ON CELLULARITY IN HOMOMORPHIC IMAGES OF BOOLEAN ALGEBRAS

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Abstract

$c_{HR}A = \{(\mu, \nu) : |A/I| = \nu \geq \omega$ and $c(A/I) = \mu$ for some ideal $I$ of $A\}$ for $A$ an infinite Boolean algebra. Special cases of the main results are: (1) If $(\omega_1, \omega_2) \in c_{HR}A$ and $(\omega, \omega_2) \notin c_{HR}A$, then $(\omega_1, \omega_1) \in c_{HR}A$. (2) There is a model with a BA $A$ such that $c_{HR}A = \{(\omega, \omega), (\omega_1, \omega_1), (\omega, \omega_2), (\omega_2, \omega_2)\}$. (3) Under GCH, there is a BA $A$ such that $c_{HR}A = \{(\omega, \omega_1), (\omega_1, \omega_1), (\omega_1, \omega_2), (\omega_2, \omega_2)\}$. (4) If $cA \geq \omega_2$ and $(\omega, \omega_2) \in c_{SR}A$, then $(\omega_1, \omega_2) \in c_{SR}A$ for the notion $c_{SR}$ analogous to $c_{HR}$.

For any infinite Boolean algebra $A$, let $c_{HR}A = \{(\mu, \nu) : |A/I| = \nu \geq \omega$ and $c(A/I) = \mu$ for some ideal $I$ of $A\}$. Here for any Boolean algebra $A$, $cA$ is the cellularity of $A$, which is defined to be the supremum of the cardinalities of families of pairwise disjoint elements of $A$. We call $c_{HR}$ the homomorphic cellularity relation of $A$. In topological terms, we are dealing with compact zero-dimensional Hausdorff spaces $X$, with

$c_{HR}X = \{(\mu, \nu) : \text{there is an infinite closed subspace } Y \text{ of } X$ with weight $\nu$ and cellularity $\mu\}$.

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It is natural to try to characterize these relations in cardinal number terms. This appears to be a difficult task, but one can give various properties of the relations. We mention some known facts; see Monk [6] for references and more details.

(1) (Shapirovskii, Shelah) If \((\lambda, (2^\kappa)^+) \in c_{HR}A\) for some \(\lambda \leq \kappa\), then \((\omega, (2^\kappa)^+) \in c_{HR}A\).

(2) (Koszmider) If \((\kappa^+, \lambda^+) \in c_{HR}A\), \(\kappa^+\) is not inaccessible, and \(\kappa^+ < \text{cf}|A|\), then there is a \(\kappa'' \geq \kappa^+\) such that \((\kappa'', |A|) \in c_{HR}A\).

(3) (Todorcević) Assuming \(V = L\), for each infinite \(\kappa\) there is a BA \(A\) such that \(c_{HR}A = \{(\lambda, \lambda) : \omega \leq \lambda \leq \kappa\} \cup \{(\kappa, \kappa^+)\}\).

(4) (Malyhin, Shapirovskii) Under MA, if \(|A| < 2^\omega\), then \(A\) has a countable homomorphic image (implying obvious things about \(c_{HR}A\)).

(5) (Koszmider) There is a model with BA's \(A, B, C, D\) having respective homomorphic cellularity relations \(\{\omega, \omega_1\}\), \(\{\omega, \omega_2\}, \{(\omega, \omega_2), (\omega_1, \omega_2)\}\), \(\{\omega, \omega_1), (\omega_1, \omega_1)\}\).

In this paper we give some more properties of these relations.

(6) If \((\omega_1, \omega_2) \in c_{HR}A\) and \((\omega, \omega_2) \notin c_{HR}A\), then \((\omega_1, \omega_1) \in c_{HR}A\). This was mentioned without proof in Monk [6]. We prove a generalization of this to higher cardinalities.

(7) There is a model with a BA \(A\) such that \(c_{HR}A = \{(\omega, \omega), (\omega_1, \omega_1), (\omega, \omega_2), (\omega_2, \omega_2)\}\). This was also mentioned without proof in Monk [6]. The model is a standard one used to adjoin a big maximal almost disjoint family of sets of integers, and we give the construction of that model, and a property it has that is crucial for this application, in a general form.

(8) Under CH, there is a BA \(A\) such that \(c_{HR}A = \{(\omega, \omega_1), (\omega_1, \omega_1), (\omega_2, \omega_2)\}\). This solves problem 8(i) of Monk [6] positively. This BA is the algebra of countable and cocountable subsets of \(\omega_2\), and we describe \(c_{HR}\) for algebras \(\langle |\kappa|^{\leq \rho}\rangle\) in general, in ZFC.
(9) Under GCH, there is a BA $A$ such that $c_{Hr}A = \{(\omega, \omega_1), (\omega_1, \omega_1), (\omega_1, \omega_2), (\omega_2, \omega_2)\}$. This solves problem 8(i) of Monk [6] positively. The BA is obtained from one of the previous algebras by adjoining a family of almost disjoint sets.

There is an analogous notion for subalgebras: $c_{Sr}A = \{(\mu, \nu) : A$ has a subalgebra of size $\nu \geq \omega$ and cellularity $\mu\}$. Concerning this notion we give one result, a special case of which is

(10) If $cA \geq \omega_2$ and $(\omega, \omega_2) \in c_{Sr}A$, then $(\omega_1, \omega_2) \in c_{Sr}A$. This solves problem 4 of Monk [6] negatively.

Results about the relations $c_{Hr}A$ and $c_{Sr}A$ are described thoroughly in Monk [6]. In particular, the situation for algebras of size at most $\omega_2$ is thoroughly discussed. After the results in the present paper, there remain six natural open problems, which can be concisely described as follows:

(1) Can one prove in ZFC that there is a BA $A$ such that $c_{Hr}A = \{(\omega, \omega), (\omega, \omega_1), (\omega_1, \omega_1), (\omega_1, \omega_2)\}$? It is consistent that such a BA exists.

(2) Can one prove in ZFC that there is a BA $A$ such that $c_{Hr}A = \{(\omega, \omega), (\omega, \omega_1), (\omega_1, \omega_1), (\omega_1, \omega_2), (\omega_2, \omega_2)\}$? Again it is consistent that such a BA exists.

(3) Is it consistent that there is a BA $A$ such that $c_{Hr}A = \{(\omega, \omega_1), (\omega_1, \omega_1), (\omega_1, \omega_2)\}$? It is consistent that no such BA exists.

(4) Is it consistent that there is a BA $A$ such that $c_{Hr}A = \{(\omega, \omega_1), (\omega_1, \omega_1), (\omega, \omega_2), (\omega_2, \omega_2)\}$? It is consistent that no such BA exists.

(5) Can one prove in ZFC that there is a BA $A$ such that $c_{Sr}A = \{(\omega, \omega), (\omega, \omega_1), (\omega_1, \omega_1), (\omega_1, \omega_2)\}$? It is consistent that such a BA exists.

(6) Can one prove in ZFC that there is a BA $A$ such that $c_{Sr}A = \{(\omega, \omega), (\omega, \omega_1), (\omega_1, \omega_1), (\omega_1, \omega_2), (\omega_2, \omega_2)\}$?
It is consistent that such a BA exists.

**Notation.** For set theory, we follow Kunen [5], with the following changes and additions. If $f : A \to B$ and $X \subseteq A$, then the $f$-image of $X$ is denoted by $f[X]$. A family of sets $\mathcal{A}$ is *almost disjoint* if $|X \cap Y| < |X|, |Y|$ for any two distinct $X, Y \in \mathcal{A}$; it is *$\mu$-almost disjoint* or *$\mu$-ad* if the intersection of any two distinct members has size less than $\mu$. A subset $X$ of a set $A$ is called *co-$\kappa$* if $|A \setminus X| < \kappa$.

For any topological space $X$, the collection of all closed and open subsets of $X$ is denoted by $\text{clop}X$.

