The spectrum of maximal independent subsets of a Boolean algebra

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Abstract

Recall that a subset $X$ of a Boolean algebra (BA) $A$ is independent if for any two finite disjoint subsets $F, G$ of $X$ we have

$$\prod_{x \in F} x \prod_{y \in G} y \neq 0.$$ 

The independence of a BA $A$, denoted by $\text{Ind}(A)$, is the supremum of cardinalities of its independent subsets. We can also consider the maximal independent subsets. The smallest size of an infinite maximal independent subset is the cardinal invariant $i(A)$, well known in the case $A = \mathcal{P}(\omega)/\text{fin}$. In this article we consider the collection of all cardinalities of infinite maximal independent subsets of a BA $A$: we call this set the spectrum of infinite maximal independent subsets, denoted by $\text{Spind}(A)$. Note that infinite maximal independent subsets exist in any BA which is not superatomic.

The main result is that any set of infinite cardinals can occur as $\text{Spind}(A)$ for some infinite BA $A$. Beyond this we give results concerning the way that $\text{Spind}(A)$ changes under various algebraic operations. However, the basic components of most algebras that we deal with are free algebras.

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For notation and facts about Boolean algebras, see [2]. Information on $\text{Ind}(A)$ can be found in Monk [5]. The invariant $i(A)$ for Boolean algebras in general is treated in [6].

For any element $a$ of a BA we let $a1 = a$ and $a0 = a$. The free BA on $\kappa$ many generators is denoted by $\text{Fr}(\kappa)$. If $A$ is freely generated by $X$ and $a \in A$, then there is...
a unique smallest finite subset $F$ of $X$ such that $a$ is in the subalgebra of $A$ generated by $F$. We denote this set $F$ by supp$(a)$. supp$(a)$ is called the support of $a$. Note that supp$(0) =$ supp$(1) = \emptyset$.

1. Elementary results

The following obvious proposition indicates the relationship of the set Spind$(A)$ to Ind$(A)$ and $i(A)$. If $\kappa \leq \lambda$ are cardinals, $[\kappa, \lambda]_{\text{card}}$ denotes the set of all cardinals $\mu$ such that $\kappa \leq \mu \leq \lambda$.

**Proposition 1.1.** Assume that $A$ is atomless.

(i) Spind$(A) \subseteq [\omega, \text{Ind}(A)]_{\text{card}}$.

(ii) sup(Spind$(A)) = \text{Ind}(A)$, and $\text{Ind}(A)$ is attained iff $\text{Ind}(A) \in \text{Spind}(A)$.

(iii) $i(A) = \min(\text{Spind}(A))$.

The following fact is used in the proof of the main result.

**Lemma 1.2.** Spind$(A) \subseteq \text{Spind}(A \times B)$.

**Proof.** Let $X$ be a maximal independent subset of $A$. Define

$$Y = \{(a, 1) : a \in X\}.$$  

Clearly $Y$ is an independent subset of $A \times B$. Now suppose that $(c, d) \notin Y$.

**Case 1:** $c \in X$. Thus $(c, d) \neq (c, 1)$, so

$$(c, d) \cdot -(c, 1) = (0, 0)$$

shows that $Y \cup \{(c, d)\}$ is dependent.

**Case 2:** $c \notin X$. Therefore there exist a finite $F \subseteq X$, an $e \in F^2$, and a $\delta \in 2$, such that $\prod_{a \in F} a^{\delta(a)} e^\delta = 0$. Choose $b \in X \setminus F$. Then

$$\prod_{a \in F} (a, 1)^{\delta(a)} \cdot (c, d)^\delta \cdot -(b, 1) = (0, 0)$$

shows again that $Y \cup \{(c, d)\}$ is dependent. \qed

Ralph McKenzie has shown that actually equality holds in Lemma 1.2; see [4].

2. The main theorem

Note first that if $A$ is superatomic, then $A$ has no infinite independent subsets, and hence Spind$(A) = \emptyset$. The following lemma treats a special case of the main result.
Lemma 2.1. Spind \( \prod_{\lambda \in \Gamma} \text{Fr}(\lambda) \) = \( \Gamma \) if \( \Gamma \) is a finite nonempty set of infinite cardinals.

Proof. \( \square \) holds by Lemma 1.2. Now suppose that \( \kappa \) is a member of the set Spind \( \prod_{\lambda \in \Gamma} \text{Fr}(\lambda) \) \( \setminus \Gamma \); we want to get a contradiction. Say that \( X \) is maximal independent with \(|X| = \kappa \). Let \( A = \{ \lambda \in \Gamma : \kappa < \lambda \} \). For each \( \lambda \in A \) let \( b_\lambda \) be a free generator of \( \text{Fr}(\lambda) \) not in the support of any element \( x_\lambda \) for \( x \in X \). Let \( b_\lambda = 0 \) if \( \lambda \in \Gamma \cap \kappa \). Then \((b_\lambda)_{\lambda \in \Gamma} \not\in X \); and so there exist a finite subset \( F \) of \( X \), an \( \varepsilon \in \mathbb{A} \), and a \( \delta \in 2 \) such that \( \prod_{x \in F} x^{\varepsilon(x)} \cdot b^\delta = 0 \). In particular we must have \( \prod_{x \in F} (x \uparrow \Delta)^{(x)} \cdot (b \uparrow \Delta)^\delta = 0 \), and hence \( \prod_{x \in F} (x \uparrow \Delta)^{(x)} = 0 \). Now \( \prod_{i \in \Gamma \setminus \Delta} \text{Fr}(\lambda) \) is open, so we can choose distinct \( x, y \in X \setminus F \) such that \( x \uparrow (\Gamma \setminus \Delta) = y \uparrow (\Gamma \setminus \Delta) \). Then

\[ x \cdot -y \cdot \prod_{z \in F} z^{\varepsilon(z)} = 0, \]

contradiction. \( \square \)

