Math. Log. Quart. 42 (1996) 537 – 550

Mathematical Logic
Quarterly

Minimum-sized Infinite Partitions of Boolean Algebras

J. Donald Monk

Department of Mathematics, University of Colorado,
Boulder, Colorado 80309-0395, U.S.A.[1)]

Abstract. For any Boolean algebra A, let $c_{mm}(A)$ be the smallest size of an infinite partition of unity in A. The relationship of this function to the 21 common functions described in MONK [4] is described, for the class of all Boolean algebras, and also for its most important subclasses. This description involves three main results: the existence of a rigid tree algebra in which c_{mm} exceeds any preassigned number, a rigid interval algebra with that property, and the construction of an interval algebra in which every well-ordered chain has size less than c_{mm}.

Mathematics Subject Classification: 06E05, 03E05.

Keywords: Partition of a Boolean Algebra, Interval algebra, Tree algebra, Rigid Boolean algebra, Mono-rigid Boolean algebra, Depth of a Boolean Algebra.

0 Introduction

A *partition* of a Boolean algebra (BA) is a partition of 1 in it, i.e., a system of nonzero pairwise disjoint elements with sum 1. With each BA A we associate the cardinal number

$$c_{mm}(A) = \min\{|X| : X \text{ is an infinite partition of } A\}.$$

The purpose of this paper is to study this function in the style of work on cardinal functions on topological spaces and Boolean algebras in general. The function has been extensively studied for the algebra $\mathcal{P}\omega/\text{fin}$; see, for example, VAN DOUWEN [2]. The function is also briefly discussed for Boolean algebras in general in MONK [4]; there the main concern was what happens to the function under algebraic operations. This material is not needed here, and will not be repeated. We are interested here in the relationship of this function to other ones, and in particular also for special classes of Boolean algebras. We will assume an acquaintance with Boolean algebras, and the notation of the Handbook of Boolean Algebras [3] will be used. The notation of MONK [4] will also be used, but it will be recalled here at the appropriate place; some theorems in that book will be used also. Three main theorems are proved: (1) There is a rigid tree algebra with c_{mm} arbitrarily large (generalizing a construction of BRENNER); (2) There exists an interval algebra A such that $\text{Depth}(A) < c_{mm}(A)$;

[1)] e-mail: monkd@euclid.colorado.edu

(3) There is a mono-rigid interval algebra with c_{mm} arbitrarily large (adapting a construction of TODORČEVIĆ). Here Depth(A), the *depth* of A, is the supremum of cardinalities of subsets of A well-ordered by the Boolean ordering. An algebra is *mono-rigid* if it has no one–one endomorphism except the identity.

1 The general case

Of course we always have $c_{mm}(A) \le c(A)$, the latter being the supremum of cardinalities of disjoint subsets of A (the *cellularity* of A). The difference can be arbitrarily large, as is seen by the interval algebra on a cardinal. The following comments take care of the relationships of c_{mm} with the main 21 functions in MONK [4].

The inequality $\chi(A) < c_{mm}(A)$ is possible, where $\chi(A)$ is the *character* of A; the *character* of an ultrafilter F of A is the smallest cardinality of a filter-generating subset of F, and the *character* of A itself is the supremum of the characters of its ultrafilters. An example with the indicated inequality is clear from the text following [4, 3.26], and the discussion of the Aleksandroff duplicate in [4, Chapter 14]. Let Aut(A) be the collection of all automorphisms of A. Then $c_{mm}(A) < |\mathrm{Aut}(A)|$ in $A = B \times C$, where A is a countable BA and B is an uncountable finite-cofinite algebra.

A rigid BA shows that $|\mathrm{Aut}(A)| < c_{mm}(A)$ is possible. Together with the above results, this shows that for BA's in general the only relationship between c_{mm} and the main functions of MONK [4] is $c_{mm} \le c$ and inequalities implied by this.

It is natural, though to ask whether one can have a rigid BA with c_{mm} arbitrarily large. A modification of BRENNERS's rigid tree algebra (described in [3]) works for this purpose, as we shall see.

