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# On the Existence of Towers in Pseudo-Tree Algebras

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**Abstract** A *tower* in a Boolean algebra (BA) is a strictly increasing sequence, of regular order type, of elements of the algebra different from 1 but with sum 1. A *pseudo-tree* is a partially ordered set $T$ such that the set $T \downarrow t = \{s \in T : s < t\}$ is linearly ordered for every $t \in T$. If that set is well-ordered, then $T$ is a *tree*. For any pseudo-tree $T$, the BA $\mathrm{Treealg}(T)$ is the algebra of subsets of $T$ generated by all of the sets $T \uparrow t = \{s \in T : t \leq s\}$. The main theorem of this note is a characterization in tree terms of when $\mathrm{Treealg}(T)$ has a tower of order type $\kappa$ (given in advance).

**Keywords** Towers · Pseudo-trees · Trees · Linearly ordered sets

For background on these notions see Monk [7, 8], Koppelberg and Monk [4], and Heindorf [2]. For general notation and reference see Koppelberg [3], Kunen [5], and Monk [6]. Pseudo-trees generalize both linearly ordered sets and trees. A BA is isomorphic to a pseudo-tree algebra iff it is isomorphic to a subalgebra of an interval algebras. Towers exist in any atomless BA, and the author has considered this case in several articles. There is a tree algebra with no towers. Perhaps the easiest example of such is the tree $T$ that has a single root $r$, and $\omega_1$ immediate successors of the root, with no other elements. For this $T$, $\mathrm{Treealg}(T)$ is isomorphic to the BA of finite and cofinite subsets of $\omega_1$, and it is easy to see that this algebra has no towers. It is also true that there are linear orders whose associated interval algebras have no towers. These are somewhat harder to construct, so we give the details. The construction depends on the following characterization of existence of towers in interval algebras, which is a slight generalization of a result in Monk [7].

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**Theorem 1** *Suppose that $L$ is an infinite linear order with first element 0, and $\kappa$ is a regular cardinal. Then the following conditions are equivalent.*

(i) $\mathrm{Intalg}(L)$ *has a tower of order type $\kappa$.*
(ii) *One of the following holds:*

   (a) *There is a $c \in L$ and a strictly decreasing sequence $\langle a_\alpha : \alpha < \kappa\rangle$ of elements of $(c, \infty)$ coinitial with $c$.*
   (b) *There is a $c \in L \cup \{\infty\}$ and a strictly increasing sequence $\langle a_\alpha : \alpha < \kappa\rangle$ of elements of $[0, c)$ cofinal in $c$.*
   (c) *There exist a strictly increasing sequence $\langle b_\alpha : \alpha < \kappa\rangle$ of elements of $L$ and a strictly decreasing sequence $\langle c_\alpha : \alpha < \kappa\rangle$ of elements of $L$ such that $b_\alpha < c_\beta$ for all $\alpha, \beta < \kappa$, and there is no element $d \in L$ such that $b_\alpha < d < c_\beta$ for all $\alpha, \beta < \kappa$.*

*Proof* (ii)⇒(i): Assume (ii)(a). Then $\langle [0, c) \cup [a_\alpha, \infty) : \alpha < \kappa\rangle$ is a tower in $\mathrm{Intalg}(L)$.

Assume (ii)(b). Then $\langle [0, a_\alpha) \cup [c, \infty) : \xi < \kappa\rangle$ is a tower in $\mathrm{Intalg}(L)$.

Assume (ii)(c). Then $\langle [0, b_\alpha) \cup [c_\alpha, \infty) : \alpha < \kappa\rangle$ is a tower in $\mathrm{Intalg}(L)$.

(i)⇒(ii): Let $\langle a_\alpha : \alpha < \kappa\rangle$ be a tower in $\mathrm{Intalg}(L)$. Write

$$a_\alpha = \left[b^\alpha_0, c^\alpha_0\right) \cup \ldots \cup \left[b^\alpha_{m_\alpha - 1}, c^\alpha_{m_\alpha - 1}\right),$$

where $0 \le b^\alpha_0 < c^\alpha_0 < \cdots < b^\alpha_{m_\alpha-1} < c^\alpha_{m_\alpha-1} \le \infty$. Clearly $b^\alpha_0 \ge b^\beta_0$ if $\alpha < \beta < \kappa$. If each $b^\alpha_0 \ne 0$, then clearly 0 is the g.l.b. of $\langle b^\alpha_0 : \alpha < \kappa\rangle$, and (ii)(a) holds with $c = 0$. Hence we may assume that $b^\alpha_0 = 0$ for all $\alpha < \kappa$. Clearly $c^\alpha_0 \le c^\beta_0$ if $\alpha < \beta < \kappa$. If $\sup_{\alpha<\kappa} c^\alpha_0 = \infty$, then (ii)(b) holds for $\infty$ and some subsequence of $\langle c^\alpha_0 : \alpha < \kappa\rangle$. So we may assume that $\sup_{\alpha<\kappa} c^\alpha_0 \ne \infty$. We now consider several cases.

**Case 1** There is an $\alpha < \kappa$ such that $c^\beta_0 = c^\alpha_0$ for all $\beta \in [\alpha, \kappa)$. Clearly then there is a subsequence of $\langle b^\alpha_1 : \alpha < \kappa\rangle$ which gives (ii)(a).

**Case 2** There is no $\alpha$ as in Case 1, but $\sup_{\alpha<\kappa} c^\alpha_0$ exists. This gives (ii)(b).

**Case 3** There is no $\alpha$ as in Case 1, but $\sup_{\alpha<\kappa} c^\alpha_0$ does not exist. Clearly then there is a subsequence of $\langle b^\alpha_1 : \alpha < \kappa\rangle$ which gives (ii)(c). □

Using this theorem, we give an example of an infinite linear order whose interval algebra does not have any towers. The construction depends on some more-or-less standard notation, which we now introduce.

A *gap* in a linear order $L$ is a pair $(A, B)$ of nonempty subsets of $L$ such that $L = A \cup B$, $\forall a \in A \forall b \in B(a < b)$, $A$ has no largest element, and $B$ has no smallest element. The *lower character* of such a gap is the least cardinality of a subset of $A$ cofinal in $A$, and similarly for *upper character*; these are both regular cardinals. The *character* of $(M, N)$ is the pair of these characters.

*Example 2* There is an infinite linear order $L$ such that $\mathrm{Intalg}(L)$ does not have a tower.

