This is an update of Monk [14]; mainly we indicate solutions of some of the problems. We give a question mark for ones which have not been checked.

Problem 7. *Does $A \leq_s B$ imply that $\mathfrak{a}(B) \leq \mathfrak{a}(A)$?*

This is answered positively by Santos [23]. We give his argument here.

Proposition 1. *If A is finite and B is infinite, then $\mathfrak{a}(A \oplus B) = \mathfrak{a}(B)$.*

Proof. By 11.6 of Koppelberg [89], $A \oplus B \cong {}^nB$, where n is the number of atoms of A. By Proposition 3.36 of Monk [14], $\mathfrak{a}({}^nB) = \mathfrak{a}(B)$. □

Proposition 2. *Let A be an infinite BA and let I be an ideal of A such that A/I is infinite. Suppose that $\mathfrak{a}(A) < \mathfrak{a}(A/I)$. Suppose that $\langle a_\alpha : \alpha < \kappa\rangle$ is an infinite partition of A of minimum size.*

Then there exist $b \in A^+$ and $E \in [\kappa]^\kappa$ such that $\langle a_\alpha \cdot b : \alpha \in E\rangle$ is an infinite partition of $A \restriction b$ of minimum size consisting of elements of I.

Proof. *Case 1.* $\sum_{\alpha<\kappa}[a_\alpha]_I = [1]_I$. Note that $\forall\alpha,\beta < \kappa[\alpha \neq \beta \rightarrow [a_\alpha]_I] \cdot [a_\beta]_I = 0]$. Since A/I is infinite and A/I has no infinite partition of size $\leq \kappa$, there is an $F \in [\kappa]^{<\omega}$ such that $\sum_{\alpha\in F}[a_\alpha]_I = [1]_I$. Thus $b \stackrel{\text{def}}{=} -\sum_{\alpha\in F} a_\alpha \in I$. Hence $\forall\alpha < \kappa[a_\alpha \cdot b \in I]$. Let $E = \{\alpha < \kappa : a_\alpha \cdot b \neq 0\}$.

$$\neg\exists E' \in [E]^{<\omega}\left[b = \sum_{\alpha\in E'}(a_\alpha \cdot b)\right] \tag{1}$$

For, assume that $E' \in [E]^{<\omega}$ and $b = \sum_{\alpha\in E'}(a_\alpha \cdot b)$. Then

$$1 = b + -b = \sum_{\alpha\in E'}(a_\alpha \cdot b) + \sum_{\alpha\in F} a_\alpha \leq \sum_{\alpha\in E'\cup F} a_\alpha,$$

contradiction. So (1) holds.

$$b = \sum_{\alpha\in E}(a_\alpha \cdot b) \tag{2}$$

In fact, otherwise there is a nonzero $c < b$ such that $\forall\alpha \in E[a_\alpha \cdot c = 0]$. Hence $\forall\alpha < \kappa[a_\alpha \cdot c = 0]$, contradicting $\langle a_\alpha : \alpha < \kappa\rangle$ being a partition of A.

By (1) and (2), $\langle a_\alpha \cdot b : \alpha \in E\rangle$ is an infinite partition of $A \restriction b$. Suppose that $|E| < \kappa$. Then $\langle a_\alpha \cdot b : \alpha \in E\rangle^\frown\langle\{b\}\rangle$ is a partition of A of size less than $\mathfrak{a}(A)$, contradiction.

Case 2. $\sum_{\alpha<\kappa}[a_\alpha]_I < [1]_I$. Then there is a $b \in A\backslash I$ such that $\forall\alpha < \kappa[[b] \cdot [a_\alpha]_I = 0]$. Let $E = \{\alpha < \kappa : a_\alpha \cdot b \neq 0\}$.

$$\forall E' \in [E]^{<\omega}\left[b \neq \sum_{\alpha\in E'}(b \cdot a_\alpha)\right]. \tag{3}$$

For, if $E' \in [E]^{<\omega}$ and $b = \sum_{\alpha\in E'}(b \cdot a_\alpha)$, then $b \in I$, contradiction.