A construction which will be used several times below is the weak product of a system \( \langle A_i : i \in I \rangle \) of BAs; by definition it is the set of all \( f \in \prod_{i \in I} A_i \) such that \( \{ i \in I : f(i) \neq 0 \} \) is finite or \( \{ i \in I : f(i) \neq 1 \} \) is finite, and it is denoted by \( \prod_{i \in I}^w A_i \).

Theorem 2.2. If \( I \) is any set of infinite cardinals, then there is a BA \( A \) such that Spind(\( A \)) = \( I \).

Proof. By Lemma 2.1 we may assume that \( I \) is infinite. Let \( \kappa \) be the smallest member of \( I \). Define

\[ A = \left( \prod_{\lambda \in I}^w \text{Fr}(\lambda) \right) \oplus \text{Fr}(\kappa). \]

Here \( \oplus \) is the free product operation. First fix any \( \lambda \in I \); we show that \( \lambda \in \text{Spind}(A) \). Let \( \langle x_\lambda : \lambda < \lambda \rangle \) enumerate free generators of \( \text{Fr}(\lambda) \). For each \( \lambda < \lambda \), define \( y_\lambda \in \prod_{\mu \in I}^w \text{Fr}(\mu) \) by defining, for any \( \mu \in I \),

\[ y_\lambda(\mu) = \begin{cases} x_\lambda & \text{if } \mu = \lambda, \\ 0 & \text{otherwise}. \end{cases} \]

Clearly \( \{ y_\lambda : \lambda < \lambda \} \) is an independent system of elements of \( A \); extend it to a maximal independent set \( X \).

(1) \( |X| = \lambda \). In fact, suppose not. Thus \( |X| > \lambda \). For each \( z \in X \) write

\[ z = \sum_{i < m_z} u_i^z \cdot v_i^z, \]

where \( u_i^z \in \prod_{\mu \in I}^w \text{Fr}(\mu) \) and \( v_i^z \in \text{Fr}(\kappa) \). Clearly each \( m_z \neq 0 \). Let \( X' \) be a subset of \( X \) of size \( \lambda^+ \) such that for some \( n \), \( m_z = n \) for all \( z \in X' \), and the two sequences

\[ \langle u_i^z(\lambda) : i < n \rangle \text{ and } \langle v_i^z : i < n \rangle \]
do not depend on the particular \( z \in X' \). Take any distinct \( w, z \in X' \), and choose \( x < \lambda \) so that \( y_x \neq w, z \). Then \( w \cdot z \cdot y_x = 0 \), contradiction. In fact,

\[
-w \cdot z \cdot y_x = \left( \sum_{i<n} u_i^w \cdot v_i^w \right) \cdot \prod_{i<n} (-u_i^x + -v_i^x) \cdot y_x
\]

\[
= \left( \sum_{i<n} \left( u_i^w \cdot v_i^w \cdot \prod_{j \in J} -u_j^y \cdot \prod_{j \in \mathbb{n} \setminus J} -v_j^y \right) \right) \cdot y_x.
\]

Now take any \( i < n \) and \( J \subseteq n \). If \( i \in J \), then

\[
u_i^w \cdot \prod_{j \in J} -u_j^y \cdot y_x = 0
\]

as desired. If \( i \notin J \), then

\[
v_i^w \cdot \prod_{j \in \mathbb{n} \setminus J} -v_j^y = 0
\]

as desired.

Thus (1) holds.

Now suppose that \( \mu \notin I \) but \( X \) is a maximal independent subset of \( A \) of size \( \mu \); we want to get a contradiction. For each \( x \in X \) write

\[
x = \sum_{i<m_x} u_i^x \cdot v_i^x
\]

with \( u_i^x \in \bigcap_{I \in \mathbb{L}} \text{Fr}(\lambda), v_i^x \in \text{Fr}(\kappa) \), and \( u_i^x \cdot v_i^x = 0 \) for distinct \( i, j \). Let \( \langle x_a : x < \kappa \rangle \) be a system of free generators of \( \text{Fr}(\kappa) \).

(2) \( \kappa < |X| \). For, suppose that \( |X| < \kappa \). Choose \( x \) so that \( x_a \) is not in the support of any \( v_i^x \). We claim that \( X \cup \{ x_a \} \) is independent (contradiction). For, suppose that \( F \) is a finite subset of \( X \), and \( x \in F \). Then we can write

\[
\prod_{x \in F} x^{\delta (x)} = \sum_{i<n} s_i \cdot t_i,
\]

where each \( s_i \) is in \( \bigcap_{I \in \mathbb{L}} \text{Fr}(\lambda) \) and each \( t_i \) is in the subalgebra of \( \text{Fr}(\kappa) \) generated by \( \bigcup_{x \in F, x < m_x} \text{supp}(v_i^x) \). Clearly then \( \prod_{x \in F} x^{\delta (x)} \cdot x_a^\delta \neq 0 \) for \( \delta = 0, 1 \), giving the indicated contradiction. So (2) holds.

