \leq |B|$ then $|\mathscr{P}(A)| \leq |\mathscr{P}(B)|$.*

Let f be a one-one function mapping A into B. For each $X \in \mathscr{P}(A)$ let $g(X) = f[X]$. So g maps $\mathscr{P}(A)$ into $\mathscr{P}(B)$. We claim that g is one-one. For, suppose that $X, Y \in \mathscr{P}(A)$ and $X \neq Y$. Say by symmetry that $x \in X\backslash Y$. Then $f(x) \in f[X]$ but $f(x) \notin f[Y]$ by one-oneness.

E12.12 *Prove the following general distributive law:*

$$\prod_{i\in I}^{c} \sum_{j\in J_i} \kappa_{ij} = \sum_{f\in P} \prod_{i\in I}^{c} \kappa_{i,f(i)},$$

where $P = \prod_{i\in I} J_i$.

The left side is the number of elements of

(1) $$\prod_{i\in I} \left(\bigcup_{j\in J_i} (\kappa_{ij} \times \{j\}) \right),$$

and the right side is the number of elements of

(2) $$\bigcup_{f\in P} \left(\prod_{i\in I}^{c} \kappa_{i,f(i)} \times \{f\} \right).$$

For each $f \in P$ let F_f be a bijection from $\prod_{i\in I} \kappa_{i,f(i)}$ onto $\prod^c_{i\in I} \kappa_{i,f(i)}$. Now given x in (1) we define $G(x)$ in (2) as follows. For each $i \in I$ we have $x_i \in \bigcup_{j\in J_i}(\kappa_{ij} \times \{j\})$, and so there is a unique $j \in J_i$ such that $x \in \kappa_{ij} \times \{j\}$; let $f_x(i)$ be this j. Thus $f_x \in P$. Now $1^{\rm st}(x_i) \in \kappa_{i,f_x(i)}$ for all $i \in I$, so $\langle 1^{\rm st}(x_i) : i \in I\rangle \in \prod_{i\in I} \kappa_{i,f_x(i)}$. Now we define

$$G(x) = (F_{f_x}(\langle 1^{\rm st}(x_i) : i \in I\rangle), f_x).$$

Clearly $G(x)$ is in (2).

Suppose that $G(x) = G(y)$. Now $f_x = 2^{\rm nd}(G(x)) = 2^{\rm nd}(G(y) = f_y$. Write $f_x = g$. Then for any $i \in I$,

$$\begin{aligned} 1^{\rm st}(x_i) &= (F_g^{-1}(1^{\rm st}(G(x))))_i \\ &= (F_g^{-1}(1^{\rm st}(G(y))))_i \\ &= 1^{\rm st}(y_i), \end{aligned}$$

and $2^{\rm nd}(x_i) = g(i) = 2^{\rm nd}(y_i)$. So $x_i = y_i$. Hence $x = y$. So G is one-one.

To show that G maps onto (2), suppose that z is a member of (2). Choose $h \in P$ such that $z \in (\prod^c_{i\in I} \kappa_{i,h(i)}) \times \{h\}$. Now $F_h^{-1}(1^{\rm st}(z)) \in \prod_{i\in I} \kappa_{i,h(i)}$, so for each $i \in I$ we can let

$$x_i = ((F_h^{-1}(1^{\rm st}(z))_i, h(i)).$$

Then x is in (1). Moreover, clearly $f_x = h$. Then $1^{\rm st}(x_i) = (F_h^{-1}(1^{\rm st}(z))_i$, hence $\langle 1^{\rm st}(x_i) : i \in I\rangle = F_h^{-1}(1^{\rm st}(z))$, and so

$$\begin{aligned} G(x) &= (F_{f_x}(\langle 1^{\rm st}(x_i) : i \in I\rangle), f_x) \\ &= (F_h(\langle 1^{\rm st}(x_i) : i \in I\rangle), h) \\ &= (F_h(F_h^{-1}(1^{\rm st}(z))), h) \\ &= (1^{\rm st}(z), 2^{\rm nd}(z)) \\ &= z, \end{aligned}$$

as desired.

E12.13 *Show that for any cardinal κ we have $\kappa^+ = \{\alpha : \alpha$ is an ordinal and $|\alpha| \le \kappa\}$.*

First suppose that $\alpha < \kappa^+$. Then $|\alpha| \le \alpha$, so $|\alpha| \le \kappa$. Now suppose that $|\alpha| \le \kappa$. Thus there is a one-one function from α into κ. If $\kappa^+ \le \alpha$, then we could also get a one-one function from κ^+ into κ, so $\kappa^+ = |\kappa^+| \le |\kappa| = \kappa$, contradiction. So $\alpha < \kappa^+$, as desired.

E12.14 *For every infinite cardinal λ there is a cardinal $\kappa > \lambda$ such that $\kappa^\lambda = \kappa$.*

Let $\kappa = 2^\lambda$. Then $\kappa^\lambda = (2^\lambda)^\lambda = 2^{\lambda\cdot\lambda} = 2^\lambda = \kappa$.

E12.15 *For every infinite cardinal λ there is a cardinal $\kappa > \lambda$ such that $\kappa^\lambda > \kappa$.*

Let $\lambda = \aleph_\alpha$. Note that $\mathrm{cf}(\aleph_{\alpha+\omega}) = \omega \le \lambda$. Let $\kappa = \aleph_{\alpha+\omega}$. Then $\kappa^\lambda > \kappa$.

E12.16 *Prove that for every $n \in \omega$, and every infinite cardinal κ, $\aleph_n^\kappa = 2^\kappa \cdot \aleph_n$.*

We prove this by induction on n. $n = 0$: $\aleph_0^\kappa = 2^\kappa = 2^\kappa \cdot \aleph_0$. Assume it for n. Then by Hausdorff's theorem,

$$\aleph_{n+1}^\kappa = ((\aleph_n)^+)^\kappa = \aleph_n^\kappa \cdot (\aleph_n)^+ = 2^\kappa \cdot \aleph_n \cdot \aleph_{n+1} = 2^\kappa \cdot \aleph_{n+1}.$$

E12.17 *Prove that* $\prod_{i\in I}^c (\kappa_i \cdot \lambda_i) = \prod_{i\in I}^c \kappa_i \cdot \prod_{i\in I}^c \lambda_i$.

We have

$$(1) \qquad \prod_{i\in I}^c (\kappa_i \cdot \lambda_i) = \left|\prod_{i\in I} (\kappa_i \cdot \lambda_i)\right|$$

and

$$(2) \qquad \prod_{i\in I}^c \kappa_i \cdot \prod_{i\in I}^c \lambda_i = \left|\left(\prod_{i\in I}^c \kappa_i\right) \times \left(\prod_{i\in I}^c \lambda_i\right)\right|$$

For each $i \in I$ let f_i be a bijection from $\kappa_i \cdot \lambda_i$ onto $\kappa_i \times \lambda_i$. Let g be a bijection from $\prod_{i\in I}^c \kappa_i$ onto $\prod_{i\in I} \kappa_i$, and let h be a bijection from $\prod_{i\in I}^c \lambda_i$ onto $\prod_{i\in I} \lambda_i$. Now we define a function F from $\prod_{i\in I}(\kappa_i \cdot \lambda_i)$ to $\left(\prod_{i\in I}^c \kappa_i\right) \times \left(\prod_{i\in I}^c \lambda_i\right)$ by setting, for any $x \in \prod_{i\in I}(\kappa_i \cdot \lambda_i)$,

$$F(x) = (g^{-1}(\langle 1^{\text{st}}(f_i(x_i)) : i \in I\rangle), h^{-1}(2^{\text{nd}}(f_i(x_i)) : i \in I\rangle)).$$

This does map into $\left(\prod_{i\in I}^c \kappa_i\right) \times \left(\prod_{i\in I}^c \lambda_i\right)$, since for any $i \in I$ we have $x_i \in \kappa_i \cdot \lambda_i$, hence $f_i(x_i) \in \kappa_i \times \lambda_i$; so $1^{\text{st}}(f_i(x_i)) \in \kappa_i$. Thus $\langle 1^{\text{st}}(f_i(x_i)) : i \in I\rangle \in \prod_{i\in I} \kappa_i$ and hence $g^{-1}(\langle 1^{\text{st}}(f_i(x_i)) : i \in I\rangle) \in \prod_{i\in I}^c \kappa_i$. Similarly $h^{-1}(2^{\text{nd}}(f_i(x_i)) : i \in I\rangle) \in \prod_{i\in I}^c \lambda_i$, so that $F(x) \in \left(\prod_{i\in I}^c \kappa_i\right) \times \left(\prod_{i\in I}^c \lambda_i\right)$.

F is one-one; for suppose that $F(x) = F(y)$. Then for any $i \in I$,

$$\begin{aligned} 1^{\text{st}}(F(x)) &= g^{-1}(\langle 1^{\text{st}}(f_i(x_i)) : i \in I\rangle); \\ g(1^{\text{st}}(F(x)) &= \langle 1^{\text{st}}(f_i(x_i)) : i \in I\rangle; \\ (g(1^{\text{st}}(F(x))_i &= 1^{\text{st}}(f_i(x_i)). \end{aligned}$$

Similarly, $(g(1^{\text{st}}(F(y))_i = 1^{\text{st}}(f_i(y_i))$. Then $1^{\text{st}}(f_i(x_i)) = 1^{\text{st}}(f_i(y_i))$ follows from the assumption that $F(x) = F(y)$. Similarly, $2^{\text{nd}}(f_i(x_i)) = 2^{\text{nd}}(f_i(y_i))$. So $f_i(x_i) = f_i(y_i)$, and hence $x_i = y_i$. This being true for all $i \in I$, we have $x = y$, as desired.

F maps onto; for suppose that $z \in \left(\prod_{i\in I}^c \kappa_i\right) \times \left(\prod_{i\in I}^c \lambda_i\right)$. Then $1^{\text{st}}(z) \in \prod_{i\in I}^c \kappa_i$, so $g(1^{\text{st}}(z)) \in \prod_{i\in I} \kappa_i$. Hence for any $i \in I$, $(g(1^{\text{st}}(z)))_i \in \kappa_i$. Similarly, $(h(2^{\text{nd}}(z))_i \in \lambda_i$. It follows that $((g(1^{\text{st}}(z)))_i, (h(2^{\text{nd}}(z))_i) \in \kappa_i \times \lambda_i$, and hence

$$f_i^{-1}(((g(1^{\text{st}}(z)))_i, (h(2^{\text{nd}}(z))_i)) \in \kappa_i \cdot \lambda_i.$$

We let $x_i = f_i^{-1}(((g(1^{\text{st}}(z)))_i, (h(2^{\text{nd}}(z))_i))$. So $x \in \prod_{i\in I}(\kappa_i \cdot \lambda_i)$, and

$$\begin{aligned} f_i(x_i) &= ((g(1^{\text{st}}(z)))_i, (h(2^{\text{nd}}(z))_i); \\ 1^{\text{st}}(f_i(x_i)) &= (g(1^{\text{st}}(z)))_i; \\ \langle 1^{\text{st}}(f_i(x_i)) : i \in I\rangle &= g(1^{\text{st}}(z)); \\ g^{-1}(\langle 1^{\text{st}}(f_i(x_i)) : i \in I\rangle) &= 1^{\text{st}}(z). \end{aligned}$$

Similarly, $h^{-1}(\langle 2^{\rm nd}(f_i(x_i)) : i \in I\rangle) = 2^{\rm nd}(z)$. So $F(x) = z$.

By (1) and (2) this completes the exercise.

E12.18 *Prove that* $\aleph_\omega^{\aleph_1} = 2^{\aleph_1} \cdot \aleph_\omega^{\aleph_0}$.

Note that $\aleph_\omega = \sum_{m\in\omega} \aleph_m$, since $\aleph_n \le \sum_{m\in\omega} \aleph_m$ for each $n \in \omega$, hence

$$\aleph_\omega \le \sum_{m\in\omega} \aleph_m \le \sum_{m\in\omega} \aleph_\omega = \omega \cdot \aleph_\omega = \aleph_\omega.$$

Hence by Theorem 12.41 we have $\aleph_\omega < \prod^c_{m\in\omega} \aleph_{m+1} \le \prod^c_{m\in\omega} \aleph_m$. So

$$\begin{aligned}
\aleph_\omega^{\aleph_1} &\le \left(\prod^c_{m\in\omega} \aleph_m\right)^{\aleph_1} \\
&= \prod^c_{m\in\omega} \aleph_m^{\aleph_1} \\
&= \prod^c_{m\in\omega} (2^{\aleph_1} \cdot \aleph_m) \quad \text{by exercise E12.16} \\
&= \prod^c_{m\in\omega} 2^{\aleph_1} \cdot \prod^c_{m\in\omega} \aleph_m \quad \text{by exercise E12.17} \\
&= 2^{\aleph_1 \cdot \aleph_0} \cdot \prod^c_{m\in\omega} \aleph_m \\
&= 2^{\aleph_1} \cdot \prod^c_{m\in\omega} \aleph_m \\
&\le 2^{\aleph_1} \cdot \prod^c_{m\in\omega} \aleph_\omega \\
&= 2^{\aleph_1} \cdot \aleph_\omega^{\aleph_0} \\
&\le \aleph_\omega^{\aleph_1}.
\end{aligned}$$

E12.19 *Prove that* $\aleph_\omega^{\aleph_0} = \prod_{n\in\omega} \aleph_n$.

By the argument at the beginning of the solution of exercise E12.18, $\aleph_\omega = \sum_{m\in\omega} \aleph_m < \prod^c_{m\in\omega} \aleph_m$. Hence

$$\aleph_\omega^{\aleph_0} \le \left(\prod^c_{m\in\omega} \aleph_m\right)^{\aleph_0} = \prod^c_{m\in\omega} \aleph_m^{\aleph_0} = \prod^c_{m\in\omega} (2^{\aleph_0} \cdot \aleph_m) = 2^{\aleph_0} \cdot \prod^c_{m\in\omega} \aleph_m = \prod^c_{m\in\omega} \aleph_m \le \prod^c_{m\in\omega} \aleph_\omega = \aleph_\omega^{\aleph_0}.$$

E12.20 *Prove that for any infinite cardinal* κ, $(\kappa^+)^\kappa = 2^\kappa$.

By Hausdorff's theorem, $(\kappa^+)^\kappa = \kappa^\kappa \cdot \kappa^+ = 2^\kappa \cdot \kappa^+ = 2^\kappa$.

E12.21 *Show that if κ is an infinite cardinal and C is the collection of all cardinals less than κ, then $|C| \le \kappa$.*

Let $\kappa = \aleph_\alpha$. Thus $C \subseteq \omega \cup \{\aleph_\beta : \beta < \alpha\}$. Hence $|C| \le \omega + |\alpha|$. Now $\omega \le \kappa$, and $|\alpha| \le \alpha \le \aleph_\alpha = \kappa$ since $\aleph$ is a normal function. Hence $|C| \le \kappa$.

E12.22 *Show that if κ is an infinite cardinal and C is the collection of all cardinals less than κ, then*

$$2^\kappa = \left(\sum_{\nu\in C} 2^\nu\right)^{\mathrm{cf}(\kappa)}.$$

First suppose that κ is a successor cardinal λ^+. Then

$$2^\kappa \le \left(\sum_{\nu\in C} 2^\nu\right)^{\kappa} = \left(\sum_{\nu\in C} 2^\nu\right)^{\mathrm{cf}(\kappa)} \le \left(\sum_{\nu\in C} 2^\lambda\right)^{\kappa} = (|C|\cdot 2^\lambda)^\kappa \le (\lambda^+\cdot 2^\lambda)^\kappa \le (2^\kappa)^\kappa = 2^\kappa,$$

as desired.

Now suppose that κ is a limit cardinal. Let $\langle \mu_\xi : \xi < \mathrm{cf}(\kappa)\rangle$ be a strictly increasing sequence of cardinals with supremum κ. Then

$$2^\kappa = 2^{\sum_{\xi<\mathrm{cf}(\kappa)}\mu_\xi} = \prod^c_{\xi<\mathrm{cf}(\kappa)} 2^{\mu_\xi} \le \left(\sum_{\nu\in C} 2^\lambda\right)^{\mathrm{cf}(\kappa)} \le (2^\kappa)^{\mathrm{cf}(\kappa)} = 2^\kappa.$$

E12.23 *Prove that for any limit ordinal τ, $\prod^c_{\xi<\tau} 2^{\aleph_\xi} = 2^{\aleph_\tau}$.*

$$2^{\aleph_\tau} = 2^{\sum_{\xi<\tau}\aleph_\xi} = \prod^c_{\xi<\tau} 2^{\aleph_\xi}.$$

E12.24 *Assume that κ is an infinite cardinal, and $2^\lambda < \kappa$ for every cardinal $\lambda < \kappa$. Show that $2^\kappa = \kappa^{\mathrm{cf}(\kappa)}$.*

If κ is a successor cardinal, then $\mathrm{cf}(\kappa) = \kappa$ and the desired conclusion is clear. Suppose that κ is a limit cardinal. Let $\langle \mu_\xi : \xi < \mathrm{cf}(\kappa)\rangle$ be a strictly increasing sequence of cardinals with supremum κ. Then

$$2^\kappa = 2^{\sum_{\xi<\mathrm{cf}(\kappa)}\mu_\xi} = \prod^c_{\xi<\mathrm{cf}(\kappa)} 2^{\mu_\xi} \le \prod^c_{\xi<\mathrm{cf}(\kappa)} \kappa = \kappa^{\mathrm{cf}(\kappa)} \le \kappa^\kappa = 2^\kappa.$$

E12.25 *Suppose that λ is a singular cardinal, $\mathrm{cf}(\lambda) = \omega$, and $2^\kappa < \lambda$ for every $\kappa < \lambda$. Prove that $2^\lambda = \lambda^\omega$.*

Let $\langle \kappa_n : n \in \omega\rangle$ be a system of cardinals less than λ with supremum λ. Then

$$2^\lambda = 2^{\sum_{n\in\omega}\kappa_n} = \prod^c_{n\in\omega} 2^{\kappa_n} \le \prod^c_{n\in\omega} \lambda = \lambda^\omega \le \lambda^\lambda = 2^\lambda.$$

E13.1 *Let $(A, +, \cdot, -, 0, 1)$ be a Boolean algebra. Show that $(A, \triangle, \cdot, 0, 1)$ is a ring with identity in which every element is idempotent. This means that $x \cdot x = x$ for all x.*

Obviously $\triangle$ is commutative, and it is associative by Proposition 13.11(iii). Clearly $x \triangle 0 = x$ for all x. Clearly $x \triangle x = 0$, so each element x has itself as additive inverse. Hence $(A, \triangle, 0)$ is an abelian group.

Clearly $\cdot$ is associative. The distributive law holds by Proposition 13.11(ii). Clearly $x \cdot 1 = x$ for all x, and clearly $x \cdot x = x$ for all x.

Hence $(A, \triangle, \cdot, 0, 1)$ is a ring with identity in which every element is idempotent.

E13.2 *Let $(A, +, \cdot, 0, 1)$ be a ring with identity in which every element is idempotent. Show that A is a commutative ring, and $(A, \oplus, \cdot, -, 0, 1)$ is a Boolean algebra, where for any $x, y \in A$, $x \oplus y = x + y + xy$ and for any $x \in A$, $-x = 1 + x$. Hint: expand $(x + y)^2$.*

$x + y = (x + y)^2 = x^2 + xy + yx + y^2 = x + xy + yx + y$, and hence $0 = xy + yx$ for any x, y. Setting $x = y$, we get $0 = x + x$, and so x is its own additive inverse. Then from $0 = xy + yx$ we see that yx is the additive inverse of xy, hence $xy = yx$. Thus the ring is commutative.

To show that $(A, \oplus, \cdot, -, 0, 1)$ is a Boolean algebra, we need to check all of the axioms.

(C): Clear.

(A): For any x, y, z,

$$\begin{aligned} x \oplus (y \oplus z) &= x + (y \oplus z) + x(y \oplus z) \\ &= x + y + z + yz + x(y + z + yz) \\ &= x + y + z + yz + xy + xz + xyz; \end{aligned}$$

Hence, using (C),

$$\begin{aligned} (x \oplus y) \oplus z &= z \oplus (x \oplus y) \\ &= z + x + y + xy + zy + zxy \\ &= \text{above}. \end{aligned}$$

(A′): obvious.

(C′): obvious.

(L):

$$x \oplus xy = x + xy + xxy = x + xy + xy = x.$$

(L′):

$$x(x \oplus y) = x(x + y + xy) = xx + xy + xxy = x + xy + xy = x.$$

(D):

$$\begin{aligned} x(y \oplus z) &= x(y + z + yz) = xy + xz + xyz; \\ xy \oplus xz &= xy + xz + xyxz = xy + xz + xyz. \end{aligned}$$

(D′): See Proposition 13.12.

(K): $x + (-x) = x + 1 + x = 1$.

(K'): $x(1 + x) = x + xx = x + x = 0$.

Thus we have a BA.

E13.3 *Show that the processes described in exercises E2.1 and E2.2 are inverses of one another.*

For each BA $(A, +, \cdot, -, 0, 1)$ let $\mathscr{R}(A, +, \cdot, -, 0, 1) = (A, \triangle, \cdot, 0, 1)$ be the associated ring, and for each ring $(A, +, \cdot, 0, 1)$ with identity in which every element is idempotent let $\mathscr{B}(A, +, \cdot, 0, 1) = (A, \oplus, \cdot, -, 0, 1)$ be the associated Boolean algebra. We want to show that $\mathscr{R}$ and $\mathscr{B}$ are inverses of each other.

First suppose that $(A, +, \cdot, -, 0, 1)$ is a BA. Let $\mathscr{R}((A, +, \cdot, -, 0, 1)) = (A, \triangle, \cdot, 0, 1)$ be the associated ring, and let $\mathscr{B}(\mathscr{R}((A, +, \cdot, -, 0, 1)) = (A, \oplus, \cdot, -', 0, 1)$ be the BA associated with that ring; we want to show that $+ = \oplus$ and $- = -'$. We have

$$\begin{aligned} x \oplus y &= x \triangle y \triangle (x \cdot y) \\ &= x \triangle (y \cdot -(x \cdot y) + x \cdot y \cdot -y) \\ &= x \triangle (y \cdot -x) \\ &= x \cdot -(y \cdot -x) + y \cdot -x \cdot -x \\ &= x + y \cdot -x \\ &= x + y \cdot x + y \cdot -x \\ &= x + y. \end{aligned}$$

Also, $-'x = 1 \triangle x = -x$.

Second, suppose that $(A, +, \cdot, 0, 1)$ is a ring with identity in which every element is idempotent, let $\mathscr{B}((A, +, \cdot, 0, 1)) = (A, +', \cdot, -', 0, 1)$ be the associated BA, and let $\mathscr{R}(\mathscr{B}((A, +, \cdot, 0, 1))) = (A, \triangle', \cdot, 0, 1)$ be the ring associated with it. We want to show that $+ = \triangle'$. We have

$$\begin{aligned} x \triangle' y &= (x \cdot -'y) +' (y \cdot -'x) \\ &= (x \cdot (1 + y)) +' (y \cdot (1 + x)) \\ &= (x + xy) +' (y + xy) \\ &= x + xy + y + xy + (x + xy)(y + xy) \\ &= x + y + xy + xy + xy + xxy + xyy + xyxy \\ &= x + y + xy + xy + xy + xy + xy + xy \\ &= x + y. \end{aligned}$$

E13.4 *Prove that a filter F is an ultrafilter iff F is maximal among the set of all filters G such that $0 \notin G$.*

$\Rightarrow$: Assume that F is an ultrafilter. Hence by definition $0 \notin F$. Suppose that $F \subset G$ with G a filter. Choose $x \in G\backslash F$. Since $x \notin F$ it follows that $-x \in F$, and hence $-x \in G$. So $0 = x \cdot -x \in G$. So F is maximal among the set of filters G such that $0 \notin G$.

$\Leftarrow$: Suppose that F is maximal among the set of filters G such that $0 \notin G$. Suppose that $a \in A$ and $a \notin F$. Let $G = \{x \in A : a \cdot y \leq x$ for some $y \in F\}$. Then G is a filter on A. In fact, obviously conditions (1) and (2) hold. For (3), suppose that $x, z \in G$. Choose $y, w \in F$ such that $x = a \cdot y$ and $z = a \cdot w$. Now $y \cdot w \in F$, and $x \cdot z = a \cdot y \cdot w$. So $x \cdot z \in G$. Thus, indeed, G is a filter on A. Clearly also $F \subseteq G$. Clearly $a \in G$ (taking $y = 1$), so $F \subset G$.

It follows by supposition that $0 \in G$. Say $0 = a \cdot y$, with $y \in F$. then $y \leq -a$, so $-a \in F$. Thus F is an ultrafilter.

<u>E13.5</u> *Prove that for any nonzero $a \in A$ there is an ultrafilter F such that $a \in F$.*

Let $\mathscr{A} = \{G : G$ is a filter in A, $a \in G$, and $0 \notin G\}$. We consider $\mathscr{A}$ as a partially ordered set under $\subseteq$. To verify the hypothesis of Zorn's lemma, suppose that $\mathscr{B}$ is a subset of $\mathscr{A}$ linearly ordered by $\subseteq$. Now $\{x \in A : a \leq x\}$ is clearly a member of $\mathscr{A}$, so we may assume that $\mathscr{B}$ is nonempty. Let $H = \bigcup \mathscr{B}$. Since $\mathscr{B}$ is nonempty, it is clear that $a \in H$. Suppose that $x \in H$ and $x \leq y$. Choose $G \in \mathscr{B}$ such that $x \in G$. Then $y \in G$ since G is a filter. So $y \in H$. Suppose that $x, y \in H$. Choose $G, G' \in \mathscr{B}$ such that $x \in G$ and $y \in G'$. By symmetry say $G \subseteq G'$. Then $x, y \in G'$, so $x \cdot y \in G'$, hence $x \cdot y \in H$. Thus we have shown that H is a filter on A. Clearly $0 \notin H$. So H is a member of $\mathscr{A}$ which is an upper bound for $\mathscr{B}$.

Thus by Zorn's lemma, $\mathscr{A}$ has a maximal member G. By exercise E13.4, G is as desired.

<u>E13.6</u> *Prove that any BA is isomorphic to a field of sets. (Stone's representation theorem) Hint: given a BA A, let X be the set of all ultrafilters on A and define $f(a) = \{F \in X : a \in F\}$.*

Let X be the collection of all ultrafilters, and let $F, G \in X$.

$F \in f(-a)$ iff $-a \in F$ iff $a \notin F$, so $f(-a) = X\backslash f(a)$.

Suppose that $F \in f(a + b)$. Then $a + b \in F$. Suppose that $F \notin f(a)$. Then $a \notin F$, so $-a \in F$, hence $-a \cdot (a + b) \in F$. Since $-a \cdot (a + b) \leq b$, also $b \in F$, so $F \in f(b)$. This shows that $f(a + b) \subseteq f(a) \cup f(b)$. On the other hand, if $F \in f(a)$, then $a \in F$; but $a \leq a + b$, so also $a + b \in F$; hence $F \in f(a + b)$. Altogether this shows that $f(a + b) = f(a) \cup f(b)$.

Suppose that $a \neq b$. Then $a \triangle b \neq 0$, so $a \triangle b \in F$ for some ultrafilter F, by exercise E13. Hence $F \in f(a \triangle b) = [f(a)\backslash f(b)] \cup [f(b)\backslash f(a)]$, and so $f(a) \neq f(b)$. So f is one-one.

<u>E13.7</u> *Suppose that F is an ultrafilter on a BA A. Let 2 be the two-element BA. (This is, up to isomorphism, the BA of all subsets of 1.) For any $a \in A$ let*

$$f(a) = \begin{cases} 1 & \textit{if } a \in F, \\ 0 & \textit{if } a \notin F. \end{cases}$$

Show that f is a homomorphism of A into 2.

$f(a \cdot b) = 1$ iff $a \cdot b \in F$ iff $a, b \in F$ iff $f(a) \cdot f(b) = 1$. Hence $f(a \cdot b) = f(a) \cdot f(b)$.

$f(-a) = 1$ iff $-a \in F$ iff $a \notin F$ iff $f(a) = 0$. Hence $f(-a) = -f(a)$.
$f(a + b) = f(-(-a \cdot -b)) = -(-f(a) \cdot -f(b)) = f(a) + f(b)$.
$f(0) = f(a \cdot -a) = f(a) \cdot -f(a) = 0$.
$f(1) = f(a + -a) = f(a) + -f(a) = 1$.

E13.8 *(A knowledge of logic is assumed.) Suppose that $\mathscr{L}$ is a first-order language and T is a set of sentences of $\mathscr{L}$. Define $\varphi \equiv_T \psi$ iff φ and ψ are sentences of $\mathscr{L}$ and $T \models \varphi \leftrightarrow \psi$. Show that this is an equivalence relation on the set S of all sentences of $\mathscr{L}$. Let A be the collection of all equivalence classes under this equivalence relation. Show that there are operations $+$, $\cdot$, $-$ on A such that for any sentences φ, ψ,*

$$[\varphi] + [\psi] = [\varphi \vee \psi];$$
$$[\varphi] \cdot [\psi] = [\varphi \wedge \psi];$$
$$-[\varphi] = [\neg\varphi].$$

Finally, show that $(A, +, \cdot, -, [\exists v_0(\neg(v_0 = v_0))], [\exists v_0(v_0 = v_0)])$ is a Boolean algebra.

$\equiv_T$ is reflexive: $T \models \varphi \leftrightarrow \varphi$ for any sentence φ.
$\equiv_T$ is symmetric: If $T \models \varphi \leftrightarrow \psi$, then $T \models \psi \leftrightarrow \varphi$.
$\equiv_T$ is transitive: If $T \models \varphi \leftrightarrow \psi$ and $T \models \psi \leftrightarrow \chi$, then $T \models \varphi \leftrightarrow \chi$.
$+$ is well-defined: If $T \models \varphi \leftrightarrow \varphi'$ and $T \models \psi \leftrightarrow \psi'$, then $T \models (\varphi \vee \psi) \leftrightarrow (\varphi' \vee \psi')$.
$\cdot$ is well-defined: If $T \models \varphi \leftrightarrow \varphi'$ and $T \models \psi \leftrightarrow \psi'$, then $T \models (\varphi \wedge \psi) \leftrightarrow (\varphi' \wedge \psi')$.
$-$ is well-defined: If $T \models \varphi \leftrightarrow \varphi'$, then $T \models \neg\varphi \leftrightarrow \neg\varphi'$.
Finally, we need to check the axioms for BAs:

(A) holds since

$$[\varphi] + ([\psi] + [\chi]) = [\varphi \vee (\psi \vee \chi)] = [(\varphi \vee \psi) \vee \chi] = ([\varphi] + [\psi]) + [\chi];$$

(A′) holds since

$$[\varphi] \cdot ([\psi] \cdot [\chi]) = [\varphi \wedge (\psi \wedge \chi)] = [(\varphi \wedge \psi) \wedge \chi] = ([\varphi] \cdot [\psi]) \cdot [\chi];$$

(C) holds since

$$[\varphi] + [\psi] = [\varphi \vee \psi] = [\psi \vee \varphi] = [\psi] + [\varphi];$$

(C′) holds since

$$[\varphi] \cdot [\psi] = [\varphi \wedge \psi] = [\psi \wedge \varphi] = [\psi] \cdot [\varphi];$$

(L) holds since

$$[\varphi] + [\varphi] \cdot [\psi]) = [\varphi \vee (\varphi \wedge \psi)] = [\varphi];$$

(L′) holds since

$$[\varphi] \cdot [\varphi] + [\psi]) = [\varphi \wedge (\varphi \vee \psi)] = [\varphi];$$

(D) holds since

$$[\varphi] \cdot ([\psi] + [\chi]) = [\varphi \wedge (\psi \vee \chi)] = [(\varphi \wedge \psi) \vee (\varphi \wedge \chi)] = [\varphi] \cdot [\psi] + [\varphi] \cdot [\chi];$$

for (D′) see Proposition 2.12; (K) holds since

$$[\varphi] + -[\varphi] = [\varphi \vee \neg\varphi] = [\exists v_0(v_0 = v_0)];$$

(K′) holds since

$$[\varphi] \cdot -[\varphi] = [\varphi \wedge \neg\varphi] = [\exists v_0(\neg(v_0 = v_0))].$$

E13.9 *(A knowledge of logic is assumed.) Show that every Boolean algebra is isomorphic to one obtained as in exercise E13.8. Hint: Let A be a Boolean algebra. Let $\mathscr{L}$ be the first-order language which has a unary relation symbol R_a for each $a \in A$. Let T be the following set of sentences of $\mathscr{L}$:*

$$\begin{aligned}
&\forall x\forall y(x = y);\\
&\forall x[R_{-a}(x) \leftrightarrow \neg R_a(x)] \quad \textit{for each } a \in A;\\
&\forall x[R_{a\cdot b}(x) \leftrightarrow R_a(x) \wedge R_b(x)] \quad \textit{for all } a, b \in A;\\
&\forall x R_1(x).
\end{aligned}$$

We follow the hint, and consider $\equiv_T$. Define $f(a) = [\forall x R_a(x)]$ for any $a \in A$. To show that f preserves $\cdot$, suppose that $a, b \in A$. Note that

$$T \models \forall x R_{a\cdot b}(x) \leftrightarrow \forall x R_a(x) \wedge \forall x R_b(x);$$

hence $f(a \cdot b) = f(a) \cdot f(b)$.

To proceed we need the following fact

(1) $T \models \forall x\varphi \leftrightarrow \varphi$ for any variable x and any formula φ.

In fact, trivially $T \models \forall x\varphi \rightarrow \varphi$, and $T \models \varphi \rightarrow \exists x\varphi$. Since $T \models x = y$, clearly $T \models \exists x\varphi \rightarrow \forall x\varphi$. So (1) holds.

Now to show that f preserves $-$, suppose that $a \in A$. Then $T \models \forall x R_{-a}(x) \leftrightarrow \forall x\neg R_a(x)$. By (1), $T \models \forall x\neg R_a(x) \leftrightarrow \neg R_a(x)$ and $T \models \neg R_a(x) \leftrightarrow \neg\forall x R_a(x)$. Putting these statements together we have $T \models \forall x R_{-a}(x) \leftrightarrow \neg\forall x R_a(x)$, and it follows that f preserves $-$.

To show that f is one-one, suppose that $a, b \in A$ and $a \neq b$; say $a \cdot -b \neq 0$. Let F be an ultrafilter on A such that $a \cdot -b \in F$. We now define an $\mathscr{L}$-structure $\mathfrak{A}$. Let $A = 1$. For each $a \in A$, let

$$R_a^{\mathfrak{A}} = \begin{cases} 1 & \text{if } a \in F,\\ 0 & \text{otherwise.}\end{cases}$$

Clearly $\mathfrak{A}$ is a model of T. Also, $\mathfrak{A} \models R_{a\cdot -b}(x)$. It follows that $[\forall x R_{a\cdot -b}(x)] \neq [\exists v_0(\neg(v_0 = v_0))]$, and so $f(a) = [\forall x R_a(x)] \neq [\forall x R_b(x)] = f(b)$, as desired.

It remains only to show that f maps onto.

(2) For any formula φ there is an $a \in A$ such that $T \models \varphi \leftrightarrow R_a(x)$.

Condition (2) is easily proved by induction on φ, using (1). Hence f is onto.

$\boxed{\text{E13.10}}$ *Let A be the collection of all subsets X of $Y \stackrel{\text{def}}{=} \{r \in \mathbb{Q} : 0 \leq r\}$ such that there exist an $m \in \omega$ and $a, b \in {}^m(Y \cup \{\infty\})$ such that $a_0 < b_0 < a_1 < b_1 < \cdots < a_{m-1} < b_{m-1} \leq \infty$ and*

$$X = [a_0, b_0) \cup [a_1, b_1) \cup \ldots \cup [a_{m-1}, b_{m-1}).$$

Note that $\emptyset \in A$ by taking $m = 0$, and $Y \in A$ since $Y = [0, \infty)$.

(i) Show that if X is as above, $c, d \in Y \cup \{\infty\}$ with $c < d$, $c \leq a_0$, then $X \cup [c, d) \in A$, and c is the first element of $X \cup [c, d)$.

(ii) Show that if X is as above and $c, d \in Y \cup \{\infty\}$ with $c < d$, then $X \cup [c, d) \in A$.

(iii) Show that $(A, \cup, \cap, \backslash, \emptyset, Y)$ is a Boolean algebra.

(i): Assume the hypothesis. If $m = 0$ the desired conclusion is clear, so suppose that $m > 0$. We consider several cases.

Case 1. $b_{m-1} \leq d$. Then $X \cup [c, d) = [c, d) \in A$.

Case 2. There is an $i < m - 1$ such that $b_i \leq d < a_{i+1}$. Then

$$X \cup [c, d) = [c, d) \cup [a_{i+1}, b_{i+1}) \cup \ldots \cup [a_{m-1}, b_{m-1}) \in A.$$

Case 3. There is an $i < m$ such that $a_i \leq d < b_i$. Then

$$X \cup [c, d) = [c, b_i) \cup [a_{i+1}, b_{i+1}) \cup \ldots \cup [a_{m-1}, b_{m-1}) \in A.$$

Case 4. $d < a_0$. Then

$$X \cup [c, d) = [c, d) \cup [a_0, b_0) \cup \ldots \cup [a_{m-1}, b_{m-1}) \in A.$$

(ii): Again we consider several cases.

Case 1. $c \leq a_0$. Then $X \cup [c, d) \in A$ by (i).

Case 2. There is an $i < m$ such that $a_i \leq c \leq b_i$. Let $X' = [a_i, b_i) \cup \ldots \cup [a_{m-1}, b_{m-1})$. Then by (i) applied to X' and $[a_i, d)$ we get $X' \cup [a_i, d) \in A$, and a_i is the least element of $X' \cup [a_i, d)$. Clearly

$$X \cup [c, d) = [a_0, b_0) \cup \ldots \cup [a_{i-1}, b_{i-1}) \cup X' \cup [a_i, d) \in A.$$

Case 3. There is an $i < m - 1$ such that $b_i < c < a_{i+1}$. Then we can apply (i) to $[a_{i+1}, b_{i+1}) \cup \ldots \cup [a_{m-1}, b_{m-1})$ and $[c, d)$ to get the desired result as in Case 2.

Case 4. $c = b_{m-1}$. Then

$$X \cup [c, d) = [a_0, b_0) \cup \ldots \cup [a_{m-1}, d) \in A.$$

Case 5. $b_{m-1} < c$. This case is clear.

(iii): From (ii) it is clear that A is closed under $\cup$. Now suppose that X is given as above. To show that also $Y \backslash X \in A$, we consider several cases.

Case 1. $m = 0$. So $X = \emptyset$, and $Y = [0, \infty) \in A$.

Case 2. $m > 0$, $0 < a_0$, and $b_{m-1} < \infty$. Then

$$Y \backslash X = [0, a_0) \cup [b_0, a_1) \cup \ldots \cup [b_{m-2}, a_{m-1}) \cup [b_{m-1}, \infty) \in A.$$

Case 3. $m > 0$, $a_0 = 0$, and $b_{m-1} < \infty$. Then

$$Y \backslash X = [b_0, a-1) \cup \ldots \cup [b_{m-2}, a_{m-1}) \cup [b_{m-1}, \infty) \in A.$$

Case 4. $m > 0$, $0 < a_0$, and $b_{m-1} = \infty$. Then

$$Y \backslash X = [0, a_0) \cup [b_0, a_1) \cup \ldots \cup [b_{m-2}, a_{m-1}) \in A.$$

Case 5. $m > 0$, $0 = a_0$, and $b_{m-1} = \infty$. Then

$$Y \backslash X = [b_0, a_1) \cup \ldots \cup [b_{m-2}, a_{m-1}) \in A.$$

Thus (iii) holds.

E13.11 *(Continuing exercise E13.10) For each $n \in \omega$ let $x_n = [n, n+1)$, an interval in $\mathbb{Q}$. Show that $\sum_{n\in\omega} x_{2n}$ does not exist in A.*

Suppose that the sum does exist. Let $X = \sum_{n\in\omega} x_{2n}$, and assume that X is as in exercise E13.10.

We claim that $b_{m-1} = \infty$. In fact, if $b_{m-1} < \infty$, then there is an $m \in \omega$ such that $b_{m-1} < 2m$; then $x_{2m} = [2m, 2m+1)$ is disjoint from X according to the form of X, but $x_{2m} \leq X$ by definition, contradiction. So our claim holds.

Now choose $m \in \omega$ so that $a_{m-1} < 2m+1$. Then $[2m+1, 2m+2) \cap x_{2n} = \emptyset$ for all n, hence $[2m+1, 2m+2) \cap X = \emptyset$. But $[2m+1, 2m+2) \subseteq [a_{m-1}, b_{m-1}) \subseteq X$, contradiciton.

E13.12 *Let A be the Boolean algebra of all subsets of some nonempty set X, under the natural set-theoretic operations. Show that if $\langle a_i : i \in I\rangle$ is a system of elements of A, then*

$$\prod_{i\in I}(a_i + -a_i) = 1 = \sum_{\varepsilon\in {}^I2} \prod_{i\in I} a_i^{\varepsilon(i)},$$

where for any y, $y^1 = y$ and $y^0 = -y$.

First note that the big products and sums are just the ordinary intersections and unions. Obviously $a_i + -a_i = a_i \cup (X\backslash a_i) = X = 1$, giving the first equality. Now suppose that $x \in X$. We define

$$\varepsilon(i) = \begin{cases} 1 & \text{if } x \in a_i, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly then $x \in a_i^{\varepsilon(i)}$ for each $i \in I$, and hence x is in the right side of the second equality, as desired.

E13.13 *Let M be the set of all finite functions $f \subseteq \omega \times 2$. For each $f \in M$ let*

$$U_f = \{g \in {}^\omega 2 : f \subseteq g\}.$$

Let A consist of all finite unions of sets U_f.

(i) Show that A is a Boolean algebra under the set-theoretic operations.

(ii) For each $i \in \omega$, let $x_i = U_{\{(i,1)\}}$. Show that

$$^{\omega}2 = \prod_{i\in\omega}(x_i + -x_i)$$

while

$$\sum_{\varepsilon\in{}^{\omega}2}\prod_{i\in\omega}x_i^{\varepsilon(i)} = \emptyset,$$

where for any y, $y^1 = y$ and $y^0 = -y$.

(i): Obviously A is closed under $\cup$. Now suppose that $a \in A$; we want to show that $({}^{\omega}2)\backslash a \in A$. Say $a = \bigcup_{f\in N} U_f$, where N is a finite subset of M. Let

$$P = \left\{g \in M : \mathrm{dmn}(g) \subseteq \bigcup_{f\in N}\mathrm{dmn}(f) \text{ and } \forall f \in N \exists i \in \mathrm{dmn}(g)\cap\mathrm{dmn}(f)[f(i) \neq g(i)]\right\}.$$

Clearly P is a finite subset of M. We claim that $({}^{\omega}2)\backslash a = \bigcup_{g\in P} U_g$. First suppose that $h \in ({}^{\omega}2)\backslash a$. Let $g = h \restriction \bigcup_{f\in N}\mathrm{dmn}(f)$. So $g \in M$ and $h \in U_g$. We claim that $g \in P$. For, suppose that $f \in N$. Then $U_f \subseteq a$, so it follows that $h \notin U_f$. So we can choose $i \in \mathrm{dmn}(f)$ such that $f(i) \neq h(i)$. Clearly $i \in \mathrm{dmn}(g)$ and $f(i) \neq g(i)$. This shows that $g \in P$, proving $\subseteq$ of our claim.

For $\supseteq$, suppose that $g \in P$ and $h \in U_g$. Suppose that $h \in a$. Choose $f \in N$ such that $h \in U_f$, hence $f \subseteq h$. But $g \subseteq H$ too, so there is an $i \in \mathrm{dmn}(g)\cap\mathrm{dmn}(f)$ such that $f(i) \neq g(i)$. But this means that $f(i) \neq h(i)$, contradicting $f \subseteq h$. We have now shown (i).

Clearly $x_i \cup (({}^{\omega}2)\backslash x_i = {}^{\omega}2$ for any $i \in \omega$. Hence ${}^{\omega}2 = \prod_{i\in\omega}(x_i + -x_i)$.

Now suppose that $\varepsilon \in {}^{\omega}2$; we want to show that $\prod_{i\in\omega} x_i^{\varepsilon(i)} = 0$, i.e., that there is no nonzero element a of A such that $a \leq x_i^{\varepsilon(i)}$ for all $i \in \omega$. suppose that a is such an element. Then there is a $g \in M$ such that $U_g \subseteq a$. Take any $i \notin \mathrm{dmn}(g)$, and let $h \in {}^{\omega}2$ be any function such that $g \subseteq h$ and $h(i) \neq \varepsilon(i)$. Then $h \in U_g$ but $h \notin x_i^{\varepsilon(i)}$, contradiction.

E13.14 *Suppose that $(P, \leq, 1)$ is a forcing order. Define*

$$p \equiv q \quad \textit{iff} \quad p, q \in P,\ p \leq q,\ \textit{and } q \leq p.$$

Show that $\equiv$ is an equivalence relation, and if Q is the collection of all $\equiv$-classes, then there is a relation $\preceq$ on Q such that for all $p, q \in P$, $[p]_\equiv \preceq [q]_\equiv$ iff $p \leq q$. Finally, show that $(Q, \preceq)$ is a partial order, i.e., $\preceq$ is reflexive on Q, transitive, and antisymmetric ($q_1 \preceq q_2 \preceq q_1$ implies that $q_1 = q_2$); moreover, $q \leq [1]$ for all $q \in Q$.

Since $\leq$ is reflexive on P, clearly also $\equiv$ is reflexive on P. Clearly $\equiv$ is symmetric. Now suppose that $p \equiv q \equiv r$. Thus $p \leq q$, $q \leq p$, $q \leq r$, and $r \leq q$. Then $p \leq r$ and $r \leq p$, so $p \equiv r$. So $\equiv$ is an equivalence relation on P.

Let

$$\preceq = \{(a, b) : \exists p, q \in P[p \leq q,\ a = [p],\ \text{and } b = [q]]\}.$$

Obviously then $p \leq q$ implies that $[p] \preceq [q]$. Now suppose that $[p] \preceq [q]$. Choose $p', q' \in P$ such that $p' \leq q'$, $[p] = [p']$, and $[q] = [q']$. Then $p \leq p'$ and $q' \leq q$, so $p \leq q$.

To show that $\preceq$ is a partial order on Q, first suppose that $a \in Q$. Write $a = [p]$. Then $p \leq p$, so $a \preceq a$. Thus $\preceq$ is reflexive on Q. Now suppose that $a \preceq b \preceq c$. Then there exist p, q, q', r such that $p \leq q$, $a = [p]$, $b = [q]$, $q' \leq r$, $b = [q']$, and $c = [r]$. Then $q \leq q'$ since $[q] = [q']$. So $p \leq q \leq q' \leq r$, hence $p \leq r$. So $a = [q] \preceq [r] = c$. This shows that $\preceq$ is transitive. Finally, suppose that $a \preceq b \preceq a$. Then there exist p, q, q', r such that $p \leq q$, $a = [p]$, $b = [q]$, $q' \leq r$, $b = [q']$, and $a = [r]$. Then $q \leq q'$ since $[q] = [q']$. Also $r \leq p$ since $[p] = [r]$, so $q \leq q' \leq r \leq p$, hence $q \leq p$. But also $p \leq q$, so $a = [p] = [q] = b$. So $\preceq$ is a partial order. Clearly $a \leq [1]$ for all $a \in Q$.

E13.15 *We say that $(P, <)$ is a partial order in the second sense iff $<$ is transitive and irreflexive. (Irreflexive means that for all $p \in P$, $p \not< p$.) Show that if $(P, <)$ is a partial order in the second sense and if we define $\preceq$ by $p \preceq q$ iff ($p, q \in P$ and $p < q$ or $p = q$), then $\mathscr{A}(P, \preceq)$ is a partial order. Furthermore, show that if $(P, \leq)$ is a partial order, and we define $p \prec q$ by $p \prec q$ iff ($p, q \in P$, $p \leq q$, and $p \neq q$), then $\mathscr{B}(P, \prec)$ is a partial order in the second sense.*

Also prove that $\mathscr{A}$ and $\mathscr{B}$ are inverses of one another.

Clearly $\preceq$ is reflexive on P. Now suppose that $x \preceq y \preceq z$. If $x = y$ or $y = z$, then $x \preceq z$ by supposition. If $x < y < z$, then $x < z$, and so $x \preceq z$. Thus $\preceq$ is transitive. Suppose that $x \preceq y \preceq x$, but $x \neq y$. Then $x < y < x$, hence $x < x$, contradiction. So $\preceq$ is antisymmetric. Hence $(P, \preceq)$ is a partial order.

Now suppose that $(P, \leq)$ is a partial order, and define $p \prec q$ by $p \prec q$ iff ($p, q \in P$, $p \leq q$, and $p \neq q$). Clearly $\prec$ is irreflexive. Suppose that $p \prec q \prec r$. Then $p \leq q \leq r$, so $p \leq r$. Suppose that $p = r$. Then $p \leq q \leq p$, so $p = q$ by antisymmetry, contradiction. Thus $p \neq r$, and so $p \prec r$. So $(P, \prec)$ is a partial order in the second sense.