For Boolean algebras we follow Koppelberg [4]. If $I$ is an ideal in a BA $A$ and $x \in I$, then $[x]_I$ is the equivalence class of $x$ under the equivalence relation determined by $I$. The subalgebra of $A$ generated by $X$ is denoted by $\langle X \rangle_A$, or simply $\langle X \rangle$ if $A$ is clear. The free algebra on $\kappa$ free generators is denoted by $\text{Fr} \kappa$. The algebra of finite and cofinite subsets of a cardinal $\kappa$ is denoted by $\text{Finco} \kappa$. The completion of an algebra $A$ is denoted by $\bar{A}$. We need a slight generalization of a result of Juhász and Shelah [2]; their result corresponds to successor $\lambda$ in Theorem 2.

Let $<$ be a binary relation on a set $X$, and let $\tau$ and $\mu$ be infinite cardinal numbers. For any subset $a$ of $X$ and any $x \in X$, let $\text{Pred}_a x = \{y \in a : y < x\}$. We say that $<$ is *$(< \tau)$-full* if for every $a \in [X]^{< \tau}$ there is an $x \in X$ such that $a = \text{Pred}_a x$. And we say that $<$ is *$\mu$-local* if for every $x \in X$ we have $|\text{Pred}_X x| \leq \mu$.

**Lemma 1.** Let $<$ be a binary relation on an infinite cardinal $\rho$ that is both $(< \tau)$-full and $\mu$-local. Then for every $\sigma < \tau$ and every almost disjoint family $\mathcal{A} \subseteq [\rho]^{\sigma}$ we have $|\mathcal{A}| \leq \rho \cdot \mu^{< \tau}$.

**Proof.** Since $<$ is $(< \tau)$-full, for every $a \in \mathcal{A}$ there is a $\xi_a < \rho$ such that $a = \text{Pred}_a \xi_a$. Thus $a \in [\text{Pred}_\rho \xi_a]^{< \tau}$. So $\mathcal{A} \subseteq \bigcup_{\xi < \rho} [\text{Pred}_\rho \xi]^{< \tau}$, and the latter has size at most $\rho \cdot \mu^{< \tau}$. \qed

**Theorem 2.** Suppose that $\kappa$ and $\lambda$ are infinite cardinals, $\lambda \leq$
Let $f$ be a homomorphism from $\langle [\kappa]^{< \lambda} \rangle_{\mathcal{P} \kappa}$ onto an infinite BA $B$. Then $|B| < 2^{< \lambda}$ or $|B|^{< \lambda} = |B|$.

**Proof.** Let $\rho = |B|$ and $C = f([\kappa]^{< \lambda})$. Thus $|C| = \rho$ too. Suppose that $2^{< \lambda} \leq \rho$.

(1) $\leq_B$ restricted to $C$ is $< \lambda$-full.

For, suppose that $a \subseteq C$ and $|a| < \lambda$. Then there is an $x \in [\kappa]^{< \lambda}$ such that $a = f[x]$. Since $\lambda$ is regular, also $b \overset{\text{def}}{=} \bigcup x \in [\kappa]^{< \lambda}$, so $f(b) \in C$. Now $a \subseteq \text{Pred}_C f(b)$. For, if $u \in a$, say $u = f(c)$ with $c \in x$. Then $c \subseteq b$, so $f(c) \leq f(b)$. Hence $a = \{ y \in a : y \leq f(b) \}$, and (1) follows.

(2) $\leq_B$ restricted to $C$ is $2^{< \lambda}$-local.

In fact, suppose that $c \in C$; say $c = f(x)$ with $x \in [\kappa]^{< \lambda}$. If $b \in C$ and $b \leq c$, say $b = f(y)$ with $y \in [\kappa]^{< \lambda}$. Then $f(y \cap x) = f(y) \cap f(x) = b$. Thus $b \in f[\mathcal{P}x]$; and $|\mathcal{P}x| \leq 2^{< \lambda}$, as desired in (2).

Now by lemma 1 we have

(3) For every $\tau < \lambda$, and every almost disjoint $\mathcal{A} \subseteq [\rho]^\tau$ we have $|\mathcal{A}| \leq \rho \cdot (2^{< \lambda})^{< \lambda} = \rho$.

Now we are ready to show that $\rho^{< \lambda} = \rho$. For, suppose that $\rho^{< \lambda} > \rho$. Since $\lambda \leq \rho$, it follows that $\rho^\tau > \rho$ for some $\tau < \lambda$; let $\tau$ be minimum with this property. Then by a well-known argument, there is an almost disjoint $\mathcal{A} \subseteq [\rho]^\tau$ of size $\rho^\tau$. This contradicts (3). $\square$

**Lemma 3.** Suppose that $\kappa$ and $\lambda$ are cardinals, $\omega \leq \lambda \leq \kappa^+$, $\lambda$ regular. Let $A = \langle [\kappa]^{< \lambda} \rangle_{\mathcal{P} \kappa}$. Let $I$ be an ideal on $A$, and assume that $|A/I| > 2^{< \lambda}$. Then

(i) $\forall a \in I(|a| < \lambda)$.

(ii) Suppose that $\mathcal{A} \subseteq A$, $\forall a \in \mathcal{A}(|a| < \lambda)$, $\langle [a]_I : a \in \mathcal{A} \rangle$ is pairwise disjoint, and $\mathcal{A}$ is maximal with these properties. Then $\sum_{a \in \mathcal{A}} [a]_I = 1$.

(iii) Continuing (ii), $|A/I| \leq |\bigcup \mathcal{A}|^{< \lambda}$. 


(iv) \(|A/I| \leq c(A/I)^{<\lambda}\).
(v) \(2^{<\lambda} < c(A/I)\).

**Proof.** For (i), suppose that \(a \in I\) and \(|-a| < \lambda\). Then the mapping \(x \mapsto [x]_I\) for \(x \subseteq -a\) is a homomorphism from \(\mathcal{P}(-a)\) onto \(A/I\). But \(|\mathcal{P}(-a)| \leq 2^{<\lambda}\), contradicting \(|A/I| > 2^{<\lambda}\).

For (ii), suppose not: say \([b]_I = [\emptyset]_I\), while \([b]_I \cdot [a]_I = [\emptyset]_I\) for all \(a \in \mathcal{A}\). Then for all \(c \in [b]^{<\lambda}\) we have \([c]_I = 0\). Hence \(|b| \geq \lambda\), so \(|-b| < \lambda\). So \([c]_I = [c \setminus b]_I\) for all \(c \in [\kappa]^{<\lambda}\). Hence \([c]_I : c \in [\kappa]^{<\lambda}\} = \{[c]_I : c \in [-b]^{<\lambda}\} \) has size at most \(\mu^{<\lambda}\), where \(\mu = |-b|\). And \(\mu < \lambda\), so \(\mu^{<\lambda} \leq 2^{<\lambda}\). Hence \(|A/I| \leq 2^{<\lambda}\), contradiction.

For (iii), note that if \(b \in [\kappa \setminus \mathcal{A}]^{<\lambda}\), then \(b \in I\) by the maximality of \(\mathcal{A}\). So

\[
\{[b]_I : b \in A, |b| < \lambda\} = \{[b \cap \mathcal{A}]_I : |b| < \lambda\},
\]
so (iii) holds.

For (iv), note that if \(c(A/I) < \lambda\), then \(|\mathcal{A}| < \lambda\) by regularity of \(\lambda\), and so \(|\mathcal{A}|^{<\lambda} \leq 2^{<\lambda}\), and (iii) gives a contradiction. So \(\lambda \leq c(A/I)\). Hence \(|\mathcal{A}| \leq c(A/I)\). Then (iii) yields (iv).

Finally, (v) follows from (iv) and the hypothesis. \(\square\)

**Theorem 4.** Suppose that \(\omega < \rho < \kappa\). Let \(A = \langle [\kappa]^{<\rho} \rangle_{\mathcal{A}}\). Then \(c_{\mathcal{H}}(A) = S \cup T \cup U\), where

\[
S = \{ (\mu, \nu) : \omega \leq \mu \leq \nu \leq 2^\rho, \nu^\omega = \nu \};
T = \{ (\mu, \mu^\rho) : 2^\rho < \mu \leq \kappa \};
U = \{ (\mu, \kappa^\rho) : 2^\rho < \mu, \mu^\rho = \kappa^\rho, \kappa < \mu \}.
\]

**Proof.** First suppose that \((\mu, \nu) \in S\). The mapping \(a \mapsto a \cap \rho\) gives a homomorphism of \(A\) onto \(\mathcal{P}\rho\). Since \(\mathcal{P}\rho\) has an independent subset of size \(2^\rho\), there is a homomorphism of \(\mathcal{P}\rho\) onto an algebra \(B\) such that \(\text{Fr}v \leq B \leq \text{Fr}v\). Since \(\nu^\omega = \nu\), we have \(|B| = \nu\). Now there is a homomorphism of \(B\) onto an
algebra \( C \) such that \( \text{Fr} \nu \times \text{Fin} \mu \leq C \leq \text{Fr} \nu \times \text{Fin} \mu \). Thus \( |C| = \nu \) and \( c(C) = \mu \), so \((\mu, \nu) \in c_{\text{HR}}(A)\).