It follows that $\langle a_\alpha \cdot b : \alpha \in E\rangle$ is a partition of $A \upharpoonright b$. As in Case 1, $|E| = \kappa$. □

Theorem 3. *If $A \leq_s B$, then $\mathfrak{a}(B) \leq \mathfrak{a}(A)$.*

Proof. By Monk [14] Proposition 2.28 let I_0 and I_1 be ideals of A such that $I_0 \cap I_1 = \{0\}$ and $B \cong (A/I_0) \times (A/I_1)$. Wlog $B = (A/I_0) \times (A/I_1)$.

Case 1. One of $A/I_0, A/I_1$ is finite; say by symmetry that A/I_0 is finite. By Proposition 1, $\mathfrak{a}(B) = \mathfrak{a}(A/I_1)$, Suppose that $\kappa = \mathfrak{a}(A) < \mathfrak{a}(A/I_1(A)$. Let $\langle a_\alpha : \alpha < \kappa\rangle$ be an infinite partition of A. Choose b and E as in Proposition 2. Since A/I_0 is finite, there exist $\alpha < \beta$ in E such that $b \cdot a_\alpha \equiv_{I_0} b \cdot a_\beta$ while $b \cdot a_\alpha \neq b \cdot a_\beta$. Hence $0 \neq (b \cdot a_\alpha)\triangle(b \cdot a_\beta) \in I_0 \cap I_1$, contradiction.

Case 2. A/I_0 and A/I_1 are both infinite. By Proposition 3.36 of Monk [14] we have $\mathfrak{p}(B) = \min\{\mathfrak{p}(A/I_0), \mathfrak{p}(A/I_1)\}$. Suppose that $\mathfrak{p}(A) < \mathfrak{p}(B)$. Let $\langle a_\alpha : \alpha < \kappa\rangle$ be an infinite partition of A. Let b and E be given by Proposition 2 Then $\langle a_\alpha \cdot b : \alpha \in E\rangle^\frown\{-b\}$ is a partition of A, with each $a_\alpha \cdot b \in I_0$. Let $\langle a'_\alpha : \alpha < \lambda\rangle$ be a one-one enumeration of $\{a_\alpha \cdot b : \alpha \in E\} \cup \{-b\}$. Applying Proposition 2 to this partition and I_1 we get b' and E' such that $\langle a'_\alpha \cdot b' : \alpha \in E'\rangle$ is an infinite partition of $A \upharpoonright b'$ of minimum size consisting of elements of I_1. Thus $\sum_{\alpha\in E'}(a'_\alpha \cdot b') = b'$ and $\sum_{\alpha<\lambda} a'_\alpha = 1$. Choose $\alpha < \lambda$ such that $a'_\alpha = a_\beta \cdot b$ for some β. Then $0 \neq a'_\alpha \cdot b' = a_\alpha \cdot b \cdot b' \in I_0 \cap I_1$, contradiction. □

Problem 8. *Is it true that for all infinite BAs A, B one has*

$$\mathfrak{a}(A \oplus B) = \min\{\mathfrak{a}(A), \mathfrak{a}(B)\}?$$

This problem is still open, but there are several partial results concerning it.

?Kurilić [17] showed that $\omega \leq \min\{\mathfrak{a}(A), \mathfrak{a}(B), \mathfrak{p}(A), \mathfrak{p}(B) \leq \mathfrak{a}(A \oplus B)$

However note that Fact 3.3 in that paper is false; a counter example is given by setting each $K_i = \{i\}$.

In Santos [23] the following is shown:

$$?\min\{\min\{\mathfrak{a}(A), \mathfrak{a}(B)\}, \max\{\mathfrak{p}(A), \mathfrak{p}(B)\}\} \leq \mathfrak{a}(A \oplus B).$$

In Santos [25] the following is shown. If A and B are homogeneous BAs, then

$$?\min\{\mathfrak{a}(A), \mathfrak{a}(B), \max\{\mathfrak{s}(A), \mathfrak{s}(B)\}\} \leq \mathfrak{a}(A \oplus B).$$

?If $\omega_1 \leq \mathfrak{a}(A), \mathfrak{a}(B)$, then

$$\min\{\mathfrak{a}(A, \mathfrak{a}(B), \max\{\mathfrak{b}(A), \mathfrak{b}(B)\}\} \leq \mathfrak{a}(A \oplus B).$$

Problem 37. *Does $A \leq_s B$ imply that* $\mathrm{tow}_{\mathrm{spect}}(A) \subseteq \mathrm{tow}_{\mathrm{spect}}(B)$ *or* $\mathrm{tow}(B) \leq \mathrm{tow}(A)$*?*

?This was answered in part positively by Santos [23].