Next, suppose that $(P, <)$ is a partial order in the second sense, and let $\mathscr{A}(P, <) = (P, \preceq)$. Furthermore, let $\mathscr{B}(\mathscr{A}(P, <)) = (P, <')$. Then

$$p <' q \text{ iff } (p \preceq q \text{ and } p \neq q) \text{ iff } ((p < q \text{ or } p = q) \text{ and } p \neq q) \text{ iff } p < q.$$

Thus $\mathscr{B}(\mathscr{A}(P, <)) = (P, <)$.

Finally, suppose that $(P, \leq)$ is a partial order. Let $\mathscr{B}(P, \leq) = (P, \prec)$, and let $\mathscr{A}(\mathscr{B}(P, \leq)) = (P, \leq')$. Then

$$p \leq' q \text{ iff } (p \prec q \text{ or } p = q) \text{ iff } ((p \leq q \text{ and } p \neq q) \text{ or } p = q) \text{ iff } p \leq q.$$

E13.16 *Show that if $(P, \leq, 1)$ is a forcing order and we define $\prec$ by $p \prec q$ iff ($p, q \in P$, $p \leq q$ and $q \not\leq p$), then $(P, \prec)$ is a partial order in the second sense. Give an example where this partial order is not isomorphic to the one derived from $(P, \leq, 1)$ by the procedure of exercise E13.14.*

$\prec$ is irreflexive, since $x \prec x$ would imply that $x \not\leq x$, a contradiction. For transitivity, suppose that $x \prec y \prec z$. Then $x \leq y$ and $y \leq z$, so $x \leq z$. Also, $y \not\leq x$ and $z \not\leq y$. Suppose that $z \leq x$. Then $y \leq z \leq x$ and hence $y \leq x$, contradiction. Hence $z \not\leq x$, and so $x \prec z$. Thus $(P, \prec)$ is a partial order in the second sense.

For the example, let X be any infinite set, and let $\leq$ be $X\times X$. Fix $1\in X$. So $(X,\leq,1)$ is a quasiorder. The partial order constructed in exercise E13.14 has only one element, while the partial order of the present exercise has X, an infinite set, as its underlying set. Note that $\prec$ is empty.

$\boxed{\text{E13.17}}$ *Prove that if $(P,\preceq,1)$ is a forcing order such that the mapping e from P into $\mathrm{RO}(P)$ is one-one, then $(P,\preceq)$ is a partial order. Give an example of a forcing order such that e is not one-one. Give an example of an infinite forcing order Q such that e is not one-one, while for any $p,q\in Q$, $p\leq q$ iff $e(p)\subseteq e(q)$.*

Suppose that P is a forcing order such that e is one-one, and $p\leq q\leq p$. Then $P\downarrow p=P\downarrow q$, and hence $e(p)=e(q)$. So $p=q$. Hence $(P,\leq)$ is a partial order.

For an example of a forcing order such that e is not one-one, take any simple ordering with greatest element; see the remarks preceding Proposition 13.21.

For the final example, take any infinite set Q, and take the forcing order $(Q,Q\times Q,q)$ for any element $q\in Q$. So e is the constant function with value Q. For any $p,q\in Q$ we have $p\leq q$ and $e(p)\subseteq e(q)$, so these statements are equivalent trivially.

$\boxed{\text{E13.18}}$ *(Continuing E13.14.) Let $\mathbb{P}=(P,\leq,1)$ be a forcing order, and let $\mathbb{Q}=(Q,\preceq,[1])$ be as in exercise E13.14. Show that there is an isomorphism f of $RO(\mathbb{P})$ onto $RO(\mathbb{Q})$ such that $f\circ e_{\mathbb{P}}=e_{\mathbb{Q}}\circ\pi$, where $\pi:P\to Q$ is defined by $\pi(p)=[p]$ for all $p\in P$.*

We will apply Theorem 13.22. For any $p\in P$ let $j(p)=e_{\mathfrak{Q}}(\pi(p))$. Thus $j:P\to\mathrm{RO}(\mathfrak{Q})$.

Suppose that $0\neq X\in\mathrm{RO}(\mathfrak{Q})$. By Theorem 13.20(i), choose $q\in Q$ such that $e_{\mathfrak{Q}}(q)\leq X$. Say $q=[p]$. Then $j(p)=e_{\mathfrak{Q}}(\pi(p))\leq X$. So $j[P]$ is dense in $\mathrm{RO}(\mathfrak{Q})$.

Suppose that $p,q\in P$ and $p\leq q$. Then $[p]\preceq[q]$, and so $j(p)\leq j(q)$ by Theorem 13.20(ii).

Suppose that $p,q\in P$. If $p\not\perp q$, choose $r\leq p,q$. Then $j(r)\leq j(p),j(q)$, so $j(p)\cap j(q)\neq\emptyset$. If $j(p)\cap j(q)\neq\emptyset$, then by Theorem 13.20(iii), $\pi(p)\not\perp\pi(q)$. So there is an $r\in Q$ such that $r\leq\pi(p),\pi(q)$. Say $r=\pi(s)$. Then $s\leq p,q$, so $p\not\perp q$.

This verifies the hypotheses of Theorem 13.22, and the desired conclusion follows.

Solutions to exercises in chapter 14

$\boxed{\text{E14.1}}$ *Write out all the elements of V_α for $\alpha=0,1,2,3,4$.*

$V_0=\emptyset$.

$V_1=\mathscr{P}(V_0)=\mathscr{P}(\emptyset)=\{\emptyset\}=1$.

$V_2=\mathscr{P}(V_1)=\mathscr{P}(\{\emptyset\})=\{\emptyset,\{\emptyset\}\}=2$.

$V_3=\mathscr{P}(V_2)=\mathscr{P}(\{\emptyset,\{\emptyset\}\})=\{\emptyset,\{\emptyset\},\{\{\emptyset\}\},\{\emptyset,\{\emptyset\}\}\}$. Note that V_3 has four elements, but it is not equal to 4, since, for example, $\{\{\emptyset\}\}\in V_3\backslash 4$.

For V_4, it helps to use the usual abbreviations for natural numbers. Thus $V_3=\{0,1,2,\{1\}\}$. We list out the subsets of V_3 with 0,1,2,3,4 elements:

$V_4=\mathscr{P}(V_3)=\{0$ 0 elements; 1 of these

$\{0\},\{1\},\{2\},\{\{1\}\}$ 1 element, 4 of these

$\{0,1\},\{0,2\},\{0,\{1\}\},\{1,2\},\{1,\{1\}\},\{2,\{1\}\}$ 2 elements, 6 of these
$\{0,1,2\},\{0,1,\{1\}\},\{0,2,\{1\}\},\{1,2,\{1\}\}$ 3 elements, 4 of these
$\{0,1,2,\{1\}\}$ 4 elements, 1 of these.}

E14.2 *Define by recursion*

$$S(\alpha) = \bigcup_{\beta<\alpha} \mathscr{P}(S(\beta))$$

for every ordinal α. *Prove that* $V_\alpha = S(\alpha)$ *for every ordinal* α.

First we apply the recursion theorem 8.7. Define $\mathbf{G} : \mathbf{On} \times \mathbf{V} \to \mathbf{V}$ by setting, for any ordinal α and any set x,

$$\mathbf{G}(\alpha, x) = \begin{cases} \bigcup_{\beta<\alpha} \mathscr{P}(x(\beta)) & \text{if } x \text{ is a function with domain } \alpha, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then obtain $\mathbf{F} : \mathbf{On} \to \mathbf{V}$ by Theorem 6.7: for any ordinal α, $\mathbf{F}(\alpha) = \mathbf{G}(\alpha, \mathbf{F} \restriction \alpha)$. Thus $\mathbf{F}(\alpha) = \bigcup_{\beta<\alpha} \mathscr{P}(\mathbf{F}(\beta))$.

We prove that $V_\alpha = S(\alpha)$ for all α by induction:

$$\begin{aligned}
S(0) &= \bigcup_{\beta<0} \mathscr{P}(S(\beta)) = \emptyset = V_0; \\
S(\alpha+1) &= \bigcup_{\beta<\alpha+1} \mathscr{P}(S(\beta)) \\
&= \bigcup_{\beta<\alpha} \mathscr{P}(S(\beta)) \cup \mathscr{P}(S(\alpha)) \\
&= V_\alpha \cup \mathscr{P}(V_\alpha) \quad \text{(inductive hypothesis)} \\
&= V_{\alpha+1} \quad \text{(using Theorem 14.5(ii));} \\
S(\alpha) &= \bigcup_{\beta<\alpha} \mathscr{P}(S(\beta)) \quad \text{(with } \alpha \text{ limit)} \\
&= \bigcup_{\gamma<\alpha} \bigcup_{\beta<\gamma} \mathscr{P}(S(\beta)) \\
&= \bigcup_{\gamma<\alpha} V_\gamma \quad \text{(inductive hypothesis)} \\
&= V_\alpha.
\end{aligned}$$

E14.3 *Determine exactly the ranks of the following sets in terms of the ranks of the sets entering into their definitions. In some cases the rank is not completely determined by the ranks of the constituents; in such cases, describe all possibilities.*

(i) $\{x\}$	*(iv)* $x \cup y$	*(vii)* $\bigcup x$	*(x)* $\mathbf{R}^{-1}$
(ii) $\{x, y\}$	*(v)* $x \cap y$	*(viii)* $\mathrm{dmn}(\mathbf{R})$	
(iii) (x, y)	*(vi)* $x \backslash y$	*(ix)* $\mathscr{P}(x)$	

(i): Let $\alpha = \text{rank}(x)$. Thus $x \in V_{\alpha+1}$. Hence $\{x\} \subseteq V_{\alpha+1}$, so $\{x\} \in \mathscr{P}(V_{\alpha+1}) = V_{\alpha+2}$. So $\text{rank}(\{x\}) \le \alpha+1$. Suppose that $\text{rank}(\{x\}) < \alpha+1$. Then $\{x\} \in V_{\alpha+1} = \mathscr{P}(V_\alpha)$, hence $\{x\} \subseteq V_\alpha$, so $x \in V_\alpha$, and so $\text{rank}(x) < \alpha$, contradiction. Thus $\text{rank}(\{x\}) = \alpha+1$.

(ii): Let $\alpha = \text{rank}(x)$ and $\beta = \text{rank}(y)$. We claim that $\text{rank}(\{x,y\}) = \max(\alpha,\beta)+1$. To prove this, by symmetry it suffices to assume that $\alpha \le \beta$. Hence $x, y \in V_{\beta+1}$, so $\{x,y\} \subseteq V_{\beta+1}$, so $\{x,y\} \in \mathscr{P}(V_{\beta+1}) = V_{\beta+2}$. Hence $\text{rank}(\{x,y\}) \le \beta+1$. Suppose that $\text{rank}(\{x,y\}) < \beta+1$. Then $\{x,y\} \in V_{\beta+1} = \mathscr{P}(V_\beta)$, so $\{x,y\} \subseteq V_\beta$. Hence $y \in V_\beta$, contradiction.

(iii): $\text{rank}((x,y)) = \max(\text{rank}(x), \text{rank}(y)) + 2$ by (i) and (ii).

(iv): Let $\text{rank}(x) = \alpha$ and $\text{rank}(y) = \beta$. We claim that $\text{rank}(x \cup y) = \max(\alpha,\beta)$. To prove this, by symmetry we may assume that $\alpha \le \beta$. Thus $x, y \in V_{\beta+1}$, so $x, y \subseteq V_\beta$, hence $x \cup y \subseteq V_\beta$ and so $x \cup y \in V_{\beta+1}$. Hence $\text{rank}(x \cup y) \le \beta$. Suppose that $\text{rank}(x \cup y) < \beta$. Then $x \cup y \in V_\beta$, so by exercise E14.2, $x \cup y \subseteq V_\gamma$ for some $\gamma < \beta$. So $y \subseteq V_\gamma$, hence $y \in V_{\gamma+1} \subseteq V_\beta$, contradiction.

(v): Let α and β be as in (iv). We claim that $\text{rank}(x \cap y) \le \min(\text{rank}(x), \text{rank}(y))$, and actually can be anything $\le$ this minimum. To prove this, by symmetry assume that $\alpha \le \beta$. Then $x \cap y \subseteq x \subseteq V_\alpha$, so $\text{rank}(x \cap y) \le \alpha$.

For the second assertion, suppose that γ is any ordinal, and suppose that $\delta \le \gamma$. We define two sets x and y such that $\min(\text{rank}(x), \text{rank}(y)) = \gamma$ while $\text{rank}(x \cap y) = \delta$. Let $x = \delta \cup \{\gamma\}$ and $y = \gamma$. Then $\text{rank}(x) = \gamma+1$, $\text{rank}(y) = \gamma$, and $\text{rank}(x \cap y) = \text{rank}(\delta) = \delta$.

(vi): Let $\alpha = \text{rank}(x)$. We claim that $\text{rank}(x \backslash y) \le \alpha$, and it can be anything $\le \alpha$. In fact, $x \backslash y \subseteq x \subseteq V_\alpha$, so $x \backslash y \in V_{\alpha+1}$, and so $\text{rank}(x \backslash y) \le \alpha$.

For the second assertion, let $\beta \le \alpha$; we define x, y so that $\text{rank}(x) = \alpha$ while $\text{rank}(x \backslash y) = \beta$. Let $x = \alpha$ and $y = \alpha \backslash \beta$.

(vii): Let $\text{rank}(x) = \alpha$. We claim that $\text{rank}\left(\bigcup x\right) = \bigcup \alpha$. (Thus it is α if α is limit or 0, and it is β if $\alpha = \beta+1$.) To prove this, first we show that $\bigcup x \subseteq V_{\bigcup \alpha}$. For, suppose that $y \in \bigcup x$. Say $y \in z \in x$. Now $x \in V_{\alpha+1}$, so $x \subseteq V_\alpha$. Hence $z \in V_\alpha$. Hence by exercise E14.2, $z \subseteq V_\beta$ for some $\beta < \alpha$. So $y \in V_\beta$. Now $\beta \subseteq \bigcup \alpha$, so $\beta \le \bigcup \alpha$. Hence $y \in V_{\bigcup \alpha}$, as desired.

It follows that $\bigcup x \in V_{(\bigcup \alpha)+1}$, and hence $\text{rank}\left(\bigcup x\right) \le \bigcup \alpha$. Suppose that $\bigcup x \in V_{\bigcup \alpha}$. Then by exercise E14.2 $\bigcup x \subseteq V_\gamma$ for some $\gamma < \bigcup \alpha$. Say $\gamma < \delta < \alpha$. Then $x \subseteq V_\delta$. In fact, if $y \in x$, then $y \subseteq \bigcup x \subseteq V_\gamma$, hence $y \in V_{\gamma+1} \subseteq V_\delta$. So, indeed, $x \subseteq V_\delta$. Hence $x \in V_{\delta+1} \subseteq V_\alpha$, contradiction. This proves that $\text{rank}\left(\bigcup x\right) = \bigcup \alpha$.

(viii): Let $\alpha = \text{rank}(R)$. We claim, first of all, that $\text{rank}(\text{dmn}(R)) \le \bigcup\bigcup \alpha$. For, take any $x \in \text{dmn}(R)$. Choose y such that $(x,y) \in R$. So $x \in \{x\} \in (x,y) \in R$; it follows that $x \in \bigcup\bigcup R$. So $\text{dmn}(R) \subseteq \bigcup\bigcup R$, and so $\text{rank}(\text{dmn}(R)) \le \text{rank}(\bigcup\bigcup R) = \bigcup\bigcup \alpha$ by (vii).

Next we claim that if $\beta \le \bigcup\bigcup \alpha$, then there is a set R such that $\text{rank}(R) = \alpha$ while $\text{rank}(\text{dmn}(R)) = \beta$. To give the examples here, we consider two cases.

Case 1. $\beta = 0$. Then we take $R = \alpha$. We use the easy fact that no ordinal is an ordered pair. [(a,b) has at most two elements, and is a nonempty set. So the only possibilities for (a,b) to be an ordinal are $(a,b) = 1$ or $(a,b) = 2$. Since $(a,b) = \{\{a\}, \{a,b\}\}$, neither case is really possible, since the members of (a,b) are nonempty.]

Case 2. $0 < \beta$. Let $R = \{(\xi, 0) : \xi < \beta\} \cup \alpha$. To see that this works, note that $\text{dmn}(R) = \beta = \text{rank}(\beta)$. But we also need to see that $\text{rank}(R) \le \alpha$. For this we consider several subcases.

Subcase 2.1. α is a limit ordinal. Then $\bigcup\bigcup\alpha = \alpha$, $\text{rank}(\{(\xi,0):\xi<\beta\}) \le \alpha$

Subcase 2.2. $\alpha = \gamma+1$ for some limit ordinal γ. Since $\beta \le \gamma$, clearly $\text{rank}(\{(\xi,0):\xi<\beta\}) \le \gamma < \alpha$

Subcase 2.3. $\alpha = \gamma+2$ for some limit ordinal γ. Similar to Subcase 2.2.

Subcase 2.4. $\alpha = \gamma + n$ for some limit ordinal γ and some $n \in \omega\backslash 3$. Then $\beta \le \gamma + n - 2$, and so $\text{rank}(\{(\xi,0):\xi<\beta\}) \le \gamma+n$

Subcase 2.5. $0 < \beta \le \alpha - 2$, with $\alpha \in \omega\backslash 3$. Again clearly ok.

These are all of the possibilities.

(ix): Let $\text{rank}(x) = \alpha$. We claim that $\text{rank}(\mathscr{P}(x)) = \alpha+1$. Now $x \in V_{\alpha+1}$, so $x \subseteq V_\alpha$. Hence $y \subseteq V_\alpha$ for every $y \subseteq x$. So $\mathscr{P}(x) \subseteq \mathscr{P}(V_\alpha) = V_{\alpha+1}$; hence $\mathscr{P}(x) \in V_{\alpha+2}$. This shows that $\text{rank}(\mathscr{P}(x)) \le \alpha+1$. Suppose that $\mathscr{P}(x) \in V_{\alpha+1}$. Hence $x \in \mathscr{P}(x) \subseteq V_\alpha$, so $x \in V_\alpha$, contradiction. Hence $\text{rank}(\mathscr{P}(x)) = \alpha+1$.

(x): We claim that for all ordinals α, β, $\exists R[\text{rank}(R) = \alpha \text{ and } \text{rank}(R^{-1}) = \beta]$ iff $\beta \le \alpha$ and one of the following conditions holds:

(1) $\beta = \gamma + 3$ for some ordinal γ.

(2) β is a limit ordinal.

To prove this, we first note that if a and b have ranks ξ, η respectively, then (a,b) has rank $\max(\xi,\eta)+2$ by (iii). Hence if S is a collection of ordered pairs, then $\text{rank}(S) = \sup\{\text{rank}(s)+1 : s \in S\}$, and hence $\text{rank}(S)$ is either a limit ordinal (if $\{\text{rank}(s)+1 : s \in S\}$ does not have a greatest element) or it is of the form $\gamma+3$. It follows that if $\text{rank}(R) = \alpha$ and $\text{rank}(R^{-1}) = \beta$, then $\beta \le \alpha$ and (1) or (2) holds.

Now suppose that $\beta \le \alpha$. If $\beta = \gamma+3$ for some γ, let $R = \{(\gamma,\gamma)\} \cup \alpha$; then $\text{rank}(R) = \alpha$ and $\text{rank}(R^{-1}) = \beta$. If β is a limit ordinal, let $R = \{(\gamma,\gamma) : \gamma < \beta\} \cup \alpha$; then $\text{rank}(R) = \alpha$ and $\text{rank}(R^{-1}) = \beta$.

E14.4 *Define $x\mathbf{R}y$ iff $(x,1) \in y$. Show that $\mathbf{R}$ is well-founded and set-like on $\mathbf{V}$.*

Suppose that X is a nonempty set. Choose $x \in X$ of smallest rank. Suppose that $y \in X$ and yRx. Thus $(y,1) \in x$, so $y \in \{y\} \in (y,1) \in x$, and hence $\text{rank}(y) < \text{rank}(x)$, contradiction. Hence $\mathbf{R}$ is well-founded on $\mathbf{V}$.

For any $y \in \mathbf{V}$ we have $\text{pred}_{\mathbf{VR}}(y) = \{x : (x,1) \in y\}$. Note that if $(x,1) \in y$ then $x \in \{x\} \in \{\{x\},\{x,1\}\} = (x,1) \in y$, so $x \in \bigcup\bigcup y$. So $\text{pred}_{\mathbf{VR}}(y) = \{x \in \bigcup\bigcup y : (x,1) \in y\}$, and hence $\text{pred}_{\mathbf{VR}}(y)$ is a set. Thus $\mathbf{R}$ is set-like on $\mathbf{V}$.

E14.5 *(Continuing E14.4) By recursion let $\check{y} = \{(\check{x},1) : x \in y\}$ for any set y. Let $\mathbf{F}$ be the Mostowski collapsing function for $\mathbf{R},\mathbf{V}$ in exercise E14.4. Prove that $\mathbf{F}(\check{y}) = y$ for every set y.*

First we show that the function $\check{}$ exists. Let $\mathbf{S} = \{(x,y) : x \in y\}$. So $\mathbf{S}$ is well-founded and set-like on $\mathbf{V}$. Define $\mathbf{G} : \mathbf{V} \times \mathbf{V} \to \mathbf{V}$ by setting, for any sets y, f,

$$\mathbf{G}(y,f) = \begin{cases} \{(f(x),1) : x \in y\} & \text{if } f \text{ is a function with domain } \text{pred}_{\mathbf{VS}}(y), \\ \emptyset & \text{otherwise.} \end{cases}$$

Then let $\mathbf{H}$ be obtained from $\mathbf{G}$ by Theorem 5.7: for any set y, $\mathbf{H}(y) = \mathbf{G}(y, \mathbf{H} \restriction \text{pred}_{\mathbf{VS}}(y))$. Hence for any set y, $\mathbf{H}(y) = \{(\mathbf{H}(x), 1) : x \in y\}$, as desired.

Now for the main part of the exercise, suppose that it is not true. So there is a y such that $\mathbf{F}(\check{y}) \neq y$. Let $W = \{w \in \text{trcl}(y \cup \{y\}) : \mathbf{F}(\check{w}) \neq w\}$. Thus $y \in W$, so $W \neq \emptyset$. By the foundation axiom, choose $z \in W$ with $z \cap W = \emptyset$. Then

$$\begin{aligned}\mathbf{F}(\check{z}) &= \{\mathbf{F}(s) : s\mathbf{R}\check{z}\} \quad \text{(definition of } \mathbf{F}\text{)}\\ &= \{\mathbf{F}(s) : (s,1) \in \check{z}\} \quad \text{(definition of } \mathbf{R}\text{)}\\ &= \{\mathbf{F}(\check{t}) : t \in z\} \quad \text{(definition of } \check{z}\text{)}\\ &= \{t : t \in z\} \quad \text{(since } z \cap W = \emptyset\text{)}\\ &= z,\end{aligned}$$

contradiction.

E14.6 *Define $x\mathbf{R}y$ iff $x \in \text{trcl}(y)$. Show that $\mathbf{R}$ is well-founded and set like on $\mathbf{V}$.*

Clearly $x\mathbf{R}y$ implies that $\text{rank}(x) < \text{rank}(\text{trcl}(y)) = \text{rank}(y)$, so $\mathbf{R}$ is well-founded. For any $a \in \mathbf{V}$, the class $\{b \in \mathbf{V} : b\mathbf{R}a\} = \{b : b \in \text{trcl}(a)\} = \text{trcl}(a)$, a set.

E14.7 *Let $\mathbf{F}$ be the Mostowski collapsing function for $\mathbf{R}, \mathbf{V}$. Show that $\mathbf{F}(x) = \text{rank}(x)$ for every set x.*

Suppose not, and by the foundation axiom let x be a set such that $\mathbf{F}(x) \neq \text{rank}(x)$ while $\mathbf{F}(y) = \text{rank}(y)$ for every $y \in x$. Note that $\mathbf{F}(x)$ is transitive: if $u \in v \in \mathbf{F}(x)$, choose $y \in \text{trcl}(x)$ such that $v = F(y)$. Then choose $z \in \text{trcl}(y)$ such that $u = \mathbf{F}(z)$. Then $z \in \text{trcl}(x)$, and so $u \in \mathbf{F}(y)$. It follows that $\mathbf{F}(x)$ is an ordinal. Now

$$\begin{aligned}\mathbf{F}(x) &= \{\mathbf{F}(y) : y \in \mathbf{V} \text{ and } y\mathbf{R}x\}\\ &= \{\mathbf{F}(y) : y \in \text{trcl}(x)\}\\ &= \{\text{rank}(y) : y \in \text{trcl}(x)\}.\end{aligned}$$

Now if $y \in \text{trcl}(x)$, then $\text{rank}(y) < \text{rank}(\text{trcl}(x)) = \text{rank}(x)$. So $\mathbf{F}(x) \leq \text{rank}(x)$. Conversely, if $\alpha \in \text{rank}(x)$, then $\alpha \in \text{rank}(\text{trcl}(x))$ and hence $\alpha \leq \text{rank}(y)$ for some $y \in \text{trcl}(x)$. Since $\mathbf{F}(x)$ is an ordinal, it follows that $\alpha \in \mathbf{F}(x)$.

Thus $\mathbf{F}(x) = \text{rank}(x)$, contradiction.

E14.8 *Prove that if a is transitive, then $\{\text{rank}(b) : b \in a\}$ is an ordinal.*

Let $S = \{\text{rank}(b) : b \in a\}$, and let α be the least ordinal not in S. Thus $\alpha \subseteq S$. We claim that $\alpha = S$. Suppose not, and let β be the least member of $S \backslash \alpha$; so $\alpha < \beta$. Thus there is a $b \in a$ such that $\beta = \text{rank}(b)$. Now a is transitive, so if $c \in b$ then $c \in a$ and hence $\text{rank}(c) \in S$ with $\text{rank}(c) < \beta$, so $\text{rank}(c) < \alpha$. Hence $\beta = \sup_{c \in b}(\text{rank}(c) + 1) \subseteq \alpha$, contradiction.

E14.9 *Show that for any set a we have $\text{rank}(\text{trcl}(a)) = \text{rank}(a)$.*

We use the notation in the proof of Theorem 14.6. First we prove that $\mathrm{rank}(d_m) = \mathrm{rank}(a)$ for every $m \in \omega$, by induction on m. Obviously $\mathrm{rank}(d_0) = \mathrm{rank}(a)$. For the inductive step we use Theorem 14.8:

$$\begin{aligned}
\mathrm{rank}(a) &\leq \mathrm{rank}(d_{m+1}) \\
&= \max(\mathrm{rank}(d_m), \mathrm{rank}(\bigcup d_m)) \\
&= \max(\mathrm{rank}(a), \bigcup_{b \in d_m} \mathrm{rank}(b)) \\
&\leq \max(\mathrm{rank}(a), \bigcup_{b \in d_m} \mathrm{rank}(d_m)) \\
&= \max(\mathrm{rank}(a), \mathrm{rank}(a)) \\
&= \mathrm{rank}(a).
\end{aligned}$$

Now $\mathrm{rank}(\mathrm{trcl}(a)) = \mathrm{rank}(a)$ by Exercise E14.3(vii).

E14.10 *For any infinite cardinal κ, let $H(\kappa)$ be the set of all x such that $|trcl(x)| < \kappa$. Prove that $V_\omega = H(\omega)$. ($H(\omega)$ is the collection of all hereditarily finite sets.) Hint: $V_\omega \subseteq H(\omega)$ is easy. For the other direction, suppose that $x \in H(\omega)$, let $t = trcl(x)$, and let $S = \{rank(y) : y \in t\}$. Show that S is an ordinal.*

First we show $V_\omega \subseteq H(\omega)$. Suppose that $x \in V_\omega$. Say $x \in V_n$ with $n \in \omega$. Then $n = m+1$ for some m, and so $x \subseteq V_m$. Hence by Theorem 14.6, $\mathrm{trcl}(x) \subseteq V_m$. So $|\mathrm{trcl}(x)| \leq |V_m| < \omega$. It follows that $x \in H(\omega)$, as desired.

Conversely, suppose that $x \in H(\omega)$. Let $t = \mathrm{trcl}(x)$. So t is finite. Let $S = \{\mathrm{rank}(y) : y \in t\}$. Thus S is finite, since t is. S is a set of ordinals; we claim that it actually is an ordinal. It suffices to show that S is transitive. Suppose that $\beta \in \alpha \in S$; we may assume that α is minimum with this property. Write $\alpha = \mathrm{rank}(y)$ with $y \in t$. Now by Theorem 14.8(iv), $\mathrm{rank}(y) = \sup_{z \in y}(\mathrm{rank}(z) + 1)$. It follows that there is a $z \in y$ such that $\beta < \mathrm{rank}(z) + 1$. Now $z \in t$ since t is transitive. Thus with $\gamma = \mathrm{rank}(z)$, we have $\gamma \in S$, and hence we cannot have $\beta < \gamma$, as this would contradict the minimality of α. So $\beta = \gamma \in S$, proving our claim: S is a finite ordinal.

For each $y \in t$ we have $y \in V_{S+1}$, hence $x \subseteq t \subseteq V_{S+1}$, so $x \in \mathscr{P}(V_{S+1}) = V_{S+2} \subseteq V_\omega$, as desired.

E14.11 *Which axioms of ZFC are true in* **On***?*

Extensionality holds, by Theorem 14.11.

Comprehension fails. Let $\varphi(x, z)$ be the formula $\exists y(y \in x)$. Suppose that $[\exists w \forall x(x \in w \leftrightarrow x \in 2 \wedge \exists y(y \in x))]^{\mathbf{On}}$. So, choose an ordinal w such that $\forall x \in \mathbf{On}(x \in w \leftrightarrow x \in 2 \wedge \exists y \in \mathbf{On}(y \in x))$. Thus $\forall x(x \in w \leftrightarrow x \in 2 \wedge x \neq \emptyset)$. So $1 \in w$ but $0 \notin w$, contradicting w being an ordinal.

Pairing holds, by Theorem 14.13.

Union holds, by Theorem 14.14

Power set holds, by Theorem 14.15. In fact, given an ordinal x, any subset of x which is an ordinal is $\leq x$ itself, so we can take $y = x$.

Replacement holds. To prove this, we use Theorem 14.16. So, suppose that φ is a formula with free variables among $x, y, A, w_1, \ldots, w_n$, and suppose that $A, w_1, \ldots, w_n$ are ordinals such that

$$\forall x \in A \exists! y[y \in \mathbf{On} \wedge \varphi^{\mathbf{On}}(x, y, A, w_1, \ldots, w_n)].$$

For each $x \in A$, let y_x be the unique ordinal such that $\varphi^{\mathbf{On}}(x, y_x, A, w_1, \ldots, w_n)$. Let $\alpha = \sup_{x \in A} y_x$. Then clearly $\{y \in \mathbf{On} : \exists x \in A \varphi^{\mathbf{On}}(x, y, A, w_1, \ldots, w_n)\} \subseteq \alpha$, as desired.

Foundation holds, by Theorem 14.17.

Infinity holds. This is clear if we write the axiom out:

$$\begin{aligned}&\exists x[\exists y(y \in x \wedge \forall z(z \notin y) \wedge \\ &\quad \forall y \in x \exists z \in x \forall w(w \in z \leftrightarrow w \in y \vee w = y)]\end{aligned}$$

Clearly ω works for x.

Axiom of choice. This holds trivially. Given an ordinal $\mathscr{A}$, if $\mathscr{A} = 0$, then the conclusion holds if we take $\mathscr{B}$ to be any ordinal; and if $\mathscr{A} \neq 0$, the hypothesis of the implication in the axiom is false.

E14.12 *Show that the power set operation is absolute for V_α for α limit.*

We first show that the power set operation is defined in V_α. We take the following formula as the official definition for this operation:

$$\forall z \in y[z \subseteq x] \wedge \forall z[z \subseteq x \rightarrow z \in y].$$

Now given $x \in V_\alpha$, we have $\text{rank}(\mathscr{P}(x)) = \text{rank}(x)+1$ by the solution for exercise E14.3(ix). Hence $\mathscr{P}(x) \in V_\alpha$. Clearly $y = \mathscr{P}(x)$ satisfies the above formula. The uniqueness condition is clear.

Since every subset of x is in V_α by Theorem 14.8(iii), clearly $\mathscr{P}^{V_\alpha}(x) = \mathscr{P}(x)$. See Proposition 14.21.

E14.13 *Let M be a countable transitive model of ZFC. Show that the power set operation is not absolute for M.*

Note that $\omega \in M$ by Theorem 14.28. If the power set operation is absolute for M, then $\mathscr{P}^M(\omega) = \mathscr{P}(\omega) \in M$. But $|\mathscr{P}(\omega)| = 2^\omega$ and M is transitive, so M is uncountable, contradiction.

E14.14 *Show that V_ω is a model of ZFC $-$ Inf.*

Extensionality: true by Theorem 14.11.

Comprehension: assume that φ has free variables among $x, z, w_1, \ldots, w_n$ and we are given $z, w_1, \ldots, w_n \in V_\omega$. Let

$$A = \{x \in z : \varphi^{V_\omega}(x, z, w_1, \ldots, w_n)\}.$$

Choose $m \in \omega$ so that $z, w_1, \ldots, w_n \in V_m$. Then $A \subseteq z \subseteq V_m$, so $A \in \mathscr{P}(V_m) = V_{m+1} \subseteq V_\omega$. So $A \in V_\omega$. Hence the comprehension axiom holds in V_ω, by Theorem 14.12.

Pairing: if $x, y \in V_\omega$, choose n such that $x, y \in V_n$. Then $\{x, y\} \subseteq V_n$, so $\{x, y\} \in \mathscr{P}(V_n) = V_{n+1} \subseteq V_\omega$. So $\{x, y\} \in V_\omega$. By Theorem 14.13, the pairing axiom holds in V_ω.

Union: if $x \in V_\omega$, choose n such that $x \in V_n$. Then $\bigcup x \subseteq V_n$, so $\bigcup x \in \mathscr{P}(V_n) = V_{n+1} \subseteq V_\omega$. By Theorem 14.14, the union axiom holds in V_ω.

Power set: if $x \in V_\omega$, choose m such that $x \in V_n$. Then $y \subseteq V_n$ for each $y \subseteq x$, so $\mathscr{P}(x) \subseteq \mathscr{P}(V_n) = V_{n+1} \in V_{n+1} \subseteq V_\omega$. So the power set axiom holds by Theorem 14.15.

Replacement: preparing to use Theorem 14.16, let φ be a formula with free variables among $x, y, A, w_1, \ldots, w_n$, suppose that $A, w_1, \ldots, w_n \in V_\omega$, and also assume that

$$\forall x \in A \exists! y[y \in V_\omega \wedge \varphi^{V_\omega}(x, y, A, w_1, \ldots, w_n)].$$

Choose $m \in \omega$ such that $A, w_1, \ldots, w_n \in V_m$. For each $x \in A$, let y_x be the unique set such that

$$y_x \in V_\omega \wedge \varphi^{V_\omega}(x, y_x, A, w_1, \ldots, w_n)],$$

and let $p_x \in \omega$ be such that $y_x \in V_{p_x}$. Now A is finite, so there is a $q \in \omega$ such that $m < q$ and $p_x < q$ for every $x \in A$. It follows that

$$\{y \in V_\omega : \exists x \in A \varphi(x, y, A, w_1, \ldots, w_n)\} = \{y_x : x \in A\} \subseteq V_q \in V_\omega,$$

as desired.

Finally, foundation holds by Theorem 14.17.

E14.15 *Show that the formula $\exists x(x \in y)$ is not absolute for all nonempty sets, but it is absolute for all nonempty transitive sets.*

Let $A = \{\{\emptyset\}\}$. Then $\exists x(x \in \{\emptyset\})$ holds in V, but not in A, since there is no $a \in A$ such that $a \in \{\emptyset\}$.

Now suppose that B is a nonempty transitive set, and $y \in B$. Then y has an element iff it has an element in B, and so $\exists x(x \in y)$ iff $\exists x \in B(x \in y)$ iff $(\exists x(x \in y))^B$. So the formula is absolute for B. (Note that $\exists x(x \in y)$ is not quite a Δ_0 formula.)

E14.16 *Show that the formula $\exists z(x \in z)$ is not absolute for every nonempty transitive set.*

Take the transitive set 2. Then $\exists z(1 \in z)$, but this does not hold in 2, since there is no $z \in 2$ such that $1 \in z$.

E14.17 *A formula is Σ_1 iff it has the form $\exists x \varphi$ with φ a Δ_0 formula; it is Π_1 iff it has the form $\forall x \varphi$ with φ a Δ_0 formula.*

(i) Show that "X is countable" is equivalent on the basis of ZF to a Σ_1 formula.

(ii) Show that "α is a cardinal" is equivalent on the basis of ZF to a Π_1 formula.

(i) Basically the following statement proves this:

$$X \text{ is countable iff } \exists f[f \text{ is a one-one function and } \mathrm{rng}(f) \subseteq \omega].$$

(ii) Basically the following statement proves this:

$$\alpha \text{ is a cardinal iff } \forall f[\alpha \text{ is an ordinal and } \forall \beta \in \alpha(f : \beta \to \alpha \to f \text{ is not onto})].$$

However, in each case we have more work to do, since the "insides" were not proved to be Δ_0 in this chapter; all we can really use are the definitions. For brevity, we say "is Δ_0" instead of "is equivalent under ZF to a Δ_0 formula".

(1) "x is an ordinal" is Δ_0. For,

$$x \text{ is an ordinal iff } x \text{ is transitive and } \forall y \in x[y \text{ is transitive}].$$

(2) "x is a successor ordinal" is Δ_0. For,

$$x \text{ is a successor ordinal iff } x \text{ is an ordinal and } \exists y \in x(x = y \cup \{y\})].$$

(3) "n is a natural number" is Δ_0. For,

$$\begin{aligned}&n \text{ is a natural number iff } (n = \emptyset \text{ or } n \text{ is a successor ordinal})\\ &\text{and } (\forall y \in n[y = \emptyset \text{ or } y \text{ is a successor ordinal}]).\end{aligned}$$

(4) "a is an ordered pair" is Δ_0. For,

$$a \text{ is an ordered pair iff } \exists x \in a \exists y \in a \exists u \in x \exists v \in y[x = \{u\} \text{ and } y = \{u, v\} \text{ and } a = \{x, y\}]$$

(5) "R is a relation" is Δ_0. For,

$$R \text{ is a relation iff } \forall a \in R[a \text{ is an ordered pair}]$$

(6) If $\varphi(a, f)$ is Δ_0, then so is $\forall a \in \bigcup f \varphi(a, f)$. For,

$$\forall a \in \bigcup f \varphi(a, f) \text{ iff } \forall x \in f \forall a \in x \varphi(a, f).$$

(7) "f is a function" is Δ_0. For,

$$\begin{aligned}&f \text{ is a function iff } f \text{ is a relation and } \forall a \in f \forall b \in f \forall x \in \bigcup a \forall y \in \bigcup a \forall z \in \bigcup b\\ &[a = (x, y) \text{ and } b = (x, z) \to y = z].\end{aligned}$$

(8) "f is a one-one function" is Δ_0. For,

$$\begin{aligned}&f \text{ is a one-one function iff } f \text{ is a function and } \forall a \in f \forall b \in f \forall x \in \bigcup a \forall y \in \bigcup a \forall z \in \bigcup b\\ &[a = (x, y) \text{ and } b = (z, y) \to x = z].\end{aligned}$$

(9) "x is in the range of f" is Δ_0. For,

$$x \text{ is in the range of } f \text{ iff } \exists a \in f \exists y \in \bigcup a[a = (y, x)].$$

(10) "x is in the domain of f" is Δ_0. For,

$$x \text{ is in the domain of } f \text{ iff } \exists a \in f \exists y \in \bigcup a[a = (x, y)].$$

(11) $x = \text{dmn}(f)$ is Δ_0. For,

$$x = \text{dmn}(f) \text{ iff } \forall y \in x(y \text{ is in the domain of } f) \text{ and}$$
$$\forall a \in f \forall y \in a \forall u \in a(u \text{ is in the domain of } f \text{ implies that } u \in x).$$

Now we can give the details on (i) and (ii):

$$X \text{ is countable iff } \exists f[f \text{ is a one-one function and } X = \text{dmn}(f)$$
$$\text{and } \forall a \in f \forall x \in \bigcup a[x \text{ is in the range of } f \text{ implies that } x \text{ is a natural number}]].$$

$$\alpha \text{ is a cardinal iff } \forall f\ [\alpha \text{ is an ordinal and}$$
$$\forall \beta \in \alpha[f \text{ is a function and } \beta = \text{dmn}(f) \text{ implies that } \exists x \in \alpha(x \text{ is not in the range of } f)]$$

E14.18 *Prove that if κ is an infinite cardinal, then $H(\kappa) \subseteq V_\kappa$.*

Let $a \in H(\kappa)$. Let $\alpha = \{\text{rank}(b) : b \in \text{trcl}(a)\}$. Thus α is an ordinal by exercise E14.8. By exercise E14.9 and Theorem 14.8(iv) $\text{rank}(a) = \text{rank}(\text{trcl}(a)) \leq \alpha + 1$. Now $|\text{trcl}(a)| < \kappa$, and $\langle \text{rank}(b) : b \in \text{trcl}(a) \rangle$ maps $\text{trcl}(a)$ onto α, so $|\text{rank}(a)| \leq |\text{trcl}(a)| < \kappa$. Hence $\text{rank}(a) < \kappa$, as desired.

E14.19 *Prove that for κ regular, $H(\kappa) = V_\kappa$ iff $\kappa = \omega$ or κ is inaccessible.*

$\Rightarrow$: Assume that $H(\kappa) = V_\kappa$ and κ is uncountable. Suppose that $\lambda < \kappa \leq 2^\lambda$. Thus $\lambda \in V_{\lambda+1}$, and so $\mathscr{P}(\lambda) \in V_{\lambda+2} \subseteq V_\kappa$. But $|\mathscr{P}(\lambda) = 2^\lambda \geq \kappa$, so $\mathscr{P}(\lambda) \notin V_\kappa$.

$\Leftarrow$: $\kappa = \omega$ implies that $H(\kappa) = V_\kappa$ by exercise E14.14. Now suppose that κ is inaccessible. By exercise E14.18 we have $H(\kappa) \subseteq V_\kappa$. Now suppose that $S \in V_\kappa$. Let $\alpha = \text{rank}(S)$. Then also $\alpha = \text{rank}(\text{trcl}(S))$ by exercise E14.14. So $\text{trcl}(S) \in V_{\alpha+1}$, hence $\text{trcl}(S) \subseteq V_{\alpha+1}$ and so $|\text{trcl}(S)| \leq |V_{\alpha+1}| = \beth_\beta$ with $\omega + \beta = \alpha + 1$. Now $\beta < \kappa$ and $\beth_\beta < \beth_\kappa = \kappa$, so $S \in H(\kappa)$.

E14.20 *Assume that κ is an infinite cardinal. Prove the following:*
(a) $H(\kappa)$ is transitive.
(b) $H(\kappa) \cap \mathbf{On} = \kappa$.
(c) If $x \in H(\kappa)$, then $\bigcup x \in H(\kappa)$.
(d) If $x, y \in H(\kappa)$, then $\{x, y\} \in H(\kappa)$.
(e) If $y \subseteq x \in H(\kappa)$, then $y \in H(\kappa)$.
(f) If κ is regular and x is any set, then $x \in H(\kappa)$ iff $x \subseteq H(\kappa)$ and $|x| < \kappa$.

(a): Suppose that $x \in y \in H(\kappa)$. Then $|\text{trcl}(y)| < \kappa$. Moreover, $x \in \text{trcl}(y)$, hence $x \subseteq \text{trcl}(y)$, hence $\text{trcl}(x) \subseteq \text{trcl}(y)$. So $|\text{trcl}(x)| < \kappa$.

(b): If $\alpha \in H(\kappa) \cap \mathbf{ON}$, then $|\alpha| = |\text{trcl}(\alpha)| < \kappa$, and so $\alpha < \kappa$. Conversely, if $\alpha \in \kappa$, then $|\text{trcl}(\alpha)| = |\alpha| < \kappa$, so $\alpha \in H(\kappa)$.

(c): If $y \in \bigcup x$, then $\exists z[y \in z \in x]$, hence $\exists z[y \in z \in \text{trcl}(x)]$, hence $y \in \text{trcl}(x)$. Thus $\bigcup x \subseteq \text{trcl}(x)$, hence $\text{trcl}(\bigcup x) \subseteq \text{trcl}(x)$. So $\bigcup x \in H(\kappa)$.

(d): Note that $\text{trcl}(x) \cup \{x\}$ is transitive and contains $\{x\}$. If b is any transitive set containing $\{x\}$, then $x \in b$, hence $\text{trcl}(x) \subseteq b$, so that $\text{trcl}(x) \cup \{x\} \subseteq b$. This proves that $\text{trcl}(\{x\}) = \text{trcl}(x) \cup \{x\}$. Hence $|\text{trcl}(\{x\})| < \kappa$.

Similarly, $\text{trcl}(\{x,y\}) = \text{trcl}(\{x\}) \cup \text{trcl}(\{y\})$, so $|\text{trcl}(\{x,y\})| < \kappa$. So $\{x,y\} \in H(\kappa)$.

(e): obvious.

(f): Suppose that κ is regular and x is any set. If $x \in H(\kappa)$, then $x \subseteq H(\kappa)$ by (a), and $|x| \le |\text{trcl}(x)| < \kappa$. Now suppose that $x \subseteq H(\kappa)$ and $|x| < \kappa$. Now clearly $\text{trcl}(x) = x \cup \bigcup_{y\in x} \text{trcl}(y)$. Hence $|\text{trcl}(x)| \le 1 + \sum_{y\in x} |\text{trcl}(y)| < \kappa$; so $x \in H(\kappa)$.

E14.21 *Show that if κ is regular and uncountable, then $H(\kappa)$ is a model of all of the ZFC axioms except possibly the power set axiom.*

Extensionality: true since $H(\kappa)$ is transitive.

Comprehension: using Theorem 14.12, it suffices to take a formula φ with free variables among $x, z, w_1, \ldots, w_n$, assume that $z, w_1, \ldots, w_n \in H(\kappa)$, and prove that $\{x \in z : \varphi^{H(\kappa)}(x, z, w_1, \ldots, w_n) \in H(\kappa)$. This is true since the indicated set is a subset of z; hence its transitive closure is a subset of $\text{trcl}(z)$, which has size less than κ.

Pairing: Given $x, y \in H(\kappa)$, clearly $\{x,y\} \in H(\kappa)$.

Union: given $x \in H(\kappa)$, clearly $\bigcup x \subseteq \text{trcl}(x)$, hence $\text{trcl}(\bigcup x) \subseteq \text{trcl}(x)$, and so $\bigcup x \in H(\kappa)$.

Replacement: Suppose that φ is a formula with free variables among those in the list $x, y, A, w_1, \ldots, w_n$, $A, w_1, \ldots, w_n \in H(\kappa)$, and

$$\forall x \in A \exists! y[y \in H(\kappa) \wedge \varphi^{H(\kappa)}(x, y, A, w_1, \ldots, w_n)].$$

So for each $x \in A$ let $y_x \in H(\kappa)$ such that $\varphi^{H(\kappa)}(x, y, A, w_1, \ldots, w_n)$ holds. Let $Y = \{y_x : x \in A\}$. Then $\text{trcl}(Y) = \bigcup_{x\in A} \text{trcl}(y_x)$, and this has size less than κ since κ is regular. Clearly $\{z \in H(\kappa) : \exists x \in A \varphi^{H(\kappa)}(x, y, A, w_1, \ldots, w_n)\} \subseteq Y$.

Infinity: obviously $\omega \in H(\kappa)$; see Theorem 14.27.

Foundation: true since $H(\kappa)$ is transitive.

Axiom of choice: obvious.

Solutions to exercises in chapter 15

E15.1 *Let I and J be sets with I infinite and $|J| > 1$, and let $\mathbb{P} = (P, \le, \emptyset)$, where P is the collection of all finite functions contained in $I \times J$ and $\le$ is $\supseteq$ restricted to P. Show that $\mathbb{P}$ satisfies the condition of Lemma 15.2.*

Suppose that $p \in P$. Pick any $i \in I \backslash \text{dmn}(p)$, and let j, k be distinct elements of J. Then $p \subseteq p \cup \{(i,j)\}$, $p \subseteq p \cup \{(i,k)\}$, and these two extensions of p are incompatible.

E15.2 *Show that if the condition in the hypothesis of Lemma 15.2 fails, then there is a $\mathbb{P}$-generic filter G over M such that $G \in M$, and G intersects every dense subset of P (not only those in M). [Cf. Lemma 15.1.]*

Let p be such that for all q, r, if q and r are $\le p$ then they are compatible. Define

$$G = \{q : \exists r[r \le q \text{ and } (r \le p \text{ or } p \le r)].$$

We claim that G is as desired. Clearly $G \in M$.

Note that $p \in G$, by taking $r = p$.