*Proof* We start out with a dense linear order $M$ such that every point of $M$ has character $(\omega_1, \omega_1)$, the gaps of $M$ have characters $(\omega, \omega_1)$ or $(\omega_1, \omega)$, $M$ has no first or last element, and $M$ has cofinality and coinitiality $\omega_1$. The existence of $M$ follows from a theorem in Hausdorff [1]. We replace each element of $M$ by $\omega^* + \omega$, put $\omega$ to the left of the result, and $\omega^*$ to the right. For definiteness let $a$ and $b$ be one-one functions with domain $\omega$ such that $\mathrm{rng}(a) \cap \mathrm{rng}(b) = \emptyset$ and $(\mathrm{rng}(a) \cup \mathrm{rng}(b)) \cap (M \times \mathbb{Z}) = \emptyset$. Let

$$L = \mathrm{rng}(a) \cup \mathrm{rng}(b) \cup (M \times \mathbb{Z}),$$

and for $m, n \in \omega$, $p, q \in \mathbb{Z}$, and $c, d \in M$ define

$$\begin{array}{lll}
a_m < a_n & \text{iff} \quad m < n; & \\
b_m < b_n & \text{iff} \quad n < m; & \\
a_m < (c, p); & & \\
(c, p) < (c, q) & \text{iff} \quad p < q; & \\
(c, p) < (d, q) & \text{iff} \quad c < d, & \text{when } c \neq d; \\
a_m < b_n; & & \\
(c, p) < b_m. & &
\end{array}$$

Clearly this gives a linear order, and the elements do not have infinite characters. Thus it suffices by the preceding theorem to show that the characters of the gaps of $L$ are $(\omega, \omega_1)$ or $(\omega_1, \omega)$.

Suppose that $(A, B)$ is a gap in $L$. Then there are these possibilities:

**Case 1** $A = \{a_n : n \in \omega\}$. Then the character of $(A, B)$ is $(\omega, \omega_1)$.

**Case 2** There is a $c \in M$ such that $A = \{x \in L : x < (c, m)$ for every $m \in \mathbb{Z}\}$. Then the character of $(A, B)$ is $(\omega_1, \omega)$.

**Case 3** There is a $c \in M$ such that $A = \{x \in L : x < (d, m)$ for every $d > c$ and every $m \in \mathbb{Z}\}$. Then the character of $(A, B)$ is $(\omega, \omega_1)$.

**Case 4** There is a gap $(C, D)$ in $M$ such that $A = \{(c, m) : c \in C, m \in \mathbb{Z}\} \cup \mathrm{rng}(a)$. Then the character of $(A, B)$ is the same as the character of $(C, D)$, and thus is $(\omega, \omega_1)$ or $(\omega_1, \omega)$.

**Case 5** $A = \{x \in L : x < b_m$ for all $m \in \omega\}$. Then the character of $(A, B)$ is $(\omega_1, \omega)$. □

Now we turn to trees and pseudo-trees. The main result of this note depends upon the following well-known fact.

**Lemma 3** *Suppose that $\kappa$ is an uncountable regular cardinal and $T$ is a tree of height $\kappa$ with each level finite. Then $T$ has a branch of order type $\kappa$.*

*Proof* Assume the hypotheses. Give each $\mathrm{Lev}_\alpha(T)$ the discrete topology, and consider the product space $\prod_{\alpha<\kappa}\mathrm{Lev}_\alpha(T)$. It is compact. For each finite nonempty $F\subseteq\kappa$ let

$$C_F=\left\{f\in\prod_{\alpha<\kappa}\mathrm{Lev}_\alpha(T):\forall\alpha,\beta\in F[\alpha<\beta\Rightarrow f(\alpha)<f(\beta)]\right\}.$$

Each such set is closed. For if $f\notin C_F$, then there exist $\alpha,\beta\in F$ such that $\alpha<\beta$ and $f(\alpha)\not< f(\beta)$. The set $U\stackrel{\text{def}}{=}\left\{g\in\prod_{\alpha<\kappa}\mathrm{Lev}_\alpha(T):g(\alpha)=f(\alpha)\text{ and }g(\beta)=f(\beta)\right\}$ is then an open neighborhood of $f$ which is disjoint from $C_F$.

Each $C_F$ is nonempty. For let $\alpha$ be the largest member of $F$, choose $a\in\mathrm{Lev}_\alpha(T)$, and for each $\beta\in F$ with $\beta\le\alpha$ let $f(\beta)$ be the unique member of $\mathrm{Lev}_\beta(T)$ which is $\le a$. Extend $f$ in any way to a member of $\prod_{\alpha<\kappa}\mathrm{Lev}_\alpha(T)$. Then $f\in C_F$.

If $\mathsf{F}$ is a nonempty finite set of nonempty finite subsets of $\kappa$, then $\bigcap_{F\in\mathsf{F}}C_F$ is nonempty, since $C_{\bigcup\mathsf{F}}$ is a subset of each of them.

Hence by compactness, the intersection of all sets $C_F$ is nonempty. □

Now we turn to the main theorem. Since for any pseudo-tree $T$ there is another pseudo-tree $S$ with a minimum element such that $\mathrm{Treealg}(T)\cong\mathrm{Treealg}(S)$, we restrict ourselves to pseudo-trees with a minimum element.

**Theorem 4** *Let $\kappa$ be a regular cardinal and let $T$ be a pseudo-tree with a minimum element. Then the following conditions are equivalent.*

(i) $\mathrm{Treealg}(T)$ *has a tower* $\langle a_\alpha:\alpha<\kappa\rangle$.

(ii) *There is a sequence $\langle x_\alpha:\alpha<\kappa\rangle$ of elements of $T$ such that $x_\alpha\le x_\beta$ whenever $\alpha<\beta<\kappa$, either the sequence is strictly increasing or has a constant value, and one of the following conditions holds:*

  (a) *$\langle x_\alpha:\alpha<\kappa\rangle$ is strictly increasing, and there is a finite set $F$ of incomparable elements of $T$ such that $x_\alpha<v$ for every $v\in F$, and $\forall w\in T[\forall\alpha<\kappa(x_\alpha<w)\Rightarrow\exists v\in F(v\le w)]$.*

  (b) *There exist countable sets $Y,Z$ and for each $y\in Y$ a strictly decreasing sequence $\langle t_{y\alpha}:\alpha<\kappa\rangle$ of elements of $T$, such that $Z\subseteq T$, and the following conditions hold:*

    (I) *$x_\beta<t_{y\alpha}$ for each $y\in Y$ and all $\alpha,\beta<\kappa$.*

    (II) *$x_\beta<z$ for each $z\in Z$ and each $\beta<\kappa$.*

    (III) *The members of $Z$ are pairwise incomparable.*

    (IV) *If $y$ and $z$ are distinct members of $Y$ and $\alpha,\beta<\kappa$, then $t_{y\alpha}$ and $t_{z\beta}$ are incomparable.*

