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Remarks on continuum cardinals on Boolean algebras

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We give some results concerning various generalized continuum cardinals. The results answer some natural questions which have arisen in preparing a new edition of [5]. To make the paper self-contained we define all of the cardinal functions that enter into the theorems here. There are many problems concerning these new functions, and we formulate some of the more important ones.

1 Notation

For set-theoretical notation we follow [3]. We follow [2] for Boolean algebraic notation, and Monk [5] for more specialized notation concerning cardinal functions on Boolean algebras. $\mathrm{Fr}(\kappa)$ is the free Boolean algebra on κ generators. $\overline{A}$ is the completion of A.

If L is a linear order, then $\mathrm{Intalg}(L)$ is the interval algebra over L (perhaps after adjoining a first element to L). Any element x of $\mathrm{Intalg}(L)$ has the form $[a_0, b_0) \cup \cdots \cup [a_{m-1}, b_{m-1})$, with $a_0 < b_0 < \cdots < b_{m-1} \leq \infty$. (Here ∞ is not in L.) The intervals $[a_i, b_i)$ are called the *components* of x.

Now we define some notions entering into the definitions of our cardinal functions: A *tower* in A is a subset T of A well-ordered by the Boolean ordering in a limit ordinal type and with sum 1. We say that X is *weakly dense* in A if for all $a \in A$ there is an $x \in X^+$ such that $x \leq a$ or $x \leq -a$. A *weak partition* of A is a system $\langle b_\xi : \xi < \alpha \rangle$ of pairwise disjoint elements with sum 1; it is not assumed that all b_ξ are nonzero. We say that $X \subseteq A$ is *independent* if for all $F, G \in [X]^{<\omega}$ we have $\big[F \cap G = \varnothing \rightarrow \prod_{x\in F} x \cdot \prod_{x\in G} -x \neq 0\big]$; we say that it is *ideal independent* if for all $x \in X$ and all $F \in [X\backslash\{x\}]^{<\omega}$ we have that $x \not\leq \sum_{y\in F} y$. A set $X \subseteq A^+$ is *dense* in an ultrafilter D of A if for all $a \in D$ there is an $x \in X$ such that $x \leq a$. It is not assumed that $X \subseteq D$. A subset X of A^+ *splits* A if for every $a \in A$ for which $A\restriction a$ is infinite there is an $x \in X$ such that $a \cdot x \neq 0 \neq a \cdot -x$. A *free sequence* in a Boolean algebra A is a sequence $\langle a_\xi : \xi < \alpha \rangle$ of elements of A such that

$$\forall F, G \in [\alpha]^{<\omega}\left[\forall \xi \in F \forall \eta \in G[\xi < \eta] \longrightarrow \prod_{\xi\in F} a_\xi \cdot \prod_{\eta\in G} -a_\eta \neq 0\right].$$

Such a sequence is *maximal* if there is no $b \in A$ such that $\langle a_\xi : \xi < \alpha \rangle^\frown \langle b \rangle$ is still a free sequence.

The functions considered in this paper are as follows.

$$\begin{aligned}
c(A) &= \sup\{|X| : X \text{ is a disjoint subset of } A\};\\
\mathfrak{a}(A) &= \min\{|X| : X \text{ is an infinite partition of unity in } A\};\\
\mathfrak{p}(A) &= \min\{|Y| : \textstyle\sum Y = 1 \text{ and } \sum F \neq 1 \text{ for every finite } F \subseteq Y\};\\
\mathfrak{t}(A) &= \min\{|T| : T \text{ is a tower in } A\};\\
\mathfrak{r}(A) &= \min\{|X| : X \text{ is weakly dense in } A\};\\
\mathfrak{i}(A) &= \min\{|X| : X \subseteq A \text{ is maximal independent}\};
\end{aligned}$$

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$$\begin{aligned}
\mathfrak{u}(A) &= \min\{|X| : X \text{ generates a nonprincipal ultrafilter on} A\};\\
\mathfrak{l}(A) &= \min\{|X| : X \text{ is a maximal chain in } A\};\\
s_{\rm mm}(A) &= \min\{|X| : X \text{ is an infinite maximal ideal independent subset of } A\};\\
\mathfrak{s}(A) &= \min\{|X| : X \text{ weakly splits } A\};\\
\pi\chi_{\rm inf}(A) &= \min\{|X| : X \text{ is dense in some ultrafilter on } A\};\\
\mathfrak{f}(A) &= \min\{|\alpha| : \text{ there is a maximal free sequence of infinite length } \alpha\}.
\end{aligned}$$

The functions c and $\pi\chi_{\rm inf}$ are discussed in [5]; $s_{\rm mm}$ in [8]; $\mathfrak{l}$ in [7]; $\mathfrak{f}$ in [9]; the others in [6].

