Notes on pseudo-tree algebras
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The purpose of these notes is to give a self-contained proof that any subalgebra of an interval algebra is isomorphic to a pseudo-tree algebra. This was stated by Purisch [94]. Simple results from Engelking [77] concerning connected spaces will be used.

Definitions

A *continuum* is a compact connected Hausdorff space. If X is a continuum, a *cutpoint* of X is a point x such that $X\backslash\{x\}$ is not connected. An *arc* is a continuum with exactly two non-cutpoints. If x, y, z are distinct points in a space X, we say that z *separates* x from y if there are disjoint open subsets U, V of X such that $X\backslash\{z\} = U \cup V$, $x \in U$, and $y \in V$. A *dendron* is a compact connected Hausdorff space X such that if x, y are distinct points of X then there is a point z of X which separates x from y. A subset X of a space Y is a *T-set* in Y provided that X is closed in Y and the boundary of each component of $Y\backslash X$ consists of exactly two points. A *strong T-set* in Y is a T-set X in Y such that each component of $Y\backslash X$ is homeomorphic to $(0, 1)$. A continuum X is *rim-finite* if for every $x \in X$ and every open neighborhood U of x there is an open neighborhood W of x such that $W \subseteq V$ and W has a finite boundary. A continuum X is *locally connected* provided that for every point $x \in X$ and every open neighborhood U of x there is a connected set $C \subseteq U$ such that $x \in \mathrm{int}C$. A continuum X is *hereditarily locally connected* if every subcontinuum of X is locally connected. Given a continuum K and distinct points $x, y \in K$, we say that K is an *irreducible continuum* between x and y if K has no proper subcontinuum containing x and y.

1. The exponential

The results of this section are well-known. Let X be any topological space. $\mathrm{Exp}X$ is the collection of all non-empty closed subspaces of X. We topologize it by taking the collection of sets of the following form as a base:

$$\mathscr{V}(U_1, \ldots, U_m) \stackrel{\text{def}}{=} \{F \in \mathrm{Exp}X : F \subseteq U_1 \cup \ldots \cup U_m \text{ and } F \cap U_i \neq 0 \text{ for all } i\},$$

where $U_1, \ldots, U_m$ are open in X.

Theorem 1.1. *Let X be a compact Hausdorff space. Then* $\mathrm{Exp}X$ *is also Hausdorff and compact.*

Proof. Hausdorff: suppose that F and G are distinct non-empty closed sets. Say $x \in F\backslash G$. Let W and U be disjoint open sets such that $x \in W$ and $G \subseteq U$. Thus $G \in \mathscr{V}(U)$, $F \in \mathscr{V}(X, W)$, and $\mathscr{V}(U) \cap \mathscr{V}(X, W) = 0$.

Compact: First we note some facts. For each open U in X let $T_U = \{F \in \mathrm{Exp}X : F \cap U \neq 0\}$. This is an open set, since $T_U = \mathscr{V}(X, U)$. Next,

(1) $\{\mathscr{V}(U) : U \text{ open}\} \cup \{T_U : U \text{ open}\}$ is a subbase for the topology on $\mathrm{Exp}X$.

For, $\mathscr{V}(U_1, \ldots, U_m) = \mathscr{V}(U_1 \cup \ldots \cup U_m) \cap T_{U_1} \cap \ldots \cap T_{U_m}$.

Now to prove compactness of Exp$X$, suppose that $\mathscr{O}$ is a cover of Exp$X$ by subbase members in (1). *Case 1.* $\{W : T_W \in \mathscr{O}\}$ covers X. Choose $W_1, \ldots, W_m$ with each $T_{W_i} \in \mathscr{O}$ such that $X = W_1 \cup \ldots \cup W_m$. Then $\{T_{W_1}, \ldots, T_{W_m}\}$ covers Exp$X$, as desired. *Case 2.* $\{W : T_W \in \mathscr{O}\}$ does not cover X. Let $Y = \bigcup_{T_W \in \mathscr{O}} W$. So Y is a proper open subset of X, and hence $X\backslash Y \in \text{Exp}X$. Therefore $X\backslash Y \in \mathscr{V}(U)$ for some $\mathscr{V}(U) \in \mathscr{O}$. Thus $X\backslash Y \subseteq U$, so $X\backslash U \subseteq Y$. Since $X\backslash U$ is compact, there exist $W_1, \ldots, W_m$ with each $T_{W_i} \in \mathscr{O}$ such that $X\backslash U \subseteq W_1 \cup \ldots \cup W_m$. Hence $\{\mathscr{V}(U)\} \cup \{T_{W_1}, \ldots, T_{W_m}\}$ covers Exp$X$, as is easily verified. □

Theorem 1.2. *Let X be a continuum. Then the collection of all nonempty subcontinua of X is a closed subspace of* Exp$X$.

Proof. Let $C \in \text{Exp}X$, C not a continuum; we want to find a neighborhood of C with all members non-continua. Let U be a nonempty proper clopen subset of C. Then U and $C\backslash U$ are disjoint closed subsets of X; let V and W be disjoint open sets in X containing these sets respectively. Then $C \in \mathscr{V}(V, W)$, and all members of $\mathscr{V}(V, W)$ are non-continua. □

2. General results

Many of the results and proofs in this section can be found in Willard [70].

Lemma 2.1. *Suppose that X is a continuum, V is a nonempty open set in X, and $\overline{V} \neq X$. Suppose that A is a component of $\overline{V}$. Then $A \cap (\overline{V}\backslash V) \neq 0$.*

Proof. Assume the contrary. Note that $\overline{V}\backslash V \neq 0$ since X is connected. Now A is a quasicomponent, so for each $y \in \overline{V}\backslash V$ let W_y be a clopen subset of $\overline{V}$ such that $A \subseteq W_y$ and $y \notin W_y$. Say $\overline{V}\backslash W_{y_1}, \ldots, \overline{V}\backslash W_{y_m}$ covers $\overline{V}\backslash V$. Then $W_{y_1} \cap \ldots \cap W_{y_m}$ is a nonempty closed subset of X. Let

$$K = (\overline{V}\backslash W_{y_1}) \cup \ldots \cup (\overline{V}\backslash W_{y_m}) \cup X\backslash\overline{V}.$$

Thus K is nonempty, disjoint from $W_{y_1} \cap \ldots \cap W_{y_m}$, and has union X with $W_{y_1} \cap \ldots \cap W_{y_m}$. So we get a contradiction by showing that K is closed. Note that

$$\overline{K} = (\overline{V}\backslash W_{y_1}) \cup \ldots \cup (\overline{V}\backslash W_{y_m}) \cup \overline{X\backslash\overline{V}}.$$

Hence it suffices to show that

$$\overline{X\backslash\overline{V}} \subseteq K.$$

But if $x \in \overline{X\backslash\overline{V}}\backslash(X\backslash\overline{V})$, then clearly

$$x \in \overline{V}\backslash V \subseteq (\overline{V}\backslash W_{y_1}) \cup \ldots \cup (\overline{V}\backslash W_{y_m}) \subseteq K,$$

as desired. □

Lemma 2.2. If X is a rim-finite continuum and K is a subcontinuum of X, then K is rim-finite.

Proof. Let $x \in K$, $x \in U$, U open in K. Say $U = K \cap V$, V open in X. Then choose an open $W \subseteq V$, $x \in W$, so that $\overline{W} \backslash W$ is finite. Clearly $\overline{W \cap K} \backslash (W \cap K)$ (closure in K) is a subset of $\overline{W} \backslash W$, as desired. □

The following lemma is well-known.

Lemma 2.3. *Suppose that X is a connected Hausdorff space, $x \in X$, and A and B are disjoint open subsets of X such that $X \backslash \{x\} = A \cup B$. Then $A \cup \{x\}$ and $B \cup \{x\}$ are connected.*

Proof. By symmetry it suffices to show that $A \cup \{x\}$ is connected. Suppose not: let U and V be disjoint nonempty open sets with union $A \cup \{x\}$. Say $x \in V$. Then U, being open in the open subset A, is open in X. But also U is closed in $A \cup \{x\}$, which, as the complement of B, is closed; so U is closed in X. Hence U is nonempty and clopen in X, and $X \neq U$, contradicting the connectivity of X. □

Lemma 2.4. *Suppose that X is a connected Hausdorff space, $x \in X$, and C is a subset of $X \backslash \{x\}$ which is clopen in it. Then $C \cup \{x\}$ is connected.*

Proof. C is open in X, and $(X \backslash \{x\}) \backslash C$ is open in $X \backslash \{x\}$, hence in X. So the desired result follows by Lemma 2.3. □

Lemma 2.5. *Let X be any Hausdorff space and let $x \in X$. Then the component containing x is a subset of the quasicomponent of x.*

Proof. Let C be the component with $x \in C$. If U is a clopen set with $x \in U$, then $U \cap C = 0$ or $C \subseteq U$. Since $x \in C \cap U$, it follows that $C \subseteq U$. So C is included in the quasicomponent of x. □

Now we recall that a linearly ordered space is compact iff the linear ordering is complete; see Koppelberg [89], Proposition 15.6. The following well-known result will be needed.