Let T be a tree. If C is an initial chain of T, an *immediate successor* of C is an element $t \in T$ such that $s < t$ for all $s \in C$, while if $u < t$, then $u \le s$ for some $s \in C$. We let SC be the set of all immediate successors of C. In case $C = \{t\}$, we write St instead of SC. We use ht$T$ for the height of a tree T. If T is a tree and $t \in T$ we let $T \uparrow t = \{s \in t : t \le s\}$.

Lemma 1. *Suppose that T is a tree with a single root, λ is an infinite cardinal, and the following conditions hold:*

(i) $|\mathrm{S}\,t| \ge \lambda$ *for all* $t \in T$;

(ii) $|\mathrm{S}\,C| \ge \lambda$ *for every bounded nonempty initial chain* C *of* T;

(iii) $\mathrm{cf}(\mathrm{ht}\,T) \ge \omega_1$.

Then Treealg T *has no infinite partition of size* $< \lambda$.

Proof. Suppose that P is an infinite partition of size $< \lambda$. We may assume that each element $p \in P$ has the form

$$p = (T \uparrow t_p) \setminus \textstyle\bigcup_{s \in \Gamma_p} (T \uparrow s),$$

where Γ_p is a finite subset of $(T \uparrow t_p) \setminus \{t_p\}$. Note that $t_p \ne t_q$ for $p \ne q$.

(1) If $p \in P$ and $s \in \Gamma_p$, then there is a $q \in P$ such that $t_q = s$.

Suppose that (1) fails. Then

(1a) for every $q \in P$ such that $(T \uparrow s) \cap q \ne \emptyset$ there is a unique $t'_q \in \mathrm{S}\,s$ such that $t'_q \le t_q$.

For, say $s \leq u \in q$. Thus $t_p, t_q \leq u$, so t_p is comparable with t_q. If $t_q < t_p$, then $t_p \in p \cap q$, a contradiction. So $t_p < t_q$ and (1a) holds. Since $|P| < \lambda$, by (i) choose $v \in \mathrm{S}\, s$ such that $v \neq t'_q$ for all $q \in P$ such that $(T \uparrow s) \cap q \neq \emptyset$. But there is a $q \in P$ such that $(T \uparrow v) \cap q \neq \emptyset$, and then $(T \uparrow s) \cap q \neq \emptyset$ and $t'_q = v$, a contradiction. So (1) holds.

Let r be the root of T.

(2) There is a $p \in P$ such that $t_p = r$ and $\Gamma_p \neq \emptyset$.

Suppose that $t_p \neq r$ for all $p \in P$. For each $p \in P$ let $t'_p \in \mathrm{S}\, r$ be such that $t'_p \leq t_p$. By (i) choose $v \in \mathrm{S}\, r$ such that $v \neq t'_p$ for all $p \in P$. Then $(T \uparrow v) \cap p = \emptyset$ for all $p \in P$, a contradiction. It follows that $t_p = r$ for some $p \in P$. Clearly $\Gamma_p \neq \emptyset$ for such a p.

Let $\Delta = \{p \in P : \Gamma_p \neq \emptyset\}$.

(3) Δ is infinite.

For, suppose that Δ is finite. If $p \in \Delta$ and $s \in \Gamma_p$, choose $q_{ps} \in P$ such that $t_{q_{ps}} = s$, by (1). Then

(4) $T = \bigcup_{p \in \Delta} p \cup \bigcup_{p \in \Delta,\, s \in \Gamma_p} q_{ps}$,

which contradicts P being infinite. To prove (4), suppose that $t \in T$ and $t \notin \bigcup_{p \in \Delta} p$. By (2) choose $u \in P$ so that $t_u = r$. Then $u \in \Delta$, so $t_u < t$. Choose $p \in \Delta$ such that $t_p < t$ and t_p is maximum among all $q \in \Delta$ such that $t_q < t$. Since $t \notin p$, we have $t \in (T \uparrow s)$ for some $s \in \Gamma_p$. Now $t_{q_{ps}} = s$, so either $t = s \in q_{ps}$ as desired, or $s < t$. In the latter case we have $t_p < s$ and hence $q_{ps} \notin \Delta$ by the choice of p. So $q_{ps} = (T \uparrow s)$ and $t \in q_{ps}$, as desired. Thus (3) holds.

