20. Powers of regular cardinals

In this chapter we continue Chapter 12, and describe in more detail the possibilities for \(2^\kappa\) when \(\kappa\) is regular, where the results are fairly complete. The case of singular \(\kappa\) is more involved, and there are still important open problems.

To obtain upper bounds on the size of powers the following concept will be used. Suppose that \(P\) is a quasi-order and \(\sigma \in V^P\). A nice name for a subset of \(\sigma\) is a member of \(V^P\) of the form

\[
\bigcup_{\pi \in \text{dom}(\sigma)} (\{\pi\} \times A_\pi),
\]

where each \(A_\pi\) is an antichain in \(P\).

**Lemma 20.1.** If \(\sigma\) is a \(P\)-name and \((\pi, p) \in \sigma\), then \(p \Vdash \pi \in \sigma\).

**Proof.** Let \(G\) be generic with \(p \in G\). Then by Theorem 11.6, \(\pi^G \in \sigma^G\).

**Proposition 20.2.** Let \(M\) be a c.t.m. of ZFC, \(P \in M\) a quasi-order, and \(\sigma \in M^P\).

(i) For any \(\mu \in M^P\) there is a nice name \(\tau \in M^P\) for a subset of \(\sigma\) such that

\[
(\ast)\quad 1 \Vdash \tau = \mu \cap \sigma.
\]

(ii) If \(G\) is \(P\)-generic over \(M\) and \(a \subseteq \sigma^G\) in \(M[G]\), then \(a = \tau^G\) for some nice name \(\tau\) for a subset of \(\sigma\).

**Proof.** Assume the hypotheses of the proposition.

(i): Assume also that \(\mu \in M^P\). For each \(\pi \in \text{dom}(\sigma)\) let \(A_\pi \subseteq P\) be such that

1. \(p \Vdash (\pi \in \mu \wedge \pi \in \sigma)\) for all \(p \in A_\pi\).
2. \(A_\pi\) is an antichain of \(P\).
3. \(A_\pi\) is maximal with respect to (1) and (2).

Moreover, we do this definition inside \(M\), so that \(\langle A_\pi : \pi \in \text{dom}(\sigma)\rangle \in M\). Now let

\[
\tau = \bigcup_{\pi \in \text{dom}(\sigma)} (\{\pi\} \times A_\pi).
\]

To prove (\(\ast\)), suppose that \(G\) is \(P\)-generic over \(M\); we want to show that \(\tau^G = \mu^G \cap \sigma^G\).

First suppose that \(a \in \mu^G \cap \sigma^G\). Choose \((\pi, p) \in \sigma\) such that \(p \in G\) and \(a = \pi^G\). By Lemma 20.1, \(p \Vdash \pi \in \sigma\).

(4) \(A_\pi \cap G \neq \emptyset\).

For, suppose that \(A_\pi \cap G = \emptyset\). By Lemma 11.14(i), there is a \(q \in G\) such that \(q \perp r\) for all \(r \in A_\pi\). Now since \(\pi^G \in \mu^G\), by Corollary 11.21 there is a \(q' \in G\) such that \(q' \Vdash \pi \in \mu\). Let \(r \in G\) with \(r \leq q, q'\). Then \(r \Vdash (\pi \in \mu \wedge \pi \in \sigma)\). It follows that \(A_\pi \cup \{r\}\) is an antichain, contradicting (3). Thus (4) holds.
By (4), take \( q \in A_\pi \cap G \). Then \((\pi, q) \in \tau \) and \( q \in G \), so \( a = \pi_G \in \tau_G \). Thus we have shown that \( \mu_G \cap \sigma_G \subseteq \tau_G \).

Now suppose that \( a \in \tau_G \). Choose \((\pi, p) \in \tau \) such that \( p \in G \) and \( a = \pi_G \). Thus \( p \in A_\pi \), so by (1), \( p \Vdash (\pi \in \mu \land \pi \in \sigma) \). By the definition of forcing, \( a = \pi_G \in \mu_G \cap \sigma_G \).

This shows that \( \tau_G \subseteq \mu_G \cap \sigma_G \). Hence \( \tau_G = \mu_G \cap \sigma_G \).