Theorem 2.6. *If X is a compact linearly ordered space and K is closed in X, then the relative topology on K is the same as the order topology on K (order inherited from X).*

Proof. Without loss of generality, $|K| > 1$. First suppose that U is open in K in the relative topology; say $U = K \cap V$, with V open in X. Let $x \in U$. Say $x \in (u, v) \subseteq V$.

Case 1. x is neither the least nor the greatest element of K. If there exist $s, t \in K$ such that $u \leq s < x < t \leq v$, then $x \in (s, t)_K \subseteq U$, as desired. Suppose that there is an $s \in K$ such that $u \leq s < x$, but there is no $t \in K$ such that $x < t \leq v$. Since x is not the greatest element of K, there is a $w \in K$ such that $x < w$; so $v < w$. Let $w = \inf\{w \in K : x < w\}$. Since K is closed, we have $w \in K$. Then $x \in (s, w)_K \subseteq U$, as desired. Other possibilities are treated similarly.

Case 2. not Case 1: similarly.

Second, suppose that U is open in the order topology of K. Let

$$V = \bigcup\{(s, t)_X : s, t \in K \cup \{-\infty, \infty\} \text{ and } (s, t)_K \subseteq U\}.$$

Then V is open in X and $U = K \cap V$. $\square$

Theorem 2.7. *Let X be a continuum, and suppose that $\mathscr{K}$ is a nonempty collection of nonempty subcontinua of X directed by $\supseteq$. Then $\bigcap_{K\in\mathscr{K}} K$ is a nonempty continuum.*

Proof. Clearly $\bigcap_{K\in\mathscr{K}} K$ is closed and nonempty. Suppose that $\bigcap_{K\in\mathscr{K}} K = H \cup L$ with $x \in H$, $y \in L$, H and L disjoint and closed. Let U and V be disjoint open sets containing H and L respectively. Now for any $K \in \mathscr{K}$ we have $K \cap U \neq 0 \neq K \cap L$, so by the connectedness of K we can choose $x_K \in K\backslash(U \cup V)$. The net $\langle x_K : K \in \mathscr{K}\rangle$ has a cluster point $z \in X$. If W is any neighborhood of z and $K \in \mathscr{K}$, then there is some neighborhood S of z with $S \subseteq W$ and some $K' \subseteq K$ such that $x_{K'} \in W$. This implies that $z \in \overline{K} = K$. Thus $z \in \bigcap_{K\in\mathscr{K}} K \subseteq U \cup V$. But all x_K are outside of $U \cup V$, contradiction. $\square$

Lemma 2.8. *Suppose that K is a continuum and p is a cutpoint of K; say $K\backslash\{p\} = U \cup V$ with U and V disjoint nonempty open sets. Then both of the sets U and V contain non-cutpoints of K.*

Proof. By symmetry it suffices to consider only U. Suppose that every point of U is a cutpoint; for each $x \in U$, write $K\backslash\{x\} = U_x \cup V_x$ with U_x and V_x disjoint nonempty open sets. If for some $x \in U$ the sets U_x and V_x both meet $V \cup \{p\}$, then they separate it, contradicting Lemma 2.3. So one of them, say U_x, is disjoint from $V \cup \{p\}$ and hence is included in U.

(1) If $x \in U$ and $y \in U_x$, then $x \notin U_y$, and $U_y \cup \{y\} \subseteq U_x \cup \{x\}$.

In fact, suppose that $x \in U_y$. then

$$V_y \cup \{y\} = [(V_y \cup \{y\}) \cap U_x] \cup [(V_y \cup \{y\}) \cap V_x],$$

and both the latter sets are nonempty, contradiction. So, $x \notin U_y$. Now suppose that $U_y \cap V_x \neq 0$. Then

$$U_y \cup \{y\} = [(U_y \cup \{y\}) \cap U_x] \cup [(U_y \cup \{y\}) \cap V_x],$$

and both the latter sets are nonempty, contradiction. Thus (1) holds.

Now define $\langle q_\alpha : \alpha < |X|^+\rangle$ by induction, as follows. Let $q_0 \in U$ be arbitrary. Having defined $q_\alpha \in U$, choose $q_{\alpha+1} \in U_{q_\alpha}$. If q_α has been defined for all $\alpha < \lambda$, where λ is a limit ordinal, pick $q_\lambda \in \bigcap_{\alpha<\lambda}(U_{q_\alpha} \cup \{q_\alpha\})$, using Lemma 2.7. Clearly $\alpha < \beta$ implies that $U_{q_\alpha} \cup \{q_\alpha\} \supseteq U_{q_\beta} \cup \{q_\beta\}$. Now the sequence $\langle q_\alpha : \alpha < |X|^+\rangle$ repeats; say $\alpha < \beta$ and $q_\beta = q_\alpha$. Then $q_\alpha = q_\beta \in U_{q_{\alpha+1}} \cup \{q_{\alpha+1}\}$. Now $q_{\alpha+1} \in U_{q_\alpha}$, so obviously $q_\alpha \neq q_{\alpha+1}$, and $q_\alpha \notin U_{q_{\alpha+1}}$ by (1). This is a contradiction. $\square$

Theorem 2.9. *If K is a continuum with at least 2 elements, then K has at least two non-cutpoints.*

Proof. Without loss of generality K has a cutpoint. So the result follows by Lemma 2.8. $\square$

Lemma 2.10. *If Y is a closed subspace of X, X compact Hausdorff, $y \in Y$, P an open subset of X, $Y \cap P = 0$, $\overline{P} \cap Y = \{y\}$, and $Y \cup P$ is connected, then Y is connected.*

Proof. Suppose that Y is not connected. Say $0 \neq U \subset Y$ with U clopen in Y. Then there are disjoint open subsets V, W in X such that $U \subseteq V$ and $Y \backslash U \subseteq W$. Say $y \in U$. Then $Y \cup P \subseteq (V \cup P) \cup (W \backslash \overline{P})$, the latter being disjoint open sets with non-empty intersection with $Y \cup P$, contradiction. $\square$

The next lemma is well-known. Its proof will be used later.

Lemma 2.11. *If K is a compact linearly ordered space, then K is a subspace of a compact connected linearly ordered space I.*

Proof. Recall that the ordering on K is complete. If $x <\cdot\ y$, insert $(0,1)$ between x and y; this gives I. Clearly I is complete and dense. Suppose $0 \neq U \subset I$, U clopen. Choose $a \notin U$. Without loss of generality there is a $b \in U$ with $b < a$. Let $c = \sup\{b \in U : b < a\}$. Then $c \in U$ since U is closed. So $c < a$. But then c is the inf of elements of $I \backslash U$, so $c \in I \backslash U$, contradiction.

Clearly K is closed in I, so K is a subspace of I. $\square$

The next lemma is an exercise in some topology textbooks.

Lemma 2.12. *Suppose that f is a closed continuous function from a locally connected Hausdorff space X onto a Hausdorff space Y. Then Y is locally connected.*

Proof. Let U be an open set in Y, and $y \in U$. We want to find a connected subset W of Y such that $W \subseteq U$ and $y \in \mathrm{int} W$. For each x such that $fx = y$ choose a connected subset C_x of X such that $C_x \subseteq f^{-1}[U]$ and $x \in \mathrm{int} C_x$. Then $y \in \bigcap_{fx=y} f[C_x]$, and so $\bigcup_{fx=y} f[C_x]$ is a connected subset of U. Let $V = \bigcup_{fx=y} \mathrm{int} C_x$. The proof is completed by showing

$$y \in Y \backslash f[X \backslash V] \subseteq \bigcup_{fx=y} f[C_x].$$

To prove the first part, suppose on the contrary that $y = fu$ with $u \in X \backslash V$. But $u \in \mathrm{int} C_u \subseteq V$, contradiction. For the inclusion, suppose that $z \in Y \backslash f[X \backslash V]$. Write $z = fw$. Then $w \in V$; say $w \in \mathrm{int} C_x$ with $fx = y$. So $z \in f[C_x]$, as desired. $\square$

The following lemma is well-known.

Lemma 2.13. *Suppose that X is locally connected, V is open in X, and W is a component of the subspace V. Then W is open.*

Proof. Let $y \in W$; so $y \in V$. Let C be connected, $C \subseteq V$, $y \in \mathrm{int} C$. Now $y \in W \cap C$, so $W \cup C$ is connected, hence $C \subseteq W$. So $y \in \mathrm{int} C \subseteq W$, showing that W is open. $\square$

For the next result we need some notation not mentioned at the beginning. If X is a space and $x, y \in X$, then a *simple chain connecting x and y* is a sequence $U_0, \ldots, U_{m-1}$ of open sets of X such that $x \in U_0$, $x \notin U_i$ for $i > 0$, $y \in U_{m-1}$, $y \notin U_i$ for $i < m-1$, $U_i \cap U_{i+1} \neq 0$ for all $i < m-1$, and $U_i \cap U_j = 0$ if $i + 1 < j$.

