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MAXIMAL IRREDUNDANCE AND MAXIMAL IDEAL INDEPENDENCE IN BOOLEAN ALGEBRAS

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Introduction. Recall that a subset X of an algebra A is *irredundant* iff $x \notin \langle X\backslash\{x\}\rangle$ for all $x \in X$, where $\langle X\backslash\{x\}\rangle$ is the subalgebra generated by $X\backslash\{x\}$. By Zorn's lemma there is always a maximal irredundant set in an algebra. This gives rise to a natural cardinal function $\mathrm{Irr}_{\mathrm{mm}}(A) = \min\{|X|: X$ is a maximal irredundant subset of $A\}$. The first half of this article is devoted to proving that there is an atomless Boolean algebra A of size 2^ω for which $\mathrm{Irr}_{\mathrm{mm}}(A) = \omega$.

A subset X of a BA A is *ideal independent* iff $x \notin \langle X\backslash\{x\}\rangle^{\mathrm{id}}$ for all $x \in X$, where $\langle X\backslash\{x\}\rangle^{\mathrm{id}}$ is the ideal generated by $X\backslash\{x\}$. Again, by Zorn's lemma there is always a maximal ideal independent subset of any Boolean algebra. We then consider two associated functions. A spectrum function

$$\mathrm{s}_{\mathrm{spect}}(A) = \{|X|: X \text{ is a maximal ideal independent subset of } A\}$$

and the least element of this set, $\mathrm{s}_{\mathrm{mm}}(A)$. We show that many sets of infinite cardinals can appear as $\mathrm{s}_{\mathrm{spect}}(A)$. The relationship of s_{mm} to similar "continuum cardinals" is investigated. It is shown that it is relatively consistent that $\mathrm{s}_{\mathrm{mm}}(\mathfrak{P}(\omega)/\mathrm{fin}) < 2^\omega$.

We use the letter s here because of the relationship of ideal independence with the well-known cardinal invariant *spread*; see Monk [5]. Namely, $\sup\{|X|: X$ is ideal independent in $A\}$ is the same as the spread of the Stone space $\mathrm{Ult}(A)$; the spread of a topological space X is the supremum of cardinalities of discrete subspaces.

Notation. Our set-theoretical notation is standard, with some possible exceptions, as follows. limord is the class of all limit ordinals, and reg is the class of all regular cardinals. If α and β are ordinals, then $[\alpha, \beta]_{\mathrm{card}}$ is the collection of all cardinals κ such that $\alpha \leq \kappa \leq \beta$; similarly $[\alpha, \beta]_{\mathrm{reg}}$ for the collection of all regular cardinals in this interval; and similarly for other intervals (half open, rays, etc.).

We follow Koppelberg [2] for Boolean algebraic notation, and Monk [5] for more specialized notation concerning cardinal functions on BAs. $\mathrm{Fr}(\kappa)$ is the free BA on κ generators. $\overline{A}$ is the completion of A. In several places we use the following construction. Let $\langle A_i : i \in I\rangle$ be a system of BAs, with I infinite. The *weak product* $\prod^{\mathrm{w}}_{i\in I} A_i$ consists of all members x of the full product such that one of the two sets

$$\{i \in I: x_i \neq 0\} \quad \text{or} \quad \{i \in I: x_i \neq 1\}$$

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is finite; the corresponding set is then called the *support* of x, and is denoted by $\mathrm{supp}(x)$; x is called *of type* I or II respectively.

If L is a linear order, then $\mathrm{Intalg}(L)$ is the interval algebra over L (perhaps after adjoining a first element to L).

For some results concerning $\mathrm{s_{mm}}$ we assume known the definitions of some other "continuum cardinals"; see Monk [6].

§1. **Irredundance.** The background for consideration of $\mathrm{Irr_{mm}}(A)$ is provided by the easy result of McKenzie, given as Proposition 4.23 in Koppelberg [2], that $\langle X\rangle$ is dense in A for any maximal irredundant subset X of A. Thus we have

THEOREM 1.1. $\pi(A) \leq \mathrm{Irr_{mm}}(A)$.

Here $\pi(A)$ is the smallest size of a dense subset of A.

PROPOSITION 1.2. *For any infinite cardinal* κ, *if* A *is a subalgebra of* $\mathfrak{P}(\kappa)$ *containing* $\mathrm{Intalg}(\kappa)$, *then* $\mathrm{Irr_{mm}}(A) = \kappa$.

PROOF. $\geq$ holds by Theorem 1.1, so we just need to exhibit a maximal irredundant set of size κ. Let

$$X = \{[0,\alpha) \colon 0 < \alpha < \kappa\}.$$

we claim that X is as desired. In fact, it is well-known and easy to see that X is irredundant.

Now suppose that $a \in A\backslash\langle X\rangle$; we want to show that $X \cup \{a\}$ is redundant. We may assume that $a \neq \emptyset, \kappa$. If $0 \notin a$, let α be the least member of a. Then $[0,\alpha+1)\backslash a = [0,\alpha)$, so that $[0,\alpha) \in \langle (X \cup \{a\})\backslash\{[0,\alpha)\}\rangle$. If $0 \in a$, let α be the least member of $\kappa\backslash a$. Then $[0,\alpha+1) \cap a = [0,\alpha)$, leading to the same conclusion. ⊣

Thus we have examples of atomic BAs A such that $\mathrm{Irr_{mm}}(A) = \pi(A) < \mathrm{Irr}(A)$. ($\mathrm{Irr}(A)$ is the supremum of cardinalities of irredundant subsets of A.)

THEOREM 1.3. *There is an atomless BA* A *such that* $\mathrm{Irr_{mm}}(A) = \omega = \pi(A) < 2^\omega = |A|$.

PROOF. We construct A as a subalgebra of $\overline{\mathrm{Fr}(\omega)}$. Let $\langle x_i : i \in \omega\rangle$ be a system of free generators of $\mathrm{Fr}(\omega)$. Now we make some definitions, working in $\overline{\mathrm{Fr}(\omega)}$ (recall here that for any element x of a BA, x^1 is x and x^0 is $-x$):

$$\begin{aligned}
N &= \{\varepsilon \in {}^{<\omega}2 \colon \mathrm{dmn}(\varepsilon) > 0 \text{ and } \varepsilon(\mathrm{dmn}(\varepsilon)-1) = 1\},\\
M &= \{\varepsilon \in N \colon \forall m < \mathrm{dmn}(\varepsilon) - 1(\varepsilon(m) = 0)\},\\
y_\varepsilon &= \prod_{i<\mathrm{dmn}(\varepsilon)} x_i^{\varepsilon(i)} \quad \text{for each } \varepsilon \in {}^{<\omega}2,\\
A &= \Big\langle \mathrm{Fr}(\omega) \cup \Big\{ \sum_{\varepsilon\in P} y_\varepsilon \colon P \subseteq M\Big\}\Big\rangle,\\
z_m &= \sum\{y_\varepsilon \colon \varepsilon \in M, \mathrm{dmn}(\varepsilon) \leq m\} \quad \text{for each } m \in \omega\backslash 1,\\
X &= \{y_\varepsilon \colon \varepsilon \in N\backslash M\} \cup \{z_m \colon m \in \omega\backslash 1\}.
\end{aligned}$$

Thus N is the set of all nonempty finite sequences of 0's and 1's that have 1 as their last entry, and M is the set of all members of N which are 0 except for that last entry. Clearly for any $\varepsilon, \delta \in {}^{<\omega}2$, either ε and δ are comparable under inclusion, and then y_ε and y_δ are comparable, or ε and δ are incomparable, and then $y_\varepsilon \cdot y_\delta = 0$. In particular, $\langle y_\varepsilon : \varepsilon \in M\rangle$ is a system of pairwise disjoint elements, and hence $|A| = 2^\omega$. Since $\mathrm{Fr}(\omega)$ is a dense subalgebra of A, it follows that A is atomless. We claim that X is a maximal irredundant subset of A, which will complete the proof. We prove this in several steps.