(3) Suppose that \( x, y \) are distinct members of \( X \), \( m_x = m_y \), and \( v_i^x = v_i^y \) for all \( i < m_x \). Then

\[
x \cdot y = \sum_{i<m_x} u_i^x \cdot -u_i^y \cdot v_i^x.
\]
For,

\[ x \cdot -y = \left( \sum_{i \leq m_x} u_i^x \cdot v_i^x \right) \cdot \prod_{i \leq m_y} (-u_i^y + v_i^y) \]

\[ = \sum_{i \leq m_x} \left( u_i^x \cdot \prod_{j \leq m_y} v_j^x \cdot (-u_j^y + v_j^y) \right) \]

\[ = \sum_{i \leq m_x} u_i^x \cdot -u_i^y \cdot v_i^x. \]

So (3) holds.

Now for each \( x \in X \) and \( i \leq m_x \) let

\[ \delta_{ix} = \begin{cases} 0 & \text{if } \{ \lambda \in I : u_i^x(\lambda) \neq 0 \} \text{ is finite}, \\
1 & \text{if } \{ \lambda \in I : u_i^x(\lambda) \neq 1 \} \text{ is finite} \end{cases} \]

and let \( F_{ix} = \{ \lambda \in I : u_i^x(\lambda) \neq \delta_{ix} \} \). Now let \( Y \) be an uncountable subset of \( X \) such that the following conditions hold:

(4) There is an \( n \) such that \( m_x = n \) for all \( x \in Y \).

(5) \( \delta_{ix} = \delta_i \) for all \( x \in Y \) and all \( i \leq n \).

(6) \( v_i^x = v_i \) for all \( x \in Y \) and all \( i \leq n \).

(7) For each \( i \leq n \), \( \langle F_{ix} : x \in Y \rangle \) is a \( \Lambda \)-system, say with kernel \( G_i \).

Let \( H = \bigcup_{i \leq n} G_i \). Then

(8) If \( x, s, t, w \) are distinct members of \( Y \), \( i \leq n \), and \( \lambda \in I \setminus H \), then \( (u_i^x \cdot u_i^s \cdot -u_i^t \cdot -u_i^w)(\lambda) = 0 \).

For, if \( \delta_i = 0 \), then since \( F_{ix} \cap F_{is} = G_i \) we get \( (u_i^x \cdot u_i^s)(\lambda) = 0 \), and if \( \delta_i = 1 \) similarly \( (-u_i^t \cdot -u_i^w)(\lambda) = 0 \), so (8) holds.

Now for each \( x \in X \) let

\[ x^H = \sum_{i \leq m_x} (u_i^x \mid H) \cdot v_i^x, \]

considered as a member of \((\prod_{\lambda \in H} Fr(\lambda)) \oplus Fr(\kappa)\). Now we claim

(9) \( \langle x^H : x \in X \rangle \) is independent in \((\prod_{\lambda \in H} Fr(\lambda)) \oplus Fr(\kappa)\).

To prove this, suppose that \( K \) and \( L \) are disjoint finite subsets of \( X \) and

(10) \[ \prod_{y \in K} y^H \cdot \prod_{y \in L} -y^H = 0; \]

we want to get a contradiction. Let \( P = \prod_{y \in K} m_y \) and \( N = \{(y, i) : y \in L \text{ and } i \leq m_y \} \).

Then

(11) \[ \prod_{y \in K} y^H \cdot \prod_{y \in L} -y^H \]

\[ = \prod_{y \in K} \left( \sum_{i \leq m_y} (u_i^y \mid H) \cdot v_i^y \right) \cdot \prod_{y \in L} \prod_{i \leq m_y} [(-u_i^y \mid H) + v_i^y] \]
\[
\sum_{f \in P, \ M \subseteq N} \left( \prod_{y \in K} \left( u_{f(y)}^y \uparrow H \right) \cdot v_{f(y)}^y \right) \cdot \prod_{(y, j) \in M} (-u_j^y \uparrow H) \cdot \prod_{(y, j) \in N \setminus M} -v_j^y.
\]

Now choose distinct \(x, s, t, w \in Y \setminus (K \cup L)\). Then, using (3),

\[
(12) \quad x \cdot s \cdot -t \cdot -w \cdot \prod_{y \in K} y \cdot \prod_{y \in L} -y
\]

\[
= \sum_{i < n, f \in P, \ M \subseteq N} \left( u_i^y \cdot u_i^y \cdot -u_i^w \cdot -u_i^w \cdot v_i \right.
\]

\[
\left. \cdot \prod_{y \in K} (u_{f(y)}^y \cdot v_{f(y)}^y) \cdot \prod_{(y, j) \in M} -u_j^y \cdot \prod_{(y, j) \in N \setminus M} -v_j^y \right).
\]

Next,

\[
(13) \quad \text{If } f \in P, \ M \subseteq N, \ \lambda \in H, \text{ and}
\]

\[
\prod_{y \in K} v_{f(y)}^y \cdot \prod_{(y, j) \in N \setminus M} -v_j^y \neq 0,
\]

then

\[
\left( \prod_{y \in K} u_{f(y)}^y \cdot \prod_{(y, j) \in M} -u_j^y \right)(\lambda) = 0.
\]

In fact, under the hypothesis of (13), using (10) and (11) we get

\[
\prod_{y \in K} (u_{f(y)}^y \uparrow H) \cdot \prod_{(y, j) \in M} (-u_j^y \uparrow H) = 0,
\]

and so the conclusion of (13) follows.