Lemma 2.14. *If X is connected and $\mathscr{U}$ is an open cover of X, then any two points $a, b \in X$ can be connected by a simple chain consisting of elements of $\mathscr{U}$.*

Proof. Let Z be the set of all points of X connected to a by a simple chain of members of $\mathscr{U}$. Then clearly $a \in Z$ and Z is open. Since X is connected, it suffices now to show that Z is closed. So, suppose that $z \in \overline{Z}$.

Choose $U \in \mathscr{U}$ such that $z \in U$, and then choose $x \in U \cap Z$. Let $U_0, \ldots, U_{m-1}$ be a simple chain of members of $\mathscr{U}$ connecting a to x. If $z \in U_k$ for some k, then the smallest such k produces a simple chain $U_0, \ldots, U_k$ of members of $\mathscr{U}$ connecting a to z, hence $z \in Z$ as desired. So, suppose that $z \notin U_k$ for all $k < m$. Now $U \cap U_{m-1} \neq 0$; choose the smallest l such that $U \cap U_l \neq 0$. Then $U_0, \ldots, U_l, U$ is a simple chain connecting a to z, again yielding $z \in Z$. □

Corollary 2.15. *If X is a locally connected continuum and K is an open connected subspace of X, then for any two $x, y \in K$ there is a continuum $L \subseteq K$ such that $x, y \in L$.*

Proof. Let $\mathscr{U} = \{U : U$ is open and connected, and $\overline{U} \subseteq K\}$. Then $\mathscr{U}$ covers K. In fact, let $a \in K$. Choose V open with $a \in V$ and $\overline{V} \subseteq K$. Let L be the component of a in V. Hence L is open by Lemma 2.13; it is connected, and $\overline{L} \subseteq \overline{V} \subseteq K$, as desired. Now we apply Lemma 2.14 to get a chain $U_0, \ldots, U_{m-1}$ of members of $\mathscr{U}$ connecting x to y. Then $\overline{U}_0 \cup \ldots \cup \overline{U}_{m-1}$ is the desired continuum containing x, y and contained in K. □

Theorem 2.16. *If K is a continuum and $A \subseteq K$, then there is an irreducible continuum L such that $A \subseteq L \subseteq K$.*

Proof. Apply Zorn's lemma to the collection of all subcontinua of K that contain A; see Theorem 2.7. □

Let X be a connected space, and let $a, b \in X$. Then we set

$$E(a, b) = \{a, b\} \cup \{p \in X : p \text{ separates } a \text{ from } b\}.$$

We define a relation $\leq$ on $E(a, b)$ by declaring that $p_1 \leq p_2$ iff $p_1 = a$, $p_1 = p_2$, or p_1 separates a from p_2. For the next few results it is convenient to have the following notation. We write $\mathscr{S}(X, p, U, V, a, b)$ to mean that $X \backslash \{p\} = U \cup V$, U and V are disjoint open sets, $a \in U$, and $b \in V$.

Lemma 2.17. *For any continuum X and any distinct $a, b \in X$, $\leq$ is a linear order on $E(a, b)$.*

Proof. Suppose that $p_1 \leq p_2 \leq p_1$. Suppose $p_1 \neq p_2$. Clearly then $a \neq p_1, p_2$. Thus p_1 separates a from p_2, and p_2 separates a from p_1. Say $\mathscr{S}(X, p_1, U, V, a, p_2)$ and $\mathscr{S}(X, p_2, U', V', a, p_1)$. Then $U \cap U'$ and $V \cup V'$ are disjoint and open, both nonempty, and with union X, contradiction.

Next, suppose that $p_1 \leq p_2 \leq p_3$. In order to show that $p_1 \leq p_3$, we may assume that these three elements are all distinct and different from a. So p_1 separates a from p_2, and p_2 separates a from p_3. Say $\mathscr{S}(X, p_1, U, V, a, p_2)$ and $\mathscr{S}(X, p_2, U', V', a, p_3)$. Then $\mathscr{S}(X, p_1, U \cap U', V \cup (V' \backslash \{p_1\}), a, p_3)$.

Now suppose that r and s are distinct points of $E(a, b)$; we want to show that $r \leq s$ or $s \leq r$. If $r = a$, then $r \leq s$, so we may assume that $r \neq a$, and similarly that $s \neq a$. If $s = b$, then $r \leq s$ since $r \in E(a, b)$; so we may assume that $s \neq b$, and similarly that $r \neq b$.

Say $\mathscr{S}(X,r,U,V,a,b)$ and $\mathscr{S}(X,s,U',V',a,b)$.
Case 1. $s \in V$. Then $r \leq s$.
Case 2. $s \in U$. Now $V \cup \{r\}$ is connected by Lemma 2.3, and $V \cup \{r\} \subseteq X \backslash \{s\}$, so $V \cup \{r\} \subseteq U'$ or $V \cup \{r\} \subseteq V'$. Since $b \in V \cap V'$, we actually must have $V \cup \{r\} \subseteq V'$. Thus $r \in V'$, so $s \leq r$. $\square$

Lemma 2.18. *If X is a continuum and $a, b \in X$, then any subset of $E(a,b)$ open in the interval topology is open in the relative topology.*

Proof. It suffices to take any $r \in E(a,b) \backslash \{a,b\}$ and show that $(-\infty, r)$ and (r, ∞) are open in the relative topology. Say $\mathscr{S}(X,r,U,V,a,b)$.

(1) $U \cap E(a,b) = (-\infty, r)$.

To prove this, first let $x \in U \cap E(a,b)$. If $x = a$ there is no problem, so assume that $x \neq a$. We want to show that x separates a from r. Say $\mathscr{S}(X,x,U',V',a,b)$. If $r \in V'$, then $x \in (-\infty, r)$, so assume that $r \in U'$. Now $V \cup \{r\} \subseteq U' \cup V'$, $V \cup \{r\}$ is connected, and $(V \cup \{r\}) \cap U' \neq 0$, so $V \cup \{r\} \subseteq U'$. Hence $b \in U'$, contradiction.

Conversely, suppose that $x \in (-\infty, r)$ in $E(a,b)$. If $r = a$, then $r \in U \cap E(a,b)$. Assume that $x \neq a$. Now $x < r$, so x separates a from r. Say $\mathscr{S}(X,x,U',V',a,r)$. If $x \in V$, then $U' \cup \{x\} \subseteq V$ and so $a \in V$, contradiction. So $x \in U$. This proves (1).

(2) $V \cap E(a,b) = (r, \infty)$.

This is true since, by (1),

$$\begin{aligned} V \cap E(a,b) &= E(a,b) \backslash (U \cup \{r\}) \\ &= (E(a,b) \backslash U) \backslash \{r\} \\ &= (E(a,b) \backslash (-\infty, r)) \backslash \{r\} \\ &= (r, \infty). \end{aligned}$$

$\square$

Lemma 2.19. *If X is an arc with endpoints a, b, then the order on $E(a,b)$ is dense.*

Proof. Suppose that $x <\cdot\ y$ in this order. Then X is the union of the disjoint nonempty open sets $(-\infty, y)$ and (x, ∞), contradiction. $\square$

Corollary 2.20. *Every arc is a dendron.*

Proof. Let X be an arc with endpoints a, b. Suppose $x, y \in X$, $x \neq y$. Say $x < y$. By Lemma 2.19, choose z with $x < z < y$. Then $\mathscr{V}(X, z, (-\infty, z), (z, \infty), x, y)$, so z separates x from y. $\square$

Lemma 2.21. *If X is a compact Hausdorff space, C is closed in X, and $\mathscr{T}$ is a topology on C such that $U \in \mathscr{T}$ implies that U is open in the relative topology, then $\mathscr{T}$ coincides with the relative topology.*

Proof. The identity on C is a continuous function from C with the relative topology onto C under $\mathscr{T}$. Hence it is a homeomorphism. $\square$

Lemma 2.22. *An arc is an irreducible continuum between its endpoints.*

Proof. Let A be an arc with endpoints a, b. Suppose that B is a proper subcontinuum of A with $a, b \in B$. Say $c \in A \backslash B$. Then $c \neq a, b$ and c is a non-cutpoint of A, as is easily checked. $\square$

3. Dendrons

(following the van Mill, Wattel article)

In this section, D is an arbitrary dendron, unless otherwise stated.

Lemma 3.1. *Let $x \in D$. Then the components and quasicomponents of $D\backslash\{x\}$ coincide, and every component is open in D. Hence every component is clopen in $D\backslash\{x\}$, and for every component C of $D\backslash\{x\}$, $D\backslash C$ is connected.*

Proof. For one direction of the first statement, see Lemma 2.5. Now we show that any quasi-component Q of $D\backslash\{x\}$ is open in D. Suppose that this is not true. Let $q \in Q$ such that q is in the closure of $D\backslash(Q \cup \{x\})$. By the dendron property, let z be a point separating q and x. Then

(1) $z \in Q$.

In fact, suppose not. Then there is a set U clopen in $D\backslash\{x\}$ such that $Q \subseteq U$ and $z \notin U$. Let $V = (D\backslash\{x\})\backslash U$. So U and V are disjoint open sets in D with union $D\backslash\{x\}$. So by Lemma 1, $U \cup \{x\}$ is connected. Now $q \in Q \subseteq U$, so both q and x are in $U \cup \{x\}$, while $z \notin U \cup \{x\}$. This contradicts z separating q and x. So (1) holds.