    (V) *If $y\in Y$, $\alpha<\kappa$, and $z\in Z$, then $t_{y\alpha}$ and $z$ are incomparable.*

    (VI) *If $\kappa$ is uncountable, then $Y$ and $Z$ are finite, and $Y$ is nonempty.*

    (VII) *If $\kappa$ is uncountable, then there is a $y\in Y$ such that $\neg\exists v\in T\forall\delta,\beta<\kappa[x_\beta<v<t_{y\alpha}]$.*

    (VIII) *If $\kappa$ is uncountable and $x_\beta<v$ for all $\beta<\kappa$, then one of the following holds:*

      (A) *There is a $z\in Z$ such that $z\le v$.*

      (B) *There is a $y\in Y$ such that $v<t_{y\alpha}$ for all $\alpha<\kappa$.*

      (C) *There exist $y\in Y$ and $\alpha<\kappa$ such that $t_{y\alpha}\le v$.*

(IX) *If $\kappa$ is uncountable, $x_\beta < v$ for all $\beta < \kappa$, and there is a $y \in Y$ such that $v < t_{y\alpha}$ for all $\alpha < \kappa$, then there is a $y \in Y$ such that $v < t_{y\alpha}$ for all $\alpha < \kappa$, while there is no $w$ such that $v < w < t_{y\alpha}$ for all $\alpha < \kappa$.*

(X) *If $\kappa = \omega$ and $Y = \emptyset$, then $Z$ is infinite.*

(XI) *If $\kappa = \omega$ and $F$ is a finite set of elements of $T$ each greater than each $x_\beta$ for $\beta < \kappa$, then one of the following conditions holds:*

(A) *There is an $s \in Z$ such that $\forall x \in F(x \not\leq s)$.*

(B) *There exist $y \in Y$ and $l \in \omega$ such that $\forall x \in F(x \not\leq t_{yl})$.*

(XII) *If $\kappa = \omega$, $x_\beta < w$ for each $\beta < \kappa$, and $F$ is a finite subset of $(T \uparrow w)\backslash\{w\}$, then one of the following conditions holds:*

(A) *There is an $s \in Z$ such that $w$ and $s$ are comparable and $\forall x \in F(x \not\leq s)$.*

(B) *There exist $y \in Y$ and $l \in \omega$ such that $w$ and $t_{yl}$ are comparable and $\forall x \in F(x \not\leq t_{yl})$.*

*Proof* (i)⇒(ii): Suppose that $\langle a_\alpha : \alpha < \kappa\rangle$ is a tower in Treealg($T$). For each $\alpha < \kappa$ write $a_\alpha$ in full normal form:

$$\begin{aligned} a_\alpha &= \textstyle\bigcup_{t\in M_\alpha} e_{\alpha t};\\ e_{\alpha t} &= (T \uparrow t)\backslash \textstyle\bigcup_{s\in N_{\alpha t}} (T \uparrow s); \end{aligned}$$

where $M_\alpha$ is a finite subset of $T$, $N_{\alpha t}$ is a finite set of pairwise incomparable elements of $(T \uparrow t)\backslash\{t\}$, $e_{\alpha t} \cap e_{\alpha r} = \emptyset$ for $t \neq r$, and $t \notin N_{\alpha r}$ for $t \neq r$.

We now break the rest of the proof that (i)⇒(ii) into these cases:

*Case 1.* There is an element $r \in T$ such that $r \notin a_\alpha$ for all $\alpha < \kappa$.
*Subcase 1.1.* $\kappa$ is uncountable.
*Subcase 1.2.* $\kappa = \omega$.
*Case 2.* $\forall r \in T \exists \alpha < \kappa[r \in a_\alpha]$.

**Case 1** There is an element $r \in T$ such that $r \notin a_\alpha$ for all $\alpha < \kappa$. Let $x_\beta = r$ for all $\beta < \kappa$.

(1) If $\alpha < \kappa$ and $r < v \in a_\alpha$, then there is a $t \in M_\alpha$ such that $r < t \leq v$.

For, choose $t \in M_\alpha$ such that $v \in e_{\alpha t}$. Then $t \leq v$. Since also $r < v$, it follows that $t$ and $r$ are comparable. If $t \leq r$, then $r \in [t, v] \subseteq e_{\alpha t} \subseteq a_\alpha$, contradiction. Hence $r < t$, and (1) holds.

(2) There exist $\alpha < \kappa$ and $t \in M_\alpha$ with $r < t$.

In fact, choose $\alpha < \kappa$ such that $(T \uparrow r) \cap a_\alpha \neq \emptyset$. Say $s \in (T \uparrow r) \cap a_\alpha$. Then $r < s \in a_\alpha$, so by (1), there is a $t \in M_\alpha$ such that $r < t$.

We may assume that actually there is a $t \in M_0$ such that $r < t$. For each $\alpha < \kappa$ let $M'_\alpha = \{t \in M_\alpha : r < t\}$.

(3) If $\alpha < \beta < \kappa$ and $t \in M'_\alpha$, then there is an $s \in M'_\beta$ such that $s \leq t$.

In fact, the hypothesis implies that $t \in a_\alpha \subseteq a_\beta$. So (3) follows from (1).

Now for each $\alpha < \kappa$, let $M''_\alpha$ be the set of all minimal elements of $M'_\alpha$.

(4) If $\alpha < \beta < \kappa$ and $t \in M''_\alpha$, then there is a unique $s \in M''_\beta$ such that $s \le t$.

This is clear from (3).

For each $\alpha < \kappa$ and $t \in M''_\alpha$ we define a function $f_{\alpha t} : \kappa\backslash\alpha \to T$ by defining $f_{\alpha t}(\beta)$ to be the $s \in M''_\beta$ such that $s \le t$, for every $\beta \in \kappa\backslash\alpha$. Clearly if $\alpha \le \beta < \gamma < \kappa$ then $f_{\alpha t}(\gamma) \le f_{\alpha t}(\beta)$. Let

$$\begin{aligned} Y &= \{(\alpha, t) : \alpha < \kappa, t \in M''_\alpha,\ f_{\alpha t} \text{ is not eventually constant}\} \text{ and} \\ Z &= \{(\alpha, t) : \alpha < \kappa, t \in M''_\alpha,\ f_{\alpha t} \text{ is eventually constant}\}. \end{aligned}$$

For $(\alpha, t) \in Z$, let $s_{\alpha t}$ be the eventual value of $f_{\alpha t}$.