2 Arbitrary atomless Boolean algebras

The known relationships between our functions for atomless Boolean algebras are indicated in diagram 1, on the next page.

The relations $\pi\chi_{\rm inf} \le s_{\rm mm}$ and $\mathfrak{t} \le \mathfrak{l}$ are new in this paper. We also give an example with $\mathfrak{f} < \mathfrak{i}$.

The following proposition improves [8, Proposition 2.12], which states that $\mathfrak{r}(A) \le s_{\rm mm}(A)$ for any atomless Boolean algebra A.

Proposition 2.1 $\pi\chi_{\rm inf}(A) \le s_{\rm mm}(A)$ *for any atomless Boolean algebra* A.

Proof. Let X be maximal ideal independent with $|X| = s_{\rm mm}(A)$. Suppose that $2 \le m < \omega$. Now $\pi\chi_{\rm inf}(A)$ is the least size of a set which is m-dense for all $m \ge 2$; cf. [1]. We claim that

$$\left\{ y \cdot \prod_{x\in F} -x : y \in X, F \in [X\backslash\{y\}]^{<\omega} \right\}$$

is m-dense. For, suppose that $\langle b_i : i < m\rangle$ is a weak partition of A. For any $i < m$ we have two possibilities:

(1) There is a finite $F \in [X]^{<\omega}$ such that $b_i \le \sum F$.
(2) There exist $x \in X$ and a finite $F \subseteq X\backslash\{x\}$ such that $x \le b_i + \sum F$.

If (2) holds for some i, clearly the desired conclusion follows. If (1) holds for all i, ideal independence of X is contradicted. □

Some related open problems are:

Problem 2.2 *Is there an atomless Boolean algebra A such that* $\mathfrak{i}(A) < \pi\chi_{\rm inf}(A)$*?*

Problem 2.3 *Is there an atomless Boolean algebra A such that* $s_{\rm mm}(A) < \mathfrak{i}(A)$*?*

Example 2.4 Let κ be any cardinal greater that ω_1, and define $A = \{x \in {}^{\omega_1}\mathrm{Fr}(\kappa) : \{\alpha < \omega_1 : x_\alpha \neq 0\}$ is countable or $\{\alpha < \omega_1 : x_\alpha \neq 1\}$ is countable$\}$. Then $\mathfrak{u}(A), \mathfrak{f}(A) \le \omega_1$, and $\mathfrak{i}(A) \ge \kappa$. In fact, $F := \{x \in A : \{\alpha < \omega_1 : x_\alpha \neq 1\}$ is countable$\}$ is an ultrafilter such that $\mathfrak{u}(F) = \omega_1$. For each $\alpha < \omega_1$, let

$$a_\alpha(\beta) = \begin{cases} 0 & \text{if} \quad \beta \le \alpha,\\ 1 & \text{if} \quad \alpha < \beta.\end{cases}$$

Then $\langle a_\alpha : \alpha < \omega_1\rangle$ is strictly decreasing, with $1 > a_0$. It generates F, so it is a maximal free sequence. Thus $\mathfrak{f}(A) \le \omega_1$.

Now suppose that $X \subseteq A$ is independent with $|X| < \kappa$; we assume that X is maximal, and try to get a contradiction. We may assume that $\{\alpha < \omega_1 : x_\alpha \neq 0\}$ is countable for each $x \in X$. Fix $Y \in [X]^\omega$. Let $M = \{\alpha < \omega_1 : x_\alpha \neq 0 \text{ for some } x \in Y\}$; so M is countable. Let $a \in A$ be such that a_α is a free generator of $\mathrm{Fr}(\kappa)$ not in the support of any element x_α with $x \in \langle X\rangle$ and $0 < x_\alpha < 1$, for each $\alpha \in M$, and let $a_\alpha = 0$ if $\alpha \notin M$. By the maximality of X, choose a finite $G \subseteq X$ an $\varepsilon \in {}^G2$, and a $\delta \in 2$ such that $a^\delta \cdot \prod_{x\in G} x^{\varepsilon(x)} = 0$. Choose $y \in Y\backslash G$. Then $y \cdot \prod_{x\in G} x^{\varepsilon(x)} \neq 0$. Then there is an $\alpha \in M$ such that $(y \cdot \prod_{x\in G} x^{\varepsilon(x)})_\alpha \neq 0$. Hence $(a^\delta \cdot y \cdot \prod_{x\in G} x^{\varepsilon(x)})_\alpha \neq 0$, contradiction.