Now by Lemma 2.4,

$$\bigcup\{C \cup \{x\} : C \text{ clopen in } D\backslash\{x\} \text{ and } C \cap Q = 0\} = D\backslash Q$$

is connected. But q is in the closure of $D\backslash Q$, so $\{q\} \cup (D\backslash Q)$ is connected; it contains both x and q, but not z. This contradicts z separating x and q. It follows that Q is open.

Now take any quasicomponent Q. It is connected, for if A is a subset of Q which is clopen in Q, then it is open in $D\backslash\{x\}$ since Q is open. But also Q is closed in $D\backslash\{x\}$, so A is closed in $D\backslash\{x\}$. So A is clopen in $D\backslash\{x\}$, and hence $A = 0$ or $A = Q$, as desired.

Thus the components and quasicomponents of $D\backslash\{x\}$ coincide, and every component is open in D. Since components are closed, every component is clopen in $D\backslash\{x\}$. Finally, for any component C of $D\backslash\{x\}$, the set $(D\backslash\{x\})\backslash C$ is clopen in $D\backslash\{x\}$, and so by Lemma 2.4, $D\backslash C = ((D\backslash\{x\})\backslash C) \cup \{x\}$ is connected. $\square$

Now we define

$$\mathscr{U} D = \{U \subseteq D : \exists x \in D(U \text{ is a component of } D\backslash\{x\}\}.$$

Members of $\mathscr{U} D$ are called *cutpoint components* of D. We also define

$$\mathscr{J} D = \{D\backslash U : U \in \mathscr{U} D\}.$$

Lemma 3.2. *$\mathscr{J} D$ consists of connected sets.*

Proof. By Lemma 3.1. $\square$

Lemma 3.3. *$\mathscr{U}D$ is an open subbase for the topology of D.*

Proof. Suppose that V is open in D and $x \in V$. For each $y \in D\backslash V$, let z_y separate x and y. Then choose components U_y and W_y of $D\backslash\{z_y\}$ such that $x \in U_y$ and $y \in W_y$; clearly these are disjoint. They are open by Lemma 3.1. Let $\{W_{y_0}, \ldots, W_{y_{m-1}}\}$ cover $D\backslash V$. Then $x \in U_{y_0} \cap \ldots \cap U_{y_{m-1}} \subseteq V$, as desired. $\square$

Lemma 3.4. *If C is a component of $D\backslash\{x\}$, then $x \in \mathrm{cl}C$.*

Proof. C is closed in $D\backslash\{x\}$. If $x \notin \mathrm{cl}C$, then C is closed in D, hence clopen in D, contradicting D connected. $\square$

A collection $\mathscr{L}$ of subsets of a set X is *cross-free* if for all $L_0, L_1 \in \mathscr{L}$ we have $L_0 \subseteq L_1$, $L_1 \subseteq L_0$, $L_0 \cap L_1 = 0$, or $L_0 \cup L_1 = X$.

Lemma 3.5. *$\mathscr{U}D$ is cross-free.*

Proof. Let $U_1, U_2 \in \mathscr{U}D$; say that U_i is a component of $D\backslash\{x_i\}$, $i = 1, 2$. If $x_1 = x_2$, then obviously U_1 and U_2 are either disjoint or equal. Assume that $x_1 \neq x_2$. *Case 1.* $x_1 \in U_2$ and $x_2 \in U_1$. If C is any component of $D\backslash\{x_1\}$ which does not contain x_2, then by Lemma 3.4, $C \cup \{x_1\}$ is connected, and hence so is $C \cup \{x_1\} \cup U_2$. Since $x_2 \notin C$, it follows that $C \subseteq U_2$. Hence $U_1 \cup U_2 = D$. *Case 2.* $x_1 \notin U_2$. Then U_2 is a connected subset of $D\backslash\{x_1\}$, and hence is either contained in or disjoint from U_1. *Case 3.* $x_2 \notin U_1$. Symmetric to Case 2. $\square$

Corollary 3.6. *$\mathscr{J}D$ is cross-free.* $\square$

A collection $\mathscr{L}$ of subset of a set X is *binary* provided that for all $\mathscr{M} \subseteq \mathscr{L}$ with $\bigcap \mathscr{M} = 0$, there exist $M, N \in \mathscr{M}$ such that $M \cap N = 0$.

Lemma 3.7. *$\mathscr{J}D$ is binary.*

Proof. Suppose not. Say that $\mathscr{M}$ is a subset of $\mathscr{J}D$, any two members of $\mathscr{M}$ have nonempty intersection, but $\bigcap \mathscr{M} = 0$. Since the members of $\mathscr{M}$ are closed, it follows that some finite subset $\mathscr{N}$ of $\mathscr{M}$ has empty intersection. Choose such a subset $\mathscr{N}$ with the smallest number of elements, and pick $M \in \mathscr{N}$. Now any two elements of $\mathscr{N}$ have nonempty intersection, and neither is contained in the other, by minimality; so by Corollary 3.6 we conclude that their union is D. Hence $M \cup \bigcap_{N \in \mathscr{N}, N \neq M} N = D$, while $M \cap \bigcap_{N \in \mathscr{N}, N \neq M} N = 0$. This contradicts D being connected. $\square$

A collection L of subsets of a set X is called *normal* if for all disjoint $T_0, T_1 \in \mathscr{L}$ there are $S_0, S_1 \in \mathscr{L}$ such that $T_0 \cap S_1 = 0 = T_1 \cap S_0$ and $S_0 \cup S_1 = X$. Note that this condition is automatic if $T_0 = X$ or $T_1 = X$.

Lemma 3.8. *$\mathscr{J}D$ is normal.*

Proof. Let T_0, T_1 be nonempty disjoint members of $\mathscr{J}D$. Since T_0 is closed and nonempty and D is connected, choose $t_0 \in T_0 \cap \overline{X\backslash T_0}$. Similarly, there is a $t_1 \in T_1 \cap \overline{X\backslash T_1}$.

Since D is Hausdorff, there hence exist basic closed sets B_0 and B_1 such that $t_0 \notin B_1$, $t_1 \notin B_0$, and $B_0 \cup B_1 = D$. Hence we can write

$$B_0 = F_0 \cup \ldots \cup F_m \quad \text{and} \quad B_1 = F_{m+1} \cup \ldots \cup F_n$$

for suitable $F_i \in \mathscr{J} D$. What is important here is:

(*) We have distinct points $t_0 \in T_0 \backslash \overline{X \backslash T_0}$, $t_1 \in T_1 \backslash \overline{X \backslash T_1}$, and $F_0, \ldots, F_m, \ldots, F_n \in \mathscr{J} D$ such that $F_0 \cup \ldots \cup F_m \cup \ldots \cup F_n = D$, $t_0 \notin F_{m+1} \cup \ldots \cup F_n$, and $t_1 \notin F_0 \cup \ldots \cup F_m$.

Without loss of generality, no F_i is contained in any F_j for $i \neq j$.

(1) There is exactly one i such that $t_0 \in F_i$.

For, there is at least one since $F_0 \cup \ldots \cup F_m \cup \ldots \cup F_n = D$. Suppose that $t_0 \in F_i \cap F_j$, $i \neq j$. Thus $i, j \leq m$; hence $t_1 \notin F_i \cup F_j$. It follows by Corollary 3.6 that $F_i \subseteq F_j$ or $F_j \subseteq F_i$, contradiction. So (1) holds. Similarly,

(2) There is exactly one j such that $t_1 \in F_j$.

Say by (1) and (2) that $t_0 \in F_0$ and $t_1 \in F_n$.

(3) If $0 < i < n$, then $F_0 \cap F_i = 0$.

For, suppose that $t_2 \in F_0 \cap F_i$. Now $t_1 \notin F_0 \cup F_i$, so Corollary 3.6 yields $F_0 \subseteq F_i$ or $F_i \subseteq F_0$, contradiction. So (3) holds. Similarly

(4) If $0 < i < n$, then $F_i \cap F_n = 0$.

(5) $F_0 \cup F_n = D$.

In fact, if this is not true, then the reductions indicated imply that $F_0 \cap F_n = 0$; but then we get a decomposition of D into three nonempty closed sets F_0, F_n, and $\bigcup_{0<i<n} F_i$, contradiction.