To check (1), suppose that $q, r \in G$; we want to find $s \in G$ with $s \leq q, r$. Choose $t \leq q$ such that $t \leq p$ or $p \leq t$, and choose $u \leq r$ such that $u \leq p$ or $p \leq u$. If $t \leq p$ and $u \leq p$, then by the choice of p there is a $v \leq t, u$. Then $v \leq u \leq p$, and $v \leq t \leq q$, hence $v \in G$, and $v \leq q$ and $v \leq u \leq r$, as desired.

If $t \leq p$ and $p \leq u$, then $t \leq u \leq r$, so $t \in G$ and $t \leq q, r$, as desired.

If $p \leq t$ and $u \leq p$, then $u \leq r$ implies that $u \in G$, and $u \leq p \leq t \leq q$, as desired.

Finally, if $p \leq t, u$, then $p \leq q, r$ and $p \in G$, as desired. So (1) holds.

(2) is obvious.

Now suppose that D is dense. Choose $q \in D$ such that $q \leq p$. Then $q \in G$, as desired.

E15.3 *Assume the hypothesis of Lemma 15.2. Show that there does not exist a $\mathbb{P}$-generic filter over M which intersects every dense subset of P (not only those which are in M). Hint: Take G generic, and show that $\{p \in P : p \notin G\}$ is dense. Thus in the definition of generic filter, the condition on dense sets being in M is necessary.*

Let D be the set indicated in the hint. Let p be any element of P, and choose incompatible $q, r \leq p$ by the hypothesis of Lemma 15.2. Then it is not true that both $q, r \in G$, as desired; this checks that D is dense. Obviously $G \cap D = \emptyset$, proving the assertion of the exercise.

E15.4 *Show that if $\mathbb{P}$ satisfies the condition of Lemma 15.2, then it has uncountably many dense subsets.*

(Solution due to Josh Sanders) For each $p \in P$ let $p(0), p(1)$ be members of P such that $p(0), p(1) \leq p$ and $p(0) \perp p(1)$; thus $p(0), p(1) < p$. We now define an element p_f for each finite sequence f of 0s and 1s, by recursion on the domain of f. Let $p_\emptyset$ be any element of P. Having defined p_f, let $p_{f0} = p_f(0)$ and $p_{f1} = p_f(1)$. For each $f \in {}^\omega 2$ let $K_f = \{p_{f\restriction m} : m \in \omega\}$. Now if $f, g \in {}^\omega 2$ and $f \neq g$, choose m minimum such that $f(m) \neq g(m)$. Clearly then $p_{f\restriction(m+1)} \in K_f \backslash K_g$. Thus $K_f \neq K_g$ for distinct $f, g \in {}^\omega 2$. For each $f \in {}^\omega 2$ let $D_f = P \backslash K_f$. So $D_f \neq D_g$ for $f \neq g$.

We claim that each D_f is dense; this will prove the statement of the exercise. To see this, take any $q \in P$. If $q \in D_f$, then there is nothing to prove. Suppose that $q \notin D_f$. Thus $q \in K_f$. Say $q = p_{f\restriction m}$. Let $\varepsilon = 1 - f(m)$. Then $p_{(f\restriction m)\varepsilon} \in D_f$ and $p_{(f\restriction m)\varepsilon} \leq q$, as desired.

E15.5 *Assume the hypothesis of Lemma 15.2. Show that there are 2^ω filters which are $\mathbb{P}$-generic over M.*

Let $D_0, D_1, \ldots$ list all of the dense subsets of $\mathbb{P}$ which are in M. We now define elements r_f and p_f in $\mathbb{P}$ for f a finite sequence of 0's and 1's, by recursion on the length of f. Let $p_\emptyset = 1$ and choose $r_\emptyset \in D_0$ so that $r_\emptyset \leq p_\emptyset$. Now suppose that p_f and r_f have been defined, with f having domain $n \in \omega$. Choose p_{f0} and p_{f1} both $\leq r_f$ so that $p_{f0} \perp p_{f1}$. Then choose $r_{f0} \leq p_{f0}$ with $r_{f0} \in D_{n+1}$, and choose $r_{f1} \leq p_{f1}$ with $r_{f1} \in D_{n+1}$.

For any $f \in {}^\omega 2$ let

$$G_f = \{q \in \mathbb{P} : p_{f\restriction n} \leq q \text{ for some } n \in \omega\}.$$

Clearly G_f is $\mathbb{P}$-generic; and there are 2^ω such filters.

E15.6 *Let $\mathbb{P} = (\{1\}, \leq, 1)$. Prove that the collection of all $\mathbb{P}$-names is a proper class.*

Clearly the following two facts hold with no assumption about P.

(1) If σ is a P-name, then so is $\{(\sigma, 1)\}$.

(2) If A is a set of P-names, then $\{(\sigma, 1) : \sigma \in A\}$ is a P-name.

Now let A be the collection of all P-names, and suppose that A is a set. By (2), also $\tau \stackrel{\text{def}}{=} \{(\sigma, 1) : \sigma \in A\}$ is a P-name, and by (1), $\{(\tau, 1)\}$ is a P-name. So $\sigma \stackrel{\text{def}}{=} \{(\tau, 1)\} \in A$. Thus

$$\tau \in \{\tau\} \in \{\{\tau\}, \{\tau, 1\}\} = (\tau, 1) \in \{(\tau, 1)\} = \sigma \in \{\sigma\} \in \{\{\sigma\}, \{\sigma, 1\}\} = (\sigma, 1) \in \tau,$$

contradiction.

E15.7 *Show that $p \Vdash \sigma = \tau$ iff the following two conditions hold.*
(i) For every $(\xi, q) \in \sigma$ and every $r \leq p, q$ one has $r \Vdash \xi \in \tau$.
(ii) For every $(\xi, q) \in \tau$ and every $r \leq p, q$ one has $r \Vdash \xi \in \sigma$.

First assume that $p \Vdash \sigma = \tau$. For (i), suppose that $(\xi, q) \in \sigma$ and $r \leq p, q$. Let G be $\mathbb{P}$-generic over M with $r \in G$. Then also $p, q \in G$, so $\xi_G \in \sigma_G$ and $\sigma_G = \tau_G$. Hence $\xi_G \in \tau_G$, as desired. (ii) is similar.

Second assume the two conditions. To show that $p \Vdash \sigma = \tau$, let G be $\mathbb{P}$-generic over M with $p \in G$. Suppose that $x \in \sigma_G$. Then there is a $(\xi, q) \in \sigma$ such that $q \in G$ and $x = \xi_G$. Choose $r \in G$ such that $r \leq p, q$. By (i) we have $\xi_G \in \tau_G$. This shows that $\sigma_G \subseteq \tau_G$. The other inclusion is similar.

E15.8 *Assume that $\mathbb{P} \in M$, $p, q \in P$, and $p \perp q$. Show that $\{\tau \in M^P : p \Vdash \tau = \check{\emptyset}\}$ is a proper class in M.*

In M we define members τ_α of $M^{\mathbb{P}}$ by recursion, such that $p \Vdash \tau_\alpha = \check{0}$, and such that the ranks increase. let $\tau_0 = \check{0}$. Having defined τ_α, let $\tau_{\alpha+1} = \{(\tau_\alpha, q)\}$. Note that if G is generic and $p \in G$, then $q \notin G$, and so $(\tau_{\alpha+1})_G = \emptyset$; so $p \Vdash \tau_{\alpha+1} = \emptyset$. For λ a limit ordinal, let $\tau_\lambda = \{(\tau_\alpha, q) : \alpha < \lambda\}$. Clearly $p \Vdash \tau_\lambda = \check{0}$.

E15.9 *A forcing order is separative iff it is antisymmetric ($p \leq q \leq p$ implies that $p = q$), and for all p, q, if $p \not\leq q$ then there is an $r \leq p$ such that $r \perp q$. Show that the forcing order of exercise E15.1 is separative.*

If $p \supseteq q \supseteq p$, then $p = q$. Now suppose that $p \not\supseteq q$. Choose $(i, j) \in q \backslash p$. If $i \in \text{dmn}(p)$, then $p \perp q$, as desired. If $i \notin \text{dmn}(p)$, let $k \in J \backslash \{j\}$ and define $r = p \cup \{(i, k)\}$. Then $r \supseteq p$ and $r \perp q$.

E15.10 *Assume that $\mathbb{P} \in M$ is separative and $p, q, r \in P$. Prove that the following two conditions are equivalent:*
(i) $p \Vdash \{(\{(\emptyset, q)\}, r)\} = \check{1}$.
(ii) $p \leq r$ and $p \perp q$.

$\Rightarrow$: Assume that

(1) $$p \Vdash \{(\{(0, q)\}, r)\} = \check{1}.$$

Suppose that $p \not\leq r$. By the definition of separative, choose s such that $s \leq p$ and $s \perp r$. Let G be $\mathbb{P}$-generic over M with $s \in G$. Then $\{(\{(0,q)\},r)\}_G = 0$, contradiction. Thus $p \leq r$. Suppose that p and q are compatible; say $t \leq p,q$. Let G be $\mathbb{P}$-generic over M with $t \in G$. Then $q \in G$, so $\{(0,q)\}_G = \{0\}$. Also $r \in G$, so $\{(\{(0,q)\},r)\}_G = \{\{0\}\} \neq 1$, contradiction.

$\Leftarrow$: Suppose that $p \leq r$ and $p \perp q$. Suppose that G is $\mathbb{P}$-generic over M and $p \in G$. Then $q \notin G$, so $\{(0,q)\}_G = 0$. Also, $r \in G$, so $\{(\{(0,q)\},r)\}_G = \{\{(0,q)\}_G\} = \{0\} = 1$, as desired.

E15.11 *Suppose that $f : A \to M$ with $f \in M[G]$. Show that there is a $B \in M$ such that $f : A \to B$. Hint: let $f = \tau_G$ and $B = \{b : \exists p \in P[p \Vdash \check{b} \in \operatorname{rng}(\tau)]\}$.*

In the hint, the definition of B takes place in M; so $B \in M$. Suppose that b is in the range of f. Thus $\check{b}_G = b \in \operatorname{rng}(\tau_G)$, so we can choose $p \in B$ such that $p \Vdash \check{b} \in \operatorname{rng}(\tau)$. So $b \in B$, as desired.

E15.12 *Assume that $\mathbb{P} \in M$ and α is a cardinal of M. Then for any $\mathbb{P}$-generic G over M the following conditions are equivalent:*

(1) For all $B \in M$, ${}^{\alpha}B \cap M = {}^{\alpha}B \cap M[G]$.
(2) ${}^{\alpha}M \cap M = {}^{\alpha}M \cap M[G]$.

(1)$\Rightarrow$(2): Assume (1). Obviously $\subseteq$ in (2) holds. Now suppose that $f \in {}^{\alpha}M \cap M[G]$. By exercise E15.11 choose $B \in M$ such that $f : \alpha \to B$. So by (1), $f \in M$, as desired.

(2)$\Rightarrow$(1): Assume (2). Then $\subseteq$ in (1) is clear. Suppose that $f \in {}^{\alpha}B \cap M[G]$. Then $f \in {}^{\alpha}M \cap M[G]$ since M is transitive, so $f \in M$ by (2).

E15.13 *Suppose that $\mathbb{P} \in M$ is a forcing order satisfying the condition of Lemma 15.2. Assume that*

$$M = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n \subseteq \cdots \quad (n \in \omega),$$

where $M_{n+1} = M_n[G_n]$ for some G_n which is $\mathbb{P}$-generic over M_n, for each $n \in \omega$. Show that the power set axiom fails in $\bigcup_{n\in\omega} M_n$.

Assume that $R = \bigcup_{n\in\omega} M_n$ does satisfy the power set axiom. Then $R \models \exists y \forall z(z \subseteq P \to z \in y)$. Choose $y \in R$ so that $R \models \forall z(z \subseteq P \to z \in y)$. Say $y \in M_n$. Then $R \models G_n \subseteq P \to z \in y$. By absoluteness, $R \models G_n \subseteq P$. So $R \models G_n \in y$, hence $G_n \in y \in M_n$. This contradicts Lemma 15.2.

E15.14 *Prove that the following conditions are equivalent:*

$$[\![\varphi(\sigma_0,\ldots,\sigma_{m-1}) \leftrightarrow \psi(\sigma_0,\ldots,\sigma_{m-1})]\!] = 1$$
$$[\![\varphi(\sigma_0,\ldots,\sigma_{m-1})]\!] = [\![\psi(\sigma_0,\ldots,\sigma_{m-1})]\!].$$

We omit the parameters $\sigma_0,\ldots,\sigma_{m-1}$. First assume that $[\![\varphi \leftrightarrow \psi]\!] = 1$. Then

$$0 = -[\![\varphi \leftrightarrow \psi]\!]$$

$$\begin{aligned}
&= -[\![(\varphi \to \psi) \wedge (\psi \to \varphi)]\!] \\
&= -[\![\neg(\varphi \wedge \neg\psi) \wedge \neg(\psi \wedge \neg\varphi)]\!] \\
&= -([\![\neg(\varphi \wedge \neg\psi)]\!] \cdot [\![\neg(\psi \wedge \neg\varphi)]\!] \\
&= -(-[\![\varphi \wedge \neg\psi) \cdot -[\![\psi \wedge \neg\varphi]\!]) \\
&= [\![\varphi \wedge \neg\psi]\!] + [\![\psi \wedge \neg\varphi]\!] \\
&= ([\![\varphi]\!] \cdot -[\![\psi]\!]) + ([\![\psi]\!] \cdot -[\![\psi]\!].
\end{aligned}$$

It follows that $[\![\varphi]\!] = [\![\psi]\!]$.

Conversely, if $[\![\varphi]\!] = [\![\psi]\!]$, then we can reverse the above equations to get $-[\![\varphi \leftrightarrow \psi]\!] = 0$, so that $[\![\varphi \leftrightarrow \psi]\!] = 1$.

E15.15 *Prove that* $[\![\sigma = \tau]\!] \cdot [\![\tau = \rho]\!] \leq [\![\sigma = \rho]\!]$.

Suppose not. Then $[\![\sigma = \tau]\!] \cdot [\![\tau = \rho]\!] \cdot -[\![\sigma = \rho]\!] \neq 0$. By Theorem 9.20(i) choose p so that $e(p) \leq [\![\sigma = \tau]\!] \cdot [\![\tau = \rho]\!] \cdot -[\![\sigma = \rho]\!]$. Then $(p \Vdash^* \sigma = \tau)^M$, $(p \Vdash^* \tau = \rho)^M$, and $p \Vdash^* (\neg\sigma = \rho)^M$. Let G be $\mathbb{P}$-generic over M. Then by Theorem 15.19, $\sigma_G = \tau_G$, $\tau_G = \rho_G$, and $\sigma_G \neq \rho_G$, contradiction.

E15.16 *Prove that if ZFC* $\models \varphi$ *then* $[\![\varphi]\!] = 1$, *for any sentence* φ.

Suppose that $[\![\varphi]\!] \neq 1$. Thus $-[\![\varphi]\!] \neq 0$, so by Theorem 13.20(i) choose p so that $e(p) \leq -[\![\varphi]\!] = [\![\neg\varphi]\!]$. Thus $(p \Vdash^* \neg\varphi)^M$. Let G be $\mathbb{P}$-generic over M. Then by Theorem 15.19 we have $\neg\varphi^{M[G]}$.

Solutions to exercises in chapter 16

E16.1 *Show that* $\mathrm{Fn}(\omega_1, 2, \omega_1)$ *preserves cardinals* $\geq \omega_2$.

Clearly $|\mathrm{Fn}(\omega_1, 2, \omega_1)| = \omega_1$, so $\mathrm{Fn}(\omega_1, 2, \omega_1)$ is ω_2-cc. Hence the result follows by Proposition 16.5.

E16.2 *A system* $\langle A_i : i \in I\rangle$ *of sets is an indexed* Δ*-system iff there is a set* r *(again called the root such that* $A_i \cap A_j = r$ *for all distinct* $i, j \in I$. *Note that in an indexed system* $\langle A_i : i \in I\rangle$ *it is possible to have distinct* $i, j \in I$ *such that* $A_i = A_j$*; in fact, all of the* A_i*'s could be equal, in which case the system is already an indexed* Δ*-system.*

Prove that if κ *is an uncountable regular cardinal and* $\langle A_i : i \in I\rangle$ *is a system of finite sets with* $|I| \geq \kappa$, *then there is a* $J \in [I]^\kappa$ *such that* $\langle A_i : i \in J\rangle$ *is an indexed* Δ*-system.*

We may assume that $|I| = \kappa$. Define $i \equiv j$ iff $i, j \in I$ and $A_i = A_j$. Thus $\equiv$ is an equivalence relation on I. If some equivalence class K has κ elements, then $\langle A_i : i \in K\rangle$ is an indexed Δ system with $|K| = \kappa$, as desired; the kernel is A_i for any $i \in K$. Suppose that every equivalence class has size less than κ. Then there are κ equivalence classes. Let $J \subseteq I$ have one element from each equivalence class, and let $\mathscr{A} = \{A_j : j \in J\}$. Then $\mathscr{A}$ is a collection of finite sets, and $|\mathscr{A}| = \kappa$. Hence by Theorem 16.6 let $\mathscr{B} \in [\mathscr{A}]^\kappa$ be a Δ-system. For each $B \in \mathscr{B}$ there is an $i_B \in J$ such that $A_{i_B} = B$. Let $K = \{i_B : B \in \mathscr{B}\}$. Then $\langle A_j : j \in K\rangle$ is an indexed Δ-system, and $|K| = \kappa$ since the function i is clearly one-one.

$\boxed{\text{E16.3}}$ *Here we work only in ZFC (or in a fixed model of it). Suppose that* $(X,<)$ *is a linear order. Let* P *be the set of all pairs* (p,n) *such that* $n\in\omega$ *and* $p\subseteq X\times n$ *is a finite function. Define* $(p,n)\le(q,m)$ *iff* $m\le n$, $\mathrm{dmn}(q)\subseteq\mathrm{dmn}(p)$, $\forall x\in\mathrm{dmn}(q)[p(x)\cap m=q(x)$, *and*

$$\forall x,y\in\mathrm{dmn}(q),\ \textit{if } x<y \textit{ then } p(x)\backslash p(y)\subseteq m.$$

Show that $\mathbb{P}$ *has ccc.*

Suppose that $\mathscr{A}$ is an uncountable subset of P. By the Δ-system theorem, we may assume that $\langle\mathrm{dmn}(p):(p,n)\in\mathscr{A}\rangle$ is a Δ-system, say with root r. We may also assume that $p\restriction r=q\restriction r$ whenever $(p,n),(q,m)\in\mathscr{A}$. Now suppose that $(p,n),(q,m)\in\mathscr{A}$. Let s be the maximum of m and n. Clearly $p\cup q$ is a function, and so $(p\cup q,s)\in P$. We claim that $(p\cup q,s)\le(p,n),(q,m)$, as desired. By symmetry it suffices to show that $(p\cup q,s)\le(q,m)$. Suppose that $x\in\mathrm{dmn}(q)$. Then $(p\cup q)(x)\cap m=q(x)\cap m=q(x)$. If $x,y\in\mathrm{dmn}(q)$ and $x<y$, then $(p\cup q)(x)\backslash(p\cup q)(y)=q(x)\backslash q(y)\subseteq m$.

$\boxed{\text{E16.4}}$ *Continuing exercise E16.3, suppose that we are working in a c.t.m.* M *of ZFC. Let* G *be* $\mathbb{P}$*-generic over* M*. For each* $x\in X$ *let*

$$a_x=\bigcup\{p(x):(p,n)\in G\textit{ for some } n\in\omega,\textit{ with } x\in\mathrm{dmn}(p)\}.$$

Thus $a_x\subseteq\omega$*. Show that if* $x<y$*, then* $a_x\backslash a_y$ *is finite.*

For each $z\in X$ let $D_z=\{(p,n):z\in\mathrm{dmn}(p)\}$. Given any $(q,m)\in P$, if $z\notin\mathrm{dmn}(q)$ clearly $(q\cup\{(z,0)\},m)\in P$, $(q\cup\{(z,0)\},m)\in D_z$, and $(q\cup\{(z,0)\},m)\le(q,m)$. So D_z is dense.

Choose $(p,n)\in D_x\cap G$ and $(q,m)\in D_y\cap G$. Say $(p,n),(q,m)\ge(r,s)\in G$. We claim then that $a_x\backslash a_y\subseteq s$. Let $i\in a_x\backslash a_y$. Say $i\in u(x)$ with $(u,t)\in G$ and $x\in\mathrm{dmn}u$. Say $(u,t),(r,s)\ge(v,z)\in G$. Thus $v(x)\backslash v(y)\subseteq s$. Now $i\in u(x)$, so $i\in v(x)$. Also, $i\notin v(y)$ since $i\notin a_y$. So $i\in s$, as desired.

$\boxed{\text{E16.5}}$ *Continuing exercises E16.3 and E16.4, show that if* $x<y$*, then* $a_y\backslash a_x$ *is infinite. Hint: for each* $i<\omega$ *let*

$$E^i=\{(p,n):x,y\in\mathrm{dmn}(p)\textit{ and } |p(y)\backslash p(x)|\ge i\},$$

and show that E^i *is dense.*

For each $i<\omega$ let

$$E^i=\{(p,n):x,y\in\mathrm{dmn}(p)\text{ and }|p(y)\backslash p(x)|\ge i\}.$$

We claim that E^i is dense. Let (q,n) be given. Wlog $x,y\in\mathrm{dmn}(q)$. Say $\mathrm{dmn}(q)$ is

$$u_0<\cdots<u_j=x<\cdots<u_{m-1}.$$

Let $\mathrm{dmn}(r)=\mathrm{dmn}(q)$, $r(u_t)=q(u_t)$ for $t\le j$; choose $w>n$ with $|w-n|=i$, and let $r(u_t)=q(u_t)\cup(w\backslash n)$ for $j<t$. Then $(q,n)\ge(r,w)\in E^i$, as desired.

Now for any $i \in \omega$ we show that $|a_y \backslash a_x| \geq i$. Choose $(p,n) \in E^i \cap G$. We claim that $p(y)\backslash p(x) \subseteq a_y \backslash a_x$ (as desired). Let $j \in p(y)\backslash p(x)$. So $j \in a_y$, and $j < n$ since $p(y) \subseteq n$. Suppose that $j \in a_x$. Say $y \in q(x)$, with $(q,v) \in G$ and $x \in \mathrm{dmn}(q)$. Say $(p,n),(q,v) \geq (r,s) \in G$. Then $j \in r(x)$ since $j \in q(x)$. Hence $j \in r(x) = p(x) \cap n$, so $j \in p(x)$, contradiction.

E16.6 *Define a set $\mathscr{A}$ of finite sets with $|\mathscr{A}| = \omega$ while there is no Δ-system $\mathscr{B} \in [\mathscr{A}]^\omega$.*

Let $\mathscr{A} = \omega$. Suppose that $\mathscr{B} \in [\mathscr{A}]^\omega$ is a Δ-system, say with kernel r. Choose $m \in \omega$ with $n < m$ for all $n \in r$. Then $m \cap (m+1) = m \neq r$, contradiction.

E16.7 *Let κ be singular. Define a set $\mathscr{A}$ of finite sets with $|\mathscr{A}| = \kappa$ while there is no Δ-system $\mathscr{B} \in [\mathscr{A}]^\kappa$.*

Suppose that κ is uncountable and singular. Let $\langle \lambda_\xi : \xi < \mathrm{cf}(\kappa)\rangle$ be a strictly increasing continuous sequence of cardinals with supremum κ. Let $\mathscr{A} = \{\{\lambda_\xi, \alpha\} : \xi < \mathrm{cf}(\kappa), \lambda_\xi < \alpha < \lambda_{\xi+1}\}$. Clearly $|\mathscr{A}| = \kappa$. Suppose that $\mathscr{B} \in [\mathscr{A}]^\kappa$ is a Δ-system, say with kernel G. If $G = \emptyset$, then for each $\xi < \mathrm{cf}(\kappa)$, $\mathscr{B}$ has at most one member in $[\lambda_\xi, \lambda_{\xi+1})$, and so $|\mathscr{B}| \leq \mathrm{cf}(\kappa) < \kappa$, contradiction. Let $\alpha \in G$. Say $\lambda_\xi \leq \alpha < \lambda_{\xi+1}$. Now B has some member F not in $[\lambda_\xi, \lambda_{\xi+1})$, as otherwise $|\mathscr{B}| \leq \lambda_{\xi+1}$. Then $G \not\subseteq F$, contradiction.

Solutions to exercises in Chapter 17

E17.1 *Show that Theorem 17.2 does not extend to ω_1. Hint: consider $\omega_1 \times \mathbb{Q}$ and $\omega_1^* \times \mathbb{Q}$, both with the lexicographic order, where ω_1^* is ω_1 under the reverse order ($\alpha <^* \beta$ iff $\beta < \alpha$).*

Clearly each of these linear orders is dense with no first or last element. In fact, concerning the first linear order, suppose that $(\xi,r) < (\eta,s)$. If $\xi < \eta$, then $(\xi,r) < (\xi,r+1) < (\eta,s)$. If $\xi = \eta$, then $r < s$, and so

$$(\xi,r) < \left(\xi, \frac{r+s}{2}\right) < (\xi,s).$$

Thus $\omega_1 \times \mathbb{Q}$ is dense. It does not have a first or last element, since if (ξ,r) is given, then $(0,r-1) < (\xi,r) < (\xi,r+1)$. The other order is treated similarly.

We claim that these two linear orders are not isomorphic. Suppose to the contrary that f is an isomorphism of $\omega_1 \times \mathbb{Q}$ onto $\omega_1^* \times \mathbb{Q}$. For all $\xi < \omega_1$ let $f(\xi,0) = (\alpha(\xi), q(\xi))$. Now if $\xi < \eta < \omega_1$, then $\alpha(\xi) \geq \alpha(\eta)$. Hence

(1) There is a $\rho < \omega_1$ such that for all $\xi \in [\rho,\omega_1)$ we have $\alpha(\xi) = \alpha(\rho)$.

In fact, suppose not. Thus for every $\rho < \omega_1$ there is a $\xi \in (\rho,\omega_1)$ such that $\alpha(\xi) < \alpha(\rho)$. Define $\langle \rho_n : n \in \omega\rangle$ by recursion as follows. Let $\rho_0 = 0$. If ρ_n has been defined, choose $\rho_{n+1} > \rho_n$ such that $\alpha(\rho_{n+1}) < \alpha(\rho_n)$. Then $\langle \alpha(\rho_n) : n \in \omega\rangle$ is a strictly decreasing sequence of ordinals, contradiction.

Thus (1) holds. Now $\langle q(\xi) : \rho \leq \xi < \omega_1\rangle$ is a strictly increasing sequence of rationals, contradicting the fact that $|\mathbb{Q}| = \omega$.

E17.2 *For any infinite cardinal κ, consider ${}^\kappa 2$ under the lexicographic order, as for H_α. Show that it is a complete linear order.*

Let $A \subseteq {}^\kappa 2$; we want to find a lub for A. We define $f \in {}^\kappa 2$ by recursion, as follows: for any $\alpha < \kappa$,

$$f(\alpha) = \begin{cases} 1 & \text{if there is a } g \in A \text{ such that } g \restriction \alpha = f \restriction \alpha \text{ and } g(\alpha) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $g \in A$ and $f < g$. Then $f \restriction \chi(f,g) = g \restriction \chi(f,g)$, $f(\chi(f,g)) = 0$, and $g(\chi(f,g)) = 1$. This contradicts the definition of f. (We are using $\chi(f,g)$ in the same way as for H_α.) Thus $g \leq f$ for all $g \in A$.

Suppose that h is an upper bound for A and $h < f$. Thus $h \restriction \chi(h,f) = f \restriction \chi(h,f)$, $h(\chi(h,f)) = 0$, and $f(\chi(h,f)) = 1$. By the definition of f, there is a $g \in A$ such that $\chi(g,f) = \chi(h,f)$ and $g(\chi(g,f) = 1$. But then $\chi(g,h) = \chi(h,f)$ too, and it follows that $h < g$. This contradicts h being an upper bound for A.

Thus f is the desired lub of A.

E17.3 *Suppose that κ and λ are cardinals, with $\omega \leq \lambda \leq \kappa$. Let μ be minimum such that $\kappa < \lambda^\mu$. Take the lexicographic order on ${}^\mu\lambda$, as for H_α. Show that this gives a dense linear order of size λ^μ with a dense subset of size κ.*

Note that obviously $\mu \leq \kappa$. Clearly ${}^\mu\lambda$ is a linear order. Let

$$D = \{f \in {}^\mu\lambda : \text{ there is a } \xi < \mu \text{ such that } f(\xi) = 1 \text{ and } f(\eta) = 0 \text{ for all } \eta \in (\xi, \mu)\}.$$

Clearly $|D| \leq \kappa$. We can show that ${}^\mu\lambda$ is dense, and D is dense in it, by finding an element of D between any two elements $f < g$ of ${}^\mu\lambda$. For brevity let $\alpha = \chi(f,g)$. Now define h by:

$$h(\beta) = \begin{cases} f(\beta) & \text{if } \beta \leq \alpha; \\ f(\alpha+1)+1 & \text{if } \beta = \alpha+1; \\ 1 & \text{if } \beta = \alpha+2; \\ 0 & \text{if } \beta \in (\alpha+2, \mu). \end{cases}$$

Clearly h is as desired.

If $|D| < \kappa$, we can simply add κ elements to it to satisfy the requirement that $|D| = \kappa$.

E17.4 *Show that $\mathscr{P}(\omega)$ under $\subseteq$ contains a chain of size 2^ω. Hint: remember that $|\omega| = |\mathbb{Q}|$.*

Let $f : \omega \to \mathbb{Q}$ be a bijection. For each real number r, let $a_r = f^{-1}[\{s \in \mathbb{Q} : s < r\}]$. Clearly $r < t$ implies that $a_r \subseteq a_t$; and in fact $a_r \subset a_t$ since there is a rational number u such that $r < u < t$, and so $f^{-1}(u) \in a_t \backslash a_r$. Now a is an isomorphic embedding by Proposition 4.14.

E17.5 *A subset S of a linear order L is weakly dense iff for all $a, b \in L$, if $a < b$ then there is an $s \in S$ such that $a \leq s \leq b$. Show that the following conditions are equivalent for any cardinals κ, λ such that $\omega \leq \kappa \leq \lambda$:*

(i) There is a linear order of size λ with a weakly dense subset of size κ.

(ii) $\mathscr{P}(\kappa)$ has a chain of size λ.

$\Rightarrow$: Clearly we may assume that $\kappa < \lambda$. Assume that L is a linear order with a weakly dense subset D of size κ. Let f be a bijection from D to κ. For each $r \in L \backslash D$ let

$a_r = f[\{d \in D : d < r\}]$. thus $a_r \in \mathscr{P}(\kappa)$, and $a_r \subseteq a_s$ if $r, s \in L\backslash D$ with $r < s$. Also, by the weak denseness of D, there is a $d \in D$ such that $r \leq d \leq s$. Since $r, s \in L\backslash D$, we have $r < d < s$, and so $d \in a_s\backslash a_r$. Thus $r < s$ implies that $a_r \subset a_s$. The other implication follows from this one. Note that $|L\backslash D| = \lambda$.

$\Leftarrow$: Again we may assume that $\kappa < \lambda$. Let L be a chain in $\mathscr{P}(\kappa)$ of size λ. For each $\alpha < \kappa$ let

$$x_\alpha = \bigcup\{a : a \in L,\ \alpha \notin a\}.$$

(1) If $\alpha, \beta < \kappa$, then $x_\alpha \subseteq x_\beta$ or $x_\beta \subseteq x_\alpha$.

For, suppose that $\gamma \in x_\alpha\backslash x_\beta$ and $\delta \in x_\beta\backslash x_\alpha$. Say $\gamma \in a \in L$ with $\alpha \notin a$ and $\delta \in b \in L$ with $\beta \notin b$. Also, $\beta \in a$ and $\alpha \in b$, since $\gamma \notin x_\beta$ and $\delta \notin x_\alpha$. Say by symmetry $a \subseteq b$. Then $\beta \in b$, contradiction.

(2) If $\alpha \in \kappa$ and $a \in L$, then $a \subseteq x_\alpha$ or $x_\alpha \subseteq a$.

For, suppose that $\alpha \in \kappa$ and $a \in L$. If $\alpha \notin a$, then $a \subseteq x_\alpha$. Suppose that $\alpha \in a$. If $\alpha \notin b \in L$, we then must have $b \subset a$, since a and b are comparable. Thus $x_\alpha \subseteq a$ in this case. So (2) holds.

By (1) and (2), the set $M \stackrel{\text{def}}{=} L \cup \{x_\alpha : \alpha \in \kappa\}$ is a chain. Its size is clearly λ. We claim that $\{x_\alpha : \alpha < \kappa\}$ is weakly dense in it, which will finish the proof (expanding $\{x_\alpha : \alpha < \kappa\}$ to a set of size κ if necessary). For, suppose that $a, b \in L$ and $a \subset b$. Choose $\alpha \in b\backslash a$. Then clearly $a \subseteq x_\alpha$. Also, $x_\alpha \subseteq b$ by the proof of (2).

[E17.6] *Suppose that L_i is a linear order with at least two elements, for each $i \in \omega$. Let $\prod_{i\in\omega} L_i$ have the lexicographic order. Show that it is not a well-order.*

For each $i \in \omega$ let $a_i < b_i$ be elements of L_i. For each $j \in \omega$ define f^j by setting, for each $i \in \omega$,

$$f^j(i) = \begin{cases} a_i & \text{if } i \leq j \text{ or } j+1 < i; \\ b_i & \text{if } i = j+1. \end{cases}$$

Then $f^0 > f^1 > \cdots$.

[E17.7] *Suppose that L is a ccc dense linear order. Show that L has a dense subset of size $\leq \omega_1$. Hint: let $\prec$ be a well-order of L, and let*

$$N = \{p \in L : \textit{ there is an open set } U \textit{ in } L \textit{ such that } p \textit{ is the } \prec\textit{-first element of } U\},$$

and show that N is dense in L and has size at most ω_1.

We may assume that $|L| \geq \omega_1$. Let $\prec$ be a well-order of L, and define

$$N = \{p \in L : \text{ there is an open } U \subseteq L \text{ such that } p \text{ is the } \prec\text{-first element of } U\}.$$

Then N is dense in L. For, if $a, b \in L$ and $a < b$, let p be the $\prec$-first element of (a, b). Then $p \in N$ and $a < p < b$, as desired.

Thus it suffices to show that $|N| \leq \omega_1$. Suppose, to the contrary, that $|N| > \omega_1$.

For each $p \in N$ let

$$I_p = \bigcup\{(a,b) : a, b \in L,\ a < b,\ \text{and } p \text{ is the } \prec\text{-first element of } (a,b)\}.$$

Then

(1) For each $p \in N$, the set I_p is a nonempty convex open subset of L.

In fact, it is clearly nonempty and open. If $u < w < v$ with $u, v \in I_p$, choose $a, b, c, d \in L$ such that $u \in (a,b)$, $v \in (c,d)$, and p is the $\prec$-first element of both (a,b) and (c,d). Clearly then p is the $\prec$-first element of $(\min(a,c), \max(b,d))$, and $w \in (\min(a,c), \max(b,d))$, so $w \in I_p$. Thus (1) holds.

(2) If $p \prec q$, then $I_p \cap I_q = \emptyset$ or $I_q \subset I_p$.

In fact, suppose that $I_p \cap I_q \neq \emptyset$; say $x \in I_p \cap I_q$. Choose $a, b, c, d \in L$ such that $x \in (a,b)$, p is the $\prec$-first element of (a,b), $x \in (c,d)$, and q is the $\prec$-first element of (c,d). Since $x \in (a,b) \cap (c,d)$, it follows that $(a,b) \cup (c,d) = (\min(a,c), \max(b,d))$, and p is the $\prec$-first element of this interval. Thus

$(\min(a,c), \max(b,d)) \subseteq I_p$.

Now let t be any member of I_q. Say that $t \in (e,f)$, with q the $\prec$-first element of (e,f). Now q is also the $\prec$-first element of (c,d), so $(c,d) \cup (e,f) = (\min(c,e), \max(d,f))$, and q is the $\prec$-first element of this interval. q is also a member of $(\min(a,c), \max(b,d))$, so $(\min(a,c), \max(b,d)) \cup (\min(c,e), \max(d,f)) = (\min(a,c,e), \max(b,d,f))$, and p is the $\prec$-first element of this interval. Since t is in this interval, it follows that $t \in I_p$. This proves (2).

Let $\langle p_\alpha : \alpha < \omega_2 \rangle$ list the first ω_2 elements of N under $\prec$. Let $M = \{I_p : p \in N\}$. Now we construct by recursion two sequences $\langle A_\alpha : \alpha < \omega_1 \rangle$ and $\langle \beta_\alpha : \alpha < \omega_1 \rangle$. Let $\beta_0 = 0$, and let A_0 be a subset of ω_2 with $0 \in A_0$ such that $\{I_{p_\gamma} : \gamma \in A_0\}$ is a maximal collection of pairwise disjoint members of M. Thus A_0 is countable. Suppose now that we have constructed A_γ and β_γ for all $\gamma < \alpha$, where $\alpha < \omega_1$, so that each A_γ is a countable subset of ω_2. Let β_α be any ordinal less than ω_2 and greater than each ordinal in $\bigcup_{\gamma<\alpha} A_\gamma$. Then let A_α be maximal subject to the following conditions:

(3) $A_\alpha \subseteq [\beta_\alpha, \omega_2)$.

(4) $\beta_\alpha \in A_\alpha$.

(5) $\{I_{p_\gamma} : \gamma \in A_\alpha\}$ is pairwise disjoint.

This finishes the construction. Now take any δ greater than each ordinal in $\bigcup_{\alpha<\omega_1} A_\alpha$, and with $p_\varepsilon < p_\delta$ for all $\varepsilon \in \bigcup_{\alpha<\omega_1} A_\alpha$. For each $\alpha < \omega_1$ there is a $\gamma_\alpha \in A_\alpha$ such that $I_{p_\delta} \cap I_{p_{\gamma_\alpha}} \neq \emptyset$. Hence by (2) we have $I_{p_\delta} \subset I_{p_{\gamma_\alpha}}$. Hence if $\alpha < \beta < \omega_1$ then $I_{p_{\gamma_\beta}} \subset I_{p_{\gamma_\alpha}}$. For brevity let $q(\alpha) = p_{\gamma_\alpha}$ for each $\alpha < \omega_1$. So $I_{q(\beta)} \subset I_{q(\alpha)}$ if $\alpha < \beta < \omega_1$. Now for each limit ordinal $\mu < \omega_1$ choose $\xi(\mu), \eta(\mu), \rho(\mu)$ such that $\xi(\mu) \in I_{q(\mu)} \backslash I_{q(\mu+1)}$, $\eta(\mu) \in I_{q(\mu+1)} \backslash I_{q(\mu+2)}$, and $\rho(\mu) \in I_{q(\mu+2)} \backslash I_{q(\mu+3)}$. Then $\xi(\mu), \eta(\mu), \rho(\mu)$ are distinct elements all in $I_{q(\mu)} \backslash I_{q(\mu+3)}$. Hence two of them, say ε_μ and θ_μ, with $\varepsilon_\mu < \theta_\mu$, are on the same side of $I_{q(\mu+3)}$, say the left side. It follows that $\langle(\varepsilon_\mu, \theta_\mu) : \mu \text{ limit} < \omega_1\rangle$ is a system of pairwise disjoint sets, contradiction.

E17.8 *Let $\langle L_i : i \in I\rangle$ be a system of linear orders, with I itself an ordered set. Show that if each L_i is dense without first or last elements, then also $\sum_{i\in I} L_i$ is dense without first or last elements.*

Suppose that $(i, a) < (j, b)$. If $i < j$, let c be an element of L_i such that $a < c$. Then $(i, a) < (i, c) < (j, b)$. If $i = j$, let c be an element of L_i such that $a < c < b$. Then $(i, a) < (i, c) < (i, b)$. Thus $\sum_{i\in I} L_i$ is dense. Given (i, a), choose $c, d \in L_i$ with $c < a < d$. Then $(i, c) < (i, a) < (i, d)$. Thus $\sum_{i\in I} L_i$ is does not have a first or last element.

E17.9 *Let κ be any infinite cardinal number. Let L_0 be a linear order similar to $\omega^*+\omega+1$; specifically, let it consist of a copy of $\mathbb{Z}$ followed by one element a greater than every integer, and let L_1 be a linear order similar to $\omega^* + \omega + 2$; say it consists of a copy of $\mathbb{Z}$ followed by two elements $a < b$ greater than every integer. For any $f \in {}^\kappa 2$ let*

$$M_f = \sum_{\alpha<\kappa} L_{f(\alpha)}.$$

Show that if $f, g \in {}^\kappa 2$ then M_f and M_g are not isomorphic.

Conclude that there are exactly 2^κ linear orders of size κ up to isomorphism.

Let $f, g \in {}^\kappa 2$. We assume that M_f is isomorphic to M_g and show that $f = g$. Let F be an isomorphism of M_f onto M_g. Clearly

(1) For all $x \in M_f \cup M_g$ the following conditions are equivalent:

(a) x does not have an immediate predecessor.

(b) $x = (a, \xi)$ for some $\xi < \kappa$.

Now for any $x \in M_f$, x has an immediate predecessor iff $F(x)$ has an immediate predecessor, as is easily seen. We claim then that $F(a, \xi) = (a, \xi)$ for all $\xi < \kappa$. We prove this by transfinite induction. Suppose that $F(a, \eta) = (a, \eta)$ for all $\eta < \xi$. Now $F(a, \xi)$ does not have an immediate predecessor, so by (1) it has the form (a, ρ) for some ρ. We cannot have $\rho < \xi$, since this would contradict F being one-one, by the supposition. If $\xi < \rho$, then we would have $F^{-1}(a, \xi) < F^{-1}(a, \rho) = (a, \xi)$, again contradicting the inductive assumption. Thus $F(a, \xi) = (a, \xi)$, finishing the inductive proof.

Next we claim

(2) For any $x \in M_f$ the following conditions are equivalent:

(a) x does not have an immediate predecessor, but it has an immediate successor y which in turn does not have an immediate successor.

(b) $x = (a, \xi)$ for some ξ such that $f(\xi) = 1$.

This is obvious, and a similar condition for M_g holds.

Now the property given in (2)(a) is preserved under isomorphisms, so by the above, for any $\xi < \kappa$,

$$\begin{aligned} f(\xi) = 1 \quad &\text{iff} \quad (a, \xi) \text{ satisfies (2)(a)} \\ &\text{iff} \quad F(a, \xi) \text{ satisfies (2)(a)} \\ &\text{iff} \quad g(\xi) = 1. \end{aligned}$$

Thus $f = g$, as desired.

The required conclusion in the exercise is clear.

E17.10 *Let κ be an uncountable cardinal. Let L_0 be a linear order similar to $\eta+1+\eta\cdot\omega_1^*$; specifically consisting of a copy of the rational numbers in the interval $(0,1]$ followed by $\mathbb{Q}\times\omega_1$, where $\mathbb{Q}\times\omega_1$ is ordered as follows: $(r,\alpha)<(s,\beta)$ iff $\alpha>\beta$, or $\alpha=\beta$ and $r<s$. Let L_1 be a linear order similar to $\eta\cdot\omega_1+1+\eta\cdot\omega_1^*$; specifically, we take L_1 to be the set*

$$\{(q,\alpha,0): q\in\mathbb{Q}, \alpha<\omega_1\}\cup\{(0,0,1)\}\cup\{(q,\alpha,2): q\in\mathbb{Q}, \alpha<\omega_1\},$$

with the following ordering:

$$\begin{aligned}
&(q,\alpha,0)<(r,\beta,0) \quad \textit{iff} \quad \alpha<\beta, \text{ or } \alpha=\beta \text{ and } q<r;\\
&(q,\alpha,0)<(0,0,1)<(r,\beta,2) \text{ for all relevant } q,r,\alpha,\beta;\\
&(q,\alpha,2)<(r,\beta,2) \quad \textit{iff} \quad \alpha>\beta, \text{ or } \alpha=\beta \text{ and } q<r.
\end{aligned}$$

For each $f\in{}^\kappa 2$ let

$$M_f=\sum_{\alpha<\kappa}L_{f(\alpha)}.$$

Show that each M_f is a dense linear order without first or last elements, and if $f,g\in{}^\kappa 2$ and $f\neq g$, then M_f and M_g are not isomorphic.

Conclude that for κ uncountable there are exactly 2^κ dense linear orders without first or last elements, of size κ, up to isomorphism.

For L_0, if q is an element in the initial $(0,1]$ part, then $q-1<q$; so L_0 has no least element. If (q,α) is an element in the second part, then $(q,\alpha)<(q+1,0)$; so L_0 has no greatest element. For denseness, suppose that $x<y$ in L_0. If both are in the first part, clearly there is a z such that $x<z<y$. Suppose that x is in the first part and $y=(q,\alpha)$ is in the second part. Then $x<(q-1,\alpha)<y$. Finally, suppose that $x=(r,\beta)$ and $y=(s,\gamma)$ are both in the second part. If $\beta>\gamma$, then $x<(r+1,\beta)<y$. If $\beta=\gamma$, then $r<s$ and the desired element is clear.

For L_1, given any element $(q,\alpha,0)$ in the first part, we have $(q-1,\alpha,0)<(q,\alpha,0)$, so L_1 does not have a least element. If $(q,\alpha,2)$ is any element in the third part, then $(q,\alpha,2)<(q+1,\alpha,2)$, so L_2 does not have a greatest element. Suppose that $x<y$ in L_1. We can consider several cases.

Case 1. $x=(q,\alpha,0)$, $y=(r,\beta,0)$, and $\alpha<\beta$. Then $x<(q+1,\alpha,0)<y$.

Case 2. $x=(q,\alpha,0)$, $y=(r,\alpha,0)$. Then $q<r$; so with $q<t<r$ we have $x<(t,\alpha,0)<y$.

Case 3. $x=(q,\alpha,0)$, $y=(0,0,1)$. Then $x<(q+1,\alpha,0)<y$.

Case 4. $x=(q,\alpha,0)$, $y=(r,\beta,2)$. Then $x<(0,0,1)<y$.

Case 5. $x=(0,0,1)$, $y=(q,\alpha,2)$. Then $x<(q-1,\alpha,2)<y$.

Case 6. $x=(q,\alpha,2)$, $y=(r,\beta,2)$, and $\alpha>\beta$. Then $x<(q+1,\alpha,2)<y$.

Case 7. $x=(q,\alpha,2)$, $y=(r,\alpha,2)$. Thus $q<r$. Then with $q<t<r$ we have $x<(t,\alpha,2)<y$.

Thus L_1 is dense.

Before beginning the main part of the exercise, we note the following facts.

(1) Each q in the first part of L_0, except for the final 1, has character (ω, ω).

(2) The final 1 has character (ω, ω_1). A strictly decreasing sequence with limit 1 is

$$\langle (0, \alpha) : \alpha < \omega_1 \rangle.$$

(3) Every element in the second part of L_0 has character (ω, ω).

(4) Every element in the first part of L_1 has character (ω, ω).

(5) $(0, 0, 1)$ has character (ω_1, ω_1).

In fact, a strictly increasing sequence with limit $(0, 0, 1)$ is

$$\langle (0, \alpha, 0) : \alpha < \omega_1 \rangle.$$

A strictly decreasing sequence with limit $(0, 0, 1)$ is

$$\langle (0, \alpha, 2) : \alpha < \omega_1 \rangle.$$

Now we suppose that $f \in {}^{\kappa}2$. We give some properties of M_f:

(6) The character of an element (x, ξ) of M_f is equal to the character of x in $L_{f(\xi)}$.

This is true because $L_{f(\xi)}$ is a convex set in M_f, i.e., if $u < v < w$ with $u, w \in L_{f(\xi)}$, then $v \in L_{f(\xi)}$. Also, the fact that $L_{f(\xi))}$ does not have a least or greatest element is needed to see (6).

(7) For each $\xi < \kappa$ there is a unique element of M_f of the form (x_ξ, ξ) which has character (ω, ω_1) if $f(\xi) = 0$ and has character (ω_1, ω_1) if $f(\xi) = 1$; all other elements of the form (y, ξ) have character (ω, ω).

Now we can treat the main part of the exercise. Suppose that $f, g \in {}^{\kappa}2$ and M_f is isomorphic to M_g; say F is an isomorphism. The sequence

$$\langle F(x_\xi, \xi) : \xi < \kappa \rangle,$$

with x_ξ given in (7), is an increasing sequence of elements of M_g such that all other elements of M_g have character (ω, ω). In M_g there is only one sequence of order type κ consisting of elements which do not have character (ω, ω), by (7) for M_g. Hence $F(x_\xi, \xi) = (y_\xi, \xi)$, where y_ξ is defined for M_g like x_ξ was for M_f in (7). But x_ξ and y_ξ then have the same characters, and so $f(\xi) = g(\xi)$. Thus $f = g$.

The final statement of the exercise is clear.

