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A large list of small cardinal characteristics of Boolean algebras

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A large list of mostly known small cardinal characteristics of Boolean algebras is given and relationships between them (mostly known) are described. Many open problems are formulated.

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Many of the cardinal numbers defined for $\wp(\omega)/\text{fin}$ have been generalized to Boolean algebras. Besides standard ones as treated, e.g., in [2], others have been considered. Most of the functions which we discuss here have been treated with references to original papers in [19]. We discuss the relationships between these functions for the class of all Boolean algebras. We also describe what is known for $\wp(\omega)/\text{fin}$. Besides appealing to known results we also give some new theorems. Most of the functions considered here are small, in the sense that they involve taking the minimum of a set of cardinals.

Our set theory terminology follows [10] and the Boolean algebra notation follows [9]. We mention now however some notation which may not be familiar. The basic operations of a Boolean algebra are denoted by $+, \cdot, -$ rather than, e.g., $\vee, \wedge, '$. The set of nonzero elements of a set X in a Boolean algebra is denoted by X^+. The subalgebra of A generated by a set X is denoted by $\langle X\rangle$. A subset X of A is *pairwise disjoint* if and only if $x \cdot y = 0$ for all distinct $x, y \in X$. For any Boolean algebra A, $\pi(A)$ is the least size of a dense subset of A, where $X \subseteq A^+$ is dense if and only if $\forall a \in A^+ \exists x \in X[x \leq a]$. If $a \in A$, then $A \restriction a = \{x \in A : x \leq a\}$; this is a Boolean algebra under $+, \cdot, -'$, where $-'x = a \cdot -x$; 0 is the smallest element, and a the largest element. If $\langle A_i : i \in I\rangle$ is a system of Boolean algebras, the weak product $\prod^w_{i\in I} A_i$ is the subalgebra of the full product $\prod_{i\in I} A_i$ consisting of those x such that $\{i \in I : x_i \neq 0\}$ is finite or $\{i \in I : x_i \neq 1\}$ is finite. If A is generated by a set X, a monomial over X is a finite product $x_0 \cdot \ldots \cdot x_n$ where for each i, $x_i \in X$ or $-x_i \in X$. A Boolean algebra A is (κ, ∞)-distributive if and only if every set of at most κ partitions of unity has a common refinement.

We begin by defining some notions entering into the definitions of our cardinal functions.

A set $S \subseteq A$ *splits* A if and only if for all $a \in A^+$ there is an $s \in S$ such that $a \cdot s \neq 0 \neq a \cdot -s$. A *tower* in A is a subset T of A well-ordered by the Boolean ordering in a limit ordinal type and with sum 1. A set X is *weakly dense* in A if and only if for all $a \in A$ there is an $x \in X^+$ such that $x \leq a$ or $x \leq -a$. The set X is *quasi-dense* in A if and only if $0, 1 \notin X$ and for all $a \in A$ there is an $x \in X$ such that $x \leq a$ or $a \leq x$. A *weak partition* of A is a system $\langle b_\xi : \xi < \alpha\rangle$ of pairwise disjoint elements with sum 1; it is not assumed that all b_ξ are nonzero. For any integer m with $2 \leq m$, a set $X \subseteq A$ is *m-dense* if and only if for any weak partition $\langle b_i : i < m\rangle$ of A there exist an $x \in X$ and $i < m$ such that $x \leq b_i$. The set $X \subseteq A$ is *independent* if and only if for all $F, G \in [X]^{<\omega}$, if $F \cap G = \varnothing$, then $\prod_{x\in F} x \cdot \prod_{x\in G} -x \neq 0$; it is *irredundant* if and only if for all $x \in X$, we have $x \notin \langle X\backslash\{x\}\rangle$; it is *ideal independent* if and only if for all $x \in X$ and $F \in [X\backslash\{x\}]^{<\omega}$, we have that $x \not\leq \sum_{y\in F} y$. For $2 \leq n < \omega$, a set $X \subseteq A^+$ is n-independent if and only if

$$\forall R \in [X]^{<\omega} \forall \varepsilon \in {}^R 2 \left[\prod_{x\in R} x^{\varepsilon(x)} = 0 \rightarrow \exists R' \in [R]^{\leq n}\left[\varepsilon[R'] = \{1\} \text{ and } \prod R' = 0\right]\right].$$

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A set $X \subseteq A^+$ is ω-independent if and only if

$$\forall R \in [X]^{<\omega} \forall \varepsilon \in {}^R 2 \left[\prod_{x \in R} x^{\varepsilon(x)} = 0 \rightarrow \exists R' \subseteq R \left[\varepsilon[R'] = \{1\} \text{ and } \prod R' = 0 \right] \right].$$

A *free sequence* on a Boolean algebra A is a sequence $\langle a_\xi : \xi < \alpha \rangle$ of elements of A, with α an ordinal, such that for all $F, G \in [\alpha]^{<\omega}$ with $F < G$ we have $\prod_{\xi \in F} a_\xi \cdot \prod_{\xi \in G} -a_\xi \neq 0$. Here $[\alpha]^{<\omega}$ is the collection of all finite subsets of α. We write $F < G$ to mean that $\xi < \eta$ for all $\xi \in F$ and $\eta \in G$. We allow $F = \varnothing$ or $G = \varnothing$. A free sequence is *maximal* if and only if there is no $b \in A$ such that $\langle a_\xi : \xi < \alpha \rangle ^\frown \langle b \rangle$ is a free sequence, where $\langle a_\xi : \xi < \alpha \rangle ^\frown \langle b \rangle$ is the result of adjoining b at the end of the sequence $\langle a_\xi : \xi < \alpha \rangle$. A set $X \subseteq A^+$ is *dense* in an ultrafilter D of A if and only if for all $a \in D$ there is an $x \in X$ such that $x \leq a$. It is not assumed that $X \subseteq D$. An h-cof sequence is a sequence $\langle a_\xi : \xi < \alpha \rangle$ of elements different from 1 which is well-founded in the Boolean order, and is such that $\text{rank}(a_\xi) \leq \text{rank}(a_\eta)$ whenever $\xi < \eta$. A *tree* is a partially ordered set T such that for each $t \in T$ the set $\{s \in T : s < t\}$ is well-ordered. A *tree in a Boolean algebra* A is a subset of A which is a tree in the Boolean order. The tree T is *maximal* if and only if $1 \notin T$ and there is no tree T' such that $T \subset T' \subseteq A$, $1 \notin T'$, and for all $t \in T$ and all $s \in T'$, if $s \leq t$, then $s \in T$.

A fairly complete list of "small" functions on a Boolean algebra is as follows.

$$\begin{aligned}
\text{cell}(A) &= \sup\{|X| : X \subseteq A \text{ is pairwise disjoint}\};\\
\pi_{\min}(A) &= \min\{\pi(A \restriction a) : a \in A^+\};\\
\mathfrak{a}(A) &= \min\{|X| : X \text{ is an infinite partition of unity in } A\};\\
\mathfrak{p}(A) &= \min\left\{|Y| : \sum Y = 1 \text{ and } \sum F \neq 1 \text{ for every finite } F \subseteq Y\right\};\\
\mathfrak{s}(A) &= \min\{|S| : S \text{ splits } A\};\\
\text{tow}(A) &= \min\{|T| : T \text{ is a tower in } A\};\\
\mathfrak{h}(A) &= \min\{\kappa : A \text{ is not } (\kappa, \infty)\text{-distributive}\};\\
\mathfrak{r}(A) &= \min\{|X| : X \text{ is weakly dense in } A\};\\
\mathfrak{r}_m(A) &= \min\{|X| : X \text{ is } m\text{-dense in } A\} \text{ for } 2 \leq m < \omega;\\
\mathfrak{q}(A) &= \min\{|X| : X \text{ is quasi-dense in } A\};\\
\mathfrak{i}(A) &= \min\{|X| : X \subseteq A \text{ is maximal independent}\};\\
\mathfrak{u}(A) &= \min\{|X| : X \text{ generates a nonprincipal ultrafilter on } A\};\\
\mathfrak{l}(A) &= \min\{|X| : X \text{ is a maximal chain in } A\};\\
\text{irr}_{\text{mm}}(A) &= \min\{|X| : X \text{ is a maximal irredundant subset of } A\};\\
\text{s}_{\text{mm}}(A) &= \min\{|X| : X \text{ is an infinite maximal ideal independent subset of } A\};\\
\mathfrak{i}_n(A) &= \min\{|X| : X \text{ is maximal } n\text{-independent in } A\} \quad \text{for } 2 \leq n \leq \omega;\\
\mathfrak{f}(A) &= \min\{|\alpha| : \text{there is a maximal free sequence of length } \alpha\};\\
\pi\chi_{\inf}(A) &= \min\{|X| : X \text{ is dense in some nonprincipal ultrafilter on } A\};\\
\text{inc}_{\text{mm}}(A) &= \min\{|X| : X \text{ is a maximal incomparable set with } 0, 1 \notin X\};\\
\text{inc}_{\text{mm}}^{\text{tree}}(A) &= \min\{|T| : T \text{ is a maximal tree}\};\\
\text{cf}(A) &= \min\{\kappa : \text{ there is a strictly increasing sequence of subalgebras of length } \kappa \text{ with}\\
&\qquad \text{union } A\};\\
\text{alt}(A) &= \min\{\kappa : \text{ there is a strictly increasing sequence of filters of length } \kappa \text{ whose union}\\
&\qquad \text{is an ultrafilter}\};
\end{aligned}$$

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$$\text{p-alt}(A) = \min\{\kappa : \text{ there is an infinite homomorphic image of } A \text{ with a nonprincipal ultrafilter generated by } \kappa \text{ elements}\};$$

$$\text{card}_{\text{H}-}(A) = \min\{|B| : B \text{ is an infinite homomorphic image of } A\};$$

$$\text{hcof}_{\text{mm}}(A) = \min\{|\alpha| : \text{there is a maximal h-cof sequence of infinite length } \alpha\};$$

$$\text{hd}^{\text{id}}_{\text{mm}}(A) = \min\{|\alpha| : \text{there is a maximal strictly decreasing sequence of ideals of } A \text{ of infinite length } \alpha\}.$$

Clearly $\mathfrak{h}(A)$ and $\mathfrak{s}(A)$ do not exist for some Boolean algebras. The value $\mathfrak{i}(A)$ does not exist if A is superatomic.