Finally, we show that $F_n \cap T_0 = 0$ (similarly, $F_0 \cap T_1 = 0$). We have $t_0 \in T_0 \backslash F_n$ and $t_1 \in F_n \backslash T_0$. Also, t_0 has a neighborhood U disjoint from F_n, and there is some $x \in U \cap (D \backslash T_0)$, so $T_0 \cup F_n \neq D$. It follows from Corollary 3.6 that $F_n \cap T_0 = 0$. $\square$

Lemma 3.9. *If u and v are distinct elements of D, then there exist $S_0, S_1 \in \mathscr{J} D$ such that $u \in S_0 \backslash S_1$, $v \in S_1 \backslash S_0$, and $S_0 \cup S_1 = D$.*

Proof. We have $\{u\} = \bigcap\{C : u \in C,\ C \text{ basic closed}\}$, and similarly for v. For each basic closed C such that $u \in C$ there is a $D \in \mathscr{J} D$ such that $u \in D \subseteq C$. So $\{u\} = \bigcap\{C : u \in C,\ C \in \mathscr{J} D\}$, and similarly for v. Hence by Lemma 3.7 there are disjoint $T_0, T_1 \in \mathscr{J} D$ such that $u \in T_0$ and $v \in T_1$. By Lemma 3.8, say $S_0, S_1 \in \mathscr{J} D$, $T_0 \cap S_1 = 0 = T_1 \cap S_0$, and $S_0 \cup S_1 = D$. Note that $u \in S_0 \backslash S_1$ and $v \in S_1 \backslash S_0$. $\square$

For elements x, y of a dendron D we define

$$I(x, y) = \bigcap\{T : T \in \mathscr{J} D,\ x, y \in T\}.$$

Lemma 3.10. *If C is a nonempty intersection of members of $\mathscr{J}D$, then there is a retraction r of D onto C such that for any $x \in D$,*

$$\{rx\} = \bigcap_{c \in C} I(x, c) \cap C.$$

Proof. First suppose that $E \stackrel{\text{def}}{=} \bigcap_{c \in C} I(x, c) \cap C = 0$. By Lemma 3.7 this gives two cases.

Case 1. There exist $c \in C$ and $S, T \in \mathscr{J}D$ such that $x, c \in S$, $C \subseteq T$, and $S \cap T = 0$. But $c \in S \cap T$, contradiction.

Case 2. There exist $c, d \in C$ and $S, T \in \mathscr{J}D$ such that $x, c \in S$, $x, d \in T$, and $S \cap T = 0$. but $x \in S \cap T$.

Thus $E \neq 0$. Suppose that it has two distinct members e_0, e_1. By Lemma 3.9 choose $T_0, T_1 \in \mathscr{J}D$ such that $e_0 \in T_0 \backslash T_1$, $e_1 \in T_1 \backslash T_0$, and $T_0 \cup T_1 = D$. If $x \in T_0$, then $E \subseteq I(x, e_0) \subseteq T_0$, which is impossible since $e_1 \notin T_0$. Similarly $x \in T_1$ yields a contradiction. So $|E| = 1$, and we write $E = \{rx\}$.

To show that r is continuous, it is enough to show that $r^{-1}[U \cap C]$ is open for each $U \in \mathscr{U}D$. We may assume that $0 \neq U \cap C \neq C$. Write $A = D \backslash U$; so $A \in \mathscr{J}D$ and $A \cap C \neq 0$. Let $x \in r^{-1}[U \cap C]$. Now $\{rx\} \cap A = 0$ and $A \cap C \neq 0$, so by the definition of r and Lemma 3.7 there is a $c \in C$ such that $I(x, c) \cap A = 0$. Hence there is a $B \in \mathscr{J}D$ such that $x, c \in B$ and $B \cap A = 0$. By Lemma 3.8 choose $S_0, S_1 \in \mathscr{J}D$ such that $S_0 \cap A = 0 = S_1 \cap B$ and $S_0 \cup S_1 = D$. Then

(*) $x \in D \backslash S_1 \subseteq r^{-1}[U \cap C]$.

(This will finish the proof.) To prove this, first it is obvious that $x \notin S_1$, since $x \in B$. Similarly $c \notin S_1$, so $c \in S_0$. Now let u be an arbitrary member of $D \backslash S_1$. Then $u \in S_0$. So $I(u, x) \subseteq S_0$, and hence $I(u, x) \cap A = 0$ and $ru \in U \cap C$, as desired. $\square$

Corollary 3.11. *If C is a nonempty intersection of members of $\mathscr{J}D$, then C is a continuum. In particular, for any $x, y \in D$ the set $I(x, y)$ is a continuum.* $\square$

Lemma 3.12. *$E(x, y) \subseteq I(x, y)$ in a dendron.*

Proof. Suppose that $p \in E(x, y) \backslash \{x, y\}$. Say $\mathscr{V}(D, p, U, V, x, y)$. Since $x \in I(x, y) \cap U$, $y \in I(x, y) \cap V$, and $I(x, y)$ is connected (Corollary 3.11), also $p \in I(x, y)$. $\square$

Lemma 3.13. *$E(x, y) = I(x, y)$ in a dendron D.*

Proof. The inclusion $\subseteq$ holds by Lemma 3.12. Now suppose that $p \in I(x, y) \backslash E(x, y)$. In particular, $x \neq y$ and $p \neq x, y$. For distinct $u, v \in D$ let $U_u v$ be the component of u in $D \backslash \{v\}$. First we claim:

(1) $(E(x, p) \cup E(y, p)) \backslash \{p\} \subseteq E(x, y)$.

For, suppose that $q \in E(x, p) \backslash \{p\}$ (the case $q \in E(y, p) \backslash \{p\}$ is similar). We may assume that $q \notin \{x, y\}$. Then no component of $D \backslash \{q\}$ contains both x and y; for if V is such a component, then $V \cap U_p q = 0$ and $\{x, y\} \subseteq D \backslash U_p q \in \mathscr{J}D$ and so $p \in I(x, y) \subseteq D \backslash U_p q$, contradiction. Thus $q \in E(x, y)$.

Now we set

$$A_x = \bigcup_{q\in E(x,p)\backslash\{x,p\}} U_xq$$
$$A_y = \bigcup_{q\in E(y,p)\backslash\{y,p\}} U_yq$$
$$A_p = D\backslash(A_x \cup A_y \cup \{p\}).$$

(2) If $q \in E(x,p)\backslash\{x,p\}$ and $r \in E(y,p)\backslash\{y,p\}$, then $U_xq \cap U_yr = 0$.

For, suppose that $U_xq \cap U_yr \neq 0$. Now $U_xq \cap U_pq = 0$. If $y \notin U_pq$, then $x, y \in D\backslash U_pq \in \mathscr{J} D$, so $p \in I(x,y) \subseteq D\backslash U_pq$, contradiction. So $y \in U_pq$. If $r \notin U_xq$, then from the assumption $U_xq \cap U_yr \neq 0$ we see that r does not separate x from y, contradicting (1). So $r \in U_xq$. But then r does not separate y and p, contradiction. Thus (2) holds.

Now A_x and A_y are open. We claim that also A_p is open. To show this, let $a \in A_p$. Since D is a dendron, let s separate a and p. We show that $U_as \cap (A_x \cup A_y) = 0$, which establishes that A_p is open.

(3) $U_as \cap A_x = 0$.

For, suppose that $t \in U_as \cap A_x$. Choose $r \in E(x,p)\backslash\{x,p\}$ so that $t \in U_xr$. Thus

(4) $U_as \cap U_xr \neq 0$.

Now $p \notin U_as$. Also, $p \notin U_xr$ since $r \in E(x,p)$. Hence

(5) $U_as \cup U_xr \neq D$.

Next, $U_xr \subseteq A_x$, so $a \notin U_xr$. So $a \in U_as\backslash U_xr$, and hence

(6) $U_as \not\subseteq U_xr$.

Now suppose that $x \in U_as$. In particular, $s \neq x$. Since s separates a and p, it follows that s separates x and p. so $s \in E(x,p)\backslash\{x,p\}$. And $a \in U_xs \subseteq A_x$, contradiction. So $x \in U_xr\backslash U_as$, so

(7) $U_xr \not\subseteq U_as$.

The conditions (4)–(7) contradict Lemma 3.5. So (3) holds. Similarly,

(8) $U_as \cap A_y = 0$.

Therefore, A_p is open. But now we have $D\backslash\{p\} = A_x \cup A_y \cup A_p$, a decomposition into pairwise disjoint sets, of which the first two are definitely nonempty. So p separates x from y, contradiction. □

Lemma 3.14. *In a dendron, $E(a,b) \cap E(a,c) \cap E(b,c)$ is a singleton.*

Proof. Let D be the dendron. The conclusion is obvious if the elements are not distinct; so suppose that they are distinct. By Lemmas 3.7 and 3.13, the intersection is nonempty. Now suppose that x and y are distinct elements of the intersection. By Lemma

3.9 choose $S_0, S_1 \in \mathscr{J} D$ such that $x \in S_0 \backslash S_1$, $y \in S_1 \backslash S_0$, and $S_0 \cup S_1 = D$. At least two of the elements a, b, c are in one of S_0 or S_1; say $a, b \in S_0$. Then

$$E(a,b) \cap E(a,c) \cap E(b,c) \subseteq E(a,b) = I(a,b) \subseteq S_0,$$

and this is a contradiction since $y \notin S_0$. □

Lemma 3.15. *In a dendron D, if $a, b \in D$, then the order topology on $E(a,b)$ is the same as the relative topology on $E(a,b)$.*

Proof. By 2.18, 2.21, 3.11, and 3.13. □

Corollary 3.16. *Any arc A is locally connected.*

Proof. By Lemma 2.20, A is a dendron. If A has endpoints a, b, then $A = E(a,b)$. Then use Lemma 3.15. □

Lemma 3.17. *Every dendron is locally connected.*

Proof. Let D be a dendron, $x \in D$, U an open neighborhood of x. Since $\mathscr{J} D$ is a closed subbase for D, there are $T_0, \ldots, T_{n-1} \in \mathscr{J} D$ such that $x \notin \bigcup_{i<n} T_i \supseteq D \backslash U$. Now

(1) $\{x\} = \bigcap \{T \in \mathscr{J} D : x \in T\}$.