(ii): Assume the hypotheses of (ii). Write \( a = \mu_G \). Taking \( \tau \) as in (i), we have \( a = \mu_G = \mu_G \cap \sigma_G = \tau_G \), as desired. \( \square \)

**Proposition 20.3.** Suppose that \( M \) is a c.t.m. of ZFC, and in \( M \), \( P \) is a quasi-order, \( |P| = \kappa \geq \omega, P \) has the \( \lambda \)-cc, and \( \mu \) is an infinite cardinal. Suppose that \( G \) is \( P \)-generic over \( M \). Then there is a function in \( M[G] \) mapping \((\kappa^{<\lambda})^{\mu}\) onto a set containing \( \mathcal{P}(\mu)^{M[G]} \).

**Proof.** We do some calculations in \( M \). Each antichain in \( P \) has size at most \( \kappa^{<\lambda} \).

Since \( |\text{dmm}(\tilde{\mu})| \) has size \( \mu \), we thus have at most \( \nu \overset{\text{def}}{=} (\kappa^{<\lambda})^{\mu} \) nice names for subsets of \( \tilde{\mu} \).

Let \( \langle \tau_\alpha : \alpha < \nu \rangle \) enumerate all of these names. Define

\[
\pi = \{ (\text{op}(\bar{\alpha}, \tau_\alpha), 1) : \alpha < \nu \}.
\]

Now \( \pi_G \) is a function. For, if \( x \in \pi_G \), then there is an \( \alpha < \nu \) such that \( x = (\alpha, (\tau_\alpha)_G) \), by Lemma 11.22. Thus \( \pi_G \) is a relation. Now suppose that \((x, y), (x, z) \in \pi_G \). Then there exist \( \alpha, \beta < \nu \) such that \((x, y) = (\alpha, (\tau_\alpha)_G) \) and \((x, z) = (\beta, (\tau_\beta)_G) \). Hence \( \alpha = \beta \) and \( y = z \). Clearly the domain of \( \pi_G \) is \( \nu \). By Proposition 20.2, \( \mathcal{P}(\mu) \subseteq \text{rng}(\pi_G) \) in \( M[G] \), as desired. \( \square \)

Now we can prove a more precise version of Theorem 12.8.

**Theorem 20.4.** (Solovay) Let \( M \) be a c.t.m. of ZFC. Suppose that \( \kappa \) is a cardinal of \( M \) such that \( \kappa^\omega = \kappa \). Let \( P \) be the partial order \( \text{fin}(\kappa, 2) \) ordered by \( \supseteq \), and let \( G \) be \( P \)-generic over \( M \). Then \( M[G] \) has the same cofinalities and cardinals as \( M \), and \( 2^\omega = \kappa \) in \( M[G] \).

Moreover, if \( \lambda \) is any infinite cardinal in \( M \), then \( \kappa \leq (2^\lambda)^{M[G]} \leq (\kappa^\lambda)^M \).

**Proof.** By Theorem 12.8, \( M[G] \) has the same cofinalities and cardinals as \( M \) and \( \kappa \leq 2^\omega \).

Note that \( |\text{fin}(\kappa, 2)| = \kappa \) in \( M \). Hence by Proposition 20.3, for any infinite cardinal \( \lambda \) of \( M \) we have

\[
\kappa \leq (2^\omega)^{M[G]} \leq (2^\lambda)^{M[G]} \leq ((\kappa^{<\lambda})^\lambda)^M = (\kappa^\lambda)^M.
\]

Applying this to \( \lambda = \omega \) we get \( 2^\omega = \kappa \) in \( M[G] \). \( \square \)

By assuming that the ground model satisfies GCH, which is consistent by the theory of constructible sets, we can obtain a sharper result.