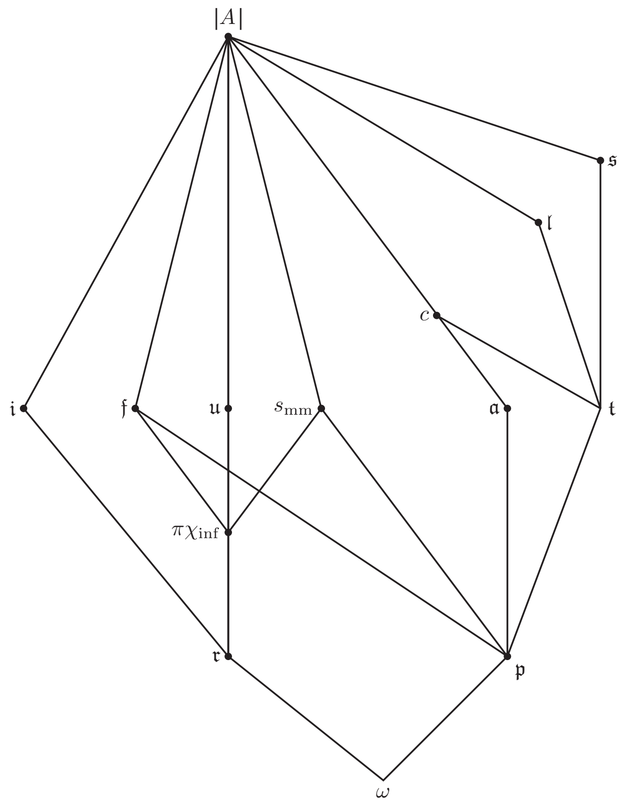


Fig. 1 Atomless Boolean algebras.

Example 2.4 provides the simplest known example of an atomless Boolean algebra A such that $\mathfrak{u}(A) < \mathfrak{i}(A)$. The first such example was, consistently, $A = \mathscr{P}(\omega)/\mathrm{fin}$. In [4], a ZFC example was given with a rather complicated proof.

In connection with $\mathfrak{f}$ and $\mathfrak{u}$ the following problem should be mentioned.

Problem 2.5 *Is* $\mathfrak{f}(A) = \mathfrak{u}(A)$ *for atomless Boolean algebras A?*

Maximal chains in Boolean algebras were investigated in [7], but the following simple connection with the well-known tower number was not mentioned.

Proposition 2.6 *If A is an atomless Boolean algebra, then* $\mathfrak{t}(A) \leq \mathfrak{l}(A)$.

Proof. Let C be a maximal chain in A. Then $1 \in C$, and $C\backslash\{1\}$ does not have a last element. Let κ be the cofinality of $C\backslash\{1\}$. Clearly $\mathfrak{t}(A) \leq \kappa$. □

In [4] it was shown that $\mathfrak{i}\left(\prod_{i\in I} A_i\right) = \min_{i\in I} \mathfrak{i}(A_i)$ whenever I is finite, with each A_i atomless. An open problem concerns whether this extends to infinite products.

Problem 2.7 *For $\langle A_i : i \in I\rangle$ a system of atomless Boolean algebras, with I infinite, is* $\mathfrak{i}\left(\prod_{i\in I} A_i\right) = \min_{i\in I} \mathfrak{i}(A_i)$*?*

The following proposition gives a small result concerning this problem.

Proposition 2.8 *If I is an infinite set and $\langle \kappa_i : i \in I\rangle$ is a system of infinite cardinals, then* $\mathfrak{i}(\prod_{i\in I} \mathrm{Fr}(\kappa_i)) = \min_{i\in I} \kappa_i$.

Proof. For, $\leq$ holds by [6, Proposition 7(ii)]. Now suppose that $X \subseteq \prod_{i\in I} \mathrm{Fr}(\kappa_i)$ is independent with $|X| < \min_{i\in I} \kappa_i$. It suffices to show that X is not maximal. Let $x \in \prod_{i\in I} \mathrm{Fr}(\kappa_i)$ be such that for every $i \in I$, x_i is a free generator of $\mathrm{Fr}(\kappa_i)$ not in the support of any element y_i for $y \in X$. Clearly $X \cup \{x\}$ is still independent. □

3 Complete Boolean algebras

Many of the continuum cardinals are somewhat mysterious for complete Boolean algebras; hence the following results are of some interest.

Theorem 3.1 *If A is an atomless σ-complete Boolean algebra, then $s_{\mathrm{mm}}(A) > \omega$.*

Proof. Suppose that $X \in [A]^\omega$ is ideal independent; we show that X is not maximal. Let $\langle x_i : i < \omega \rangle$ be a one-one enumeration of X. For each $n \in \omega$ let $y_n = x_n \cdot \prod_{m<n} -x_m$. Note that each y_n is nonzero, by ideal independence. By induction, $\sum_{n<m} y_n = \sum_{n<m} x_n$ for all $n \in \omega$, and hence $\sum_{n\in\omega} y_n = \sum_{n\in\omega} x_n$. Now let $n \in \omega$; we define an element z_n of A.