Second, suppose that \( 2^\mu < \mu \leq \kappa \). The mapping \( a \mapsto a \cap \mu \) gives a homomorphism of \( A \) onto \( ([\mu]^{\leq \rho}) \), which has size \( \mu^\rho \) and cellularity \( \mu \). So \((\mu, \mu^\rho) \in c_{\text{HR}}(A)\).

Third, suppose that \( 2^\rho < \mu = \kappa^\rho \), and \( \kappa < \mu \). Note that \( 2^\rho < \kappa \), for if \( \kappa \leq 2^\rho \) then \( \kappa^\rho \leq 2^\rho \leq \kappa^\rho \), so \( \kappa^\rho = 2^\rho < \mu \leq \mu^\rho = \kappa^\rho \), contradiction. Now let \( \nu \) be minimum such that \( \kappa \leq \nu^\rho \).

Since \( 2^\rho < \kappa \) and \( \kappa < \kappa^\rho \), it follows from Jech [1], Theorem 19, that \( \text{cf} \nu \leq \rho < \nu \) and \( \kappa^\rho = \nu^{\text{cf} \nu} \). Now if \( \sigma < \text{cf} \nu \), then \( \nu^\sigma \leq \kappa \), for

\[
\nu^\sigma = [\sigma \nu] = \bigcup_{\delta < \nu} [\sigma \delta] \leq \sum_{\delta < \nu} |\delta| \leq \kappa.
\]

Hence \( \bigcup_{\sigma < \text{cf} \nu} \sigma \nu \leq \kappa \), so there is an \( A \subseteq [\kappa]^{\text{cf} \nu} \) which is \( \text{cf} \nu \)-ad and of size \( \nu^{\text{cf} \nu} = \kappa^\rho \). Let \( I = [\kappa]^{< \text{cf} \nu} \). Then \( \langle [a]_I : a \in A \rangle \) is isomorphic to \( \text{Fin} \mu \). By the Sikorski extension theorem we get a homomorphism \( h \) of \( A \) onto \( \text{Fin} \mu \). Thus \( c(B) = \mu \), and by Theorem 2, \( |B|^\rho = |B| \).

Since \( \kappa < \mu \leq |B| \), it follows that \( \kappa^\rho \leq |B|^\rho = |B| \leq \kappa^\rho \). So \( |B| = \kappa^\rho \). Thus \((\mu, \kappa^\rho) \in c_{\text{HR}}(A)\).

Finally, suppose conversely that \((\mu, \nu) \in c_{\text{HR}}(A)\). Since \( A \) is \( \sigma \)-complete, it is well-known that \( \nu^\omega = \nu \). So if \( \nu \leq 2^\rho \), then \((\mu, \nu) \in S \). Suppose that \( 2^\rho < \nu \). By Theorem 2 \( \nu^\rho = \nu \), and by Lemma 3, \( 2^\rho < \mu \) and \( \nu \leq \mu^\rho \). Hence \( \mu^\rho \leq \nu^\rho \leq \mu^\rho \), so \( \nu = \nu^\rho = \mu^\rho \). If \( \mu \leq \kappa \), then \((\mu, \nu) \in T \). Suppose that \( \kappa < \mu \).

Then \( \kappa^\rho \leq \mu^\rho = \nu \leq \kappa^\rho \), so \( \nu = \kappa^\rho \) and \((\mu, \nu) \in U \).

Theorem 4 provides a positive solution of Problem 8(i) of Monk [6]. Namely, assume CH and let \( \kappa = \omega_2 \) and \( \rho = \omega \) in the theorem. Thus with \( A = ([\omega_2]^{\leq \omega}) \mathcal{G}_{\omega_2} \), under CH we have

\[
c_{\text{HR}}A = \{ (\omega, \omega), (\omega_1, \omega_1), (\omega_2, \omega_2) \}.
\]

Under GCH, there is a simpler description of \( ([\kappa]^{\leq \rho}) \mathcal{G}_{\kappa} \):
Corollary 5. (GCH) Suppose that $\omega \leq \rho \leq \kappa$. Let $A = \langle [\kappa]^{<\rho} \rangle$. Then

$$c_{HR}A = \{ (\mu, \nu) : \omega \leq \mu \leq \nu \leq \rho^+ , cf \nu > \omega \}$$

$$\cup \{ (\mu, \mu) : \rho^+ < \mu , \rho < cf \mu , \mu \leq \kappa \}$$

$$\cup \{ (\mu, \mu^+) : \rho^+ < \mu , cf \mu \leq \rho , \mu \leq \kappa \}$$

$$\cup \{ (\kappa^+, \kappa^+) : cf \kappa \leq \rho < \kappa \}.$$ 

It is natural to also consider the algebra $A = \langle [\kappa]^{<\lambda} \rangle$ for $\lambda$ limit. For $\lambda$ singular the situation is unclear. Note that if $cf \lambda = \omega$, it is possible that $A$ has a countable homomorphic image. For example, let $\kappa = \lambda = \aleph_\omega$. For each $n \in \omega$ let $F_n$ be an ultrafilter on the Boolean algebra $\mathcal{P}(\aleph_n)$ such that $X \in F_n$ for every $X \subseteq \aleph_n$ for which $|\aleph_n \setminus X| < \aleph_n$. Define $f(a) = \{ n \in \omega : a \cap \aleph_n \in F_n \}$ for every $a \in A$. It is easy to see that $f$ is a homomorphism from $A$ onto $\text{Fin}_\mathcal{C}$. 

For $\lambda$ regular limit (meaning that it is weakly inaccessible), we can give a complete description of the cellularity homomorphism relation. For this we need another lemma. This lemma is proved like Lemma 3.

Lemma 6. Suppose that $\kappa$ and $\lambda$ are cardinals, $\lambda$ is weakly inaccessible, $2^\mu < 2^{<\lambda}$ for all $\mu < \lambda$, and $\lambda \leq \kappa$. Let $A = \langle [\kappa]^{<\lambda} \rangle$. Let $I$ be an ideal on $A$, and assume that $|A/I| = 2^{<\lambda}$. Then

(i) $\forall a \in I (|a| < \lambda)$.

(ii) Suppose that $\mathcal{A} \subseteq A$, $\forall a \in \mathcal{A} (|a| < \lambda)$, $\langle [a]_I : a \in \mathcal{A} \rangle$ is pairwise disjoint, and $\mathcal{A}$ is maximal with these properties. Then $\sum_{a \in \mathcal{A}} [a]_I = 1$.

(iii) Continuing (ii), $|A/I| \leq [\bigcup \mathcal{A}]^{<\lambda}$.

(iv) $c(A/I) \geq \lambda$.