**Corollary 20.5.** Suppose that \( M \) is a c.t.m. of ZFC + GCH. Suppose that \( \kappa \) is an uncountable regular cardinal of \( M \). Let \( P \) be the partial order \( \text{fin}(\kappa, 2) \) ordered by \( \supseteq \), and let \( G \) be \( P \)-generic over \( M \). Then \( M[G] \) has the same cofinalities and cardinals as \( M \), and \( 2^\omega = \kappa \) in \( M[G] \).
Moreover, for any infinite cardinal $\lambda$ of $M$ we have

$$(2^\lambda)^{M[G]} = \begin{cases} 
\kappa & \text{if } \lambda < \kappa, \\
\lambda^+ & \text{if } \kappa \leq \lambda.
\end{cases}$$

**Proof.** By GCH we have $\kappa^\omega = \kappa$. Hence the hypothesis of Theorem 20.4 holds, and the conclusion follows using GCH in $M$. \qed

We give several more specific corollaries.

**Corollary 20.6.** If ZFC is consistent, then so is each of the following:

(i) $\text{ZFC } + 2^{\aleph_0} = \aleph_2$.

(ii) $\text{ZFC } + 2^{\aleph_0} = \aleph_{203}$.

(iii) $\text{ZFC } + 2^{\aleph_0} = \aleph_{\omega_1}$.

(iv) $\text{ZFC } + 2^{\aleph_0} = \aleph_{\omega_4}$.

**Corollary 20.7.** If $\text{ZFC}+$ “there is an uncountable regular limit cardinal” is consistent, so is $\text{ZFC}+$ “$2^\omega$ is a regular limit cardinal”. \qed

**Corollary 20.8.** Suppose that $M$ is a c.t.m. of ZFC. Then there is a generic extension $M[G]$ such that in it, $2^\omega = ((2^\omega)^+)^M$.

Since clearly $((2^\omega)^+)^\omega = (2^\omega)^+$ in $M$, this is immediate from Theorem 20.4. \qed

Now we turn to powers of regular uncountable cardinals, where similar results hold. We need some elementary facts about cardinals. For cardinals $\kappa, \lambda$, we define

$$\kappa^{<\lambda} = \sup_{\alpha<\lambda} |^\alpha \kappa|.$$ 

Note here that the supremum is over all ordinals less than $\lambda$, not only cardinals.

**Proposition 20.9.** Let $\kappa$ and $\lambda$ be cardinals with $\kappa \geq 2$ and $\lambda$ infinite and regular. Then $(\kappa^{<\lambda})^{<\lambda} = \kappa^{<\lambda}$.

**Proof.** Clearly $\geq$ holds. For $\leq$, by the fact that $\lambda \cdot \lambda = \lambda$ it suffices to find an injection from

$$\bigcup_{\alpha<\lambda} \left( \bigcup_{\beta<\lambda} \right)$$

into

$$\bigcup_{\alpha,\beta<\lambda} ^\alpha \times ^\beta (\kappa + 1).$$
Let \( x \) be a member of \((1)\), and choose \( \alpha < \lambda \) accordingly. Then for each \( \xi < \alpha \) there is a \( \beta_{x,\xi} < \lambda \) such that \( x(\xi) \in \beta_{x,\xi} \). Let \( \gamma_x = \sup_{\xi < \alpha} \beta_{x,\xi} \). Then \( \gamma_x < \lambda \) by the regularity of \( \lambda \). We now define \( f(x) \) with domain \( \alpha \times \gamma_x \) by setting, for any \( \xi < \alpha \) and \( \eta < \gamma_x \):

\[
(f(x))(\xi, \eta) = \begin{cases} (x(\xi))(\eta) & \text{if } \eta < \beta_{x,\xi}, \\ \kappa & \text{otherwise.} \end{cases}
\]

Then \( f \) is one-one. In fact, suppose that \( f(x) = f(y) \). Let the domain of \( f(x) \) be \( \alpha \times \gamma_x \) as above. Suppose that \( \xi < \alpha \). If \( \beta_{x,\xi} \neq \beta_{y,\xi} \), say \( \beta_{x,\xi} < \beta_{y,\xi} \), and \( (f(x))(\xi, \beta_{x,\xi}) = \kappa \) while \( (f(y))(\xi, \beta_{x,\xi}) < \kappa \), contradiction. Hence \( \beta_{x,\xi} = \beta_{y,\xi} \). Finally, take any \( \eta < \beta_{x,\xi} \). Then:

\[
(x(\xi))(\eta) = (f(x))(\xi, \eta) = (f(y))(\xi, \eta) = (y(\xi))(\eta);
\]

it follows that \( x = y \).