For, $\{x\}$ is the intersection of all finite unions C of members of $\mathscr{J} D$ such that $x \in C$, since $\mathscr{J} D$ forms a closed subbase, and (1) follows.

For each $i < n$ we have $T_i \cap \bigcap \{T \in \mathscr{J} D : x \in T\} = 0$, so by Lemma 3.7 there is a $T'_i \in \mathscr{J} D$ such that $x \in T'_i$ and $T_i \cap T'_i = 0$. Then by Lemma 3.8 we get $S^0_i, S^1_i \in \mathscr{J} D$ such that $T_i \cap S^0_i = 0 = T'_i \cap S^1_i$ and $S^0_i \cup S^1_i = X$. Hence $T'_i \subseteq S^0_i$ and $x \in \mathrm{int} S^0_i$ since $x \in T'_i \subseteq X \backslash S^1_i \subseteq S^0_i$. Let $T = \bigcap_{i<n} S^0_i$. Then T is a neighborhood of x, $T \subseteq \bigcap_{i<n} (X \backslash T_i) \subseteq U$, and by Corollary 3.11, T is connected. □

4. Some results of Tymchatyn

The following is a corollary of Lemma 2 in Tymchatyn [77].

Lemma 4.1. *Every rim-finite continuum is hereditarily locally connected.*

Proof. Suppose that X is a continuum which is rim-finite but not hereditarily locally connected. Let K be a subcontinuum of X which is not locally connected. Then by Lemma 2.2, K is still rim-finite. We work inside K from now on. Let $x \in K$, $x \in U$ open, such that there is no connected $C \subseteq U$ such that $x \in \mathrm{int} C$. Choose an open neighborhood V of x such that $\overline{V} \subseteq U$ and V has finite boundary. So, there is no connected $C \subseteq \overline{V}$ such that $x \in \mathrm{int} C$. Let B be the component of x in $\overline{V}$. So $x \notin \mathrm{int} B$. For every open neighborhood W of x choose $y_W \in V \cap W \backslash B$, and let D_W be the component of y_W in $\overline{V}$. By Lemma 2.1 it suffices to show that there are infinitely many components D_W to get a contradiction. Let $\mathscr{N}$ be the collection of all open neighborhoods of x, directed by $\supseteq$. Thus $\langle D_W : W \in \mathscr{N} \rangle$ is a net, and so has a cluster point E (in the exponential topology of nonempty closed subsets of $\overline{V}$). Note by Lemma 1.2 that E is a continuum.

(1) $x \in E$.

For, suppose not. Let Z and Y be disjoint open sets such that $x \in Z$ and $E \subseteq Y$. Choose $W \in \mathscr{N}$ such that $W \subseteq Z$ and $D_W \in \mathscr{V}(Y)$. thus $D_W \subseteq Y$. But $y_W \in W \cap D_W$, contradiction. So (1) holds.

From (1) it follows that $E \subseteq B$. The desired conclusion now follows from the following statement:

If $Z_1, \ldots, Z_m$ are open neighborhoods of x, then there is an open neighborhood W of x such that $D_W \neq D_{Z_1}, \ldots, D_{Z_m}$.

Suppose not. Now $D_{Z_1}, \ldots, D_{Z_m}$ are all disjoint from E. Let S and T be disjoint open sets such that $D_{Z_1} \cup \ldots \cup D_{Z_m} \subseteq S$ and $E \subseteq T$. Thus $E \in \mathscr{V}(T)$. So, choose $W \in \mathscr{N}$ such that $D_W \in \mathscr{V}(T)$. So $D_W \subseteq T$ but D_W is equal to some D_{Z_i}, contradiction. $\square$

The following is Lemma 3 in Tymchatyn [77]:

Lemma 4.2. *Suppose that X is a locally connected continuum which is an irreducible continuum between two distinct points $x, y \in X$. Then X is an arc.*

Proof. Let $p \in X \backslash \{x, y\}$; we shall show that p is a cutpoint; hence by Theorem 2.9, X has exactly two non-cutpoints, so that it is, indeed, an arc. Let K be the component in $X \backslash \{p\}$ of x. Since X is locally connected, K is open by Lemma 2.13. If $y \in K$, then by Lemma 2.15, there is a continuum L such that $x, y \in L$ and $L \subseteq K$; but this contradicts the irreducibility of X. So, $y \notin K$. Now

$$X \backslash \{p\} = K \cup (X \backslash (\{p\} \cup K))$$

is a decomposition of X into disjoint open sets; and $K \neq 0$, $y \in X \backslash (\{p\} \cup K)$, so p is a cutpoint. ($X \backslash (\{p\} \cup K)$ is open since it is the union of all the components of $X \backslash \{p\}$ which are different from K.) $\square$

Finally, here is Corollary 4 from Tymchatyn [77]:

Corollary 4.3. *There is an arc between every two elements of a hereditarily locally connected continuum.*

Proof. Let X be a hereditarily locally connected continuum, and let $x, y \in X$, $x \neq y$. By Theorem 2.16, let C be an irreducible continuum between x and y. By Lemma 4.2, C is an arc. $\square$

5. Strong T-sets

For the results in this section, see Bula, Nikiel, Tuncali, Tymchatyn [92], remark 5.2.

Lemma 5.1. *Suppose that Z is a strong T-set in a locally connected continuum X, P is a component of $X \backslash Z$, and u and v are distinct elements of $X \backslash P$. Then either $\forall x \in P$(x separates u and v), or $\forall x \in P$(x does not separate u and v).*

Proof. Note that P is open in X by Lemma 2.13. Suppose that $x \in P$ separates u from v and $y \in P$ does not separate u from v. Let U be the component of u and v in $X \backslash \{y\}$. Thus $x \in U$ since x separates u from v. Let s and t be the endpoints of P. Say $s < x < y < t$ in the ordering of $P \cup \{s, t\}$. Choose z such that $x < z < y$. Now $\{w : x \leq$

$w < z\}$ and U are connected, x is in both, and y is in neither, so $\{w : s \leq w < z\} \subseteq U$. By Lemma 2.10, $U\backslash\{w : s < w < z\}$ is still connected. But $u, v \in U\backslash\{w : s < w < z\}$, contradicting that x separates u and v. □

Lemma 5.2. *If Z is a strong T-set in a dendron D, P is a component of $D\backslash Z$, and $x \in P$, then $D\backslash\{x\}$ has exactly two components.*

Proof. Let u and v be the endpoints of P. Thus $u, v \in Z$. First we show

(1) If $w \in D$ separates u and v, then $w \in P$.

For, suppose on the contrary that $w \notin P$. Let U be the component of u in $D\backslash\{w\}$, and V the component of v. Now $P \cup \{u, v\}$ is connected, being homeomorphic to $[0, 1]$. Hence $P \cup \{u, v\} \cup U \cup V$ is connected, contradiction. So, (1) holds.

(2) $E(u, v)\backslash\{u, v\} = P$.

This follows by Lemma 5.1. (D is locally connected by Lemma 3.17.)

Now suppose that there are at least 3 components of $D\backslash\{x\}$. Then there is a $w \in D\backslash\{x\}$ which lies in a different component from u and v. Hence each member of P is in $E(u, v) \cap E(u, w) \cap E(v, w)$, contradicting Lemma 3.14. □

Lemma 5.3. *Let Z be a zero-dimensional compact set which is a strong T-set in a dendron D. For each component P of $D\backslash Z$, pick $x_P \in P$, and let u_P and v_P be the two components of $D\backslash\{x_P\}$. Then*

$$\{Z \cap u_P : P \text{ a component of } D\backslash Z\} \cup \{Z \cap v_P : P \text{ a component of } D\backslash Z\}$$

separates the points of Z.

Proof. Suppose u and v are distinct points of Z; we want to find a component P of $D\backslash Z$ such that x_P separates u from v. By Lemma 5.1 it suffices to show that $E(u, v)\backslash Z \neq 0$. Suppose on the contrary that $E(u, v) \subseteq Z$. Since $E(u, v)$ is connected with at least two elements, this is impossible. □

6. Treybig's lemma

The following is Lemma 4 in Treybig [64].

Theorem 6.1. *Suppose that K_1 is a compact linearly ordered space, and f_1 is a continuous function from K_1 onto a Hausdorff space S. Then exist a compact linearly ordered space K_2 and a continuous function f_2 from K_2 onto S such that*

(i) If L is a proper closed subspace of K_2, then $f_2[L] \neq S$;

(ii) If $x, y \in K_2$, $x \neq y$, and $f_2x = f_2y$, there there is a z in K_2 between x and y such that $f_2z \neq f_2x$.