Proof. Only (iv) requires additional scrutiny. If $c(A/I) < \lambda$, then $|\mathcal{A}| < \lambda$, so by the regularity of $\lambda$, $|\bigcup \mathcal{A}| < \lambda$. But then $|[\bigcup \mathcal{A}]^{<\lambda}| = |\mathcal{P}(\bigcup \mathcal{A})| < 2^{<\lambda}$, contradiction. $\square$
Theorem 7. Suppose that $\lambda$ is uncountable and weakly inaccessible and $\lambda \leq \kappa$. Let $A = [\kappa]^{<\lambda}$. Define

$$S = \{((\mu, \nu) : \omega \leq \mu \leq \nu < 2^{\lambda}, \nu^\omega = \nu\};$$
$$T = \{((\mu, \mu^{<\lambda}) : 2^{<\lambda} \leq \mu \leq \kappa)\};$$
$$U = \{((\mu, \kappa^{<\lambda}) : 2^{<\lambda} < \mu, \mu^{<\lambda} = \kappa^{<\lambda}, \kappa < \mu)\};$$
$$V = \{((\mu, 2^{<\lambda}) : \omega \leq \mu \leq 2^{<\lambda})\};$$
$$W = \{((\mu, 2^{<\lambda}) : \lambda \leq \mu \leq 2^{<\lambda})\}.$$

Then

(i) If $2^\rho = 2^{<\lambda}$ for some $\rho < \lambda$, then $c_{HR}(A) = S \cup T \cup U \cup V$;
(ii) If $2^\rho < 2^{<\lambda}$ for all $\rho < \lambda$, then $c_{HR}(A) = S \cup T \cup U \cup W$.
(iii) If $\lambda$ is strongly inaccessible, then $c_{HR}(A) = S \cup T \cup U \cup \{(\lambda, \lambda)\}$.

Proof. The proof that $S \cup T \cup U \subseteq c_{HR}(A) \subseteq S \cup T \cup U \cup V$ is very similar to the proof for Theorem 4. For example, to show that $U \subseteq c_{HR}(A)$, take $\mu$ such that $2^{<\lambda} < \mu, \mu^{<\lambda} = \kappa^{<\lambda}$, and $\kappa < \mu$. Then $2^{<\lambda} < \kappa$ by an argument like that in the proof of Theorem 4. Since $\kappa < \mu \leq \kappa^{<\lambda}$, choose $\rho$ so that $\kappa < \kappa^\rho$ and $\rho < \lambda$, and then proceed as before.

Now suppose that $\rho < \lambda$ and $2^\rho = 2^{<\lambda}$. The mapping $a \mapsto a \cap \rho$ gives a homomorphism from $A$ onto $\mathcal{P}\rho$. Then the argument at the beginning of the proof of Theorem 4 shows that $(\mu, 2^{<\lambda}) \in c_{HR}(A)$ for all $\mu \in [\omega, 2^{<\lambda}]$. This proves (i).

Next, suppose that $2^\rho < 2^{<\lambda}$ for all $\rho < \lambda$, and that $\lambda \leq \mu < 2^{<\lambda}$. Then there is a $\rho < \lambda$ such that $\mu < 2^\rho$. Write $\lambda = \Gamma_0 \cup \Gamma_1$, where $\Gamma_0 \cap \Gamma_1 = \emptyset$, $|\Gamma_0| = \lambda$, and $|\Gamma_1| = \rho$. By Theorem 4 there is a homomorphism $f$ of $\mathcal{P}\Gamma_1$ onto an algebra of size $2^\rho$ and cellularity $\mu$. Let $g(a) = (a \cap \Gamma_0, f(a \cap \Gamma_1))$ for all $a \in A$. The image of $g$ has size $2^{<\lambda}$ and cellularity $\mu$.

To get a homomorphic image of size and cellularity $2^{<\lambda}$ we have to modify this argument. Let $M$ be the set of all infinite cardinals less than $\lambda$, and let $(\Gamma_\alpha : \alpha \in M)$ be a partition of $\lambda$ with $|\Gamma_\alpha| = \alpha$ for all $\alpha \in M$. For each $\alpha \in M$ let $f_\alpha$ be a
homomorphism of $\mathcal{P}\Gamma_\alpha$ onto an algebra of size and cellularity $2^\alpha$. Then let $g(a)_\alpha = f_\alpha(a \cap \Gamma_\alpha)$ for all $a \in A$. Then the image of $g$ is as desired.

That no other pairs are in $c_{\mathcal{H}_\alpha}(A)$ follows from Lemma 6. Thus (ii) holds.

(iii) is a clear consequence of (ii).

For the next result we need a standard Boolean algebraic fact:

**Proposition 8.** Suppose that $A$ is $\kappa$-complete, and $I$ is a $\kappa$-complete maximal ideal in $A$. Suppose that $f : I \rightarrow B$ preserves $\langle \kappa \rangle$-joins, $\langle \kappa \rangle$-meets, and 0. Then $f$ extends to a unique $\kappa$-complete homomorphism $f^+ : A \rightarrow B$. Moreover, $f^+$ is one-one iff $\forall x \in I[f(x) = 0 \Rightarrow x = 0]$ and $\forall x \in I[f(x) \neq 1]$.

**Proof.** The following definition of $f^+$ is forced upon us:

$$f^+(a) = \begin{cases} f(a) & \text{if } a \in I, \\ -f(-a) & \text{if } a \notin I. \end{cases}$$

Then $f^+$ preserves $-$, since if $a \in I$, then $f^+(-a) = -f(a) = -f^+(a)$, and if $a \notin I$, then $f^+(-a) = f(-a) = --f^+(a)$.

Now we show that $f^+$ preserves $\langle \kappa \rangle$-joins. So, let $\sum_{\xi < \alpha} a_\xi$ be given, with $\alpha < \kappa$. If $\forall \xi < \alpha a_\xi \in I$, then

$$f^+ \left( \sum_{\xi < \alpha} a_\xi \right) = f \left( \sum_{\xi < \alpha} a_\xi \right) = \sum_{\xi < \alpha} f(a_\xi) = \sum_{\xi < \alpha} f^+(a_\xi).$$

Now suppose that $\exists \xi < \alpha [a_\xi \notin I]$. Let $\Gamma = \{ \xi < \alpha : a_\xi \notin I \}$. 
Then

\[
\sum_{\xi \in \Gamma} a_\xi + \left( \sum_{\xi < \alpha} a_\xi \right) = \sum_{\xi \in \Gamma} a_\xi + \left( - \left( \sum_{\xi \in \Gamma} a_\xi + \sum_{\xi \in \alpha \setminus \Gamma} a_\xi \right) \right) \\
= \sum_{\xi \in \Gamma} a_\xi + \left( - \sum_{\xi \in \Gamma} a_\xi \cdot \sum_{\xi \in \alpha \setminus \Gamma} a_\xi \right) \\
= \sum_{\xi \in \Gamma} a_\xi + \left( - \sum_{\xi \in \alpha \setminus \Gamma} a_\xi \right).
\]

Using this,

\[
\sum_{\xi < \alpha} f^+(a_\xi) + \left( - f^+ \left( \sum_{\xi < \alpha} a_\xi \right) \right) \\
= \sum_{\xi \in \Gamma} f(a_\xi) + \sum_{\xi \in \alpha \setminus \Gamma} - f(-a_\xi) + f \left( - \sum_{\xi < \alpha} a_\xi \right) \\
= f \left( \sum_{\xi \in \Gamma} a_\xi + \left( - \sum_{\xi < \alpha} a_\xi \right) \right) + \sum_{\xi \in \alpha \setminus \Gamma} - f(-a_\xi) \\
= f \left( \sum_{\xi \in \Gamma} a_\xi + \left( - \sum_{\xi \in \alpha \setminus \Gamma} a_\xi \right) \right) + \sum_{\xi \in \alpha \setminus \Gamma} - f(-a_\xi) \\
= f \left( \sum_{\xi \in \Gamma} a_\xi \right) + f \left( - \sum_{\xi \in \alpha \setminus \Gamma} a_\xi \right) + \sum_{\xi \in \alpha \setminus \Gamma} - f(-a_\xi) \\
= f \left( \sum_{\xi \in \Gamma} a_\xi \right) + \prod_{\xi \in \alpha \setminus \Gamma} f(-a_\xi) + \left( - \prod_{\xi \in \alpha \setminus \Gamma} f(-a_\xi) \right) \\
= 1.
\]
And if $\xi \in \Gamma$, then
\[
f^+(a_\xi) \cdot -f^+ \left( \sum_{\eta < \alpha} a_\eta \right) = f(a_\xi) \cdot f \left( -\sum_{\eta < \alpha} a_\eta \right).
\]
\[
= f \left( a_\xi \cdot -\sum_{\eta < \alpha} a_\eta \right)
\]
\[
= f(0) = 0.
\]

If $\xi \in \alpha \setminus \Gamma$, then
\[
f^+(a_\xi) \cdot -f^+ \left( \sum_{\eta < \alpha} a_\eta \right) = -f(-a_\xi) \cdot f \left( -\sum_{\eta < \alpha} a_\eta \right).
\]

Now $a_\xi \leq \sum_{\eta < \alpha} a_\eta$, so $-\sum_{\eta < \alpha} a_\eta \leq -a_\xi$, hence $f \left( -\sum_{\eta < \alpha} a_\eta \right) \leq f(-a_\xi)$, so $-f(-a_\xi) \cdot f \left( -\sum_{\eta < \alpha} a_\eta \right) = 0$. So we have proved that $f^+ \left( \sum_{\xi < \alpha} a_\xi \right) = \sum_{\xi < \alpha} f(a_\xi)$. So $f$ is a $\kappa$-homomorphism.