Now the direction \( \leq \) follows.

**Proposition 20.10.** For any cardinals \( \kappa, \lambda \), \( |[\kappa]^{<\lambda}| \leq \kappa^{<\lambda} \).

**Proof.** For each cardinal \( \mu < \lambda \) define \( f: \mu \times \kappa \rightarrow [\kappa]^{\leq \mu} \setminus \{\emptyset\} \) by setting \( f(x) = \text{rng}(x) \) for any \( x \in \mu \times \kappa \). Clearly \( f \) is an onto map. It follows that \( |[\kappa]^{\leq \mu}| \leq |\mu \times \kappa| \leq \kappa^{<\lambda} \). Hence

\[
|[\kappa]^{<\lambda}| = \sum_{\mu < \lambda, \kappa < \lambda} |[\kappa]^{\leq \mu}| \leq \sum_{\mu < \lambda, \kappa < \lambda} \kappa^{<\lambda} \leq \lambda \cdot \kappa^{<\lambda} = \kappa^{<\lambda}.
\]

We now define a new partial order for the remaining forcing results of this chapter. For any sets \( I, J \) and infinite cardinal \( \lambda \),

\[
\text{Fn}(I, J, \lambda) = \{ f : f \text{ is a function, } f \subseteq I \times J, \text{ and } |f| < \lambda \}.
\]

We consider this as a partial order under \( \supseteq \); the greatest element is again \( \emptyset \). We claim that this partial order has the \( (|J|^{<\lambda})^+ \)-chain condition.

**Lemma 20.11.** If \( I, J \) are sets and \( \lambda \) is an infinite cardinal, then \( \text{Fn}(I, J, \lambda) \) has the \( (|J|^{<\lambda})^+ \)-cc.

**Proof.** Let \( \theta = (|J|^{<\lambda})^+ \), and suppose that \( \{ p_\xi : \xi < \theta \} \) is a collection of elements of \( \text{Fn}(I, J, \lambda) \); we want to show that there are distinct \( \xi, \eta < \theta \) such that \( p_\xi \) and \( p_\eta \) are
compatible. We want to apply the general indexed \( \Delta \)-system theorem 16.4, with \( \kappa, \lambda, \langle A_i : i \in I \rangle \) replaced by \( \lambda, \theta, \langle \text{dom}(p_\xi) : \xi < \theta \rangle \) respectively. Obviously \( \theta \) is regular. If \( \alpha < \theta \), then \( |\alpha|^{<\lambda} \leq |\alpha|^{<\lambda} \) (by Proposition 20.10) \( \leq (|J|^{<\lambda})^{<\lambda} = |J|^{<\lambda} \) (by Proposition 20.9) \( < \theta \). Thus we can apply 16.4, and we get \( J \in [\theta]^{\theta^\lambda} \) such that \( \langle \text{dom}(p_\xi) : \xi \in J \rangle \) is an indexed \( \Delta \)-system, say with root \( r \).

Now we can give our main theorem concerning making \( 2^{\lambda} \) as large as we want, for any regular \( \lambda \) given in advance.

**Theorem 20.15.** Suppose that \( M \) is a c.t.m. of ZFC and in \( M \) we have cardinals \( \kappa, \lambda \) such that \( \lambda < \kappa, \lambda \) is regular, \( 2^{<\lambda} = \lambda \), and \( \kappa^\lambda = \kappa \). Let \( P = \text{Fn}(\kappa, 2, \lambda) \) ordered by \( \supseteq \). Then \( P \) preserves cofinalities and cardinalities. Let \( G \) be \( \mathbb{P} \)-generic over \( M \). Then

(i) \( (2^\lambda = \kappa)^{M[G]} \).

(ii) If \( \mu \) and \( \nu \) are cardinals of \( M \) and \( \omega \leq \mu < \lambda \), then \( (\nu^\mu)^M = (\nu^\mu)^{M[G]} \).