Proposition 1 *The number* $\text{tow}(A)$ *does not exist for* $A = \text{Finco}(\kappa)$ *when* κ *is an uncountable cardinal.*

P r o o f. Suppose to the contrary that $\langle a_\xi : \xi < \alpha\rangle$ is a tower in A, where α is a limit ordinal. A cofinite set has only finitely many successors, so every a_ξ is finite. We have that $\bigcup_{\xi<\omega} a_\xi$ is infinite, so $\alpha = \omega$. But then $\sum_{\xi<\alpha} a_\xi$ is countable, and so is not equal to κ. □

Proposition 2 *Let* $a \in A$ *be an atom. Then*

(i) $\{a\}$ *is weakly dense;*
(ii) $\{a\}$ *is maximal independent;*
(iii) if $0 < x < 1$*, then* $\{x, -x\}$ *is maximal ideal independent;*
(iv) if $X \subseteq A$ *is ideal independent, then* $0 \notin X$*;*
(v) $\langle a\rangle$ *is a maximal free sequence;*
(vi) $\{0\}$ *is maximal incomparable;*
(vii) $\{1\}$ *is maximal incomparable;*
(viii) $\{a, -a\}$ *is maximal incomparable;*
(ix) for $2 \le n \le \omega$*, the set* $\{a, -a\}$ *is maximal* n*-independent;*
(x) $\pi_{\min}(A) = \mathfrak{r}(A) = \mathfrak{i}(A) = \mathfrak{f}(A) = 1$*;*
(xi) $\text{inc}_{\text{mm}}(A) = \mathfrak{i}_n(A) = 2$*;*
(xii) if X *is infinite and ideal independent, then* $\langle X\rangle^{\text{id}} \ne A$*.*

P r o o f. Most of these assertions are trivial. For (viii), clearly $\{a, -a\}$ is incomparable. Now suppose that $b \in A\backslash\{a, -a\}$. Then $a \le b$ or $a \le -b$. If $a \le -b$, then $b \le -a$. So b is comparable to some member of $\{a, -a\}$.

For (ix), clearly $\{a, -a\}$ is n-independent. If $x \notin \{a, -a\}$, there are two cases.

Case 1. $a \le x$. Then $a^1 \cdot x^0 = 0$ shows that $\{a, -a, x\}$ is not n-independent.
Case 2. $a \le -x$ Then $(-a)^0 \cdot x^1 = 0$ shows that $\{a, -a, x\}$ is not n-independent.

For (xii), if $\langle X\rangle^{\text{id}} = A$, then $\sum F = 1$ for some finite $F \subseteq X$, and then $x \le \sum F$ for $x \in X\backslash F$. □

Proposition 3 *(i) If* $a \in A$*, then* $\{a\}$ *is not quasi-dense.*
(ii) If a *is an atom, then* $\{a, -a\}$ *is quasi-dense.*
(iii) If A *is atomless, then* $\mathfrak{q}(A) = \mathfrak{r}(A)$*.*
(iv) If A *has an atom, then* $\mathfrak{q}(A) = 2$*.*

P r o o f. (i) Assume that $a \in A$. If $a = 0$ or $a = 1$, obviously $\{a\}$ is not quasi-dense. Suppose that $0 \ne a \ne 1$. Then $a \not\le -a$ and $-a \not\le a$.

(ii) Suppose that a is an atom. Take any $b \in A$. If $a \le b$, this is as desired. If $a \le -b$, then $b \le -a$, as desired.

(iii) Suppose that A is atomless, and first suppose that X is weakly dense, with $|X| = \mathfrak{r}(A)$. Let $Y = (X \cup \{a : -a \in X\})\backslash\{0, 1\}$. We claim that Y is quasi-dense. For, let $a \in A$. Choose $x \in X^+$ such that $x \le a$ or $x \le -a$. Then $x \le a$ or $a \le -x$. Hence $\exists y \in Y[y \le a \text{ or } a \le y]$.

Second suppose that Y is quasi-dense. Let $X = Y \cup \{a : -a \in X\}$. We claim that Y is weakly dense. For, let $a \in A$. Choose $y \in Y$ such that $a \le y$ or $y \le a$. Then $y \le a$ or $-y \le -a$. Hence $\exists x \in X[x \le a \text{ or } x \le -a]$.

(iv) Cf. the proof of (iii). □

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Proposition 4 *If A is infinite, then* $\mathrm{inc}_{\mathrm{mm}}^{\mathrm{tree}}(A)$ *is infinite.*

Proof. Suppose to the contrary that X is a finite maximal tree contained in A. Let C be the subalgebra generated by X, and let $\langle a_i : i < m\rangle$ list all of the atoms of C. Let $J = \{i < m : a_i \text{ is not an atom of } A\}$. Thus $J \neq \varnothing$.

Case 1. There exist a maximal element b of X and an $i \in J$ such that $b \cdot a_i = 0$. Say $0 < x < a_i$. Let $c = b + x$. If $y \in X$ and $c \leq y$, then $b < y$, contradiction. If $y \in X$ and $y \leq c$, then $y \cdot a_i \leq x$; hence $y \cdot a_i = 0$, so $y \leq b$. Thus $X \cup \{c\}$ is a tree, contradiction.

Case 2. For every maximal element b of X and every $i \in J$ we have $a_i \leq b$. It follows that $J \neq m$. We claim that $\sum_{i \notin J} a_i$ is incomparable with each maximal element b of X, so that $X \cup \{\sum_{i\notin J} a_i\}$ is a tree extending X, contradiction. For, if b is a maximal element of X then there is an $i \notin J$ such that $b \cdot a_i = 0$. This implies that $\sum_{i\notin J} a_i \not\leq b$. Since $J \neq \varnothing$ and $\forall i \in J[a_i \leq b]$, we also have $b \not\leq \sum_{i \notin J} a_i$. □

Proposition 5 *We have* $\mathrm{hd}_{\mathrm{mm}}^{\mathrm{id}}(A) = \omega$ *for any infinite Boolean algebra A.*

Proof. Let $\langle b_n : n \in \omega\rangle$ be a partition in $\overline{A}$. Define

$$I_n = \{a \in A : \forall m < n[a \cdot b_m = 0]\}.$$

Clearly $I_n \supseteq I_p$ if $n < p$. Suppose that $a \in \bigcap_{n\in\omega} I_n$ and $a \neq 0$. Choose $n \in \omega$ so that $a \cdot b_n \neq 0$. Then $a \cdot b_n \in I_{n+1}$, so $a \cdot b_n = 0$, contradiction. □

Proposition 6 *We have* $\mathrm{hcof}_{\mathrm{mm}}(A) = \omega$ *for any infinite Boolean algebra A.*

Proof. First suppose that A is not atomic. Let $b \in A$ be such that $A \mid b$ is atomless, and let $\langle a_i : i \in \omega\rangle$ be a system of pairwise disjoint nonzero elements of $A \mid b$, and for each $i \in \omega$ let $\langle b^i_j : j \in \omega\rangle$ be a system of pairwise disjoint nonzero elements less than a_i. We consider the following sequence

$$\begin{array}{cl} -a_0, -a_1, -a_2, \ldots & \text{(rank 0)} \\ -a_1 + b^1_0, -a_2 + b^2_0, -a_3 + b^3_0 \ldots & \text{(rank 1)} \\ -a_2 + b^2_0 + b^2_1, -a_3 + b^3_0 + b^3_1, -a_4 + b^4_0 + b^4_1 \ldots & \text{(rank 2)} \\ \vdots & \end{array}$$

This gives an h-cof sequence of length ω^2. Suppose that c is adjoined at the end. Then c has rank ω, so it includes cofinally many sets $-a_n$. Since $-a_n + -a_m = 1$ for $n \neq m$, it follows that $c = 1$, contradiction.