Concerning the final statement, the direction $\Rightarrow$ is clear. Now suppose the indicated condition holds, and $f^+(a) = 0$. If $a \in I$, then $f(a) = f^+(a) = 0$, so $a = 0$. If $a \notin I$, then $f^+(a) = -f(-a) = 0$, so $f(-a) = 1$ and $-a \in I$, contradiction. $\square$

**Lemma 9.** Suppose that $\kappa < \lambda$, $\kappa$ is regular, $\mathcal{A} \subseteq [\kappa]^\kappa$ is almost disjoint, and $|\mathcal{A}| = \lambda$. Let $A$ be the $\kappa$-complete subalgebra of $\mathcal{P}_\kappa$ generated by $\mathcal{A} \cup \{\{\xi\} : \xi < \kappa\}$. Then $A/[\kappa]^{<\kappa} \cong ([\lambda]^{<\kappa})^{\mathcal{A}_\lambda}$.

**Proof.** Let $(X_\alpha : \alpha < \lambda)$ be a one-one enumeration of $\mathcal{A}$. Set $I = [\kappa]^{<\kappa}$. For each $\Gamma \in [\lambda]^{<\kappa}$ let $f(\Gamma) = [\bigcup_{\alpha \in \Gamma} X_\alpha]$. Clearly $f$ preserves $(< \kappa)$-joins, and $f(0) = 0$. It also preserves $(< \kappa)$-meets. For, suppose that $\Gamma_\alpha \in [\lambda]^{<\kappa}$ for all $\alpha < \gamma$, where $\gamma < \kappa$. Let $\Delta = \bigcup_{\alpha < \gamma} \Gamma_\alpha$. So $|\Delta| < \kappa$ since $\kappa$ is regular. Let
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$P$ be the set of all nonconstant $g \in \prod_{\alpha<\gamma} \Gamma_{\alpha}$. Then

$$\bigcap_{\alpha<\gamma} \bigcup_{\xi \in \Gamma_{\alpha}} X_{\xi} = \bigcup_{g \in \prod_{\alpha<\gamma} \Gamma_{\alpha}} \bigcap_{\alpha<\gamma} X_{g(\alpha)}$$

$$= \bigcup_{\xi \in \cap_{\alpha<\gamma} \Gamma_{\alpha}} \bigcup_{g \in P} \bigcap_{\alpha<\gamma} X_{g(\alpha)}.$$ 

Now

$$\bigcup_{g \in P} \bigcap_{\alpha<\gamma} X_{g(\alpha)} \subseteq \bigcup_{\{X_{\alpha} \cap X_{\beta} : \alpha, \beta \in \Delta, \alpha \neq \beta\}},$$

and the latter set has size less than $\kappa$. This shows that $f$ preserves ($<\kappa$)-meets.

Hence by Proposition 8, $f$ extends to a $\kappa$-homomorphism from $\langle [\lambda]^{<\kappa}\rangle_{\mathcal{P}_{\lambda}}$ into $A/I$. By the same proposition it is clear that $f$ is one-one. Since $f[[\lambda]^{<\kappa}]$ generates $A/I$ as a $\kappa$-complete algebra, $f$ maps onto $A/I$. $\Box$

**Theorem 10.** (GCH) Let $\mathcal{A} \subseteq [\kappa^+]^{\kappa^+}$ be $\kappa^+$-ad, with $|\mathcal{A}| = \kappa^{++}$. Let $A$ be the $\kappa^+$-complete subalgebra of $\mathcal{P}_{\kappa^+}$ generated by $\mathcal{A} \cup \{\{\alpha\} : \alpha < \kappa^+\}$. Then

$$c_{H_{\mathcal{A}}} A = \{(\mu, \nu) : \omega \leq \mu \leq \nu \leq \kappa^+, c_{f\nu} > \omega\} \cup \{(\kappa^+, \kappa^{++}), (\kappa^{++}, \kappa^{++})\}.$$ 

**Proof.** Let $\langle X_{\alpha} : \alpha < \kappa^{++}\rangle$ be a one-one enumeration of $\mathcal{A}$. Let $I = [\kappa^+]^{\leq \kappa}$. Then by Lemma 9,

(1) $A/I \cong \langle [\kappa^{++}]^{\leq \kappa}\rangle_{\mathcal{P}_{\kappa^{++}}}.$

Hence by Corollary 5, $c_{H_{\mathcal{A}}} A$ contains the set of the theorem. Suppose that $(\mu, \nu) \in c_{H_{\mathcal{A}}} A$, with $(\mu, \nu)$ not in the indicated set. Then $\nu = \kappa^{++}$ and $\mu \leq \kappa$. So $A$ has an independent subset $\mathcal{F}$ of size $\kappa^{++}$. Since $|I| = \kappa^+$, we may assume that the members of $\mathcal{F}$ are pairwise inequivalent modulo $I$, each nonzero modulo $I$. By the proof of (1), for each $a \in \mathcal{F}$ we can choose a $\Gamma_{\alpha} \in [\kappa^{++}]^\kappa$ such that $[a]_I = \bigcup_{\alpha \in \Gamma_{\alpha}} X_{\alpha}$. Then
there is a $\Delta \in [\mathcal{P}]^{++}$ such that $\langle \Gamma_a : a \in \Delta \rangle$ is a $\Delta$-system. Let $a, b, c$ be distinct members of $\Delta$. Then $|a \cdot b \cdot -c| = 0$, i.e., $|a \cdot b \cdot -c| \leq \kappa$. Hence

$$\langle a \cdot b \cdot -c \cdot d : d \in \Delta \setminus \{a, b, c\} \rangle$$

is a system of $\kappa^{++}$ independent subsets of $a \cdot b \cdot -c$, which contradicts GCH. \qed

Taking $\kappa = \omega$ in this theorem we get, under GCH, a BA $A$ such that

$$c_{HR} A = \{ (\omega, \omega_1), (\omega_1, \omega_1), (\omega_1, \omega_2), (\omega_2, \omega_2) \}.$$

This solves Problem 8(iii) of Monk [6] positively.

For the next result we need a fact about one of the standard ways of forcing a large mad family. This fact was observed by Richard Laver, and we thank him for allowing us to include the proof of the fact here.

**Theorem 11.** In a model of ZFC+$GCH$, suppose that $\kappa$ and $\lambda$ are infinite cardinals, $\kappa$ regular, $\kappa < \lambda$. Then there is an extension preserving cofinalities and cardinalities in which there is a system $\langle A_\alpha : \alpha < \lambda \rangle$ of almost disjoint members of $[\kappa]^\kappa$ with the following property:

(*) if $X \in [\kappa]^\kappa$ and $|X \cap A_\alpha| = \kappa$ for $\kappa$ many $\alpha < \lambda$, then $|X \cap A_\alpha| = \kappa$ for $\text{co-}\kappa^+$ many $\alpha < \lambda$.

**Proof.** Let $\mathbb{P}$ be the set of all functions $f$ such that there exist an $F \in [\lambda]^{<\kappa}$ and a $\nu < \kappa$ such that $f : F \times \nu \to 2$. For $f \in \mathbb{P}$ we let $F_f$ and $\nu_f$ be the $F, \nu$ mentioned, with $F_f = 0 = \nu_f$ if $f = 0$. We write $f \leq g$ iff $g \subseteq f$ and for any distinct $\alpha, \beta \in F_g$ and any $\iota \in \nu_f \setminus \nu_g$, $f(\alpha, \iota) = 0$ or $f(\beta, \iota) = 0$. Clearly

(1) $(\mathbb{P}, \leq)$ is $\kappa$-closed and satisfies the $\kappa^+$-chain condition. Consequently, forcing with $(\mathbb{P}, \leq)$ preserves cofinalities and cardinals.
(2) For any $\alpha < \lambda$, the set $\{f \in \mathcal{P} : \alpha \in F_f\}$ is dense.