Case 1. There is a $j > n$ such that $y_n \cdot x_j \neq 0$. Take the least such j and let $z_n = y_n \cdot x_j$.

Case 2. $y_n \cdot x_j = 0$ for all $j > n$. Let z_n be any element such that $0 < z_n < y_n$.

Now let $w = \sum_{n\in\omega} z_n$. We claim that $w \notin X$, and $X \cup \{w\}$ is still ideal independent.

(1) $$\forall n \in \omega \left[w \not\leq \sum_{m<n} x_m \right].$$

This is clear, since $0 \neq z_{n+1} \leq w$ and $z_{n+1} \cdot \sum_{m<n} x_m \leq y_{n+1} \cdot \sum_{m<n} x_m = 0$.

(2) $$\forall F \in [\omega]^{<\omega} \forall n \in \omega \backslash F \left[x_n \not\leq \sum_{j\in F} x_j + w \right]$$

For, suppose not, for certain F, n. Without loss of generality, $F = m\backslash\{n\}$ for some $m > n$. Now $-w = \sum_{j<\omega}(y_j \cdot -z_j) + -\sum_{j\in\omega} y_j$. Hence

(3) $$0 = x_n \cdot \prod_{\substack{j<m\\ j\neq n}} -x_j \cdot -w = x_n \cdot \prod_{\substack{j<m\\ j\neq n}} -x_j \cdot y_n \cdot -z_n.$$

Case (a) $z_n = y_n \cdot x_k$ with $k > n$ and $y_n \cdot x_k \neq 0$. Hence $-z_n = -y_n + -x_k = -x_n + \sum_{j<n} x_j + -x_k$, and so by (3),

$$0 = x_n \cdot \prod_{\substack{j<m\\ j\neq n}} -x_j \cdot y_n \cdot -z_n = x_n \cdot \prod_{\substack{j<m\\ j\neq n}} -x_j \cdot -z_n = x_n \cdot \prod_{\substack{j<m\\ j\neq n}} -x_j \cdot -x_k,$$

contradiction.

Case (b) $y_n \cdot x_k = 0$ for all $k > n$. Then by (3),

$$0 = x_n \cdot \prod_{\substack{j<m\\ j\neq n}} -x_j \cdot y_n \cdot -z_n = y_n \cdot -z_n,$$

contradiction. □

Example 3.2 There is an atomless complete Boolean algebra A such that $2^\omega = \mathfrak{l}(A) < c(A)$. Namely we take $A = {}^\kappa\overline{\mathrm{Fr}(\omega)}$, where $\kappa = (2^\omega)^+$. Clearly $c(A) = \kappa$. Now let L be a maximal chain in $\overline{\mathrm{Fr}(\omega)}$. For each $a \in L$ define $f_a \in A$ by setting $f_a(\alpha) = a$ for all $\alpha < \kappa$. We claim that $\{f_a : a \in L\}$ is a maximal chain in A. For, suppose that $g \in A\backslash\{f_a : a \in L\}$ and g is comparable with each member of $\{f_a : a \in L\}$. Say $g(0) = a$. Then a is comparable with each member of L, and so $a \in L$. Choose $\alpha < \kappa$ such that $g(\alpha) \neq a$. Now $g(\alpha)$ is comparable with each member of L, so $g(\alpha) \in L$. Say $b \in L$ and $a < b < g(\alpha)$. Then $f_b < g$. But $f_b(0) = b > a = g(0)$, contradiction.

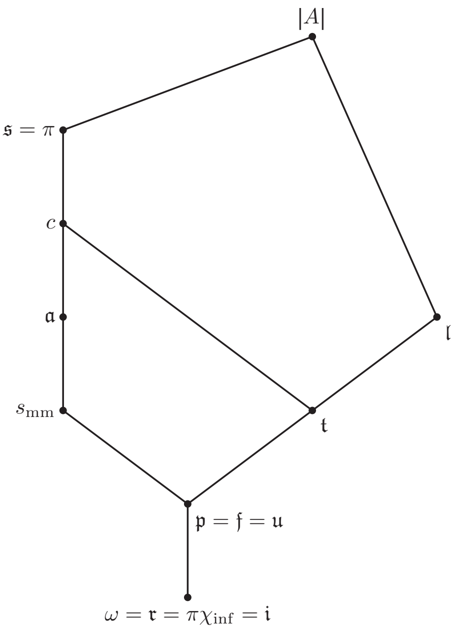


Fig. 2 Atomless interval algebras.