Second, suppose that A is atomic. Let $\langle a_i : i \in \omega + 2\rangle$ be a system of atoms of A. Consider the sequence

$$\langle a_0, a_0 + a_1, \ldots, -a_\omega \cdot -a_{\omega+1}, -a_\omega, -a_{\omega+1}\rangle.$$

This is a maximal h-cof sequence. □

These propositions take care of rather trivial matters. We now indicate in two diagrams below the relationships between the functions; one diagram for Boolean algebras in general, and another one for atomless Boolean algebras. By Proposition 2, many of our functions are trivial if the Boolean algebra has an atom. For the diagrams, we first indicate why the implications hold; references are not necessarily to the original places.

The implications given in the chain on the left of Figure 1 are proved in [19, Proposition 9.1]. Obviously $\text{p-alt}(A) \leq \mathfrak{u}(A)$. It is clear that $|A|$ is an upper bound for all of the functions considered. We have $\mathfrak{r}(A) = 1$ if A has an atom, and that $\mathfrak{r}(A) \leq \mathfrak{i}(A)$ for atomless A is given as [19, Proposition 10.21]. That $\mathfrak{r}(A) \leq \mathfrak{f}(A)$ is part of the proof of [17, Proposition 2.4]. That $\mathfrak{r}(A) \leq \pi\chi_{\mathrm{inf}}(A)$ is part of [19, Theorem 6.28].

Proposition 7 *We have* $\mathfrak{r}(A) \leq \mathrm{inc}_{\mathrm{mm}}^{\mathrm{tree}}(A)$ *for any infinite Boolean algebra A.*

Proof. Let T be a maximal tree in A. We claim that $(T \cup \{a : -a \in T\})\backslash\{0\}$ is weakly dense. For, let $b \in A$. Then b is comparable with some element of $T\backslash\{0\}$, and the desired conclusion follows. □

That $\mathfrak{r}(A) \leq \mathfrak{i}_n(A)$ for atomless A is given by [3, Corollary 6.8]. We have $\pi\chi_{\mathrm{inf}}(A) \leq \mathfrak{f}(A)$ for every infinite Boolean algebra A by (the proof of) [19, Proposition 12.20]. Clearly $\pi\chi_{\mathrm{inf}}(A) \leq \mathfrak{u}(A)$. We have $\pi\chi_{\mathrm{inf}}(A) \leq \mathfrak{s}_{\mathrm{mm}}(A)$ for any infinite A by (the proof of) [18, Proposition 2.1].

The following proposition and corollary are given as [4, Proposition 2.3]; we include a proof for completeness.

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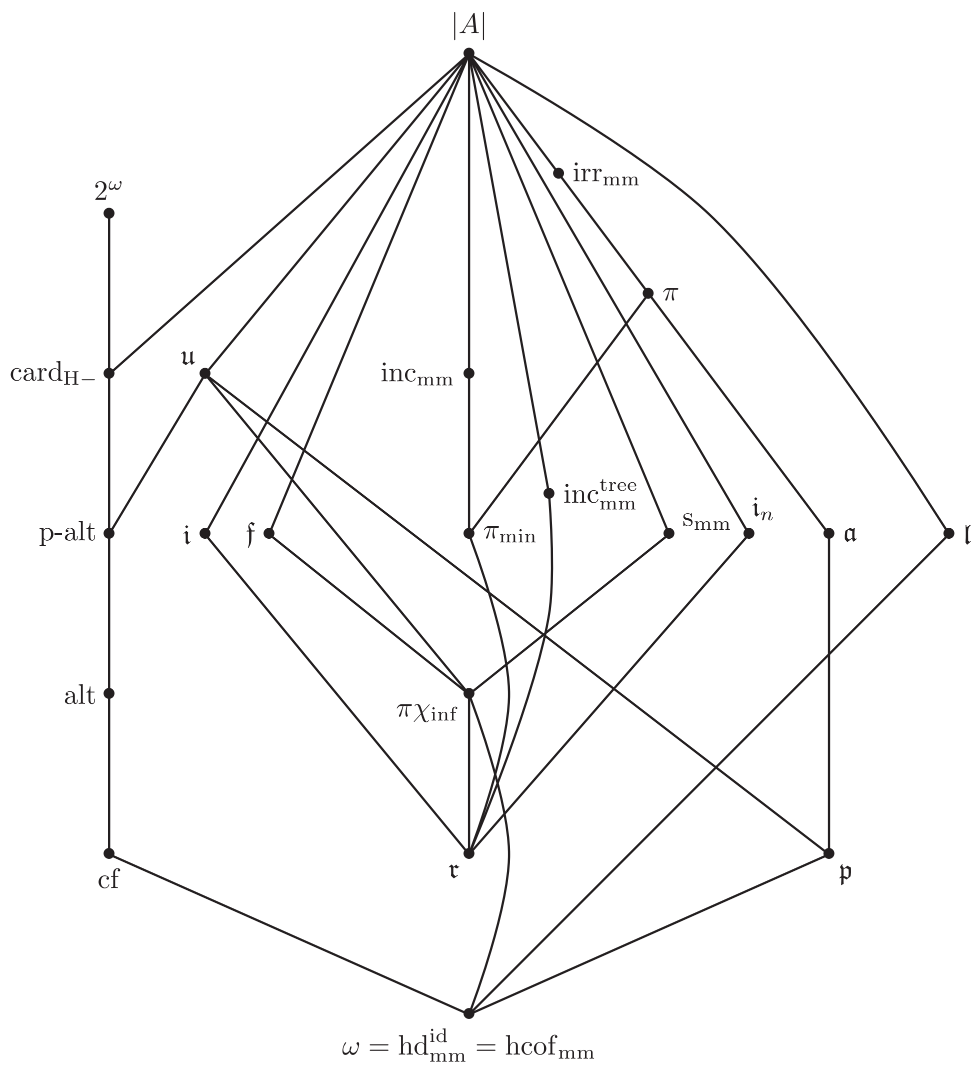

Fig. 1 Boolean algebras in general. (Note that $\mathfrak{r} = \mathfrak{r}_2 \leq \mathfrak{r}_3 \leq \cdots \leq \pi\chi_{\mathrm{inf}} \leq \mathfrak{r}_2^+$.)

Proposition 8 *If B is an atomless Boolean algebra, $X \subseteq B$ is incomparable, $0, 1 \notin X$, and $|X| < \pi_{\min}(B)$, then there is an $a \in B\backslash X$ such that $X \cup \{a\}$ is incomparable.*

Proof. Let C be the subalgebra of B generated by X; then also $|C| < \pi_{\min}(B)$. Take any $a \in X$. Then $a \neq 0, 1$. Since $|C| < \pi(B \upharpoonright -a)$, there is an $x \leq -a$ with $x \neq 0$ such that no nonzero element of C is below x. Similarly there is a $y \leq a$ with $y \neq 0$ such that no nonzero element of C is below y. Let $b = x + a \cdot -y$. Note that $a \not\leq b$, as otherwise $a \leq b \cdot -x \leq -y$, hence $y \leq -a$ and so $y = 0$, contradiction. Also, $b \not\leq a$, as otherwise $x \leq a$ and so $x = 0$, contradiction. Suppose that $c \in X$ and $c \leq b$. Then $c \cdot -a \leq x$, $c \cdot -a \in C$, and $c \cdot -a \neq 0$, contradiction. Suppose that $c \in X$ and $b \leq c$. Then $a \cdot -y \leq c$, so $a \cdot -c \leq y$, again a contradiction. So b is incomparable with each element of X. □

Corollary 9 *We have $\pi_{\min}(A) \leq \mathrm{inc}_{\mathrm{mm}}(A)$ for any atomless Boolean algebra A.*

Note by this Corollary that $\mathrm{inc}_{\mathrm{mm}}(\wp(\omega)/\mathrm{fin}) = 2^\omega$. We have $\mathfrak{p}(A) \leq \mathfrak{u}(A)$ by [14, Proposition 6], and $\mathfrak{p}(A) \leq \mathfrak{a}(A)$ by [19, Proposition 4.47(iii)]. Obviously $\pi_{\min}(A) \leq \pi(A)$ and $\mathfrak{a}(A) \leq \pi(A)$. We have $\pi(A) \leq \mathrm{irr}_{\mathrm{mm}}(A)$ by [9, Proposition 4.23]. Clearly $\omega \leq \mathfrak{l}(A)$ for any infinite Boolean algebra A; $\omega \leq \mathrm{tow}(A)$ for any atomless Boolean algebra A;.