Solutions to exercises in Chapter 18

E18.1 *Let κ be an uncountable regular cardinal, and suppose that there is a κ-Aronszajn tree. Show that there is one which is a normal subtree of ${}^{<\kappa}2$. Hint: for each $\alpha < \kappa$ let g_α be an injection of $Lev_\alpha(T)$ into ${}^{|\mathrm{Lev}_\alpha(T)|}2$ and glue these maps together.*

Let T be a κ-Aronszajn tree. By Theorem 18.7 we may assume that it is well-pruned. For $s \in T$ and $\alpha < \mathrm{ht}(s,T)$ let s^α be the unique element of height α below s. For each $\alpha < \kappa$, let g_α be an injection from $\mathrm{Lev}_\alpha(T)$ into ${}^{|\mathrm{Lev}_\alpha(T)|}2$.

We define by recursion sequences $\langle \mu_\alpha : \alpha < \kappa \rangle$ and $\langle F_\alpha : \alpha < \kappa \rangle$. Let $\mu_0 = |\mathrm{Lev}_0(T)|$, and let $F_0 = g_0$. Now suppose that μ_α, F_α have been defined so that the following conditions hold:

(1_α) $\mu_\alpha < \kappa$.

(2_α) F_α is a function with domain $\bigcup_{\beta \le \alpha} \mathrm{Lev}_\beta(T)$.

(3_α) for all $\beta, \gamma < \alpha$, if $\beta < \gamma$ then $\mu_\beta \le \mu_\gamma$ and $F_\beta \subseteq F_\gamma$.

(Clearly these conditions hold for $\alpha = 0$.) Now let $\mu_{\alpha+1} = \mu_\alpha + |\mathrm{Lev}_{\alpha+1}(T)|$ (ordinal addition). Let $F_{\alpha+1}$ be the extension of F_α such that for every $t \in \mathrm{Lev}_\alpha(T)$, every immediate successor s of t, and every $\beta < \mu_{\alpha+1}$,

$$(F_{\alpha+1}(s))(\beta) = \begin{cases} (F_\alpha(t))(\beta) & \text{if } \beta < \mu_\alpha, \\ (g_\alpha(s))(\xi) & \text{if } \beta = \mu_\alpha + \xi \text{ with } \xi < |\mathrm{Lev}_{\alpha+1}(T)|. \end{cases}$$

Clearly $(1_{\alpha+1})$–$(3_{\alpha+1})$ hold.

Now suppose that α is a limit ordinal and μ_β, F_β have been defined for all $\beta < \alpha$ so that (1_β)–(3_β) hold. Let $\nu = \bigcup_{\beta<\alpha} \mu_\beta$, $G = \bigcup_{\beta<\alpha} F_\beta$, and set $\mu_\alpha = \nu + |\mathrm{Lev}_\alpha(T)|$. Let F_α be the extension of G such that for every $s \in \mathrm{Lev}_\alpha(T)$ and every $\beta < \mu_\alpha$,

$$(F_\alpha(s))(\beta) = \begin{cases} (G(s^\gamma))(\beta) & \text{if } \beta < \mu_\gamma \text{ with } \gamma < \alpha, \\ (g_\alpha(s))(\xi) & \text{if } \beta = \nu + \xi \text{ with } \xi < |\mathrm{Lev}_\alpha(T)|. \end{cases}$$

Clearly (1_α)–(3_α) hold.

Let $H = \bigcup_{\alpha<\kappa} F_\alpha$. Clearly

(4) If $s \in \mathrm{Lev}_\alpha(T)$, then $H(s) \in {}^{\mu_\alpha}2$,

(5) If $u < s$ then $H(u) \subset H(s)$.

We prove (5) by induction on the level of s. It is vacuously true for level 0. Now suppose inductively that s has level $\alpha+1$. Say that t is the immediate predecessor of s. Let γ be the level of u. Then for any $\beta < \mu_\gamma$ we have

$$(H(s))(\beta) = (F_{\alpha+1}(s))(\beta) = (F_\alpha(t))(\beta) = (H(t))(\beta) = (H(u))(\beta).$$

Finally, suppose inductively that s has limit level α. Then for any $\beta < \gamma$ we have

$$(H(s))(\beta) = (F_\gamma(s^\gamma))(\beta) = (H(u))(\beta).$$

Hence (5) holds.

(6) If $s, t \in T$ have the same height and $s \neq t$, then $H(s) \neq H(t)$.

We prove (6) by induction on the common height α of s and t. If $\alpha = 0$ the conclusion is clear since g_0 is one-one. Suppose inductively that they both have height $\alpha+1$. Let s', t'

be their immediate predecessors. If $s' \neq t'$, then $H(s') \neq H(t')$, so $H(s) \neq H(t)$ by (5). Suppose that $s' = t'$. Then $H(s) \neq H(t)$ since g_α is one-one. Finally, suppose inductively that α is limit. Then $H(s) \neq H(t)$ since g_α is one-one. So (6) holds.

Now let $T' = \{h \in {}^{<\kappa}2 : h \subseteq H(s) \text{ for some } s \in T\}$. We claim that T' is as desired. Clearly it is a normal subtree of ${}^{<\kappa}2$. Now consider any $\alpha < \kappa$. Choose β minimum such that $\alpha \leq \mu_\beta$.

(7) If $h \in T'$ with $\text{dmn}(h) = \alpha$, then there is an $s \in T$ of height β such that $h \subseteq H(s)$.

In fact, choose $t \in T$ such that $h \subseteq H(t)$. Then $\text{dmn}(H(t)) \geq \alpha \geq \mu_\beta$, so t has height $\geq \beta$. Let $s \in T$ of height β with $s \leq t$. Then $H(s), h \subseteq H(t)$, so $h \subseteq H(s)$, as desired.

It follows from (7) that each level of T' has size less than κ. From (5) and (7) it follows that T' does not have a chain of size κ.

E18.2 *Do exercise E18.1 for κ-Suslin trees.*

We use the same construction as in exercise 18.1. Thus our new tree T' does not have any chain of size κ. Suppose that A is an antichain of size κ. For each $a \in A$ choose $s_a \in T$ such that $a \subseteq H(s_a)$. Since T is a Suslin tree, choose distinct $a, b \in A$ such that s_a and s_b are comparable. Say $s_a \leq s_b$. Then $H(s_a) \subseteq H(s_b)$. Since $a, b \subseteq H(s_b)$, it follows that a and b are comparable, contradiction.

E18.3 *Suppose that T and T' are κ-Aronszajn trees. Define an order $<$ on $T \times T'$ by $(s, s') < (t, t')$ iff $s < t$ and $s' < t'$. Show that $T \times T'$ is not a tree.*

Clearly $T \times T'$ is a partial order. Let $x_0 < x_1 < x_2$ in T and $y_0 < y_1 < y_2$ in T'. Then $(x_1, y_0) < (x_2, y_2)$ and $(x_0, y_1) < (x_2, y_2)$, but (x_1, y_0) and (x_0, y_1) are incomparable.

E18.4 *Suppose that T and T' are κ-Aronszajn trees. Let*

$$T \times' T' = \bigcup_{\alpha<\kappa} Lev_\alpha(T) \times Lev_\alpha(T');$$

$$(s, s') < (t, t') \text{ iff } (s, s'), (t, t') \in T \times' T', \ s < t, \text{ and } s' < t';$$

Show that $(T \times' T', <)$ is a κ-Aronszajn tree.

Clearly $T \times' T'$ is a partial order. Suppose that $(s, t) \in T \times' T'$. Say that s and t have height α, and let $\langle a_\xi : \sigma < \alpha\rangle$ and $\langle b_\xi : \sigma < \alpha\rangle$ are the systems of predecessors of s, t respectively. Clearly $\langle (a_\xi, b_\xi) : \xi < \alpha\rangle$ is the system of predecessors of (s, t). So we have a tree. Each level α has size $|\text{Lev}_\alpha(T) \times \text{Lev}_\alpha(T')| < \kappa$; so $T \times' T'$ is a κ-tree. If $\langle (x_\xi, y_\xi) : \xi < \beta\rangle$ is a chain in $T \times' T'$, then $\langle x_\xi : \xi < \beta\rangle$ is a chain in T, and so $\beta < \kappa$. Thus $T \times' T'$ is a κ-Aronszajn tree.

E18.5 *Assume that κ is regular and uncountable. Suppose that T is a κ-Suslin tree. With the order on $T \times' T$ given in exercise E18.4, show that $T \times' T$ is not a κ-Suslin tree. Hint: first show that for every $\alpha < \kappa$ there is an element s of T at level α such that there are incomparable $t, u > s$.*

First we claim

(1) For every $\alpha < \kappa$ there is an s of height α such that there are incomparable $t, u > s$.

In fact, otherwise we get an $\alpha < \kappa$ such that for every s of height α, the set $\{t \in T : s < t\}$ is a chain. Since all chains have size less than κ, for each s of level α choose β_s greater than the level of all t with $s < t$. Let $\gamma < \kappa$ be such that $\beta_s < \gamma$ for all s of level α. Let β be any element of T of level γ, and let s be its predecessor at level α. But $\gamma > \beta_s$, contradiction. Thus (1) holds.

Now we construct elements $s_\alpha, t_\alpha, u_\alpha$ for $\alpha < \kappa$ by recursion. Suppose that they have been constructed for all $\beta < \alpha$. Let

$$\gamma = \max(\sup\{\text{ht}(t_\beta, T) : \beta < \alpha\}, \sup\{\text{ht}(u_\beta, T) : \beta < \alpha\}) + 1.$$

By (1), choose s_α of height γ with incomparable $t_\alpha, u_\alpha > s_\alpha$ of some common level $> \gamma$.

We claim that $\langle (t_\alpha, u_\alpha) : \alpha < \kappa \rangle$ is an antichain in $T \times' T$. For, suppose that $\beta < \alpha$ and $(t_\beta, u_\beta) < (t_\alpha, u_\alpha)$. So $t_\beta < t_\alpha$ and $u_\beta < u_\alpha$. Since $t_\beta, s_\alpha < t_\alpha$, we have $t_\beta < s_\alpha$ (since s_α is at a higher level than t_β). Similarly, $u_\beta < s_\alpha$. So t_β and u_β are comparable, contradiction.

[E18.6] *A tree T is everywhere branching iff every $t \in T$ has at least two immediate successors. Show that every everywhere branching tree has at least 2^ω branches.*

We define a branch b_f for every $f \in {}^\omega 2$ by defining elements a_h for every $h \in {}^{<\omega}2$ by recursion on $\text{dmn}(h)$. Let $a_\emptyset$ be a root of the tree. Suppose that a_h has been defined for every $h \in {}^n 2$. For each $h \in {}^n 2$, let a_{h0} and a_{h1} be two immediate successors of a_h. This finishes the definition of the a_h's. Now let b_f be an extension of $\langle a_{f \restriction n} : n \in \omega \rangle$ to a branch. Clearly this is as desired.

[E18.7] *Show that the hypothesis that all levels are finite is necessary in König's theorem.*

For each $n \in \omega$ let $f_n \in {}^{n+1}\omega$ be any function such that $f_n(0) = n$, and let T be the tree consisting of all $g \in {}^{<\omega}\omega$ such that $g \subseteq f_n$ for some n. Then two elements $g, h \in T$ are comparable iff they are both contained in the same f_n. If C is a maximal chain in T, it must be a subset of some f_n, and hence is finite.

[E18.8] *Show that if κ is singular with $\text{cf}(\kappa) = \omega$, then there is no κ-Aronszajn tree with all levels finite.*

Let $\langle \alpha_n : n \in \omega \rangle$ be a strictly increasing sequence of ordinals with supremum κ. Suppose that T is a κ-Aronszajn tree with all levels finite. Define

$$T' = \{t \in T : \text{there is an } n \in \omega \text{ such that } t \text{ has height } \alpha_n\}.$$

Then T', with the order induced by T, is a tree of height ω with all levels finite. Hence by König's theorem it has an infinite branch B. Let $B' = \{t \in T : t \leq s$ for some $s \in T'$. Then B' is a branch in T of size κ, contradiction. □

[E18.9] *Prove that if κ is singular and there is a $\text{cf}(\kappa)$-Aronszajn tree, then there is a κ-Aronszajn tree with all levels of power less than $\text{cf}(\kappa)$.*

Let T be a cf(κ)-Aronszajn tree. Let $\langle\mu_\alpha : \alpha < \mathrm{cf}(\kappa)\rangle$ be a strictly increasing continuous sequence of cardinals with supremum κ, and with $\mu_0 = 0$. We define

$$T' = \{(t,\beta) : \text{there is an } \alpha < \mathrm{cf}(\kappa) \text{ such that } t \in \mathrm{Lev}_\alpha(T) \text{ and } \mu_\alpha \leq \beta < \mu_{\alpha+1};$$
$$(t,\beta) < (t',\beta') \text{ iff } t < t', \text{ or } t = t' \text{ and } \beta < \beta'.$$

Clearly this gives a partial order. To show that it is a tree, suppose that $(t,\beta) \in T'$. We define a function f from β into the set of predecessors of (t,β) as follows. Let $\mathrm{ht}(t) = \alpha$. Suppose that $\gamma < \beta$. then there is a δ such that $\mu_\delta \leq \gamma < \mu_{\delta+1}$. Clearly $\delta \leq \alpha$. We define $f(\gamma) = (t',\gamma)$, where t' is the predecessor of t at level δ. Clearly then $f(\gamma) < (t,\beta)$. Clearly also f is order preserving and maps onto the set of all predecessors of (t,β). Thus T' is a tree. For each $\beta < \kappa$, say with $\mu_\alpha \leq \beta < \mu_{\beta+1}$ we have

$$\mathrm{Lev}_\beta(T') = \{(t,\beta) : t \in \mathrm{Lev}_\alpha(T)\}.$$

Thus each level of T' has size less than cf(κ). If B is a branch of size κ, then for each $\beta < \kappa$ it has an element of height at least β, and hence for each $\alpha < \mathrm{cf}(\kappa)$ it has an element whose first coordinate has height at least α. These first coordinates are linearly ordered. This contradicts T being a cf(κ)-Aronszajn tree.

Thus T' is as desired.

E18.10 *Show that for every infinite cardinal κ there is an eventually branching tree T of height κ such that for every subset S of T, if S is a tree under the order induced by T and every element of S has at least two immediate successors, then S has height ω.*

The idea is to put a copy of ${}^{<\omega}2$ on top of longer and longer chains. More precisely, define

$$T = \{(\alpha,\xi,\emptyset) : \alpha < \kappa, \xi < \alpha\} \cup \{(\alpha,\alpha,f) : \alpha < \kappa, f \in {}^{<\omega}2\};$$
$$(\alpha,\xi,f) < (\beta,\eta,g) \text{ iff } \alpha = \beta \text{ and either } \xi < \eta, \text{ or } \xi = \eta = \alpha \text{ and } f \subset g.$$

Clearly T is a tree. The height of an element $(\alpha,\xi,\emptyset)$ is ξ, and the height of an element (α,α,f) is $\alpha + n$, where $f \in {}^n2$. In particular, T has height κ.

Now suppose that S is as indicated in the exercise, and take any element (α,ξ,f) of S. (α,ξ,f) is a root of S iff $\alpha = \xi$ and $f = \emptyset$. It follows that all the non-root elements of S have the form (α,α,f), and so in S the height of every element is finite.

E18.11 *Show that if κ is an uncountable regular cardinal and T is a κ-Aronszajn tree, then T has a subset S such that under the order induced by T, S is a well-pruned κ-Aronszajn tree in which every element has at least two immediate successors.*

By Theorem 18.7, we may assume that T is well-pruned. Now we construct a strictly increasing sequence $\langle\alpha_\xi : \xi < \kappa\rangle$ of ordinals less than κ. Let $\alpha_0 = 0$. Suppose that α_ξ has been defined. Now T is eventually branching. (See the remark before Theorem 18.7.) Hence for each $t \in \mathrm{Lev}_{\alpha_\xi}(T)$ there is an ordinal $\beta_t > \alpha_\xi$ such that t has at least two successors at level β_t. Let $\alpha_{\xi+1}$ be any ordinal less than κ such that $\beta_t < \alpha_{\xi+1}$ for all $t \in \mathrm{Lev}_{\alpha_\xi}(T)$. Note by the well-prunedness condition, each $t \in \mathrm{Lev}_{\alpha_\xi}$ has at least two successors at level $\alpha_{\xi+1}$. Finally, suppose that η is a limit ordinal less than κ, and α_ξ has been constructed for all $\xi < \eta$. Let $\alpha_\eta = \sup_{\xi<\eta} \alpha_\xi$.

Let $S = \bigcup_{\xi<\kappa} \mathrm{Lev}_{\alpha_\xi}(T)$. Clearly S is as desired.

Solutions to exercises in Chapter 19

E19.1 *Assume that κ is an uncountable regular cardinal and $\langle A_\alpha : \alpha < \kappa\rangle$ is a sequence of subsets of κ. Let $D = \triangle_{\alpha<\kappa} A_\alpha$. Prove the following:*

(i) For all $\alpha < \kappa$, the set $D\backslash A_\alpha$ is nonstationary.

(ii) Suppose that $E \subseteq \kappa$ and for every $\alpha < \kappa$, the set $E\backslash A_\alpha$ is nonstationary. Show that $E\backslash D$ is nonstationary.

(i): For any $\beta < \kappa$,

$$\begin{aligned} D\backslash A_\beta &= \{\alpha < \kappa : \forall\gamma < \alpha(\alpha \in A_\gamma) \text{ and } \alpha \notin A_\beta\} \\ &\subseteq \{\alpha < \kappa : \alpha \le \beta\}; \end{aligned}$$

Hence $D\backslash A_\beta$ is nonstationary.

(ii): For each $\beta < \kappa$ let C_β be club in κ such that $(E\backslash A_\beta) \cap C_\beta = \emptyset$. Let F be the diagonal intersection of the C_β's; thus

$$F = \{\gamma < \kappa : \forall\alpha < \gamma(\gamma \in C_\alpha)\}.$$

Thus F is club. We claim that $F \cap (E\backslash D) = \emptyset$ (as desired). For, suppose that $\gamma \in F \cap (E\backslash D)$. Since $\gamma \notin D$, there is a $\beta < \gamma$ such that $\gamma \notin A_\beta$. Since $\gamma \in F$, we have $\gamma \in C_\beta$. Since $(E\backslash A_\beta) \cap C_\beta = \emptyset$, this is a contradiction.

E19.2 *Let $\kappa > \omega$ be regular. Show that there is a sequence $\langle S_\alpha : \alpha < \kappa\rangle$ of stationary subsets of κ such that $S_\beta \subseteq S_\alpha$ whenever $\alpha < \beta < \kappa$, and $\triangle_{\alpha<\kappa} S_\alpha = \{0\}$. Hint: use Theorem 19.12.*

By Theorem 19.12, let $\langle A_\alpha : \alpha < \kappa\rangle$ be pairwise disjoint stationary subsets of κ. Then $A_\alpha\backslash(\alpha+1)$ is also stationary. Let $S_\alpha = \bigcup_{\beta>\alpha}(A_\beta\backslash(\beta+1))$. Then S_α is stationary and $\alpha < \beta$ implies that $S_\beta \subseteq S_\alpha$. Let D be the diagonal intersection of the S_α's:

$$D = \{\gamma < \kappa : \forall\alpha < \gamma(\gamma \in S_\alpha)\}.$$

Thus $0 \in D$. Suppose that $0 \neq \gamma \in D$. Then $0 < \gamma$, so $\gamma \in S_0$. Hence there is a $\beta > 0$ such that $\gamma \in A_\beta\backslash(\beta+1)$. Hence $\beta < \gamma$. So $\gamma \in S_\beta$. Choose $\delta > \beta$ such that $\gamma \in A_\delta\backslash(\delta+1)$. So $\gamma \in A_\beta \cap A_\delta = \emptyset$, contradiction.

E19.3 *Suppose that κ is uncountable and regular, and for each limit ordinal $\alpha < \kappa$ we are given a function $f_\alpha \in {}^\omega\alpha$. Suppose that S is a stationary subset of κ. Let $n \in \omega$. Show that there exist a $t \in {}^n\kappa$ and a stationary $S' \subseteq S$ such that for all $\alpha \in S'$, $f_\alpha \restriction n = t$.*

We define sequences $\langle S_0, S_1, \ldots, S_n\rangle$ of stationary subsets of S and $\langle \beta_0, \ldots, \beta_{n-1}\rangle$ of ordinals less than κ. Let $S_0 = S$. Suppose that S_i has been defined, $i < n$. Let $g(\alpha) = f_\alpha(i)$ for all $\alpha \in S_i$. Then g is a regressive function, and hence there exist a stationary subset S_{i+1} of S_i and an ordinal β_i such that $g(\alpha) = \beta_i$ for all $\alpha \in S_{i+1}$. This finishes the construction.

If $\alpha \in S_n$, then for any $i < n$ we have $\alpha \in S_{i+1}$, and hence $f_\alpha(i) = \beta_i$. Hence we can let $t(i) = \beta_i$ for all $i < n$, and the property of the exercise holds.

E19.4 *Suppose that* $\mathrm{cf}(\kappa) > \omega$, *$C \subseteq \kappa$ is club of order type* $\mathrm{cf}(\kappa)$, *and* $\langle c_\beta : \beta < \mathrm{cf}(\kappa) \rangle$ *is the strictly increasing enumeration of C. Let $X \subseteq \kappa$. Show that X is stationary in κ iff* $\{\beta < \mathrm{cf}(\kappa) : c_\beta \in X\}$ *is stationary in* $\mathrm{cf}(\kappa)$.

Assume $X \subseteq \kappa$. Let $X' = \{\beta < \mathrm{cf}(\kappa) : c_\beta \in X\}$.

$\Rightarrow$: Assume that X is stationary in κ. We want to show that X' is stationary in $\mathrm{cf}(\kappa)$. Let D' be club in $\mathrm{cf}(\kappa)$. Define $D = \{c_\beta : \beta \in D'\}$. We claim that D is club in κ. For closure, suppose that $\alpha < \kappa$ is a limit ordinal and $D \cap \alpha$ is unbounded in α. Let $\gamma = \bigcup\{\beta \in D' : c_\beta < \alpha\}$.

(1) $\gamma < \mathrm{cf}(\kappa)$.

For, since C is unbounded in κ, there is a $\beta < \mathrm{cf}(\kappa)$ such that $\alpha < c_\beta$. Now if $\varepsilon \in D'$ and $c_\varepsilon < \alpha$ we have $c_\varepsilon < c_\beta$, hence $\varepsilon < \beta$. Therefore $\gamma \le \beta < \mathrm{cf}(\kappa)$, proving (1).

(2) γ is a limit ordinal, and $D' \cap \gamma$ is unbounded in γ.

For, suppose that $\delta < \gamma$. Choose $\beta \in D'$ such that $c_\beta < \alpha$ and $\delta < \beta$. Since $D \cap \alpha$ is unbounded in α, choose $c_\varepsilon \in D \cap \alpha$ such that $c_\beta < c_\varepsilon$. Then $\delta < \beta < \varepsilon \le \gamma$. Since $\beta \in D'$, this proves (2).

(3) $c_\gamma = \alpha$.

For, suppose that $\delta < c_\gamma$. Now $c_\gamma = \bigcup_{\beta<\gamma} c_\beta$ by (2), so there is a $\beta < \gamma$ such that $\delta < c_\beta$. By the definition of γ, there is a $\beta' \in D'$ such that $\beta < \beta'$ and $c_{\beta'} < \alpha$. Thus $\delta < c_\beta < c_{\beta'} < \alpha$, so $\delta < \alpha$. This proves that $c_\gamma \le \alpha$. On the other hand, suppose that $\delta < \alpha$. Since $D \cap \alpha$ is unbounded in α, there is a $\beta \in D'$ such that $\delta < c_\beta < \alpha$. Thus $\beta \le \gamma$, so $\delta < c_\gamma$. This proves that $\alpha \le c_\gamma$, and finishes the proof of (3).

Now by (2) we have $\gamma \in D'$, and hence (3) yields $\alpha \in D$, as desired; we have now proved that D is closed in κ.

To show that D is unbounded in κ, let $\alpha < \kappa$ be arbitrary. Choose $\beta < \mathrm{cf}(\kappa)$ such that $\alpha < c_\beta$. Since D' is unbounded in $\mathrm{cf}(\kappa)$, choose $\beta' \in D'$ such that $\beta < \beta'$. Thus $\alpha < c_\beta < c_{\beta'} \in D$, as desired.

So we have shown that D is club in κ. Since X is stationary in κ, choose $\beta \in D'$ such that $c_\beta \in X$. thus $\beta \in D' \cap X'$, as desired.

$\Leftarrow$: Assume that X' is stationary in $\mathrm{cf}(\kappa)$. Let D be club in κ, and let $D' = \{\beta < \mathrm{cf}(\kappa) : c_\beta \in D\}$. We claim that D' is club in $\mathrm{cf}(\kappa)$. To show that it is closed, suppose that $\alpha < \mathrm{cf}(\kappa)$ is a limit ordinal and $D' \cap \alpha$ is unbounded in α. We claim that $D \cap c_\alpha$ is unbounded in c_α. For, suppose that $\gamma < c_\alpha$. Then there is a $\beta < \alpha$ such that $\gamma < c_\beta$. Choose $\delta \in D' \cap \alpha$ such that $\beta < \delta$; this is possible since $D' \cap \alpha$ is unbounded in α. Thus $\gamma < c_\beta < c_\delta \in D \cap c_\alpha$, as desired. So D' is closed in $\mathrm{cf}(\kappa)$. To show that it is unbounded, suppose that $\alpha < \mathrm{cf}(\kappa)$. Now $C \cap D$ is club in κ, so there is a β such that $c_\alpha < c_\beta \in D$. So $\alpha < \beta \in D'$. This shows that D' is unbounded in $\mathrm{cf}(\kappa)$. Hence D' is club in $\mathrm{cf}(\kappa)$.

Choose $\beta \in D' \cap X'$. Then $c_\beta \in D \cap X$, as desired.

E19.5 *Suppose that κ is regular and uncountable, and $S \subseteq \kappa$ is stationary. Also, suppose that every $\alpha \in S$ is an uncountable regular cardinal. Show that*

$$T \stackrel{\text{def}}{=} \{\alpha \in S : S \cap \alpha \text{ is non-stationary in } \alpha\}$$

is stationary in κ. Hint: given a club C in κ, let C' be the set of all limit points of C and let α be the least element of $C' \cap S$; show that $\alpha \in T \cap C$.

Assume the notation of the exercise. Let C be club in κ; we want to show that $T \cap C \neq \emptyset$. Let C' be the set of all limit points of C, i.e., the set of all limit ordinals $\alpha \in \kappa$ such that $C \cap \alpha$ is unbounded in α. Clearly $C' \subseteq C$, and C' is club in κ. Since S is stationary in κ, let α be the least element of $S \cap C'$. Clearly $C' \cap \alpha$ is closed in α; we claim that it is also unbounded in α. For, suppose that $\beta < \alpha$. Now $C \cap \alpha$ is unbounded in α, so we can construct a sequence $\langle \gamma_i : i < \omega \rangle$ of members of C such that $\beta < \gamma_0 < \gamma_1 < \cdots < \alpha$. Let $\delta = \sup_{i \in \omega} \gamma_i$. Then $\delta \in C'$, and $\delta < \alpha$ since α is uncountable and regular. So $C' \cap \alpha$ is club in α. Now $S \cap C' \cap \alpha = \emptyset$ by the minimality of α, so $C' \cap \alpha$ is a club in α which shows that $S \cap \alpha$ is non-stationary in α. So $\alpha \in T \cap C$, as desired.

E19.6 *Suppose that κ is uncountable and regular, and $\kappa \leq |A|$. Suppose that C is a closed subset of $[A]^{<\kappa}$ and D is a directed subset of C with $|D| < \kappa$. (Directed means that if $x, y \in D$ then there is a $z \in D$ such that $x \cup y \subseteq z$.) Show that $\bigcup D \in C$. Hint: use induction on $|D|$.*

Proof. If D is finite, then $\bigcup D \in D$; so $\bigcup D \in C$. Suppose that $|D| = \omega$; say $D = \{x_n : n \in \omega\}$. For each $n \in \omega$ choose $y_n \in D$ so that $\{x_m : m \leq n\} \cup \{y_m : m < n\} \subseteq y_n$. Then $\bigcup_{n \in \omega} y_n \in C$ since C is closed, and $\bigcup D = \bigcup_{n \in \omega} y_n$.

Now suppose inductively that $|D| > \omega$. Let $|D| = \kappa$ and write $D = \{x_\alpha : \alpha < \kappa\}$. For all $y, z \in D$ let $f(y, z) \in D$ be such that $y, z \subseteq f(y, z)$. We now define $\langle E_\alpha : \alpha < \kappa \rangle$ by recursion. Suppose that E_β has been defined for all $\beta < \alpha$ so that $E_\beta \subseteq D$, E_β is directed, and $|E_\beta| \leq |\beta| + \omega$. Let $F_0 = \{x_\alpha\} \cup \bigcup_{\beta < \alpha} E_\beta$. So $|F_0| \leq |\alpha| + \omega$. Let $F_{n+1} = F_n \cup \{f(x, y) : x, y \in F_n\}$. Then $|F_{n+1} \leq |\alpha| + \omega$. Let $E_\alpha = \bigcup_{n \in \omega} F_n$. Then $E_\alpha \subseteq D$, E_α is directed, and $|E_\alpha| \leq |\alpha| + \omega$.

By the inductive hypothesis, $y_\alpha \stackrel{\text{def}}{=} \bigcup E_\alpha$ is in C, and $y_\alpha \subseteq y_\beta$ for $\alpha < \beta$. Hence $\bigcup D = \bigcup_{\alpha < \kappa} y_\alpha \in C$.

E19.7 *Let κ be uncountable and regular, and $\kappa \leq |A|$. If $f : [A]^{<\omega} \to [A]^{<\kappa}$ let $C_f = \{x \in [A]^{<\kappa} : \forall s \in [x]^{<\omega}[f(s) \subseteq x]\}$. Show that C_f is club in $[A]^{<\kappa}$.*

Suppose that $x_\xi \in C_f$ for all $\xi < \alpha$, with $\alpha < \kappa$ and $x_\xi \subseteq x_\eta$ for $\xi < \eta$. Clearly $\bigcup_{\xi < \alpha} x_\xi \in C_f$ if α is a successor ordinal. Suppose that α is a limit ordinal. Take any $s \in \left[\bigcup_{\xi < \alpha} x_\xi\right]^{<\omega}$. Then there is a $\xi < \alpha$ such that $s \in [x_\xi]^{<\omega}$, and hence $f(s) \in x_\xi \subseteq \bigcup_{\eta < \zeta} x_\eta$. Thus C_f is closed.

To show that C_f is unbounded, let $y \in [A]^{<\kappa}$. Define $z_0 = y$ and $z_{n+1} = z_n \cup \{f(s) : s \in [z_n]^{<\omega}\}$. By induction, $z_n \in [A]^{<\kappa}$ for all $n \in \omega$. Now $\bigcup_{n \in \omega} z_n \in C_f$, showing that C_f is unbounded.

E19.8 *(Continuing Exercise E19.7) Let κ be uncountable and regular, and $\kappa \le |A|$. Let D be club in $[A]^{<\kappa}$. Show that there is an $f : [A]^{<\omega} \to [A]^{<\kappa}$ such that $C_f \subseteq D$. Hint: show that there is an $f : [A]^{<\omega} \to D$ such that $\forall e \in [A]^{<\omega}[e \subseteq f(e)]$ and $\forall e_1, e_2 \in [A]^{<\omega}[e_1 \subseteq e_2 \to f(e_1) \subseteq f(e_2)]$.*

We claim that there is an $f : [A]^{<\omega} \to D$ such that $\forall e \in [A]^{<\omega}[e \subseteq f(e)]$ and $\forall e_1, e_2 \in [A]^{<\omega}[e_1 \subseteq e_2 \to f(e_1) \subseteq f(e_2)]$. We define f by induction on $|e|$. Let $f(\emptyset)$ be any member of D. Suppose that $f(e)$ has been defined for all $e \in [A]^{<\omega}$ such that $|e| < m$, and suppose that $e \in [A]^{<\omega}$ with $|e| = m$. Let $f(e)$ be a member of D such that $e \subseteq f(e)$ and $f(e\backslash\{a\}) \subseteq f(e)$ for all $a \in e$. Clearly f is as desired.

Now we show that $C_f \subseteq D$. Let $x \in C_f$. Note that $\{f(e) : e \in [x]^{<\omega}\}$ is directed and has union x. Hence $x \in D$ by Exercise 19.6.

E19.9 *Let κ be uncountable and regular, $\kappa \le |A|$, and $A \subseteq B$. If $Y \in [A]^{<\kappa}$, let $Y^B = \{x \in [B]^{<\kappa} : x \cap A \in Y\}$. Show that if Y is club in $[A]^{<\kappa}$, then Y^B is club in $[B]^{<\kappa}$.*

Assume the hypotheses. Suppose that $\alpha < \kappa$ and $\langle x_\xi : \xi < \alpha\rangle$ is a sequence of members of Y^B with $x_\xi \subseteq x_\eta$ for $\xi < \eta$. Then $x_\xi \cap A \in Y$ for all $\xi < \alpha$, and $x_\xi \cap A \subseteq x_\eta \cap A$ for $\xi < \eta$. Hence $\bigcup_{\xi<\alpha}(x_\xi \cap A) \in Y$. So $\bigcup_{\xi<\alpha} x_\xi \in Y^B$. Thus Y^B is closed.

To show that Y^B is unbounded, let $b \in [B]^{<\kappa}$. Then $b \cap A \in [A]^{<\kappa}$, so there is a $c \in Y$ such that $b \cap A \subseteq c$. Then $b \subseteq c \cup (b\backslash A)$, and $(c \cup (b\backslash A)) \cap A = c \in Y$. So $c \cup (b\backslash A) \in Y^B$.

E19.10 *Let κ be uncountable and regular, $\kappa \le |A|$, and $A \subseteq B$. If $Y \in [B]^{<\kappa}$, let $Y \restriction A = \{y \cap A : y \in Y\}$. Show that if Y is stationary in $[B]^{<\kappa}$ then $Y \restriction A$ is stationary in $[A]^{<\kappa}$.*

Assume the hypotheses. Suppose that C is club in $[A]^{<\kappa}$. Then by Exercise E19.9, C^B is club in $[B]^{<\kappa}$. Choose $y \in Y \cap C^B$. Then $y \cap A \in (Y \restriction A) \cap C$.

E19.11 *With κ, A, B as in exercise E19.9, suppose that $f : [B]^{<\omega} \to [B]^{<\kappa}$. For each $e \in [A]^{<\omega}$ define*

$$\begin{aligned} x_0(e) &= e; \\ x_{n+1}(e) &= x_n(e) \cup \{f(s) : s \in [x_n(e)]^{<\omega}\}; \\ w(e) &= \bigcup_{n\in\omega} x_n(e). \end{aligned}$$

Also, for each $y \in [A]^{<\kappa}$ let $v(y) = \bigcup\{w(e) : e \in [y]^{<\omega}\}$.

Prove that $w(e) \in C_f$ for all $e \in [A]^{<\omega}$ and $v(y) \in C_f$ for all $y \in [A]^{<\kappa}$.

If $z \in [w(e)]^{<\omega}$, then $z \in [x_n(e)]^{<\omega}$ for some n, and hence $f(z) \in x_{n+1}(e) \subseteq w(e)$. Thus $w(e) \in C_f$. Note that if $e_1 \subseteq e_2$ then $x_n(e_1) \subseteq x_n(e_2)$ for all n (by induction), and so $w(e_1) \subseteq w(e_2)$. Now suppose that $z \in [v(y)]^{<\omega}$. Then there is a finite $F \subseteq [y]^{<\omega}$ such that $z \subseteq \bigcup_{e\in F} w(e)$. Let $e' = \bigcup_{e\in F} e$. Then $\bigcup_{e\in F} w(e) \subseteq w(e')$. So $z \in [w(e')]^{<\omega}$. It follows from the first part of this solution that $f(z) \in w(e') \subseteq v(y)$. Thus $v(y) \in C_f$.

E19.12 *With κ, A, B as in exercise E19.9, suppose that S is stationary in $[A]^{<\kappa}$. Show that S^B is stationary in $[B]^{<\kappa}$. Hint: use exercises E19.8 and E19.11.*

Assume the hypotheses. Suppose that D is club in $[B]^{<\kappa}$. By Exercise 19.8 there is an $f : [B]^{<\omega} \to [B]^{<\kappa}$ such that $C_f \subseteq D$. For any $e \in [A]^{<\omega}$ let $g(e) = w(e) \cap A$. We claim that $C_f \restriction A = C_g$. Suppose that $y \in C_f$, so that $y \cap A \in C_f \restriction A$. To show that $y \cap A \in C_g$, let $e \in [y \cap A]^{<\omega}$. Then $x_n(e) \subseteq y$ by induction on n. Since $x_n(e) = e$, it is true for $n = 0$. Suppose that $x_n(e) \subseteq y$. If $s \in [x_n(e)]^{<\omega}$, then $s \in [y]^{<\omega}$, so $f(s) \subseteq y$ since $y \in C_f$. Hence $x_{n+1}(e) \subseteq y$. It follows that $w(e) \subseteq y$, and so $g(e) \subseteq y \cap A$. This shows that $y \cap A \in C_g$, and proves that $C_f \restriction A \subseteq C_g$. Now suppose that $y \in C_g$. Hence $\forall e \in [y]^{<\omega}[g(e) \subseteq y]$. We claim that $v(y) \cap A = y$. For, if $e \in [y]^{<\omega}$, then $g(e) \subseteq y$, i.e., $w(e) \cap A \subseteq y$. So $v(y) \cap A \subseteq y$. If $a \in y$, then $a \in w(\{a\}) \subseteq v(y)$; so $v(y) \cap A = y$. This shows that $C_g \subseteq C_f \restriction A$. Thus $C_g = C_f \restriction A$.

Choose $z \in C_g \cap S$. Then $z \in (C_f \restriction A) \cap S$, so there is a $y \in C_f$ such that $z = y \cap A$. Thus $y \cap A \in S$, so $y \in S^B \cap D$.

Solutions to exercises in Chapter 20

E20.1 *Suppose that $\kappa^\omega > \kappa$. Show that there is a family $\mathscr{A}$ of subsets of κ, each of size ω, with $|\mathscr{A}| = \kappa^+$ and the intersection of any two members of $\mathscr{A}$ is finite.*

Let $K = {}^{<\omega}\kappa$. Thus $|K| = \kappa$. Let F be a bijection from K to κ. For each $f \in {}^\omega\kappa$ let

$$X_f = F[\{f \restriction m : m \in \omega\}].$$

Clearly each X_f has size ω. If $f, g \in {}^\omega\kappa$ and $f \neq g$, choose $p \in \omega$ such that $f(p) \neq g(p)$. Then

$$\begin{aligned} X_f \cap X_g &= F[\{f \restriction m : m \in \omega\}] \cap F[\{g \restriction m : m \in \omega\}] \\ &= F[\{f \restriction m : m \in \omega\} \cap \{g \restriction m : m \in \omega\}] \\ &\subseteq F[\{f \restriction m : m \leq p\}] \end{aligned}$$

and hence $X_f \cap X_g$ is finite. Since $\kappa^\omega \geq \kappa^+$, the desired result follows.

E20.2 *Suppose that κ is any infinite cardinal, and λ is minimum such that $\kappa^\lambda > \kappa$. Show that there is a family $\mathscr{A}$ of subsets of κ, each of size λ, with the intersection of any two members of $\mathscr{A}$ being of size less than λ, and with $|\mathscr{A}| = \lambda^+$.*

Let $K = {}^{<\lambda}\kappa$. Thus $|K| = \kappa$. Let F be a bijection from K to κ. For each $f \in {}^\lambda\kappa$ let

$$X_f = F[\{f \restriction m : m \in \lambda\}].$$

Clearly each X_f has size λ. If $f, g \in {}^\lambda\kappa$ and $f \neq g$, choose $p \in \lambda$ such that $f(p) \neq g(p)$. Then

$$\begin{aligned} X_f \cap X_g &= F[\{f \restriction m : m \in \lambda\}] \cap F[\{g \restriction m : m \in \lambda\}] \\ &= F[\{f \restriction m : m \in \lambda\} \cap \{g \restriction m : m \in \lambda\}] \\ &\subseteq F[\{f \restriction m : m \leq p\}] \end{aligned}$$

and hence $X_f \cap X_g$ has size less than λ. Since $\kappa^\lambda > \kappa$, the desired result follows.

E20.3 *Suppose that κ is uncountable and regular. Show that there is a family $\mathscr{A}$ of subsets of κ, each of size κ with the intersection of any two members of $\mathscr{A}$ of size less than κ, and with $|\mathscr{A}| = \kappa^+$. Hint: (1) show that there is a partition of κ into κ subsets, each of size κ; (2) Use Zorn's lemma to start from (1) and produce a maximal almost disjoint set; (3) Use a diagonal construction to show that the resulting family must have size $> \kappa$.*

First of all, recall that κ can be partitioned into κ sets, each of size κ. Namely, if $f : \kappa \times \kappa \to \kappa$ is a bijection, let $X_\alpha = f[\{(\alpha, \beta) : \beta < \kappa\}]$; then clearly $\langle X_\alpha : \alpha < \kappa \rangle$ is as claimed.

Thus we can apply Zorn's lemma to get a maximal collection $\mathscr{A} \subseteq [\kappa]^\kappa$ such that the members of $\mathscr{A}$ are pairwise almost disjoint, and $|\mathscr{A}| \geq \kappa$.

Hence we just have to get a contradiction from the assumption that $|\mathscr{A}| = \kappa$. Making this assumption, let $\langle Y_\alpha : \alpha < \kappa \rangle$ be a one-one enumeration of $\mathscr{A}$. Note that for any $\alpha < \kappa$,

$$Y_\alpha \backslash \bigcup_{\beta<\alpha} Y_\beta = Y_\alpha \backslash \bigcup_{\beta<\alpha} (Y_\alpha \cap Y_\beta)$$

has size κ. This enables us to define by recursion a sequence $\langle z_\alpha : \alpha < \kappa \rangle$ like this: having defined z_β for all $\beta < \alpha$, choose

$$z_\alpha \in Y_\alpha \backslash \left(\{z_\beta : \beta < \alpha\} \cup \bigcup_{\beta<\alpha} Y_\beta \right).$$

Then $Z \stackrel{\text{def}}{=} \{z_\alpha : \alpha < \kappa\}$ is a set of size κ, and for any $\alpha < \kappa$,

$$Z \cap Y_\alpha \subseteq \{z_\beta : \beta \leq \alpha\},$$

so that $|Z \cap Y_\alpha| < \kappa$. This contradicts the maximality of $\mathscr{A}$.

E20.4 *Prove that if $\mathscr{F}$ is an uncountable family of finite functions each with range $\subseteq \omega$, then there are distinct $f, g \in \mathscr{F}$ such that $f \cup g$ is a function.*

We apply Theorem 20.4 to the indexed system $\langle \text{dmn}(f) : f \in \mathscr{F} \rangle$ and get an uncountable subset $\mathscr{G}$ of $\mathscr{F}$ such that $\langle \text{dmn}(f) : f \in \mathscr{G} \rangle$ is an indexed Δ-system; say that $\text{dmn}(f) \cap \text{dmn}(g) = D$ for all distinct $f, g \in \mathscr{G}$. Then

$$\mathscr{G} = \bigcup_{h \in {}^D\omega} \{f \in \mathscr{G} : f \restriction D = h\};$$

since the index set ${}^D\omega$ is countable and $\mathscr{G}$ is uncountable, there exist an $h \in {}^D\omega$ for which there are two distinct $f, g \in \mathscr{G}$ such that $f \restriction D = g \restriction D = h$. Then $f \cup g$ is a function.

E20.5 *(Double Δ-system theorem) Suppose that κ is a singular cardinal with $\text{cf}(\kappa) > \omega$. Let $\langle \lambda_\alpha : \alpha < \text{cf}(\kappa) \rangle$ be a strictly increasing sequence of successor cardinals with supremum κ, with $\text{cf}(\kappa) < \lambda_0$, and such that for each $\alpha < \text{cf}(\kappa)$ we have $(\sum_{\beta<\alpha} \lambda_\beta)^+ \leq \lambda_\alpha$. Suppose that $\langle A_\xi : \xi < \kappa \rangle$ is a system of finite sets. Then there exist a set $\Gamma \in [\text{cf}(\kappa)]^{\text{cf}(\kappa)}$, a* *sequence $\langle\, \Xi_\alpha : \alpha \in \Gamma\rangle$ of subsets of κ, a sequence $\langle F_\alpha : \alpha \in \Gamma\rangle$ of finite sets, and a finite set G, such that the following conditions hold:*

(i) $\langle\, \Xi_\alpha : \alpha \in \Gamma\rangle$ is a pairwise disjoint system, and $|\Xi_\alpha| = \lambda_\alpha$ for every $\alpha \in \Gamma$.

(ii) $\langle A_\xi : \xi \in \Xi_\alpha\rangle$ is a Δ-system with root F_α for every $\alpha \in \Gamma$.

(iii) $\langle F_\alpha : \alpha \in \Gamma\rangle$ is a Δ-system with root G.

(iv) If $\xi \in \Xi_\alpha$, $\eta \in \Xi_\beta$, and $\alpha \neq \beta$, then $A_\xi \cap A_\eta = G$.

Let $\kappa = \bigcup_{\alpha<\mathrm{cf}(\kappa)} \Xi'_\alpha$ where the Ξ'_α's are pairwise disjoint, with $|\Xi'_\alpha| = \lambda_\alpha$ for every $\alpha < \mathrm{cf}(\kappa)$. For each $\alpha < \mathrm{cf}(\kappa)$ let $\Xi''_\alpha \in [\Xi'_\alpha]^{\lambda_\alpha}$ be such that $\langle A_\eta : \eta \in \Xi''_\alpha\rangle$ is a Δ-system, say with root F_α. Choose $\Gamma \in [\mathrm{cf}(\kappa)]^{\mathrm{cf}(\kappa)}$ such that $\langle F_\alpha : \alpha \in \Gamma\rangle$ is a Δ-system, say with root G. For each $\alpha \in \Gamma$ let

$$B_\alpha = \bigcup\left\{\bigcup_{\xi\in\Xi''_\beta} A_\xi : \beta \in \Gamma, \beta < \alpha\right\}.$$

We claim

(1) $|B_\alpha| < \lambda_\alpha$.

In fact,

$$|B_\alpha| \leq \sum_{\substack{\beta<\alpha\\ \beta\in\Gamma}} \left|\bigcup_{\xi\in\Xi''_\beta} A_\xi\right| \leq \sum_{\substack{\beta<\alpha\\ \beta\in\Gamma}} \lambda_\beta < \lambda_\alpha.$$

So (1) holds. Now for any $\alpha \in \Gamma$,

$$\Xi''_\alpha = \bigcup_{J\in[B_\alpha]^{<\omega}} \{\xi \in \Xi''_\alpha : A_\xi \cap B_\alpha = J\},$$

so by (1) there is a $C_\alpha \in [B_\alpha]^{<\omega}$ such that $\Xi'''_\alpha \stackrel{\text{def}}{=} \{\xi \in \Xi''_\alpha : A_\xi \cap B_\alpha = C_\alpha\}$ has size λ_α. Note that $C_\alpha \subseteq F_\alpha$, since for distinct $\xi, \eta \in \Xi'''_\alpha$ we have $C_\alpha = A_\xi \cap A_\eta \cap B_\alpha \subseteq F_\alpha$. Next note that $\langle A_\xi \backslash F_\alpha : \xi \in \Xi'''_\alpha\rangle$ is a system of pairwise disjoint sets; hence for each $\beta \in \Gamma$, the set $\{\xi \in \Xi'''_\alpha : (A_\xi\backslash F_\alpha) \cap F_\beta \neq 0\}$ is finite. Since $|\Gamma| = \mathrm{cf}(\kappa) < \lambda_\alpha$, it follows that the set

$$\Xi_\alpha \stackrel{\text{def}}{=} \Xi'''_\alpha \backslash \{\xi \in \Xi''_\alpha : (A_\xi\backslash F_\alpha) \cap F_\beta \neq \emptyset \text{ for some } \beta \in \Gamma\}.$$

has size λ_α.

Now we can verify the conditions of the exercise. Conditions (i)–(iii) are clear. Now suppose that $\xi \in \Xi_\alpha$, $\eta \in \Xi_\beta$, and $\alpha \neq \beta$. Say $\beta < \alpha$. Suppose that $\gamma \in A_\xi \cap A_\eta \backslash G$; we want to get a contradiction. Since $F_\alpha \cap F_\beta = G$, we have two possibilities.

Case 1. $\gamma \notin F_\alpha$. But $\gamma \in A_\xi \cap B_\alpha = C_\alpha \subseteq F_\alpha$, contradiction.

Case 2. $\gamma \in F_\alpha \backslash F_\beta$. Thus $\gamma \in (A_\eta \backslash F_\beta) \cap F_\alpha$, contradicting the definition of Ξ_β.