(5) If $(\alpha, t), (\beta, s) \in Z$ and $s_{\alpha t} \ne s_{\beta s}$, then $s_{\alpha t}$ and $s_{\beta t}$ are incomparable.

For choose $\gamma > \alpha, \beta$ so that $f_{\alpha t}(\gamma) = s_{\alpha t}$ and $f_{\beta s}(\gamma) = s_{\beta s}$. Then $s_{\alpha t}$ and $s_{\beta s}$ both belong to $M''_\gamma$, and so they are incomparable.

We now define

$$(\alpha, s) \equiv (\beta, t) \quad \text{iff} \quad (\alpha, s), (\beta, t) \in Y \text{ and } \exists\gamma > \alpha, \beta\forall\delta > \gamma[f_{\alpha s}(\delta) = f_{\beta t}(\delta)].$$

Clearly $\equiv$ is an equivalence relation on $Y$. Clearly

(6) If $(\alpha, s) \not\equiv (\beta, t)$, then $\forall\gamma \ge \alpha, \beta[f_{\alpha s}(\gamma) \ne f_{\beta t}(\gamma)]$.

Let $C \subseteq Y$ choose one element from each equivalence class. For each $(\alpha, s) \in C$ let $\langle\gamma_{\alpha s}(\delta) : \delta < \kappa\rangle$ be a strictly increasing sequence of ordinals less than $\kappa$ such that $\alpha \le \gamma_{\alpha s}(0)$ and $\langle f_{\alpha s}(\gamma_{\alpha s}(\delta)) : \delta < \kappa\rangle$ is strictly decreasing.

(7) If $(\alpha, s), (\beta, t)$ are distinct members of $C$ and $\delta, \varepsilon < \kappa$, then $f_{\alpha s}(\gamma_{\alpha s}(\delta))$ and $f_{\beta t}(\gamma_{\beta t}(\varepsilon))$ are incomparable.

For, let $\theta = \max(\gamma_{\alpha s}(\delta), \gamma_{\beta t}(\varepsilon))$. Suppose that $f_{\alpha s}(\gamma_{\alpha s}(\delta)) \le f_{\beta t}(\gamma_{\beta t}(\varepsilon))$. Then $f_{\alpha s}(\theta) \le f_{\alpha s}(\gamma_{\alpha s}(\delta)) \le f_{\beta t}(\gamma_{\beta t}(\varepsilon))$ and also $f_{\beta t}(\theta) \le f_{\beta t}(\gamma_{\beta t}(\varepsilon))$, so $f_{\alpha s}(\theta)$ and $f_{\beta t}(\theta)$ are comparable. Since they are both in $M''_\theta$, they must be equal. But then $(\alpha, s) \equiv (\beta, t)$, contradiction.

(8) If $(\alpha, s) \in C$ and $(\beta, t) \in Z$, and if $\delta < \kappa$, then $f_{\alpha s}(\delta)$ and $s_{\beta t}$ are incomparable.

This condition is obvious.

**Subcase 1.1** $\kappa$ is uncountable.

We may assume that there is a natural number $m$ such that $|M''_\alpha| = m$ for all $\alpha < \kappa$.

For any $\alpha < \kappa$, let $M'''_\alpha$ be a subset of $M''_\alpha$ satisfying the following conditions:

(a) If $t \in M'''_\alpha$, then $(\alpha, t) \in Z$.
(b) If $t$ and $u$ are distinct elements of $M'''_\alpha$, then $s_{\alpha t} \ne s_{\alpha u}$.
(c) If $t \in M''_\alpha$ and $(\alpha, t) \in Z$, then there is a $u \in M'''_\alpha$ such that $s_{\alpha t} = s_{\alpha u}$.

Again we may assume that all of these sets $M'''_\alpha$ have the same size, say $n$. Now if $t \in M'''_0$, then there is a $\beta_t < \kappa$ such that $f_{0t} \upharpoonright (\kappa\backslash\beta_t)$ is constant. Letting $\alpha = \sup_{t \in M'''_0} \beta_t$, we then see that $M'''_\alpha = M'''_\gamma$ for all $\gamma \in \kappa\backslash\alpha$. Thus we may assume that all sets $M'''_\alpha$ are equal.

(9) $|M_0'''| < m$.

In fact, suppose that $|M_0'''| = m$. Then $M_0''' = M_0''$. It follows that $(T \uparrow r) \setminus \bigcup_{s \in M_0'''} (T \uparrow s)$ is a nonempty set disjoint from each $a_\alpha$. It is nonempty since $r$ is a member of it. If $u \in a_\alpha \cap (T \uparrow r) \setminus \bigcup_{s \in M_0'''} (T \uparrow s)$, by (1) choose $t \in M_\alpha$ such that $r < t \leq u$. So $t \in M_\alpha'$, and so we can choose $s \in M_\alpha''$ such that $s \leq t$. Hence $s \in M_0'''$, and this contradicts the choice of $u$. So (9) holds.

(10) For the equivalence relation $\equiv$ above, there are at most $m - |M_0'''|$ equivalence classes.

For, suppose that $|C| > m - |M_0'''|$. Choose $\gamma$ greater than each $\gamma_{\alpha s}(0)$ such that $(\alpha, s) \in C$ for some $s$. Then $\{f_{\alpha s}(\gamma) : (\alpha, s) \in C\} \cup M_0'''$ is a set of more than $m$ elements of $M_\gamma''$, contradiction.

(11) $\exists(\alpha, s) \in C \neg \exists v \forall \delta < \kappa [r < v < f_{\alpha s}(\gamma_{\alpha s}(\delta))]$.

For, suppose not. For every $(\alpha, s) \in C$ choose $v_{\alpha s}$ such that $\forall \delta < \kappa [r < v_{\alpha s} < f_{\alpha s}(\gamma_{\alpha s}(\delta))]$. Choose $\varphi < \kappa$ such that

$$a_\varphi \cap \left[ (T \uparrow r) \setminus \left( \bigcup_{(\alpha,s) \in C} (T \uparrow v_{\alpha s}) \cup \bigcup_{w \in M_0'''} (T \uparrow w) \right) \right] \neq \emptyset,$$

and let $u$ be a member of this set. By (1), choose $v \in M_\varphi$ such that $r < v \leq u$. Then choose $w \in M_\varphi''$ such that $w \leq v$. Now $w \leq v \leq u$, so $w \notin M_0'''$. Hence $(\varphi, w) \in Y$. Choose $(\alpha, s) \in C$ such that $(\varphi, w) \equiv (\alpha, s)$. Choose $\psi > \varphi, \alpha$ such that $f_{\varphi w}(\psi) = f_{\alpha s}(\psi)$. Then

$$v_{\alpha s} \leq f_{\alpha s}(\gamma_{\alpha s}(\psi)) = f_{\varphi w}(\gamma_{\alpha s}(\psi)) \leq w \leq u,$$

contradiction. Thus (11) holds.