(1) $\langle X\rangle = \mathrm{Fr}(\omega)$.

In fact, clearly $X \subseteq \mathrm{Fr}(\omega)$, so $\subseteq$ holds. For the other inclusion, note first that if $\varepsilon \in M$, with domain m, then $y_\varepsilon = z_m \cdot -z_{m-1}$ if $m > 1$, and $y_\varepsilon = z_1$ if $m = 1$; hence $y_\varepsilon \in \langle X\rangle$ for every $\varepsilon \in N$. Now for any $n \in \omega$ we have

$$1 = \sum_{\varepsilon \in {}^n2} \prod_{i<n} x_i^{\varepsilon(i)}, \quad \text{and hence} \quad x_n = \sum_{\varepsilon \in {}^n2} \prod_{i<n} (x_i^{\varepsilon(i)} \cdot x_n) = \sum_{\varepsilon \in {}^n2 \cap N} y_\varepsilon \in \langle X\rangle.$$

This proves (1).

(2) $\sum_{\varepsilon \in M} y_\varepsilon = 1$.

To prove this, it suffices to show that for any $\delta \in {}^{<\omega}2$ there is an $\varepsilon \in M$ such that $y_\delta \cdot y_\varepsilon \neq 0$. If $\delta(i) = 1$ for some i, choose the least such i and let ε be the member of M with domain $i + 1$. Then $0 \neq y_\delta = y_\delta \cdot y_\varepsilon$. If $\delta(i) = 0$ for all $i < \mathrm{dmn}(\delta)$, let ε be the member of M with domain $\mathrm{dmn}(\delta) + 1$. Then $0 \neq y_\varepsilon = y_\delta \cdot y_\varepsilon$.

(3) Suppose that F and G are finite subsets of N. Then the following are equivalent:
 (a) $\prod_{\varepsilon \in F} y_\varepsilon \cdot \prod_{\delta \in G} -y_\delta = 0$.
 (b) $F \neq \emptyset$, and one of the following holds:
  (A) There are distinct $\varepsilon_1, \varepsilon_2 \in F$ which are incompatible.
  (B) $\rho \stackrel{\text{def}}{=} \bigcup F$ is a function, $\rho \in F$, and if $p \geq \mathrm{dmn}(\varepsilon)$ for each $\varepsilon \in F \cup G$, then for every $\sigma \in {}^p2$, if $\rho \subseteq \sigma$ then there is a $\delta \in G$ such that $\delta \subseteq \sigma$.

To prove (3), first suppose that (a) holds. Suppose that $F = \emptyset$. Let $\varepsilon \in M$ with domain greater than the domains of all y_δ for $\delta \in G$. Then $y_\varepsilon \cdot y_\delta = 0$ for all $\delta \in G$, so that $0 \neq y_\varepsilon \leq \prod_{\delta \in G} -y_\delta$, contradiction. So $F \neq \emptyset$.

Now suppose that (b)(A) fails. Then ρ as defined is a function, $\rho \in F$, and $\prod_{\varepsilon \in F} y_\varepsilon = y_\rho$. Thus by (a) we have $y_\rho \leq \sum_{\delta \in G} y_\delta$. Let p be as in (b)(B), and suppose that $\sigma \in {}^p2$ and $\rho \subseteq \sigma$, but $\delta \nsubseteq \sigma$ for all $\delta \in G$. Take a homomorphism of $\mathrm{Fr}(\omega)$ into 2 which takes each x_i with $i < p$ to $\sigma(i)$. Then y_ρ goes to 1, but each $y_\delta, \delta \in G$, goes to 0, contradicting the above inequality. So (b)(B) holds.

Conversely, assume (b). Clearly (b)(A) implies (a). Now assume (b)(B), and let p be any integer as indicated there. Then

$$\prod_{\varepsilon \in F} y_\varepsilon \cdot \prod_{\delta \in G} -y_\delta = y_\rho \cdot \prod_{\delta \in G} -y_\delta.$$

For every $\sigma \in {}^p2$ such that $\rho \subseteq \sigma$ choose $\delta_\sigma \in G$ such that $\delta_\sigma \subseteq \sigma$. Then

$$y_\rho = \sum\{y_\sigma : \rho \subseteq \sigma \in {}^p2\} \leq \sum\{y_{\delta_\sigma} : \rho \subseteq \sigma \in {}^p2\},$$

and (a) follows.

(4) If G is a finite subset of N, then $\prod_{\delta \in G} -y_\delta \neq 0$.

This is immediate from (3).

(5) If G is a finite subset of N, $\rho \in N$, and $\prod_{\delta \in G} -y_\delta \cdot y_\rho = 0$, then $\delta \subseteq \rho$ for some $\delta \in G$; in case $\rho \in M$, we have $\rho \in G$.

Assume the hypothesis of (5). Let p be $\geq$ the domains of all these functions, and let σ extend ρ to a function with domain p by adding 0's. Then by (3) we have $\delta \subseteq \sigma$ for some $\delta \in G$. Since δ ends with 1, we must actually have $\delta \subseteq \rho$. If $\rho \in M$, then ρ has only zeros except for its last entry, and hence $\rho = \delta \in G$.

(6) If $\rho \in N$, m is a positive integer, and $y_\rho \cdot z_m \neq 0$, then there is a $\delta \in M$ with $\mathrm{dmn}(\delta) \leq m$ such that $\delta \subseteq \rho$; so $y_\rho \leq z_m$.

For, choose $\delta \in M$ with $\mathrm{dmn}(\delta) \leq m$ such that $y_\rho \cdot y_\delta \neq 0$. If $\mathrm{dmn}(\rho) < \mathrm{dmn}(\delta)$, then $y_\rho \cdot y_\delta = 0$ since δ has all zero values except for its last one. So $\mathrm{dmn}(\delta) \leq \mathrm{dmn}(\rho)$, and hence $\delta \subseteq \rho$ since $y_\delta \cdot y_\rho \neq 0$. So (6) holds.

(7) X is irredundant.

To prove (7), first suppose that $\varepsilon \in N \backslash M$ and $y_\varepsilon \in \langle X \backslash \{y_\varepsilon\} \rangle$. Then there exist $n \in \omega$, $F, G \in {}^n([N \backslash (M \cup \{\varepsilon\})]^{<\omega})$ and $H, K \in {}^n([\omega \backslash 1]^{<\omega})$ such that

$$y_\varepsilon = \sum_{i<n} \Big(\prod_{\delta \in F_i} y_\delta \cdot \prod_{\delta \in G_i} -y_\delta \cdot \prod_{m \in H_i} z_m \cdot \prod_{m \in K_i} -z_m \Big),$$

where each summand is nonzero, and $|F_i|, |H_i|, |K_i| \leq 1$. Now take any $i < n$. Then

$$\prod_{\delta \in F_i} y_\delta \cdot \prod_{\delta \in G_i} -y_\delta \cdot \prod_{m \in H_i} z_m \cdot \prod_{m \in K_i} -z_m \cdot -y_\varepsilon = 0. \qquad (*)$$

Hence by (4) we have $F_i \neq \emptyset$ or $H_i \neq \emptyset$.

(8) $F_i \neq \emptyset$.

For, suppose that $F_i = \emptyset$. Then by the above remark, $H_i \neq \emptyset$. It follows that there is a $\rho \in M$ such that

$$\prod_{\delta \in G_i} -y_\delta \cdot y_\rho \cdot \prod_{m \in K_i} -z_m \neq 0 \text{ while } \prod_{\delta \in G_i} -y_\delta \cdot y_\rho \cdot \prod_{m \in K_i} -z_m \cdot -y_\varepsilon = 0.$$

Hence by (5) we have $\rho = \varepsilon$, contradicting $\varepsilon \notin M$. So (8) holds.