E20.6 *Suppose that $\mathscr{F}$ is a collection of countable functions, each with range $\subseteq 2^\omega$, and with $|\mathscr{F}| = (2^\omega)^+$. Show that there are distinct $f, g \in \mathscr{F}$ such that $f \cup g$ is a function.*

Let $\kappa=\omega_1$, $\lambda=(2^\omega)^+$, and apply Theorem 20.4 with $\langle A_i : i\in I\rangle$ replaced by $\langle \mathrm{dmn}(f) : f\in\mathscr{F}\rangle$. We get $J\in[\mathscr{F}]^\lambda$ such that $\langle \mathrm{dmn}(f) : f\in J\rangle$ is an indexed Δ-system, say with root r. Now

$$J=\bigcup_{h:r\to 2^\omega}\{f\in J : f\restriction r=h\},$$

and $|{}^r(2^\omega)|=2^\omega$, so there is an $h : r\to 2^\omega$ such that $K\stackrel{\text{def}}{=}\{f\in J : f\restriction r=h\}$ has size $(2^\omega)^+$. For any two $f,g\in K$, the set $f\cup g$ is a function.

E20.7 *For any infinite cardinal κ, any linear order of size at least $(2^\kappa)^+$ has a subset of order type κ^+ or one similar to $(\kappa^+,>)$.*

Let L be a linear order of size $(2^\kappa)^+$, and let $\prec$ be a well-order of L. Define $f : [L]^2\to 2$ by setting, for any $\{a,b\}\in[L]^2$, say with $a<b$,

$$f(\{a,b\})=\begin{cases}0 & \text{if } a\prec b,\\ 1 & \text{if } b\prec a.\end{cases}$$

By the Erdös-Rado theorem, i.e., Corollary 8.7, let $A\in[L]^{\kappa^+}$ such that A is homogeneous for f. If f takes the value 0 on $[A]^2$, then A is well-ordered under $<$, and since its size is κ^+, it has a subset of order type κ^+. Similarly if f takes the value 1 on $[A]^2$.

E20.8 *For any infinite cardinal κ, any tree of size at least $(2^\kappa)^+$ has a branch or an antichain of size at least κ^+.*

Suppose that T is a tree of size at least $(2^\kappa)^+$. Let S be a subset of T of size $(2^\kappa)^+$. Define $f : [S]^2\to 2$ by setting, for distinct $s,t\in S$,

$$f(\{s,t\})=\begin{cases}1 & \text{if } s \text{ and } t \text{ are comparable},\\ 0 & \text{otherwise.}\end{cases}$$

By the Erdös-Rado theorem, let $X\subseteq S$ be homogeneous for f of size κ^+. So if f has constant value 1 on $[X]^2$, then X is a chain of size κ^+, hence extends to a branch of size at least κ^+, while if f has constant value 0 on $[X]^2$, then X is an antichain of size κ^+.

E20.9 *Any uncountable tree either has an uncountable branch or an infinite antichain.*

Suppose that T is an uncountable tree. Let $S\in[T]^{\aleph_1}$. We define $f : [S]^2\to 2$ by setting, for any distinct $s,t\in S$,

$$f(\{s,t\})=\begin{cases}1 & \text{if } s \text{ and } t \text{ are comparable},\\ 0 & \text{otherwise.}\end{cases}$$

Then the desired conclusion follows from the Dushnik-Miller theorem.

E20.10 *Suppose that m is a positive integer. Show that any infinite set X of positive integers contains an infinite subset Y such that one of the following conditions holds:*

(i) The members of Y are pairwise relatively prime.

(ii) There is a prime $p<m$ such that for any two $a,b\in Y$, p divides $a-b$.

(iii) If a, b are distinct members of Y, then a, b are not relatively prime, but the smallest prime divisor of $a - b$ is at least equal to m.

Let $p_0, \ldots, p_{i-1}$ list all of the primes $< m$, in order. Thus $i = 0$ if $m = 1$. Define $f : [X]^2 \to i + 2$ by setting, for any distinct $x, y \in X$,

$$f(\{x, y\}) = \begin{cases} i & \text{if } x \text{ and } y \text{ are relatively prime,} \\ j & \text{if } j < i \text{ and } p_j \text{ is the smallest prime dividing } x - y, \\ i + 1 & \text{otherwise.} \end{cases}$$

Applying Ramsey's theorem in the form

$$\omega \to (\underbrace{\omega, \ldots, \omega}_{i+1 \text{ times}})^2,$$

we get an infinite homogeneous subset Y of X. If $f[[Y]^2] = \{i\}$, then any two members of Y are relatively prime. If $f[[Y]^2] = \{j\}$ with $j < i$, then p_j divides $x - y$ for any two members x, y of Y. If $f[[Y]^2] = \{i + 1\}$, then for any two members x, y of Y, the least prime dividing $x - y$ is at least as big as m.

E20.11 *Suppose that X is an infinite set, and $(X, <)$ and $(X, \prec)$ are two well-orderings of X. Show that there is an infinite subset Y of X such that for all $y, z \in Y$, $y < z$ iff $y \prec z$.*

Define $f : [X]^2 \to 2$ by setting, for any distinct $x, y \in X$, say with $x < y$,

$$f(\{x, y\}) = \begin{cases} 1 & \text{if } x \prec y \\ 0 & \text{otherwise.} \end{cases}$$

By Ramsey's theorem, let Y be an infinite subset of X which is homogeneous for f. If $f[[Y]^2 = \{0\}$, then $x < y$ implies $x \succ y$ for all distinct $x, y \in Y$. Now since Y is infinite and $<$ is a well-order of Y, the order type of Y under $<$ is an infinite ordinal. Hence there is a system $\langle y_n : n \in \omega \rangle$ of elements of Y such that $y_0 < y_1 < \cdots$. Hence $y_0 \succ y_1 \succ \cdots$, contradicting the fact that $\prec$ is a well-order.

Hence $f[[Y]^2 = 1$. This means that for any distinct $x, y \in Y$ we have $x < y$ iff $x \prec y$, as desired.

E20.12 *Let S be an infinite set of points in the plane. Show that S has an infinite subset T such that all members of T are on the same line, or else no three distinct points of T are collinear.*

Define $f : [S]^3 \to 2$ by

$$f(\{s, t, u\}) = \begin{cases} 1 & \text{if } s, t, u \text{ are on a line,} \\ 0 & \text{otherwise.} \end{cases}$$

Let T be an infinite subset of S homogeneous for f. If $f[[T]^2] = \{1\}$, then all points of T are on a line. If $f[[T]^2] = \{0\}$, then no three points of T are on a line.

E20.13 *We consider the following variation of the arrow relation. For cardinals $\kappa, \lambda, \mu, \nu$, we define*

$$\kappa \to [\lambda]^\mu_\nu$$

to mean that for every function $f : [\kappa]^\mu \to \nu$ there exist an $\alpha < \nu$ and a $\Gamma \in [\kappa]^\lambda$ such that $f[[\Gamma]^\mu] \subseteq \nu \backslash \{\alpha\}$. In coloring terminology, we color the μ-element subsets of κ with ν colors, and then there is a set which is anti-homogeneous for f, in the sense that there is a color α and a subset of size λ all of whose μ-element subsets do not get the color α.

Prove that for any infinite cardinal κ,

$$\kappa \not\to [\kappa]^\kappa_{2^\kappa}.$$

Hint: (1) Show that there is an enumeration $\langle X_\alpha : \alpha < 2^\kappa \rangle$ of $[\kappa]^\kappa$ in which every member of $[\kappa]^\kappa$ is repeated 2^κ times. (2) Show that $|[\kappa]^\kappa| = 2^\kappa$. (3) Show that there is a one-one $\langle Y_\alpha : \alpha < 2^\kappa \rangle$ such that $Y_\alpha \in [X_\alpha]^\kappa$ for all $\alpha < 2^\kappa$. (4) Define $f : [\kappa]^\kappa \to 2^\kappa$ so that for all $\alpha < 2^\kappa$ one has

$$f(Y_\alpha) = o.t.(\{\beta < \alpha : X_\beta = X_\alpha\}).$$

We follow the hint.

(1) There is an enumeration $\langle X_\alpha : \alpha < 2^\kappa \rangle$ of $[\kappa]^\kappa$ in which every member of $[\kappa]^\kappa$ is repeated 2^κ times.

In fact, let $f : 2^\kappa \to [\kappa]^\kappa$ be a surjection and let $g : 2^\kappa \times 2^\kappa \to 2^\kappa$ be a bijection. For each $\alpha < 2^\kappa$ let $X_\alpha = f(1^{\text{st}}(g^{-1}(\alpha)))$. Then for all $\alpha, \beta < 2^\kappa$,

$$X_{g(\alpha,\beta)} = f(1^{\text{st}}(g^{-1}(g(\alpha, \beta)))) = f(1^{\text{st}}(\alpha, \beta)) = f(\alpha),$$

and (1) follows.

(2) $|[\kappa]^\kappa| = 2^\kappa$.

To prove (2), let $f : \kappa \times \kappa \to \kappa$ be a bijection. For each $g \in {}^\kappa 2$ let

$$Z_g = \bigcup_{\substack{\alpha<\kappa,\\ g(\alpha)=1}} f[\kappa \times \{\alpha\}].$$

Then $|Z_g| = \kappa$ provided that g is not identically 0, and $Z_g \neq Z_h$ if $g \neq h$. (If $g(\alpha) \neq h(\alpha)$, say $g(\alpha) = 1$ and $h(\alpha) = 0$; then $f[\kappa \times \alpha] \subseteq Z_g$ but $f[\kappa \times \{\alpha\}] \cap Z_h = \emptyset$.) Thus $2^\kappa \leq |[\kappa]^\kappa| + 1 \leq 2^\kappa + 1$ with cardinal addition, and so (2) follows.

(3) There is a one-one $\langle Y_\alpha : \alpha < 2^\kappa \rangle$ such that $Y_\alpha \in [X_\alpha]^\kappa$ for all $\alpha < 2^\kappa$.

We construct Y_α by recursion. If Y_β has been constructed for all $\beta < \alpha$, where $\alpha < 2^\kappa$, choose $Y_\alpha \in [X_\alpha]^\kappa \backslash \{Y_\beta : \beta < \alpha\}$; this is possible by (2). So (3) holds.

Now we define $f : [\kappa]^\kappa \to 2^\kappa$ so that for all $\alpha < 2^\kappa$ one has

$$f(Y_\alpha) = \text{o.t.}(\{\beta < \alpha : X_\beta = X_\alpha\}).$$

This defines f on $\{Y_\alpha : \alpha < 2^\kappa\}$, and it can be extended to all of $[\kappa]^\kappa$ in any fashion.

Now we show that f is the desired counterexample. For, suppose that $\beta < 2^\kappa$, $\Gamma \in [\kappa]^\kappa$, and $f[[\Gamma]^\kappa] \subseteq 2^\kappa \backslash \{\beta\}$. Choose $\alpha < 2^\kappa$ such that $X_\alpha = \Gamma$ and $\{\gamma < \alpha : X_\gamma = \Gamma\}$ has order type β. Then $Y_\alpha \in [\Gamma]^\kappa$ and $f(Y_\alpha) = \beta$, contradiction.

Solutions to exercises in Chapter 21

E21.1 *Assume* MA(κ). *Suppose that X is a compact Hausdorff space, and any pairwise disjoint collection of open sets in X is countable. Suppose that U_α is dense open in X for each $\alpha < \kappa$. Show that $\bigcap_{\alpha<\kappa} U_\alpha \neq \emptyset$.*

Let P consist of all nonempty open subsets of X, with $\subseteq$ as the order. Note that for $U, V \in P$, U is compatible with V iff $U \cap V \neq \emptyset$. Hence ccc holds for P. For each $\alpha < \kappa$ let $D_\alpha = \{p \in P : \overline{p} \subseteq U_\alpha\}$. We claim that D_α is dense in the sense of P. For, suppose that $p \in P$. Since U_α is (topologically) dense, we have $p \cap U_\alpha \neq \emptyset$. By regularity of the space there is a nonempty open set q such that $\overline{q} \subseteq p \cap U_\alpha$. Thus $q \in D_\alpha$ and $q \subseteq p$, as desired.

So, we apply MA(κ) and obtain a filter G intersecting each D_α. Because G is a filter, it has the fip as a collection of open sets. Hence by compactness, $\bigcap_{p\in G} \overline{p} \neq \emptyset$. For any $\alpha < \kappa$ there is a $p \in G \cap D_\alpha$, and hence $\overline{p} \subseteq U_\alpha$. This implies that $\bigcap_{p\in G} \overline{p} \subseteq \bigcap_{\alpha<\kappa} U_\alpha$.

E21.2 *A partial order P is said to have ω_1 as a precaliber iff for every system $\langle p_\alpha : \alpha < \omega_1\rangle$ of elements of P there is an $X \in [\omega_1]^{\omega_1}$ such that for every finite subset F of X there is a $q \in P$ such that $q \leq p_\alpha$ for all $\alpha \in F$.*

Show that MA(ω_1) implies that every ccc partial order P has ω_1 as a precaliber.

Hint: for each $\alpha < \omega_1$ let

$$W_\alpha = \{q \in P : \exists\beta > \alpha(q \text{ and } p_\alpha \text{ are compatible})\}.$$

Show that there is an $\alpha < \omega_1$ such that $W_\alpha = W_\beta$ for all $\beta > \alpha$, and apply MA(ω_1) to W_α.

Let $\langle p_\alpha : \alpha < \omega_1\rangle$ be a system of elements of P; we want to come up with a set X as indicated. For each $\alpha < \omega_1$ let

$$W_\alpha = \{q \in P : \exists\beta > \alpha(q \text{ and } p_\alpha \text{ are compatible})\}.$$

Clearly if $\alpha < \beta < \omega_1$ then $W_\beta \subseteq W_\alpha$. Now we claim

$$(1) \qquad \exists\alpha\forall\beta \in (\alpha, \omega_1)[W_\alpha = W_\beta].$$

In fact, otherwise we get a strictly increasing sequence $\langle \alpha_\xi : \xi < \omega_1\rangle$ of ordinals such that $W_{\alpha_{\xi+1}} \subset W_{\alpha_\xi}$ for all $\xi < \omega_1$. Choose $q_\xi \in W_{\alpha_\xi} \backslash W_{\alpha_{\xi+1}}$ for all $\xi < \omega_1$. Then there is an ordinal β_ξ such that $\alpha_\xi < \beta_\xi \leq \alpha_{\xi+1}$ and q_ξ and p_{β_ξ} are compatible; say $r_\xi \leq q_\xi, p_{\beta_\xi}$. We claim that r_ξ and r_η are incompatible for $\xi < \eta < \omega_1$ (contradicting ccc for P). For, if $s \leq r_\xi, r_\eta$, then q_ξ and p_{β_η} are compatible, and hence $q_\xi \in W_{\alpha_{\xi+1}}$, contradiction. Thus (1) holds.

We are going to apply MA(ω_1) to W_α. The dense sets are as follows. For each $\beta \in (\alpha, \omega_1)$, let

$$D_\beta = \{q \in W_\alpha : \exists\gamma \in (\beta, \omega_1)[q \leq p_\gamma]\}.$$

To prove density, suppose that $r \in W_\alpha$. Then, since $W_\alpha = W_\beta$ it follows that r and p_γ are compatible for some $\gamma > \beta$, as desired.

So, let G be a filter on W_α intersecting each set D_β. It follows that there exist a strictly increasing sequence $\langle \beta_\xi : \xi < \omega_1 \rangle$ and a sequence $\langle q_\xi : \xi < \omega_1 \rangle$ such that $q_\xi \le p_{\beta_\xi}$ with $q_\xi \in G$ for all $\xi < \omega_1$. Clearly then $\{p_{\beta_\xi} : \xi < \omega_1\}$ has the desired property.

E21.3 *Call a topological space X ccc iff every collection of pairwise disjoint open sets in X is countable. Show that $\prod_{i\in I} X_i$ is ccc iff $\forall F \in [I]^{<\omega}[\prod_{i\in F} X_i$ is ccc]. Hint: use the Δ-system theorem.*

$\Rightarrow$: Suppose that $\prod_{i\in I} X_i$ is ccc and $F \in [I]^{<\omega}$. Also suppose that $\mathscr{A}$ is a pairwise disjoint collection of open sets in $\prod_{i\in F} X_i$. Then

$$\mathscr{A}' \stackrel{\text{def}}{=} \left\{ \left\{ x \in \prod_{i\in I} X_i : \langle x_i : i \in T\rangle \in U \right\} : U \in \mathscr{A} \right\}$$

is a collection of pairwise disjoint open sets in $\prod_{i\in I} X_i$, and hence $\mathscr{A}'$ is countable. Clearly this implies that $\mathscr{A}$ is countable.

$\Leftarrow$: Suppose that $\forall F \in [I]^{<\omega}[\prod_{i\in F} X_i$ is ccc], and $\mathscr{B}$ is a collection of pairwise disjoint open sets in $\prod_{i\in I} X_i$. Suppose that $\mathscr{B}$ is uncountable; we want to get a contradiction. We may assume that each $U \in \mathscr{B}$ is basic open, which means that there exist a finite set F_U and an open set V_U in $\prod_{i\in F} X_i$ such that $U = \{x \in \prod_{i\in I} X_i : \langle x_i : i \in F_U\rangle \in V_U\}$.

E21.4 *Assuming* MA(ω_1), *show that any product of ccc spaces is ccc.*

By exercise E21.3 it suffices to show that any product of two ccc spaces X, Y is ccc. Suppose that $\mathscr{A}$ is an uncountable collection of pairwise disjoint open subsets of $X \times Y$; we want to get a contradiction. We may assume that each member of $\mathscr{A}$ has the form $U \times V$, with U open in X and V open in Y. Let $\langle U_\alpha \times V_\alpha : \alpha < \omega_1\rangle$ be a one-one enumeration of a subset of $\mathscr{A}$. Let P be the partially ordered set consisting of all nonempty open subsets of X, ordered by $\subseteq$. Thus by exercise 21.2, P has ω_1 as a precaliber. Hence let $M \in [\omega_1]^{\omega_1}$ be such that for every finite subset F of M there is a $V \in P$ such that $V \subseteq U_\alpha$ for all $\alpha \in F$. Take any distinct $\alpha, \beta \in M$. Then $U_\alpha \cap U_\beta \ne \emptyset$, while $(U_\alpha \times V_\alpha) \cap (U_\beta \times V_\beta) = \emptyset$, so $V_\alpha \cap V_\beta = \emptyset$. This contradicts ccc for Y.

E21.5 *Assume* MA(ω_1). *Suppose that P and Q are ccc partially ordered sets. Define $\le$ on $P \times Q$ by setting $(a, b) \le (c, d)$ iff $a \le c$ and $b \le d$. Show that $<$ is a ccc partial order on $P \times Q$. Hint: use exercise E21.2.*

Suppose that $\langle (p_\alpha, q_\alpha) : \alpha < \omega_1\rangle$ is a system of elements of $P \times Q$; we want to find distinct $\alpha, \beta < \omega_1$ such that (p_α, q_α) and (p_β, q_β) are compatible. By exercise 21.2, let $\Gamma \in [\omega_1]^{\omega_1}$ be such that for every finite subset F of Γ there is an $r \in P$ such that $r \le p_\alpha$ for all $\alpha \in F$. Since Q has ccc, there exist distinct $\alpha, \beta \in \Gamma$ such that q_α and q_β are compatible. Also, p_α and p_β are compatible. So (p_α, q_α) and (p_β, q_β) are compatible.

E21.6 *We define $<^*$ on ${}^\omega\omega$ by setting $f <^* g$ iff $f, g \in {}^\omega\omega$ and $\exists n \forall m > n(f(m) < g(m)$. Suppose that MA($\kappa$) holds and $\mathscr{F} \in [{}^\omega\omega]^\kappa$. Show that there is a $g \in {}^\omega\omega$ such that $f <^* g$*

for all $f \in \mathscr{F}$. Hint: let P be the set of all pairs (p, F) such that p is a finite function mapping a subset of ω into ω and F is a finite subset of $\mathscr{F}$. Define $(p, F) \le (q, G)$ iff $q \subseteq p$, $G \subseteq F$, and

$$\forall f \in G \forall n \in \mathrm{dmn}(p) \backslash \mathrm{dmn}(q)[p(n) > f(n)].$$

$\mathbb{P}$ is ccc: assume that X is an uncountable subset of $\mathbb{P}$. There are only countably many finite functions from ω to ω, so there are distinct $(p, F), (p, G) \in \mathbb{P}$. Then $(p, F \cup G) \le (p, F), (p, G)$. So X is not an incompatible set.

For each $h \in \mathscr{F}$ let $D_h = \{(p, F) \in \mathbb{P} : h \in F\}$. Then D_h is dense, since for any $(p, F) \in \mathscr{P}$ we have $(p, F \cup \{h\}) \le (p, F)$.

For each $n \in \omega$ let $E_n = \{(p, F) : n \in \mathrm{dmn}(p)\}$. Then E_n is dense. For, suppose that $(p, F) \in \mathbb{P}$ and $n \notin \mathrm{dmn}(p)$. Choose $m \in \omega$ greater than each member of $\{f(n) : f \in F\}$. Then clearly $(p \cup \{(n, m)\}, F) \le (p, F)$.

Let G be a filter on $\mathbb{P}$ intersecting each D_h and E_n. Let $g = \bigcup_{(p,F)\in F} p$. Then g is a function since G is a filter. Moreover, $g \in {}^\omega\omega$ since $G \cap E_n \ne \emptyset$ for all n. Now let $f \in \mathscr{F}$. Choose $(p, F) \in G \cap D_f$. Thus $p \in F$. We claim that if m is greater than each member of $\mathrm{dmn}(p)$ then $g(m) > f(m)$ (as desired).

Since $m \in \mathrm{dmn}(g)$, choose $(q, H) \in G$ such that $m \in \mathrm{dmn}(q)$. Choose $(r, K) \in G$ with $(r, K) \le (p, F), (q, H)$. Then $m \in \mathrm{dmn}(q) \subseteq \mathrm{dmn}(r)$, so $m \in \mathrm{dmn}(r)$. Hence $m \in \mathrm{dmn}(r) \backslash \mathrm{dmn}(p)$, and $(r, K) \le (p, F)$, so $g(m) = r(m) > f(m)$.

E21.7 *Let $\mathscr{B} \subseteq [\omega]^\omega$ be almost disjoint of size κ, with $\omega \le \kappa < 2^\omega$. Let $\mathscr{A} \subseteq \mathscr{B}$ with $\mathscr{A}$ countable. Assume MA(κ). Show that there is a $d \subseteq \omega$ such that $|d \cap x| < \omega$ for all $x \in \mathscr{A}$, and $|x \backslash d| < \omega$ for all $x \in \mathscr{B} \backslash \mathscr{A}$. Hint: Let $\langle a_i : i \in \omega \rangle$ enumerate $\mathscr{A}$. Let*

$$\mathbb{P} = \{(s, F, m) : s \in [\omega]^{<\omega}, F \in [\mathscr{B} \backslash \mathscr{A}]^{<\omega}, \text{ and } m \in \omega\};$$
$$(s', F', m') \le (s, F, m) \text{ iff } s \subseteq s', F \subseteq F', m \le m', \text{ and}$$
$$\forall x \in F \left[\left(x \backslash \bigcup_{i\in m} a_i\right) \cap s' \subseteq s\right].$$

Show that $\mathbb{P}$ satisfies ccc. To apply MA(κ), one needs various dense sets. The most complicated is defined as follows. Let $\mathscr{D} = \{(s, F, m, i, n) : (s, F, m) \in \mathbb{P}, i < m$, and $n \in a_i \backslash s\}$. Clearly $|\mathscr{D}| = \kappa$. For each $(s, F, m, i, n) \in \mathscr{D}$ let

$$E_{(s,F,m,i,n)} = \{(s', F', m') \in \mathbb{P} : (s, F, m) \text{ and } (s', F', m') \text{ are incompatible or } (s', F', m') \le (s, F, m) \text{ and } n \in s'\}.$$

$\mathbb{P}$ is ccc: assume that X is an uncountable subset of $\mathbb{P}$. There are only countably many finite functions from ω to ω, so there are distinct $(p, F), (p, G) \in \mathbb{P}$. Then $(p, F \cup G) \le (p, F), (p, G)$. So X is not an incompatible set.

For each $x \in \mathscr{B} \backslash \mathscr{A}$ let $D_x = \{(s, F, m) \in \mathbb{P} : x \in F\}$. Then D_x is dense, since clearly $(s, F \cup \{x\}, m) \le (s, F, m)$ for any $(s, F, m) \in \mathbb{P}$.

Let $\mathscr{D} = \{(s, F, m, i, n) : (s, F, m) \in \mathbb{P}, i < m$, and $n \in a_i \backslash s\}$. Clearly $|\mathscr{D}| = \kappa$. For each $(s, F, m, i, n) \in \mathscr{D}$ let

$$E_{(s,F,m,i,n)} = \{(s', F', m') \in \mathbb{P} : (s, F, m) \text{ and } (s', F', m') \text{ are incompatible or } (s', F', m') \le (s, F, m) \text{ and } n \in s'\}.$$

Now $E_{(s,F,m,i,n)}$ is dense for each $(s,F,m,i,n) \in \mathscr{D}$. For, suppose that (s',F',m') is given. We may assume that (s,F,m) and (s',F',m') are compatible; say $(s'',F'',m'') \le (s,F,m),(s',F',m')$. We may assume that $n \notin s''$. We claim that $(s'' \cup \{n\},F'',m'') \le (s'',F'',m'')$ (as desired). This is true since for any $x \in F''$ we have $n \notin (x \backslash \bigcup_{j\in m''} a_j)$, by virtue of $n \in a_i$ and $i \in m \le m''$.

Next, for any $i < \omega$ let $H_i = \{(s,F,m) \in \mathbb{P} : i < m\}$. Then H_i is dense, since $(s,F,i+1) \le (s,F,m)$ for any $m \le i$.

Now let G be a filter on $\mathbb{P}$ intersecting all of these dense sets. Let $d = \bigcup_{(s,F,m)\in G} s$. Take any $x \in \mathscr{B}\backslash\mathscr{A}$, and choose $(s,F,m) \in G \cap D_x$; so $x \in F$. We claim that $x \cap d \subseteq \bigcup_{i\in m}(x \cap a_i) \cup s$, so that $x \cap d$ is finite. For, suppose that $n \in x \cap d$. choose $(s',F',m') \in G$ such that $n \in s'$. Take $(s'',F'',m'') \in G$ with $(s'',F'',m'') \le (s,F,m),(s',F',m')$. Then $n \in s''$. Since $(x \backslash \bigcup_{i\in m} a_i) \cap s'' \subseteq s$ and $n \in x \cap s''$, it follows that $n \in \bigcup_{i\in m} a_i \cup s$, as desired.

Next, take any $i < \omega$. Choose $(s,F,m) \in G \cap H_i$. Thus $i < m$. We claim that $a_i \backslash s \subseteq d$, so that $a \backslash d$ is finite. To prove this, take any $n \in a_i \backslash s$. Then $(s,F,m,i,n) \in \mathscr{D}$, so we can choose $(s',F',m') \in G \cap E_{(s,F,m,i,n)}$. Since (s,F,m) and (x',F',m') are compatible as elements of G, it follows that $(s',F',m') \le (s,F,m)$ and $n \in s'$. Thus $n \in d$, as desired. This proves the claim.

$\boxed{\text{E21.8}}$ *[The condition that $\mathscr{A}$ is countable is needed in exercise E21.7.] Show that there exist $\mathscr{A},\mathscr{B}$ such that $\mathscr{B}$ is an almost disjoint family of infinite subsets of ω, $\mathscr{A} \subseteq \mathscr{B}$, $|\mathscr{A}| = |\mathscr{B}\backslash\mathscr{A}| = \omega_1$, and there does not exist a $d \subseteq \omega$ such that $|x\backslash d| < \omega$ for all $x \in \mathscr{A}$, and $|x \cap d| < \omega$ for all $x \in \mathscr{B}\backslash\mathscr{A}$. Hint: construct $\mathscr{A} = \{a_\alpha : \alpha < \omega_1\}$ and $\mathscr{B}\backslash\mathscr{A} = \{b_\alpha : \alpha < \omega_1\}$ by constructing a_α, b_α inductively, making sure that the elements are infinite and pairwise almost disjoint, and also $a_\alpha \cap b_\alpha = \emptyset$, while for $\alpha \ne \beta$ we have $a_\alpha \cap b_\beta \ne \emptyset$.*

First we show that the hint works. Suppose that we have constructed $\langle a_\alpha : \alpha < \omega_1\rangle$ and $\langle b_\alpha : \alpha < \omega_1\rangle$, so that they are infinite and pairwise almost disjoint, with the additional indicated property. Suppose that d exists as indicated. Wlog $\forall\alpha < \omega_1(a_\alpha\backslash d = F$ and $b_\alpha \cap d = G)$. Choose $m \in a_0 \cap b_1$. If $m \in d$, then $m \in b_1 \cap d = G \subseteq b_0$, so $m \in a_0 \cap b_0$, contradiction. If $m \notin d$, then $m \in a_0\backslash d = F \subseteq a_1$, so $m \in a_1 \cap b_1$, contradiction.

Now we do the construction indicated in the hint. Let $\langle c_i : i < \omega\rangle$ be a system of pairwise disjoint infinite subsets of ω. Let $\langle c_{i,j} : j < \omega\rangle$ be a one-one enumeration of c_i. Then we define for each $m \in \omega$

$$\begin{aligned} a_m &= c_{2m} \cup \{c_{2n+1,0} : n < m\};\\ b_m &= c_{2m+1} \cup \{c_{2n,0} : n < m\}. \end{aligned}$$

Clearly this defines infinite pairwise almost disjoint subsets of ω, $a_m \cap b_m = \emptyset$ for all $m \in \omega$, and for $n < m$ we have $c_{2n,0} \in a_n \cap b_m$ and $c_{2n+1,0} \in a_m \cap b_n$.

Now suppose that a_β and b_β have been defined for all $\beta < \alpha$, with $\omega \le \alpha < \omega_1$. Let f be a function from ω onto α. By recursion, choose

$$x_m \in b_{f(m)} \backslash \left(\bigcup_{n<m} b_{f(n)} \cup \bigcup_{n<m} a_{f(n)} \cup \{x_n : n < m\} \cup \{y_n : n < m\} \right);$$

$$y_m \in a_{f(m)} \backslash \left(\bigcup_{n<m} b_{f(n)} \cup \bigcup_{n<m} a_{f(n)} \cup \{x_n : n \leq m\} \cup \{y_n : n < m\} \right).$$

Let $a_\alpha = \{x_m : m \in \omega\}$ and $b_\alpha = \{y_m : m \in \omega\}$. Then $a_\alpha \cap b_\alpha = \emptyset$ and

$$\begin{aligned} x_m \in a_\alpha \cap b_{f(m)} &\subseteq \{x_0, \ldots, x_m\}; \\ y_m \in b_\alpha \cap a_{f(m)} &\subseteq \{y_0, \ldots, y_m\}; \\ a_\alpha \cap a_{f(m)} &\subseteq \{x_0, \ldots, x_m\}; \\ b_\alpha \cap b_{f(m)} &\subseteq \{y_0, \ldots, y_m\}. \end{aligned}$$

Hence the construction is complete.

E21.9 *Suppose that $\mathscr{A}$ is a family of infinite subsets of ω such that $\bigcap F$ is infinite for every finite subset F of $\mathscr{A}$. Suppose that $|\mathscr{A}| \leq \kappa$. Assuming MA($\kappa$), show that there is an infinite $X \subseteq \omega$ such that $X \backslash A$ is finite for every $A \in \mathscr{A}$. Hint: use Theorem 21.5.*

Let $\mathscr{A}' = \{X \subseteq \omega : \omega \backslash X \in \mathscr{A}\}$, and let $\mathscr{B} = \{\omega\}$. Clearly the hypothesis of Theorem 21.5 holds for $\mathscr{A}'$ and $\mathscr{B}$. Hence by 21.5 there is a $d \subseteq \omega$ such that $|d \cap X| < \omega$ for all $X \in \mathscr{A}'$ and $|d| = |d \cap \omega| = \omega$. Clearly d is as desired.

E21.10 *Show that MA(κ) is equivalent to MA(κ) restricted to ccc partial orders of cardinality $\leq \kappa$. Hint: Assume the indicated special form of MA(κ), and assume given a ccc partially ordered set P and a family $\mathscr{D}$ of at most κ dense sets in P; we want to find a filter on P intersecting each member of $\mathscr{D}$. We introduce some operations on P. For each $D \in \mathscr{D}$ define $f_D : P \to P$ by setting, for each $p \in P$, $f_D(p)$ to be some element of D which is $\leq p$. Also we define $g : P \times P \to P$ by setting, for all $p, q \in P$,*

$$g(p,q) = \begin{cases} p & \textit{if } p \textit{ and } q \textit{ are incompatible,} \\ r & \textit{with } r \leq p, q \textit{ if there is such an } r. \end{cases}$$

Here, as in the definition of f_D, we are implicitly using the axiom of choice; for g, we choose any r of the indicated form.

We may assume that $\mathscr{D} \neq \emptyset$. Choose $D \in \mathscr{D}$, and choose $s \in D$. Now let Q be the intersection of all subsets of P which have s as a member and are closed under all of the operations f_D and g. We take the order on Q to be the order induced from P. Apply the special form to Q.

We assume the indicated special form of MA(κ), and assume given a ccc partially ordered set P and a family $\mathscr{D}$ of at most κ dense sets in P; we want to find a filter on P intersecting each member of $\mathscr{D}$. We introduce some operations on P. For each $D \in \mathscr{D}$ define $f_D : P \to P$ by setting, for each $p \in P$, $f_D(p)$ to be some element of D which is $\leq p$. Also we define $g : P \times P \to P$ by setting, for all $p, q \in P$,

$$g(p,q) = \begin{cases} p & \text{if } p \text{ and } q \text{ are incompatible,} \\ r & \text{with } r \leq p, q \text{ if there is such an } r. \end{cases}$$

Here, as in the definition of f_D, we are implicitly using the axiom of choice; for g, we choose any r of the indicated form.

We may assume that $\mathscr{D} \neq \emptyset$. Choose $D \in \mathscr{D}$, and choose $s \in D$. Now let Q be the intersection of all subsets of P which have s as a member and are closed under all of the operations f_D and g. We take the order on Q to be the order induced from P.

(1) $|Q| \leq \kappa$.

To prove this, we give an alternative definition of Q. Define

$$\begin{aligned} R_0 &= \{s\}; \\ R_{n+1} &= R_n \cup \{g(a,b) : a, b \in R_n\} \cup \{f_D(a) : D \in \mathscr{D} \text{ and } a \in R_n\}. \end{aligned}$$

Clearly $\bigcup_{n\in\omega} R_n = Q$. By induction, $|R_n| \leq \kappa$ for all $n \in \omega$, and hence $|Q| \leq \kappa$, as desired in (1).

We also need to check that Q is ccc. Suppose that X is a collection of pairwise incompatible elements of Q. Then these elements are also incompatible in P, since $x, y \in X$ with x, y compatible in P implies that $g(x,y) \leq x, y$ and $g(x,y) \in Q$, so that x, y are compatible in Q. It follows that X is countable. So Q is ccc.

Now if $D \in \mathscr{D}$, then $D \cap Q$ is dense in Q. In fact, take any $q \in Q$. Then $f_D(q) \in Q$ and $f_D(q) \leq q$, as desired.

Now we can apply our special case of MA(κ) to Q and $\{D \cap Q : D \in \mathscr{D}\}$; we obtain a filter G on Q such that $G \cap D \cap Q \neq \emptyset$ for all $D \in \mathscr{D}$. Let

$$G' = \{p \in P : q \leq p \text{ for some } q \in G\}.$$

We claim that G' is the desired filter on P intersecting each $D \in \mathscr{D}$.

Clearly if $p \in G'$ and $p \leq r$, then $r \in G'$.

Suppose that $p_1, p_2 \in G'$. Choose $q_1, q_2 \in G$ such that $q_i \leq p_1$ for each $i = 1, 2$. Then there is an $r \in G$ such that $r \leq q_1, q_2$. Then $r \in G'$ and $r \leq p_1, p_2$. So G' is a filter on P.

Now take any $D \in \mathscr{D}$. Then as proved above, $D \cap Q$ is dense in Q. It follows that $G \cap D \cap Q \neq \emptyset$; say $q \in G \cap D \cap Q$. Then $q \in G' \cap D$, as desired.

E21.11 *Define $x \subset^* y$ iff $x, y \subseteq \omega$, $x \backslash y$ is finite, and $y \backslash x$ is infinite. Assume* MA(κ), *and suppose that $L, <)$ is a linear ordering of size at most κ. Show that there is a system $\langle a_x : x \in L \rangle$ of infinite subsets of ω such that for all $x, y \in L$, $x < y$ iff $a_x \subset^* a_y$. Hint: let P consist of all pairs (p, n) such that $n \in \omega$, p is a function whose domain is a finite subset of L, and $\forall x \in \text{dmn}(p)[p(x) \subseteq n]$. Define $(p, n) \leq (q, m)$ iff $m \leq n$, $\text{dmn}(q) \subseteq \text{dmn}(p)$, $\forall x \in \text{dmn}(q)[p(x) \cap m = q(x)]$, and $\forall x, y \in \text{dmn}(q)[x < y \rightarrow p(x) \backslash p(y) \subseteq m]$.*

First note that it is enough to do the construction so that $\forall x, y \in L[x < y \rightarrow a_x \subset^* a_y]$. In fact, knowing this, if $a_x \subset^* a_y$, then we must have $x < y$, as otherwise $y \leq x$ and hence $a_y \subset^* a_x$ or $a_y = a_x$, both of which are ruled out by $a_x \subset^* a_y$.

Now we show that P has ccc. Suppose that $\mathscr{A}$ is an uncountable subset of P. By the indexed Δ-system theorem, Theorem 16.4, there is an uncountable subset $\mathscr{B}$ of $\mathscr{A}$ such that

$\langle \mathrm{dmn}(p) : (p,n) \in \mathscr{B}\rangle$ is an indexed Δ-system. Say $M \in [L]^{<\omega}$ with $\mathrm{dmn}(p) \cap \mathrm{dmn}(q) = M$ for any distinct $(p,n),(q,m) \in \mathscr{B}$. Now let $Q = {}^M\omega$. Then

$$\mathscr{B} = \bigcup_{f\in Q} \{(p,n) \in \mathscr{B} : \forall x \in M[p(x) = f(x)]\}.$$

Since Q is countable, there exist a $\mathscr{C} \in [\mathscr{B}]^{<\omega}$ and an $f \in Q$ such that $\forall (p,n) \in \mathscr{C} \forall x \in M[p(x) = f(x)]$. Now take two distinct members (p,n) and (q,m) of $\mathscr{C}$. Say $m \le n$. We now define a function r. $\mathrm{dmn}(r) = \mathrm{dmn}(p) \cup \mathrm{dmn}(q)$. For any $x \in \mathrm{dmn}(r)$,

$$r(x) = \begin{cases} p(x) & \text{if } x \in \mathrm{dmn}(p), \\ q(x) & \text{if } x \in \mathrm{dmn}(q) \backslash \mathrm{dmn}(p). \end{cases}$$

Clearly $(r,n) \in P$. We claim $(r,n) \le (p,m),(q,n)$; this will show that P has ccc. First we show that $(r,n) \le (p,m)$. We have $m \le n$ and $\mathrm{dmn}(p) \subseteq \mathrm{dmn}(r)$. Take any $x \in \mathrm{dmn}(p)$. Then $r(x) \cap m = p(x) \cap m = p(x)$. Suppose that $x,y \in \mathrm{dmn}(p)$ and $x < y$. Then $r(x)\backslash r(y) = p(x)\backslash p(y) \subseteq m$. Therefore $(r,n) \le (p,m)$. Second we show that $(r,n) \le (q,n)$. We have $n \le n$ and $\mathrm{dmn}(q) \subseteq \mathrm{dmn}(r)$. Take any $x \in \mathrm{dmn}(q)$. If also $x \in \mathrm{dmn}(p)$ then $x \in M$, and $r(x) \cap n = p(x) \cap n = f(x) \cap n = q(x) \cap n = q(x)$. Suppose that $x,y \in \mathrm{dmn}(q)$ and $x < y$.

Case 1. $x,y \in \mathrm{dmn}(p)$. Then $r(x)\backslash r(y) = p(x)\backslash p(y) \subseteq m \le n$.

Case 2. $x \in \mathrm{dmn}(p)$ and $y \notin \mathrm{dmn}(p)$. Then $x \in M$, and so $r(x)\backslash r(y) = p(x)\backslash q(y) = f(x)\backslash q(y) = q(x)\backslash q(y) \subseteq n$.

Case 3. $x \notin \mathrm{dmn}(p)$ and $y \in \mathrm{dmn}(p)$. Similar to Case 2.

Case 4. $x,y \notin \mathrm{dmn}(p)$. The conclusion is clear.

This finishes the proof that P has ccc.

Now after defining certain dense sets we are going to take a filter G with respect to them and then define

$$a_x = \bigcup_{\substack{(p,n)\in G \\ x\in \mathrm{dmn}(p)}} p(x)$$

for each $x \in L$.

To show that we can define a_x for each $x \in L$ we consider $D_x \stackrel{\text{def}}{=} \{(p,n) \in P : x \in \mathrm{dmn}(p)\}$. Then D_x is dense, since given $(p,n) \in P$ with $x \notin \mathrm{dmn}(p)$, we may assume that $n > 0$, and we have $(p \cup \{(x,0)\},n) \in P$ and $(p \cup \{(x,0)\},n) \le (p,n)$, as is easily checked.

To show that for a given $x \in L$ the set a_x is infinite, we consider for each $i \in \omega$ the set

$$E_{ix} \stackrel{\text{def}}{=} \{(p,n) : x \in \mathrm{dmn}(p) \text{ and } i < n \text{ and } p(x) \not\subseteq i\}.$$

To show that this set is dense, let $(q,m) \in P$. By the argument for D_x we may assume that $x \in \mathrm{dmn}(q)$. Let $n = \max(i+1,m)$. Let p have domain $\mathrm{dmn}(q)$, and for each $y \in \mathrm{dmn}(q)$ let $p(y) = q(y) \cup \{n\}$. Clearly $(p,n+1) \in P$. We have $m < n+1$ and $\mathrm{dmn}(q) = \mathrm{dmn}(p)$. For any $y \in \mathrm{dmn}(q)$ we have $p(y) \cap m = q(y)$. Now suppose that $y,z \in \mathrm{dmn}(q)$ and $y < z$. Then $p(y)\backslash p(z) = q(y)\backslash q(z) \subseteq m$. So $(p,n+1) \le (q,m)$, and clearly $(p,n+1) \in E_{ix}$.

Next, for any $x, y \in L$ such that $x < y$ and any $i \in \omega$ let

$$F_{ixy} = \{(p,n) \in P : x, y \in \mathrm{dmn}(p) \text{ and } \exists j \geq i[j \in p(y)\backslash p(x)]\}.$$

To show that this set is dense, let (q, m) be given. Wlog $x, y \in \mathrm{dmn}(q)$. Let $n = \max(i + 1, m)$. Let $\mathrm{dmn}(p) = \mathrm{dmn}(q)$, and for $z \in \mathrm{dmn}(q)$ let

$$p(z) = \begin{cases} q(z) & \text{if } z \neq y, \\ q(z) \cup \{n\} & \text{if } y \leq z. \end{cases}$$

Clearly $(p, n+1) \in P$. Since $p(x) = q(x) \subseteq m$ and $n \geq m$, we have $n \in p(y)\backslash p(x)$. If $z \in \mathrm{dmn}(q)$, then $p(z) \cap m = q(z)$. Suppose that $u, v \in \mathrm{dmn}(q)$ and $u < v$. Then $p(u)\backslash p(v) \subseteq m$; this is only questionable if $y \leq u$, and this case is clear. Thus $(p, n+1) \leq (q, m)$.

Now we take a filter G with respect to all of these dense sets. So we can define a_x for $x \in L$ as above, and each a_x is infinite. Now suppose that $x, y \in L$ and $x < y$.

Using D_x and D_y, let $(p, n) \in G$ with $x, y \in \mathrm{dmn}(p)$. We claim that $a_x\backslash a_y \subseteq n$. We prove that $a_x\backslash n \subseteq a_y$. Suppose that $i \in a_x\backslash n$. Choose $(q, m) \in G$ such that $x \in \mathrm{dmn}(q)$ and $i \in q(x)$. Then choose $(r, s) \in G$ such that $(r, s) \leq (p, n), (q, m)$. Now $i \in q(x)$, so $i \in r(x)$. Since $x < y$ and $x, y \in \mathrm{dmn}(p)$, we have $r(x)\backslash r(y) \subseteq n$. Now $i \geq n$, so it follows that $i \in r(y)$. Hence $i \in a_y$, as desired.

It remains only to show that $a_y\backslash a_x$ is infinite. Suppose it is finite; say $a_y\backslash a_x \subseteq i$ with $i \in \omega$. Choose $(p, n) \in F_{ixy} \cap G$. Thus $x, y \in \mathrm{dmn}(p)$ and there is a $j \geq i$ such that $j \in p(y)\backslash p(x)$. Thus $j \in a_y$. Since $a_y\backslash i \subseteq a_x$, we have $j \in a_x$. Choose $(q, m) \in G$ such that $x \in \mathrm{dmn}(q)$ and $j \in q(x)$. Choose $(r, s) \leq (p, n), (q, m)$. Then $j \in q(x)$, so $j \in r(x)$. Now $r(x) \cap n = p(x)$, so $j \in p(x)$, contradiction.

E21.12 *If $\mathscr{A}, \mathscr{B}$ are nonempty countable subsets of $[\omega]^\omega$ and $a \subseteq^* b$ whenever $a \in \mathscr{A}$ and $b \in \mathscr{B}$, then there is a $c \in [\omega]^\omega$ such that $a \subseteq^* c \subseteq^* b$ whenever $a \in \mathscr{A}$ and $b \in \mathscr{B}$.*

Write $\mathscr{A} = \{a_n : n \in \omega\}$ and $\mathscr{B} = \{b_n : n \in \omega\}$. Let

$$c = \bigcup_{n\in\omega} \left[\left(\bigcup_{m\leq n} a_m \right) \cap \bigcap_{m\leq n} b_m \right].$$

Now suppose that $p \in \omega$. Then

$$\begin{aligned} a_p\backslash c &= \bigcap_{n\in\omega} \left[a_p \cap \left(\bigcap_{m\leq n} (\omega\backslash a_m) \cup \bigcup_{m\leq n} (\omega\backslash b_m) \right) \right] \\ &= \bigcap_{n<p} \left[a_p \cap \left(\bigcap_{m\leq n} (\omega\backslash a_m) \cup \bigcup_{m\leq n} (\omega\backslash b_m) \right) \right] \\ &\quad \cap \bigcap_{n\geq p} \left[a_p \cap \left(\bigcap_{m\leq n} (\omega\backslash a_m) \cup \bigcup_{m\leq n} (\omega\backslash b_m) \right) \right] \end{aligned}$$

$$
\begin{aligned}
&= \bigcap_{n<p}\left[a_p \cap \left(\bigcap_{m\le n}(\omega\backslash a_m) \cup \bigcup_{m\le n}(\omega\backslash b_m)\right)\right]\\
&\quad\cap \bigcap_{n\ge p}\left[a_p \cap \bigcup_{m\le n}(\omega\backslash b_m)\right]\\
&\subseteq a_p \cap \bigcup_{m\le p}(\omega\backslash b_m),
\end{aligned}
$$

and this last set is finite.

Furthermore,

$$
\begin{aligned}
c\backslash b_p &= \bigcup_{n<p}\left[\left(\bigcup_{m\le n} a_m\right) \cap \bigcap_{m\le n} b_m \cap (\omega\backslash b_p)\right]\\
&\subseteq \left(\bigcup_{m<p} a_m\right)\backslash b_p,
\end{aligned}
$$

and this last set is finite.

The set c is infinite, as otherwise $a_0 = (a_0 \cap c) \cup (a_0\backslash c)$ would be finite.

E21.13 *Suppose that $\mathscr{A}$ is a nonempty countable family of members of $[\omega]^\omega$, and $\forall a, b \in \mathscr{A}[a \subseteq^* b$ or $b \subseteq^* a]$. Also suppose that $\forall a \in \mathscr{A}[a \subset^* d]$, where $d \in [\omega]^\omega$. Then there is a $c \in [\omega]^\omega$ such that $\forall a \in \mathscr{A}[a \subseteq^* c \subset^* d]$.*

If $\exists a \in \mathscr{A}\forall b \in \mathscr{A}[b \subseteq^* a]$, then the conclusion is obvious. So suppose that no such a exists. Then there is a sequence $\langle a_n : n \in \omega\rangle$ of elements of $\mathscr{A}$ such that $a_n \subset^* a_m$ for $n < m$, and the sequence is cofinal in $\mathscr{A}$ in the $\subseteq^*$-sense. Let $\mathscr{C} = \{a_0\} \cup \{a_{m+1}\backslash a_m : m \in \omega\} \cup \{\omega\backslash d\}$. Then $\mathscr{C}$ is an almost disjoint family, except that possibly $\omega\backslash d$ is finite. By Theorem 11.1 and Corollary 11.6, let $e \subseteq \omega$ be infinite and almost disjoint from each member of $\mathscr{C}$. Let $c = d\backslash e$. Then for any $n \in \omega$,

$$
\begin{aligned}
a_{n+1}\backslash c &= (a_{n+1}\backslash d) \cup (a_{n+1} \cap e)\\
&\subseteq (a_{n+1}\backslash d) \cup \left[\bigcup_{i\le n}(a_{i+1}\backslash a_i) \cup a_0\right] \cap e,
\end{aligned}
$$

and the last set is finite. Thus $a_{n+1} \subseteq^* c$, hence $b \subseteq^* c$ for all $b \in \mathscr{A}$.