Problem 38. *Are there BAs A, B such that $A \leq_m B$ and* $\mathrm{tow}(B) < \mathrm{tow}(A)$*?*

?This was answered positively by Santos [23].

Problem 44. *Does $A \leq_s B$ imply that $\mathfrak{p}_{\mathrm{spect}}(A) \subseteq \mathfrak{p}_{\mathrm{spect}}(B)$?*
?This was answered in part by Santos [13].

Problem 45. *Are there BAs A, B such that $A \leq_m B$ and $\mathfrak{p}(B) < \mathfrak{p}(A)$?*

? This was answered positively by Santos [23].

Problem 46. *Is it true that for all infinite BAs A, B we have $\mathfrak{p}(A \oplus B) = \min\{\mathfrak{p}(A), \mathfrak{p}(B)$?*

This was answered positively by Santos [23]. We give his argument here.

Proposition 4. *For all infinite BAs A, B we have $\mathfrak{p}(A \oplus B) = \min\{\mathfrak{p}(A), \mathfrak{p}(B)\}$*

Proof. $\leq$ is clear. For $\geq$, assume that $X \subseteq A \oplus B$, $\prod X = 0$, and $\forall F \in [X]^{<\omega}[\prod F \neq 0]$, with $|X| = \mathfrak{p}(A \times B) < \min\{\mathfrak{p}(A), \mathfrak{p}(B)\}$. For each $x \in X$ write $x = \sum_{i \in m_x}(a_{ix} \cdot b_{ix})$, with each $a_{ix} \in A$ and each $b_{ix} \in B$. For each $F \in [X]^{<\omega}$ let

$$C_F = \left\{ f \in \prod_{x \in X} m_x : \prod_{x \in F}(a_{f(x),x} \cdot b_{f(x)x}) \right\} \neq 0.$$

We claim that each C_F is nonempty. For,

$$0 \neq \prod F = \prod_{x \in F} \sum_{i \in m_x}(a_{ix} \cdot b_{ix}) = \sum_{f \in \prod_{x \in F} m_x} \left(\prod_{x \in F}(a_{f(x)x} \cdot b_{f(x)x}) \right),$$

so there is an $f \in \prod_{x \in F} m_x$ such that $\prod_{x \in F}(a_{f(x)x} \cdot b_{f(x)x}) \neq 0$. Thus any extension of f to a member of $\prod_{x \in X} m_x$ is in C_f. So C_f is nonempty.

Now each C_f is closed in $\prod_{xinX} m_x$. For, suppose that $f \in \prod_{x \in X} m_x \backslash C_F$. Thus $\prod_{x \in F}(a_{f(x)x} \cdot b_{f(x))x}) = 0$. Then f is in the open set $\{g \in \prod_{x \in X} m_x : g \restriction F = f\}$, and this set is disjoint from C_F.

Now let $f \in \bigcap_{F \in [X]^{<\omega}} C_F$. Then $\forall F \in [X]^{<\omega}(\prod_{x \in F}(a_{f(x)x} \cdot b_{f(x)x})) \neq \emptyset$. Hence $\forall F \in [X]^{<\omega}[\prod_{x \in F} a_{f(x)x} \neq 0]$, so $\prod_{x \in X} a_{f((x)x} \neq 0$. Similarly $\prod_{x \in X} b_{f((x)x} \neq 0$. Hence

$$0 \neq \prod_{x \in X} a_{f((x)x} \cdot \prod_{x \in X} b_{f((x)x} \leq \prod X = 0.$$

contradiction. □

Problem 48. *Is*

$$\mathfrak{p}(A) = \min\{|X| : X \textit{ is a maximal ramification set in } A\}?$$

?This is solved negatively in Santos [23].

Problem 49. *Is it consistent that* $\mathfrak{p}(\mathscr{P}(\omega)/\text{fin} < \text{tow}(\mathscr{P}(\omega)/\text{fin}$*?*

?This is answered positively by Malliaris, Shelah [17].

Problem 52. *Is* $\text{spl}(A \oplus B) = \min\{\text{spl}(A), \text{spl}(B)\}$*?*

?This is answered positively by Santos [23].