Henceforth we assume that $F_i = \{\rho_i\}$.

(9) If $H_i = \{m\}$, then $y_{\rho_i} \leq z_m$.

This follows from (6). Because of (9), we may assume that $H_i = \emptyset$.

(10) $\varepsilon \subset \rho_i$.

In fact, we now have

$$y_{\rho_i} \cdot \prod_{\delta \in G_i} -y_\delta \cdot \prod_{m \in K_i} -z_m \neq 0 = y_{\rho_i} \cdot \prod_{\delta \in G_i} -y_\delta \cdot \prod_{m \in K_i} -z_m \cdot -y_\varepsilon,$$

so the desired conclusion follows by (5) and the assumption that $\rho_i \neq \varepsilon$.

Now we can finish the proof of the first possibility in (7) as follows. We have

$$y_\varepsilon = \sum_{i<n} \Big(y_{\rho_i} \cdot \prod_{\delta \in G_i} -y_\delta \cdot \prod_{m \in K_i} -z_m \Big) \leq \sum_{i<n} y_{\rho_i} \leq y_\varepsilon,$$

so $y_\varepsilon = \sum_{i<n} y_{\rho_i}$. Now $\varepsilon \subset \rho_i$ for each i by (10). So if we take a homomorphism of $\mathrm{Fr}(\omega)$ into 2 which maps each x_i with $i < \mathrm{dmn}(\varepsilon)$ to $\varepsilon(i)$ and otherwise takes the value 0, the above equality becomes $1 = 0$, contradiction.

Now suppose that $q \in \omega\backslash 1$ and $z_q \in \langle X\backslash\{z_q\}\rangle$. Then there exist $n \in \omega$, $F, G \in {}^n([N\backslash M]^{<\omega})$, and $H, K \in {}^n([\omega\backslash\{q\}]^{<\omega})$ such that

$$z_q = \sum_{i<n}\Big(\prod_{\delta\in F_i} y_\delta \cdot \prod_{\delta\in G_i} -y_\delta \cdot \prod_{m\in H_i} z_m \cdot \prod_{m\in K_i} -z_m\Big),$$

where each summand is nonzero, and $|F_i|, |H_i|, |K_i| \le 1$. Note by (4) that $F_i \neq \emptyset$ or $H_i \neq \emptyset$. If $F_i = \emptyset$, let $H_i = \{m_i\}$, and if $F_i \neq \emptyset$, let $F_i = \{\rho_i\}$.

Now take any $i < n$. Then

$$\prod_{\delta\in F_i} y_\delta \cdot \prod_{\delta\in G_i} -y_\delta \cdot \prod_{m\in H_i} z_m \cdot \prod_{m\in K_i} -z_m \cdot -z_q = 0. \qquad (**)$$

(11) If $F_i = \emptyset = K_i$, then $m_i < q$.

For, if $\rho \in M$ and $\mathrm{dmn}(\rho) = m_i$, then $\rho \notin G_i$; hence from

$$\prod_{\delta\in G_i} -y_\delta \cdot y_\rho \cdot -z_q = 0$$

we get by (5) that $m_i \le q$; so $m_i < q$.

(12) If $F_i = \emptyset$ and $K_i = \{r\}$, then $m_i < q$.

For, since the i^{th} summand is nonzero, we have $r < m_i$. Hence the argument for (11) works.

(13) If $F_i \neq \emptyset$, then we may assume that $H_i = \emptyset$.

This is clear from (6).

(14) If $F_i = \{\rho_i\}$ and $H_i = \emptyset$, then ρ_i is a proper extension of some $\tau \in M$ such that $\mathrm{dmn}(\tau) \le q$, and $y_{\rho_i} < z_q$.

For, we have

$$y_{\rho_i} \cdot \prod_{\delta\in G_i} -y_\delta \cdot \prod_{m\in K_i} -z_m \cdot -z_q = 0.$$

By (5) we get a $\tau \in M$ with $\mathrm{dmn}(\tau) \le q$ such that $\tau \subseteq \rho_i$. Since $\rho_i \notin M$, we have $\tau \subset \rho_i$. So $y_{\rho_i} < y_\tau \le z_q$, as desired.

Now we can finish the proof of (7) in our second case. Let $R = \{i < n\colon F_i = \emptyset\}$. Then

$$\begin{aligned} z_q &= \sum_{i\in R}\Big(\prod_{\delta\in G_i} -y_\delta \cdot z_{m_i} \cdot \prod_{r\in K_i} -z_r\Big) \\ &\quad + \sum_{i\in n\backslash R}\Big(y_{\rho_i} \cdot \prod_{\delta\in G_i} -y_\delta \cdot \prod_{m\in H_i} z_m \cdot \prod_{r\in K_i} -z_r\Big) \\ &\le \sum_{i\in R} z_{m_i} + \sum_{i\in n\backslash R} y_{\rho_i} \\ &\le z_q. \end{aligned}$$

Hence

$$z_q = \sum_{i\in R} z_{m_i} + \sum_{i\in n\backslash R} y_{\rho_i}.$$

Here we have $m_i < q$ for all $i \in R$, and each ρ_i is a proper extension of some $\sigma \in M$ with $\mathrm{dmn}(\sigma) \le q$. Now map x_{q-1} to 1 and all other generators to 0. Then

z_q goes to 1 but the right side of the above equation goes to 0, contradiction. This completes the proof of (7).

(15) If Q is a subset of M and $a \in \mathrm{Fr}(\omega)$, then there is an m such that one of the following conditions holds:
(1) $a \cdot y_\varepsilon = y_\varepsilon$ for all $\varepsilon \in Q$ such that $\mathrm{dmn}(\varepsilon) \geq m$.
(2) $a \cdot y_\varepsilon = 0$ for all $\varepsilon \in Q$ such that $\mathrm{dmn}(\varepsilon) \geq m$.

For, write $a = \sum_{\delta \in P} \prod_{i<n} x_i^{\delta(i)}$ for some $n \in \omega$ and some $P \subseteq {}^n 2$, and let $m = n+1$. Then (1) holds if the all 0 function is in P, and (2) holds otherwise.

(16) A consists of all elements of the form

$$\sum_{\varepsilon \in Q} y_\varepsilon + a$$

such that Q is a subset of M and $a \in \mathrm{Fr}(\omega)$.

To prove (6) first note that the set Y of all such elements is clearly a subset of A and contains the set of generators in the definition of A. Clearly Y is closed under $+$. So it suffices to show that Y is closed under $-$:

$$\begin{aligned} -\Big(\sum_{\varepsilon \in Q} y_\varepsilon + a\Big) &= -a \cdot -\sum_{\varepsilon \in Q} y_\varepsilon \\ &= -a \cdot \sum_{\varepsilon \in M \setminus Q} y_\varepsilon. \end{aligned}$$

Now by (15), choose m such that either $-a \cdot y_\varepsilon = y_\varepsilon$ for all $\varepsilon \in M \setminus Q$ with domain at least m, or $-a \cdot y_\varepsilon = 0$ for all $\varepsilon \in M \setminus Q$ with domain at least m. Hence in the first case we have

$$-a \cdot \sum_{\varepsilon \in M \setminus Q} y_\varepsilon = \sum_{\varepsilon \in R} y_\varepsilon + \Big(-a \cdot \sum_{\varepsilon \in M \setminus (Q \cup R)} y_\varepsilon\Big)$$

where R is the set of all $\varepsilon \in M \setminus Q$ with domain at least m, and in the second case $-a \cdot \sum_{\varepsilon \in M \setminus Q} y_\varepsilon$ is in $\mathrm{Fr}(\omega)$.

(17) X is maximal irredundant in A.