Since $c \subseteq d$, we have $c \subseteq^* d$. Also, $d\backslash c = d \cap e$, and this is infinite since $e\backslash d$ is finite. Thus $c \subset^* d$

Note that c is infinite, since $a \subseteq^* c$ for all $a \in \mathscr{A}$.

E21.14 *If $a, b \in [\omega]^\omega$ and $a \subset^* b$, then there is a $c \in [\omega]^\omega$ such that $a \subset^* c \subset^* b$.*

Write $b\backslash a = d \cup e$ with d, e infinite and disjoint. Let $c = a \cup d$.

E21.15 *Suppose that $\mathscr{A}$ and $\mathscr{B}$ are nonempty countable subsets of $[\omega]^\omega$, $\forall x, y \in \mathscr{A}[x \subseteq^* y$ or $y \subseteq^* x]$, $\forall x, y \in \mathscr{B}[x \subseteq^* y$ or $y \subseteq^* x]$, and $\forall x \in \mathscr{A} \forall y \in \mathscr{B}[a \subset^* b]$. Then there is a $c \in [\omega]^\omega$ such that $a \subset^* c \subset^* b$ for all $a \in \mathscr{A}$ and $b \in \mathscr{B}$.*

By Exercise 21.12 choose $d \subseteq \omega$ such that $\forall a \in \mathscr{A} \forall b \in \mathscr{B}[a \subseteq^* d \subseteq^* b]$. Thus either $\forall a \in \mathscr{A}[a \subset^* d]$ or $\forall b \in \mathscr{B}[d \subset^* b]$.

Case 1. $\forall a \in \mathscr{A}[a \subset^* d]$. By Exercise 21.13 choose $e \subseteq \omega$ such that $\forall a \in \mathscr{A}[a \subseteq^* e \subset^* d]$. By Exercise 11.14 choose $c \in [\omega]^\omega$ such that $e \subset^* c \subset^* d$.

Case 2. $\forall b \in \mathscr{B}[d \subset^* b]$. Then $\forall b \in \mathscr{B}[(\omega \backslash b) \subset^* (\omega \backslash d)]$. By exercise E21.13 choose $e \subseteq \omega$ such that $\forall b \in \mathscr{B}[(\omega \backslash b) \subseteq^* e \subset^* (\omega \backslash d)]$. By exercise E21.14 choose $c \subseteq \omega$ such that $e \subset^* c \subset^* (\omega \backslash d)$. Then $\forall a \in \mathscr{A} \forall b \in \mathscr{B}[a \subset^* (\omega \backslash c) \subset^* b]$.

E21.16 *Suppose that $a_m \in [\omega]^\omega$ for all $m \in \omega$, $a_m \subset^* a_n$ whenever $m < n \in \omega$, $b \in [\omega]^\omega$, and $a_m \subset^* b$ for all $m \in \omega$. Then there is a $c \in [\omega]^\omega$ such that $\forall m \in \omega[a_m \subset^* c \subset^* b]$ and c is near to $\{a_n : n \in \omega\}$.*

By Exercise 21.15 choose $d \subseteq \omega$ such that $\forall m \in \omega[a_n \subset^* d \subset^* b]$. Now for each $m \in \omega$, $\bigcup_{i \leq m}(a_m \cap a_{m+1})$ is finite, and $a_{m+1} \backslash \bigcup_{i \leq m} a_i = a_{m+1} \backslash \bigcup_{i \leq m}(a_{m+1} \cap a_i)$, so $a_{m+1} \backslash \bigcup_{i \leq m} a_i$ is infinite. Choose $e_m \subseteq a_{m+1} \backslash \bigcup_{i \leq m} a_i$ such that $|e_m| = m$. Let $c = d \backslash \bigcup_{m \in \omega} e_m$. Thus $c \subseteq^* d \subset^* b$.

If $n \in \omega$, then

$$a_n \backslash c = (a_n \backslash d) \cup \bigcup_{m \in \omega} (a_n \cap e_m) = (a_n \backslash d) \cup \bigcup_{m < n} (a_n \cap e_m),$$

and this last set is finite. Hence $a_n \subseteq^* c$. Since n is arbitrary, it follows that $a_n \subset^* c$ for all $n \in \omega$.

Also for any $m \in \omega$ we have $a_{m+1} \backslash c \supseteq a_{m+1} \cap e_m = e_m$, and so $|a_{m+1} \backslash c| \geq m$. It follows that for any $n \in \omega$, $\{a_m : a_m \backslash c \subseteq n\} \subseteq \{a_0, \ldots, a_n\}$. So c is near to $\{a_m : m \in \omega\}$.

E21.17 *Suppose that $\mathscr{A} \subseteq [\omega]^\omega$, $\forall x, y \in \mathscr{A}[x \subset^* y$ or $y \subset^* x]$, $b \in [\omega]^\omega$, $\forall x \in \mathscr{A}[x \subset^* b]$, and $\forall a \in \mathscr{A}[b$ is near to $\{d \in \mathscr{A} : d \subset^* a\}]$.*

Then there is a $c \in [\omega]^\omega$ such that $\forall a \in \mathscr{A}[a \subset^* c \subset^* b]$ and c is near to $\mathscr{A}$.

Proof. We consider several cases.

Case 1. $\exists a \in \mathscr{A} \forall d \in \mathscr{A}[d \subseteq^* a]$. By Exercise 21.14, choose c such that $a \subset^* c \subset^* b$. Choose $n \in \omega$ such that $c \backslash b \subseteq n$. Then for any $m \in \omega$ and any $d \in \mathscr{A}$, if $d \backslash c \subseteq m$ then $d \backslash b \subseteq (d \backslash c) \cup (c \backslash b) \subseteq \max(m, n)$. Hence

$$\{d \in \mathscr{A} : d \backslash c \subseteq m\} \subseteq \{a\} \cup \{d \in \mathscr{A} : d \subset^* a \text{ and } d \backslash b \subseteq \max(m, n)\},$$

and the later set is finite, since b is near to $\{d \in \mathscr{A} : d \subset^* a\}$. Thus c is as desired.

Case 2. $\forall a \in \mathscr{A} \exists d \in \mathscr{A}[a \subset^* d]$ and b is near to $\mathscr{A}$. By Exercise 21.15, choose c so that $\forall a \in \mathscr{A}[a \subset^* c \subset^* b]$. Choose $n \in \omega$ such that $c \backslash b \subseteq n$. Then for any $m \in \omega$ and any $d \in \mathscr{A}$, if $d \backslash c \subseteq m$ then $d \backslash b \subseteq (d \backslash c) \cup (c \backslash b) \subseteq \max(m, n)$. Hence

$$\{d \in \mathscr{A} : d \backslash c \subseteq m\} \subseteq \{a\} \cup \{d \in \mathscr{A} : d \backslash b \subseteq \max(m, n)\},$$

and the later set is finite, since b is near to $\mathscr{A}$. Thus c is as desired.

Case 3. $\forall a \in \mathscr{A} \exists d \in \mathscr{A}[a \subset^* d]$ and b is not near to $\mathscr{A}$. For each $m \in \omega$ let $\mathscr{B}_m = \{a \in \mathscr{A} : a \backslash b \subseteq m\}$. Since b is not near to $\mathscr{A}$, choose m so that $\mathscr{B}_m$ is infinite. Note that $p < q \rightarrow \mathscr{B}_p \subseteq \mathscr{B}_q$. Hence $\mathscr{B}_n$ is infinite for every $n \geq m$. Now we claim

$$\forall n \geq m \forall a \in \mathscr{A} \exists d \in \mathscr{B}_n[a \subseteq^* d]. \tag{1}$$

In fact, otherwise we get $n \geq m$ and $a \in \mathscr{A}$ such that $\forall d \in \mathscr{B}_n[d \subset^* a]$. Now b is near to $\{d \in \mathscr{A} : d \subset^* a\}$ by a hypothesis of the lemma, so $\{d \in \mathscr{A} : d \subset^* a$ and $d \backslash b \subseteq n\}$ is finite. But $\mathscr{B}_n \subseteq \{d \in \mathscr{A} : d \subset^* a$ and $d \backslash b \subseteq n\}$, contradiction. So (1) holds.

Next we claim

$$\forall n \geq m \forall d \in \mathscr{B}_n[\{e \in \mathscr{B}_n : e \subset^* d\} \text{ is finite}]. \tag{2}$$

In fact, suppose that $n \geq m$, $d \in \mathscr{B}_n$ and $\{e \in \mathscr{B}_n : e \subset^* d\}$ is infinite. Since b is near to $\{a \in \mathscr{A} : a \subset^* d\}$, the set $\{a \in \mathscr{A} : a \subset^* d$ and $a \backslash b \subseteq n\}$ is finite. But $\{e \in \mathscr{B}_n : e \subset^* d\} \subseteq \{a \in \mathscr{A} : a \subset^* d$ and $a \backslash b \subseteq n\}$, contradiction. So (2) holds.

From (2) it follows that $\mathscr{B}_n$ has order type ω under $\subset^*$, for each $n \geq m$. Now clearly $\mathscr{A} = \bigcup_{p \in \omega} \mathscr{B}_p$, so $\mathscr{A}$ is countable.

Now by Exercise 21.16, choose c_m such that $\forall d \in \mathscr{B}_m[d \subset^* c_m \subset^* b]$ and c_m is near to $\mathscr{B}_m$. By (1), $a \subset^* c_m$ for each $a \in \mathscr{A}$. Now suppose that $n \geq m$ and c_n has been defined so that $a \subset^* c_n$ for each $a \in \mathscr{A}$. Again by Exercise 21.16 choose c_{n+1} such that $\forall d \in \mathscr{B}_{n+1}[d \subset^* c_{n+1} \subset^* c_n]$ and c_{n+1} is near to $\mathscr{B}_{n+1}$. Thus we have

$$\forall a \in \mathscr{A}[a \subset^* \cdots \subset^* c_{n+1} \subset^* c_n \subset^* \cdots \subset^* c_m \subset^* b].$$

By Exercise 21.15, choose d so that $\forall a \in \mathscr{A} \forall n \geq m[a \subset^* d \subset^* c_n]$. We claim that d is near to $\mathscr{A}$, completing the proof. For, let $n \in \omega$. Let $p = \max(m, n)$, and choose $q \geq p$ such that $d \backslash c_p \subseteq q$. Then

$$\begin{aligned} \{a \in \mathscr{A} : a \backslash d \subseteq n\} &\subseteq \{a \in \mathscr{A} : a \backslash d \subseteq p\} \\ &= \{a \in \mathscr{B}_p : a \backslash d \subseteq p\} \\ &\subseteq \{a \in \mathscr{B}_p : a \backslash c_p \subseteq q\}, \end{aligned}$$

where the last inclusion holds since $a \backslash c_p = (a \backslash d) \cup (d \backslash c_p)$. The last set is finite since c_p is near to $\mathscr{B}_p$, as desired.

E21.18 *(The Hausdorff gap) There exist sequences $\langle a_\alpha : \alpha < \omega_1 \rangle$ and $\langle b_\alpha : \alpha < \omega_1 \rangle$ of members of $[\omega]^\omega$ such that $\forall \alpha, \beta < \omega_1[\alpha < \beta \rightarrow a_\alpha \subset^* a_\beta$ and $b_\beta \subset^* b_\alpha]$, $\forall \alpha, \beta < \omega_1[a_\alpha \subset^* b_\beta]$, and there does not exist a $c \subseteq \omega$ such that $\forall \alpha < \omega_1[a_\alpha \subset^* c$ and $c \subset^* b_\alpha]$.*

We construct by recursion $a_\alpha, b_\alpha \subseteq \omega$ for $\alpha < \omega_1$ so that $a_\alpha \subset^* b_\alpha$, $\alpha < \beta \rightarrow a_\alpha \subset^* a_\beta$ and $b_\beta \subset^* b_\alpha$, and for all $\alpha < \omega_1$, b_β is near to $\{a_\alpha : \alpha < \beta\}$.

Let $a_0 = \emptyset$, $b_0 = \omega$. Suppose that a_α and b_α have been constructed for all $\alpha < \beta$ so that $a_\alpha \subset^* b_\alpha$, $\alpha < \gamma < \beta \rightarrow a_\alpha \subset^* a_\gamma$ and $b_\gamma \subset^* b_\beta$, and $\alpha < \beta \rightarrow b_\alpha$ is near to $\{a_\gamma : \gamma < \alpha\}$. By exercise 21.15 choose c such that $\forall \alpha < \beta[a_\alpha \subset^* c \subset^* b_\alpha]$. Suppose that

$\alpha < \beta$. We claim that c is near to $\{a_\gamma : \gamma < \alpha\}$. In fact, suppose that $m \in \omega$. Choose $n \geq m$ such that $c \backslash b_\alpha \subseteq n$. Now for any $\gamma < \alpha$ we have $a_\gamma \backslash b_\alpha \subseteq (a_\gamma \backslash c) \cup (c \backslash b_\alpha)$, so

$$\{a_\gamma : \gamma < \alpha \text{ and } a_\gamma \backslash c \subseteq m\} \subseteq \{a_\gamma : \gamma < \alpha \text{ and } a_\gamma \backslash b_\alpha \subseteq n\},$$

and the latter set is finite since b_α is near to $\{a_\gamma : \gamma < \alpha\}$. Thus indeed c is near to $\{a_\gamma : \gamma < \alpha\}$. Now by exercise E21.17 there is a b_β such that $\forall \alpha < \beta[a_\alpha \subset^* b_\beta \subset^* c]$ and b_β is near to $\{a_a : \alpha < \beta\}$. By exercise E21.15 choose a_β so that $\forall \alpha < \beta[a_\alpha \subset^* a_\beta \subset^* b_\beta]$. This finishes the construction.

Now suppose that $d \subseteq \omega$ and $\forall \alpha < \omega_1[a_\alpha \subset^* d \subset^* b_\alpha]$. Now $\omega_1 = \bigcup_{m \in \omega}\{\alpha < \omega_1 : a_\alpha \backslash d \subseteq m\}$, so we can choose $m \in \omega$ such that $|\{\alpha < \omega_1 : a_\alpha \backslash d \subseteq m\}| = \omega_1$. Hence there is an $\alpha < \omega_1$ such that $\{\beta < \alpha : a_\beta \backslash d \subseteq m\}$ is infinite. Choose $p \geq m$ such that $d \backslash b_\alpha \subseteq p$. Now $a_\beta \backslash b_\alpha \subseteq (a_\beta \backslash d) \cup (d \backslash b_\alpha)$, so $\{\beta < \alpha : a_\beta \backslash d \subseteq m\} \subseteq \{\beta < \alpha : a_\beta \backslash b_\alpha \subseteq p\}$, contradicting b_α near to $\{a_\beta : \beta < \alpha\}$.

Solutions to exercises in Chapter 22

E22.1 *Let κ be an uncountable regular cardinal. We define $S < T$ iff S and T are stationary subsets of κ and the following two conditions hold:*

(1) $\{\alpha \in T : \mathrm{cf}(\alpha) \leq \omega\}$ is nonstationary in κ.

(2) $\{\alpha \in T : S \cap \alpha$ is nonstationary in $\alpha)\}$ is nonstationary in κ.

Prove that if $\omega < \lambda < \mu < \kappa$, all these cardinals regular, then $E^\kappa_\lambda < E^\kappa_\mu$, where

$$E^\kappa_\lambda = \{\alpha < \kappa : \mathrm{cf}(\alpha) = \lambda\},$$

and similarly for E^κ_μ.

First of all, $\{\alpha \in E^\kappa_\mu : \mathrm{cf}(\alpha) \leq \omega\}$ is empty, so of course it is nonstationary in κ.

For (2), let $C = (\mu, \kappa)$. We claim that

$$\{\alpha \in E^\kappa_\mu : E^\kappa_\lambda \cap \alpha \text{ is nonstationary in } \alpha\} \cap C = \emptyset;$$

this will prove (2). In fact, suppose that α is in the indicated intersection. Let D be club in α such that $E^\kappa_\lambda \cap D = \emptyset$. Now $\alpha \in E^\kappa_\mu$, so $\mathrm{cf}(\alpha) = \mu$. Define α_ξ for all $\xi < \lambda$ as follows. Let α_0 be the least member of D. If $\alpha_\xi \in D$ has been defined, take any member $\alpha_{\xi+1}$ of D greater than α_ξ. If ξ is limit less than λ, let $\alpha_\xi = \bigcup_{\eta < \xi} \alpha_\eta$. Then $\alpha_\xi \in D$ because D is closed. Now let $\beta = \bigcup_{\xi < \lambda} \alpha_\xi$. Then $\beta \in D$ since D is closed, and $\mathrm{cf}(\beta) = \lambda$. So $\beta \in E^\kappa_\lambda \cap D$, contradiction.

E22.2 *Continuing exercise E22.1: Assume that κ is uncountable and regular. Show that the relation $<$ is transitive.*

Suppose that $A < B < C$. Then by definition

(1) $\{\alpha \in C : \mathrm{cf}(\alpha) \leq \omega\}$ is nonstationary in κ.

(2) $\{\alpha \in B : \alpha \cap A$ is nonstationary$\}$ is nonstationary in κ.

(3) $\{\alpha \in C : \alpha \cap B$ is nonstationary$\}$ is nonstationary in κ.

We want to show

$$\{\alpha \in C : \alpha \cap A \text{ is nonstationary}\} \text{ is nonstationary.}$$

Our assumptions give us clubs M, N in κ such that

$$\{\alpha \in B : \alpha \cap A \text{ is nonstationary}\} \cap M = \emptyset \text{ and}$$
$$\{\alpha \in C : \alpha \cap B \text{ is nonstationary}\} \cap N = \emptyset.$$

Let M' be the set of all limits of members of M; so also M' is club in κ. Now it suffices to show that

$$\{\alpha \in C : \alpha \cap A \text{ is nonstationary}\} \cap M' \cap N = \emptyset.$$

So, suppose that $\alpha \in C \cap M' \cap N$; we show that $\alpha \cap A$ is stationary in α. To this end, let P be club in α, and let P' be the set of all of its limit points. Now $\alpha \in C \cap N$, so $\alpha \cap B$ is stationary. Since $\alpha \in M'$, it follows that $\alpha \cap M$ is club in α. So $M \cap P'$ is club in α, and so we can choose $\beta \in \alpha \cap B \cap M \cap P'$. Now $\beta \in B \cap M$, so $\beta \cap A$ is stationary in β. Since $\beta \in P'$, it follows that $P \cap \beta$ is club in β. So $\beta \cap A \cap P \neq \emptyset$, hence $A \cap P \neq \emptyset$, as desired.

E22.3 *If κ is an uncountable regular cardinal and S is a stationary subset of κ, we define*

$$Tr(S) = \{\alpha < \kappa : \mathrm{cf}(\alpha) > \omega \text{ and } S \cap \alpha \text{ is stationary in } \alpha\}.$$

Suppose that A, B are stationary subsets of an uncountable regular cardinal κ and $A < B$. Show that $Tr(A)$ is stationary.

Assume the conditions of the exercise. Thus by definition, $\{\alpha \in B : \mathrm{cf}(\alpha) \leq \omega\}$ is nonstationary in κ, and also $\{\alpha \in B : A \cap \alpha$ is non-stationary in $\alpha\}$ is non-stationary in κ. Hence there is a club C in κ such that $C \cap \{\alpha \in B : \mathrm{cf}(\alpha) \leq \omega\} = \emptyset$ and also $C \cap \{\alpha \in B : A \cap \alpha$ is non-stationary in $\alpha\} = \emptyset$. Thus $B \cap C \subseteq \mathrm{Tr}(A)$, and it follows that $\mathrm{Tr}(A)$ is stationary in κ.

E22.4 *(Real-valued measurable cardinals) We describe a special kind of measure. A measure on a set S is a function $\mu : \mathscr{P}(S) \to [0, \infty)$ satisfying the following conditions:*

(1) $\mu(\emptyset) = 0$ and $\mu(S) = 1$.

(2) If $\mu(\{s\}) = 0$ for all $s \in S$,

(3) If $\langle X_i : i \in \omega\rangle$ is a system of pairwise disjoint subsets of S, then $\mu(\bigcup_{i\in\omega} X_i) = \sum_{i\in\omega} \mu(X_i)$. (The X_i's are not necessarily nonempty.)

Let κ be an infinite cardinal. Then μ is κ-additive iff for every system $\langle X_\alpha : \alpha < \gamma\rangle$ of nonempty pairwise disjoint sets, wich $\gamma < \kappa$, we have

$$\mu\left(\bigcup_{\alpha<\gamma} X_\alpha\right) = \sum_{\alpha<\gamma} \mu(X_\alpha).$$

Here this sum (where the index set γ might be uncountable), is understood to be

$$\sup_{\substack{F\subseteq\gamma,\\ F\text{ finite}}}\sum_{\alpha\in F}\mu(X_\alpha).$$

We say that an uncountable cardinal κ is real-valued measurable iff there is a κ-additive measure on κ. Show that every measurable cardinal is real-valued measurable. Hint: let μ take on only the values 0 and 1.

Suppose that κ is measurable. Thus κ is uncountable, and there is a κ-complete nonprincipal ultrafilter U on κ. Now for any $X\subseteq\kappa$ we define

$$\mu(X)=\begin{cases}1 & \text{if } X\in U,\\ 0 & \text{otherwise.}\end{cases}$$

Conditions (1) and (2) in the definition of measure are clear. We can check (3) and κ-additivity simultaneously, by assuming that $\langle X_\alpha:\alpha<\beta\rangle$ is a system of pairwise disjoint subsets of κ, with $\beta<\kappa$. If $\mu(\bigcup_{\alpha<\beta}X_\alpha)=0$, clearly $\mu(X_\alpha)=0$ for all $\alpha<\beta$, and so

$$\mu\left(\bigcup_{\alpha<\gamma}X_\alpha\right)=\sum_{\alpha<\gamma}\mu(X_\alpha).$$

Suppose that $\mu(\bigcup_{\alpha<\beta}X_\alpha)=1$. Thus $\bigcup_{\alpha<\beta}X_\alpha\in U$. If $X_\alpha\notin U$ for all $\alpha<\beta$, then $\kappa\backslash X_\alpha\in U$ for all $\alpha<\beta$, and hence by κ-completeness,

$$\kappa\backslash\left(\bigcup_{\alpha<\beta}X_\alpha\right)=\bigcap_{\alpha<\beta}(\kappa\backslash X_\alpha)\in U,$$

contradiction. Hence $X_\alpha\in U$ for some $\alpha<\beta$. There can be only one such α, since if $\gamma\neq\alpha$ and $X_\gamma\in U$, then $\emptyset=X_\alpha\cap X_\gamma\in U$, contradiction. Hence again

$$\mu\left(\bigcup_{\alpha<\gamma}X_\alpha\right)=\sum_{\alpha<\gamma}\mu(X_\alpha).$$

E22.5 *Suppose that μ is a measure on a set S. A subset A of S is a μ-atom iff $\mu(A)>0$ and for every $X\subseteq A$, either $\mu(X)=0$ or $\mu(X)=\mu(A)$. Show that if κ is a real-valued measurable cardinal, μ is a κ-additive measure on κ, and $A\subseteq\kappa$ is a μ-atom, then $\{X\subseteq A:\mu(X)=\mu(A)\}$ is a κ-complete nonprincipal ultrafilter on A. Conclude that κ is a measurable cardinal if there exist such μ and A.*

Let F be the indicated set. Obviously $A\in F$. Suppose that $X\in F$ and $X\subseteq Y\subseteq A$. Then

$$\mu(A)=\mu(X\cup(Y\backslash X)\cup(A\backslash Y))=\mu(X)+\mu(Y\backslash X)+\mu(A\backslash Y)=\mu(A)+\mu(Y\backslash X)+\mu(A\backslash Y),$$

and so $\mu(A\backslash Y) = 0$. Hence $\mu(A) = \mu((A\backslash Y \cup Y) = \mu(A\backslash Y) + \mu(Y) = \mu(Y)$. So $Y \in F$. Now suppose that $Y, Z \in F$. Then

$$\mu(A) = \mu(Y) = \mu(Y \cap Z) + \mu(Y\backslash Z) \quad \text{and}$$
$$\mu(A) = \mu(Z) = \mu(Y \cap Z) + \mu(Z\backslash Y).$$

It follows that $\mu(Y\backslash Z) = \mu(Z\backslash Y)$. If $\mu(Y\backslash Z) = \mu(A)$, then also $\mu(Z\backslash Y) = \mu(A)$, and hence

$$2\mu(A) = \mu(Y\backslash Z) + \mu(Z\backslash Y) = \mu((Y\backslash Z) \cup (Z\backslash Y)) \leq \mu(A),$$

contradiction. So $\mu(Y\backslash Z) = 0$, and hence $\mu(A) = \mu(Y \cap Z)$. It follows that $Y \cap Z \in F$. So, F is a filter.

Clearly $\emptyset \notin F$, so F is proper.

If $X \subseteq A$, then $\mu(A) = \mu(X) + \mu(A\backslash X)$, and hence $\mu(X) = \mu(A)$ or $\mu(A\backslash X) = \mu(A)$. So $X \in F$ or $A\backslash X \in F$. Thus F is an ultrafilter.

Finally, for κ-completeness, suppose that $\mathscr{A} \in [F]^{<\kappa}$ Suppose that $\bigcap \mathscr{A} \notin F$. Then $A\backslash \bigcap \mathscr{A} \in F$. Let $\langle X_\alpha : \alpha < \lambda \rangle$ be an enumeration of $\mathscr{A}$. For each $\alpha < \lambda$ let $Y_\alpha = \bigcap_{\beta<\alpha} X_\beta \backslash X_\alpha$.

$$\bigcup_{\alpha<\lambda} Y_\alpha = \bigcup_{\alpha<\lambda} (A\backslash X_\alpha) \tag{1}$$

In fact, $\subseteq$ is clear. Suppose that $\xi \in \bigcup_{\alpha<\lambda}(A\backslash X_\alpha)$, and choose $\alpha < \lambda$ minimum such that $\xi \in (A\backslash X_\alpha)$. Then $\xi \in Y_\alpha$. So (1) holds.

Clearly the Y_α's are pairwise disjoint. So from (1) we get

$$\begin{aligned}
\mu(A) &= \mu\left(A\backslash \bigcap \mathscr{A}\right) \\
&= \mu\left(\bigcup_{\alpha<\lambda} (A\backslash X_\alpha)\right) \\
&= \mu\left(\bigcup_{\alpha<\lambda} Y_\alpha\right) \\
&= \sum_{\alpha<\lambda} \mu(Y_\alpha),
\end{aligned}$$

and hence there is a $\alpha < \lambda$ such that $\mu(Y_\alpha) = 1$. Hence $\mu(A\backslash X_\alpha) = \mu(A)$ also, contradiction.

Hence F is κ-complete.

Since all members of F have size κ by κ-completeness and nonprincipality, it follows that $|A| = \kappa$. So κ is a measurable cardinal.

E22.6 *Prove that if κ is real-valued measurable then either κ is measurable or $\kappa \leq 2^\omega$. Hint: if there do not exist any μ-atoms, construct a binary tree of height at most ω_1.*

Let μ be a κ-additive measure on κ. By exercise E22.5, if there is a μ-atom, then κ is measurable. So, suppose that there does not exist any μ-atoms. We construct a tree under $\supset$ by constructing the levels L_α, as follows. $L_0 = \{\kappa\}$. Suppose that L_α has been constructed, and that it is a nonempty collection of subsets of κ each of positive measure. For each $X \in L_\alpha$ let Y_X be a subset of X such that $0 < \mu(Y_X) < \mu(X)$; such a set exists since X is not a μ-atom. Then we define

$$L_{\alpha+1} = \{Y_X, X\backslash Y_X : X \in L_\alpha\}.$$

If α is a limit ordinal and L_β has been constructed for every $\beta < \alpha$, then we define

$$L_\alpha = \left\{ \bigcap_{\beta<\alpha} Z_\beta : Z_\beta \in L_\beta \text{ for all } \beta < \alpha \text{ and } \mu\left(\bigcap_{\beta<\alpha} Z_\beta\right) > 0 \right\},$$

except that if $L_\alpha = \emptyset$ the construction stops.

Clearly this gives a tree. Let α be the least ordinal such that L_α is not defined. So α is a limit ordinal.

(1) $\alpha \leq \omega_1$, and in fact, if $\langle Z_\beta : \alpha < \gamma\rangle$ is a branch of the tree, thus with $Z_\beta \subset Z_\delta$ if $\delta < \beta < \gamma$, then γ is countable.

In fact, we have $\mu(Z_\beta\backslash Z_{\beta+1}) > 0$ for every $\beta < \gamma$, and the sets $Z_\beta\backslash Z_{\beta+1}$ are pairwise disjoint. If $\gamma \geq \omega_1$, then

$$\gamma = \bigcup_{n\in\omega} \left\{\beta < \gamma : \mu(Z_\beta\backslash Z_{\beta+1}) > \frac{1}{n+1}\right\},$$

and hence there would be an $n \in \omega$ such that

$$\left\{\beta < \gamma : \mu(Z_\beta\backslash Z_{\beta+1}) > \frac{1}{n+1}\right\}$$

is uncountable, which is not possible. So (1) holds.

Similarly each level of our tree is countable. It follows that the tree has at most 2^ω branches.

Let $\mathscr{B}$ be the collection of all branches in this tree, and for each $B \in \mathscr{B}$ let $W_B = \bigcap_{X\in B} X$. Let $\mathscr{C} = \{W_B : B \in \mathscr{B}\}\backslash\{\emptyset\}$. Now clearly $|\mathscr{C}| \leq 2^\omega$, and $\mathscr{C}$ consists of measure 0 sets.

(2) $\kappa = \bigcup\mathscr{C}$.

In fact, if $\alpha \in \kappa$, then $B = \{X \in T : \alpha \in X\}$ is a branch, and so $\alpha \in W_B$.

From (2) it follows that $\kappa \leq 2^\omega$, since the measure μ is κ-additive and $\mu(\kappa) = 1$. In fact, $2^\omega < \kappa$ would imply by (2) that $\mu(\kappa) = 0$, contradiction.

E22.7 *Let κ be a regular uncountable cardinal. Show that the diagonal intersection of the system $\langle(\alpha+1,\kappa) : \alpha < \kappa\rangle$ is the set of all limit ordinals less than κ.*

For any $\beta \in \kappa$,

$$\begin{aligned}\beta \in \triangle_{\alpha<\kappa}(\alpha+1,\kappa) \quad &\text{iff} \quad \forall\alpha<\beta[\beta\in(\alpha+1,\kappa)]\\ &\text{iff} \quad \forall\alpha<\beta[\alpha+1<\beta]\\ &\text{iff} \quad \beta \text{ is a limit ordinal.}\end{aligned}$$

E22.8 *Let F be a filter on a regular uncountable cardinal κ. We say that F is normal iff it is closed under diagonal intersections. Suppose that F is normal, and $(\alpha,\kappa) \in F$ for every $\alpha < \kappa$. Show that every club of κ is in F. Hint: use exercise E22.7.*

Let C be a club, and let $\langle \alpha_\xi : \xi < \kappa \rangle$ be the strictly increasing enumeration of C, and let D be the set of all limit ordinals less than κ. By exercise E22.7 suffices to show that

$$D \cap \triangle_{\xi<\kappa}(\alpha_\xi,\kappa) \subseteq C.$$

So, take any $\beta \in D \cap \triangle_{\xi<\kappa}(\alpha_\xi,\kappa)$. Thus β is a limit ordinal, and $\forall \xi < \beta[\beta \in (\alpha_\xi,\kappa)]$, i.e., $\forall \xi < \beta[\alpha_\xi < \beta]$. Now $\xi \le \alpha_\xi$ for all ξ, so $C \cap \beta$ is unbounded in β. Hence $\beta \in C$.

E22.9 *Let F be a proper filter on a regular uncountable cardinal κ. Show that the following conditions are equivalent.*

(i) F is normal

(ii) For any $S_0 \subseteq \kappa$, if $\kappa\backslash S_0 \notin F$ and f is a regressive function defined on S_0, then there is an $S \subseteq S_0$ with $\kappa\backslash S \notin F$ and f is constant on S.

(i)⇒(ii): Assume (i), and suppose that $S_0 \subseteq \kappa$, $\kappa\backslash S_0 \notin F$, and f is a regressive function on S_0. Suppose that the conclusion fails. Then for every $\gamma < \kappa$ we have $\kappa\backslash f^{-1}[\{\gamma\}] \in F$, as otherwise we could take $S = f^{-1}[\{\gamma\}]$. By (i), take $\beta \in \triangle_{\gamma<\kappa}(\kappa\backslash f^{-1}[\{\gamma\}])$. Then $\forall \gamma < \beta[\beta \in \kappa\backslash f^{-1}[\{\gamma\}]$; in particular, $\beta \in \kappa\backslash f^{-1}[\{f(\beta)\}]$, contradiction.

(ii)⇒(i): Assume (ii), and suppose that $\langle a_\alpha : \alpha < \kappa \rangle$ is a system of members of F. Suppose that $\triangle_{\alpha<\kappa} a_\alpha \notin F$. Now $\forall \alpha \in \kappa\backslash\triangle_{\alpha<\kappa} a_\alpha \exists \beta < \alpha[\alpha \notin a_\beta]$. This gives us a regressive function f defined on $\kappa\backslash\triangle_{\alpha<\kappa} a_\alpha$ such that for every α in that set, $\alpha \notin a_{f(\alpha)}$. Hence by (ii) choose $S \subseteq \kappa\backslash\triangle_{\alpha<\kappa} a_\alpha$ such that f is constant on S, say with value γ, with $\kappa\backslash S \notin F$. Since $a_\gamma \in F$, we have $a_\gamma \not\subseteq \kappa\backslash S$. Choose $\beta \in a_\gamma \cap S$. Then $\beta \notin a_{f(\beta)}$ gives a contradiction.

E22.10 *A probability measure on a set S is a real-valued function μ with domain $\mathscr{P}(S)$ having the following properties:*

(i) $\mu(\emptyset) = 0$ and $\mu(S) = 1$.

(ii) If $X \subseteq Y$, then $\mu(X) \le \mu(Y)$.

(iii) $\mu(\{a\}) = 0$ for all $a \in S$.

(iv) If $\langle X_n : n \in \omega \rangle$ is a system of pairwise disjoint sets, then $\mu(\bigcup_{n\in\omega} X_n) = \sum_{n\in\omega} \mu(X_n)$. (Some of the sets X_n might be empty.)

Prove that there does not exist a probability measure on ω_1. Hint: consider an Ulam matrix.

Suppose that μ is a probability measure on ω_1. Let $f = \langle f_\rho : \rho < \omega_1\rangle$ be a family of injections $f_\rho : \rho \to \omega$. Define the function $A : \omega \times \omega_1 \to \mathscr{P}(\omega_1)$ by setting, for any $\xi < \omega$ and $\alpha < \omega_1$,

$$A^\xi_\alpha = \{\rho \in \omega_1\backslash(\alpha+1) : f_\rho(\alpha) = \xi\}.$$

Take any $\alpha < \omega_1$. Since $\bigcup_{n\in\omega} A^n_\alpha = \omega_1\backslash(\alpha+1)$, $A^n_\alpha \cap A^m_\alpha = \emptyset$ for $\alpha \neq \beta$, and $\mu(\omega_1\backslash(\alpha+1)) = 1$, choose $n(\alpha) \in \omega$ such that $\varphi(A^{n(\alpha)}_\alpha) > 0$. Then there exist $M \in [\omega_1]^{\omega_1}$ and $m \in \omega$ such that $n(\alpha) = m$ for every $\alpha \in M$. Then $\langle A^m_\alpha : \alpha \in M\rangle$ is a system of pairwise disjoint sets each of positive measure, contradiction.

E22.11 *Show that if κ is a measurable cardinal, then there is a normal κ-complete nonprincipal ultrafilter on κ. Hint: Let D be a κ-complete nonprincipal ultrafilter on κ. Define $f \equiv g$ iff $f, g \in {}^\kappa\kappa$ and $\{\alpha < \kappa : f(\alpha) = g(\alpha)\} \in D$. Show that $\equiv$ is an equivalence relation on ${}^\kappa\kappa$. Show that there is a relation $\prec$ on the collection of all $\equiv$-classes such that for all $f, g \in {}^\kappa\kappa$, $[f] \prec [g]$ iff $\{\alpha < \kappa : f(\alpha) < g(\alpha)\} \in D$. Here for any function $h \in {}^\kappa\kappa$ we use $[h]$ for the equivalence class of h under $\equiv$. Show that $\prec$ makes the collection of all equivalence classes into a well-order. Show that there is a $\prec$ smallest equivalence class x such that $\forall f \in x\forall\gamma < \kappa[\{\alpha < \kappa : \gamma < f(\alpha)\} \in D$. Let $E = \{X \subseteq \kappa : f^{-1}[X] \in D\}$. Show that E satisfies the requirements of the exercise.*

$\equiv$ is reflexive: $\kappa = \{\alpha < \kappa : f(\alpha) = f(\alpha)\}$, hence $f \equiv f$.

$\equiv$ is symmetric: Assume that $f \equiv g$. Thus $\{\alpha < \kappa : f(\alpha) = g(\alpha)\} \in D$. Hence $\{\alpha < \kappa : g(\alpha) = f(\alpha)\} = \{\alpha < \kappa : f(\alpha) = g(\alpha)\} \in D$. Hence $g \equiv f$.

$\equiv$ is transitive: Assume that $f \equiv g \equiv h$. Thus $\{\alpha < \kappa : f(\alpha) = g(\alpha)\} \in D$ and $\{\alpha < \kappa : g(\alpha) = h(\alpha)\} \in D$. Hence

$$\{\alpha < \kappa : f(\alpha) = g(\alpha)\} \cap \{\alpha < \kappa : g(\alpha) = h(\alpha)\} \in D;$$

since

$$\{\alpha < \kappa : f(\alpha) = g(\alpha)\} \cap \{\alpha < \kappa : g(\alpha) = h(\alpha)\} \subseteq \{\alpha < \kappa : f(\alpha) = h(\alpha)\},$$

we get $\{\alpha < \kappa : f(\alpha) = h(\alpha)\} \in D$, so $f \equiv h$.

Now define

$$x \prec y \quad \text{iff} \quad \exists f, g[x = [f] \text{ and } y = [g] \text{ and } \{\alpha < \kappa : f(\alpha) < g(\alpha)\} \in D].$$

(1) $\forall f, g \in {}^\kappa\kappa[[f] \prec [g]$ iff $\{\alpha < \kappa : f(\varphi) < g(\alpha)\} \in D]$.

In fact, $\Leftarrow$ is immediate from the definition. Now suppose that $[f] \prec [g]$. Choose $f', g' \in {}^\kappa\kappa$ such that $[f] = [f']$, $[g] = [g']$, and $\{\alpha < \kappa : f'(\alpha) < g'(\alpha)\} \in D$. Then

$$\{\alpha < \kappa : f(\alpha) = f'(\alpha)\} \cap \{\alpha < \kappa : f'(\alpha) < g'(\alpha)\} \cap \{\alpha < \kappa : g(\alpha) = g'(\alpha)\} \\ \subseteq \{\alpha < \kappa : f(\alpha) < g(\alpha)\};$$

the left side is in D, hence also the right side is in D, so $\{\alpha < \kappa : f(\alpha) < g(\alpha)\} \in D$. Thus (1) holds.

$\prec$ *is irreflexive*: $\{\alpha<\kappa : f(\alpha)<f(\alpha)\}=\emptyset\notin D$, so $[f]\not\prec[f]$.

$\prec$ *is transitive*: Assume that $[f]\prec[g]\prec[h]$. Then

$$\{\alpha<\kappa : f(\alpha)<g(\alpha)\}\cap\{\alpha<\kappa : g(\alpha)<h(\alpha)\}\subseteq\{\alpha<\kappa : f(\alpha)<h(\alpha)\};$$

the left side is in D, hence also the right side is in D, so $[f]\prec[h]$.

$\prec$ *is a linear order*: Suppose that $f,g\in{}^\kappa\kappa$ are such that $[f]\neq[g]$ and $[f]\not\prec[g]$. Now

$$\kappa=\{\alpha<\kappa : f(\alpha)<g(\alpha)\}\cup\{\alpha<\kappa : f(\alpha)=g(\alpha\}\cup\{\alpha<\kappa : g(\alpha)<f(\alpha)\};$$

The first two sets are not in D, so the third one is in D, and hence $[g]\prec[f]$.

$\prec$ *is a well-order*: Suppose not. Then we get a sequence $\langle f^m : m\in\omega\rangle$ of members of ${}^\kappa\kappa$ such that $[f_{m+1}]\prec[f_m]$ for all $m\in\omega$. Thus $\{\alpha<\kappa : f_{m+1}(\alpha)<f_m(\alpha)\}\in D$ for all $m\in\omega$. It follows that

$$\bigcap_{m\in\omega}\{\alpha<\kappa : f_{m+1}(\alpha)<f_m(\alpha)\}\in D;$$

taking any element α in this intersection, we get $\ldots f_{m+1}(\alpha)<f_m(\alpha)\ldots$, contradiction.

Now let $k(\alpha)=\alpha$ for all $\alpha<\kappa$. Then for any $\gamma<\kappa$ we have

$$\{\alpha<\kappa : \gamma<k(\alpha)\}=\{\alpha<\kappa : \gamma<\alpha\}=\kappa\backslash(\gamma+1)\in D.$$

It follows that we can take the smallest equivalence class $[f]$ such that for any $\gamma<\kappa$ we have $\{\alpha<\kappa : \gamma<f(\alpha)\}\in D$. Now we let $E=\{X\subseteq\kappa : f^{-1}[X]\in D\}$. We claim that E is as desired in the exercise.

$\emptyset\notin E$: This is true since $f^{-1}[\emptyset]=\emptyset\notin D$.

If $X\subseteq Y\subseteq\kappa$ and $X\in E$, then $Y\in E$: In fact, assume that $X\subseteq Y\subseteq\kappa$ and $X\in E$. Then $f^{-1}[X]\subseteq f^{-1}[Y]$ and $f^{-1}[X]\in D$, so $f^{-1}[y]\in D$, so that $Y\in E$.

If $X,Y\in E$, then $X\cap Y\in E$: In fact, $f^{-1}[X\cap Y]=f^{-1}[X]\cap f^{-1}[y]$, so this is clear.

If $X\subseteq\kappa$, then $X\in E$ or $(\kappa\backslash X)\in E$: For, suppose that $X\notin E$. Then $f^{-1}[X]\notin D$, so $f^{-1}[\kappa\backslash X]=(\kappa\backslash f^{-1}[X])\in D$, and hence $(\kappa\backslash X)\in E$.

E is nonprincipal: for any $\alpha<\kappa$ we have $\{\beta<\kappa : \alpha<f(\beta)\}\in D$, and $\{\beta<\kappa : \alpha<f(\beta)\}\subseteq\{\beta<\kappa : \alpha\neq f(\beta)\}$, so $\{\beta<\kappa : \alpha\neq f(\beta)\}\in D$, hence $\{\beta<\kappa : \alpha=f(\beta)\}\notin D$, hence $f^{-1}[\{\alpha\}]\notin D$ and so $\{\alpha\}\notin E$.

E is κ-complete: Suppose that $\langle X_\alpha : \alpha<\beta\rangle$ is a system of subsets of κ, with $\beta<\kappa$ and with $[X_\alpha]\in E$ for all $\alpha<\beta$. Thus $f^{-1}[X_\alpha]\in D$ for all $\alpha<\beta$. Since $f^{-1}\left[\bigcap_{\alpha<\beta}X_\alpha\right]=\bigcap_{\alpha<\beta}f^{-1}[X_\alpha]\in D$, it follows that $\bigcap_{\alpha<\beta}X_\alpha\in E$.

E is normal: We apply exercise 22.9. Suppose that $S_0\in E$ and g is regressive on S_0. Note that $f^{-1}[S_0]\in D$. Let $h=g\circ f$. Then for any $\alpha\in f^{-1}[S_0]$ we have $h(\alpha)<f(\alpha)$, so that $[h]\prec[f]$. By the definition of f it then follows that there is a $\gamma<\kappa$ such that $\{\alpha<\kappa : \gamma<h(\alpha)\}\notin D$. Hence $\{\alpha<\kappa : h(\alpha)\leq\gamma\}\in D$. Now

$$\{\alpha<\kappa : h(\alpha)\leq\gamma\}=\bigcup_{\delta\leq\gamma}\{\alpha<\kappa : h(\alpha)=\delta\},$$

and so there is a $\delta \le \gamma$ such that $\{\alpha < \kappa : h(\alpha) = \delta\} \in D$. Now $\{\alpha < \kappa : h(\alpha) = \delta\} = h^{-1}[\{\delta\}] = f^{-1}[g^{-1}[\{\delta\}]$, so $g^{-1}[\{\delta\}] \in E$. This checks the condition of exercise 22.9.

Solutions to exercises in Chapter 23

E23.1 *In the ordering $<_{\mathbf{L}}$ determine the first four sets and their order. Hint: use Lemma 23.2*

Recall that $\mathbf{L}$ is well-ordered by ordering each set L_α, placing the elements of $L_{\alpha+1}$ after all of the elements of L_α. We have $L_0 = \emptyset$. By 23.23(viii) we have $L_1 = V_1 = \{\emptyset\}$, $L_2 = V_2 = \{\emptyset, \{\emptyset\}\}$, $L_3 = V_3 = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$. Thus the first four elements of $\mathbf{L}$ are $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}$. We have $\emptyset <_{\mathbf{L}} \{\emptyset\}$ and $\{\emptyset\} <_{\mathbf{L}}$ both $\{\{\emptyset\}\}$ and $\{\emptyset, \{\emptyset\}\}$. We just need to determine the relative order of $\{\{\emptyset\}\}$ and $\{\emptyset, \{\emptyset\}\}$.

(1) $\mathbf{Df}(L_2, \emptyset, 1) = \mathscr{P}({}^1L_2)$.

For, clearly $\subseteq$ holds. We can see $\supseteq$ by applying Lemma 23.2:

For φ equal to $v_0 = v_0$ we get ${}^1L_2 \in \mathbf{Df}(L_2, 1)$.

For φ equal to $\neg(v_0 = v_0)$ we get $\emptyset \in \mathbf{Df}(L_2, 1)$.

For φ equal to $\exists v_1(v_0 \in v_1)$ we get $\{\langle\emptyset\rangle\} \in \mathbf{Df}(L_2, 1)$.

For φ equal to $\exists v_0(v_0 \in v_1)$ we get $\{\langle\{\emptyset\}\rangle\} \in \mathbf{Df}(L_2, 1)$.

This proves (1).

Now $L_2 = \{x \in L_2 : \emptyset^\frown\langle x\rangle \in {}^1L_2\}$ and $\{\{\emptyset\}\} = \{x \in L_2 : \emptyset^\frown\langle x\rangle \in \{\langle\{\emptyset\}\rangle\}\}$. It follows that in determining the order of our two elements we have $n(L_2) = n(\{\{\emptyset\}\}) = 0$. It follows that $s(L_2) = s(\{\{\emptyset\}\}) = \emptyset$. Now we need to determine $R(L_2)$ and $R(\{\{\emptyset\}\})$. Since ${}^1L_2 = \text{Diag}_=(\emptyset, 1, 0, 0)$, we have ${}^1L_2 \in \mathbf{Df}'(0, L_2, \emptyset, 1)$. Clearly $\{\{\emptyset\}\} \notin \mathbf{Df}'(0, L_2, \emptyset, 1)$, so it follows that ${}^1L_2 <_{4L_2 1} \{\langle\{\{\emptyset\}\}\rangle\}$. Hence L_2 preceeds $\{\{\emptyset\}\}$ in the order $<_{\mathbf{L}}$.

Thus the first four sets are, in order, $\emptyset$, $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$, and $\{\{\emptyset\}\}$.

E23.2 *Suppose that $\mathbf{M}$ is a nonempty transitive class satisfying the comprehension axioms, and also $\forall x \subseteq \mathbf{M} \exists y \in \mathbf{M}[x \subseteq y]$. Show that $\mathbf{M}$ is a model of ZF.*

Extensionality holds by Theorem 10.11.

Foundation holds by Theorem 10.17

Comprehension is given.

Pairing: given $x, y \in \mathbf{M}$, we have $\{x, y\} \subseteq \mathbf{M}$, hence there is a $z \in \mathbf{M}$ such that $\{x, y\} \subseteq z$. So pairing holds by Theorem 10.13

Union: Suppose that $x \in \mathbf{M}$. Then $\bigcup x \subseteq \mathbf{M}$, so choose $z \in \mathbf{M}$ such that $\bigcup x \subseteq z$. Hence union holds by Theorem 10.14

Power set: Clear by Theorem 10.15 and the hypothesis.