(5) Suppose that $p \in P$ and $\{q \in P : t_p \leq t_q \text{ and } \Gamma_q \neq \emptyset\}$ is infinite. Then there is a $q \in P$ such that $t_q \in \Gamma_p$ and $\{x \in P : t_q \leq t_x \text{ and } \Gamma_x \neq \emptyset\}$ is infinite.

For, suppose that $q \in P$, $t_p < t_q$, and $\Gamma_q \neq \emptyset$. Since $p \cap q = \emptyset$, say $s \in \Gamma_p$ and $s \leq t_q$. This proves:

$$\{q \in P : t_p \leq t_q \text{ and } \Gamma_q \neq \emptyset\} \subseteq \{p\} \cup \bigcup_{s \in \Gamma_p} \{x \in P : s \leq t_x \text{ and } \Gamma_x \neq \emptyset\},$$

and (5) follows, using (1).

By (5) and (2) there is a sequence $\langle p_0, p_1, \ldots \rangle$ of members of P such that $t_{p_0} = r$ and $t_{p_{i+1}} \in \Gamma_{p_i}$ for all $i < \omega$. Let $C = \{u : u \leq p_i \text{ for some } i < \omega\}$. Thus C is an initial chain without last element, of cofinality ω, and hence it is bounded by (iii). Suppose that $s \in \mathrm{S}\, C$. Choose $q_s \in p$ such that $(T \uparrow s) \cap q_s \neq \emptyset$. Say $s \leq u \in q_s$. Thus $s, t_{q_s} \leq u$. If $t_{q_s} < s$, then $t_{q_s} < t_{p_i}$ for some $i < \omega$. Take the smallest such i. Then $i > 0$ and $t_{p_{i-1}} \leq t_{q_s} < t_{p_i}$. So $t_{q_s} \in p_{i-1} \cap q_s$, and so $p_{i-1} = q_s$. Now $u \in p_{i-1}$, $t_{p_i} \leq s \leq u$, and $t_{p_i} \in \Gamma_{p_{i-1}}$, a contradiction.

It follows that $s \leq t_{q_s}$ for any $s \in \mathrm{S}\, C$. Hence $s \mapsto q_s$ defines a one–one function from $\mathrm{S}\, C$ into P, so $|P| \geq \lambda$ by (iii), a contradiction. □

Theorem 2. *For every infinite cardinal κ there is a rigid tree algebra A such that* $\mathrm{c_{mm}}(A) \geq \kappa$.

Proof. We may assume that κ is regular and uncountable. There clearly is a tree T having the following properties:

(1) T has a single root.

(2) T has height ω_1.

(3) $|\mathrm{S}\,t| \geq \kappa$ for all $t \in T$.

(4) If s and t are distinct members of T, then $|\mathrm{S}\,s| \neq |\mathrm{S}\,t|$.

(5) If C is an initial chain of T of length less than ω_1, then $|\mathrm{S}\,C| > \sup\{|\mathrm{S}\,t| : t \in C\}$.

(6) If $s < t$, then $|\mathrm{S}\,s| < |\mathrm{S}\,t|$.

Let $A = \mathrm{Treealg}\,T$. Now we show

(7) A is rigid.

For, suppose not. Let g be a nonidentity automorphism of A. Then there is a nonzero $a \in A$ such that $a \cap ga = \emptyset$. We may assume that $a = (T \uparrow s)$ for some $s \in T$. Write

$$g(T \uparrow s) = \bigcup_{j<m}[(T \uparrow t_j) \setminus \bigcup_{u \in \Gamma_j}(T \uparrow u)],$$

where Γ_j is a finite subset of $(T \uparrow t_j) \setminus \{t_j\}$ for each $j < m$. Clearly

(8) For each $j < m$, there is a partition of $(T \uparrow t_j) \setminus \bigcup_{u \in \Gamma_j}(T \uparrow u)$ which has $|\mathrm{S}\,t_j|$ elements.

Now we consider two cases.