Suppose that $d \in A \setminus \langle X \rangle$. By (16), write

$$d = \sum_{\varepsilon \in Q} y_\varepsilon + e,$$

where Q is a subset of M and $e \in \mathrm{Fr}(\omega)$. Since $d \notin \langle X \rangle = \mathrm{Fr}(\omega)$, the set Q is infinite and co-infinite. Now write $e = \sum_{\delta \in T} \prod_{i<m} x_i^{\delta(i)}$ with $T \subseteq {}^m 2$. Let $\zeta \in {}^m 2$ be the constantly 0 function. If $\zeta \in T$, then $y_\varepsilon \leq e$ for all $\varepsilon \in N$ with $\mathrm{dmn}(\varepsilon) > m$, so $d \in \mathrm{Fr}(\omega) = \langle X \rangle$, contradiction. Thus $\zeta \notin T$. It follows that $y_\varepsilon \cdot e = 0$ for all $\varepsilon \in M$ such that $\mathrm{dmn}(\varepsilon) > m$. Hence $e \cdot -z_m = 0$. Choose $p > m+2$ so that Q has a member with domain p but none with domain $p+1$. Then

$$d \cdot -z_m \cdot z_{p+1} = \sum \{y_\rho \colon \rho \in Q, m < \mathrm{dmn}(\rho) \leq p\}.$$

Hence $d \cdot -z_m \cdot z_{p+1} + z_{p-1} = z_p$. This proves (17). $\dashv$

It is possible that Theorem 1.3 can be generalized. The following problem represents the maximum possible generalization.

PROBLEM 1. *Is* $\mathrm{Irr}_{\mathrm{mm}}(A) = \pi(A)$ *for every infinite BA? In particular, we do not know whether this is true for the following algebras*:

(i) *The completion of the denumerable atomless BA.*
(ii) *The interval algebra on* $\mathbb{R}$.

The following minor results are somewhat relevant to this problem.

PROPOSITION 1.4. *It is possible to have X denumerable and irredundant, $\langle X\rangle$ dense in A, $|A| = 2^\omega$, but X not maximal irredundant.*

PROOF. Take $A = \mathfrak{P}(\omega)$ and $X = \{\{m\}: m \in \omega\}$. So X is irredundant and $\langle X\rangle$ is dense in A. Let $E = \{m \in \omega: m \text{ is even}\}$. Clearly $\langle X\rangle = \mathrm{Finco}(\omega)$, and hence $E \notin \langle X\rangle$. So if $X \cup \{E\}$ is redundant, then there exist an $m \in \omega$ and pairwise disjoint $y, z, w \in \langle X\backslash\{\{m\}\}\rangle$ such that $\{m\} = (E \cap y) \cup (z\backslash Y) \cup w$. So $w = \emptyset$. Clearly y is finite with $m \notin y$, or y is cofinite; and similarly for z. So one of y, z is cofinite, and this is clearly impossible.

This example is atomic. An atomless example is as follows. Let $B = \overline{\mathrm{Fr}(\omega)}$ and $X = \{x_n: n \in \omega\}$, where $\langle x_n: n \in \omega\rangle$ is a system of free generators of $\mathrm{Fr}(\omega)$. For each $n \in \omega$ let $z_n = x_n \cdot \prod_{m<n} -x_m$, and let $y = \sum_{n\in\omega} z_{2n}$. Clearly $y \notin \mathrm{Fr}(\omega)$. Suppose that $X \cup \{y\}$ is redundant. Then there exist $m \in \omega$ and pairwise disjoint $u, v, w \in \langle X\backslash\{x_n\}\rangle$ such that $x_n = y \cdot u + -y \cdot v + w$. Since $w \le x_n$, it follows that $w = 0$. Clearly $u, v \neq 1$. Now write

$$u = \sum_{\varepsilon\in M}\prod_{m\in N} x_m^{\varepsilon(m)} \quad \text{and} \quad v = \sum_{\varepsilon\in P}\prod_{m\in N} x_m^{\varepsilon(m)},$$

where N is a finite subset of $X\backslash\{n\}$ and M, P are disjoint subsets of ${}^N2$. Since $x_n \le u + v$, we must have $u + v = 1$, and hence $M \cup P = {}^N2$. Let $\zeta \in {}^N2$ be the all 0 sequence. By symmetry, say $\zeta \in M$. Let p be an even integer greater than n and each member of N. Then $z_p \le y \cdot u \le x_n$, contradiction. ⊣

PROPOSITION 1.5. *If X is a denumerable maximal irredundant subset of $\overline{\mathrm{Fr}(\omega)}$, then we may assume that $\langle X\rangle = \mathrm{Fr}(\omega)$.*

PROOF. Since $\langle X\rangle$ is dense in $\overline{\mathrm{Fr}(\omega)}$, it is atomless, and hence is isomorphic to $\mathrm{Fr}(\omega)$. Hence there is an automorphism f of $\overline{\mathrm{Fr}(\omega)}$ such that $f[\langle X\rangle] = \mathrm{Fr}(\omega)$. ⊣

Note that $\mathrm{Irr}_{\mathrm{mm}}(\mathfrak{P}(\omega)/\mathrm{fin}) = 2^\omega$, since $\pi(\mathfrak{P}(\omega)/\mathrm{fin}) = 2^\omega$.

§2. Maximal ideal independence.

The following proposition gives a method of constructing maximal ideal independent sets.

PROPOSITION 2.1. *Suppose A is a BA and that $X \subseteq A$ is ideal independent and X generates a maximal ideal I. Then X is maximal ideal independent.*

PROOF. Let $y \in A\backslash X$. If $y \in I$, then $y \le \sum F$ for some finite $F \subseteq X$. If $-y \in I$, then $-y \le \sum F$ for some finite $F \subseteq X$, and hence $y + \sum F = 1$. ⊣

PROPOSITION 2.2. $\mathrm{s}_{\mathrm{spect}}(\mathrm{Fr}(\kappa)) = \{\kappa\}$.

PROPOSITION 2.3. $\mathrm{s}_{\mathrm{spect}}(A) \cup \mathrm{s}_{\mathrm{spect}}(B) \subseteq \mathrm{s}_{\mathrm{spect}}(A \times B)$.

PROOF. Let X be maximal ideal-independent in A. Define $Y = \{(x, 1)\colon x \in X\}$. Clearly Y is ideal-independent in $A \times B$. To show that it is maximal, suppose that $(u, v) \in A \times B$. Then there are two possibilities.

Case 1. There is a finite subset F of X such that $u \leq \sum F$. We may assume that $F \neq \emptyset$. Then $(u, v) \leq \sum_{x \in F}(x, 1)$, as desired.

Case 2. There exist an $x \in X$ and a finite subset F of $X \backslash \{x\}$ such that $x \leq u + \sum F$. Again we may assume that $F \neq \emptyset$. Then $(x, 1) \leq (u, v) + \sum_{y \in F}(y, 1)$, as desired.

Hence the proposition follows by symmetry. ⊣

COROLLARY 2.4. *If $\langle A_i\colon i \in I\rangle$ is any system of BAs, then* $\bigcup_{i \in I} \mathrm{s_{spect}}(A_i) \subseteq \mathrm{s_{spect}}\left(\prod_{i \in I} A_i\right)$ *and also* $\bigcup_{i \in I} \mathrm{s_{spect}}(A_i) \subseteq \mathrm{s_{spect}}\left(\prod^{\mathrm{w}}_{i \in I} A_i\right)$.