4 Atomless interval algebras

Theorem 4.1 *We have $s_{\mathrm{mm}} \leq \mathfrak{a}$ for atomless interval algebras.*

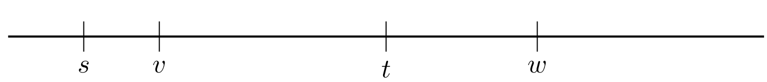
P r o o f. Let X be a partition of size $\mathfrak{a}(A)$. We may assume that each member of X has the form $[s, t)$. Clearly X is ideal independent; we claim that it is maximal ideal independent. For, take any $a \in A^+$. We may assume that

(*) $\quad [s, t) \not\subseteq [v, w)$ for every $[s, t) \in X$ and every component $[v, w)$ of a.

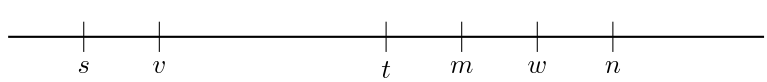
Now let $[v, w)$ be any component of a. We shall show that $[v, w)$ is contained in the union of one or two members of X. Hence a is contained in a finite union of members of X, as desired. We may assume that

(**) $\quad [v, w) \not\subseteq [s, t)$ for every $[s, t) \in X$.

Choose $[s, t) \in X$ such that $[v, w) \cap [s, t) \neq \varnothing$. Thus $\max(v, s) < \min(w, t)$. By (*) and (**) one of the following situations holds: $v < s < w < t$ or $s < v < t < w$. By symmetry we take only the case $s < v < t < w$:



Choose $[m, n) \in X$ such that $[t, w) \cap [m, n) \neq \varnothing$. By (*) we then have $t \leq m < w < n$. If $t = m$, then $[v, w) \subseteq [s, t) \cup [m, n)$, as desired. So suppose that $t < m$:



Now choose $[\alpha, \beta) \in X$ such that $[\alpha, \beta) \cap [t, m) \neq \varnothing$. Then $[\alpha, \beta) \subseteq [t, m) \subseteq [v, w)$, contradicting (*). □

The following natural question arises in connection with this result.

Problem 4.2 *Is there an atomless interval algebra A such that $s_{\mathrm{mm}}(A) < \mathfrak{a}(A)$?*

The following description of the character of ultrafilters in an interval algebra is a correction of the description on [5, p. 188].

Proposition 4.3 *Let L be a linear order with first element 0, let F be an ultrafilter on $A := \text{Intalg}(L)$, and let $T = \{a \in L : [0, a) \in F\}$ be the end segment determined by F. Then*:

(i) *If $T = [b, \infty)$ and b has an immediate predecessor c, then F is the principal ultrafilter determined by $\{c\}$.*
(ii) *If $T = [b, \infty)$ and b does not have an immediate predecessor, then the character of F is the cofinality of $[0, b)$.*
(iii) *If $T = (b, \infty)$ and b has an immediate successor, then F is the principal ultrafilter determined by $\{b\}$.*
(iv) *If $T = (b, \infty)$ and b does not have an immediate successor, then the character of F is the coinitiality of $[b, \infty)$.*
(v) *If there is no glb for T in L, then the character of F is the maximum of the left and right characters of the gap $(L\backslash T, T)$.*

Proposition 4.4 *If A is an atomless interval algebra, then $\mathfrak{u}(A) \leq \mathfrak{t}(A)$.*

Proof. By Proposition 4.3 and [6, Proposition 41]. □

In view of Proposition 4.3 we give a corrected proof for [6, Proposition 43].

Proposition 4.5 $\mathfrak{u}(A) = \mathfrak{p}(A)$ *for any atomless interval algebra.*

Proof. Since $\mathfrak{p}(A) \leq \mathfrak{u}(A)$ for any atomless Boolean algebra A, it suffices to show that $\mathfrak{u}(A) \leq \mathfrak{p}(A)$ for any atomless interval algebra. So, let L be a dense linear order with first element 0, and let $A = \text{Intalg}(L)$. Suppose that $X \subseteq A$, $\sum X = 1$, $\sum F \neq 1$ for every finite subset F of X, and $|X| = \mathfrak{p}(A)$. Without loss of generality, each member of X has the form $[u, v)$ with $u < v \leq \infty$. Clearly $\{-x : x \in X\}$ has fip, so this set is included in an ultrafilter U. It suffices now to show that the character of U is $\leq |X|$. Let

$$\begin{aligned} T &= \{v \in L : [0, v) \in U\}; \\ M &= \{w \in L : \exists v([v, w) \in X \text{ and } [w, \infty) \in U)\}; \\ N &= \{v \in L : \exists w([v, w) \in X \text{ and } [0, v) \in U\}. \end{aligned}$$