Case 1. There is a $j < m$ such that $|\mathrm{S}\,t_j| < |\mathrm{S}\,s|$. Then by (8), and because g is an isomorphism, $T \uparrow s$ has a partition with $|\mathrm{S}\,t_j|$ elements, and this contradicts the lemma.

Case 2. $|\mathrm{S}\,s| < |\mathrm{S}\,t_j|$ for all $j < m$. Now $T \uparrow s$ has a partition P of size $|\mathrm{S}\,s|$. For each $p \in P$ there is a $j < m$ such that $g(p) \cap (T \uparrow t_j) \setminus \bigcup_{u \in \Gamma_j}(T \uparrow u) \neq \emptyset$. Hence there is a $j < m$ such that $\{g(p) \cap (T \uparrow t_j) \setminus \bigcup_{u \in \Gamma_j}(T \uparrow u) : p \in P\}$ has $|P|$ nonzero elements, which form a partition of $(T \uparrow t_j) \setminus \bigcup_{u \in \Gamma_j}(T \uparrow u)$. Again, this contradicts the lemma.

Thus (7) holds.

By the lemma, $\mathrm{c_{mm}}(A) \geq \kappa$. □

2 Interval algebras

Suppose that I is a linear ordering with first element 0. A *simple partition* of $\mathrm{Intalg}(I)$ is a partition in which every element has the form $[a, b)$. Clearly

$$\mathrm{c_{mm}}(A) = \min\{|P| : P \text{ is a simple partition of } \mathrm{Intalg}(I)\}.$$

Let P be a simple partition of $\mathrm{Intalg}(I)$. We set

$$L_P = \{a \in I : [a, b) \in P \text{ for some } b \in I \cup \{\infty\}\}.$$

For each $c \in L_P$ let $c^+ \in I \cup \{\infty\}$ be such that $[c, c^+) \in P$. Thus

(*) For any $a, b \in I$, if $a < b$, then there is a $c \in L_P$ such that $c < b$ and $a < c^+$.

This will be used frequently in what follows.

Note also that if κ is an infinite cardinal and I is a κ-saturated linear order (in the model-theoretic sense), then $\mathrm{c_{mm}}(\mathrm{Intalg}(I)) \geq \kappa$. Also recall that I is κ-saturated iff for all subsets $A, B \in [I]^{<\kappa}$ with $A < B$ there is a $c \in I$ such that $A < c < B$. Here, of course, $A < B$ means that $a < b$ for all $a \in A$ and $b \in B$; similarly for $A < c < B$.

If F is an ultrafilter on a BA A, the *π-character* of F is the smallest size of a subset D of $A \setminus \{\emptyset\}$ such that for all $x \in F$ there is an $a \in D$ such that $a \leq x$; it is denoted by $\pi\chi(F)$. Then

$$\pi\chi(A) = \sup\{\pi\chi(F) : F \text{ is an ultrafilter on } A\}.$$

A description of the π-character of interval algebras can be found in [4, p. 161]. From this it follows that for $A = \mathrm{Intalg}(\kappa)$, κ an infinite cardinal, we have $\pi\chi(A) = \kappa$. But $\mathfrak{c}_{\mathrm{mm}}(A) = \omega$, as is seen by the simple partition $\{\{0\}, \{1\}, \ldots, [\omega, \infty)\}$. This inequality is important in seeing the place of $\mathfrak{c}_{\mathrm{mm}}$ among the 21 functions of MONK [4].

Another interesting cardinal function on Boolean algebras is defined as follows:

$$\chi_{\mathrm{inf}}(A) = \inf\{\chi(F) : F \text{ is an ultrafilter on } A\}.$$

Proposition 3. For every interval algebra A, $\chi_{\mathrm{inf}}(A) \leq \mathfrak{c}_{\mathrm{mm}}(A)$.