By (8), (12), and (13) we get

\[
x \cdot s \cdot -t \cdot -w \cdot \prod_{y \in K} y \cdot \prod_{y \in L} -y = 0,
\]

contradiction. This proves (9).

Now let \(L = \{ \lambda \in I : \lambda < \mu \} \) and \(K = \{ \lambda \in I : \mu < \lambda \} \). For each \(\lambda \in H \cap K\) let \(w(\lambda)\) be a free generator of \(\text{Fr}(\lambda)\) not in the support of \(u_i^x(\lambda)\) for any \(x \in X\) and \(i < m_x\), and let \(w(\lambda) = 0\) if \(\lambda \in I \setminus (H \cap K)\). Clearly \(w \not\in X\), so there exist a finite \(M \subseteq X\), an \(e \in M\), and a \(\delta \in 2\) such that \(w^{\delta} \cdot \prod_{y \in M} y^{e(y)} = 0\). Choose distinct \(x, z, s, t \in Y \setminus M\).

Now \(\prod_{x \in H \cap L} \lambda < \mu\), and \(K < \mu\), so there exist distinct \(x, \beta \in X \setminus (M \cup H \cup \{x, z, s, t\})\) such that \(m_x = m_\beta\), \(v_i^x = v_i^\beta\) for all \(i < m_x\), and for all \(i < m_x\) and all \(\lambda \in L \cap H\) we have
Now \( u_i^x(\lambda) = u_i^y(\lambda) \). Now (14) \((x \cdot z \cdot -s \cdot -t \cdot \alpha \cdot -\beta) \upharpoonright (I \setminus (H \cap K)) = 0.\)

In fact, by (8) we just need to take any \( \lambda \in H \cap L \) and show that \((\alpha \cdot -\beta)(\lambda) = 0.\)

Now by (3) we have

\[
\alpha \cdot -\beta = \sum_{i < m} u_i^x \cdot -u_i^y \cdot u_i^z.
\]

The choice of \( \alpha \) and \( \beta \) now gives \((\alpha \cdot -\beta)(\lambda) = 0.\) So (14) holds.

Now by (3) we have

\[
\sum_{i < m} u_i^x \cdot -u_i^y \cdot v_i^z = \sum_{i < m} u_i^x \cdot -u_i^y \cdot v_i^z.
\]

so by (14) and the choice of \( w \) we get

\[
\prod_{y \in M} y^{(y)} \cdot x \cdot z \cdot -s \cdot -t \cdot \alpha \cdot -\beta = 0,
\]

contradiction. \( \square \)

3. Additional results on direct products of free algebras

The following characterizes \( \text{Spind}(A) \) for \( A \) a weak product of free algebras.

**Proposition 3.1.** Suppose that \( I \) is an infinite set, and \( \langle \lambda_i : i \in I \rangle \) is a system of infinite cardinals. Then

\[
\text{Spind}\left( \prod_{i \in I} \text{Fr}(\lambda_i) \right) = \{ \lambda_i : i \in I \} \cup \{ \omega \}.
\]

**Proof.** \( \supseteq \) holds using Proposition 8 of Monk [6]. For \( \subseteq \), suppose to the contrary that \( \kappa \in \text{Spind}(\prod_{i \in I} \text{Fr}(\lambda_i)) \) and \( \kappa \not\in \{ \lambda_i : i \in I \} \cup \{ \omega \}. \) Let \( J = \{ i \in I : \lambda_i < \kappa \} \) and \( L = \{ i \in I : \kappa < \lambda_i \}. \) By Corollary 10.4 of Monk [5], \( L \neq 0. \)

Let \( X \) be a maximal independent subset of \( \prod_{i \in J} \text{Fr}(\lambda_i) \) of size \( \kappa. \) Wlog for all \( x \in X \)
the set \( F_x = \{ i \in I : x(i) \neq 0 \} \) is finite. Let \( Y \) be an uncountable subset of \( X \) such that \( \langle F_x : x \in Y \rangle \) forms a \( \Delta \)-system, say with kernel \( G. \) Obviously \( G \neq 0. \)

(\( \ast \)) \( \langle x \upharpoonright G : x \in Y \rangle \) is independent in \( \prod_{i \in G} \text{Fr}(\lambda_i). \)

In fact, suppose that \( K \subseteq [x]^{<\omega} \) and \( \varepsilon \in [2]. \) Choose distinct \( x, z \in Y \setminus K. \) Then \( xz \prod_{y \in K} y^{(y)} \neq 0, \) so \( \prod_{y \in K} \langle y \upharpoonright G \rangle^{(y)} \neq 0, \) as desired in (\( \ast \)).