In fact, given $g \in \mathcal{P}$, if $\alpha \not\in F_g$, let $F_f = F_g \cup \{\alpha\}$, $\nu_f = \nu_g$, and let $f$ extend $g$ with $f(\alpha, \iota) = 0$ for all $\iota < \nu_g$. Clearly this proves (2).

Now let $G$ be generic for $(\mathbb{P}, \leq)$ over the ground model. We then set, for any $\alpha < \lambda$,

$$\begin{align*}
A_\alpha &= \{\iota < \kappa : \exists g \in G(\alpha \in F_g, \iota < \nu_g, g(\alpha, \iota) = 1)\} \\
\Gamma_\alpha &= \{(\iota, g) : \alpha \in F_g, \iota < \nu_g, g(\alpha, \iota) = 1\}.
\end{align*}$$

Thus $\Gamma_\alpha^G = A_\alpha$.

(3) For each $\alpha < \lambda$, $|A_\alpha| = \kappa$.

In fact, it suffices to show that for any $\mu < \kappa$ the following set is dense:

$$\{g \in \mathcal{P} : \alpha \in F_g \text{ and } \exists \xi \in \kappa \setminus \mu(\xi < \nu_g \text{ and } g(\alpha, \xi) = 1)\}.$$ 

To prove this, let $f \in \mathcal{P}$. By (2) we may assume that $\alpha \in F_f$. Now let $f \subseteq g$, $F_f = F_g = \max(\nu_f + 1, \mu + 2)$, $\xi = \max(\nu_f, \mu + 1)$, with $g(\beta, \iota) = 0$ if $\nu_f \leq \iota$ and $\beta \neq \alpha$, $g(\alpha, \iota) = 0$ if $\iota \neq \xi$, and $g(\alpha, \xi) = 1$. Clearly $g \in \mathcal{P}$ and $g \leq f$, as desired in (3).

(4) $|A_\alpha \cap A_\beta| < \kappa$ for $\alpha \neq \beta$.

In fact, by (2) choose $g \in G$ such that $\alpha, \beta \in F_g$. Then, we claim, $A_\alpha \cap A_\beta = \{\iota < \nu_g : g(\alpha, \iota) = 1 = g(\beta, \iota)\}$, which will prove (4). Clearly $\supseteq$ holds. Now suppose that $\iota \in A_\alpha \cap A_\beta$. Then there is an $f \in G$ such that $f \leq g$, $\iota < \nu_f$ and $f(\alpha, \iota) = 1 = f(\beta, \iota)$. From the definition of $\leq$ it follows that $\iota < \nu_g$, and hence $f(\alpha, \iota) = g(\alpha, \iota)$ and $f(\beta, \iota) = g(\beta, \iota)$, as desired.

Now suppose that $X \in [\kappa]^\kappa$ and $|X \cap A_\alpha| = \kappa$ for $\kappa$ many $\alpha$'s. Let $\tau$ be a name for $X$. Choose $p \in G$ so that

(5) $p \vdash \forall H \in [\lambda]^{<\kappa}(\tau \setminus \bigcup_{\alpha \in H} \Gamma_\alpha) = \kappa$.

Now we claim
(6) There is a $C\in [\lambda]^{\le\kappa}$ such that $F_p \subseteq C$ and for all $q, \mu, H$, if $q \in \mathbb{P}$, $F_q \subseteq C$, $q \le p$, $\mu < \kappa$, and $H \in [C]^{<\kappa}$, then there is a $q' \le q$ such that $F_{q'} \subseteq C$ and there is a $\xi \in \kappa \setminus \mu$ such that $q' \forces \xi \in \tau \setminus \bigcup_{\beta \in H} \Gamma_\beta$.

For we construct $\langle C_\alpha : \alpha < \kappa \rangle$ by induction. Let $C_0 = F_p$. For $\alpha$ limit, let $C_\alpha = \bigcup_{\beta < \alpha} C_\beta$. Now suppose that $C_\alpha$ has been constructed, with $|C_\alpha| \le \kappa$. For $q, \mu, H$ such that $q \in \mathbb{P}$, $q \le p$, $F_q \subseteq C_\alpha$, $\mu < \kappa$, and $H \in [C_\alpha]^{<\kappa}$, there exist a $q' = q'(q, \mu, H)$ and a $\xi \in \kappa \setminus \mu$ such that $q' \le q$ and $q' \forces \xi \in \tau \setminus \bigcup_{\beta \in H} \Gamma_\beta$. Let

$$C_{\alpha+1} = C_\alpha \cup \bigcup \{F_{q'(q, \mu, H)} : q, \mu, H \text{ as above}\}.$$ 

Let $C = \bigcup_{\alpha < \kappa} C_\alpha$. Clearly $C$ is as desired in (6).

Now take any $\alpha \in \lambda \setminus C$ and $\mu < \kappa$. We finish the proof by showing

(7) $\{q : q \forces \exists \xi \in \kappa \setminus \mu \langle \xi \in \tau \setminus \Gamma_\alpha \rangle\}$ is dense below $p$.

To show this, let $r \le p$ be arbitrary. By (2), we may assume that $\alpha \in F_r$. Let $s = \tau \setminus (C \times \nu_r)$. By (6), choose $q' \le s$ and $\xi > \max(\mu, \nu_r)$ such that $F_{q'} \subseteq C$ and $q' \forces \xi \in \tau \setminus \bigcup_{\beta \in F_s} \Gamma_\beta$. Now let $F_q = F_{q'} \cup F_r$, $\nu_q = \max(\nu_{q'}, \xi + 1)$, and for any $\beta \in F_q$ and $i < \nu_q$ let

$$q(\beta, i) = \begin{cases} q'(\beta, i) & \text{if } \beta \in F_{q'} \text{ and } i < \nu_{q'}, \\ r(\beta, i) & \text{if } \beta \in F_r \setminus F_{q'} \text{ and } i < \nu_r, \\ 1 & \text{if } \beta = \alpha \text{ and } i = \xi, \\ 0 & \text{in all other cases.} \end{cases}$$

Clearly $q \in \mathbb{P}$. Since $q(\alpha, \xi) = 1$, we have $q \forces \xi \in \Gamma_\alpha$.

(8) $q \le q'$.

In fact, clearly $q' \subseteq q$. Now suppose that $\beta$ and $\gamma$ are distinct members of $F_{q'}$ and $i \in \nu_q \setminus \nu_{q'}$. Then by definition we have $q(\beta, i) = 0$ or $q(\gamma, i) = 0$, as desired; so (8) holds.

So it remains only to prove...
(9) \( q \leq r \).

For this, first note that \( F_r = (F_r \cap C) \cup (F_r \setminus C) \subseteq F_q \). And \( \nu_r \leq \nu_q \leq \nu_q \). Now suppose that \( \beta \in F_r \) and \( \iota < \nu_r \). If \( \beta \in C \), then \( r(\beta, \iota) = s(\beta, \iota) = q'(\beta, \iota) = q(\beta, \iota) \). If \( \beta \notin C \), then directly from the definition, \( q(\beta, \iota) = r(\beta, \iota) \). All of this shows that \( r \subseteq q \).

Now suppose that \( \beta \) and \( \gamma \) are distinct members of \( F_r \) and \( \iota \in \nu_q \setminus \nu_r \). To finish the proof we want to show that \( q(\beta, \iota) = 0 \) or \( q(\gamma, \iota) = 0 \).

Case 1. \( \beta, \gamma \in C \) and \( \iota < \nu_q \). Then \( \beta, \gamma \in C \cap F_r = F_s \subseteq F_q' \), so \( q(\beta, \iota) = q'(\beta, \iota) \) and \( q(\gamma, \iota) = q'(\gamma, \iota) \). Also, \( \iota \in \nu_q \setminus \nu_s \) since \( \nu_s = \nu_r \). So \( q'(\beta, \iota) = 0 \) or \( q'(\gamma, \iota) = 0 \).