Problem 70. *Are there BAs* A, B *such that* $A \leq_\sigma B$ *and* $\pi(A) > \pi(B)$*?*

?This is answered negatively by Santos [23].

Problem 71. *Are there BAs* A, B *such that* $A \leq_s B$ *and* $\pi(A) > \pi(B)$*?*

?This is answered negatively by Santos [23].

Problem 86. *Can one have* $\text{Irr}(A) < \text{Irr}(B)$ *for* $A \leq_s B$ *or* $A \leq_m (B)$*?*

This was answered negatively by Santos [23]. We give his argument bere.

Proposition 5. *Let* A *and* B *be infinite BAs. Suppose that* $A \leq_s B$*, Then* $\text{Irr}(A) = \text{Irr}(B)$.

Proof. Clearly $irr(A) \leq \text{Irr}(B)$. By Proposition 2.29 of Monk [14], there are ideals I_0, I_1 of A such that $I_0 \cap I_1 = \{0\}$ and $B \cong (A/I_0) \times (A/I_1)$. By Theorem 8.4 of Monk [14], $\text{Irr}(B) = \max\{\text{Irr}(A/I_0), \text{Irr}(A/I_1)\}$. Wlog $\text{Irr}(B) = \text{Irr}(A/I_0)$.

Case 1. $\text{Irr}(A/I_0) = \kappa^+$ for some infinite κ. Let $\langle [a_\alpha]_{I_0} : a < \kappa^+\rangle$ be an irredundant system in A/I_0. Clearly $\langle a_\alpha : a < \kappa^+\rangle$ is irredundent. Hence $\kappa^+ \leq \text{Irr}(A)$.

Case 2, $\text{Irr}(A/I_0)$ is a limit cardinal. Take any infinite $\kappa^+ < \text{Irr}(A/I_0)$. As in Case 1, $\kappa^+ \leq \text{Irr}(A)$. Hence $irr(A/I_0) \leq \text{Irr}(A)$.

Thus in any case, $\text{Irr}(A/I_0) \leq \text{Irr}(A)$. □

Problem 90. *Is* $\text{Irr}_{\text{mm}}(A) = \pi(A)$ *for every infinite BA* A*?*

This is answered consistently negatively by Kunen http://arxiv.org/1307.3533

Page 383. Theorem 11.20 is false.

Problem 126. *Is there an atomless BA* A *such that* $\text{s}_{\text{mm}}(A) < \mathfrak{i}(A)$*?*

?This is answered positively consistently by Cancino, J.; Guzman.O.; Miller, A.

Problem 149. *What is the exact place of* $\text{hd}^{\text{id}}_{\text{mm}}$ *among the other cardinal functions?*

The following result shows that $\text{hd}^{\text{id}}_{\text{mm}}$ is trivial.

Proposition 5. $\forall A[\text{hd}^{id}_{mm}(A) = \omega]$.

Proof. Let $\langle b_n : n \in \omega\rangle$ be a partition in $\overline{A}$. Define

$$I_n = \{a \in A : \forall m < n[a \cdot b_m = 0\}$$

Clearly $I_n \supseteq I_p$ if $n < p$. Suppose that $0 \neq a \in \bigcap_{n\in\omega} I_n$. Choose $n \in \omega$ such that $a \cdot b_n \neq 0$ and choose $c \in A^+$ such that $c \leq a \cdot b_n$. Now $c \in I_{n+1}$, so $c \cdot b_n = 0$, contradiction. Hence $\bigcap_{n\in\omega} I_n = 0$. Hence $\mathrm{hd}_{mm}^{id}(A) = \omega$.

Problem 156. *Is* $\mathrm{Inc}_{\mathrm{spect}}^{\mathrm{tree}}(\mathscr{P}(\kappa)) = \{\kappa, 2^\kappa\}$*?*

Campero-Arena, G.; Cancino, J; Hrusak, M.; Miranda-Perea, F. E. showed that consistently the answer is negative.

Problem 158. *Is it consistent that* $\mathrm{Inc}_{\mathrm{mm}}(\mathscr{P}(\omega)/\mathrm{fin}) \neq \mathrm{Inc}_{\mathrm{mm}}^{\mathrm{tree}}(\mathscr{P}(\omega)/\mathrm{fin})$*?*

Campero-Arena, G.; Cancino, J; Hrusak, M.; Miranda-Perea, F. E. showed that consistently the answer is yes.