Now for each \( i \in G \cap L \) let \( w(i) \) be a free generator of \( \text{Fr}(\lambda_i) \) not in the support of any \( x(i) \) with \( x \in X, \) and let \( w(i) = 0 \) if \( i \in I \setminus (G \cap L). \) Clearly \( w \not\in X, \) so there exist a finite \( K \subseteq X, \) an \( \varepsilon \in [2], \) and a \( \delta \in [2] \) such that \( w^\delta \cdot \prod_{y \in K} y^{(y)} = 0. \) Choose distinct \( x, z \in Y \setminus K. \) Now \( \prod_{i \in J \cap G} \lambda_i < \kappa, \) so there exist distinct \( u, v \in Y \setminus \{ x, z \} \) such that
$w^G \cdot x \cdot z \cdot u \cdot -v \cdot \prod_{y \in K} y^{\delta(y)} = 0$

and

$$\left( x \cdot z \cdot u \cdot -v \cdot \prod_{y \in K} y^{\delta(y)} \right) \uparrow (I \setminus G) = 0,$$

so by the choice of $w$, using (*), we get

$$x \cdot y \cdot u \cdot -v \cdot \prod_{y \in K} y^{\delta(y)} = 0,$$

contradiction. □

Concerning arbitrary infinite products of free algebras we have the following results.

**Proposition 3.2.** If $\langle \lambda_i : i \in I \rangle$ is a system of infinite cardinals with $I \neq 0$, then $\prod_{i \in I} \text{Fr}(\lambda_i)$ has a maximal independent subset of size $\prod_{i \in I} \lambda_i$.

**Proof.** This is true by Lemma 2.1 if $I$ is finite. For $I$ infinite, for each $i \in I$ let $X_i$ be a set of free generators of $\text{Fr}(\lambda_i)$ of size $\lambda_i$. Let $Y$ be a finitely distinguished subset of $\prod_{i \in I} X_i$ of size $|A| = \prod_{i \in I} \lambda_i$. (See [2, p. 197]) Clearly $Y$ is independent, and $|Y| = \prod_{i \in I} \lambda_i$. That is the size of the whole product, so the Proposition follows. □

**Corollary 3.3.** If $\langle \lambda_i : i \in I \rangle$ is a system of infinite cardinals with $I$ infinite, then for each infinite nonempty $J \subseteq I$ we have $\prod_{j \in J} \lambda_j \in \text{Spind}(\prod_{i \in I} \text{Fr}(\lambda_i))$.

The methods of proof for the above results give the following.

**Proposition 3.4.** Suppose that $\Gamma$ is a nonempty set of infinite cardinals, and $\kappa$ is an infinite cardinal not in $\Gamma$ such that

$$\prod_{\substack{i \in \Gamma, \atop \lambda_i < \kappa}} \lambda_i < \kappa.$$

Then $\kappa \notin \text{Spind}(\prod_{\lambda \in \Gamma} \text{Fr}(\lambda))$.

**Proposition 3.5.** Suppose that $\Gamma$ is an infinite set of infinite cardinals, $\kappa$ is a cardinal not in $\Gamma$,

$$\kappa \leq \prod_{\substack{i \in \Gamma, \atop \lambda_i < \kappa}} \lambda.$$
and
\[
\forall \mu < \kappa \left[ \prod_{\lambda \in \Gamma, \lambda < \mu} \lambda < \kappa \right].
\]

Then \( \kappa \) is a limit cardinal, and \( \kappa \not\in \text{Spind}(\prod_{\lambda \in \Gamma} \text{Fr}(\lambda)) \).

**Proof.** For each \( \lambda \in \Gamma \) let \( X_\lambda \) be a set of free generators of \( \text{Fr}(\lambda) \).

The two conditions clearly imply that \( \text{sup}\{\lambda \in \Gamma : \lambda < \kappa\} = \kappa \), and hence \( \kappa \) is a limit cardinal. Now suppose that \( Y \subseteq \prod_{\lambda \in \Gamma} \text{Fr}(\lambda) \) is maximal independent, with \( |Y| = \kappa \).

Write \( Y = \{ y^\beta : \beta < \kappa \} \). Now the order type of \( \{ \lambda \in \Gamma : \lambda < \kappa \} \) is \( \leq \kappa \). Let \( \langle \mu_\xi : \xi < \alpha \rangle \) enumerate this set in strictly increasing order. So, \( \alpha \) is a limit ordinal \( \leq \kappa \). We now define a member \( x \) of \( \prod_{\lambda \in \Gamma} \text{Fr}(\lambda) \), as follows. For any \( \xi < \alpha \),

\[
x_{\mu_\xi} = \begin{cases} 
0 & \text{if } \xi \text{ is a limit ordinal,} \\
\text{a member of } X_{\mu_{\mu_{\xi + 1}} \setminus \bigcup_{\beta < \mu_{\xi + 1}} \text{Supp}(y^\beta_{\mu_{\xi + 1}}) & \text{if } \xi = \eta + 1;
\end{cases}
\]