Case 2. \( \beta \in C, \iota \geq \nu_q' \). So \( q(\beta, \iota) = 0 \).

Case 3. \( \gamma \in C, \iota \geq \nu_q' \). So \( q(\gamma, \iota) = 0 \).

Case 4. \( \beta \notin C, \nu_r \leq \iota, \beta \neq \alpha \) or \( \iota \neq \xi \). Then \( q(\beta, \iota) = 0 \).

Case 5. \( \gamma \notin C, \nu_r \leq \iota, \gamma \neq \alpha \) or \( \iota \neq \xi \). Then \( q(\gamma, \iota) = 0 \).

Case 6. \( \beta \in C, \iota = \xi, \nu_r \leq \iota < \nu_q \). Then \( q(\beta, \iota) = q'(\beta, \xi) = 0 \) since \( q' \models \xi \notin \Gamma_\beta \).

Case 7. \( \gamma \in C, \iota = \xi, \nu_r \leq \iota < \nu_q \). Then \( q(\gamma, \iota) = q'(\gamma, \xi) = 0 \) since \( q' \models \xi \notin \Gamma_\gamma \).

Case 8. None of the above. So not both of \( \beta, \gamma \) are in \( C \), by Cases 1,2. Suppose one is in \( C \), the other not; say \( \beta \in C, \gamma \notin C \). Since \( \iota \geq \nu_r \), it follows that \( \gamma = \alpha \) and \( \iota = \xi \). Then \( q(\beta, \iota) = 0 \), either because \( \xi < \nu_q \) and \( q' \models \xi \notin \Gamma_\beta \), or because \( \xi \geq \nu_q \) and the definition of \( q \). So, suppose that \( \beta, \gamma \notin C \). Then one of Cases 4,5 must hold, contradiction. \( \square \)

**Theorem 12.** Let \( \langle A_\alpha : \alpha < \kappa \rangle \) be a system of infinite almost disjoint subsets of \( \omega \) such that \( \kappa > \omega \) and

\((\ast)\) For every infinite subset \( X \) of \( \omega \), if \( \{ \alpha < \kappa : X \cap A_\alpha \} \) is infinite, then it is cocountable.

Let \( A \) be the subalgebra of \( \mathcal{P}\omega \) generated by

\[ \{ A_\alpha : \alpha < \kappa \} \cup \{ \{ i \} : i < \omega \} . \]

Then \( \text{c}_{HR} A = \{ (\omega, \kappa) \} \cup \{ (\mu, \mu) : \omega \leq \mu \leq \kappa \} \).
Proof. $A/\text{fin} \cong \text{Fin}_{\kappa}$, so $\supset$ holds. Now suppose that $(\mu, \nu) \in c_{\text{HR}} A$, $\omega \leq \mu < \nu \leq \kappa$, and $(\mu, \nu) \neq (\omega, \kappa)$; we want to get a contradiction. Let $I$ be an ideal of $A$ such that $|A/I| = \nu$ and $c(A/I) = \mu$. Let $b = \{i < \omega : \{i\} \in I\}$.

(1) $\Gamma \overset{\text{def}}{=} \{\alpha < \kappa : A_\alpha \setminus b \text{ is infinite}\}$ is infinite.

For, suppose that $\Gamma$ is finite. Let $\rho$ be regular, with $\mu < \rho \leq \nu$; we are going to show that $A/I$ has a disjoint family of size $\rho$, contradiction. Now there is a $\Delta \in [\kappa]^\rho$ such that for all $\alpha \in \Delta$, $A_\alpha/I \neq 0$ and $A_\alpha \setminus b$ is finite, and for all distinct $\alpha, \beta \in \Delta$, $A_\alpha/I \neq A_\beta/I$. Let $\Omega \in [\Delta]^\rho$ be such that $\langle A_\alpha \setminus b : \alpha \in \Omega \rangle$ is a $\Delta$-system, say with kernel $K$. Now if $\langle A_\alpha \setminus K \rangle/I = 0$, then $A_\alpha/I \leq K/I$, and $\langle A/I \rangle \uparrow (K/I)$ is finite. So wlog, $\langle A_\alpha \setminus K \rangle/I \neq 0$ for all $\alpha \in \Omega$. Now if $\alpha, \beta$ are distinct members of $\Omega$, then $\langle (A_\alpha \cap A_\beta) \setminus b \rangle/K = 0$, so $\langle (A_\alpha \cap A_\beta) \rangle/K = \langle (A_\alpha \cap A_\beta) \setminus b \rangle/K$. But $A_\alpha \cap A_\beta \setminus b \in I$ since $A_\alpha \cap A_\beta$ is finite, so $\langle (A_\alpha \cap A_\beta) \rangle/K \subseteq I$. Thus $\langle (A_\alpha \setminus K) \rangle/I : \alpha \in \Omega \rangle$ is a system of $\rho$ disjoint elements, contradiction. This proves (1).

So from (*) it follows that $\Gamma$ is countable. Now the map $\alpha \mapsto A_\alpha \setminus b$ for $\alpha \in \Gamma$ is one-one. For any $x \in A$ let $g(x) = (x/I, x \setminus b)$. This is a homomorphism. If $x \in I$, then $x \setminus b = 0$, and so $g(x) = (0, 0)$. And if $g(x) = (0, 0)$, then $x \in I$. So the image of $g$ is isomorphic to $A/I$. It follows that $|A/I| = \kappa$. Hence $\omega < \mu$. Let $\langle c_\alpha/I : \alpha < \omega_1 \rangle$ be a system of nonzero pair­wise disjoint elements. Since there are only countably many finite subsets of $\omega$, wlog each $c_\alpha$ is infinite. In fact, we may assume that each $c_\alpha$ has the form

$$A_\beta \cdot -A_{\gamma_1} \cdot \ldots \cdot -A_{\gamma_m} \cdot -F,$$

where $F$ is finite and each $\gamma_i \neq \beta$. This can be written as

$$A_\beta \cdot -(A_\beta \cdot A_{\gamma_1}) \cdot \ldots \cdot -(A_\beta \cdot A_{\gamma_m}) \cdot -F,$$

and each $A_\beta \cdot A_{\gamma_i}$ is finite. So wlog $m = 0$. Thus we may assume that we have a pairwise disjoint system $\langle (A_\alpha \setminus F_\alpha) \rangle/I : \alpha \in \Delta \rangle$ of nonzero elements, each $F_\alpha$ finite, $\Delta \in [\kappa]^{\omega_1}$. 
Now we have $A \setminus b$ infinite for all $\alpha$ in a countable subset $\Delta'$ of $\Delta$. So $(A \setminus F_\alpha) \setminus b$ is infinite for each $\alpha \in \Delta'$. Now for $\alpha \neq \beta$ the set $A_\alpha \cdot F_\alpha \cdot A_\beta \cdot F_\beta$ is in $I$ and hence is a subset of $b$. So $(A \setminus F_\alpha) \setminus b : \alpha \in \Delta')$ is a system of $\omega_1$ pairwise disjoint subsets of $\omega$, contradiction. \hfill \Box

**Theorem 13.** Suppose that $(\kappa^+, \kappa^{++}) \in c_{\text{Hr}}A$ and $(\kappa, \kappa^+) \notin c_{\text{Hr}}A$. Then $(\kappa^+, \kappa^+) \in c_{\text{Hr}}A$.

**Proof.** We work in the Stone space $X$ of $A$. We may assume that $X$ has cellularity $\kappa^+$ and weight $\kappa^{++}$. Take points one apiece from a pairwise disjoint family of $\kappa^+$ open sets. If their closure has exactly $\kappa^+$ clopen sets, we are done, otherwise the closure has $\kappa^{++}$ clopen sets, and we may assume without loss of generality that the closure is all of $X$. Thus $X$ has isolated points $\{x_\alpha : \alpha < \kappa^+\}$, listed without repetitions, and they are dense in $X$. For all $\alpha \in [\kappa, \kappa^+]$ let $X_\alpha = \text{cl}\{x_\beta : \beta < \alpha\}$. Thus $X_\alpha$ is a Boolean space with $\kappa$ isolated points, which are dense in $X_\alpha$. So by the hypothesis of the theorem, $|\text{clop}X_\alpha| \leq \kappa^+$.