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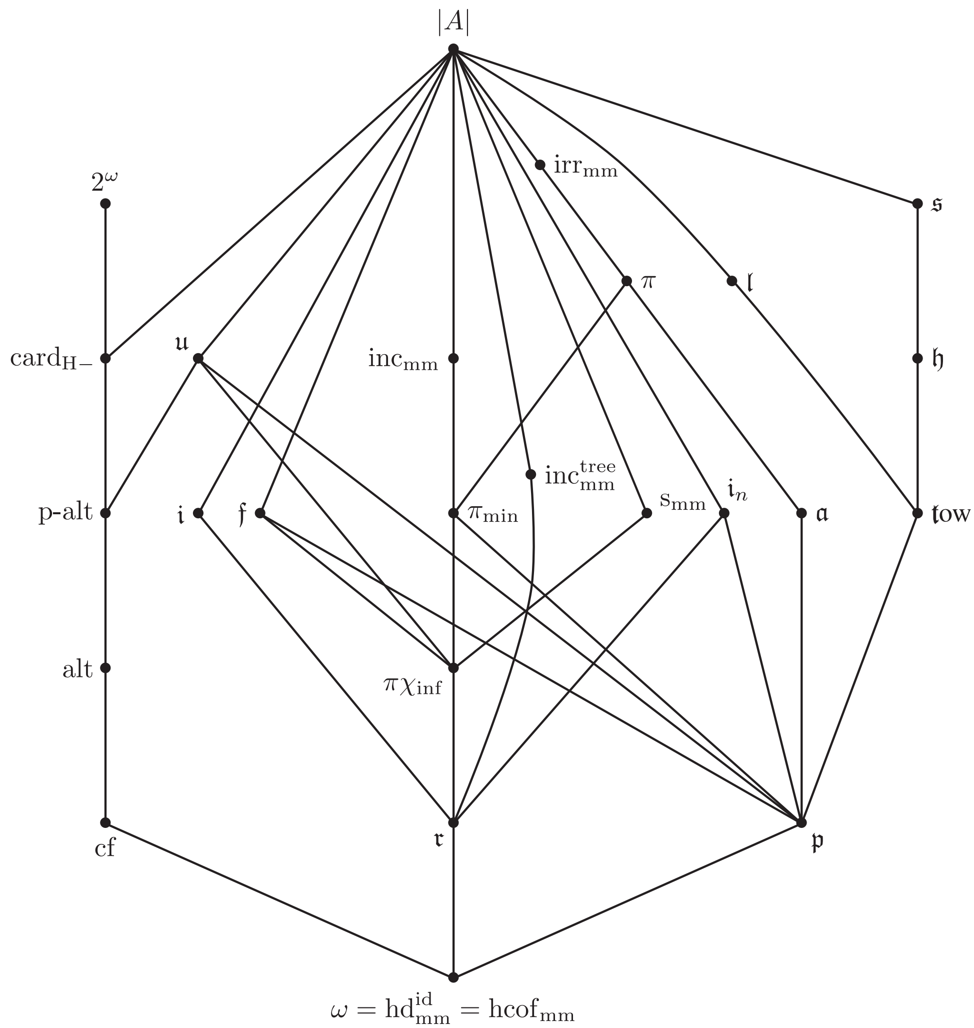

Fig. 2 Atomless Boolean algebras. (Note that $\mathfrak{r} = \mathfrak{r}_2 \leq \mathfrak{r}_3 \leq \cdots \leq \pi\chi_{\mathrm{inf}} \leq \mathfrak{r}_2^+$.)

Assume that A is atomless. We have $\omega \leq \mathfrak{r}(A)$ by [14, Proposition 1] and $\mathfrak{p}(A) \leq \mathfrak{f}(A)$ by [19, Theorem 12.21]. That $\mathfrak{p}(A) \leq \mathfrak{i}_n(A)$ is given by [3, Theorem 6.4]. We have that $\mathrm{tow}(A)$, $\mathfrak{h}$, and $\mathfrak{s}$ exist. That $\mathfrak{p}(A) \leq \mathrm{tow}(A)$ holds is given by [19, Proposition 4.47(iii)]. The fact $\mathrm{tow}(A) \leq \mathfrak{l}(A)$ is obvious. We have $\mathrm{tow}(A) \leq \mathfrak{h}(A)$ by [14, Proposition 4] and $\mathfrak{h}(A) \leq \mathfrak{s}(A)$ by [14, Proposition 5].

Proposition 10 *We have $\mathfrak{p}(A) \leq \pi_{\mathrm{min}}(A)$ for any atomless Boolean algebra A.*

Proof. Choose $a \in A^+$ so that $\pi(A \upharpoonright a) = \pi_{\mathrm{min}}(A)$. Let $X \subseteq A \upharpoonright a$ have size $\pi(A \upharpoonright a)$ with X dense in $A \upharpoonright a$. Let Y be an infinite partition of $A \upharpoonright a$ and let $X' = \{x \in X : \exists y \in Y[x \leq y]\}$. Clearly X' is dense in $A \upharpoonright a$ and $|X'| \leq |X|$. Let $Z = X' \cup \{-a\}$. Then $\sum Z = 1$ and $\sum F \neq 1$ for all finite $F \subseteq Z$. Hence $\mathfrak{p}(A) \leq \pi_{\mathrm{min}}(A)$. □

Proposition 11 *We have $\pi\chi_{\mathrm{inf}}(A) \leq \pi_{\mathrm{min}}(A)$ for every atomless Boolean algebra A.*

Proof. Let $a \in A^+$ and let X be dense in $A \upharpoonright a$, with $|X| = \pi(A \upharpoonright a) = \pi_{\mathrm{min}}(A)$. Let F be an ultrafilter on $A \upharpoonright a$ and set $G = \{x \in A : x \cdot a \in F\}$. Then G is an ultrafilter on A. Clearly X is dense in G. Thus $\pi\chi(G) \leq \pi_{\mathrm{min}}(A)$. □

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This finishes checking that the lines in the diagram are correct. Now we consider the possibility of additional relationships among the functions. It suffices to consider just crucial pairs. For example, a Boolean algebra A with $\mathrm{card}_{\mathrm{H-}}(A) < \mathrm{inc}_{\mathrm{mm}}(A)$ automatically is an example with $\mathrm{cf}(A) < \mathrm{inc}_{\mathrm{mm}}(A)$.

Proposition 12 *We have* $\mathfrak{r}(\mathrm{Fr}(\kappa)) = \kappa$ *for any infinite cardinal* κ.

Proof. Suppose that $X \subseteq \mathrm{Fr}(\kappa)$ is weakly dense, with $|X| < \kappa$. Let x be a free generator not in the support of any $y \in X$. Then $y \not\leq x$ and $y \not\leq -x$ for any $y \in X$, contradiction. □

Taking $\kappa > 2^{\omega}$ here gives a Boolean algebra A such that $\mathfrak{r}(A) > 2^{\omega}$. [19, Theorem 4.60] gives a Boolean algebra A such that $2^{\omega} < \mathfrak{p}(A)$. [19, Example 9.6] gives a Boolean algebra A such that $\mathrm{cf}(A) = \omega$, $\mathrm{alt}(A) = \text{p-alt}(A) = \omega_1$, and $\mathrm{card}_{\mathrm{H-}}(A) = 2^{\omega}$. Under $\neg$CH we then get $\text{p-alt}(A) < \mathrm{card}_{\mathrm{H-}}(A)$. On the other hand, CH implies that $\mathrm{alt}(B) = \text{p-alt}(B) = \mathrm{card}_{\mathrm{H-}}(B)$ for any infinite Boolean algebra B, by [19, Proposition 9.4]. The following two old problems remain.

Problem 13 *Is it consistent that there a Boolean algebra A such that* $\mathrm{alt}(A) < \text{p-alt}(A)$*?*

Problem 14 *Is* $\mathrm{cf}(A) \leq \omega_1$ *for every infinite Boolean algebra A?*

Turning to $\mathfrak{u}$ we have the following result.

Proposition 15 *If A is an atomless Boolean algebra with $\mathfrak{u}(A) = \omega$, then A has a countably infinite homomorphic image.*

Proof. Let D be an ultrafilter such that $\chi(D) = \omega$, Say D is generated by $\{x_i : i \in \omega\}$, where $x_0 > x_1 > \cdots$. For each $i \in \omega$ let $B_i = (A(-x_i)) \cup \{z \in A : x_i \leq z\}$. Clearly B_i is a subalgebra of A. Suppose that $z \in B_i$. If $z \leq -x_i$, then $x_{i+1} < x_i \leq -z$, hence $z \leq -x_{i+1}$. If $x_i \leq z$, then $x_{i+1} \leq z$. Thus $B_i \subseteq B_{i+1}$. We claim that $\bigcup_{i\in\omega} B_i = A$. For, suppose that $a \in A$. If $a \in D$, choose i so that $x_i \leq a$; then $a \in B_i$. If $-a \in D$, choose i so that $x_i \leq -a$; then $a \leq -x_i$, so $a \in B_i$.

Finally, we claim that D is not determined by any B_i. For, suppose that it is determined by B_i. Let E be an ultrafilter on A such that $a_i \cdot -a_{i+1} \in E$. Now $D \cap B_i = \{z \in A : x_i \leq z\} = E \cap B_i$. Clearly $E \neq D$. Now the desired conclusion follows by [8, Theorem 3]. □

It is known that $\mathrm{card}_{\mathrm{H-}}(A) \leq 2^{\omega}$ for any infinite Boolean algebra A. Hence under CH, by Proposition 11 there is no infinite Boolean algebra A such that $\mathfrak{u}(A) < \mathrm{card}_{\mathrm{H-}}(A)$. It is consistent that $\mathfrak{u}(\wp(\omega)/\mathrm{fin}) < 2^{\omega}$, and in ZFC, $\mathrm{card}_{\mathrm{H-}}(\wp(\omega)/\mathrm{fin}) = 2^{\omega}$. An atomless Boolean algebra A with $\mathfrak{u}(A) < \mathfrak{i}(A)$ is given in [13, Theorem 5].