(12) If $r < v$, then one of the following holds:

(a) There is an $s \in M_0'''$ such that $s \leq v$.
(b) There is an $(\alpha, s) \in C$ such that $v < f_{\alpha s}(\gamma_{\alpha s}(\delta))$ for all $\delta < \kappa$.
(c) There exist $(\alpha, s) \in C$ and $\delta < \kappa$ such that $f_{\alpha s}(\gamma_{\alpha s}(\delta)) \leq v$.

For, suppose that $r < v$ and (a) fails. Choose $\alpha < \kappa$ such that

$$a_\alpha \cap \left( (T \uparrow v) \setminus \bigcup_{s \in M_0'''} (T \uparrow s) \right) \neq \emptyset.$$

Say that $u$ is in this set. By (1), choose $x \in M_\alpha'$ such that $x \leq u$. Choose $x' \in M_\alpha''$ such that $x' \leq x$. We claim that $f_{\alpha x'}$ is not eventually constant. For, suppose it is, with value $s$ taken on eventually. Then $s \in M_0'''$, and $s \leq x' \leq x \leq u$, contradiction. Thus the claim holds, and $(\alpha, x') \in Y$. Choose $(\beta, s) \in C$ such that $(\alpha, x') \equiv (\beta, s)$. Then choose $\gamma$ such that $f_{\alpha x'}(\delta) = f_{\beta s}(\delta)$ for all $\delta \geq \gamma$. Thus for any $\delta \geq \gamma$ we have

$$f_{\beta s}(\gamma_{\beta s}(\delta)) = f_{\alpha x'}(\gamma_{\beta s}(\delta)) \leq x' \leq x \leq u.$$

Since also $v \leq u$, it follows that $f_{\beta s}(\gamma_{\beta s}(\delta))$ and $v$ are comparable. If $f_{\beta s}(\gamma_{\beta s}(\delta)) \leq v$ for some $\delta < \kappa$, this gives (c). If $v < f_{\beta s}(\gamma_{\beta s}(\delta))$ for all $\delta < \kappa$, we get (b).

(13) If $r < v$ and there is an $(\alpha, s) \in C$ such that $v < f_{\alpha s}(\gamma_{\alpha s}(\delta))$ for all $\delta < \kappa$, then there is a $(\beta, t) \in C$ such that $v < f_{\beta t}(\gamma_{\beta t}(\delta))$ for all $\delta < \kappa$, while there is no $w$ such that $v < w < f_{\beta t}(\gamma_{\beta t}(\delta))$ for all $\delta < \kappa$.

Suppose that (13) fails, and let

$$D = \{(\alpha, s) \in C : \forall\delta < \kappa[v < f_{\alpha s}(\gamma_{\alpha s}(\delta))]\}.$$

Since (13) fails, for each $(\alpha, s) \in D$ there is a $w_{\alpha s}$ such that $\forall\delta < \kappa[v < w_{\alpha s} < f_{\alpha s}(\gamma_{\alpha s}(\delta))]$. Also, let $E = \{x \in M_0''' : v < x\}$. Choose $\alpha < \kappa$ such that

$$a_\alpha \cap (T \uparrow v) \setminus \left( \bigcup_{(\alpha,s)\in D} (T \uparrow w_{\alpha s}) \cup \bigcup_{x\in E} (T \uparrow x) \right)$$

is nonempty, and let $u$ be a member of this set. By arguments above, one of the following holds:

(a) There is an $x \in M_0'''$ such that $x \le u$. Since also $v \le u$, it follows that $x$ and $v$ are comparable. If $v < x$, then $x \in E$, giving a contradiction. If $x \le v$, (8) is contradicted.

(b) There exist an $(\alpha, s) \in C$ and $\delta < \kappa$ such that $f_{\alpha s}(\gamma_{\alpha s}(\delta)) \le u$. Again, $f_{\alpha s}(\gamma_{\alpha s}(\delta))$ and $v$ are comparable. If $f_{\alpha s}(\gamma_{\alpha s}(\delta)) \le v$, the hypothesis of (13) is contradicted. So $v < f_{\alpha s}(\gamma_{\alpha s}(\delta))$, and in fact this is true for all $\delta < \kappa$. So $(\alpha, s) \in D$, and again we have a contradiction. So (13) holds.

**Subcase 1.2** $\kappa = \omega$

Clearly

(14) $Y$ and $Z$ are countable.

(15) If $Y = \emptyset$, then $Z$ is infinite.

For, suppose that $Y = \emptyset$ but $Z$ is finite. Choose $i \in \omega$ such that

$$a_i \cap (T \uparrow r) \setminus \bigcup_{(j,s)\in Z} (T \uparrow s_{js}) \neq \emptyset.$$

Say $u$ is a member of this set. By (1), choose $t \in M_i$ such that $r < t \le u$. So $t \in M_i'$. Let $t' \in M_i''$ with $t' \le t$. Then $(i, t') \in Z$ and $s_{it'} \le t' \le t \le u$, contradiction.

(16) If $Y$ is finite and nonempty, and $Z$ is finite, then $\exists(i, t) \in Y \neg\exists v \forall l \in \omega[r < v \le f_{it}(\gamma_{it}(l))]$.

For, assume otherwise. So for each $(i, t) \in Y$ choose $v_{it}$ such that $\forall l \in \omega[r < v_{it} \le f_{it}(\gamma_{it}(l))$. Choose $i \in \omega$ such that

$$a_i \cap (T \uparrow r) \setminus \left( \bigcup_{(i,t)\in Y} (T \uparrow v_{it}) \cup \bigcup_{(j,s)\in Z} (T \uparrow s_{js}) \right) \neq \emptyset.$$

Let $w$ be a member of this set, and by (1) choose $t \in M_i$ such that $r < t \le w$. Thus $t \in M_i'$. Choose $t' \in M_i''$ such that $t' \le t$. Now there are two possibilities.

(a) $(i, t') \in Z$. Then $s_{it'} \le t' \le t \le w$, contradiction.

(b) $(i, t') \in Y$. Say $(i, t') \equiv (j, s) \in C$. Then for some $l$, $v_{js} \leq f_{js}(l) = f_{it'}(l) \leq t' \leq t \leq w$, contradiction.

So (16) holds.