(iii) For any cardinal \( \mu \) of \( M \), if \( \mu \geq \lambda \) then \( (2^\mu)^M = (\kappa^\mu)^M \).

**Proof.** Preservation of cofinalities and cardinalities follows from Lemma 20.14. Now we turn to (i). To show that \( \kappa \leq (2^\lambda)^{M[G]} \), we proceed as in the proof of Theorem 12.1. Let \( g = \bigcup G \). So \( g \) is a function mapping a subset of \( \kappa \) into 2.
(1) For each $\alpha \in \kappa$, the set $\{ f \in \text{Fn}(\kappa, 2, \lambda) : \alpha \in \text{dom}(f) \}$ is dense in $\mathbb{P}$ (and it is a member of $M$).

In fact, given $f \in \text{Fn}(\kappa, 2, \lambda)$, either $f$ is already in the above set, or else $\alpha \notin \text{dom}(f)$ and then $f \cup \{ (\alpha, 0) \}$ is an extension of $f$ which is in that set. So (1) holds.

Since $G$ intersects each set (1), it follows that $g$ maps $\kappa$ into 2. Let (in $M$) $h : \kappa \times \lambda \to \kappa$ be a bijection. For each $\alpha < \kappa$ let $a_\alpha = \{ \xi \in \lambda : g(h(\alpha, \xi)) = 1 \}$. We claim that $a_\alpha \neq a_\beta$ for distinct $\alpha, \beta$; this will give $\kappa \leq (2^\lambda)^{M[G]}$. The set

$$\{ f \in \text{Fn}(\kappa, 2, \lambda) : \text{there is a } \xi \in \lambda \text{ such that } h(\alpha, \xi), h(\beta, \xi) \in \text{dom}(f) \text{ and } f(h(\alpha, \xi)) \neq f(h(\alpha, \xi)) \}$$

is dense in $\mathbb{P}$ (and it is in $M$). In fact, let distinct $\alpha$ and $\beta$ be given, and suppose that $f \in \text{Fn}(\kappa, 2, \lambda)$. Now $\{ \xi : h(\alpha, \xi) \in f \text{ or } h(\beta, \xi) \notin f \}$ has size less than $\lambda$, so choose $\xi \in \lambda$ not in this set. Thus $h(\alpha, \xi), h(\beta, \xi) \notin f$. Let $h = f \cup \{ (h(\alpha, \xi), 0), (h(\beta, \xi), 1) \}$. Then $h$ extends $f$ and is in the above set, as desired.

It follows that $G$ contains a member of this set. Hence $a_\alpha \neq a_\beta$. Thus we have now shown that $\kappa \leq (2^\lambda)^{M[G]}$.

For the other inequality, note by Lemma 20.11 that $\mathbb{P}$ has the $(2^{<\lambda^+})$-cc, and by hypothesis $(2^{<\lambda})^+ = \lambda^+$. By the assumption that $\kappa^\lambda = \kappa$ we also have $|P| = \kappa$. Hence by Proposition 20.3 the other inequality follows. Thus we have finished the proof of (i).

For (ii), assume the hypothesis. If $f \in M[G]$ and $f : \mu \to \nu$, then $f \in M$ by Theorem 12.10. Hence (ii) follows.

Finally, for (iii), suppose that $\mu$ is a cardinal of $M$ such that $\mu \geq \lambda$. By Proposition 20.3 with $\lambda$ replaced by $\lambda^+$ we have $(2^{\mu})^{M[G]} \leq (\kappa^\mu)^M$. Now $(\kappa^\mu)^M \leq (\kappa^\mu)^{M[G]} = ((2^\lambda)^\mu)^{M[G]} = (2^\mu)^{M[G]}$, so (iii) holds. \hfill \Box