Clearly, $M \subseteq L\backslash T$ and $N \subseteq T$. Now by Proposition 4.3 there are several cases, which are all treated similarly; we restrict ourselves to the case $T = [b, \infty)$ for some b, and b does not have an immediate predecessor. Thus the character of U is the cofinality of $[0, b)$. It suffices now to show that M is cofinal in $[0, b)$. Suppose not; take $c < b$ such that $M \cap (c, b) = \varnothing$, and also choose d with $c < d < b$. Choose $[x, y) \in X$ such that $[c, d) \cap [x, y) \neq \varnothing$. Then $\max(c, x) < \min(d, y)$, so $x < b$ and hence $[x, \infty) \in U$. Hence $-[x, y) \cap [x, \infty) = [y, \infty) \in U$. So $y \in M$ and $c < y < b$, contradiction. □

Proposition 4.6 $\mathfrak{f}(A) = \mathfrak{u}(A)$ *for A an atomless interval algebra.*

Proof. By Proposition 4.5 and the fact that $\mathfrak{p} \leq \mathfrak{f}$ in general, it suffices to prove that $\mathfrak{f}(A) \leq \mathfrak{u}(A)$. Let L be a linear order with first element 0, F an ultrafilter with $\chi(F) = \mathfrak{u}(A)$, and let T be as in Proposition 4.3. By Proposition 4.3 we have the possibilities (ii), (iv) and (v). (ii) and (iv) are treated by [9, Proposition 1.12]; also the case (v) with equal left and right characters is taken care of by this proposition. Now suppose that (v) holds with different left and right characters; say the left and right characters of the indicated gap are κ and λ. Let $\langle a_\xi : \xi < \kappa \rangle$ be strictly increasing, $\langle b_\eta : \eta < \lambda \rangle$ strictly decreasing, $a_\xi < b_\eta$ for all $\xi < \kappa$ and $\eta < \lambda$, with no element c such that $\forall \xi < \kappa \forall \eta < \lambda [a_\xi < c < b_\eta]$. By symmetry suppose that $\kappa < \lambda$. For each $\varphi < \lambda$ write $\varphi = \kappa \cdot \rho + \xi$ with $\xi < \kappa$, and set $c_\varphi = [a_\xi, b_\varphi)$. Suppose that $F, G \in [\lambda]^{<\omega}$ and $F < G$. We may assume that $G \neq \varnothing$. First suppose also that $F \neq \varnothing$. Let φ be the greatest element of F and ψ the least element of G. Then $\bigcap F$ has the form $[a_\tau, b_\varphi)$ and $\cap_{e \in G} - e$ has the form $[0, a_\sigma) \cup [b_\psi, \infty)$. So $\bigcap F \cap \bigcap_{e \in G} -e$ contains $[b_\psi, b_\varphi)$ and hence is nonempty.

The case $F = \varnothing$ is also clear by this argument.

Thus we have a free sequence. To show that it is maximal, it suffices to show that $\{c_\varphi : \varphi < \lambda\}$ generates an ultrafilter. Let x be any member of A, with $0 < x < 1$. First suppose that x has a component $[u, v)$ with $u \notin L$ and $v \in L$. Choose $\xi < \kappa$ and $\eta < \lambda$ such that $u < a_\xi$ and $b_\eta < v$. Let $\varphi = \kappa \cdot \eta + \xi$. Then $\varphi \geq \eta$, and hence

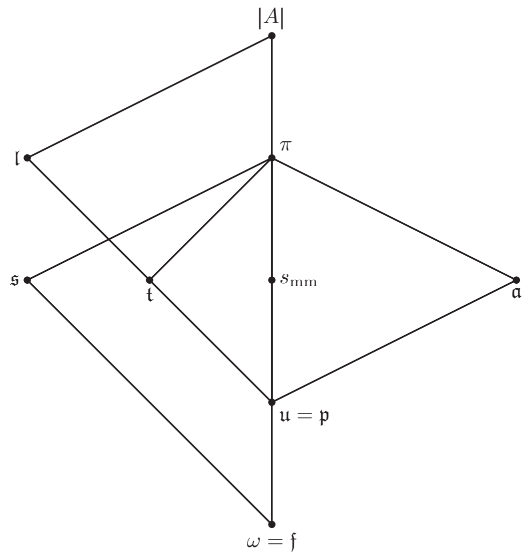


Fig. 3 Superatomic Boolean algebras.

$b_\varphi \le b_\eta$. So $c_\varphi \subseteq x$. Second, if x has no such component, then $-x$ does have such a component and so we get a $\varphi < \lambda$ such that $c_\varphi \subseteq -x$. □

5 Superatomic Boolean algebras

Consider Figure 3 for the relationships between the functions for superatomic Boolean algebras. Some of our cardinals are not defined for superatomic Boolean algebras, so they are omitted in the diagram.