Replacement: We apply Theorem 10.16 Assume that $A, w_1, \ldots, w_n \in \mathbf{M}$ and

$$\forall x \in A \exists! y \in \mathbf{M} \varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n).$$

Let

$$z = \{y \in \mathbf{M} : \exists x \in A \varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n)\}.$$

Then choose $t \in \mathbf{M}$ such that $z \subseteq t$. So replacement holds.

Infinity: By induction we show that $m \in \mathbf{M}$ for all $m \in \omega$. To show that $\emptyset \in \mathbf{M}$, take any $a \in \mathbf{M}$. By comprehension in $\mathbf{M}$ choose $z \in \mathbf{M}$ such that $\forall x[x \in z \leftrightarrow x \in a \wedge x \neq x]$. So $z = \emptyset$, as desired. Suppose $m \in \mathbf{M}$. By the pairing axiom in $\mathbf{M}$ applied to m, m, choose a set $a \in \mathbf{M}$ such that $m \in a$. Then by the compehension axiom in $\mathbf{M}$ take $b \in \mathbf{M}$ such that $\forall x[x \in b \leftrightarrow x \in a \wedge x = m]$. Thus $b = \{m\}$. By the pairing axiom in $\mathbf{M}$ choose $c \in \mathbf{M}$ such that $m, \{m\} \in c$. By the union axiom in bf M choose $d \in \mathbf{M}$ such that $\forall Y \forall x[x \in Y \in c \to x \in d]$. So $m, \{m\} \subseteq d$. By the comprehension axiom in $\mathbf{M}$ choose $e \in \mathbf{M}$ such that $\forall x[x \in e \leftrightarrow x \in d \wedge (x \in m \vee x = m)]$. Hence $e = m \cup \{m\}$. This finishes the induction. So $\omega \subseteq \mathbf{M}$. Hence by hypothesis there is a $y \in \mathbf{M}$ such that $\omega \subseteq y$. Now by comprehension in $\mathbf{M}$ let $z \in \mathbf{M}$ be such that $\forall x[x \in z \leftrightarrow x \in y \wedge x$ is a finite ordinal$]$. Thus $z = \omega$. Hence infinity holds in $\mathbf{M}$ by Theorem 10.27.

E23.3 *Show that if* $\mathbf{M}$ *is a transitive proper class model of ZF, then* $\forall x \subseteq \mathbf{M} \exists y \in \mathbf{M}[x \subseteq y]$.

Suppose that $x \subseteq \mathbf{M}$. Choose α such that $x \subseteq V_\alpha$. Thus $x \subseteq V_\alpha \cap \mathbf{M}$.

(1) If $\alpha \in \mathbf{M}$, then $V_\alpha \cap \mathbf{M} \in \mathbf{M}$.

This holds by Theorem 10.31(ii). By (1), it suffices to show that Ord $\subseteq \mathbf{M}$.

Let $\varphi(x, y)$ be the formula "rank$(x) = y$". So φ is absolute for $\mathbf{M}$ by Theorem 10.30(iv). Given an ordinal β, choose $x \in \mathbf{M}$ such that rank$(x) > \beta$; this is possible because $\mathbf{M}$ is a proper class. Now $\mathbf{M} \models \forall u \exists v \varphi(u, v)$, so choose $y \in \mathbf{M}$ such that $\varphi^{\mathbf{M}}(x, y)$. Thus $\varphi(x, y)$, so y is an ordinal greater than β. Since $\mathbf{M}$ is transitive, $\beta \in \mathbf{M}$.

E23.4 *Show that for every ordinal* $\alpha > \omega$, $|L_\alpha| = |V_\alpha|$ *iff* $\alpha = \beth_\alpha$.

Assume that $\alpha > \omega$; write $\alpha = \omega + \beta$. By Lemma 23.24, $|L_\alpha| = |\alpha|$. By Theorem 10.10(ii), $|V_\alpha| = \beth_\beta$. So $|L_\alpha| = |V_\alpha|$ iff $|\alpha| = \beth_\beta$.

Now suppose that $\alpha = \beth_\alpha$. Now if $\gamma < \alpha$, then $\omega, \gamma < \alpha$ since α is an uncountable cardinal. Hence $\omega + \gamma < \alpha$. Clearly then $\alpha = \omega + \alpha$, and so $\alpha = \beta$. It follows that

$$|L_\alpha| = |\alpha| = \beth_\alpha = \beth_\beta = |V_\alpha|.$$

Now suppose that $\alpha \neq \beth_\alpha$. If α is countable, then

$$|L_\alpha| = |\alpha| \leq \omega < \beth_\beta = |V_\alpha|,$$

as desired. Suppose that $\alpha \geq \omega_1$. Then $\beta \geq \omega_1$. Write $\beta = \omega_1 + \gamma$. Then

$$\alpha = \omega + \beta = \omega + \omega_1 + \gamma = \omega_1 + \gamma = \beta.$$

Hence $|L_\alpha| = |\alpha| \leq \alpha < \beth_\alpha = \beth_\beta = |V_\alpha|$, as desired.

E23.5 *Assume* $\mathbf{V} = \mathbf{L}$ *and* $\alpha > \omega$. *Then* $L_\alpha = V_\alpha$ *iff* $\alpha = \beth_\alpha$.

$\Rightarrow$: by exercise E23.4

$\Leftarrow$: First we claim

(*) ($\mathbf{V}=\mathbf{L}$) $V_\beta \subseteq L_{\beth_\beta}$ for every ordinal β.

We prove this by induction on β. It is obvious for $\beta = 0$, and the inductive step with β limit is obvious. Now assume it for β. Then using Theorem 23.32

$$V_{\beta+1} = \mathscr{P}(V_\beta) \subseteq \mathscr{P}(L_{\beth_\beta}) \subseteq L_{\beth_\beta^+} = L_{\beth_{\beta+1}},$$

finishing the inductive proof.

Now assume that $\alpha = \beth_\alpha$. Then $V_\alpha \subseteq L_{\beth_\alpha} = L_\alpha \subseteq V_\alpha$ using Theorem 23.23(vii).

E23.6 *Assume* $\mathbf{V}=\mathbf{L}$ *and prove that* $L_\kappa = H(\kappa)$ *for every infinite cardinal* κ.

First, $L_\omega = V_\omega = H(\omega)$. Now suppose that κ is uncountable and regular. Take any $x \in L_\kappa$. Choose $\alpha < \kappa$ such that $x \in L_\alpha$. We may assume that α is infinite. Since L_α is transitive, we also have $\mathrm{trcl}(x) \subseteq L_\alpha$. Hence $|\mathrm{trcl(x)}| \le |L_\alpha| = |\alpha| < \kappa$. So $x \in H(\kappa)$. Thus we have shown that $L_\kappa \subseteq H(\kappa)$. Suppose that $H(\kappa) \not\subseteq L_\kappa$. By the foundation axiom, choose $x \in H(\kappa)\backslash L_\kappa$ such that $x \subseteq L_\kappa$. Since $|x| < \kappa$ and κ is regular, choose $\beta < \kappa$ such that $x \subseteq L_\beta$. Hence $x \in L_{\beta^+} \subseteq L_\kappa$ by Theorem 23.32. This is a contradiction. So $L_\kappa = H(\kappa)$. This finishes the case when κ is regular.

Now suppose that κ is singular. Then

$$L_\kappa = \bigcup_{\alpha<\kappa} L_{\alpha^+} = \bigcup_{\alpha<\kappa} H(\alpha^+) = H(\kappa).$$

E23.7 *Show that if* $\varphi(y_1,\ldots,y_n,x)$ *is a formula with at most the indicated variables free, then*

$$\forall\alpha_1,\ldots,\alpha_n\forall a[\forall x[\varphi(\alpha_1,\ldots,\alpha_n,x) \leftrightarrow x = a] \rightarrow a \in \mathbf{OD}].$$

Also show that $\emptyset \in \mathbf{OD}$.

Fix $\alpha_1,\ldots,\alpha_n$ and assume that $\forall x[\varphi(\alpha_1,\ldots,\alpha_n,x) \leftrightarrow x = a]$. By Corollary 10.39 fix $\beta > \max(\alpha_1,\ldots,\alpha_n,\mathrm{rank}(a))$ so that φ is absolute for V_β. Let

$$R = \{z \in {}^{n+1}V_\beta : \varphi(z_0,\ldots,z_n)\}.$$

Let $s = \langle\alpha_1,\ldots\alpha_n\rangle \in {}^n\beta$. Then

$$\forall x \in V_\beta[s^\frown\langle x\rangle \in R \leftrightarrow x = a].$$

By the absoluteness of φ,

$$R = \{s \in {}^{n+1}V_\beta : \varphi(s_0,\ldots,s_n)^{V_\beta},$$

so $R \in \mathbf{Df}(V_\beta,\emptyset,n+1)$ by Lemma 23.2. Hence $a \in \mathbf{OD}$.

Now for the last part of the exercise, we apply the first part to the formula $y = x$ and the ordinal 0. This gives

$$\forall a[\forall x[0 = x \leftrightarrow x = a] \rightarrow a \in \mathbf{OD}],$$

which means that $0 \in \mathbf{OD}$.

E23.8 *We define $s \lhd t$ iff $s,t \in {}^{<\omega}\mathbf{ON}$ and one of the following holds:*
(i) $s = \emptyset$ and $t \neq \emptyset$;
(ii) $s,t \neq \emptyset$ and $\max(\text{rng}(s)) < \max(\text{rng}(t))$;
(iii) $s,t \neq \emptyset$ and $\max(\text{rng}(s)) = \max(\text{rng}(t))$ and $\text{dmn}(s) < \text{dmn}(t)$;
(iv) $s,t \neq \emptyset$ and $\max(\text{rng}(s)) = \max(\text{rng}(t))$ and $\text{dmn}(s) = \text{dmn}(t)$ and $\exists k \in \text{dmn}(s)[s \restriction k = t \restriction k$ and $s(k) < t(k)]$.

Prove the following:

(v) $\lhd$ well-orders ${}^{<\omega}\mathbf{ON}$.
(vi) $\forall t \in {}^{<\omega}\mathbf{ON}[\{s : s \lhd t\}$ is a set$]$.
(vii) For every infinite ordinal α we have $|{}^{<\omega}\alpha| = |\alpha|$.
(viii) For every uncountable cardinal κ, the set ${}^{<\omega}\kappa$ is well-ordered by $\lhd$ in order type κ and is an initial segment of ${}^{<\omega}\mathbf{ON}$.
(ix) ${}^{<\omega}\omega$ is well-ordered by $\lhd$ in order type ω^2.

(v): Clearly $\lhd$ is irreflexive, and $\forall s,t \in {}^{<\omega}\mathbf{ON}[s \neq t \rightarrow s \lhd t$ or $t \lhd s]$. Now suppose that $s \lhd t \lhd w$.

Case 1. $s = \emptyset$. Then $t, w \neq \emptyset$, and so $s \lhd w$.

Case 2. $s \neq \emptyset$. Then clearly $t, w \neq \emptyset$ and $\max(\text{rng}(s)) \leq \max(\text{rng}(t)) \leq \max(\text{rng}(w))$.

Subcase 2.1. $\max(\text{rng}(s)) < \max(\text{rng}(t))$. Then $\max(\text{rng}(s)) < \max(\text{rng}(w))$, and hence $s \lhd w$.

Subcase 2.2. $\max(\text{rng}(s)) = \max(\text{rng}(t))$. Thus $\text{dmn}(s) \leq \text{dmn}(t)$.

Subsubcase 2.2.1. $\max(\text{rng}(t)) < \max(\text{rng}(w)$. $\max(\text{rng}(s)) < \max(\text{rng}(w))$ follows, and so $s \lhd w$.

Subsubcase 2.2.2. $\max(\text{rng}(t)) = \max(\text{rng}(w))$. So $\text{dmn}(t) \leq \text{dmn}(w)$.

Subsubsubcase 2.2.2.1. $\text{dmn}(s) < \text{dmn}(t)$ or $\text{dmn}(t) < \text{dmn}(w)$. Then $\text{dmn}(s) < \text{dmn}(w)$ and hence $s \lhd w$.

Subsubsubcase 2.2.2.2. $\text{dmn}(s) = \text{dmn}(t) = \text{dmn}(w)$. Then there exist $k, l \in \text{dmn}(s)$ such that $s \restriction k = t \restriction k$, $s(k) < t(k)$, $t \restriction l = w \restriction l$, and $t(l) < w(l)$.

Subsubsubsubcase 2.2.2.2.1. $k < l$. Then $s \restriction k = w \restriction k$ and $s(k) < t(k) = w(k)$, so $s \lhd w$.

Subsubsubsubcase 2.2.2.2.2. $k = l$. Then $s \restriction k = w \restriction k$ and $s(k) < t(k) < w(k)$, so $s \lhd w$.

Subsubsubsubcase 2.2.2.2.3. $l < k$. Then $s \restriction l = w \restriction l$ and $s(l) = t(l) = w(l)$, so $s \lhd w$.

This completes the proof that $\lhd$ is a linear order. Now suppose that A is a nonempty subset of ${}^{<\omega}\mathbf{ON}$. If $\emptyset \in A$, then it is the least element of A. So, suppose that $\emptyset \notin A$. Let $B = \{\max(\text{rng}(s)) : s \in A\}$, and let α be the least member of B. Let $C = \{\text{dmn}(s) : s \in A, \max(\text{rng}(s)) = \alpha\}$. Thus $C \neq \emptyset$, so let m be the least member of C. Let $D = \{s(0) : s \in A, \max(\text{rng}(s)) = \alpha, \text{dmn}(s) = m\}$, and let γ_0 be the least member of D. Suppose that γ_j has been defined for all $j \leq i$ so that $i + 1 < m$ and

$$E \stackrel{\text{def}}{=} \{s \in A : \max(\text{rng}(s)) = \alpha, \text{dmn}(s) = \beta, s \restriction (i+1) = \langle \gamma_0, \ldots, \gamma_i \rangle\}$$

is nonempty. Let $F = \{s(i+1) : s \in E\}$, and let γ_{i+1} be the least element of F. By construction, there is an $s \in A$ such that $\max(\mathrm{rng}(s)) = \alpha$, $\mathrm{dmn}(s) = m$, and $s(i) = \gamma_i$ for all $i < m$. We claim that s is the least element of A. For, take any $t \in A$ with $s \neq t$. Then $\max(\mathrm{rng}(s)) = \alpha \leq \max(\mathrm{rng}(t))$; if $<$ holds here, then $s \lhd t$, as desired. So suppose $=$ holds. Then $\mathrm{dmn}(s) = m \leq \mathrm{dmn}(t)$. It $<$ holds here, then $s \lhd t$, as desired. So assume that $=$ holds. Let $i \leq m$ be maximum such that $s \restriction i = t \restriction i$. Then $i < m$ since $s \neq t$. Then by construction $s(i) < t(i)$. Hence $s \lhd t$, as desired. Therefore, $\lhd$ is a well-order.

(vi): If $t = \emptyset$, then $\{s \in {}^{<\omega}\mathrm{ON} : s \lhd t\} = \emptyset$. Now suppose that $t \neq \emptyset$. Clearly $\{s \in {}^{<\omega}\mathbf{ON} : s \lhd t\} \subseteq |^{<\omega}(\alpha+1)$, where $\alpha = \max(\mathrm{rng}(t))$.

(vii): We have

$$\left|{}^{<\omega}\alpha\right| \leq \sum_{k\in\omega} |{}^k\alpha| = \sum_{k\in\omega} |\alpha|^k = |\alpha|.$$

Also, $\xi \mapsto \{(0,\xi)\}$ is a one-one function from α into ${}^1(\alpha+1) \subseteq {}^{<\omega}(\alpha+1)$, so equality holds.

(viii): Assume that κ is an uncountable cardinal. We verify the second statement first. Suppose that $s, t \in {}^{<\omega}\mathbf{ON}$ and $s \lhd t \in {}^{<\omega}\kappa$. If $s = \emptyset$, then $s \in {}^{<\omega}\kappa$. If $s \neq \emptyset$, then $\max(\mathrm{rng}(s)) \leq \max(\mathrm{rng}(t)) \in \kappa$; so $\max(\mathrm{rng}(s)) \in \kappa$ and hence $s \in {}^{<\omega}\kappa$.

Now $|{}^{<\omega}\kappa| = \kappa$ by (vii), so $\kappa \leq |{}^{<\omega}\kappa|$. If $s \in {}^{<\omega}\kappa$, then $\{t \in {}^{<\omega}\mathbf{On} : t \lhd s\} \subseteq {}^{<\omega}(\max(\mathrm{rng}(s)))$, and so by (vii), $|\{t \in {}^{<\omega}\mathbf{On} : t \lhd s\}| \leq |\max(\omega, \mathrm{rng}(s))| < \kappa$. Hence by the preceding paragraph, the order type of ${}^{<\omega}\kappa$ is κ.

(ix): For each $m \in \omega$ let $A_m = \{s \in {}^{<\omega}\omega : s \neq \emptyset \text{ and } \max(\mathrm{rng}(s)) = m\}$. Thus $\forall m \in \omega \forall s \in A_m \forall t \in A_{m+1}[s \lhd t]$. Moreover, ${}^{<\omega}\omega = \{\emptyset\} \cup \bigcup_{m\in\omega} A_m$. For $m \in \omega$ and n a positive integer, let $B_{mn} = \{s \in A_m : \mathrm{dmn}(s) = n\}$. So $\forall m \in \omega \forall n \in \omega\backslash 1 \forall s \in B_{mn} \forall t \in B_{m,n+1}[s \lhd t]$. Each B_{mn} is finite. Hence A_m has order type ω and ${}^{<\omega}\omega$ has order type ω^2.

E23.9 *Prove that for any $m \in \omega$ and any set A, $\mathbf{Df}(A, \emptyset, n) = \{\mathrm{En}(m, A, n) : m \in \omega\}$.*

We prove $\supseteq$ by complete induction on m. Suppose that $\mathrm{En}(m', A, n) \in \mathbf{Df}(A, \emptyset, n)$ whenever $m' < m$. If $m = 0$, then $\mathrm{En}(m, A, n) = \emptyset \in \mathbf{Df}(A, \emptyset, n)$ using Lemma 23.2. Suppose that $m \neq 0$, and write $m = 2^i \cdot 3^j \cdot 5^k \cdot r$ with r not divisible by 2,3, or 5. If $r = 1$, $k = 0$, and $i, j < n$, then $\mathrm{En}(m, A, n) = \mathrm{Diag}_\in(A, n, i, j) \in \mathbf{Df}'(0, A, \emptyset, n) \subseteq \mathbf{Df}(A, \emptyset, n)$. If $r = 1$, $k = 1$, and $i, j < n$, then $\mathrm{En}(m, A, n) = \mathrm{Diag}_=(A, n, i, j) \in \mathbf{Df}'(0, A, \emptyset, n) \subseteq \mathbf{Df}(A, \emptyset, n)$. If $r = 1$ and $k = 2$, then $i < m$, hence by the inductive assumption $\mathrm{En}(i, A, n) \in \mathbf{Df}(A, \emptyset, n)$, and clearly then $\mathrm{En}(m, A, n) = {}^nA \backslash \mathrm{En}(i, A, n) \in \mathbf{Df}(A, \emptyset, n)$. If $r = 1$ and $k = 3$, then $i, j < m$, hence by the inductive assumption $\mathrm{En}(i, A, n), \mathrm{En}(i, A, n) \in \mathbf{Df}(A, \emptyset, n)$, and clearly then $\mathrm{En}(m, A, n) = \mathrm{En}(i, A, n) \cap \mathrm{En}(j, A, n) \in \mathbf{Df}(A, \emptyset, n)$. If $r = 1$ and $k = 4$, then $i < m$, hence by the inductive assumption $\mathrm{En}(i, A, n+1) \in \mathbf{Df}(A, \emptyset, n)$, and clearly then $\mathrm{En}(m, A, n) = \mathrm{Proj}(A, \mathrm{En}(i, A, n+1), n) \in \mathbf{Df}(A, \emptyset, n)$. Finally, in any other case, again $\mathrm{En}(m, i, n) = \emptyset \in \mathbf{Df}(A, \emptyset, n)$ by Lemma 23.2. This finishes the proof of $\supseteq$.

For $\subseteq$, we prove by induction on k that $\mathbf{Df}'(k, A, \emptyset, n) \subseteq \{\mathrm{En}(m, A, n) : m \in \omega\}$. For $k = 0$, take any $x \in \mathbf{Df}'(k, A, \emptyset, n)$. We have three possibilities.

Case 1. $x = \mathrm{Rel}(A, \emptyset, n, i)$. Then $x = \emptyset = \mathrm{En}(0, A, n)$.

Case 2. $x = \mathrm{Diag}_\in(A, n, i, j)$. If not$(i, j < n)$, then $x = \emptyset = \mathrm{En}(0, A, n)$. If $i, j < n$, then $x = \mathrm{En}(2^i \cdot 3^j, A, n)$.

Case 3. $x = \mathrm{Diag}_=(A,n,i,j)$. If $\mathrm{not}(i,j<n)$, then $x=\emptyset=\mathrm{En}(0,A,n)$. If $i,j<n$, then $x=\mathrm{En}(2^i\cdot 3^j\cdot 5,A,n)$.

Now suppose that $\mathbf{Df}'(k,A,\emptyset,n)\subseteq\{\mathrm{En}(m,A,n):m\in\omega\}$. Take any $x\in\mathbf{Df}'(k+1,A,\emptyset,n)$. If $x\in\mathbf{Df}'(k,A,\emptyset,n)$, then the inductive hypothesis gives the desired conclusion. Suppose that $x\notin\mathbf{Df}'(k,A,\emptyset,n)$. Then we have three possibilities.

Case 1. There is an $R\in\mathbf{Df}'(k,A,\emptyset,n)$ such that $x={}^nA\backslash R$. By the inductive hypothesis choose i so that $R=\mathrm{En}(i,A,n)$. Then $x=\mathrm{En}(2^i\cdot 5^2,A,n)$.

Case 2. There are $R,S\in\mathbf{Df}'(k,A,\emptyset,n)$ such that $x=R\cap S$. By the inductive hypothesis choose i,j so that $R=\mathrm{En}(i,A,n)$ and $S=\mathrm{En}(j,A,n)$. Then $x=\mathrm{En}(2^i\cdot 3^j\cdot 5^3,A,n)$.

Case 3. There is an $R\in\mathbf{Df}'(k,A,\emptyset,n+1)$ such that $x=\mathrm{Proj}(A,R,n)$. By the inductive hypothesis choose i so that $R=\mathrm{En}(i,A,n+1)$. Then $x=\mathrm{En}(2^i\cdot 5^4,A,n)$.

This completes the inductive proof.

E23.10 *Prove that if $\varphi(x_0,\ldots,x_{n-1})$ is a formula with free variables among $x_0,\ldots,x_{n-1}$, then there is an $m\in\omega$ such that for every set A,*

$$\{s\in{}^nA:\varphi^A(s(0),\ldots,s(n-1))\}=\mathrm{En}(m,A,n).$$

We proceed by induction on the number of quantifiers in φ, and within that, by induction on formulas in the usual sense. For brevity let $S(\varphi)$ be the set $\{s\in{}^nA:\varphi^A(s(0),\ldots,s(n-1))\}$. Then

$$\begin{aligned}S(x_i\in x_j)&=\mathrm{Diag}_\in(A,n,i,j)=\mathrm{En}(2^i\cdot 3^j,A,n);\\ S(x_i=x_j)&=\mathrm{Diag}_=(A,n,i,j)=\mathrm{En}(2^i\cdot 3^j\cdot 5,A,n);\end{aligned}$$

if $S(\psi)=\mathrm{En}(i,A,n)$, then

$$S(\neg\psi)={}^nA\backslash S(\psi)=\mathrm{En}(2^i\cdot 5^2,A,n);$$

if $S(\psi)=\mathrm{En}(i,A,n)$ and $S(\chi)=\mathrm{En}(j,A,n)$, then

$$S(\psi\wedge\chi)=S(\psi)\cap S(\chi)=\mathrm{En}(2^l\cdot 3^j\cdot 5^3,A,n).$$

The inductive step from ψ to $\exists y\psi$ requires more care. By a change of bound variable we obtain a formula χ such that $\forall y\psi$ is logically equivalent to $\forall x_n\chi$, with the free variables of χ among $x_0,\ldots,x_n$. Hence with $S(\chi)=\mathrm{En}(i,A,n+1)$,

$$\begin{aligned}S(\exists y\psi)&=\{s\in{}^nA:\exists x_n\chi(s(0),\ldots,s(n-1),x_n)\}\\ &=\mathrm{Proj}(A,S(\chi),n+1)=\mathrm{En}(2^i\cdot 5^4,A,n).\end{aligned}$$ □

E23.11 *By exercise E23.8, for each uncountable cardinal κ there is an isomorphism f_κ from $(\kappa,\in)$ onto $({}^{<\omega}\kappa,\lhd)$. Then $f_\kappa\subseteq f_\lambda$ for $\kappa<\lambda$. It follows that there is a function Enon mapping* **ON** *onto* ${}^{<\omega}$**ON** *such that $\alpha<\beta$ iff* $\mathrm{Enon}(\alpha)\lhd\mathrm{Enon}(\beta)$.

Now we define a class function Enod with domain **ON**, *as follows. For any ordinal* γ,

$$\mathrm{Enod}(\gamma)=\begin{cases} a & \textit{if there exist } s,\beta,m,n \textit{ such that } \mathrm{Enon}(\gamma)=s^\frown\langle\beta,n,m\rangle \\ & \textit{with } m,n\in\omega,\ \beta\in\mathbf{ON},\ s\in{}^{<\omega}\beta,\ \mathrm{dmn}(s)=n, \textit{ and} \\ & \forall x\in V_\beta[s^\frown x\in\mathrm{En}(m,V_\beta,n+1)\leftrightarrow x=a], \\ 0 & \textit{otherwise.}\end{cases}$$

Prove that $\mathbf{OD}=\{\mathrm{Enod}(\gamma):\gamma\in\mathbf{ON}\}$.

First suppose that $\gamma\in\mathbf{ON}$. If $\mathrm{Enod}(\gamma)=0$, then $\mathrm{Enod}(\gamma)\in\mathbf{OD}$ by the second part of exercise E23.7. Suppose that $\mathrm{Enod}(\gamma)=a$ by the first part of the definition of Enod. Choose s,β,m,n accordingly. Now $\mathrm{En}(m,V_\beta,n+1)\in\mathbf{Df}(A,\emptyset,n)$ by Exercise 23.9. So $a\in\mathbf{OD}$ by definition.

Conversely, suppose that $a\in\mathbf{OD}$. Choose β,n,s,R accordingly. By Exercise 23.9 choose m so that $R=\mathrm{En}(m,V_\beta,n{+}1)$. Let γ be such that $\mathrm{Enon}(\gamma)=s^\frown\langle\beta,m,n\rangle$. Clearly then $\mathrm{Enod}(\gamma)=a$.

E23.12 *Now we define* $\mathbf{HOD}=\{x\in\mathbf{OD}:\mathrm{trcl}(x)\subseteq\mathbf{OD}\}$.
Prove that $\mathbf{ON}\subseteq\mathbf{HOD}$ *and* **HOD** *is transitive.*

Let $\alpha\in\mathbf{ON}$. Now α is transitive, so $\mathrm{trcl}(\alpha)=\alpha$. So it suffices to show that $\alpha\in\mathbf{OD}$. We apply exercise E23.7 to the formula $y=x$ to obtain $\forall x[\alpha=x\leftrightarrow x=\alpha]\to\alpha\in\mathbf{OD}$. So $\alpha\in\mathbf{OD}$.

If $y\in x\in\mathbf{HOD}$, then $\mathrm{trcl}(x)\subseteq\mathbf{OD}$. Since $y\in\mathrm{trcl}(x)$, it follows that $y\in\mathbf{OD}$, and also $\mathrm{trcl}(y)\subseteq\mathrm{trcl}(x)$, so $\mathrm{trcl}(y)\subseteq\mathbf{OD}$. Hence $y\in\mathbf{HOD}$.

E23.13 *Show that* $(V_\alpha\cap\mathbf{HOD})\in\mathbf{HOD}$ *for every ordinal* α.

Take any ordinal α. It suffices to show that $(V_\alpha\cap\mathbf{HOD})\in\mathbf{OD}$. Applying exercise E23.7 to the formula $x=V_y\cap\mathbf{HOD}$, with $n=1$, α in place of α_1, and $V_\alpha\cap\mathbf{HOD}$ in place of a, we have

$$\forall x[x=V_\alpha\cap\mathbf{HOD}\leftrightarrow x=V_\alpha\cap\mathbf{HOD}]\to(V_\alpha\cap\mathbf{HOD})\in\mathbf{OD},$$

and the desired result follows.

E23.14 *Prove without using the axiom of choice that* **HOD** *is a model of ZFC.*

Extensionality: holds since **HOD** is transitive.

Comprehension axioms: by Theorem 10.12 it suffices to take a formula φ with free variables among $x,z,w_1,\ldots,w_n$, assume that $z,w_1,\ldots,w_n\in\mathbf{HOD}$, and prove that

$$a\stackrel{\text{def}}{=}\{x\in z:\varphi^{\mathbf{HOD}}\varphi(x,z,w_1,\ldots,w_n)\}\in\mathbf{HOD}.$$

By exercise E23.11, choose ordinals $\alpha,\beta_1,\ldots,\beta_n$ such that $z=\mathrm{Enod}(\alpha)$ and $w_i=\mathrm{Enod}(\beta_i)$ for $i=1,\ldots,n$. Let $\psi(y_0,\ldots,y_n,v)$ be the formula

$$v=\{x\in\mathrm{Enod}(y_0):\varphi^{\mathbf{HOD}}(x,\mathrm{Enod}(y_1),\ldots,\mathrm{Enod}(y_n))\}.$$

Then we apply exercise E23.7, with $\psi(y_1, \ldots, y_n, v)$ in place of $\varphi(y_1, \ldots, y_n, x)$ to obtain

$$\forall v[\psi(\alpha, \beta_1, \ldots, \beta_n, v) \leftrightarrow v = a] \rightarrow a \in \mathbf{OD}.$$

Now $\psi(\alpha, \beta_1, \ldots, \beta_n, v)$ is the formula

$$v = \{x \in \text{Enod}(\alpha) : \varphi^{\mathbf{HOD}}(x, \text{Enod}(\beta_1), \ldots, \text{Enod}(\beta_n))\},$$

which is the same as

$$v = \{x \in z : \varphi^{\mathbf{HOD}}(x, w_1, \ldots, w_n)\},$$

or simply $v = a$. So $a \in \mathbf{OD}$. Since $a \subseteq \mathbf{HOD}$, we get $a \in \mathbf{HOD}$.

Pairing: Given $x, y \in \mathbf{HOD}$, take $\alpha > \max(\text{rank}(x), \text{rank}(y))$. Then $x, y \in V_\alpha \cap \mathbf{HOD} \in \mathbf{HOD}$ by exercise E23.13. So Theorem 10.13 applies.

Union: Given $x \in \mathbf{HOD}$, take $\alpha > \text{rank}(x)$. Then $\bigcup x \in V_\alpha \cap \mathbf{HOD} \in \mathbf{HOD}$; apply Theorem 10.14

Power set: Given $x \in \mathbf{HOD}$, write $x = \text{Enod}(\alpha)$. Apply exercise E23.7 to the formula $x = \mathscr{P}(\text{Enod}(y))$; we get

$$\forall w[w = \mathscr{P}(\text{Enod}(\alpha)) \leftrightarrow w = \mathscr{P}(\text{Enod}(\alpha))] \rightarrow \mathscr{P}(\text{Enod}(\alpha)) \in \mathbf{OD},$$

and hence $\mathscr{P}(x) = \mathscr{P}(\text{Enod}(\alpha)) \in \mathbf{OD}$. Now $\mathscr{P}(x) \subseteq \mathbf{HOD}$, so $\mathscr{P}(x) \in \mathbf{HOD}$. Hence power set holds by Theorem 10.15

Replacement: to apply Theorem 10.16, let φ be a formula with free variables among $x, y, A, w_1, \ldots, w_n$, take any $A, w_1, \ldots, w_n \in \mathbf{HOD}$, and assume that

$$\forall x \in A \exists! y[y \in \mathbf{HOD} \wedge \varphi^{\mathbf{HOD}}(x, y, A, w_1, \ldots, w_n)].$$

Choose $\alpha, \beta_1, \ldots, \beta_n$ such that $A = \text{Enod}(\alpha)$ and $w_i = \text{Enod}(\beta_i)$ for $i = 1, \ldots, n$. By the replacement axiom there is a set Z such that for all $x \in A$ there is a $y \in Z$ such that $y \in \mathbf{HOD} \wedge \varphi^{\mathbf{HOD}}(x, y, A, w_1, \ldots, w_n)$. Let $a = \{y \in Z : y \in \mathbf{HOD} \wedge \exists x \in A \varphi^{\mathbf{HOD}}(x, y, A, w_1, \ldots, w_n)\}$. Then

$$a = \{y \in \mathbf{HOD} : \exists x \in A[\varphi^{\mathbf{HOD}}(x, y, A, w_1, \ldots, w_n)]\}.$$

In fact, $\subseteq$ is clear. Conversely, suppose $y \in \mathbf{HOD} \wedge \exists x \in A[\varphi^{\mathbf{HOD}}(x, y, A, w_1, \ldots, w_n)]\}$. Take such an x. Choose $y' \in Z$ such that $y' \in \mathbf{HOD} \wedge \varphi^{\mathbf{HOD}}(x, y', A, w_1, \ldots, w_n)$. By the uniqueness condition, $y = y'$. Hence $y \in a$, as desired.

Since $a \subseteq \mathbf{HOD}$ and $\mathbf{HOD}$ is transitive, it follows that $\text{trcl}(a) \subseteq \mathbf{HOD}$, and so $\text{trcl}(a) \subseteq \mathbf{OD}$. Hence it suffices to show that $a \in \mathbf{OD}$. Now let $\psi(v, s_1, \ldots, s_n, z)$ be the formula

$$z = \{y \in \mathbf{HOD} : \exists x \in \text{Enod}(v)[\varphi^{\mathbf{HOD}}(x, y, \text{Enod}(v), \text{Enod}(s_1), \ldots, \text{Enod}(s_n))]\}.$$

Now we apply exercise E23.7 to the formula $\psi(v, s_1, \ldots, s_n, z)$ in place of φ, and take the ordinals $\alpha, \beta_1, \ldots, \beta_n$; this gives

$$\forall z[\psi(\alpha, \beta_1, \ldots, \beta_n, z) \leftrightarrow z = a] \rightarrow a \in \mathbf{OD};$$

so

$$\begin{aligned}&\forall z[z = \{y \in \mathbf{HOD} : \exists x \in \mathrm{Enod}(\alpha)\\ &\quad[\varphi^{\mathbf{HOD}}(x, y, \mathrm{Enod}(\alpha), \mathrm{Enod}(\beta_1), \ldots, \mathrm{Enod}(\beta_n) \leftrightarrow z = a]\\ &\quad\to a \in \mathbf{OD},\end{aligned}$$

giving

$$\forall z[z = \{y \in \mathbf{HOD} : \exists x \in A[\varphi^{\mathbf{HOD}}(x, y, A, w_1, \ldots, w_n)\} \leftrightarrow z = a] \to a \in \mathbf{OD}.$$

Recalling the definition of a, it follows that $a \in \mathbf{OD}$.

Foundation: holds since **HOD** is transitive.

Infinity: holds since $\omega \in \mathbf{HOD}$ by exercise E23.12; see Theorem 10.27

Axiom of Choice: given $A \in \mathbf{HOD}$, we want to find a relation $R \in \mathbf{HOD}$ which well-orders A. By absoluteness (Proposition 10.32), this suffices. Let

$$R = \{(x, y) \in A \times A : \exists\xi[x = \mathrm{Enod}(\xi) \wedge \forall\eta \le \xi[y \ne \mathrm{Enod}(\eta)]].$$

Clearly R well-orders A. To see that $R \in \mathbf{HOD}$ we will apply exercise E23.7 again. Choose α so that $A = \mathrm{Enod}(\alpha)$. Let $\varphi(y, x)$ be the formula

$$x = \{(u, v) \in \mathrm{Enod}(y) \times \mathrm{Enod}(y) : \exists\xi[u = \mathrm{Enod}(\xi) \wedge \forall\eta \le \xi[v \ne \mathrm{Enod}(\eta)]\}.$$

Applying exercise E23.7, we get

$$\forall x[\varphi(\alpha, x) \leftrightarrow x = R] \to R \in \mathbf{OD}.$$

Now $\psi(\alpha, x)$ is

$$x = \{(u, v) \in A \times A : \exists\xi[u = \mathrm{Enod}(\xi) \wedge \forall\eta \le \xi[v \ne \mathrm{Enod}(\eta)]\},$$

i.e., it is $x = R$. So we conclude that $R \in \mathbf{OD}$. It remains only to check that $R \subseteq \mathbf{HOD}$. If $(u, v) \in R$, then clearly $u, v \in \mathbf{HOD}$. Hence by Theorem 10.28, $(u, v) \in \mathbf{HOD}$.

E23.15 *Show that the axiom of choice holds in $L(B)$ iff trcl($\{A\}$) can be well-ordered in $L(B)$.*

The direction $\Leftarrow$ is obvious. For $\Rightarrow$, we just need to modify a usual proof of AC in L. Note that since $C = \emptyset$ in the definition of $L(B)$, we have $\mathrm{Rel}(A, \emptyset, n, i) = \emptyset$. By the proof of Lemma 23.1, we can simply ignore $\mathrm{Rel}(A, \emptyset, n, i)$.

Let $\prec$ be a well-order of trcl($\{B\}$). Let A be any set, and n any natural number. For each $R \in \{\mathrm{Diag}_\in(A, n, i, j) : i, j < n\}$, let $(\mathrm{Ch}(0, A, n, R), \mathrm{Ch}(1, A, n, R))$ be the smallest pair (i, j), in the lexicographic order of $\omega \times \omega$ such that $i, j < n$ and $R = \mathrm{Diag}_\in(A, n, i, j)$. Now we define

$$R <_{0An} S \quad \text{iff} \quad (\mathrm{Ch}(0, A, n, R), \mathrm{Ch}(1, A, n, R)) <_{\mathrm{lex}} \mathrm{Ch}(0, A, n, S), \mathrm{Ch}(1, A, n, S)).$$

Clearly this is a well-order of $\{\mathrm{Diag}_\in(A,n,i,j) : i,j < n\}$.

In a very analogous way we can define a well-order $<_{1An}$ of $\{\mathrm{Diag}_=(A,n,i,j) : i,j < n\}$.

Now we can define a well-order $<_{2An}$ of

$$\{\mathrm{Diag}_\in(A,n,i,j) : i,j < n\} \cup \{\mathrm{Diag}_=(A,n,i,j) : i,j < n\}$$

as follows. For any R,S in this union,

$$\begin{aligned} R <_{2An} S \quad \text{iff} \quad & R,S \in \{\mathrm{Diag}_\in(A,n,i,j) : i,j < n\} \text{ and } R <_{0An} S \\ & \text{or } R \in \{\mathrm{Diag}_\in(A,n,i,j) : i,j < n\}, S \notin \{\mathrm{Diag}_\in(A,n,i,j) : i,j < n\} \\ & \text{or } R,S \notin \{\mathrm{Diag}_\in(A,n,i,j) : i,j < n\} \text{ and } R <_{1An} S. \end{aligned}$$

For the next few constructions, suppose that X and A are sets, $n \in \omega$, and we are given a well-ordering $<$ of X. Then we well-order $\{{}^nA\backslash R : R \in X\}$ by setting

$$S \prec_{0,A,n,<,X} T \quad \text{iff} \quad \exists S',T' \in X[S' < T' \text{ and } S = {}^nA\backslash S', T = {}^nA\backslash T'].$$

We well-order $\{R \cup S : R,S \in X\}$ as follows. Suppose that $U,V \in \{R \cup S : R,S \in X\}$. Let (R,S) be lexicographically smallest in $X \times X$ (using $<$) such that $U = R \cup S$, and let (R',S') be lexicographically smallest in $X \times X$ (using $<$) such that $V = R' \cup S'$. Then $U <_{1,A,n,<,X} V$ iff $(R,S) <_{\mathrm{lex}} (R',S')$.

We well-order $\{\mathrm{proj}(A,R,n) : R \in X\}$ as follows. Suppose that $U,V \in \{\mathrm{proj}(A,R,n) : R \in X\}$. Let R be $<$-minimum in X such that $U = \mathrm{proj}(A,R,n)$, and let S be $<$-minimum in X such that $V = \mathrm{proj}(A,S,n)$. Then $U <_{2,A,n,<,X} V$ iff $R < S$.

Next, for any set A and any $k,n \in \omega$ we define a well-order $<_{3kAn}$ of $\mathrm{Df}'(k,A,n)$ by induction on k. Let $<_{30An}$ be $<_{2An}$. Assume that $<_{3kAn}$ has been defined for all $n \in \omega$, and let $R,S \in \mathrm{Df}'(k+1,A,n)$. Then we define $R <_{3(k+1)An} S$ iff one of the following conditions holds:

(1) $R,S \in \mathrm{Df}'(k,A,n)$ and $R <_{3kAn} S$

(2) $R \in \mathrm{Df}'(k,A,n)$ and $S \notin \mathrm{Df}'(k,A,n)$.

(3) $R,S \notin \mathrm{Df}'(k,A,n)$, $R,S \in \{{}^nA\backslash T : T \in \mathrm{Df}'(k,A,n)\}$, and

$$R \prec_{0,A,n,<_{3kAn},\mathrm{Df}'(k,A,n)} S.$$

(4) $R,S \notin \mathrm{Df}'(k,A,n)$, $R \in \{{}^nA\backslash T : T \in \mathrm{Df}'(k,A,n)\}$, and $S \notin \{{}^nA\backslash T : T \in \mathrm{Df}'(k,A,n)\}$.

(5) $R,S \notin \mathrm{Df}'(k,A,n)$, $R,S \notin \{{}^nA\backslash T : T \in \mathrm{Df}'(k,A,n)\}$, $R,S \in \{T \cup U : T,U \in \mathrm{Df}'(k,A,n)\}$, and

$$R \prec_{1,A,n,<_{3kAn},\mathrm{Df}'(k,A,n)} S.$$

(6) $R,S \notin \mathrm{Df}'(k,A,n)$, $R,S \notin \{{}^nA\backslash T : T \in \mathrm{Df}'(k,A,n)\}$, $R \in \{T \cup U : T,U \in \mathrm{Df}'(k,A,n)\}$, and $S \notin \{T \cup U : T,U \in \mathrm{Df}'(k,A,n)\}$.

(7) $R, S \notin \mathrm{Df}'(k, A, n)$, $R, S \notin \{{}^nA \backslash T : T \in \mathrm{Df}'(k, A, n)\}$, $R, S \notin \{T \cup U : T, U \in \mathrm{Df}'(k, A, n)\}$, and

$$R \prec_{2,A,n,<_{3kA(n+1)},\mathrm{Df}'(k,A,n)} S.$$

Finally, for any set A and any natural number n, we well-order $\mathrm{Df}(A, n)$ as follows. Let $R, S \in \mathrm{Df}(A, n)$. Let k be minimum such that $R \in \mathrm{Df}'(k, A, n)$, and let l be minimum such that $S \in \mathrm{Df}'(l, A, n)$. Then we define

$$R <_{4An} S \quad \text{iff} \quad k < l, \text{ or } k = l \text{ and } R <_{3kAn} S.$$

Now we define a well-ordering $<_{B5\alpha}$ of $L_\alpha^{\{B\}\emptyset}$ by recursion. First of all, $<_{B50}=\prec$, the given well-ordering of $\mathrm{trcl}(\{B\})$. If α is a limit ordinal, then for any $x, y \in L^{\{B\}\emptyset}\alpha$ we define

$$x <_{B5\alpha} y \quad \text{iff} \quad \rho(x) < \rho(y) \vee [\rho(x) = \rho(y) \text{ and } x <_{B5\rho(x)} y].$$

Clearly this is a well-order of $L_\alpha^{\{B\}\emptyset}$.

Now suppose that a well-order $<_{B5\alpha}$ of $L_\alpha^{\{B\}\emptyset}$ has been defined. Then for each $n \in \omega$ we define the lexicographic order $<_{B6n\alpha}$ on ${}^nL_\alpha^{\{B\}\emptyset}$: for any $x, y \in {}^nL_\alpha^{\{B\}\emptyset}$,

$$x <_{B6n\alpha} y \quad \text{iff} \quad \exists k < n[x \upharpoonright k = y \upharpoonright k \text{ and } x(k) <_{B5\alpha} y(k)].$$

Clearly this is a well-order of ${}^nL_\alpha^{\{B\}\emptyset}$. Now for any $X \in L^{\{B\}\emptyset}\alpha + 1 = \mathscr{D}(L_\alpha^{\{B\}\emptyset})$, let $n(X)$ be the least natural number n such that

$$\exists s \in {}^nL_\alpha^{\{B\}\emptyset} \exists R \in \mathrm{Df}(L_\alpha^{\{B\}\emptyset}, n+1)[X = \{x \in L_\alpha^{\{B\}\emptyset} : s^\frown\langle x\rangle \in R\}].$$

Then let $s(X)$ be the least member of ${}^{n(X)}L_\alpha^{\{B\}\emptyset}$ (under the well-order $<_{B6n(X)\alpha}$) such that

$$\exists R \in \mathrm{Df}(L_\alpha^{\{B\}\emptyset}, n(X)+1)[X = \{x \in L_\alpha^{\{B\}\emptyset} : s(X)^\frown\langle x\rangle \in R\}].$$

Then let $R(X)$ be the least member of $\mathrm{Df}(L_\alpha^{\{B\}\emptyset}, n+1)$ (under $<_{B4L_\alpha^{\{B\}\emptyset}(n+1)}$) such that

$$X = \{x \in L_\alpha^{\{B\}\emptyset} : s(X)^\frown\langle x\rangle \in R(X)\}].$$

Finally, for any $X, Y \in L^{\{B\}\emptyset}\alpha + 1$ we define $X <_{B5(\alpha+1)} Y$ iff one of the following conditions holds:

(i) $X, Y \in L_\alpha^{\{B\}\emptyset}$ and $X <_{B5\alpha} Y$.

(ii) $X \in L_\alpha^{\{B\}\emptyset}$ and $Y \notin L_\alpha^{\{B\}\emptyset}$.

(iii) $X, Y \notin L_\alpha^{\{B\}\emptyset}$ and one of the following conditions holds:

(a) $n(X) < n(Y)$.

(b) $n(X) = n(Y)$ and $s(X) <_{B6n(X)\alpha} s(Y)$.

(c) $n(X) = n(Y)$ and $s(X) = s(Y)$ and $R(X) <_{4L_\alpha^{\{B\}\emptyset}(n+1)} R(Y)$.

Clearly this gives a well-order of $L_{\alpha+1}^{\{B\}\emptyset}$.

We denote the union of all the well-orders $<_{B5\alpha}$ for $\alpha \in \mathrm{On}$ by $<_{BL}$. Under $\mathrm{V} = \mathbf{L}(\mathbf{B})$ it is a well-ordering of the universe.

E23.16 *Recall from elementary set theory the following definition of the standard well-ordering of On × On:*

$$
\begin{aligned}
(\alpha,\beta) \prec (\gamma,\delta) \quad \textit{iff} \quad & (\alpha \cup \beta < \gamma \cup \delta) \\
& \textit{or } (\alpha \cup \beta = \gamma \cup \delta \textit{ and } \alpha < \gamma) \\
& \textit{or } (\alpha \cup \beta = \gamma \cup \delta \textit{ and } \alpha = \gamma \textit{ and } \beta < \delta).
\end{aligned}
$$

Prove that $\prec$ is absolute for transitive class models of ZF.

Now define $\Delta : On \to On \times On$ by recursion as follows:

$$
\Delta(0) = (0,0);
$$

$$
\Delta(\alpha+1) = \begin{cases} (\beta,\gamma+1) & \textit{if } \Delta(\alpha) = (\beta,\gamma) \textit{ and } \gamma < \beta, \\ (0,\beta+1) & \textit{if } \Delta(\alpha) = (\beta,\gamma) \textit{ and } \gamma = \beta, \\ (\beta+1,\gamma) & \textit{if } \Delta(\alpha) = (\beta,\gamma) \textit{ and } \beta+1 < \gamma, \\ (\gamma,0) & \textit{if } \Delta(\alpha) = (\beta,\gamma) \textit{ and } \beta+1 = \gamma; \end{cases}
$$

$$
\Delta(\alpha) = \prec\textit{-least}(\beta,\gamma) \textit{ such that } \forall\delta < \alpha[\Delta(\delta) \prec (\beta,\gamma)] \textit{ if } \alpha \textit{ is limit.}
$$

Prove:

(1) If $\alpha < \beta$, then $\Delta(\alpha) \prec \Delta(\beta)$.

(2) Δ maps onto On × On.

(3) Δ is absolute for transitive class models of ZF.

(4) Δ^{-1} is absolute for transitive class models of ZF.