Problem 16 *Is there a Boolean algebra A such that* $\mathfrak{u}(A) < \mathfrak{f}(A)$*?*

Note concerning this problem that if $\mathfrak{u}(A) = \omega$, then also $\mathfrak{f}(A) = \omega$. In fact, if $\mathfrak{u}(A) = \omega$, let D be a countably generated nonprincipal ultrafilter on A. Then D is generated by a descending sequence, and this sequence is a maximal free sequence by [17, Proposition 1.6]. Also note that $\text{p-alt}(\mathrm{Fr}(\omega_1)) = \omega < \mathfrak{f}(\mathrm{Fr}(\omega_1))$ by [17, Theorem 1.3].

Example 17 Let A be the weak product of ω many copies of $\mathrm{Fr}(\omega_1)$. Then $\mathfrak{u}(A) = \omega$ by [14, Proposition 8], while clearly $\pi_{\min}(A) = \omega_1$.

Problem 18 *Is there a Boolean algebra A such that* $\mathfrak{u}(A) < \mathrm{inc}^{\mathrm{tree}}_{\mathrm{mm}}(A)$*?*

By [16, Proposition 2.13] there is an atomless Boolean algebra A such that $\mathfrak{u}(A) < \mathfrak{s}_{\mathrm{mm}}(A)$.

For the next lemma and example we use the notation of [13, Theorem 5].

Lemma 19 *If $s = \langle s_0 s_1 \ldots s_m \rangle$ is a monomial over Y, then $s \in I$ if and only if one of the following holds:*

(i) $\exists\alpha < \beta < \kappa \exists i, j \leq m[s_i = x_\beta$ *and* $s_j = -x_\alpha]$.
(ii) $\exists\alpha \leq \beta < \kappa \exists\gamma < \lambda \exists i, j \leq m[s_i = x_\beta$ *and* $s_j = -y_{\alpha\gamma}]$.

Proof. "⇐": Obviously (i) implies the condition. Now assume (ii). Then

$$\begin{aligned} x_\beta \cdot -y_{\alpha\gamma} &= x_\beta \cdot -x_\alpha \cdot -y_{\alpha\gamma} + x_\beta \cdot x_\alpha \cdot -y_{\alpha\gamma} \\ &\leq x_\beta \cdot -x_\alpha + x_\alpha \cdot -y_{\alpha\gamma} \in I. \end{aligned}$$

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"⇒": Suppose that $s \in I$. Then we can write

$$s \le x_{\beta_0} \cdot -x_{\alpha_0} + \cdots + x_{\beta_{m-1}} \cdot -x_{\alpha_{m-1}} + x_{\gamma_0} \cdot -y_{\gamma_0\delta_0} + \cdots + x_{\gamma_{n-1}} \cdot -y_{\gamma_{n-1}\delta_{n-1}} \tag{1}$$

with each $\alpha_i < \beta_i$; possibly $m = 0$ or $n = 0$. We assume that m and n are minimal among all inequalities of this kind.

If $n > 0$ then each $y_{\gamma_i\delta_i}$ occurs positively or negatively in s. For, otherwise map $y_{\gamma_i\delta_i}$ to 1, fixing all other generators. We may assume that

$$\text{if } n > 0 \text{ then each } y_{\gamma_i\delta_i} \text{ occurs negatively in } s. \tag{2}$$

Otherwise multiply (1) by $y_{\gamma_i\delta_i}$ on both sides. We may also assume that

$$\text{if } n > 0 \text{ and } x_\delta \text{ occurs positively in } s, \text{ then } \gamma_i > \delta \text{ for all } i < n. \tag{3}$$

Otherwise (2) holds. We also have that

$$\text{if } n > 0, \text{ then the maximum element of } \{\beta_0, \ldots, \beta_{m-1}, \gamma_0, \ldots, \gamma_{n-1}\} \text{ is not one of the } \gamma_i\text{s}. \tag{4}$$

For, suppose not; let γ_i be maximum in $\{\beta_0, \ldots, \beta_{m-1}, \gamma_0, \ldots, \gamma_{n-1}\}$. Then γ_i is greater than each α_j. If x_{γ_i} does not occur positively or negatively in s, then mapping x_{γ_i} to 0 and fixing other generators shortens the expression, contradiction. So x_{γ_i} occurs positively or negatively in s. If it occurs negatively in s, then multiplying the inequality on both sides by $-x_{\gamma_i}$ shortens the expression, contradiction. So it occurs positively in s. But this contradicts (3). So (4) holds. We have that

$$\text{if } \beta_i \text{ is the maximum element of } \{\beta_0, \ldots, \beta_{m-1}, \gamma_0, \ldots, \gamma_{n-1}\}, \text{ then } x_{\beta_i} \text{ occurs positively in } s. \tag{5}$$

For, if x_{β_i} does not occur positively or negatively in s, we can map x_{β_i} to 0 and fix other generators, shortening (1). So x_{β_i} occurs positively or negatively in s. If it occurs negatively in s, then multiplying both sides of (1) by $-x_{\beta_i}$ shortens the expression. So x_{β_i} occurs positively in s.

From (5) and (3) it follows that $n = 0$. Furthermore,

$$\text{if } \alpha_i \text{ is the smallest element of } \{\alpha_0, \ldots, \alpha_{m-1}. \text{ then } x_{\alpha_i} \text{ occurs negatively in } s. \tag{6}$$

For, if x_{α_i} does not occur positively or negatively in s, then mapping x_{α_i} to 1 and fixing other generators shortens (1). So x_{α_i} occurs positively or negatively in s. If it occurs positively, then multiplying both sides of (1) by x_{α_i} shortens (1). Hence x_{α_i} occurs negatively in s, and (6) holds.

Now (i) holds. □

Example 20 The algebra constructed in [13, Theorem 5] has $\mathfrak{u}(A) = \kappa$ and $\mathfrak{i}_n(A) = \lambda$ for any $n \ge 2$, where $\kappa < \lambda$ are arbitrary regular cardinals. To show that $\mathfrak{i}_n(A) = \lambda$ for this algebra, suppose that $X \subseteq A$ is n-independent with $|X| < \lambda$. Let M be a set of monomials over Y such that every member of X is a finite sum of elements $[m]$ with $m \in M$. Choose β so that $y_{0\beta}$ and $-y_{0\beta}$ are not in M. Then $X \cup \{[-y_{0\beta}]\}$ is clearly n-independent. Thus X is not maximal.

By [14, Proposition 21] there is an atomless Boolean algebra A such that $\mathfrak{u}(A) < \mathfrak{a}(A)$. [14, Example 20] gives an atomless Boolean algebra A with $\mathfrak{u}(A) < \mathrm{tow}(A)$. If $A = \overline{\mathrm{Fr}(\omega)}$, then $\omega = \mathfrak{i}(A) < \mathrm{cf}(A)$ by [14, Proposition 26].

Problem 21 *Is there a Boolean algebra A such that* $\mathfrak{i}(A) < \pi\chi_{\mathrm{inf}}(A)$*?*

Note that $\pi\chi_{\mathrm{inf}}(A) \le \mathfrak{i}(A)$ if A is complete and atomless, by [1, Theorem 1.7]. For $A = {}^{\omega}\mathrm{Fr}(\omega_1)^{\mathrm{w}}$ we have $\mathfrak{i}(A) = \omega < \omega_1 = \pi_{\min}(A)$, by [14, Proposition 9].

Problem 22 *Is there a Boolean algebra A such that* $\mathfrak{i}(A) < \mathrm{inc}^{\mathrm{tree}}_{\mathrm{mm}}(A)$*?*

Problem 23 *Is there a Boolean algebra A such that* $\mathfrak{i}(A) < \mathfrak{i}_n(A)$*?*

By [14, Example 18] there is an atomless Boolean algebra A such that $\mathfrak{i}(A) < \mathfrak{p}(A)$. By [17, Proposition 1.11] we have $\mathfrak{f}(\wp(\omega)) = \omega$. Thus $\mathfrak{f}(A) < \mathrm{cf}(A)$ for $A = \wp(\omega)$.

Problem 24 *Is there an atomless Boolean algebra A such that* $\mathfrak{f}(A) < \mathrm{cf}(A)$*?*

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[17, Example 2.6] describes an atomless Boolean algebra A such that $\mathfrak{f}(A) < \mathfrak{i}(A)$. The algebra $A \stackrel{\text{def}}{=} {}^{\omega_1}\mathrm{Fr}(\omega_2)^{\mathrm{w}}$ is such that $\mathfrak{f}(A) = \omega_1$, by [17, Proposition 1.15], while clearly $\pi_{\min}(A) = \omega_2$.

Problem 25 *Is there a Boolean algebra A such that* $\mathfrak{f}(A) < \mathrm{inc}_{\mathrm{mm}}^{\mathrm{tree}}(A)$*?*

By [16, 2.13] and [17, 1.6] there is an atomless Boolean algebra A such that $\mathfrak{f}(A) < \mathrm{s}_{\mathrm{mm}}(A)$. By [17, Proposition 1.6], Example 17 is such that $\mathfrak{f}(A) < \mathfrak{i}_n(A)$. By [17, Example 2.7] there is a Boolean algebra A such that $\mathfrak{f}(A) < \mathrm{tow}(A)$. By a result of Scherer, $\mathrm{inc}_{\mathrm{mm}}(A) = \omega$ for $A = \overline{\mathrm{Fr}(\omega)}$. Thus $\mathrm{inc}_{\mathrm{mm}}(A) < \mathrm{cf}(A)$.