THEOREM 2.5. *If K is a nonempty finite set of infinite cardinals, then*

$$\mathrm{s_{spect}}\left(\prod_{\lambda \in K} \mathrm{Fr}(\lambda)\right) = K.$$

PROOF. $\supseteq$ holds by Corollary 2.3. Suppose that $\kappa \in \mathrm{s_{spect}}\left(\prod_{\lambda \in K} \mathrm{Fr}(\lambda)\right) \backslash K$. Let $L = \{\lambda \in K\colon \lambda < \kappa\}$ and $M = K \backslash L$. Assume that $L \neq \emptyset$; some obvious changes should be made in the following argument if $L = \emptyset$. Let X be a maximal independent subset of $\prod_{\lambda \in K} \mathrm{Fr}(\lambda)$ of size κ. For each $\lambda \in M$ let u_λ be a free generator of $\mathrm{Fr}(\lambda)$ not in the subalgebra generated by $\{x_\lambda\colon x \in X\}$. Now $|\prod_{\lambda \in L} \mathrm{Fr}(\lambda)| < \kappa$, so there is a $q \in \prod_{\lambda \in L} \mathrm{Fr}(\lambda)$ such that $X' \stackrel{\text{def}}{=} \{x \in X\colon x \restriction L = q\}$ has size greater than $\max(L)$. Let $f = q \cup \langle u_\lambda\colon \lambda \in M\rangle$. So $f \in \prod_{\lambda \in K} \mathrm{Fr}(\lambda)$ and $f \notin X$ (since clearly $M \neq \emptyset$). Hence $X \cup \{f\}$ is ideal-dependent. This gives two possibilities.

Case 1. There is a finite $F \subseteq X$ such that $f \leq \sum F$. It follows that $(\sum F)_\lambda = 1$ for all $\lambda \in M$. Choose $g \in X' \backslash F$. Then $g \leq \sum F$, contradiction.

Case 2. There exist a finite $F \subseteq X$ and a $g \in X \backslash F$ such that $g \leq f + \sum F$. For any $\lambda \in M$ we have $g_\lambda \cdot -u_\lambda \cdot -(\sum F)_\lambda = 0$, and hence $g_\lambda \cdot -(\sum F)_\lambda = 0$. Choose $h \in X' \backslash (F \cup \{g\})$. Then $g \leq h + \sum F$, contradiction. ⊣

The following simple proposition shows that there is an obstruction to using weak products in order to extend Theorem 2.5 to the infinite case.

PROPOSITION 2.6. *If $\langle A_i\colon i \in I\rangle$ is a system of BAs, with I infinite, then*

$$|I| \in \mathrm{s_{spect}}\left(\prod^{\mathrm{w}}_{i \in I} A_i\right).$$

PROOF. For each $i \in I$ let f^i be the member of $\prod^{\mathrm{w}}_{i \in I} A_i$ which takes the value 1 at i and the value 0 at all other places. Clearly $\{f^i\colon i \in I\}$ is maximal ideal-independent. ⊣

PROPOSITION 2.7. *Suppose that K is an infinite set of infinite cardinals such that $|K| \leq \min(K)$. Then there is a BA A such that $K \subseteq \mathrm{s_{spect}}(A)$ and $\mathrm{s_{spect}}(A) \cap \mathrm{reg} \subseteq K$.*

PROOF. Let $\mu = \min(K)$, let λ map μ onto K, and let $A = \prod^{\mathrm{w}}_{\alpha < \mu} \mathrm{Fr}(\lambda_\alpha)$. We claim that A is as desired.

The first inclusion in the proposition holds by Proposition 2.3. Now suppose that $\kappa \in (\mathrm{s_{spect}}(A) \cap \mathrm{reg}) \backslash K$. Let X be maximal ideal-independent of size κ. Let

$L = \{\alpha < \mu : \kappa < \lambda_\alpha\}$, and let $M = \mu \backslash L$. For each $\alpha \in L$ let u_α be a free generator of $\mathrm{Fr}(\lambda_\alpha)$ not in $\langle\{x_\alpha : x \in X\}\rangle$.

(1) $M \neq \emptyset$.

For, suppose that $M = \emptyset$. Then $\kappa < \lambda_\alpha$ for each $\alpha < \mu$, and so $\kappa < \min(K) = \mu$.

(2) Some $x \in X$ has type II.

For, suppose not. Now $\bigcup_{x \in X} \mathrm{supp}(x)$ has size less than $\min(K) = \mu$, so we can choose $\alpha < \mu$ not in this union. Let y take the value u_α at α and 0 elsewhere. Clearly $y \notin X$ and $X \cup \{y\}$ is still ideal-independent, contradiction. So (2) holds.

We take x as in (2). Now let $y_\alpha = u_\alpha$ for all $i \in \mathrm{supp}(x)$, and $y_\alpha = 0$ otherwise. Then $y \notin X$, so $X \cup \{y\}$ is ideal-dependent.

Case 1. $y \leq \sum F$ for some finite $F \subseteq X$. We may assume that $x \in F$. Now for $\alpha \in \mathrm{supp}(x)$ we have $u_\alpha \leq (\sum F)_\alpha$, and hence $(\sum F)_\alpha = 1$. Since $x \in F$, it follows that $\sum F = 1$, contradiction.

Case 2. There exist a finite $F \subseteq X$ and a $g \in X \backslash F$ such that $g \leq y + \sum F$. It follows easily that $g \leq \sum F$, contradiction.

This proves (1).

In particular, $\kappa > \mu$. Since κ is regular, it follows that there is a $G \in [\mu]^{<\omega}$ such that $X' \stackrel{\text{def}}{=} \{x \in X : \mathrm{supp}(x) = G\}$ has size κ. Now $|\prod_{\alpha \in G \cap M} \mathrm{Fr}(\lambda_\alpha)| < \kappa$, so there is a $q \in \prod_{\alpha \in G \cap M} \mathrm{Fr}(\lambda_\alpha)$ such that $Y \stackrel{\text{def}}{=} \{x \in X' : x \restriction (G \cap M) = q\}$ has size κ. Note also that $G \cap L \neq \emptyset$, as otherwise $G = G \cap M$ and hence $|X'| < \kappa$, contradiction. Let $Y' = \{y \in Y : y \text{ has type I}\}$ and $Y'' = Y \backslash Y'$. Now define

$$y_\alpha = \begin{cases} u_\alpha & \text{if } i \in G \cap L, \\ q_\alpha & \text{if } i \in G \cap M, \\ 0 & \text{otherwise.} \end{cases}$$

Since $G \cap L \neq \emptyset$, we have $y \notin X$. So $X \cup \{y\}$ is ideal-dependent. This gives two cases.

Case 1. There is a finite $F \subseteq X$ such that $y \leq \sum F$. Then $(\sum F) \restriction (G \cap L) = 1$. If $|Y'| = \kappa$, choose $g \in Y'$ such that $g \notin F$. Then $g \leq \sum F$, contradiction. If $|Y''| = \kappa$, choose distinct $g, h \in Y'' \backslash F$. Then $g \leq \sum F + h$, contradiction.

Case 2. There exist a finite $F \subseteq X$ and a $g \in X \backslash F$ such that $g \leq y + \sum F$. Then $g \restriction (G \cap L) \leq (\sum F) \restriction (G \cap L)$ and also $g \restriction (\mu \backslash G) \leq (\sum F) \restriction (\mu \backslash G)$. Choose $h \in Y \backslash (F \cup \{g\})$. Then $g \leq h + \sum F$, contradiction. ⊣

COROLLARY 2.8. *If K is an infinite set of regular cardinals and $|K| \leq \min(K)$, then there is a BA A such that* $\mathrm{s}_{\mathrm{spect}}(A) \cap reg = K$.

PROBLEM 2. *Is the assumption $|K| \leq \min(K)$ in Theorem 2.7 necessary?*

PROBLEM 3. *How can Theorem 2.7 be extended to singular cardinals in K?*

We now concentrate on s_{mm}. From 2.3 we have the following problem.