Example 5.1 There is a superatomic Boolean algebra A such that $\mathfrak{l}(A) < \mathfrak{s}(A)$ and $s_{\rm mm}(A) < \mathfrak{s}(A)$. In fact, [6, Example 48] works here. Recall that $A = \langle X \cup \{\{\alpha\} : \alpha < \omega_1\rangle^{\mathscr{P}(\omega_1)}$ with X a partition of ω_1 into ω_1 sets each of size ω. It was shown in that article that $\mathfrak{s}(A) = \omega_1$. We now exhibit a countable maximal chain in A.

Let $\langle x_\alpha : \alpha < \omega_1\rangle$ be a one-one enumeration of X, and for each $\alpha < \omega_1$ let $\langle \beta_{\alpha m} : m \in \omega\rangle$ be a one-one enumeration of x_α. For $m, n \in \omega$ define

$$y_{mn} = \{\beta_{ij} : i < m, j \in \omega\} \cup \{\beta_{mj} : j < n\};$$
$$z_{mn} = \omega_1 \backslash (\{\beta_{\omega+i,j} : i < m, j \in \omega\} \cup \{\beta_{\omega+m,j} : j < n\}).$$

Clearly all of these elements are in A. Moreover:

$$y_{00} = \varnothing;\ y_{m,n+1} = y_{mn} \cup \{\beta_{mn}\};\ y_{m0} = \bigcup_{n<m, j\in\omega} y_{nj};$$
$$z_{00} = \omega_1;\ z_{m,n+1} = z_{mn} \backslash \{\beta_{mn}\};\ z_{m0} = \bigcap_{n<m, j\in\omega} z_{nj};\ y_{mn} \subseteq z_{pq}.$$

From this it follows that if $w \in A$ is different from all y_{mn}, z_{pq} but comparable to all of them, then $x_n \subseteq w$ and $x_{\omega+n} \cap w = \varnothing$ for all $n \in \omega$, contradiction.

Now we show that $s_{\rm mm}(A) = \omega$. Take any $x \in X$. Then it is easy to check that $\{\{\alpha\} : \alpha \in x\} \cup \{\omega_1 \backslash x\}$ is maximal ideal independent.

The following problem is open.

Problem 5.2 *Is there a superatomic Boolean algebra A such that $\mathfrak{a}(A) < s_{\rm mm}(A)$?*

Example 5.3 There is a superatomic Boolean algebra A such that $\mathfrak{t}(A) < s_{\rm mm}(A)$. We use a construction from [6], the basic idea of which is due to Mati Rubin. First we describe that construction. Suppose that $\kappa < \lambda$ are uncountable regular cardinals. In the well-ordered set κ we insert an unordered set of elements of size λ directly below each limit ordinal, giving a partially ordered set T (which is not a tree). Notationally we have for each limit ordinal $\gamma < \kappa$ a new sequence $\langle x_{\gamma\delta} : \delta < \lambda \rangle$ of elements; these new elements are related to the elements of κ as follows:

$$\begin{aligned} \alpha < x_{\gamma\delta} &\quad \text{iff} \quad \alpha < \gamma; \\ x_{\gamma\delta} < \alpha &\quad \text{iff} \quad \gamma \leq \alpha; \\ x_{\gamma\delta} < x_{\gamma'\delta'} &\quad \text{iff} \quad \gamma < \gamma'. \end{aligned}$$

Now we let A be the algebra of subsets of T generated by all cones $[a, \infty)$ for $a \in T$. It is easy to see that every element $x \in A$ can be written in the form

$$x = F \cup ([c_0, d_0) \cup \ldots \cup [c_{m-1}, d_{m-1})) \backslash G,$$

where:

(a) F and G are finite sets of $x_{\gamma\delta}$'s.
(b) No c_i is an $x_{\gamma\delta}$, and no d_i is an $x_{\gamma\delta}$.
(c) $c_0 < d_0 < c_1 < \cdots < c_{m-1} < d_{m-1} \leq \infty$.
(d) $G \subseteq [c_0, d_0) \cup \ldots \cup [c_{m-1}, d_{m-1})$.
(e) $F \cap ([c_0, d_0) \cup \ldots \cup [c_{m-1}, d_{m-1})) = \varnothing$.

It was shown in [6] that $\mathfrak{t}(A) = \kappa$. Now we show that $s_{\rm mm}(A) > \kappa$. Suppose that $s_{\rm mm}(A) \leq \kappa$; we want to get a contradiction. Say that X is maximal ideal independent with $\omega \leq |X| \leq \kappa$. It is easy to check that every atom of A is below some member of X, and that each singleton $\{x_{\alpha\gamma}\}$ is in A and hence is an atom of A. For each $x \in X$ write

$$x = F_x \cup ([c_{0x}, d_{0x}) \cup \ldots \cup [c_{m_x-1,x}, d_{m_x-1,x})) \backslash G_x,$$

with conditions as above.