Clearly $\prec$ is absolute, by the basic absoluteness results.

For (1), fix α; we go by induction on β. Assume that the implication "$\alpha < \beta$ implies that $\Delta(\alpha) \prec \Delta(\beta)$" holds for all $\beta < \gamma$; we prove it for γ. If $\gamma \leq \alpha$, the implication vacuously holds. Now suppose that $\alpha < \gamma$. If $\gamma = \beta + 1$, then $\alpha \leq \beta$, and hence $\Delta(\alpha) \preceq \Delta(\beta)$ either trivially if $\alpha = \beta$, or by the inductive hypothesis if $\alpha < \beta$. Clearly $\Delta(\beta) < \Delta(\beta+1)$, so $\Delta(\alpha) < \Delta(\gamma)$. The case of γ limit is clear.

For (2), suppose to the contrary that $\mathrm{rng}(\Delta) \subset \mathrm{On} \times \mathrm{On}$, and let (β,γ) be the $\prec$-least pair of ordinals not in $\mathrm{rng}(\Delta)$. Then the following list of all possiblities gives a contradiction in each case:

(1) $\beta = \gamma = 0$.

(2) $\beta = \delta + 1$, $\gamma = 0$; then (β,γ) is the immediate successor of $(\delta,\delta+1)$.

(3) β limit, $\gamma = 0$; then (β,γ) is the least upper bound of $\{(\delta,\beta) : \delta < \beta\}$.

(4) $\beta = 0$, $\gamma = \delta + 1$; then (β,γ) is the immediate successor of (δ,δ).

(5) $\beta = \varepsilon + 1 < \delta + 1 = \gamma$; then (β,γ) is the immediate successor of (ε,γ).

(6) $\beta = \delta + 1 = \gamma$; then (β,γ) is the immediate successor of $(\delta+1,\delta)$.

(7) $\beta = \varepsilon + 1 > \delta + 1 = \gamma$; then (β,γ) is the immediate successor of (β,δ).

(8) β limit, $\beta < \gamma = \delta + 1$; then (β,γ) is the lub of $\{(\delta,\gamma) : \delta < \beta\}$.

(9) β limit, $\beta > \delta + 1 = \gamma$; then (β,γ) is the immediate successor of (β,δ).

(10) $\beta = 0$, γ limit; then (β,γ) is the lub of $\{(\delta,0) : \delta < \gamma\}$.

(11) $\beta = \delta + 1 < \gamma$ with γ limit; then (β, γ) is the immediate successor of (δ, γ).
(12) $\beta = \delta + 1 > \gamma$ with γ limit; then (β, γ) is the lub of $\{(\beta, \varepsilon) : \varepsilon < \gamma\}$.
(13) β and γ are limit, with $\beta < \gamma$. Then (β, γ) is the lub of $\{(\delta, \gamma) : \delta < \beta\}$.
(14) $\beta = \gamma$ limit; then (β, γ) is the lub of $\{(\delta, \beta) : \delta < \beta\}$.
(15) β and γ are limit, $\gamma < \beta$; then (β, γ) is the lub of $\{(\beta, \delta) : \delta < \gamma\}$.

(3) is clear by the theorem about absoluteness of recursive definitions.

(4) holds on general grounds; the inverse of a one-one absolute function is clearly absolute.

E23.17 *Suppose that M is a transitive class model of ZFC, and every set of ordinals is in M. Show that $M = V$. Hint: take any set X. Let $\kappa = |trcl(\{X\})|$, and let $f : \kappa \to trcl(\{X\})$ be a bijection. Define $\alpha E \beta$ iff $\alpha, \beta < \kappa$ and $f(\alpha) \in f(\beta)$. Use exercise E23.16 to show that $E \in M$. Take the Mostowski collapse of (κ, E) in M, and infer that $X \in M$.*

We follow the hint. Let Δ be as in exercise E23.16. Then $\Delta^{-1}[E]$ is a set of ordinals, so by supposition it is in M. It follows that E itself is in M. Now we work in M. Clearly E is well-founded, since $\in$ is, and obviously E is set-like. Hence we can apply the Mostowski collapse; let G be the Mostowski collapsing function.

Now working outside M, we claim that $G = f$ (hence $f \in M$). Since E is clearly well-founded (outside M too), we can suppose that $G \neq f$ and take the E-least $\alpha \in \kappa$ such that $G(\alpha) \neq f(\alpha)$. Then for any $x \in M$,

$$\begin{aligned}
x \in G(\alpha) \quad &\text{iff} \quad \exists \beta \in \kappa[x = G(\beta) \text{ and } \beta E \alpha \quad \text{(definition of } G)\\
&\text{iff} \quad \exists \beta \in \kappa[x = f(\beta) \text{ and } \beta E \alpha \quad \text{(minimality of } \alpha)\\
&\text{iff} \quad \exists \beta \in \kappa[x = f(\beta) \text{ and } f(\beta) \in f(\alpha) \quad \text{(definition of } E)\\
&\text{iff} \quad x \in f(\alpha) \quad \text{(transitivity of rng}(f)));
\end{aligned}$$

Thus $G(\alpha) = f(\alpha)$, contradiction.

Hence our claim holds: $G = f$. So $f \in M$. Since $X \in \text{trcl}(\{X\}) = \text{rng}(f)$, and $\text{rng}(f) \in M$ by absoluteness, if follows that $X \in M$.

E23.18 *Show that if $X \subseteq \omega_1$ then CH holds in $L(X)$. Hint: show that if $A \in \mathscr{P}(\omega)$ in $L(X)$, then there are $\alpha, \beta < \omega_1$ such that $X \in L_\alpha(X \cap \beta)$.*

We apply a reflection principle, Theorem 10.41. Let φ_1 be enough of ZFC to prove that "$x \in L_y(z)$" is absolute for transitive models of φ_1. Let φ_2 be the extensionality axiom, let φ_3 be the formula "$x \in L_y(z)$", and let φ_4 be the formula "x is an ordinal". Now $A \in L_\alpha(X)$ for some ordinal α. By that reflection principle, let B be a countable set such that $\{A, \alpha, X, \omega_1\} \subseteq B \subseteq L(X)$ and $\varphi_1, \varphi_2, \varphi_3 \varphi_4$ are absolute for B, V. Let C be the Mostowski collapse of B, with collapsing function G. Then $(A \in L_\alpha(X))^B$, since φ_3 is absolute for B, V. Hence also $(G(A) \in L_{G(\alpha)}(G(X))^C$, since G is an isomorphism. So $G(A) \in L_{G(\alpha)})G(X)$ since C, being isomorphic to B, is a model of φ_1. Now $A \subseteq \omega$, so clearly $G(A) = A$. Let γ be the least ordinal not in C. An easy argument using the transitivity of C shows that $C \cap \text{Ord} = \gamma$.

(1) If $\delta \in \gamma$, then $G(\delta) = \delta$.

In fact, suppose that $\delta \in \gamma$ is $\in$-minimal such that $G(\delta) \neq \delta$. If $\varepsilon \in G(\delta)$, choose $\beta \in \delta$ such that $\varepsilon = G(\beta)$. Since $\beta \in \delta$, we have $G(\beta) = \beta$; so $\varepsilon = \beta \in \delta$. This shows that $G(\delta) \subseteq \delta$. If $\beta \in \delta$, then $G(\beta) = \beta$ and $G(\beta) \in G(\delta)$; so $\beta \in G(\delta)$. This shows that $\delta \subseteq G(\delta)$. so $G(\delta) = \delta$, contradiction. Hence (1) holds.

Now α is an ordinal, so $(\alpha \text{ is an ordinal})^B$ since φ_4 is absolute for B, V. Hence $(G(\alpha) \text{ is an ordinal})^C$ since G is an isomorphism. Hence $G(\alpha)$ is an ordinal, by absoluteness since C is transitive. Morover, C is countable and $G(\alpha) \in C$, so $G(\alpha)$ is countable. This is part of the desired conclusion. Here is the other part:

$G(X) = X \cap \gamma$.

For, suppose that $y \in G(X)$. Choose $\beta \in X$ such that $y = G(\beta)$. Now $\delta \stackrel{\text{def}}{=} G(\omega_1)$ is an ordinal since B is a model of φ_4. Since $G(X) \subseteq G(\omega_1)$, it follows that $\beta < \delta < \gamma$. So by (1) we have $y = G(\beta) = \beta \in X \cap \gamma$.

On the other hand, if $\beta \in X \cap \gamma$, then by (1), $\beta = G(\beta) \in G(X)$. This finishes the proof of the statement of the hint.

It follows that in $L(X)$ we have $\mathscr{P}(\omega) \subseteq \bigcup_{\alpha,\beta<\omega_1} L_\alpha(X \cap \beta)$. Now it is clear (by induction) that $|L_\alpha(X \cap \beta)| \leq \max(\omega, |\alpha|, |\beta|)$, so CH follows.

E23.19 *Show that if $X \subseteq \omega_1$ then GCH holds in $L(X)$.*

By induction it is clear that $|L_\alpha(X)| \leq \max(\alpha, \omega, |X|)$ for any ordinal α. Next, we claim

(*) There is a sentence φ which is a finite conjunction of members of $\mathrm{ZF} + \mathrm{V} = \mathbf{L}(X)$ such that

$$ZFC \vdash \forall M[M \text{ transitive } \wedge \varphi^M \to M = L_{o(M)}(X)].$$

Proof. Let φ be a conjunction of $\mathrm{V} = \mathbf{L}(X)$ together with enough of ZF to prove that $\langle L_\alpha(X) : \alpha \in \mathrm{On}\rangle$ is absolute, and also enough to prove that there is no largest ordinal. Then for any transitive set M, if φ^M, then $o(M)$ is a limit ordinal, $(\forall x(x \in \mathbf{L}(X)))^M$ and hence $M = \mathbf{L}(X)^M$, and

$$M = \mathbf{L}(X)^M = \{x \in M : (\exists\alpha(x \in L(\alpha)(X)))^M\} = \bigcup_{\alpha \in M} L_\alpha(X) = L_{o(M)}(X),$$

as desired in (*).

(**) If $\mathrm{V} = \mathbf{L}(X)$, then for every infinite ordinal α we have $\mathscr{P}(L_\alpha(X)) \subseteq L_{\omega_1 \cup \alpha^+}$.

For, let φ be as in (*). Assume that $\mathrm{V} = \mathbf{L}(X)$ and α is an infinite ordinal. Take any $A \in \mathscr{P}(L_\alpha(X))$. Let $Y = L_\alpha(X) \cup \{A\}$. Clearly X is transitive. Now $|Y| \leq |\omega_1 \cup \alpha|$. Now by a reflection theorem, let M be a transitive set such that $X \subseteq M$, $|M| = |\alpha|$, and $\varphi^M \leftrightarrow \varphi^V$. But φ^V actually holds, so φ^M holds. Hence $M = L_{o(M)}$ by (**). Now $o(M) = M \cap \mathrm{On}$, and $|M| = |\omega_1 \cup \alpha|$, so $o(M) < \omega_1 \cup \alpha^+$. Hence $A \in X \subseteq M = L_{o(M)}(X) \subseteq L_{\omega_1 \cup \alpha^+}(X)$.

Clearly (**) and exercise E23.18 imply GCH.

Solutions to exercises in Chapter 24

E24.1 *Show that fin(ω, ω_1) collapses ω_1 to ω, but preserves cardinals $\geq \omega_2$.*

For collapsing, see the beginning of Chapter 12.

Clearly $\mathrm{fin}(\omega, \omega_1)$ has size ω_1, and hence satisfies ω_2-cc. So the second assertion follows from Proposition 12.5.

E24.2 *Suppose that κ is an uncountable regular cardinal of M, and $\mathscr{P} \in M$ is a κ-cc quasi-order. Assume that C is club in κ, with $C \in M[G]$. Show that there is a $C' \subseteq C$ such that $C' \in M$ and C' is club in κ. Hint: in $M[G]$ let $f : \kappa \to \kappa$ be such that $\forall \alpha < \kappa[\alpha < f(\alpha) \in C]$. Apply Theorem 12.4.*

By Theorem 12.4 let $F \in M$ be such that $F : \kappa \to \mathscr{P}(\kappa)$, $f(\alpha) \in F(\alpha)$ for all $\alpha < \kappa$, and $(|F(\alpha)| < \kappa)^M$ for all $\alpha < \kappa$. We now define $g : \kappa \to \kappa$ by recursion. Let $g(0) = \sup(F(0))$. If $g(\alpha)$ has been defined, let $g(\alpha + 1) = \max(g(\alpha) + 1, \sup(F(g(\alpha))))$. For β limit, let $g(\beta) = \bigcup_{\alpha<\beta} g(\alpha)$.

(1) If β is limit, then $g(\beta) \in C$.

For, it suffices to show that $g(\beta) \cap C$ is unbounded in $g(\beta)$, since $g(\beta)$ is clearly a limit ordinal. Suppose that $\gamma < g(\beta)$. Choose $\alpha < \beta$ such that $\gamma < g(\alpha)$. Then $\gamma < g(\alpha) < f(g(\alpha)) \in F(g(\alpha))$. Now $f(g(\alpha)) \leq \sup(F(g(\alpha))) \leq g(\alpha + 1) < g(\beta)$. So we have $\gamma < f(g(\alpha)) < g(\beta)$ with $f(g(\alpha)) \in C$, as desired.

It follows that $C' \stackrel{\text{def}}{=} \{g(\beta) : \beta \text{ limit}\}$ is as desired.

E24.3 *Suppose that κ is an uncountable regular cardinal of M, and $\mathscr{P} \in M$ is a κ-cc quasi-order. Assume that $S \in M$ is stationary in κ, in the sense of M. Show that it remains stationary in $M[G]$.*

Let $C \in M[G]$ be club in κ. By exercise E24.2, let $C' \in M$ be club in κ with $C' \subseteq C$. Then $\emptyset \neq C' \cap S \subseteq C \cap S$.

E24.4 *Suppose that κ is an uncountable regular cardinal of M, and $\mathscr{P} \in M$ is a κ-closed quasi-order. Assume that $S \in M$ is stationary in κ, in the sense of M. Show that it remains stationary in $M[G]$.*

Suppose that $\mathbb{P}$ is κ-closed, C is club in κ, $C \in M[G]$, and $S \cap C = \emptyset$. Let $f \in M[G]$ be the strictly increasing enumeration of C. Say $\tau_G = f$. Choose $q \in P$ such that

$$q \Vdash \tau : \kappa \to \kappa \text{ is strictly increasing, continuous, and such that } \forall \alpha \in \check{\kappa}[\tau(\alpha) \notin \check{S}]$$

Now by induction define in M sequences $\langle p_\alpha : \alpha < \kappa \rangle$, $\langle z_\alpha : \alpha < \kappa \rangle$ such that each $p_\alpha \in P$, each $z_\alpha \in \kappa \backslash S$, $p_0 = q$, $p_\alpha \Vdash \tau(\check{\alpha}) = \check{z}_\alpha$, and $p_\beta \leq p_\alpha$ for $\alpha < \beta < \kappa$; this is possible by κ-closure. We claim that $\mathrm{rng}(z)$ is club in κ with $\mathrm{rng}(z) \cap S = \emptyset$ (contradiction). Clearly $\mathrm{rng}(z) \cap S = \emptyset$.

(1) If $\alpha < \beta < \kappa$, then $z_\alpha < z_\beta$.

In fact,

$$q \Vdash \forall \gamma < \check{\kappa} \forall \delta < \check{\kappa}[\gamma < \delta \to \tau(\gamma) < \tau(\delta)],$$

so

$$p_\beta \Vdash \tau(\check{\alpha}) < \tau(\check{\beta}) \wedge \tau(\check{\alpha}) = \check{z}_\alpha \wedge \tau(\check{\beta}) = \check{z}_\beta;$$

hence $p_\beta \Vdash \check{z}_\alpha < \check{z}_\beta$, so $z_\alpha < z_\beta$, as desired in (1).

(2) If $\alpha < \kappa$ is limit, then $z_\alpha = \bigcup_{\beta<\alpha} z_\beta$.

In fact,

$$q \Vdash \forall\gamma < \kappa \left[\gamma \text{ limit } \rightarrow \tau(\gamma) = \bigcup_{\delta<\gamma} \tau(\delta)\right];$$

Hence p_α forces this too. Let H be $\mathbb{P}$-generic over M with $p_\alpha \in H$, and let $g = \tau_H$. Now $p_\alpha \Vdash \tau(\check{\delta}) = \check{z}_\delta$ for each $\delta \le \alpha$, so $z_\alpha = g(\alpha) = \bigcup_{\delta<\alpha} g(\delta) = \bigcup_{\delta<\alpha} z_\delta$, as desired in (2).

Now it follows that $\text{rng}(z)$ is club in κ.

E24.5 *Prove that if ZFC is consistent, then so is* ZFC + GCH + $\neg(V = L)$.

Let M be a c.t.m. of ZFC + GCH; it exists by the theory of constuctibility. Let $\mathbb{P} = \text{fin}(\omega_1, 2)$, and let G be $\mathbb{P}$-generic over M. By Lemma 11.2, $G \notin M$. By Lemma 11.13, M and $M[G]$ have the same ordinals, so, since L is absolute, $L^M = L^{M[G]} \subseteq M$, and so $M[G]$ is not a model of $V = L$.

For GCH, let λ be any cardinal of M. Note that our quasi-order is ccc, so that cardinals are preserved. Then by theorem 24.4,

$$\lambda^+ \le (2^\lambda)^{M[G]} \le (\omega_1^\lambda)^M = (2^\lambda)^M = (\lambda^+)^M = (\lambda^+)^{M[G]}.$$

Solutions to exercises for Chapter 25

E25.1 *Show that for any infinite cardinal κ, the partial order fn$(\kappa, 2, \omega)$ is isomorphic to fn$(\kappa \times \omega, 2, \omega)$.*

Let $f : \kappa \times \omega \to \kappa$ be a bijection. Now we define $F : \text{fn}(\kappa, 2, \omega) \to \text{fn}(\kappa \times \omega, 2, \omega)$ as follows. Let $a \in \text{Fn}(\kappa, 2, \omega)$. So a is a finite function contained in $\kappa \times 2$. Let $F(a) = a \circ f^{-1}$. Thus $F(a)$ is a function whose domain is

$$\{x \in \text{dmn}(f^{-1}) : f^{-1}(x) \in \text{dmn}(a)\} = f[\text{dmn}(a)],$$

and this is a finite subset of $\kappa \times \omega$. Obviously $F(a)$ is a function which is a subset of $(\kappa \times \omega) \times 2$. So $F(a) \in \text{Fn}(\kappa \times \omega, 2, \omega)$.

If $a, b \in \text{fin}(\kappa, 2, \omega)$, clearly $a \subseteq b$ iff $F(a) \subseteq F(b)$. In particular, F is one-one. Clearly $F(\emptyset) = \emptyset$.

So it remains only to show that F maps onto $\text{fin}(\kappa \times \omega, 2, \omega)$. Let $b \in \text{fin}(\kappa \times \omega, 2, \omega)$. Then clearly $b \circ f \in \text{fin}(\kappa, 2)$ and $F(b \circ f) = b \circ f \circ f^{-1} = b$, as desired.

E25.2 *Prove that if $\mathbb{P}$ and $\mathbb{Q}$ are isomorphic forcing orders, then RO$(\mathbb{P})$ and RO$(\mathbb{Q})$ are isomorphic Boolean algebras.*

This is abstract generalized nonsense.

Let f be an isomorphism from $\mathbb{P}$ onto $\mathbb{Q}$, and let $e^{\mathbb{P}}$ and $e^{\mathbb{Q}}$ be the embeddings of $\mathbb{P}$ into $\text{RO}(\mathbb{P})$ and of $\mathbb{Q}$ into $\text{RO}(\mathbb{Q})$ given in Chapter 9. Then the following conditions are clear:

(1) $(e^{\mathbb{Q}} \circ f)[P]$ is dense in $\mathrm{RO}(\mathbb{P})$.

(2) For all $p, q \in P$, if $p \leq q$ then $e^{\mathbb{Q}}(f(p))) \leq e^{\mathbb{Q}}(f(q))$.

(3) For all $p, q \in P$, $p \perp q$ iff $e^{\mathbb{P}}(f(p)) \cdot e^{\mathbb{P}}(f(q)) = 0$.

Hence it follows from Theorem 9.22 that there is an isomorphism g from $\mathrm{RO}(\mathbb{P})$ into $\mathrm{RO}(\mathbb{Q})$ such that $g \circ e^{\mathbb{P}} = e^{\mathbb{Q}} \circ f$.

By symmetry we get an isomorphism h from $\mathrm{RO}(\mathbb{Q})$ into $\mathrm{RO}(\mathbb{P})$ such that $h \circ e^{\mathbb{Q}} = e^{\mathbb{P}} \circ f^{-1}$.

Hence $h \circ g \circ e^{\mathbb{P}} = h \circ e^{\mathbb{Q}} \circ f = e^{\mathbb{P}} \circ f^{-1} \circ f = e^{\mathbb{P}}$. It follows from Theorem 9.22 that $h \circ g$ is the identity on $\mathrm{RO}(\mathbb{P})$.

Similarly, $g \circ h$ is the identity on $\mathrm{RO}(\mathbb{Q})$. So g is the desired isomorphism.

E25.3 *Give an example of non-isomorphic forcing orders $\mathbb{P}$ and $\mathbb{Q}$ such that RO($\mathbb{P}$) and RO($\mathbb{Q}$) are isomorphic Boolean algebras.*

Let $\mathbb{P}$ be $\mathrm{fn}(\omega, 2, \omega)$, let $A = \mathrm{RO}(\mathbb{P})$, and let $\mathbb{Q} = (A \backslash \{0\}, \geq, 1)$. For each $a \in Q$ let $j(a) = a$. Then the conditions of Theorem 9.22 are clear, and so $\mathrm{RO}(\mathbb{Q})$ is isomorphic to A. So we just need to show that $\mathbb{P}$ and $\mathbb{Q}$ are not isomorphic. Suppose to the contrary that f is an isomorphism from $\mathbb{P}$ onto $\mathbb{Q}$. Let $p = \{(0,0)\}$. Now $p \neq \emptyset$, so $f(p) \neq 1$, and hence $-f(p) \neq 0$, so that $-f(p) \in Q$. Choose $q \in P$ such that $f(q) = -f(p)$. Let r be any member of $\mathrm{fin}(\omega, 2)$ such that $q \subset r$. Hence $r < q$, and so $f(r) < f(q) = -f(p)$, and so $f(p) < -f(r)$. Since $f(r) \neq 1$, we have $-f(r) \neq 0$, so we can choose $s \in P$ such that $f(s) = -f(r)$. Thus $f(p) < f(s)$, so $p < s$. This means that $s \subset p$. Since p is a singleton, it follows that $s = \emptyset$, hence $1 = f(s) = -f(r)$, so $f(r) = 0$, contradiction.

E25.4 *For any system $\langle \mathbb{P}_i : i \in I \rangle$, we define the weak product $\prod_{i \in I}^{w} \mathbb{P}_i$ as follows: the underlying set is $\{f \in \prod_{i \in I} P_i : \{i \in I : f(i) \neq 1\}$ is finite$\}$, with $f \leq g$ iff $f(i) \leq_{\mathbb{P}_i} g(i)$ for all $i \in I$. Prove that for any infinite cardinal κ, the forcing order fin$(\kappa, 2)$ is isomorphic to $\prod_{\alpha < \kappa}^{w} \mathbb{P}_\alpha$, where each $\mathbb{P}_\alpha$ is equal to fin$(\omega, 2)$.*

Let F be a bijection from $\kappa \times \omega$ onto κ. Now for each $f \in \mathrm{fin}(\kappa, 2)$ we define $G(f) \in \prod_{\alpha < \kappa} \mathrm{fin}(\omega, 2)$ as follows. For any $\alpha < \kappa$, $(G(f))_\alpha$ has domain $\{i \in \omega : F(\alpha, i) \in \mathrm{dmn}(f)\}$. Clearly this is a finite set. For any $i \in \mathrm{dmn}((G(f))_\alpha$ we set $((G(f))_\alpha)(i) = f(F(\alpha, i))$. Clearly $\{\alpha < \kappa : \mathrm{dmn}(G(f))_\alpha \neq \emptyset\}$ is finite, so $G(f) \in \prod_{\alpha < \kappa}^{w} \mathrm{fin}(\omega, 2)$. Clearly $f \subseteq g$ implies that $G(f) \geq G(g)$.

Conversely, for each $x \in \prod_{\alpha < \kappa}^{w} \mathrm{fin}(\omega, 2)$ we define

$$H(x) = \{(\alpha, i) : \exists \beta < \kappa \exists j < \omega [\alpha = F(\beta, j),\ j \in \mathrm{dmn}(x_\beta),\ \text{and } x_\beta(j) = i]\}.$$

Now $H(x)$ is a function. For, suppose that $(\alpha, i), (\alpha, k) \in H(x)$ Then with $\alpha = F(b, j)$ we must have $j \in \mathrm{dmn}(x_\beta)$ and $x_\beta(j) = i$ and $x_\beta(j) = k$, so that $i = k$. The domain of $H(x)$ is

$$\{F(\beta, j) : j \in \mathrm{dmn}(x_\beta)\};$$

there are only finitely many β such that $x_\beta \neq \emptyset$, and for each β the set $\mathrm{dmn}(x_\beta)$ is finite, so $H(x)$ is finite. So $H(x) \in \mathrm{fin}(\kappa, 2)$. Clearly $x \leq y$ implies that $H(x) \supseteq H(y)$.

Now suppose that $f \in \text{fin}(\kappa, 2)$. we claim that $H(G(f)) = f$. For, if $\beta < \kappa$ and $j < \omega$, then

$$\begin{aligned} F(\beta, j) \in \text{dmn}(H(G(f)) \quad &\text{iff} \quad j \in \text{dmn}((G(f))_\beta \\ &\text{iff} \quad F(\beta, j) \in \text{dmn}(f), \end{aligned}$$

and for any $F(\beta, j) \in \text{dmn}(f)$,

$$(H(G(f)))(F(\beta, j)) = (G(f))_\beta(j) = f(F(\beta, j)).$$

Hence $H(G(f)) = f$, as claimed.

Finally, let $x \in \prod^w_{\alpha<\kappa} \text{fin}(\omega, 2)$; we claim that $G(H(x)) = x$. (This completes the solution.) For, if $\beta < \kappa$ and $j \in \omega$, then

$$\begin{aligned} j \in \text{dmn}((G(H(x))_\beta) \quad &\text{iff} \quad F(\beta, j) \in \text{dmn}(H(x)) \\ &\text{iff} \quad j \in \text{dmn}(x_\beta), \end{aligned}$$

and for any $j \in \text{dmn}(x_\beta)$ we have

$$(G(H(x))_\beta(j) = (H(x))(F(\beta, j)) = x_\beta(j).$$

Thus $G(H(x)) = x$.

E25.5 *We expand the language of set theory by adding an individual constant $\emptyset$. An Urelement is an object a such that $a \neq \emptyset$ but a does not have any elements. (Plural is Urelemente.) A set is an object x which is either $\emptyset$ or has an element. Both of these are just definitions, formally like this:*

$$\begin{aligned} Ur(a) &\leftrightarrow a \neq \emptyset \wedge \forall x(x \notin a); \\ Set(x) &\leftrightarrow x = \emptyset \vee \exists y(y \in x). \end{aligned}$$

Now we let ZFU be the following set of axioms in this language:

All the axioms of ZF except extensionality and foundation.

$\forall x[\neg(x \in \emptyset)]$.

$\forall x, y[Set(x) \wedge Set(y) \wedge \forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y]$.

$\forall x[Set(x) \wedge x \neq \emptyset \rightarrow \exists y \in x \forall z(z \in x \rightarrow z \notin y)]$.

We also reformulate the axiom of choice for ZFU; it is the following statement:

$$\begin{aligned} \forall \mathscr{A}\{&Set(\mathscr{A}) \wedge \forall x \in \mathscr{A}[Set(x) \wedge x \neq \emptyset] \\ &\wedge \forall x \in \mathscr{A} \forall y \in \mathscr{A}[x \neq y \rightarrow \forall z[\neg(z \in x \wedge z \in y)]] \\ &\rightarrow \exists \mathscr{B} \forall x \in \mathscr{A} \exists! y(y \in x \wedge y \in \mathscr{B})\}. \end{aligned}$$

We let ZFCU be all of these axioms.

One can adapt most of elementary set theory to use these axioms; browsing through the first few chapters should convince one of this.

In this exercise, give a new definition of ordinal.

Also, show that if we add the axiom $\neg\exists a[Ur(a)]$ we get a theory equivalent to ZF.

We define x is an ordinal to mean that x is a transitive set of transitive sets, just as before. But note that "set" is taken in the new sense; so an ordinal, by definition, does not have any Urelemente as elements.

Suppose that we add $\neg\exists a[Ur(a)]$ to these new axioms. Since $Set(a)$ is just the negation of this, we are now assuming that everything is a set. So the new extensionality axiom reduces to the old one, and the new foundation axiom reduces to the old one. The first new axiom implies that $\emptyset$ is the same as the empty set given by the comprehension axiom. Thus all of the new axioms are derivable in ZF (including $\neg\exists a[Ur(a)]$), and ZF is a consequence of $\neg\exists a[Ur(a)]$ plus the new axioms plus the part of ZF allowed in the new framework.

E25.10

(1) If $\gamma < \beta$, then $W_\gamma \subseteq W_\beta$.

An obvious induction on β, with γ fixed, proves (1).

(2) If $x \in y \in W_\beta \backslash U$, then $x \in W_\beta$.

We prove this by induction on β. It vacuously holds for $\beta = 0$. Now assume that it holds for β, and $x \in y \in W_{\beta+1} \backslash U$. If $y \in W_\beta$, then $x \in W_\beta \subseteq W_{\beta+1}$ by the induction hypothesis. Otherwise, $y \in \mathscr{P}(W_\beta)$, hence $y \subseteq W_\beta$, and obviously $x \in W_\beta$. Finally, the case β limit is clear.

(3) $W_\beta \cap V_\alpha = \emptyset$ for all β.

We prove this by induction on β. It is clear for $\beta = 0$. Assume it for β, and suppose that $x \in W_{\beta+1}$. If $x \in W_\beta$, then $x \notin V_\alpha$ by the inductive hypothesis. Otherwise, $\emptyset \neq x \in \mathscr{P}(W_\alpha)$. Choose $y \in x$. Now $x \subseteq W_\beta$, so $y \in W_\beta$. By the inductive hypothesis, $y \notin V_\alpha$, so also $x \notin V_\alpha$. This takes care of the successor case, and the limit case in the induction is clear.

(4) $W_\beta = \{x \in W : \text{rank}_W(x) < \beta\}$.

In fact, first suppose that $x \in W_\beta$. Let $\gamma = \text{rank}_W(x)$. If $\gamma = -1$, obviously $\text{rank}_W(x) < \beta$. Suppose that $\gamma \geq 0$. Then $x \notin W_\gamma$, so $\gamma < \beta$. This proves $\subseteq$.

Now suppose that $x \in W$ and $\gamma \stackrel{\text{def}}{=} \text{rank}_W(x) < \beta$. Then $x \in W_{\gamma+1} \subseteq W_\beta$, as desired.

(5) If $x, y \in W$ and $x \in y$, then $\text{rank}_W(x) < \text{rank}_W(y)$.

In fact, assume the hypotheses. Then $y \notin U$, as otherwise $x \in V_\alpha$, contradicting (3). Hence $\beta \stackrel{\text{def}}{=} \text{rank}_W(y) > -1$. So $y \in W_{\beta+1} \backslash W_\beta$, and hence $y \in \mathscr{P}(W_\beta)$. So $y \subseteq W_\beta$ and hence $x \in W_\beta$. Therefore $\text{rank}_W(x) < \beta$, as desired.

(6) If $x \in W \backslash U$, then $\text{rank}_W(x) = \sup_{y \in x}(\text{rank}_W(y) + 1)$.

For, suppose that $x \in W \backslash U$. Then $x \neq \emptyset$ by (3). If $y \in x$, then $y \in W$ by (2), and hence $\text{rank}_W(y) < \text{rank}_W(x)$ by (5). Hence $\geq$ holds. Let γ be the right side of (6). If $y \in x$, then

$\text{rank}_W(y) < \gamma$, and so $y \in W_{\text{rank}_W(y)+1} \subseteq W_\gamma$. This shows that $x \subseteq W_\gamma$. So $x \in \mathscr{P}(W_\gamma)$. Since also $x \neq \emptyset$, as noted above, it follows that $x \in W_{\gamma+1}$, and hence $\text{rank}_W(x) \leq \gamma$, giving $\leq$.

(7) If $x \in W$, then $\text{rank}(x) = \alpha + 1 + \text{rank}_W(x)$.

We prove (7) by $\in$-induction. So, suppose that the implication is true for all members of x, with $x \in W$. If $x \in U$ the conclusion is clear. Suppose that $x \notin U$. Then

$$\begin{aligned} \alpha + 1 + \text{rank}_W(x) &= \alpha + 1 + \sup_{y \in x}(\text{rank}_W(y) + 1) \quad \text{by (6)} \\ &= \sup_{y \in x}(\alpha + 1 + \text{rank}_W(y) + 1) \\ &= \sup_{y \in x}(\text{rank}(y) + 1) \quad \text{by the inductive hypothesis} \\ &= \text{rank}(x). \end{aligned}$$

Here the last step uses a property of rank found in the M6730 notes.

(8) If $a \in W \backslash U$, then $W \cap a \neq \emptyset$.

In fact, by induction if $a \in W_\beta \backslash U$ then $W \cap a \neq \emptyset$, so (8) holds.

(9) For any $a \in W$ we have $Ur^W(a)$ iff $a \in U \backslash \{Z\}$.

In fact, suppose that $Ur^W(a)$. Thus $a \neq Z$ and $x \notin a$ for all $x \in W$. Then $a \in U$ by (8). Thus $a \in U \backslash \{Z\}$. Conversely, if $a \in U \backslash \{Z\}$, then there is no $x \in W$ such that $x \in a$, for every member of a is in V_α, and (3) applies. Hence $Ur^W(a)$.

(10) For any $a \in W$ we have $Set^W(a)$ iff $a \notin U \backslash \{Z\}$.

This is clear from (9).

E25.11 The first axiom obviously holds, since every member of Z is in V_α; so (3) applies.

(11) New extensionality holds.

For, suppose that $x, y \in W$, $Set^M(x)$, $Set^M(y)$, and for all $z \in W$, $z \in x$ iff $z \in y$. Then by (10), $x, y \notin U$. Suppose that $z \in x$. Then by (2), $z \in W$, and hence $z \in y$. Similarly $y \subseteq x$. So $x = y$. This proves (11).

(12) New foundation holds.

For, suppose that $x \in W$, $Set^W(x)$, and $x \neq Z$. Then there is a $y \in W$ such that $y \in x$. Choose such a y of smallest W-rank. Suppose that $z \in W$ and $z \in x$. Then $z \notin y$ by (5) and the definition of y.

(13) Comprehension holds.

For, let φ be a formula with free variables among $x, z, w_1, \ldots, w_n$, and suppose that $z, w_1, \ldots, w_n \in W$. If $z \in U$, let $y = Z$. Then for any $x \in W$, $x \notin y$, and also it is not true that $x \in z \wedge \varphi^W$, so the desired equivalence holds. Now suppose that $z \notin U$. Let $y' = \{x \in z : \varphi^W\}$. If $y' = \emptyset$, then clearly again $y = Z$ works for our equivalence. If

$y' \neq \emptyset$, let $y = y'$. Say $z \in W_\beta$. If $x \in y$, then $x \in z \in W_\beta \backslash U$, and so $x \in W_\beta$ by (2). So $y \in \mathscr{P}(W_\beta)\backslash\{\emptyset\}$, and hence $y \in W_{\beta+1}$. Clearly

$$\forall x \in W[x \in y \leftrightarrow x \in z \wedge \varphi^W]$$

as desired. So (13) holds.

(14) Pairing holds.

For, given $x, y \in W$, choose β so that $x, y \in W_\beta$. Then $\{x, y\} \in \mathscr{P}(W_\beta)\backslash\{\emptyset\}$, so $\{x, y\} \in W_{\beta+1}$ and the desired conclusion follows.

(15) Union holds.

For, let $\mathscr{A} \in W$ be given. We define

$$A = \begin{cases} (\bigcup \mathscr{A}) \cap W & \text{if } (\bigcup \mathscr{A}) \cap W \neq \emptyset, \\ Z & \text{otherwise.} \end{cases}$$

We claim that $A \in W$. This is clear if $(\bigcup \mathscr{A}) \cap W = \emptyset$, so suppose that $(\bigcup \mathscr{A}) \cap W \neq \emptyset$. For each $a \in A$ choose β_a such that $a \in W_{\beta_a}$. Let $\gamma = \sup_{a \in A} \beta_a$. Then $A \subseteq W_\gamma$ and $A \neq \emptyset$, so $A \in W_{\gamma+1}$. Thus, indeed, $A \in W$.

Now suppose that $x, Y \in W$ and $x \in Y \in \mathscr{A}$. So $x \in \bigcup \mathscr{A} \cap W$, so $x \in A$, as desired.

(16) Power set holds.

To prove this, note that this axiom involves the defined notion $\subseteq$. We claim:

(17) For any $x, y \in W$, $(x \subseteq y)^W$ iff $x \in U$ or $x \subseteq y$.

In fact, assume that $x, y \in W$. First suppose that $(x \subseteq y)^W$ and $x \notin U$. Suppose that $z \in x$. Then $z \in W$ by (2), and so $z \in y$ since $(x \subseteq y)^W$. Thus $x \subseteq y$.

Conversely, if $x \in U$, then $(x \subseteq y)^W$ by (3). Suppose that $x \subseteq y$. Then obviously $(x \subseteq y)^W$.

Thus (17) holds.

Now for the power set axiom, let x be given. Define $y = U \cup (\mathscr{P}(x) \cap W)$ Now suppose that $z \in W$ and $(z \subseteq x)^W$. By (17) there are two possibilities. If $z \in U$, obviously $z \in y$. If $z \subseteq x$, again obviously $z \in y$. So (16) is proved.

(18) Infinity holds.

To prove (18), we first define by recursion a sequence $\langle u_n : n \in \omega \rangle$ of members of W. Let $u_0 = Z$ and $u_{n+1} = (u_n \cup \{u_n\})^W$. Let $\Omega = \text{rng} u$. Now each u_n is in W, so there is a β_n such that $u_n \in W_{\beta_n}$. Let $\gamma = \sup_{n \in \omega} \beta_n$. Thus $\Omega \subseteq W_\gamma$. Also, obviously $\Omega \neq \emptyset$. Hence $\Omega \in W_{\gamma+1}$. Clearly Ω is what is needed in the infinity axiom.

(19) Replacement holds.

For, suppose that φ is a formula with free variables among $x, y, A, w_1, \ldots, w_n$, assume that $A, w_1, \ldots, w_n \in W$, and suppose that $(\forall x \in A \exists! y \varphi)^W$. Written out more fully, this last supposition is:

$$\begin{aligned} \forall x \in W[x \in A \rightarrow \exists y \in W[\varphi(x, y, A, w_1, \ldots, w_n)^W \wedge \\ \forall z \in W[\varphi(x, z, A, w_1, \ldots, w_n)^W \rightarrow y = z]]]. \end{aligned}$$

Thus $\forall x \in A \cap W \exists! y[y \in W \wedge \varphi^W]$. Hence by the replacement axiom we obtain a set Y' such that $\forall x \in A \cap W \exists y \in Y'[y \in W \wedge \varphi^W]$. Let

$$u = \{(x, y) : x \in A \cap W \text{ and } y \in Y' \cap W \text{ and } \varphi^W\}.$$

Thus u is a function with domain $A \cap W$. Define

$$Y = \begin{cases} \text{rng}(u) & \text{if rng}(u) \neq \emptyset, \\ Z & \text{otherwise.} \end{cases}$$

We claim that $Y \in W$. This is clear if $\text{rng}(u) = \emptyset$, so suppose that $\text{rng}(u) \neq \emptyset$. For each $y \in \text{rng}(u)$ choose β_u such that $y \in W_{\beta_u}$. Let $\gamma = \bigcup_{y \in \text{rng}(u)} \beta_u$. Clearly then $\text{rng}(u) \in \mathscr{P}(W + \gamma) \backslash \{\emptyset\}$, so $\text{rng}(u) \in W_{\gamma+1}$. This proves that $Y \in W$.

If $x \in A \cap W$, then $u(x) \in Y$ and $\varphi(x, u(x), A, w_1, \ldots, w_n)^W$, as desired.

E25.12 We extend f by recursion, denoting extension by f^+. For any $a \in W$,

$$f^+(a) = \begin{cases} f(a) & \text{if } a \in U \backslash \{Z\}, \\ Z & \text{if } a = Z, \\ \{x : \exists b \in a[x = f^+(b)]\} & \text{if } a \notin U. \end{cases}$$

Note that if $a \in W \backslash U$, then $a \subseteq W$, by (2). To show that f is one-one and onto, it suffices by symmetry to show that $f^+ \circ (f^{-1})^+$ is the identity on W. We prove that $f^+((f^{-1})^+(a)) = a$ for all $a \in W$, by induction on a. This is clear if $a \in U$. Now suppose that $a \in W \backslash U$. Then

$$\begin{aligned} f^+((f^{-1})^+(a)) &= f^+(\{x : \exists b \in a[x = (f^{-1})^+(b)]\}) \\ &= \{y : \exists z \in \{x : \exists b \in a[x = (f^{-1})^+(b)]\}[y = f^+(z)]\} \\ &= \{y : \exists b \in a[y = f^+((f^{-1})^+(b))]\} \\ &= \{y : \exists b \in a[y = b]\} \quad \text{(induction hypothesis)} \\ &= a. \end{aligned}$$

So $f : W \to W$ is a bijection, and it takes Z to Z. If $a, b \in W$ and $a \in b$, obviously $f^+(a) \in f^+(b)$. Conversely, suppose that $f^+(a) \in f^+(b)$. Then $a = (f^{-1})^+(f^+(a)) \in (f^{-1})^+(f^+(b)) = b$. Thus f is the desired isomorphism.

E25.13 We define by recursion

$$\begin{aligned} D_0 &= a; \\ D_{n+1} &= D_n \cup \{b \in W : b \in c \in d \text{ for some } c \in W \cap D_n\}; \\ E &= \bigcup_{n \in \omega} D_n. \end{aligned}$$

By induction, $D_n \subseteq W$ for each $n \in \omega$, and hence $E \subseteq W$. Clearly $a \subseteq E$. Hence $E \neq \emptyset$, and it follows easily, by an argument used several times above, that $E \in W$. Now suppose that $b, c \in W$ and $b \in c \in E$. Choose n such that $c \in D_n$. It follows that $b \in D_{n+1} \subseteq E$. Thus E is W-transitive.

For the second part of the problem, it suffices to prove that if T is the W-transitive closure of a, then $f^+[T]$ is the W-transitive closure of $f^+(a)$. First, $f^+(a) \subseteq f^+[T]$. For, let $x \in f^+(a)$. Then we can write $x = f^+(b)$ with $b \in a \cap W$. Hence $b \in T$, and so $x = f^+(b) \in f^+[T]$. So $f^+(a) \subseteq f^+[T]$. Now suppose that S is any W-transitive set such that $f^+(a) \subseteq S$. Let $R = \{x : x \in W \text{ and } f^+(x) \in S\}$. Now $a \subseteq R$. For, if $y \in a$, then $f^+(y) \in f^+(a) \subseteq S$, and hence $y \in R$.

Suppose that $F \in W$ is W-transitive and $a \subseteq F$. By an easy induction, $D_n \subseteq F$ for every $n \in \omega$, and hence $E \subseteq F$.

E25.14 (i): Let $a \in U$. Then the W-transitive closure of $\{a\}$ is clearly just $\{a\}$ itself. If $a = Z$, clearly a is symmetric. If $a \neq Z$, then letting $F = \{a\}$ in the definition shows that a is symmetric.

(ii): Since a is symmetric, let F be a finite subset of $U\backslash\{Z\}$ such that $g^+(a) = a$ for every permutation g of $U\backslash\{Z\}$ which is the identity on F. Let $G = f[F]$. So G is a finite subset of $U\backslash\{Z\}$. If g is a permutation of $U\backslash\{Z\}$ which is the identity on G, Then for any $u \in F$ we have $(f^{-1} \circ g \circ f)(u) = f^{-1}(g(f(u)) = f^{-1}(f(u)) = u$. So $f^{-1} \circ g \circ f$ is the identity on F, and so $(f^{-1})^+(g^+(f^+(a))) = a = f^{-1}(f(a))$, and hence $g^+(f^+(a)) = f^+(a)$. This shows that $f^+(a)$ is symmetric.

(iii): By E25.13 and (ii), f^+ maps H into H. By considering f^{-1}, it is clear that $f^+ \restriction H$ is a permutation of H. By E25.12 it is an automorphism.

(iv): We prove this by induction on φ. It is obvious for atomic formulas, and the induction hypothesis for $\vee$ is clear. Suppose that $(\neg\varphi)^H$ holds. If $\varphi^H(f^+(v_0), \ldots, f^+(v_{n-1}))$ holds, applying the inductive hypothesis to f^{-1} gives a contradiction. Finally, suppose that $(\exists x\varphi(x, v_0, \ldots, v_{n-1}))^H$ holds. Choose $x \in H$ such that $\varphi^H(x, v_0, \ldots, v_{n-1})$. Then $\varphi^H(f^+(x), f^+(v_0), \ldots, f^+(v_{n-1}))$, hence $(\exists x\varphi(f^+(x), f^+(v_0), \ldots, f^+(v_{n-1}))^H$.

(v): We go through all of the axioms. First note that if a is hereditarily symmetric and not in U, then its elements are also hereditarily symmetric. Hence Ur and Set are absolute for H, W.

• $\forall x[\neg(x \in Z)]$. This holds since $H \subseteq W$.

• New extensionality. This holds by the initial remark in (ii).

• New foundation. This also holds by that initial remark.

• Comprehension. Let φ be a formula with free variables among $x, z, w_1, \ldots, w_n$, and suppose that $z, w_1, \ldots, w_n \in H$. Let $y = \{x \in z : \varphi^H\}$. Note that every member of z is in H, so φ^H makes sense here. Clearly it suffices to show that $y \in H$. Since each member of y is in H, it suffices to show that y is symmetric. Let $F, G_1, \ldots, G_n$ be finite subsets of $U\backslash\{Z\}$ such that for every permutation f of $U\backslash\{Z\}$, if $f \restriction F$ is the identity then $f^+(z) = z$, and for each $i = 1, \ldots, n$, if $f \restriction G_i$ is the identity then $f^+(w_i) = w_i$. Now take any permutation f of $U\backslash\{Z\}$ which is the identity on $F \cup G_1 \cup \ldots \cup G_n$. We claim that $f^+(y) = y$, which will show that y is symmetric. Take any $u \in f^+(y)$. Then we can write $u = f^+(x)$ with $x \in y$. So $x \in z$ and φ^H holds. Hence by (iv), also $\varphi^H(f^+(x), f^+(z), f^+(w_1), \ldots, f^+(w_n))$

holds. Since $f^+(z) = z$ and $f^+(w_i) = w_i$ for each i, it follows that $u = f^+(x) \in y$. Thus $f^+(y) \subseteq y$. Hence $y = (f^{-1})^+(f^+(y)) \subseteq f^+(y)$ and so $y = f^+(y)$, as desired.

• Pairing. If $x, y \in H$, clearly $\{x, y\} \in H$.

• Union. Given $\mathscr{A} \in H$, let $A = \{x \in W : \exists Y \in \mathscr{A}[x \in Y]\}$. Clearly A is as desired.

• Power set. Given $x \in H$, let

$$y = \{z : \forall w \in W[w \in z \to w \in x\}.$$

Clearly y is as desired.

• Infinity. By induction it is clear that each u_n, defined in the proof of infinity in W given above, is in H. Hence so is Ω defined there, and so infinity holds.

• Replacement. This is clear by the proof of replacement in W.

E25.15 (a): Clear, by induction on m.

(b): Again clear, by induction on n, with m fixed.

(c): We follow the hint. Note that certain simple notions like unordered and ordered pairs, relations, and functions, are absolute in the usual sense. Also note, by induction, that $g^+(\bar{i}) = \bar{i}$ for every permutation g of $U\backslash\{Z\}$ and every $i \in \omega$.

Choose a finite subset F of $U\backslash\{Z\}$ such that $g^+(f) = f$ for any permutation g of $U\backslash\{Z\}$ which is the identity on F. Clearly $\langle f(\bar{i}) : i \in \omega\rangle$ is a one-one function, by (b) and absoluteness. Choose $u \in \{f(\bar{i}) : i \in \omega\}\backslash F$, and choose $v \in U\backslash F$ with $u \neq v$. Say $u = f(\bar{i})$. Let g be a permutation of $U\backslash\{Z\}$ which is the identity on F and takes u to v. Now $(\bar{i}, u) \in f$, so

$$(\bar{i}, v) = (\bar{i}, g(u)) = (g^+(\bar{i}), g^+(f(\bar{i}))) \in g^+(f) = f.$$

Hence $f(\bar{i}) = v$, contradiction.