Notes on Cardinal invariants on Boolean algebras
(Second revised edition)

(1) Problem 49 has been solved by Malliaris and Shelah, who showed that $p(\mathcal{P}(\omega)/\text{fin}) = \text{tow}(\mathcal{P}(\omega)/\text{fin})$ in ZFC. See Malliaris, M.; Shelah, S. Cofinality spectrum theorems in model theory, set theory, and general topology. Publication 998 of Shelah.

(2) Problem 90 has been solved by Kunen, who showed assuming $2^{\aleph_1} = \aleph_2$ that there is an atomic BA $A$ such that $\pi(A) = \aleph_1 < \text{Irr}_{\text{mm}}(A)$. See Kunen, K. Irredundant sets in atomic Boolean algebras. http://arxiv.org/abs/1307.3533

(3) Problem 126 was answered, consistently: there is a model with $s_{\text{mm}}(\mathcal{P}(\omega)/\text{fin}) < i(\mathcal{P}(\omega)/\text{fin})$. See Cancino, J.; Guzmán, O.; Miller, A. Irredundant generators.

(4) Problem 149 is solved by the following result, which shows that $\text{hd}_{\text{mm}}$ is trivial:

**Proposition.** $\text{hd}_{\text{mm}}(A) = \omega$ for any infinite BA $A$.

**Proof.** Let $\langle b_n : n \in \omega \rangle$ be a partition in $\overline{A}$. Define

$$I_n = \{a \in A : \forall m < n[a \cdot b_m = 0]\}.$$ 

Clearly $I_n \supseteq I_p$ if $n < p$. Suppose that $a \in \bigcap_{n \in \omega} I_n$ and $a \neq 0$. Choose $n \in \omega$ so that $a \cdot b_n \neq 0$, and choose $c \in A^+$ so that $c \leq a \cdot b_n$. Then $c \in I_{n+1}$, so $c \cdot b_n = 0$, contradiction.

(5) Problems 156, 157 and 158 were answered by M. Hrusak. He showed that if $|X| < \min\{\pi(B \upharpoonright a) : a \in B^+\}$ with $0, 1 \notin X$, $X$ incomparable, then there is an element incomparable with each member of $X$. In particular, $\text{Inc}_{\text{mm}}(\mathcal{P}(\omega)/\text{fin}) = 2^\omega$.

http://mathoverflow.net/questions/154708/families-of-pairwise-incomparable-subsets-of-the-integers

We give the argument here. Let $C$ be the subalgebra of $B$ generated by $X$; then also $|C| < \min\{\pi(B \upharpoonright a) : a \in B, a \neq 0\}$. Take any $a \in X$. Then $a \neq 0, 1$. Since $|C| < \pi(B \upharpoonright -a)$, there is an $x \leq -a$ with $x \neq 0$ such that no nonzero element of $C$ is below $x$. Similarly there is a $y \leq a$ with $y \neq 0$ such that no nonzero element of $C$ is below $y$. Let $b = x + a \cdot -y$. Now $a \not\leq b$, for if $a \leq b$ then $a = b \cdot a = a \cdot -y \leq -y$, hence $y = a \cdot y = 0$, contradiction. Also, $b \not\leq a$, as otherwise $x \leq a$, hence $x = 0$, contradiction. Now suppose that $c \in X$ and $c \leq b$. Then $c \cdot -a \leq x$, $c \cdot -a \in C$, and $c \cdot -a \neq 0$, contradiction. Suppose that $c \in X$ and $b \leq c$. Then $a \cdot -y \leq c$, so $a \cdot -c \leq y$, again a contradiction. So $b$ is incomparable with each element of $X$.

Also, Hrusak showed that it is consistent with $\neg \text{CH}$ that there is a maximal tree in both $\mathcal{P}(\omega)$ and $\mathcal{P}(\omega)/\text{fin}$ of size $\omega_1$.  


Problem 160 has a negative solution, at least for atomless BAs. Namely, we claim that $h\text{-cof}_{\text{mm}}(A) = \omega$ for any atomless BA $A$. For, let $\langle a_i : i \in \omega \rangle$ be a system of pairwise disjoint nonzero elements of $A$, and for each $i \in \omega$ let $\langle b_j^i : j \in \omega \rangle$ be a system of pairwise disjoint nonzero elements less than $a_i$. We consider the following sequence

$$-a_0, -a_1, -a_2, \ldots \text{(rank 0)}$$
$$-a_1 + b_0^1, -a_2 + b_0^2, -a_3 + b_0^3 \ldots \text{(rank 1)}$$
$$-a_2 + b_0^2 + b_1^2, -a_3 + b_0^3 + b_1^3, -a_4 + b_0^4 + b_1^4 \ldots \text{(rank 2)}$$

$\ldots$.\ldots$

This gives an $h$-cof sequence of length $\omega^2$. Suppose that $c$ is adjoined at the end. Then $c$ has rank $\omega$, so it includes cofinally many sets $-a_n$. Since $-a_n + -a_m = 1$ for $n \neq m$, it follows that $c = 1$, contradiction.