**Case 1.** $Y := \bigcup_{\alpha \in [\kappa, \kappa^+)} X_\alpha$ is closed. Then $\bigcup_{\alpha \in [\kappa, \kappa^+)} \text{clop}X_\alpha$ is a network for $Y$. Hence $Y$ has weight $\kappa^+$. Since $\{x_\alpha : \alpha < \kappa^+\}$ is its set of isolated points, and this set is dense in $Y$, the conclusion of the theorem holds.

**Case 2.** $Y$ is not closed. Let $g \in \text{cl}Y \setminus Y$. Then $g \notin \text{cl}Z$ for all $Z \in [Y]^s$, so the tightness of $Y$ is at least $\kappa^+$. Let $\langle y_\alpha : \alpha < \kappa^+ \rangle$ be a convergent free sequence (by Juhasz, Szentmiklossy [3]). Say it converges to $z$. Let $Z = \text{cl}\{y_\alpha : \alpha < \kappa^+\}$. Note that each $y_\alpha$ is isolated in $Z$, and the $y_\alpha$'s are dense in $Z$. So it suffices to show that $Z$ has weight $\kappa^+$. Let $W_\alpha = \text{cl}\{y_\beta : \beta < \alpha\}$ for all $\alpha \in [\kappa, \kappa^+)$. Thus $W_\alpha$ is clopen in $Z$ by freeness. Clearly $\bigcap_{\alpha \in [\kappa, \kappa^+)} (Z \setminus W_\alpha) = \{z\}$. So $\{Z \setminus W_\alpha : \alpha \in [\kappa, \kappa^+)\}$ is a neighborhood basis for $z$. Now by hypothesis, each $W_\alpha$ has weight at most $\kappa^+$; let $B_\alpha$ be a base for $W_\alpha$ with $|B_\alpha| \leq \kappa^+$. Then

$$\bigcup_{\alpha \in [\kappa, \kappa^+)} B_\alpha \cup \{Z \setminus W_\alpha : \alpha < \kappa^+\}$$
is a network for $Z$, so $Z$ has weight $\kappa^+$, as desired.

This proof generalizes to give the following result:

If $\kappa^+ < \nu$, $\text{cof}\nu \neq \kappa^+$, $(\kappa^+, \nu) \in c_{Hr}A$, and $(\kappa, \nu) \notin c_{Hr}A$, then $(\kappa^+, \mu) \in c_{Hr}A$ for some $\mu < \nu$.

**Problem.** Is it necessary to assume that $\text{cof}\nu \neq \kappa^+$ in the foregoing result? Finally, a result on $c_{Sr}$:

**Theorem 14.** For every infinite cardinal $\kappa$, and every $BA A$, if $cA \geq \kappa^{++}$ and $(\kappa, \kappa^{++}) \in c_{Sr}A$, then $(\kappa^+, \kappa^{++}) \in c_{Sr}A$.

**Proof.** Suppose not. Let $B$ be a subalgebra of size $\kappa^{++}$ with cellularity $\kappa$.

1. There is an $a \in A$ such that $B \upharpoonright a$, which by definition is $\{b \cdot a : b \in B\}$, has cellularity $\kappa^{++}$.

To see this, let $X$ be pairwise disjoint of size $\kappa^+$. Then $\langle B \cup X \rangle$ is of size $\kappa^{++}$ and has cellularity greater than $\kappa$, so its cellularity is $\kappa^{++}$; let $Y$ be a pairwise disjoint subset of size $\kappa^{++}$. We may assume that each element $y \in Y$ has the form $y = b_y \cdot a_y$ with $b_y \in B$ and $a_y \in \langle X \rangle$. Since $|X| < \kappa^{++}$, we may in fact suppose that each $a_y$ is equal to some element $a$, as desired in (1).

Choose such an $a$, and let $X \in [B]^{\kappa^{++}}$ be such that $\langle x \cdot a : x \in X \rangle$ is a system of nonzero pairwise disjoint elements. Let $Y$ be a subset of $X$ of size $\kappa^+$, and let

$$C = \langle \{x \cdot a : x \in Y\} \cup \{x \cdot -a : x \in X \setminus Y\}\rangle.$$ 

Now define $x \equiv y$ iff $x, y \in X \setminus Y$ and $x \cdot -a = y \cdot -a$. Then

2. Every $\equiv$-class has size at most $\kappa$.

For, suppose that $|x/ \equiv| > \kappa$. For any $y \in (x/ \equiv) \setminus \{x\}$ we have

$$y \cdot -x = y \cdot -x \cdot a + y \cdot -x \cdot -a$$

$$= y \cdot a \cdot -(x \cdot a) + x \cdot -x \cdot -a$$

$$= y \cdot a.$$
This means that $B$ has a pairwise disjoint subset of size greater than $\kappa$, contradiction. So (2) holds.

From (2) it follows that $|C| = \kappa^{++}$. Thus we must have $cC = \kappa^{++}$. Hence by the argument for (1), there is a $d \in \langle \{x \cdot a : x \in Y\}\rangle$ and a

$$Z \in \langle\langle\{x \cdot -a : x \in X \setminus Y\}\rangle\rangle^{++}$$

such that $\langle z \cdot d : z \in Z \rangle$ is a system of nonzero pairwise disjoint elements. We may assume that each $z \in Z$ has the form

$$x_{z,0} \cdot -a \cdot \ldots \cdot x_{z,m-1} \cdot -a \cdot (-y_{z,0} + a) \cdot \ldots \cdot (-y_{z,n-1} + a),$$

where each $x_{z,i}$ and $y_{z,j}$ is in $X \setminus Y$, and $m$ and $n$ do not depend on $z$.

Now since $\langle\langle x \cdot a : x \in Y\rangle\rangle$ is isomorphic to $\text{Fin}(\kappa^+)$, there are two cases.

Case 1. $d = \sum_{x \in F} x \cdot a$ for some finite $F \subseteq Y$. Then we may assume that in fact $d = x \cdot a$ for some $x \in Y$. In this case we have $m = 0$, and then each $z \cdot d$ is just equal to $d$, contradiction.

Case 2. $d = -\sum_{x \in F} (x \cdot a)$ for some finite $F \subseteq Y$. Thus $d = -a + a \cdot -\sum_{x \in F} x$. If $m = 0$, then each $z \cdot d$ is $\geq a \cdot -\sum_{x \in F} x$, so these elements are not disjoint, contradiction. Thus $m > 0$.

Hence $z \cdot d = z$ for each $z \in Z$. For each $z \in Z$ write $e_z = x_{z,0} \cdot \ldots \cdot x_{z,m-1}$ and $c_z = e_z \cdot -y_{z,0} \cdot \ldots \cdot -y_{z,n-1}$. Define $z \equiv w$ iff $z, w \in Z$ and $e_z = e_w$. If $z \neq w$, then

$$c_z \cdot c_w = c_z \cdot c_w \cdot a + c_z \cdot c_w \cdot -a = z \cdot w = 0.$$

Since $c_z \in B$ for each $z \in Z$, it follows that there are at most $\kappa$ \not\cong-class members. Thus we may assume that all of the $e_z$'s are the same. Thus for any $z \in Z$ we have

$$z = x_0 \cdot \ldots \cdot x_{m-1} \cdot -y_{z,0} \cdot \ldots \cdot -y_{z,n-1} \cdot -a,$$

$$c_z = x_0 \cdot \ldots \cdot x_{m-1} \cdot -y_{z,0} \cdot \ldots \cdot -y_{z,n-1}.$$
Note that $c_z \cdot a = x_0 \cdot \ldots \cdot x_{m-1} \cdot a$. So if $z \neq w$, then
\[
\begin{align*}
  c_z \cdot -c_w &= c_z \cdot -c_w \cdot a + c_z \cdot -c_w \cdot -a \\
  &= c_z \cdot a \cdot -(c_w \cdot a) + c_z \cdot -a \cdot -(c_w \cdot -a) \\
  &= z \cdot -w = z.
\end{align*}
\]
So if we fix $w \in Z$, then $\langle c_z \cdot -c_w : z \in Z \setminus \{w\} \rangle$ is a system of $\kappa^{++}$ nonzero pairwise disjoint elements of $B$, a contradiction. \hfill \Box

References


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