Problem 26 *Is there a Boolean algebra A such that* $\mathrm{inc}_{\mathrm{mm}}(A) < \mathfrak{i}(A)$*?*

Problem 27 *Is there a Boolean algebra A such that* $\mathrm{inc}_{\mathrm{mm}}(A) < \mathrm{inc}_{\mathrm{mm}}^{\mathrm{tree}}(A)$*?*

Problem 28 *Is there a Boolean algebra A such that* $\mathrm{inc}_{\mathrm{mm}}(A) < \mathfrak{i}_n(A)$*?*

Problem 29 *Is there a Boolean algebra A such that* $\mathrm{inc}_{\mathrm{mm}}(A) < \mathfrak{p}(A)$*?*

Problem 30 *Is there a Boolean algebra A such that* $\mathrm{inc}_{\mathrm{mm}}(A) < \mathrm{tow}(A)$*?*

Note that for $A = \overline{\mathrm{Fr}(\omega)}$ we have $\mathrm{inc}_{\mathrm{mm}}(A) = \omega$ and $\mathfrak{l}(A) = 2^\omega$.

Problem 31 *Is there a Boolean algebra A such that* $\mathrm{inc}_{\mathrm{mm}}^{\mathrm{tree}}(A) < \mathrm{cf}(A)$*?*

Let $A = \overline{\mathrm{Fr}(\omega)}$. If $\mathrm{inc}_{\mathrm{mm}}^{\mathrm{tree}}(A) = \omega$, this solves Problem 33. If $\mathrm{inc}_{\mathrm{mm}}^{\mathrm{tree}}(A) > \omega$, this solves Problem 29.

Problem 32 *Is there a Boolean algebra A such that* $\mathrm{inc}_{\mathrm{mm}}^{\mathrm{tree}}(A) < \mathfrak{i}(A)$*?*

Problem 33 *Is there a Boolean algebra A such that* $\mathrm{inc}_{\mathrm{mm}}^{\mathrm{tree}}(A) < \pi\chi_{\mathrm{inf}}(A)$*?*

Problem 34 *Is there a Boolean algebra A such that* $\mathrm{inc}_{\mathrm{mm}}^{\mathrm{tree}}(A) < \mathfrak{p}(A)$*?*

Problem 35 *Is there a Boolean algebra A such that* $\mathrm{inc}_{\mathrm{mm}}^{\mathrm{tree}}(A) < \mathfrak{i}_n(A)$*?*

Problem 36 *Is there a Boolean algebra A such that* $\mathrm{inc}_{\mathrm{mm}}^{\mathrm{tree}}(A) < \mathfrak{l}(A)$*?*

Problem 37 *Is there a Boolean algebra A such that* $\mathrm{irr}_{\mathrm{mm}}(A) < \mathrm{cf}(A)$*?*

Problem 38 *Is there a Boolean algebra A such that* $\mathrm{irr}_{\mathrm{mm}}(A) < \mathfrak{i}(A)$*?*

Problem 39 *Is there a Boolean algebra A such that* $\mathrm{irr}_{\mathrm{mm}}(A) < \pi\chi_{\mathrm{inf}}(A)$*?*

Problem 40 *Is there a Boolean algebra A such that* $\mathrm{irr}_{\mathrm{mm}}(A) < \mathrm{inc}_{\mathrm{mm}}(A)$*?*

Problem 41 *Is there a Boolean algebra A such that* $\mathrm{irr}_{\mathrm{mm}}(A) < \mathrm{inc}_{\mathrm{mm}}^{\mathrm{tree}}(A)$*?*

Problem 42 *Is there a Boolean algebra A such that* $\mathrm{irr}_{\mathrm{mm}}(A) < \mathfrak{i}_n(A)$*?*

We describe a partial result relevant to Problem 44.

Proposition 43 *If* κ *is an infinite cardinal, then* $\mathfrak{i}_n(\mathrm{Fr}(\kappa)) = \kappa$.

P r o o f . Suppose that $X \subseteq \mathrm{Fr}(\kappa)$ is n-independent, with $|X| < \kappa$. Let y be a free generator not in the support of any element of X. Suppose that $R \in [X]^{<\omega}$, $\varepsilon \in {}^R 2$, $\delta \in 2$, and $\prod_{a\in R} a^{\varepsilon(a)} \cdot y^\delta = 0$. Then $\prod_{a\in R} a^{\varepsilon(a)} = 0$, and so there is an $R' \subseteq R$ with $|R'| \le n$ such that $\varepsilon[R'] = \{1\}$ and $\prod_{a\in R'} a = 0$. □

Now by this proposition, the algebra $A = \mathrm{Fr}(\omega_1)$ is such that $\mathfrak{a}(A) < \mathfrak{i}_n(A)$.

Problem 44 *Is there a Boolean algebra A such that* $\mathrm{irr}_{\mathrm{mm}}(A) < \mathrm{tow}(A)$*?*

Concerning the relationship between π and $\mathrm{irr}_{\mathrm{mm}}$ we mention three results. By an easy argument given as [16, Proposition 1.2], if κ is an infinite cardinal and A is a subalgebra of $\wp(\kappa)$ containing $\mathrm{Intalg}(\kappa)$, then $\mathrm{irr}_{\mathrm{mm}}(A) = \kappa$. Another result of [16] is that there is an atomless Boolean algebra A such that $\mathrm{irr}_{\mathrm{mm}}(A) = \omega = \pi(A) < 2^\omega = |A|$. [11] showed under $2^{\aleph_1} = \aleph_2$ that there is an atomic Boolean algebra A such that $\pi(A) = \aleph_1 < \mathrm{irr}_{\mathrm{mm}}(A)$.

Problem 45 *Is there a Boolean algebra A such that* $\mathrm{s}_{\mathrm{mm}}(A) < \mathrm{cf}(A)$*?*

Relevant to Problem 46 is the result of [20] that, under CH there is an atomless Boolean algebra A such that $\mathrm{s}_{\mathrm{mm}}(A) < \mathfrak{u}(A)$. In unpublished work, [5] showed that it is consistent that $\mathrm{s}_{\mathrm{mm}}(\wp(\omega)/\mathrm{fin}) < \mathfrak{i}(\wp(\omega)/\mathrm{fin})$.

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Problem 46 *Can one construct in* ZFC *a Boolean algebra A such that* $\mathrm{s}_{\mathrm{mm}}(A) < \mathfrak{i}(A)$*?*

Problem 47 *Is there a Boolean algebra A such that* $\mathrm{s}_{\mathrm{mm}}(A) < \mathfrak{f}(A)$*?*

Example 17 gives an atomless Boolean algebra A such that $\mathrm{s}_{\mathrm{mm}}(A) < \pi_{\min}(A)$, by [16, Proposition 2.6].

Problem 48 *Is there a Boolean algebra A such that* $\mathrm{s}_{\mathrm{mm}}(A) < \mathrm{inc}^{\mathrm{tree}}_{\mathrm{mm}}(A)$*?*

Problem 49 *Is there a Boolean algebra A such that* $\mathrm{s}_{\mathrm{mm}}(A) < \mathfrak{i}_n(A)$*?*

By [16, Theorem 2.16] there is an atomless Boolean algebra A such that $\mathrm{s}_{\mathrm{mm}}(A) < \mathfrak{a}(A)$. A very similar proof gives an atomless Boolean algebra A such that $\mathrm{s}_{\mathrm{mm}}(A) < \mathrm{tow}(A)$.

Problem 50 *Is there a relationship between* $\mathfrak{i}_m(A)$ *and* $\mathfrak{i}_n(A)$ *for* $m \neq n$*?*

We have $\mathfrak{i}_n(\overline{\mathrm{Fr}(\omega)}) = \omega$. In fact, let X be a maximal n-independent subset of $\mathrm{Fr}(\omega)$ containing all the free generators of $\mathrm{Fr}(\omega)$. If $y \in \overline{\mathrm{Fr}(\omega)}\backslash\mathrm{Fr}(\omega)$, then there is a monomial z such that $z \leq y$. Then $z \cdot -y$ shows that $X \cup \{y\}$ is not n-independent. This example gives an atomless Boolean algebra A such that $\mathfrak{i}_n(A) < \mathrm{cf}(A)$.

Problem 51 *Is there an atomless Boolean algebra A such that* $\mathfrak{i}_n(A) < \mathfrak{i}(A)$*?*

Problem 52 *Is there an atomless Boolean algebra A such that* $\mathfrak{i}_n(A) < \pi\chi_{\inf}(A)$*?*

For $A = \overline{\mathrm{Fr}(\omega)}$ we have $\mathfrak{i}_n(A) < \pi_{\min}(A)$; cf. above, after Problem 50.