PROBLEM 4. *Is* $\mathrm{s}_{\mathrm{mm}}(A \times B) = \min(\mathrm{s}_{\mathrm{mm}}(A), \mathrm{s}_{\mathrm{mm}}(B))$?

The first part of the proof of Theorem 2 of McKenzie, Monk [4] shows that we cannot have atomless A, B such that $\mathrm{s}_{\mathrm{mm}}(A \times B) = \omega < \min(\mathrm{s}_{\mathrm{mm}}(A), \mathrm{s}_{\mathrm{mm}}(B))$, giving a partial solution of this problem.

By Corollary 2.4 and Proposition 2.6 we have:

COROLLARY 2.9. (i) $\mathrm{s_{mm}}(\prod_{i\in I} A_i \leq \min_{i\in I}\mathrm{s_{mm}}(A_i)$.

(ii) *For I infinite,* $\mathrm{s_{mm}}\prod^{\mathrm{w}}_{i\in I}(A_i) \leq \min(|I|, \min_{i\in I}\mathrm{s_{mm}}(A_i))$.

THEOREM 2.10. *There is a BA A such that* $\mathrm{s_{mm}}(A) = \omega < \mathrm{u}(A)$.

PROOF. Let $A = {}^{\omega}\mathrm{Fr}(\omega_1)$. So $\mathrm{s_{mm}}(A) = \omega$ by Corollary 2.9. By Proposition 9(iii) of Monk [6] we have $\mathrm{u}(A) \geq \kappa$, where κ is the smallest cardinality of a subset of $\mathfrak{P}(\omega)$ which generates a nonprincipal ultrafilter on $\mathfrak{P}(\omega)$. So it suffices to assume that $\{x_i : i \in \omega\}$ is a collection of subsets of ω which generates a nonprincipal ultrafilter D on $\mathfrak{P}(\omega)$, and get a contradiction. If X is an infinite, co-infinite subset of ω, then either X or $\omega\backslash X$ is in D. It follows that not all x_i are cofinite. We may assume that x_0 is not cofinite. Now each intersection $\bigcap_{j\leq i} x_j$ is not cofinite, so we can choose distinct

$$m_i, n_i \in \omega\backslash\Big(\bigcap_{j\leq i} x_j \cup \{m_j, n_j : j < i\}\Big).$$

Let $y = \{m_i : i < \omega\}$. Then clearly $y, \omega\backslash y \notin D$, contradiction. ⊣

PROPOSITION 2.11. $\mathrm{s_{mm}}(\mathrm{Finco}(\kappa)) = \kappa$ *for every infinite cardinal* κ.

PROOF. Since $\{\{\alpha\} : \alpha < \kappa\}$ is clearly maximal ideal independent, we just need to get a contradiction upon assuming that X is maximal ideal independent with $\omega \leq |X| < \kappa$. If all members of X are finite, then it is clearly not maximal. So there is a member of X of the form $\kappa\backslash F$ with $F \subseteq \kappa$ finite. Suppose that there are infinitely many finite members of X. Then there are two distinct finite members G, H of X such that $F \cap G = F \cap H$. Then $G \subseteq (X\backslash F) \cup H$, contradicting maximality of X. Thus X has only finitely many finite members. Hence it has infinitely many cofinite members. Let $\mathfrak{A} = \{G \in [\kappa]^{<\omega} : \kappa\backslash G \in X\}$. Among the finite intersections of members of $\mathfrak{A}$ there is a minimal one; call it Y, and say $Y = \bigcap\mathfrak{B}$ with $\mathfrak{B}$ a finite subset of $\mathfrak{A}$. Take any member $G \in \mathfrak{A}\backslash\mathfrak{B}$. Then $\bigcap\mathfrak{B} \subseteq G$, hence $X\backslash G \subseteq \bigcup_{H\in\mathfrak{B}}(X\backslash H)$, contradicting X ideal independent. ⊣

PROPOSITION 2.12. $\mathfrak{r}(A) \leq \mathrm{s_{mm}}(A)$ *for any BA A.*

PROOF. Suppose that X is maximal ideal-independent. Let

$$Y = X \cup \{-\textstyle\sum F : F \in [X]^{<\omega}\} \cup \{b \cdot -\sum F : b \notin F,\ F \cup \{b\} \in [X]^{<\omega}\}.$$

Clearly the members of Y are nonzero. We claim that Y is weakly dense in A. For, suppose that $a \in A\backslash X$. Then $X \cup \{a\}$ is no longer ideal independent, so we have two cases.

Case 1. $a \leq \sum F$ for some $F \in [X]^{<\omega}$. Then $-\sum F \leq -a$, as desired.

Case 2. There exist a finite subset F of X and a $b \in X\backslash F$ such that $b \leq \sum F + a$. Then $b \cdot -\sum F \leq a$, as desired. ⊣

THEOREM 2.13. *There is a BA A such that* $\mathrm{u}(A) < \mathrm{s_{mm}}(A)$.

PROOF. We modify the proof of Lemma 21 of Monk [6]. The construction depends upon the following step:

(1) Suppose that B is a BA, $\langle a_\alpha : \alpha < \omega_1\rangle$ is a strictly decreasing sequence of elements of B generating an ultrafilter F, and $\langle b_\alpha : \alpha < \mu\rangle$ is a sequence of distinct elements of B with $\omega \leq \mu \leq \omega_1$ such that $\{b_\alpha : \alpha < \mu\}$ is ideal independent. Then

there is an extension C of B such that $\langle a_\alpha\colon \alpha < \omega_1\rangle$ still generates an ultrafilter in C, while $\{b_\alpha\colon \alpha < \mu\}$ is not maximal ideal independent in C.

To prove (1), let $B(x)$ be a free extension of B. For each $\beta < \omega_1$ let

$$I_\beta = \langle\{b_\alpha \cdot x\colon \alpha < \mu\} \cup \{a_\beta \cdot x\}\rangle^{\text{id}}.$$

Clearly $B \cap I_\beta = \{0\}$ for all $\beta < \omega_1$.

(2) There is an $\beta < \omega_1$ such that $x \notin I_\beta$.

To prove (2) we consider two cases.

Case 1. There is an $\alpha < \mu$ such that $b_\alpha \in F$. Say $a_\beta \leq b_\alpha$. Suppose that $x \in I_\beta$. Then we can write

$$x \leq b_{\alpha_0} \cdot x + \cdots + b_{\alpha_{m-1}} \cdot x + a_\beta \cdot x. \tag{3}$$

Choose $\gamma < \mu$ such that $\gamma \neq \alpha_0, \ldots, \alpha_{m-1}, \alpha$. Mapping x to b_γ and pointwise fixing A yields $b_\gamma \leq b_{\alpha_0} + \cdots + b_{\alpha_{m-1}} + b_\alpha$, contradicting ideal independence.

Case 2. $-b_\alpha \in F$ for all $\alpha < \mu$. For each $\alpha < \mu$ choose $\gamma_\alpha < \omega_1$ such that $a_{\gamma_\alpha} \leq -b_\alpha$.

Subcase 2.1. $\{\gamma_\alpha\colon \alpha < \mu\}$ is bounded in ω_1, say by β. Thus $a_\beta \leq -b_\alpha$ for all $\alpha < \mu$. If $x \in I_\beta$, then we obtain (3) again. Choose $\alpha < \mu$ with $\alpha \neq \alpha_0, \ldots, \alpha_{m-1}$. Mapping x to b_α and pointwise fixing A we obtain $b_\alpha \leq b_{\alpha_0} + \cdots + b_{\alpha_{m-1}}$, again contradicting ideal independence.