\( x_\lambda \in X_\lambda \setminus \bigcup_{\beta < \kappa} \text{Supp}(y^\beta_\lambda) \) for \( \kappa < \lambda \). Clearly \( x \not\in Y \). Hence there exist a finite subset \( F \) of \( \kappa \), an \( \varepsilon \in F^2 \), and a \( \delta \in 2 \) such that \( \prod_{\beta \in F} (y^\beta_\lambda y^\delta_\lambda) = 0 \). Choose \( \xi < \alpha \) such that \( \beta \leq \mu_\xi \) for all \( \beta \in F \). Now

\[
\kappa \setminus F = \bigcup_{w \in \prod_{\eta \leq \xi} \text{Fr}(\mu_\eta)} \{ \beta \in \kappa \setminus F : y^\beta \upharpoonright (\Gamma \cap \mu_{\xi + 1}) = w \}
\]

and \( \prod_{\eta \leq \xi} \mu_\eta < \kappa \), so there are distinct \( \gamma, \delta \in \kappa \setminus F \) such that

\[
y^\gamma \upharpoonright (\Gamma \cap \mu_{\xi + 1}) = y^\delta \upharpoonright (\Gamma \cap \mu_{\xi + 1}).
\]

Hence \( (y^\gamma - y^\delta) \upharpoonright (\Gamma \cap \mu_{\xi + 1}) = 0 \). It follows that there is a \( \lambda \in \Gamma \) with \( \mu_{\xi + 1} \leq \lambda \) such that \( (y^\gamma - y^\delta) \prod_{\beta \in F} (y^\beta_\lambda y^\delta_\lambda) \neq 0 \). So \( \langle \mu_{\xi + 1} \rangle_\lambda \not\in \text{Spind}(\lambda) \), but \( \prod_{\beta \in F} (y^\beta_\lambda y^\delta_\lambda) \cdot x^\lambda_\delta = 0 \), contradicting the definition of \( x \). \( \square \)

**Corollary 3.6.** If \( \kappa \) is a strong limit cardinal and \( \kappa \not\in \Gamma \), then \( \kappa \) is not in the set \( \text{Spind}(\prod_{\lambda \in \Gamma} \text{Fr}(\lambda)) \).

**Corollary 3.7.** If the order type of \( \Gamma \cap \kappa \) is \( \omega \), \( \text{sup}(\Gamma \cap \kappa) = \kappa \), and \( \kappa \not\in \Gamma \), then \( \kappa \not\in \text{Spind}(\prod_{\lambda \in \Gamma} \text{Fr}(\lambda)) \).

**Corollary 3.8.** (GCH) \( \text{Spind}(\prod_{i < \omega_1} \text{Fr}(\aleph_i)) = \{ \aleph_i : i < \omega \} \cup \{ \aleph_\omega+1 \} \).

The following consistency result clarifies these results.

**Proposition 3.9.** It is consistent that if \( A = \prod_{2 < \omega} \text{Fr}(\aleph_2) \), then \( A \) has a maximal independent subset of size \( \aleph_{\omega_1} \).
Proof. Take a model in which \(2^\omega = \aleph_{\omega_1}\). By Holz et al. [1, 1.6.15 (a) and exercise, 9 p. 78] we have
\[
\prod_{x < \omega} \aleph_x = \aleph_{\omega_0}^\omega = \aleph_{\omega_1}.
\]
Now apply Corollary 3.2. \(\square\)

Problem 1. Is it true that for any infinite set \(\Gamma\) of infinite cardinals one has
\[
\text{Spind} \left( \prod_{\lambda \in \Gamma} \text{Fr}(\lambda) \right) = \Gamma \cup \left\{ \prod_{\lambda \in A} \lambda : A \subseteq \Gamma \right\}?
\]

4. On free products

Theorem 4.1. If \(\langle A_i : i \in I \rangle\) is a system of BAs each with at least 4 elements, and if \(I\) is infinite, then \(\text{Spind}(\oplus_{i \in I} A_i) \subseteq [|I|, \infty)_\text{card}\). \(\square\)

Corollary 4.2. If \(|A| \leq \kappa\), then \(\text{Spind}(A \oplus \text{Fr}(\kappa)) = \{\kappa\}\).

Theorem 4.3. If \(\lambda \leq \kappa\) for all \(\lambda \in \text{Spind}(A)\), then \(\text{Spind}(A \oplus \text{Fr}(\kappa)) = \{\kappa\}\).

Proof. Suppose that \(X\) is maximal independent, \(\kappa < |X|\). For each \(x \in X\) write
\[
x = \sum_{i < m_x} a_{ix} \cdot b_{ix}
\]
with \(a_{ix} \in A\), \(b_{ix} \in \text{Fr}(\kappa)\), and \(b_{ix}b_{ix} = 0\) for \(i \neq j\). Let \(Y\) be a subset of \(X\) of size \(\kappa^+\) such that \(m_x = m\) is constant for \(x \in Y\), and so is \(\langle b_{ix} : i < m \rangle\). Now for each \(i < m\), the system \(\langle a_{ix} : x \in Y \rangle\) is dependent in \(A\). So by induction we can define pairwise disjoint finite \(F_i \subseteq Y\) for \(i < m\) along with \(\varepsilon_i \in F_i^2\) such that
\[
\prod_{x \in F_i} a_{ix}^{\varepsilon_i(x)} = 0
\]
for each \(i < m\). Let \(G = \bigcup_{i < m} F_i\). Then choose \(y \in Y \setminus G\). Let \(\delta = \bigcup_{i < m} \varepsilon_i\). Then for each \(i < m\),
\[
y \cdot b_i \cdot \prod_{x \in G} x^{\delta(x)} \leq y \cdot b_i \cdot \prod_{x \in F_i} a_{ix}^{\varepsilon_i(x)} = 0,
\]
so
\[
y \cdot \prod_{x \in G} x^{\delta(x)} = 0,
\]
contradiction. \(\square\)
5. Mixed products