(1) For every limit $\alpha < \kappa$ there exist an $x \in X$ and an $i < m_x$ such that $c_{ix} < \alpha \leq d_{ix}$.

In fact, each $x_{\alpha\beta}$ is a member of some $x \in X$, and so there exist $\beta < \lambda$, $x \in X$ and $i < m_x$ such that $x_{\alpha\beta} \in [c_{ix}, d_{ix})$, as otherwise we would have $\{x_{\alpha\beta} : \beta < \lambda\} \subseteq \bigcup_{x \in X} F_x$, which is not possible, on cardinality grounds. Hence the conclusion of (1) holds.

(2) If $x \in X$ and $i < m_x$, then there is a finite $H \subseteq X$ such that $[0, d_{ix}) \subseteq \bigcup H$.

We prove (2) by induction on an α such that $c_{ix} = \alpha$. The conclusion is obvious if $\alpha = 0$. If $\alpha = \beta + 1$, then there is some $y \in X$ such that $\beta \in y$, and so there is a $j < m_y$ such that $c_{jy} \leq \beta < d_{jy}$. Apply the inductive hypothesis to c_{jy} to get a finite $H \subseteq X$ such that $[0, d_{jy}) \subseteq \bigcup H$, and then $[0, d_{ix}) \subseteq \bigcup H \cup x$. Now suppose that α is a limit ordinal. By (1), choose $y \in X$ and $j < m_y$ such that $c_{jy} < c_{ix} \leq d_{jy}$. Apply the inductive hypothesis to c_{jy} to get a finite $H \subseteq X$ such that $[0, d_{jy}) \subseteq \bigcup H$. Then $[0, d_{ix}) \subseteq \bigcup H \cup x$. Hence (2) holds. In particular, $d_{mx-1,x} \neq \infty$ for all $x \in X$.

Now by Fodor's theorem and (1), there exist $\Gamma \in [\kappa]^\kappa$ and $\beta < \kappa$ such that Γ is a set of limit ordinals, and for each $\alpha \in \Gamma$ there exist $x(\alpha) \in X$ and $i(\alpha) < m_{x(\alpha)}$ such that $\beta = c_{i(\alpha)x(\alpha)} < \alpha \leq d_{i(\alpha)x(\alpha)}$. By (2), we can find a finite $H \subseteq X$ such that $[0, \beta + 1) \subseteq \bigcup H$. Choose $\alpha \in \Gamma$ such that $d_{m_x-1,x} < \alpha$ for all $x \in H$. Note that for any $x \in H$ we have $d_{m_x-1,x} < \alpha \leq d_{i(\alpha),x(\alpha)} \leq d_{m_{x(\alpha)}-1,x(\alpha)}$ and hence $x \neq x(\alpha)$. Thus $x(\alpha) \notin H$. Take any $\gamma \in \Gamma$ with $d_{m_{x(\alpha)}-1,x(\alpha)} < \gamma$. Since $d_{m_{x(\alpha)}-1,x(\alpha)} < \gamma \leq d_{i(\gamma),x(\gamma)} \leq d_{x(\gamma)-1,x(\gamma)}$, we have $x(\alpha) \neq x(\gamma)$. Then, since $d_{m_{x(\alpha)}-1,x(\alpha)} < \gamma$, we have $x(\alpha) \subseteq [0, \gamma) \subseteq [0, \beta) \cup [c_{i(\gamma)x(\gamma)}, d_{i(\gamma)x(\gamma)}) \subseteq \bigcup H \cup x(\gamma)$, contradiction.

Example 5.4 There a superatomic Boolean algebra A such that $s_{\rm mm}(A) < \mathfrak{t}$. Namely, we take the algebra A of [6, Example 49], with $\kappa = \omega_2$ and $\lambda = \omega_1$. It is shown in [6] that $\mathfrak{t}(A) = \omega_2$. Now consider the following set:

$$\{\{x_{\omega\beta}\} : \beta < \lambda\} \cup \{\{a_i\} : i < \omega\} \cup \{[a_\omega, \infty)\}.$$

This is a partition of A, and hence it is ideal independent. To show that it is maximal ideal independent, let $x \in A$, and write x as in (1) of Example 49. If $a_\omega \leq c_0$, then $x \leq b + [a_\omega, \infty)$, where b is the sum of all members of F of the form $x_{\omega\beta}$. If $c_0 < a_\omega$, then $\{c_0\} \subseteq x$. This proves maximality.

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