Subcase 2.2. $\{\gamma_\alpha\colon \alpha < \mu\}$ is unbounded in ω_1. Then there is a strictly increasing sequence $\langle \alpha_\xi\colon \xi < \omega_1\rangle$ of countable ordinals such that $\langle \gamma_{\alpha_\xi}\colon \xi < \omega_1\rangle$ is strictly increasing. Let

$$\Xi_\beta = \{\gamma < \mu\colon a_\beta \cdot b_\gamma = 0\}$$

for all $\beta < \omega_1$. So $\beta < \delta < \omega_1$ implies that $\Xi_\beta \subseteq \Xi_\delta$. Now $\alpha_\xi \in \Xi_{\gamma_{\alpha_\xi}}$ for all $\xi < \omega_1$. Hence $\Xi_{\gamma_{\alpha_\omega}}$ is infinite. Let $\beta = \gamma_{\alpha_\omega}$, and suppose that $x \in I_\beta$. Then we obtain (3) again. Choose $\gamma \in \Xi_\beta \backslash \{\alpha_0, \ldots, \alpha_{m-1}\}$. Then mapping x to b_γ and fixing A pointwise again contradicts ideal independence.

Thus we have now established (2), and we take β as indicated there.

Let $C = B(x)/I_\beta$. We denote members of C by $[u]$ with $u \in B(x)$. Clearly $\langle a_\alpha\colon \alpha < \omega_1\rangle$ still generates an ultrafilter in C. We claim that $\{[b_\alpha]\colon \alpha < \mu\} \cup \{[x]\}$ is ideal independent, so that $\{[b_\alpha]\colon \alpha < \mu\}$ is not maximal ideal independent in C. In fact, obviously $[x]$ is not in the ideal generated by $\{[b_\alpha]\colon \alpha < \mu\}$. Suppose that $\alpha < \mu$, $F \in [\mu\backslash\{\alpha\}]^{<\omega}$, and $[b_\alpha] \leq [x] + \sum_{\gamma\in F}[b_\gamma]$. Then we can write

$$b_\alpha \cdot -x \cdot \prod_{\gamma\in F} -b_\gamma \leq b_{\alpha_0} \cdot x + \cdots + b_{\alpha_{m-1}} \cdot x + a_\beta \cdot x.$$

Mapping x to 0 and fixing A pointwise, we then get $b_\alpha \cdot \prod_{\gamma\in F} -b_\gamma = 0$, contradicting the ideal independence of $\{b_\alpha\colon \alpha < \mu\}$.

This proves (1).

Now the construction of A proceeds from the step (1) as follows. Define A_α for $\alpha < \omega_2$ by induction. Let $A_0 = \text{Intalg}(\omega_1)$, and $a_\alpha = [\alpha, \infty)$ for each $\alpha < \omega_1$. If A_α has been defined so that $\langle a_\alpha\colon \alpha < \omega_1\rangle$ generates an ultrafilter in A_α, apply (1) many times to get an extension $A_{\alpha+1}$ in which $\langle a_\alpha\colon \alpha < \omega_1\rangle$ still generates an ultrafilter, while every infinite ideal independent subset of A_α fails to be maximal in $A_{\alpha+1}$. For α limit $\leq \omega_2$ let $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$. Clearly A_{ω_2} is as desired. ⊣

PROPOSITION 2.14. *If* $s_{\text{mm}}(A) = \omega$, *then* $\mathfrak{a}(A) = \omega$.

PROOF. Let $X = \{x_i : i < \omega\}$ be maximal ideal independent. For each $i < \omega$ let $a_i = x_i \cdot \prod_{j<i} -x_j$. Then $\sum_{i<\omega} a_i = \sum_{i<\omega} x_i = 1$. Thus $\langle a_i : i < \omega\rangle$ is a partition of unity. ⊣

LEMMA 2.15. *Suppose that* $\mathrm{Fr}(\omega_1)$ *is a subalgebra of* A *such that* $I \stackrel{\text{def}}{=} \langle\{x_\alpha : \xi < \omega_1\rangle^{\text{id}}_A$ *is a maximal ideal of* A, *where* $\langle x_\alpha : \alpha < \omega_1\rangle$ *is a system of free generators of* $\mathrm{Fr}(\omega_1)$. *Also suppose that* $X \stackrel{\text{def}}{=} \{x_\alpha : \alpha < \omega_1\}$ *is maximal ideal independent in* A. *Suppose that* Y *is an infinite partition of unity in* A, *with* $|Y| \le \omega_1$.

Then A *has an extension* B *such that* X *is still maximal ideal independent in* B, $\langle\{x_\alpha : \xi < \omega_1\rangle^{\text{id}}_B$ *is a maximal ideal of* B, *and* Y *is not a partition of unity in* B.

PROOF. The main part of the proof is in establishing the following claim.

CLAIM. There is a $b \in X$ such that $b \not\le \sum F$ for all $F \in [Y]^{<\omega}$.

We suppose that the claim does not hold. Thus

(1) For every $b \in X$ there is a finite $F_b \subseteq Y$ such that $b \le \sum F_b$.

Then

(2) $y \in I$ for all $y \in Y$.

For, suppose that $y \in Y$ and $-y \in I$. Thus there is a finite $G \subseteq X$ such that $-y \le \sum G$. Then $1 = y + \sum G = y + \sum_{b\in G} F_b$, contradiction. So (2) holds.

Thus for every $y \in I$ we can choose a finite $G_y \subseteq X$ such that $y \le G_y$.

Now if $|Y| < \omega_1$, choose x_α not in the support of any element of $\bigcup_{y\in Y} G_y$. Now $x_\alpha \le \sum F_{x_\alpha} \le \sum_{y\in F_{x_\alpha}} G_y$, contradiction. Thus $|Y| = \omega_1$.

Let $\Gamma \in [\omega_1]^{\omega_1}$ be such that $\langle F_{x_\alpha} : \alpha \in \Gamma\rangle$ is a Δ-system, say with kernel H. Then if α and β are distinct elements of Γ we have

$$x_\alpha \cdot x_\beta \le \left(\sum F_{x_\alpha}\right) \cdot \left(\sum F_{x_\beta}\right) \le \sum_{y\in H} G_y. \tag{3}$$

Choose distinct $\alpha, \beta \in \Gamma$ so that $x_\alpha, x_\beta \notin \bigcup_{y\in G} G_y$. Then (3) gives a contradiction.

This proves the claim.

Choose $b \in X$ in accordance with the claim. Let $A(x)$ be a free extension of A, and let J be the ideal of $A(x)$ generated by

$$\{y \cdot x : y \in Y\} \cup \{x \cdot -b\}.$$

Clearly $A \cap J = \{0\}$. If $x \in J$, then we can write

$$x \le y_1 \cdot x + \cdots + y_m \cdot x + x \cdot -b.$$

Mapping x to 1 yields $b \le y_1 + \cdots + y_m$, contradicting the choice of b.

Thus $A(x)/J$ is as desired. ⊣

THEOREM 2.16. *There is a BA* A *such that* $s_{\text{mm}}(A) = \omega_1$ *and* $\mathfrak{a}(A) = \omega_2$.

PROOF. This is obtained by an obvious iteration from Lemma 2.15. ⊣

Some further facts about s_{mm} are as follows.

1. For $A = \mathrm{Fr}(\kappa)$ with κ an uncountable cardinal, we have $\mathfrak{a}(A) = \text{length}_{\text{mm}}(A) = \mathfrak{s}(A) = \omega < \kappa = s_{\text{mm}}(A)$. (This is easy to see.)

2. In the algebra B of example 17 of Monk [6] one has $s_{\text{mm}}(B) = \omega < \kappa = \mathfrak{s}(B)$.

3. In the algebra B of example 20 of Monk [6] one has $\mathfrak{s}_{\mathrm{mm}}(B) \leq \kappa < \mathfrak{t}(B)$, and also $\mathfrak{r}(B) = \mathfrak{i}(B) = \omega$.