(17) If $r < u$ then one of the following holds:

(a) There is a $(i, t) \in Z$ such that $s_{it}$ and $u$ are comparable.
(b) $f_{js}(l) \leq u$ for some $(j, s) \in C$ and $l \in \omega$.
(c) There is a $(j, s) \in C$ such that $u < f_{js}(l)$ for all $l \in \omega$.

To prove this, choose $i \in \omega$ and $v \in a_i \cap (T \uparrow u)$. By (1), choose $t \in M_i$ with $r \leq t \leq v$. So $t \in M_i'$. Choose $t' \in M_i''$ such that $t' \leq t$. Again there are two possibilities.

(I) $(i, t') \in Z$. Then $s_{it'} \leq t' \leq t \leq v$ and $u \leq v$, so (a) holds.
(II) $(i, t') \in Y$. Choose $(j, s) \in C$ such that $(i, t') \equiv (j, s)$. Then for some $l$, $f_{js}(l) = f_{it'}(l) \leq t' \leq v$ and also $u \leq v$, so (b) or (c) holds.

(18) Suppose that $F$ is a finite subset of $(T \uparrow r)\backslash\{r\}$. Then one of the following conditions holds:

(a) There is an $(i, t) \in Z$ such that $\forall x \in F(x \not\leq s_{it})$.
(b) There exist $(j, s) \in C$ and $l \in \omega$ such that $\forall x \in F(x \not\leq f_{js}(l))$.

For, suppose not. Choose $i \in \omega$ and

$$u \in a_i \cap (T \uparrow r)\backslash \bigcup_{x \in F} (T \uparrow x).$$

By (1) we get $t \in M_i$ with $r < t \leq u$. This easily gives a $t' \in M_i''$ with $t' \leq t$. There are two possibilities.

(I) $(i, t') \in Z$. Since (a) fails, choose $x \in F$ such that $x \leq s_{it'}$. Then $x \leq s_{it'} \leq u$, contradiction.
(II) $(i, t') \in Y$. Choose $(s, t) \in C$ such that $(i, t') \equiv (s, t)$. Then for some $l$ we have $f_{it'}(l) = f_{s,t}(l)$. Since (b) fails, choose $x \in F$ such that $x \leq f_{s,t}(l)$. Then $x \leq f_{st}(l) = f_{it'}(l) \leq u$, contradiction.

(19) Suppose that $r < w$ and $F$ is a finite subset of $(T \uparrow w)\backslash\{w\}$. Then one of the following conditions holds:

(a) There is an $(i, t) \in Z$ such that $w$ and $s_{it}$ are comparable and $\forall x \in F(x \not\leq s_{it})$.
(b) There exist $(j, s) \in C$ and $l \in \omega$ such that $f_{js}(l)$ and $w$ are comparable and $\forall x \in F(x \not\leq f_{js}(l))$.

The proof is similar to that of (18), and will be omitted.
Note that $C$ and $\{s_{0t} : t \in M_0'''\}$ play the role of $Y$ and $Z$ mentioned in the theorem.

**Case 2** $\forall r \in T \exists \alpha < \kappa [r \in a_\alpha]$. Let $r$ be the minimum element of $T$. We may assume that $r \in a_\alpha$ for all $\alpha < \kappa$. Hence $r \in M_\alpha$ for all $\alpha < \kappa$. Note that $N_{\alpha r} \neq \emptyset$, for all $\alpha < \kappa$.

(20) If $\alpha < \beta$ and $t \in N_{\beta r}$, then there is an $s \in N_{\alpha r}$ such that $s \leq t$.

Otherwise $\forall s \in N_{\alpha r}(s \not\leq t)$, so $t \in e_{\alpha r} \subseteq a_\alpha \subseteq a_\beta$, contradiction.

For each $\alpha < \kappa$ let $P_\alpha = \{(\alpha, s) : s \in N_{\alpha r}\}$. Define $(\alpha, s) < (\beta, t)$ iff $(\alpha, s) \in P_\alpha$, $(\beta, s) \in P_\beta$, $\alpha < \beta$, and $s \leq t$. This gives an ordinary tree of height $\kappa$ with $P_\alpha$ the set of elements of level $\alpha$, for each $\alpha < \kappa$. Then there is a branch $\langle(\alpha, s_\alpha) : \alpha < \kappa\rangle$, by Lemma 1 if $\kappa$ is uncountable, and by König's tree lemma if $\kappa = \omega$.

(21) $\langle s_\alpha : \alpha < \kappa\rangle$ is not eventually constant.

Suppose it is eventually constant with eventual value $t$. By the initial assumption of this case, choose $\alpha < \kappa$ such that $t \in a_\alpha$. Take $\beta \geq \alpha$ with $s_\beta = t$. Then $t \in N_{\beta r} \cap a_\beta$, contradiction.

By (21), let $\langle\gamma(\alpha) : \alpha < \kappa\rangle$ be a strictly increasing sequence of ordinals less than $\kappa$ such that $\langle s_{\gamma(\alpha)} : \alpha < \kappa\rangle$ is strictly increasing.

Now let $S = \{t \in T : \forall\alpha < \kappa[s_\alpha < t]\}$. If there is a finite set $V$ of incomparable elements of $S$ such that $\forall w \in S \exists v \in V[v \leq w]$, then (ii)(a) holds. So suppose there is no such set:

(22) There does not exist a finite set $V$ of incomparable elements of $S$ such that $\forall w \in S \exists v \in V[v \leq w]$.

In particular, $S$ is nonempty. Then by the assumption of case 2, there is an $\alpha < \kappa$ such that $a_\alpha \cap S \neq \emptyset$. Hence we may assume that $\forall\alpha < \kappa[a_\alpha \cap S \neq \emptyset]$.

(23) If $\alpha < \kappa$ and $t \in a_\alpha \cap S$, then there is a $u$ such that $t \in e_{\alpha u}$, $u \leq t$, and $u \in S \cap M_\alpha$.

For, take any $u \in M_\alpha$ such that $t \in e_{\alpha u}$. Then $u \leq t$. Take any $\beta \geq \alpha$. since $t \in S$, also $s_\beta < t$. So $u$ and $s_\beta$ are comparable. If $u \leq s_\beta$, then $s_\beta \in [u, t] \subseteq e_{\alpha u} \subseteq a_\alpha \subseteq a_\beta$, contradiction. So $s_\beta < u$. Since $\beta$ is arbitrary, $u \in S$, as desired.

(24) $\forall\alpha < \kappa[S \cap M_\alpha \neq \emptyset]$.

This is true by (23), since $\forall\alpha < \kappa[a_\alpha \cap S \neq \emptyset]$.