**Corollary 20.16.** Suppose that $M$ is a c.t.m. of ZFC+$GCH$, and in $M$ we have cardinals $\kappa, \lambda$, both regular, with $\lambda < \kappa$. Let $P = \text{Fn}(\kappa, 2, \lambda)$ ordered by $\supseteq$. Then $P$ preserves cofinalities and cardinalities. Let $G$ be $\mathbb{P}$-generic over $M$. Then for any infinite cardinal $\mu$,

$$(2^\mu)^{M[G]} = \begin{cases} \mu^+ & \text{ if } \mu < \lambda, \\ \kappa & \text{ if } \lambda \leq \mu < \kappa, \\ \mu^+ & \text{ if } \kappa \leq \mu. \end{cases}$$

**Proof.** Immediate from Theorem 20.15. \hfill \Box

Theorem 20.15 gives quite a bit of control over what can happen to powers $2^\kappa$ for $\kappa$ regular. We can apply this theorem to obtain a considerable generalization of it.

**Theorem 20.17.** Suppose that $n \in \omega$ and $M$ is a c.t.m. of ZFC. Also assume the following:

(i) $\lambda_1 < \cdots < \lambda_n$ are regular cardinals in $M$.
(ii) $\kappa_1 \leq \cdots \leq \kappa_n$ are cardinals in $M$.
(iii) $(\text{cf}(\kappa_i) > \lambda_i)^M$ for each $i = 1, \ldots, n$.
(iv) $(2^{<\lambda_i} = \lambda_i)^M$ for each $i = 1, \ldots, n$.

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Then there is a c.t.m. \( N \supseteq M \) with the same cofinalities and cardinals such that:

(i) \( (2^\lambda_i = \kappa_i)^N \) for each \( i = 1, \ldots, n \).

(ii) \( (2^\mu)^N = (\kappa^\mu_i)^M \) for all \( \mu > \lambda_n \).

Proof. The statement vacuously holds for \( n = 0 \). Suppose that it holds for \( n - 1 \), and the hypothesis holds for \( n \), where \( n \) is a positive integer. Let \( \mathbb{P}_n = \text{Fn}(\kappa_n, 2, \lambda_n) \). Then by Lemma 20.11, \( \mathbb{P}_n \) has the \((2^{<\lambda_n})^+\)-cc, i.e., by (iv) it has the \( \lambda^+_n \)-cc. By Lemma 20.12 it is \( \lambda_n \)-closed. So by Proposition 20.14, \( \mathbb{P}_n \) preserves all cofinalities and cardinalities. Let \( G \) be \( \mathbb{P}_n \)-generic over \( M \). By Theorem 20.15, \( (2^{\lambda_n} = \kappa_n)^M[G] \), \( (2^\mu)[G] = (\kappa^\mu_i)^M \) for all \( \mu > \lambda_n \), and also conditions (i)-(v) hold for \( M[G] \) for \( i = 1, \ldots, n - 1 \). Hence by the inductive hypothesis, there is a c.t.m. \( N \) with \( M[G] \subseteq N \) such that

(1) \( (2^\lambda_i = \kappa_i)^N \) for each \( i = 1, \ldots, n - 1 \).

(2) \( (2^\mu)^N = (\kappa^\mu_i)^M[G] \) for all \( \mu > \lambda_n - 1 \).

In particular,

\[
(2^{\lambda_n})^N = (\kappa_n^{\lambda_n})^M[G] \leq (\kappa_n^{\lambda_n})^M[G] = ((2^{\lambda_n})^{\lambda_n})^M[G] = (2^{\lambda_n})^M[G] = \kappa_n = (2^{\lambda_n})^M[G] \leq (2^{\lambda_n})^N.
\]

Thus \( (2^{\lambda_n})^N = \kappa_n \). Furthermore, if \( \mu > \lambda_n \) then

\[
(2^\mu)^N = (\kappa^\mu_i)^M[G] \leq (\kappa^\mu_i)^M[G] = ((2^{\lambda_n})^\mu)^M[G] = (2^\mu)^M[G] = (\kappa^\mu_i)^N = ((2^{\lambda_n})^\mu)^N = (2^\mu)^N.
\]

It follows that \( (2^\mu)^N = (\kappa^\mu_i)^M \). This completes the inductive proof.

Corollary 20.18. Suppose that \( n \in \omega \) and \( M \) is a c.t.m. of ZFC + GCH. Also assume the following:

(i) \( \lambda_1 < \cdots < \lambda_n \) are regular cardinals in \( M \).