Problem 53 *Is there an atomless Boolean algebra A such that* $\mathfrak{i}_n(A) < \mathrm{inc}^{\mathrm{tree}}_{\mathrm{mm}}(A)$*?*

Problem 54 *Is there an atomless Boolean algebra A such that* $\mathfrak{i}_n(A) < \mathfrak{a}(A)$*?*

Problem 55 *Is there an atomless Boolean algebra A such that* $\mathfrak{i}_n(A) < \mathrm{tow}(A)$*?*

Problem 56 *Is there a Boolean algebra A such that* $\mathfrak{l}(A) < \mathrm{cf}(A)$*?*

For $A = \mathrm{Fr}(\omega_1)$ we have $\mathfrak{l}(A) = \omega < \omega_1 = \mathfrak{r}(A)$, by [14, Example 12].

Problem 57 *Is there a Boolean algebra A such that* $\mathfrak{l}(A) < \mathfrak{a}(A)$*?*

Problem 58 *Is there a Boolean algebra A such that* $\mathfrak{l}(A) < \mathfrak{h}(A)$*?*

For an atomless Boolean algebra A such that $\mathfrak{h}(A) < \mathfrak{s}(A)$, cf. [14, Example 14]. Let $A = \overline{\mathrm{Fr}(\omega_1)}$. Then $\mathfrak{s}(A) = \omega < \mathrm{cf}(A)$ by citean. The same example has $\mathfrak{r}(A) = \omega_1$. For an atomless Boolean algebra A such that $\mathfrak{s}(A) < \mathfrak{a}(A)$ cf. [13, Theorem 6].

Problem 59 *Is there an atomless Boolean algebra A such that* $\mathfrak{s}(A) < \mathfrak{l}(A)$*?*

Now we specialize these considerations to the Boolean algebra $\wp(\omega)/\mathrm{fin}$; cf. Figure 3. Here we add two functions special for this situation. For functions $f, g \in {}^\omega\omega$ we define $f \leq^* g$ if and only if $\exists m \forall n \geq m[f(m) \leq g(m)]$. A set $X \subseteq {}^\omega\omega$ is *dominating* if and only if for every $f \in {}^\omega\omega$ there is a $g \in X$ such that $f \leq^* g$. The set X is *bounded* if and only if there is a $f \in {}^\omega\omega$ such that $g \leq^* f$ for all $g \in X$. Then we let $\mathfrak{b}$ be the smallest size of an unbounded subset of ${}^\omega\omega$, and $\mathfrak{d}$ the smallest size of a dominating subset of ${}^\omega\omega$.

Many of the relations indicated in the diagram are proved in [2], and references are given for many consistency results. We indicate some additional facts. That $\mathfrak{p} = \mathfrak{t}$ is proved in [12]. The relative consistency of $\mathrm{s}_{\mathrm{mm}} < \mathfrak{c}$ is proved in [16]; of $\mathfrak{i}_n < \mathfrak{c}$ in [3]; of $\mathrm{inc}^{\mathrm{tree}}_{\mathrm{mm}} < \mathfrak{c}$ in [4] $\pi\chi_{\inf} = \mathfrak{r}$ by [1]. We have $\mathrm{inc} = \mathfrak{c}$ by [4]; cf. Corollary 9. That $\mathfrak{l} = \mathfrak{c}$ is given by an argument of Scherer. Obviously $\pi_{\min} = \mathfrak{c}$ and $\mathrm{card}_{\mathrm{H}-} = \mathfrak{c}$. We have $\mathfrak{d} \leq \mathrm{s}_{\mathrm{mm}}$ by [5] and $\omega_1 = \mathrm{alt} = \text{p-alt}$ by [6, 11.1 & 12.7].

It is relatively consistent that $\mathfrak{f} < \mathfrak{c}$. First, taking a countable model of CH, there is a P-point in this model, and this ultrafilter is generated by a decreasing ω_1-chain. In a countable support iteration of this model using Miller forcing, the ultrafilter generates an ultrafilter in the extension. By [17, Proposition 1.6], $\mathfrak{f} = \omega_1$ in the extension. The continuum is $\aleph_2$ in the extension.

We now give a detailed relative consistency proof for $\mathfrak{f} < \mathfrak{c}$ in which $\mathfrak{f} = \omega_1$ while the continuum is arbitrarily large. The forcing we use is a version of Mathias-Příkrý forcing. The following lemma is probably folklore.

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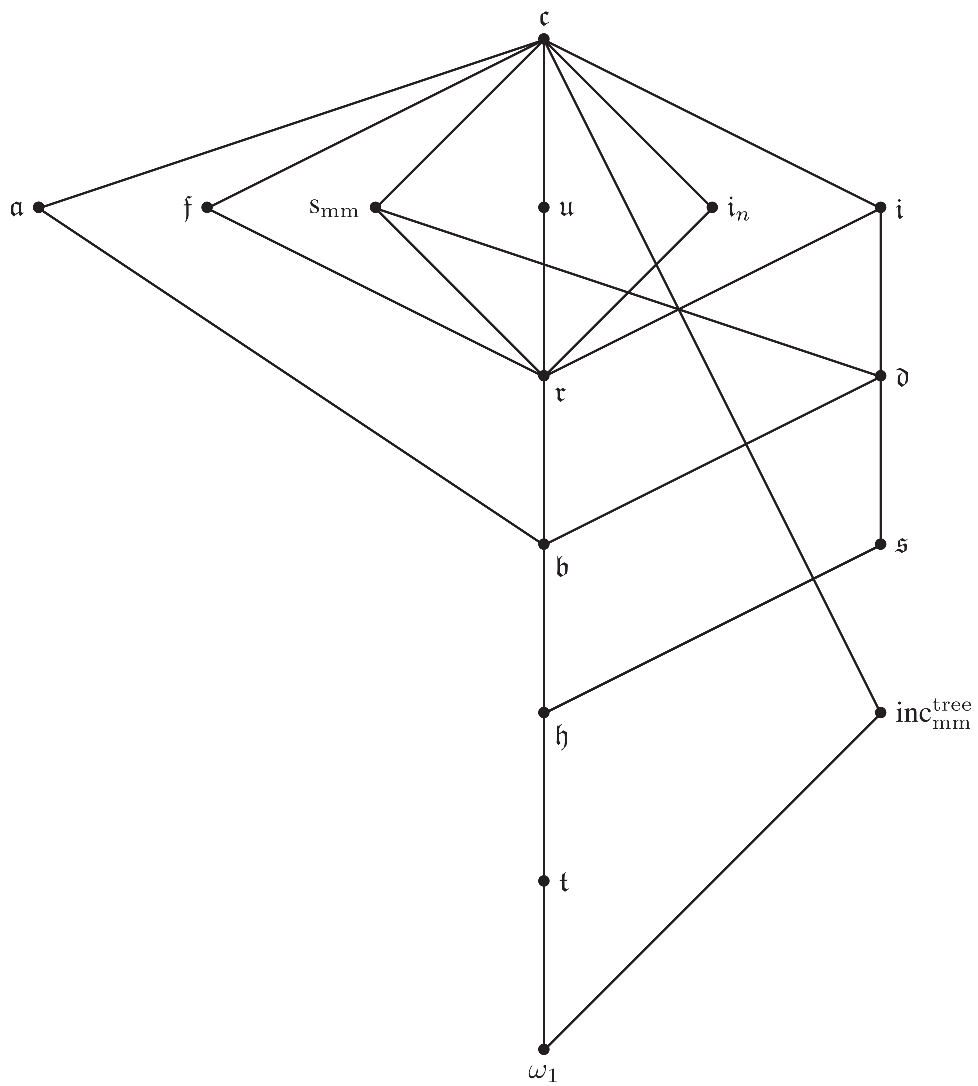

Fig. 3 $\wp(\omega)/\mathrm{fin}$

Lemma 60 *Let M be a c.t.m. of ZFC, and suppose that I is an ideal in $\wp(\omega)^M$ containing all singletons. Define $P = \{(b, y) : b \in I,\ y \in [\omega]^{<\omega}\}$ and $(b, y) \leq (b', y')$ if and only if $b \supseteq b'$, $y \supseteq y'$, and $y \cap b' \subseteq y'$. Then P is c.c.c. Let G be P-generic over M, and define $d = \bigcup_{(b,y)\in G} y$. Then the following conditions hold:*

(i) If $c \subseteq \omega$ and $c \notin I$, then $c \cap d$ is infinite.
(ii) If $c \subseteq \omega$ and $c \notin I$ then $c \backslash d$ is infinite.
(iii) If $b \in I$, then $b \cap d$ is finite.

Proof. Assume the hypotheses. Clearly P is c.c.c. For (i) and (ii), suppose that $c \subseteq \omega$ and $c \notin I$. For each $n \in \omega$ let $E_n := \{(b, y) : \exists m > n[m \in c \cap y\}$. To show that E_n is dense, let $(b, y) \in P$. Then $c\backslash b$ is infinite, as otherwise $c \subseteq b \cup (c\backslash b) \in I$. Choose $m > n$ with $m \in c\backslash b$. Then $(b, y \cup \{m\}) \in E_n$ and $(b, y \cup \{m\}) \leq (b, y)$, showing that E_n is dense. The denseness of each set E_n clearly implies (i).