Now for each $\alpha < \kappa$ let

$$b_\alpha = \bigcup_{t \in S \cap M_\alpha} e_{\alpha t}.$$

Note that each $b_\alpha$ is a subset of $S$. Now

(25) If $\alpha < \beta$, then $b_\alpha \subseteq b_\beta$.

For, suppose that $u \in b_\alpha$. So also $u \in a_\alpha \subseteq a_\beta$. Hence by (23), $u \in e_{\beta v}$ for some $v \in S \cap M_\beta$. So $u \in b_\beta$.

(26) $\forall\alpha < \kappa \exists\beta \in (\alpha, \kappa)[b_\alpha \subset b_\beta]$.

Suppose to the contrary that $\alpha < \kappa$ and $\forall\beta \in (\alpha, \kappa)[b_\alpha = b_\beta]$. Let $P$ be the set of all minimal elements of $S \cap M_\alpha$. By (22), choose $w \in S$ such that $\forall v \in P[v \not\leq w]$. By the assumption of case 2, choose $\delta > \alpha$ such that $w \in a_\delta$. Now by (23) choose $x$ such that $x \leq w$, $w \in e_{\delta x}$, and $x \in S \cap M_\delta = S \cap M_\alpha$. Choose $y \in P$ such that $y \leq x$. Then $y \leq x \leq w$, contradiction. So (26) holds.

Let $\langle\xi(\alpha) : \alpha < \kappa\rangle$ be a strictly increasing sequence of ordinals less than $\kappa$ such that $\langle b_{\xi(\alpha)} : \alpha < \kappa\rangle$ is strictly increasing.

Let $T'$ be the tree consisting of $S$ together with a new minimum $z$, with the induced ordering from $T$. Note that $b_\alpha \in \mathrm{Treealg}(T')$ for every $\alpha < \kappa$, and for all $\alpha < \kappa$, $z \notin b_\alpha$.

(27) $\langle b_{\xi(\alpha)} : \alpha < \kappa \rangle$ is a tower in $\mathrm{Treealg}(T')$.

To prove this, suppose that

$$c = (T' \uparrow w) \backslash \bigcup_{x \in W} (T' \uparrow x)$$

with $w \in T'$, $W$ a finite set of incomparable elements of $(T \uparrow w) \backslash \{w\}$. Note that $T' \uparrow x = T \uparrow x$ for all $x \in W$. We want to find $\alpha < \kappa$ such that $c \cap b_{\xi(\alpha)} \neq \emptyset$. Now $W \subseteq S$, so by (22) choose $v \in S$ such that $\forall x \in W[x \not\leq v]$. Choose $\delta < \kappa$ such that $v \in a_\delta$. By (23), choose $y$ so that $v \in e_{\delta y}$, $y \leq v$, and $y \in S \cap M_\delta$. If $w = z$, then $y \in b_\delta \cap c$, as desired. Hence suppose that $z < w$. So $w \in S$ and $T' \uparrow w = T \uparrow w$. Hence we can choose $\varphi$ and $p \in a_\varphi \cap c$. By (23), choose $u$ such that $p \in e_{\varphi u}$, $u \leq p$, and $u \in S \cap M_\varphi$. Hence $p \in b_\varphi \cap c$. Thus (27) holds.

We can now apply Case 1 to finish the proof of (ii).

(ii)$\Rightarrow$(i): We consider several cases.

**Case 1** (ii)(a) holds. For each $\alpha < \kappa$ let

$$a_\alpha = [T \backslash (T \uparrow x_\alpha)] \cup \bigcup_{y \in F} (T \uparrow y).$$

Clearly $\langle a_\alpha : \alpha < \kappa \rangle$ is strictly increasing. To show that it is a tower, it suffices to take an element $b$ of the form

$$b = (T \uparrow w) \backslash \bigcup_{z \in W} (T \uparrow z),$$

where $W$ is a finite incomparable family of elements of $(T \uparrow w) \backslash \{w\}$, and find $\alpha < \kappa$ such that $a_\alpha \cap b \neq \emptyset$. If $x_\alpha \not\leq w$ for some $\alpha < \kappa$, then $w \in a_\alpha \cap b$. Suppose that $x_\alpha \leq w$ for all $\alpha < \kappa$. Then by (ii)(a) choose $y \in F$ such that $y \leq w$. Thus $b \subseteq a_0$, as desired.

**Case 2** (ii)(a) fails, and $\kappa$ is uncountable. For each $\alpha < \kappa$ let

$$a_\alpha = [T \backslash (T \uparrow x_\alpha)] \cup \bigcup_{y \in Y} (T \uparrow t_{y\alpha}) \cup \bigcup_{v \in Z} (T \uparrow v);$$

this gives a tower of order type $\kappa$. In fact, $\langle a_\alpha : \alpha < \kappa \rangle$ is clearly strictly increasing. Now take a nonzero element $b$ as above. If $x_\alpha \not\leq w$ for some $\alpha < \kappa$, clearly $w \in b \cap a_\alpha$. Hence we may assume that $x_\alpha \leq w$ for all $\alpha < \kappa$. Suppose that $x_\alpha = w$ for all $\alpha < \kappa$. By (ii)(b)(VII), choose $y \in Y$ such that $\neg \exists v \in T \forall \delta, \beta < \kappa [x_\beta < v < t_{y\delta}]$. Thus for every $z \in W$ we can choose $\alpha_z < \kappa$ such that $z \not\leq t_{y\alpha_z}$. Let $\beta = \max(\alpha_z : z \in W)$. Thus for all $z \in W$, $z \not\leq t_{y\beta}$. So $t_{y\beta} \in b \cap a_\beta$.

Since $\langle x_\alpha : \alpha < \kappa \rangle$ is strictly increasing or constant, the only other alternative here is that $x_\alpha < w$ for all $\alpha < \kappa$, so we assume this. If (ii)(b)(VIII)(A) holds, then $b \subseteq a_0$. If (ii)(b)(VIII)(C) holds for $y \in Y$, $\alpha < \kappa$, then $b \subseteq a_\alpha$. Suppose now that (ii)(b)(VIII)B) holds. By (ii)(b)(IX)) choose $y \in Y$ such that $\forall \delta < \kappa [w < t_{y\delta}]$ and there is no $v$ such that $\forall\delta < \kappa[w < v < t_{y\delta}]$. Then for each $z \in W$ there is a $\delta_z < \kappa$ such that $z \nless t_{y\delta_z}$. Let $\gamma = \sup_{z\in W} \delta_z$. Then $\forall z \in W[z \nleq t_{y\gamma}]$. So $t_{y\gamma} \in b \cap a_\gamma$.

Hence we have shown that $\langle a_\alpha : \alpha < \kappa\rangle$ is a tower.