(ii) \( \kappa_1 \leq \cdots \leq \kappa_n \) are cardinals in \( M \).

(iii) \( (\text{cf}(\kappa_i) > \lambda_i)^M \) for each \( i = 1, \ldots, n \).

Then there is a c.t.m. \( N \supseteq M \) with the same cofinalities and cardinals such that:

(iv) \( (2^\lambda_i = \kappa_i)^N \) for each \( i = 1, \ldots, n \).

(v) \( (2^\mu)^N = (\kappa^\mu_i)^M \) for all \( \mu > \lambda_n \).

Corollary 20.19. If ZFC is consistent, then so are each of the following:

(i) \( \text{ZFC} + 2^{\aleph_0} = 2^{\aleph_1} = \aleph_3 \).

(ii) \( \text{ZFC} \cup \{2^{\aleph_n} = \aleph_{n+2} : n < 100\} \).

(iii) \( \text{ZFC} \cup \{2^{\aleph_n} = \aleph_{n+1} : n < 300\} \).

(iv) \( \text{ZFC} \cup \{2^{\aleph_n} = \aleph_{n+2} : n < 33\} \).

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Corollary 20.20. If it is consistent with ZFC that there is an uncountable regular limit cardinal, then the following is consistent:

$$\text{ZFC} \cup \{2^{\aleph_n} \text{ is the first regular limit cardinal: } n < 1000\}.$$  \hfill \square

Theorem 20.17 can itself be generalized; the following is the ultimate generalization, in some sense. We do not give the proof.

**Theorem.** (Easton) Suppose that $M$ is a c.t.m. of ZFC, and that in $M$ $E$ is a class function whose domain is the class of all regular cardinals, and whose range is contained in the class of cardinals of $M$. Also assume the following in $M$:

(i) For any regular cardinal $\lambda$, $\text{cf}(E(\lambda)) > \lambda$.

(ii) If $\lambda < \kappa$ are regular cardinals, then $E(\lambda) \leq E(\kappa)$.

Then there is a generic extension $M[G]$ of $M$ preserving cofinalities and cardinals such that in $M[G]$, $2^{\lambda} = E(\lambda)$ for every regular $\lambda$.

Note that we have always been concerned with $2^{\lambda}$ for $\lambda$ regular; $2^{\lambda}$ when $\lambda$ is singular can be computed on the basis of what has been done for regular cardinals. It is difficult to directly do something about $2^{\lambda}$ when $\lambda$ is singular, and there are even hard open problems remaining concerning this. PCF theory applies to these questions. (PCF = “possible cofinalities”)

**EXERCISES**

E20.1. Show that $\text{fin}(\omega, \omega_1)$ collapses $\omega_1$ to $\omega$, but preserves cardinals $\geq \omega_2$.

E20.2. Suppose that $\kappa$ is an uncountable regular cardinal of $M$, and $\mathcal{P} \in M$ is a $\kappa$-cc quasi-order. Assume that $C$ is club in $\kappa$, with $C \in M[G]$. Show that there is a $C' \subseteq C$ such that $C' \in M$ and $C'$ is club in $\kappa$. Hint: in $M[G]$ let $f : \kappa \rightarrow \kappa$ be such that $\forall \alpha < \kappa[\alpha < f(\alpha) \in C]$. Apply Theorem 12.4.

E20.3. Suppose that $\kappa$ is an uncountable regular cardinal of $M$, and $\mathcal{P} \in M$ is a $\kappa$-cc quasi-order. Assume that $S \in M$ is stationary in $\kappa$, in the sense of $M$. Show that it remains stationary in $M[G]$.

E20.4. Suppose that $\kappa$ is an uncountable regular cardinal of $M$, and $\mathcal{P} \in M$ is a $\kappa$-closed quasi-order. Assume that $S \in M$ is stationary in $\kappa$, in the sense of $M$. Show that it remains stationary in $M[G]$.

E20.5. Prove that if ZFC is consistent, then so is ZFC + GCH + $\neg(V = L)$.