Proof. Let

$$\mathscr{H} = \{C : C \text{ is a closed subspace of } K_1 \text{ and } f_1[C] = X\}.$$

Suppose that $\mathscr{J}$ is a nonempty subset of $\mathscr{H}$ linearly ordered by $\subseteq$. Then for any $y \in S$, the set $\mathscr{J} \cup f^{-1}[\{y\}]$ has fip, so $\bigcap \mathscr{J} \cup f^{-1}[\{y\}] \neq 0$. Hence $\bigcap \mathscr{J} \in \mathscr{H}$. So by Zorn's lemma let K be a $\subseteq$-minimal element of $\mathscr{H}$.

For all $x \in K$ let

$$K_x = \{y \in K : (y < x \text{ and } |f_1[[y,x] \cap K]| = 1) \text{ or } (x \le y \text{ and } |f_1[[x,y] \cap K]| = 1\}.$$

Then

(1) K_x is closed in K.

For, suppose that $y \in K \backslash K_x$. Say $y < x$. Thus $|f_1[[y,x] \cap K]| > 1$.

Case 1. $f_1 y \neq f_1 x$. Let U and V be disjoint neighborhoods of these in S. Then $y \in f_1^{-1}[U] \cap K \subseteq K \backslash K_x$.

Case 2. $f_1 y = f_1 x$. Then there is a $w \in [y,x] \cap K$ such that $f_1 w \neq f_1 x$. So $y \in (-\infty, w) \cap K \subseteq K \backslash K_x$.

So, (1) holds. The following two statements are clear:

(2) If $u, v \in K_x$ and $u < v$, then $[u,v] \cap K \subseteq K_x$.
(3) If $K_x \neq K_y$, then $K_x < K_y$ or $K_y < K_x$.

Let $K_2 = \{K_x : x \in K\}$. Declare U to be open in K_2 iff $\bigcup U$ is open in K.

(4) This topology on K_2 is the same as its order topology inherited from K_1.

To see this, first suppose that U is open in the above sense. Suppose that $K_x \in U$. Thus $K_x \subseteq \bigcup U$. By compactness and (1), $K_x \subseteq (a_1, b_1) \cup \ldots \cup (a_n, b_n)$ with each $(a_i, b_i) \subseteq \bigcup U$, all a_i, b_i in K (using Theorem 2.6). We may assume that $K_x \cap (a_1, b_1) \neq 0$, and $(a_i, b_i) \cap (a_j, b_j) = 0$ if $i \neq j$. It follows that $a_1, b_1 \notin K_x$. Hence $K_{a_1} < K_x < K_{b_1}$. So $K_x \in (K_{a_1}, K_{b_1})$. If $K_y \in (K_{a_1}, K_{b_1})$, then $K_y \subseteq (a_1, b_1) \subseteq \bigcup U$. Hence $K_y \in U$, since otherwise $K_y \cap \bigcup U = 0$.

Now suppose that U is open in the order topology. Suppose that $x \in \bigcup U$. Thus $K_x \in U$. Say $K_x \in (K_y, K_z) \subseteq U$. Then $x \in (y, z) \subseteq \bigcup U$, as desired. Thus (4) holds.

Now for any $x \in K$ let $f_2 K_x = f_1 x$. Clearly f_2 maps K_2 onto S. Let $\pi : K \to K_2$ be the natural mapping. So $f_2 \circ \pi = f_1$, and f_2 is continuous by Engelking 2.4.2.

If L is a proper closed subspace of K_2, let $M = \pi^{-1}[L]$. Then $f_2[L] = f_1[M] \neq S$.

Finally, suppose that $K_x \neq K_y$ and $f_2 K_x = f_2 K_y$. Say $K_x < K_y$. Then $x \notin K_y$, so $|f_1[[x,y] \cap K]| > 1$. Say $x < z < y$, $z \in K$, and $f_1 z \neq f_1 x$. Thus $K_x < K_z < K_y$ and $f_2 K_z \neq f_2 K_x$, as desired. $\square$

7. Nikiel's theorem

The following theorem is Theorem 2 in Nikiel [86].

Theorem 7.1. *If X is a Hausdorff space which is a continuous image of a compact linearly ordered space, then there is a continuum Y such that Y is a continuous image of an arc, $X \subseteq Y$, and X is a strong T-set in Y.*

Proof. By Theorem 6.1 we may assume that f is a continuous function from K onto X, where K is a compact linearly ordered space and

(1) if L is a proper closed subspace of K, then $f[L] \neq X$.
(2) If $x, y \in K$, $x \neq y$, and $fx = fy$ then there is a z between x and y such that $fz \neq fx$.

Let I be obtained from K by the procedure of the proof of Lemma 2.11. For $x, y \in I$, define $x \sim y$ iff $x = y$, or else $x, y \in K$ and $fx = fy$. Let Y be the quotient space $I/\sim$ and $F : I \to I/\sim$ the quotient map. Define $g : X \to I/\sim$ by setting $gfy = y/\sim$ for all $y \in K$; clearly g is well-defined and one-one. Clearly

(3) Y is connected.

(4) g is a homeomorphism from X onto $g[X]$ (the latter as a subspace of Y).

To prove (4) it suffices to show that if U is closed in Y, then $g^{-1}[U \cap g[X]]$ is closed in X. Now $F^{-1}[U] \cap K$ is closed in K, and hence $f[F^{-1}[U] \cap K]$ is closed in X. So it suffices to show that $g^{-1}[U \cap g[X]] = f[F^{-1}[U] \cap K]$. Let $x \in g^{-1}[U \cap g[X]]$. Say $x = fy$, $y \in K$. Then $gx = gfy = (y/\sim) = Fy \in U$, so $y \in F^{-1}[U] \cap K$ and $x \in f[F^{-1}[U] \cap K]$. Conversely, suppose that $x \in f[F^{-1}[U] \cap K]$. Say $x = fy$ with $y \in F^{-1}[U] \cap K$. Thus $gx = gfy = y/\sim = Fy \in U$, so $x \in g^{-1}[U \cap g[X]]$. so (4) holds.

(5) $g[X]$ is a strong T-set in Y.

$g[X]$ is closed in Y by (4). Now $g[X] = \{x/\sim : x \in K\}$, so $Y \backslash g[X]$ is a disjoint union of copies of $(0, 1)$ whose endpoints are distinct members of $g[X]$ since in the construction of I from K the intervals $(0, 1)$ were added between points x, y with $x \lessdot y$, hence by (2) with $fx \neq fy$. $\square$

The following lemma is Lemma 4.3 of Nikiel [88].

Theorem 7.2. *If X is a locally connected continuum and A is a zero-dimensional strong T-set in X, then X is rim-finite.*

Proof. Let $x \in X$, U an open neighborhood of x. If $x \in X \backslash A$, then x is in a copy of $(0, 1)$ and hence has an open neighborhood contained in U with two-element boundary (see Lemma 2.13). Suppose that $x \in A$.

(1) There is an open neighborhood $V \subseteq U$ of x such that $\mathrm{bd}V \cap A = 0$.

For, choose W clopen in A such that $W \subseteq U$. Then W and $(A \backslash W) \cup (X \backslash U)$ are disjoint closed sets in X. Let V, S be disjoint open sets containing these two closed sets, respectively. Hence $V \subseteq U$. Suppose $y \in \mathrm{bd}V \cap A$. Then $y \in \overline{V}$, so $y \in X \backslash S$. Hence $y \in W \backslash V$, contradiction. So (1) holds.

(2) $\mathrm{bd}V$ intersects only finitely many components of $X \backslash A$.

For, if it intersects infinitely many of them, we get a sequence $\langle x_n : n < \omega \rangle$ of elements of $\mathrm{bd}V$ in distinct components of $X \backslash A$. A cluster point then has to be in $\mathrm{bd}V \cap A$, contradiction.

Let W be the component of V containing x. Then W is open by Lemma 2.13. Moreover,

(3) $\mathrm{bd}W \subseteq \mathrm{bd}V$.

For, let $y \in \mathrm{bd}W$. Thus $y \in \overline{W} \backslash W$. Hence $y \in \overline{V}$. If $y \in V$, then $W \cup \{y\}$ is connected, hence $y \in W$, contradiction; so $y \notin V$. Hence $y \in \mathrm{bd}V$, and (3) holds.

Thus by the above we have $\mathrm{bd}W \cap A = 0$, and $\mathrm{bd}W$ intersects $X\backslash A$ in only finitely many components. We claim, then, that $\mathrm{bd}W$ is finite. To prove this, we may assume that W is not contained in a single component of $X\backslash A$. Suppose that I is a component of $X\backslash A$ and $\mathrm{bd}W$ intersects I in three points $u < v < w$. Then $W \cap (u, w)$ and $W\backslash[u, w]$ is a decomposition of W into disjoint nonempty open sets, contradiction. $\square$

The following result from Nikiel [89] is the main result of these notes.

Theorem 7.3. *If X is a zero-dimensional Hausdorff space which is a continuous image of a compact linearly ordered space, then X can be embedded as a strong T-set in a dendron.*

Proof. By Theorem 7.1, X can be embedded as a strong T-set in a continuum Y which is a continuous image of an arc. By Corollary 3.16 and Lemma 2.12, Y is locally connected. By Theorem 7.2, Y is rim-finite.