4. C. Bruns has shown that $\mathfrak{p} \leq \mathfrak{s}_{\mathrm{mm}}$. (Unpublished)

5. Recall that under MA one has $\mathfrak{r}(\mathfrak{P}(\omega)/\mathrm{fin}) = 2^\omega$; see Blass [1]. Hence the same is true of $\mathfrak{s}_{\mathrm{mm}}$.

These examples leave only one question concerning the relationship of $\mathfrak{s}_{\mathrm{mm}}$ to the cardinals of Monk [6]:

PROBLEM 5. *Is there an atomless BA A such that $\mathfrak{s}_{\mathrm{mm}}(A) < \mathfrak{i}(A)$?*

Now we show that it is consistent to have $\mathfrak{s}_{\mathrm{mm}}(\mathfrak{P}(\omega)/\mathrm{fin})$ less than 2^ω. The argument is a modification of exercises (A12), (A13) in chapter VIII of Kunen [3]; the essential argument is given in the following lemma.

LEMMA 2.17. *Let M be a c.t.m. of ZFC. Suppose that κ is an infinite cardinal and $\langle a_i : i < \kappa\rangle$ is a system of infinite subsets of ω such that $\langle [a_i] : i < \kappa\rangle$ is ideal independent, where $[x]$ denotes the equivalence class of x modulo the ideal fin of $\mathfrak{P}(\omega)$. Then there is a generic extension $M[G]$ of M using a ccc partial order such that in $M[G]$ there is a $d \subseteq \omega$ with the following two properties:*

(i) *$\langle [a_i] : i < \kappa\rangle^\frown\langle[\omega\backslash d]\rangle$ is ideal independent.*

(ii) *If $x \in (\mathfrak{P}(\omega) \cap M)\backslash(\{a_i : i < \kappa\} \cup \{\omega\backslash d\})$, then $\langle [a_i] : i < \kappa\rangle^\frown\langle[\omega\backslash d], [x]\rangle$ is not ideal independent.*

PROOF. Let I be the ideal of $\mathfrak{P}(\omega)/\mathrm{fin}$ generated by $\{[a_i]\cdot[a_j] : i < j < \kappa\}$, and let A be the quotient algebra $(\mathfrak{P}(\omega)/\mathrm{fin})/I$. Let f be the natural homomorphism of $\mathfrak{P}(\omega)$ onto A. Note that $f(a_i) \neq 0$ for all $i < \kappa$, by ideal independence. Let B be the subalgebra of A generated by $\{f(a_i) : i < \kappa\}$. Thus B is an atomic BA with $\{f(a_i) : i < \kappa\}$ its set of atoms. By Sikorski's extension theorem, let $h : \mathfrak{P}(\omega) \to \overline{B}$ extend f, where $\overline{B}$ is the completion of B.

Let $P = \{(b, y) : b \in \ker(h)$ and $y \in [\omega]^{<\omega}\}$. We define $(b, y) \leq (b', y')$ iff $b \supseteq b'$, $y \supseteq y'$, and $y \cap b' \subseteq y'$. Clearly this gives a ccc partial order of P. Let G be any P-generic filter over M, and let $d = \bigcup_{(b,y)\in G} y$.

(1) If R is a finite subset of κ and $i \in \kappa\backslash R$, then $a_i \cap \bigcap_{j\in R}(\omega\backslash a_j) \cap d$ is infinite.

In fact, let R and i be as in the hypothesis of (1). For any natural number n let

$$E_n = \left\{(b, y) \in P : \exists m > n\left[m \in a_i \cap \bigcap_{j\in R}(\omega\backslash a_j) \cap y\right]\right\}.$$

Clearly it suffices to show that each such set E_n is dense in P. Suppose that $(b, y) \in P$. Then $(a_i \cap \bigcap_{j\in R}(\omega\backslash a_j))\backslash b$ is infinite. For, if it is a finite set c, then

$$a_i \subseteq \bigcup_{j\in R} a_j \cup b \cup c,$$

and upon applying h we would get $h(a_i) \leq \sum_{j\in R} h(a_j)$, which is clearly impossible. Thus the indicated set is infinite. We can hence choose m in it with $m > n$. Clearly $(b, y \cup \{m\}) \in E_n$ and $(b, y \cup \{m\}) \leq (b, y)$, proving (1).

(2) If R is a finite subset of κ, then $\omega\backslash\left(d \cup \bigcup_{i\in R} a_i\right)$ is infinite.

In fact, let R be a finite subset of κ. For any natural number n let

$$F_n = \left\{(b, y) \in P \colon \exists m > n\left[m \in b \backslash \left(y \cup \bigcup_{i \in R} a_i\right)\right]\right\}.$$

We claim that F_n is dense in P. For, suppose that $(b, y) \in P$. Then the set $\omega \backslash \left(y \cup \bigcup_{i \in R} a_i\right)$ is infinite. For, if it is a finite set c, then we get

$$\omega = c \cup y \cup \bigcup_{i \in R} a_i,$$

and applying h we get $1 = \sum_{i \in R} h(a_i)$, which is clearly impossible. Choose $(b, y) \in F_n \cap G$, and then choose $m > n$ such that $m \in b \backslash \left(y \cup \bigcup_{i \in R} a_i\right)$. We claim that $m \notin d$; by the arbitrariness of n, this will prove (2). Suppose that $m \in d$. Choose $(c, z) \in G$ with $m \in z$. Then choose $(d, w) \in G$ with $(d, w) \leq (b, y), (c, z)$. Thus $m \in w$ since $m \in z$. Also, $m \in b \backslash y$. This contradicts $(d, w) \leq (b, y)$. Hence (2) holds.

(3) $\langle [a_i] \colon i < \kappa \rangle^\frown \langle [\omega \backslash d] \rangle$ is ideal independent.

For, suppose not. There are two possibilities.

Case 1. There are a finite subset R of κ and an $i \in \kappa \backslash R$ such that $[a_i] \leq [\omega \backslash d] + \sum_{j \in R} [a_j]$. This contradicts (1).

Case 2. There is a finite subset R of κ such that $[\omega \backslash d] \leq \sum_{i \in R} [a_i]$. This contradicts (2).

Thus (3) holds.

(4) If $b \in \ker(h)$, then $b \cap d$ is finite.

In fact, clearly $\{(c, y) \in P \colon b \subseteq c\}$ is dense in P, so we can choose $(c, y) \in G$ such that $b \subseteq c$. Then $b \cap d \subseteq y$ (as desired). For, suppose that $m \in b \cap d$. Choose $(e, z) \in G$ such that $m \in z$. Then choose $(r, w) \in G$ such that $(r, w) \leq (e, z), (c, y)$. Then $m \in w \cap c \subseteq y$.

(5) If $x \in (\mathfrak{P}(\omega) \cap M) \backslash (\{a_i \colon i < \kappa\} \cup \{\omega \backslash d\})$, then $\langle [a_i] \colon i < \kappa \rangle^\frown \langle [\omega \backslash d], [x] \rangle$ is not ideal independent.

To prove this, we consider two cases. First, if $x \in \ker(h)$, then $[x] \leq [\omega \backslash d]$ by (4), as desired. Second, if $x \notin \ker(h)$, choose $i < \kappa$ such that $h(a_i) \leq h(x)$. So $a_i \backslash x \in \ker(h)$, and so by (4) we get $[a_i] \leq [x] + [\omega \backslash d]$, as desired. ⊣

THEOREM 2.18. *It is consistent with* $2^\omega > \omega_1$ *that* $\mathrm{s_{mm}}(\mathfrak{P}(\omega)/\mathrm{fin}) = \omega_1$.

PROOF. Start with a c.t.m. M of ZFC + $2^\omega > \omega_1$. Iterate the construction of Lemma 2.17 ω_1 times, obtaining a generic filter G over M. Then $M[G]$ is as desired, using Lemma 5.14 of Chapter VIII of Kunen [3]. ⊣

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