**Case 3** $\kappa = \omega$ and $Y$ and $Z$ are both infinite. Write $Y = \{y_i : i \in \omega\}$ and $Z = \{s_i : i \in \omega\}$. For each $i \in \omega$ let

$$a_i = [T\backslash(T \uparrow x_i)] \cup \bigcup_{j\le i}(T \uparrow t_{y_j i}) \cup \bigcup_{j\le i}(T \uparrow s_j).$$

Clearly $\langle a_i : i \in \omega\rangle$ is strictly increasing. To show that it is a tower, suppose that $b$ is as above.

First suppose that $w = x_i$ for each $i \in \omega$. By (ii)(b)(XI), there are two possibilities.

(a) There is an $i < \omega$ such that $\forall x \in W(x \nleq s_i)$. Then $s_i \in a_i \cap b$.
(b) There exist an $i < \omega$ and $j \in \omega$ such that $\forall x \in W(x \nleq t_{y_i j})$. Then $t_{y_i j} \in a_k \cap b$, where $k = \max(i, j)$.

Second suppose that $x_i < w$ for all $i \in \omega$. By (ii)(b)(XII) there are two possibilities.

(a) There is an $i \in \omega$ such that $s_i$ and $w$ are comparable, and $\forall x \in W(x \nleq s_i)$. If $s_i \le w$, then $w \in a_i \cap b$. If $w < s_i$, then $s_i \in a_i \cap b$.
(b) There exist $i < \omega$ and $j \in \omega$ such that $w$ and $t_{y_i j}$ are comparable, and $\forall x \in W(x \nleq t_{y_i j})$. If $t_{y_i j} \le w$, then $w \in a_k \cap b$, with $k = \max(i, j)$. If $w < t_{y_i j}$, then $t_{y_i j} \in a_k \cap b$, with $k = \max(i, j)$.

**Case 4** $\kappa = \omega$, $Y$ is finite and nonempty and $Z$ infinite, with notation as above, define

$$a_i = [T\backslash(T \uparrow x_i)] \cup \bigcup_{y\in Y}(T \uparrow t_{yi}) \cup \bigcup_{j\le i}(T \uparrow s_j).$$

**Case 5** $\kappa = \omega$, $Y = \emptyset$ and $Z$ infinite, with notation as above, define

$$a_i = [T\backslash(T \uparrow x_i)] \cup \bigcup_{j\le i}(T \uparrow s_j).$$

**Case 6** $\kappa = \omega$, $Y$ is finite and nonempty and $Z$ finite. Define

$$a_i = [T\backslash(T \uparrow x_i)] \cup \bigcup_{y\in Y}(T \uparrow t_{yi}) \cup \bigcup_{s\in Z}(T \uparrow s).$$

□

Now we specialize this result to trees. Since in a tree there are no infinite decreasing chains, the set $Y$ in the theorem must be empty, and the conditions (and proof) simplify to give the following corollary.

**Corollary 5** *Let $T$ be an infinite tree. Then the following conditions are equivalent:*

(i) Treealg$(T)$ *has a tower.*
(ii) *One of the following conditions holds:*

  (a) *$T$ has an element with exactly $\omega$ immediate successors.*

(b) *$T$ has a chain of countable limit length with at most $\omega$ immediate successors.*
(c) *For some uncountable regular cardinal $\kappa$, $T$ has an chain of order type $\kappa$ with only finitely many immediate successors.*

Recall that for any BA $A$, $\mathfrak{t}_{\mathrm{spect}}(A)$ is the collection of all cardinalities of towers of $A$. The following theorem gives a tree construction of a result from Monk [8].

**Theorem 6** *If $K$ is a nonempty set of regular cardinals, then there is an atomless tree algebra $A$ such that $\mathfrak{t}_{\mathrm{spect}}(A) = K$.*

*Proof* Let $\lambda$ be the smallest member of $K$. Let $T_\omega$ be the tree ${}^{<\omega}\omega_1$. For each uncountable regular cardinal, let $T_\kappa$ be the tree determined by the following conditions. $T_\kappa$ has a unique root. $T_\kappa$ has height $\kappa$. Each element of $T_\kappa$, and each initial chain of $T$ of limit ordinal type has exactly $\omega_1$ immediate successors. Now the tree desired in the theorem is formed by adjoining a new root beneath the disjoint union of the following trees: $\omega_1$ copies of $T_\lambda$ and, for each $\kappa \in K\backslash\{\lambda\}$, a copy of $T_\kappa$. By the above theorems, $T$ is as desired. □

Although these theorems are fairly definitive, they use unusual trees—trees that have many elements directly above certain infinite chains. Let us call a tree $T$ *limit-normal* iff every initial chain of $T$ of limit ordinal length has at most one immediate successor. Concerning towers in such trees we have the following results.

**Corollary 7** *Suppose that $T$ is a limit-normal tree with a single root. Then the following conditions are equivalent:*

(i) $\mathrm{Treealg}(T)$ *has a tower.*
(ii) $\mathrm{Treealg}(T)$ *has a tower of order type $\omega$.*
(iii) *$T$ has an element with exactly $\omega$ immediate successors or $T$ has infinite height.*

**Corollary 8** *Suppose that $T$ is a limit-normal tree with a single root, with* $\mathrm{Treealg}(T)$ *atomless. Then* $\mathfrak{t}_{\mathrm{spect}}(\mathrm{Treealg}(T))$ *has the form $[\omega, \lambda)_{\mathrm{reg}}$ for some $\lambda > \omega$.*

## References

1. Hausdorff, F.: Grundzüge einer theorie der geordneten Mengen. Math. Ann. **65**, 435–482 (1908)
2. Heindorf, L.: On subalgebras of Boolean interval algebras. Proc. Am. Math. Soc. **125**, 2265–2274 (1997)
3. Koppelberg, S.: General theory of Boolean algebras. In: Monk, J.D., Bonnet, R. (eds.) Handbook of Boolean Algebras, vol. I, xix+312 pp. North-Holland, Amsterdam (1989)
4. Koppelberg, S., Monk, J.D.: Pseudo-trees and Boolean algebras. Order **8**, 359–374 (1991)
5. Kunen, K.: Set Theory, xvi+313 pp. North-Holland, Amsterdam (1980)
6. Monk, J.D.: Cardinal Invariants on Boolean Algebras, ix+298 pp. Birkhäuser, Boston (1996)
7. Monk, J.D.: Continuum cardinals generalized to Boolean algebras. J. Symb. Log. **66**, 1928–1958 (2001)
8. Monk, J.D.: Towers and maximal chains in Boolean algebras. Algebra Univers. **56**, 337–347 (2007)