**Proposition 5.1.** Suppose that $I$ and $J$ are sets of infinite cardinals, with $J$ infinite, and $\kappa$ is an infinite cardinal. Assume that $\mu < \kappa < \lambda$ for all $\mu \in I$ and $\lambda \in J$. Furthermore, assume that $|J| < \kappa$.

Then

$$ \kappa \notin \Spind \left( \prod_{\mu \in I} \Fr(\mu) \times \prod_{\lambda \in J} \Fr(\lambda) \right). $$

**Proof.** Suppose the contrary, and let $X$ be a maximal independent set of size $\kappa$. Wlog for all $x \in X$ the set $F_x \overset{\text{def}}{=} \{ \lambda \in J : x_1(\lambda) \neq 0 \}$ is finite. Here $x = (x_0, x_1)$ for each $x \in \prod_{\mu \in I} \Fr(\mu) \times \prod_{\lambda \in J} \Fr(\lambda)$. Now

$$ X = \bigcup_{G \in [J]^{<\omega}} \{ x \in X : F_x = G \}, $$

so we can choose $G \in [J]^{<\omega}$ such that $\{ x \in X : F_x = G \}$ is infinite. Obviously $G \neq 0$.

Now for each $\lambda \in G$ let $w(\lambda)$ be a free generator of $\Fr(\lambda)$ not in the support of any $x_1(\lambda)$ with $x \in X$ and $0 < x_1(\lambda) < 1$, and let $w(\lambda) = 0$ for all $\lambda \in J \setminus G$. Clearly then $(1, w) \notin X$, so we can choose a finite $K \subseteq X$, an $\varepsilon \in K^2$, and a $\theta \in 2$ such that $(1, w)^\theta \prod_{y \in K} y^{(\varepsilon)(\theta)}(1) = (0, 0)$. By the choice of $w$ we then get

(1) If $\lambda \in G$, then $\left( \prod_{y \in K} y^{(\varepsilon)(\theta)} \right)(\lambda) = 0$.

Now fix $x \in X \setminus K$, and choose $\delta \in 2$ so that $H \overset{\text{def}}{=} \{ \mu \in I : x_0(\mu) \neq 0 \}$ is finite. Choose $v, z \in X \setminus (K \cup \{ x \})$ such that $v_0 \upharpoonright H = z_0 \upharpoonright H$. It follows that

(2) $x_0^v - z_0 = 0$.

Next, choose $y \in X \setminus (K \cup \{ x, v, z \})$ such that $F_y = G$. Then by (1) and (2) we obtain

$$ y \cdot x^\delta \cdot v \cdot -z \cdot \prod_{y \in K} y^{(\varepsilon)(\theta)} = 0, $$

contradiction. □

**Proposition 5.2.** If $\langle x_\alpha : x < \kappa \rangle$ and $\langle y_\alpha : x < \nu \rangle$ are systems of infinite cardinals, with both $\kappa$ and $\nu$ infinite, then

$$ \omega \in \Spind \left( \left( \prod_{\alpha < \kappa} \Fr(x_\alpha) \right) \oplus \left( \prod_{\alpha < \nu} \Fr(y_\alpha) \right) \right). $$

**Proof.** An element of a weak product is of type 1 it is 0 except for finitely many places; otherwise it is of type 2.

For each $x < \kappa$ let $\langle x_{ij} : i < \omega \rangle$ be an independent system of elements of $\Fr(x_\alpha)$, and for each $\alpha < \nu$ let $\langle y_{ij} : i < \omega \rangle$ be an independent system of elements of $\Fr(y_\alpha)$. 
Now for $n \in \omega$, $\alpha < \kappa$, and $\beta < \nu$ we define

$$w_n(\alpha) = \begin{cases} x_{\alpha \beta - \beta - 1} & \text{if } \alpha < n, \\ 1 & \text{if } \alpha = n, \\ 0 & \text{if } n < \alpha \end{cases}$$

and

$$z_n(\beta) = \begin{cases} y_{\beta \mu - \beta - 1} & \text{if } \beta < n, \\ 1 & \text{if } \beta = n, \\ 0 & \text{if } n < \beta. \end{cases}$$

By the proof of Proposition 8 of Monk [6], the set

$$X = \{w_n : n \in \omega\} \cup \{z_n : n \in \omega\}$$

is independent in the free product. We claim that it is maximal independent. For, take any element $w$ of the free product. Then we can write

$$w = \sum_{i < m} u_i \cdot v_i,$$

$$-w = \sum_{i < n} u'_i \cdot v'_i,$$

where $u_i, u'_i \in \prod_{x < \kappa} \text{Fr}(\lambda_x)$ and $v_i, v'_i \in \prod_{x < \nu} \text{Fr}(\mu_x)$.