Next, define for any $n \in \omega H_n := \{(b, y) \in P : \exists m > n[m \in b \cap c\backslash y]\}$. To show that H_n is dense, let $(b, y) \in P$ be given. Since every finite subset of ω is in I, the set c is infinite. Choose $m \in c\backslash y$ with $m > n$. Then $(b \cup \{m\}, y) \in H_n$ and $(b \cup \{m\}, y) \leq (b, y)$. This shows that H_n is dense.

Now given $n \in \omega$, choose $(b, y) \in H_n \cap G$, and then choose $m > n$ such that $m \in b \cap c\backslash y$. We claim that $m \notin d$. For, suppose that $m \in d$; say $m \in y'$ with $(b', y') \in G$. Choose $(b'', y'') \in G$ such that $(b'', y'') \leq (b, y), (b', y')$.

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Thus $y'' \cap b \subseteq y$ and $y'' \cap b' \subseteq y'$. Now $m \in y'$, so $m \in y''$; also $m \in b$, so $m \in y$, contradiction. This finishes the proof of (ii).

For (iii), suppose that $b \in I$. Now the set $\{(c, y) \in P : b \subseteq c\}$ is clearly dense, so choose $(c, y) \in G$ such that $b \subseteq c$. We claim that $b \cap d \subseteq y$. In fact, suppose that $m \in b \cap d$. Say $(e, z) \in G$ with $m \in z$. Choose $(u, v) \in G$ such that $(u, v) \leq (c, y), (e, z)$. So $v \cap c \subseteq y$ and $v \cap e \subseteq z$. Now $m \in z \subseteq v$, and $m \in b \subseteq c$, so $m \in v \cap c \subseteq y$, as desired; (iii) holds. □

Lemma 61 *Let M be a c.t.m. of* ZFC. *Suppose that α is an infinite ordinal, and $\langle a_\xi : \xi < \alpha\rangle$ is a system of infinite subsets of ω such that $\langle [a_\xi] : \xi < \alpha\rangle$ is a free sequence in $\wp(\omega)/\mathrm{fin}$, where $[a_\xi]$ denotes the equivalence class of a_ξ modulo the ideal* fin.

Then there is a generic extension $M[G]$ of M using a c.c.c. partial order such that in $M[G]$ there exist infinite $d, e \subseteq \omega$ with the following properties:

(i) $\langle [a_\xi] : \xi < \alpha\rangle^\frown\langle [\omega\backslash d], [e]\rangle$ *is a free sequence.*
(ii) *If* $x \in (\wp(\omega) \cap M)\backslash(\{a_\xi : \xi < \alpha\} \cup \{\omega\backslash d, e\})$, *then* $\langle [a_\xi] : \xi < \alpha\rangle^\frown\langle [\omega\backslash d], [e], [x]\rangle$ *is not a free sequence.*

Proof. For each $\xi \leq \alpha$, the set $\{[a_\eta] : \eta < \xi\} \cup \{-[a_\eta] : \xi \leq \eta < \alpha\}$ has the fip, by the free sequence property, and we let F_ξ be an ultrafilter on $\wp(\omega)/\mathrm{fin}$ containing this set.

Let $I = \{x : -[x] \in F_\xi \text{ for all } \xi \leq \alpha\}$. Clearly I is an ideal on $\wp(\omega)$ and $\{m\} \in I$ for all $m \in \omega$. We have that

$$\text{if } \xi < \eta < \alpha, \text{ then } [a_\eta]_I < [a_\xi]_I. \tag{7}$$

In fact, suppose that $\xi < \eta < \alpha$. If $\nu \leq \alpha$ and $[a_\eta] \cdot -[a_\xi] \in F_\nu$, then $\eta < \nu$, hence $\xi < \nu$ and so $[a_\xi] \in F_\nu$, contradiction. Hence $-([a_\eta] \cdot -[a_\xi]) \in F_\nu$ for all $\nu \leq \alpha$, and so $[a_\eta]_I \leq [a_\xi]_I$. Now suppose that $[a_\eta]_I = [a_\xi]$. Then $a_\xi \cdot -a_\eta \in I$, so in particular $-[a_\xi] + [a_\eta] \in F_{\xi+1}$. Since also $[a_\xi] \in F_{\xi+1}$, it follows that $[a_\eta] \in F_{\xi+1}$. But $\xi < \eta$, contradiction. So (7) holds.

$[a_0]_I \neq 1$. This holds since $-[a_0] \in F_0$, and hence $(\omega\backslash a_0) \notin I$.

If $\alpha = \beta + 1$, then $[a_\beta]_I \neq 0$. This is true since $[a_\beta] \in F_\alpha$, and hence $[a_\beta] \notin I$.

Now let J be an ideal in $\wp(\omega)$ which is maximal subject to the following conditions: $I \subseteq J$. If $\xi < \eta < \alpha$, then $a_\xi\backslash a_\eta \notin J$. $\omega\backslash a_0 \notin J$. If $\alpha = \beta + 1$, then $a_\beta \notin J$. Clearly then we have:

For any $x \subseteq \omega$ one of the following conditions holds. (8)

(a) $x \in J$.

(b) There exist $\xi < \eta < \alpha$ such that $a_\xi \cdot -a_\eta \cdot -x \in J$.

(c) $-a_0 \cdot -x \in J$.

(d) $\alpha = \beta + 1$ and $a_\beta \cdot -x \in J$.

Also we have that

$$\text{if } F, K \in [\alpha]^{<\omega} \text{ and } F < K, \text{ then } \bigcap_{\xi\in F} a_\xi \cap \bigcap_{\eta\in K} -a_\eta \notin J. \tag{9}$$

Now we apply Lemma 60 to the ideal J to obtain a generic extension $M[G]$ such that, with $d = \bigcup_{(b,y)\in G} y$, the following conditions hold:

$$\text{if } c \subseteq \omega \text{ and } c \notin J, \text{ then } c \cap d \text{ is infinite;}$$
$$\text{if } c \subseteq \omega \text{ and } c \notin J \text{ then } c\backslash d \text{ is infinite;} \tag{10}$$
$$\text{if } b \in J, \text{ then } b \cap d \text{ is finite.} \tag{11}$$

Hence by (9) we get

(a) If $F, K \in [\alpha]^{<\omega}$ and $F < K$, then $\bigcap_{\xi\in F} a_\xi \cap \bigcap_{\eta\in K} -a_\eta \cap d$ is infinite.
(b) If $F, K \in [\alpha]^{<\omega}$ and $F < K$, then $\bigcap_{\xi\in F} a_\xi \cap \bigcap_{\eta\in K} -a_\eta\backslash d$ is infinite.

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Now let K be the ideal in $\wp(\omega)^{M[G]}$ generated by J. If F is a finite subset of α, then $\bigcap_{\xi\in F} a_\xi \cap (\omega\backslash d) \notin K$. In fact, otherwise we get a $c \in J$ such that $\bigcap_{\xi\in F} a_\xi \cap (\omega\backslash d) \subseteq c$, and so $(\bigcap_{\xi\in F} a_\xi\backslash c) \cap (\omega\backslash d) = \varnothing$. But clearly $(\bigcap_{\xi\in F} A_\xi\backslash c) \notin J$, so this contradicts (10). Similarly,

$$\text{if } F, L \in [\alpha]^{<\omega} \text{ and } F < L, \text{ then } \bigcap_{\xi\in F} a_\xi \cap \bigcap_{\eta\in L} -a_\eta \cap d \notin K. \tag{12}$$

Now we apply Lemma 60 with I replaced by K to obtain a generic extension $M[G][H]$ and an infinite subset e of ω such that $\langle [a_\xi : \xi < \alpha\rangle^\frown\langle[\omega\backslash d], [e]\rangle$ is a free sequence and the following condition holds: if $b \in K$, then $b \cap e$ is finite. If $x \in (\wp(\omega) \cap M)\backslash(\{a_\xi : \xi < \alpha\} \cup \{\omega\backslash d, e\}$, then $\langle [a_\xi] : \xi < \alpha\rangle^\frown\langle[\omega\backslash d], [e], [x]\rangle$ is not a free sequence. To prove this, we consider cases.

Case 1. $x \in K$. Then $x \cap e$ is finite by (12), as desired.

Case 2. $x \notin K$. Then $x \notin J$, and so by (8) we have three subcases.

Subcase 2.1. There exist $\xi < \eta < \alpha$ such that $a_\xi \cdot -a_\eta \cdot -x \in J$. Then by (11), $a_\xi \cdot -a_\eta \cdot -x \cdot d$ is finite, as desired.

Subcase 2.2. $-a_0 \cdot -x \in J$. Then by (11), $-a_0 \cdot -x \cdot d$ is finite, as desired.

Subcase 2.3. $\alpha = \beta + 1$ and $a_\beta \cdot -x \in J$. Then by (11), $a_\beta \cdot -x \cdot d$ is finite, as desired. □

Theorem 62 *It is consistent with $2^\omega > \omega_1$ that $\mathfrak{f} = \omega_1$.*

Proof. We start with a model M of $2^\omega > \omega_1$ and a countable free sequence. Then we iterate ω_1 steps, using Lemma 61 at the successor steps. This builds a free sequence of length ω_1 in the final model. It is maximal by [10, VIII Lemma 5.14] and Lemma 61. □

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