Let $\langle P_\alpha : \beta < \kappa\rangle$ be a one-one enumeration of all of the components of $Y\backslash X$. Now we define a sequence $\langle Y_\beta : \beta < \kappa\rangle$ of subcontinua of Y all containing X. Let $Y_0 = Y$. If Y_δ has been constructed, if $Y_\delta\backslash P_\delta$ is connected let $Y_{\delta+1} = Y_\delta\backslash P_\delta$, and otherwise let $Y_{\delta+1} = Y_\delta$. If $\gamma < \kappa$ is limit and Y_δ has been defined for all $\delta < \gamma$, let $Y_\delta = \bigcap_{\delta<\gamma} Y_\delta$. So Y_δ is a continuum by Theorem 2.7.

Let $Z = \bigcap_{\beta<\kappa} Y_\beta$. So Z is a subcontinuum of Y, and $X \subseteq Z$. Clearly X is a strong T-set in Z. We now show that Z is a dendron. By Lemma 2.2, Z is rim-finite. Now Z is hereditarily locally connected by Lemma 4.1. So by Corollary 4.3, there is an arc I in Z with endpoints x, y. Since X is zero-dimensional, we cannot have $I \subseteq X$. Say $z \in I\backslash X$. Say $z \in P_\beta$, $\beta < \kappa$. Thus $P_\beta \subseteq Z \subseteq Y_{\beta+1}$, so by construction, $Y_\beta\backslash P_\beta$ is not connected.

(1) $Z\backslash P_\beta$ is not connected.

For, $Y_\beta\backslash P_\beta$ is obtained fron $Z\backslash P_\beta$ by adjoining certain connected sets $P_\gamma\cup\{a_\gamma\}$, $\beta < \gamma < \kappa$, with $a_\gamma \in X \subseteq Z\backslash P_\beta$, and (1) follows.

Let U and V be nonempty disjoint open sets in $Z\backslash P_\beta$ such that $U\cup V = Z\backslash P_\beta$. Write $\mathrm{bd}P_\beta = \{a, b\}$ with a, b distinct elements of X.

(2) $U \cap \mathrm{bd}P_\beta \neq 0$.

For, otherwise $Z = U \cup (V \cup P_\beta)$ shows that Z is not connected. Similarly:

(3) $V \cap \mathrm{bd}P_\beta \neq 0$.

Say $a \in U$ and $b \in V$. Let $<$ be an order of $\mathrm{cl}P_\beta = P_\beta \cup \{a, b\}$ which gives the relative topology on it with $a < b$. For any $u \in P_\beta$ let

$$U_u^< = U \cup \{t \in \mathrm{cl}P_\beta : a \le t < u\},$$
$$V_v^< = V \cup \{t \in \mathrm{cl}P_\beta : u < t \le b\}.$$

(4) $U_u^<$ and $V_u^<$ are nonempty disjoint open sets in Z with $U_u^< \cup V_u^< = Z\backslash\{u\}$.

To check that $U_u^<$ is open, suppose that $s \in U_u^<$. If $s \in P_\beta$, then $s \in (a, u)_< \subseteq U_u^<$, as desired. If $s \notin P_\beta$, choose W oepn in Z such that $W \cap (Z\backslash P_\beta) = U$. Note that $\{t \in \mathrm{cl}P_\beta : u \le t\}$ is closed, so $W\backslash\{t \in \mathrm{cl}P_\beta : u \le t\}$ is open, and

$$s \in W\backslash\{t \in \mathrm{cl}P_\beta : u \le t\} \subseteq U_u^<,$$

as desired. The other parts of (4) are clear, or similar.

(5) If $a \leq u < z$ and $u \in I$, then $[u, z]_< \subseteq I$.

For, suppose that $u < s < z$ and $s \notin I$. then $I = (I \cap U_s^<) \cup (I \cap V_s^<)$ contradicts I connected. Similarly:

(6) If $z < u \leq b$ and $u \in I$, then $[z, u]_< \subseteq I$.

(7) If $U \cap I \neq 0$, then $[a, z]_< \subseteq I$.

One gets $(a, z]_< \subseteq I$ as for (5), and then $a \in I$ too since I is closed. Similarly: (8) If $V \cap I \neq 0$, then $[z, b]_< \subseteq I$.

Now we consider four cases.

Case 1. $x, y \in Z \subseteq P_\beta$. Then

(9) $(x, z)_I \cap \mathrm{bd} P_\beta \neq 0$.

Otherwise $(x, z)_I = [(x, z)_I \cap P_\beta] \cup [(x, z)_I \backslash \overline{P}_\beta]$, contradiction, since $(x, z)_I$ is connected. Similarly:

(10) $(z, y)_I \cap \mathrm{bd} P_\beta \neq 0$.

Say $x <_I a <_I z <_I b <_I y$. Now we form $U' = U_z^<$ and $V' = V_a^<$. Suppose that x and y are in the same component K of $Z \backslash \{z\}$. Now $(z, \infty)_I$ is connected and $y \in K \cap (z, \infty)_I$, so $(z, \infty)_I \subseteq K$. Similarly, $(-\infty, z)_I \subseteq K$. But then $K = (u' \cap K) \cup (V' \cap K)$ gives a contradiction.

Case 2. $x \in P_\beta$, $y \notin P_\beta$. Then as for (10),

(11) $(z, y)_I \cap \mathrm{bd} P_\beta \neq 0$.

Say $z <_I b <_I y$. Now let

$$U' = U \cup (P_\beta \backslash [z, b]_I),$$
$$V' = V \cup (z, b]_I.$$

Then as before, V' is open. To show that U' is open, suppose that $s \in U'$. If $s \in P_\beta$, then $s \in P_\beta \backslash [z, b]_I \subseteq U'$, and $P_\beta \backslash [z, b]_I$ is open. If $s \notin P_\beta$, choose W open in Z such that $W \cap (Z \backslash P_\beta) = U$. Now $W \backslash [z, b]_I$ is open, and $s \in W \backslash [z, b]_I \subseteq U'$. This gives a contradiction.

Case 3. $x \notin P_\beta$, $y \in P_\beta$. This is similar to Case 2.

Case 4. $x, y \in P_\beta$. This case is easy. $\square$

8. Proof of the main theorem

Suppose that Z is a compact zero-dimensional Hausdorff space which is the continuous image of a compact linearly ordered space. Then Z can be embedded as a strong T-set in a dendron Q. Let $\mathbf{K}(Q \backslash Z)$ be the collection of all components of $Q \backslash Z$. Fix $A \in \mathbf{K}(Q \backslash Z)$. Let v be one of the components of $Q \backslash \{x_A\}$, and for each $N \in \mathbf{K}(Q \backslash Z)$ different from A let u_N be the component of $Q \backslash \{x_P\}$ which does not contain x_A, and let v_N be the other component. We claim that $\{v \cap Z\} \cup \{u_N \cap Z : N \neq A\}$ is a ramification set generating clop$Z$. It is a collection of clopen sets generating clop$Z$. Suppose that $u_N \cap v \cap Z \neq 0$.

Then $u_N \cup v$ is connected and x_A is not in it, so $u_N \subseteq v$. Now take distinct $N, P \neq A$. *Case 1.* $x_N \in u_P$. Now $x_A \in v_N \cap v_P$, so $v_N \cup v_P$ is connected. Also, $x_N \notin v_N \cup v_P$. It follows that $v_P \subseteq v_N$, and hence $u_N \subseteq u_P$. *Case 2.* $x_N \notin u_P$. Suppose that $u_P \cap u_N \cap Z \neq 0$. Then $u_P \cup u_N$ is connected and x_N is not in it, so $u_P \subseteq u_N$.

References

Bula, W., Nikiel, J., Tuncali, H. M., Tymchatyn, E. D. [92] *Continuous images of ordered compacta are regular supercompact.* Topology and its Applications 45, 203–221.

Engelking, R. [77] **General topology**. Polish Scientific Publishers, Monografie Matematyczne, v. 60, 626pp.

Koppelberg, S. [89] **Handbook of Boolean algebras, vol. 1.** North-Holland, xix + 312pp.

van Mill, J., Wattel, E. [81] *Dendrons.* Topology and Order Structures, part I, Math. Centre Tracts 142, Math. Centrum, Amsterdam, 59–82.

Nikiel, J. [86] *Some problems on continuous images of compact ordered spaces.* Questions and Answers in General Topology 4, 117–129.

Nikiel, J. [88] *Images of arcs—a nonseparable version of the Hahn-Mazurkiewicz theorem.* Funda. Math. 129, 91–120.

Nikiel, J. [89] *Orderability properties of a zero-dimensional space which is a continuous image of an ordered compactum.* Topology and its Applications 31, 269–276.

Purisch, S. [94] In Topology Proceedings, 17, Problem section.

Treybig, L. B. [64] *Concerning continuous images of compact ordered spaces.* Proc. Amer. Math. Soc. 15, 866–871.

Tymchatyn, E. D. [77] *The Hahn-Mazurkiewicz theorem for finitely Suslinian continua.* General Topology and its Appl. 7, 123–127.

Willard, S. [70] **General topology.** Addison-Wesley, xii + 369pp.