**Case 1:** For every $i < m$, $u_i$ is of type 1 or $v_i$ is of type 1. Choose $k \in \omega$ such that for all $i \in \omega$, if $u_i$ is of type 1 then $u_i(n) = 0$ for each natural number $n \geq k$, and if $v_i$ is of type 1 then $v_i(n) = 0$ for each natural number $n \geq k$. Then

$$y_{k+1} \cdot \prod_{p \leq k} -y_p \cdot z_{k+1} \cdot \prod_{p \leq k} -z_p \cdot w = 0,$$

as desired.

**Case 2:** There is an $i < \omega$ such that both $u_i$ and $v_i$ are of type 2. Then Case 1 applies to $-w$. \(\square\)

**Proposition 5.3.** If $I$ and $J$ are nonempty finite sets of infinite cardinals, then

$$i \left( \prod_{\lambda \in I} \text{Fr}(\lambda) \right) \oplus \left( \prod_{\lambda \in J} \text{Fr}(\lambda) \right) = \max(\min(I), \min(J)).$$

**Proof.** Wlog $\min(|I|) \leq \min(|J|)$. Let $\mu = \min(I)$ and $\nu = \min(J)$. For all $\lambda \in I \cup J$ let $\langle x_\lambda^\mu : \lambda < \mu \rangle$ be a system of free generators of $\text{Fr}(\lambda)$. If $|X| < \nu$, clearly $X$ is not maximal independent. Now define, for $\alpha < \mu$ and $\lambda \in I$

$$y_{\alpha}(\lambda) = \begin{cases} x_{\alpha}^{\mu} & \text{if } \lambda = \mu, \\ 0 & \text{otherwise}. \end{cases}$$
Similarly define, for $\alpha < \nu$ and $\lambda \in J$
\[
z_\alpha(\lambda) = \begin{cases} 
x_\alpha^\nu & \text{if } \lambda = \nu, \\
0 & \text{otherwise.} 
\end{cases}
\]

Clearly $\{y_\alpha : \alpha < \mu\} \cup \{z_\alpha : \alpha < \nu\}$ is independent; extend it to a maximal independent subset $X$. So $|X| \geq \nu$; suppose that $|X| > \nu$. For each $w \in X$ write
\[w = \sum_{i<m_w} u_i^w \cdot v_i^w\]
with each $u_i^w \in \prod_{\lambda \in I} \Fr(\lambda)$ and each $v_i^w \in \prod_{\lambda \in J} \Fr(\lambda)$. Let $X'$ be a subset of $X$ of size $\nu^+$ such that for $w \in X'$ we have $m_w$ constant, $\langle u_i^w : i < m_w \rangle$ constant, $\langle v_i^w : i < m_w \rangle$ constant. A contradiction is reached as in the proof of 2.1. $\square$

Familiar arguments also given.

**Proposition 5.4.** If $I$ and $J$ are sets of infinite cardinals, with $J$ finite then
\[i \left( \left( \prod_{\lambda \in I} \Fr(\lambda) \right) \oplus \left( \prod_{\lambda \in J} \Fr(\lambda) \right) \right) = \max(\min(I), \min(J)).\]

6. Particular algebras

The following elementary result leads to a natural problem.

**Proposition 6.1.** Let $\kappa$ be an infinite cardinal and $A = \Fr(\kappa)$, the completion of $\Fr(\kappa)$. Then $\kappa, \kappa^{\omega_1} \in \Spind(A) \subseteq [\kappa, \kappa^{\omega_1}]_{\text{card}}$.

**Problem 2.** Let $A = \Fr(\omega)$. Consistently, what are the possibilities for the set $\Spind(A)$? In particular, is there a model with $2^\omega$ arbitrarily large in which $\Spind(A) = \{\omega, 2^\omega\}$? Or in which $\Spind(A) = [\omega, 2^\omega]_{\text{card}}$?

Several consistency results are known concerning $\Spind(A)$ where $A = \P(\omega)_{\text{fin}}$. Kunen [3, Theorem 2.6, p 258], shows by Cohen forcing that it is consistent to have $2^\omega$ large and $\Spind(A) = \{2^\omega\}$. In exercise (A13), page 289, he shows that it is consistent to have $2^\omega$ large and $\omega_1 \in \Spind(A)$. In the model of Shelah [7] we have $2^\omega = \omega_2$ and $\Spind(\P(\omega)_{\text{fin}}) = \{\omega_1, \omega_2\}$. On the other hand, in Shelah [8] a model is constructed in which $i(\P(\omega)_{\text{fin}})$, itself large, is much smaller than the continuum, which can be arbitrarily large.

These results appear to leave the following problem open.

**Problem 3.** Let $A = \P(\omega)_{\text{fin}}$. Is there a model in which $2^\omega$ is arbitrarily large and $\Spind(A) = \{\omega_1, 2^\omega\}$? Or in which $2^\omega$ is arbitrarily large and $\Spind(A) = [\omega_1, 2^\omega]_{\text{card}}$?
References