Proof. Suppose that $A = \mathrm{Intalg}(I)$, and P is a simple partition of A. For each terminal segment M of I with $0 \notin M$, let $\chi_{\mathrm{L}}M$ be the left character of M, i.e., the cofinality of $I \setminus M$, and let $\chi_{\mathrm{R}}M$ be the right character of M, i.e., the coinitiality of M if $M \neq \emptyset$. The terminal segments of I not having 0 as a member are in one–one correspondence with the ultrafilters on A, by [3]. By [4, p. 188], the character of the ultrafilter with associated terminal segment M is the maximum of κ and λ, where $\kappa = \chi_{\mathrm{L}}M$ and $\lambda^* = \chi_{\mathrm{R}}M$.

Now we define $c_0, c_1, \ldots \in L_P$. Fix any $c_0 \in L_P$. Then we define

$$c_{m+1} = \begin{cases} c_m^+ & \text{if } c_m^+ \in L_P, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Case 1. c_m is defined for all $m \in \omega$.

Subcase 1.1. The c_m's are cofinal in I. Then the empty terminal segment has character $(\omega^*, 1)$ and the desired conclusion is clear.

Subcase 1.2. The c_m's are not cofinal in I. Let

$$M = \{x \in I : c_m < x \text{ for all } m \in \omega\}.$$

Then the terminal segment M has left character ω. Assume that the right character of M is κ^*, κ infinite. We define a sequence $\langle d_\alpha : \alpha < \kappa\rangle$ of distinct elements of L_P; this will prove the Proposition in this subcase. Let $\langle e_\alpha : \alpha < \kappa\rangle$ be a strictly decreasing sequence of elements of I coinitial in M. Choose $d_0 \in L_P$ so that $d_0 < e_0$ and $e_1 < d_0^+$; this is possible by $(*)$ above. Now clearly $d_0^+ \in M$. Hence also $d_0 \in M$. Suppose now that we have defined d_α for all $\alpha < \beta$, all members of M. There is an e_β with $e_\beta < d_\alpha$ for all $\alpha < \beta$. Choose $d_\beta \in L_P$ so that $d_\beta < e_\beta$ and $e_{\beta+1} < d_\beta^+$. As above, $d_\beta \in M$.

Case 2. There is an m such that c_m is defined but c_{m+1} is undefined.

Subcase 2.1. $c_m^+ < \infty$. Let $M = \{x \in I : c_m^+ < x\}$. This is a terminal segment with left character 1. One can assume that the right character is infinite and proceed as in Subcase 1.2.

Subcase 2.2. $c_m^+ = \infty$. We now define $d_0 = c_0$ and

$$d_{m+1} = \begin{cases} u & \text{if } u \in L_P \text{ and } u^+ = d_m, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Since L_P is infinite, d_m is defined for all $m \in \omega$. Let

$$M = \{x : d_m < x \text{ for some } m \in \omega\}.$$

Then M is a terminal segment with right character ω. Assuming that the left character is infinite, one can proceed as in Subcase 1.2. □

Our next result settles a natural question concerning the place of c_{mm} in a diagram of the cardinal functions for interval algebras.

Theorem 4. *There is an interval algebra A such that* $\mathrm{Depth}(A) < c_{mm}(A)$.

Proof. Let $I = \{f \in {}^{\omega_1}2 : \exists\alpha\,[f(\alpha) = 1 \wedge (\forall\beta > \alpha)(f(\beta) = 0)]\}$, lexicographically ordered. For properties of this order, see COMFORT and NEGREPONTIS [1] or ROSENSTEIN [6]. A *gap* in I is an ordered pair (A, B) such that $A \cup B = I$, $A < B$, A has no largest element, and B has no smallest element. For every gap G of character type (ω_1, ω_1^*), we introduce new elements $g_{G0} < g_{G1}$ at that gap, forming an order J. More explicitly,

$$J = I \cup \{g_{G0}, g_{G1} : G \text{ is a gap of character type } (\omega_1, \omega_1^*)\},$$

and for $x, y \in I$, gaps $G = (A, B)$, $H = (C, D)$, $G \neq H$ of this type, and $\varepsilon = 0, 1$, $\delta = 0, 1$ we define

$$\begin{array}{ll}
x < y & \text{iff} \quad x < y \text{ in I;} \\
x < g_{G\varepsilon} & \text{iff} \quad x \in A; \\
g_{G\varepsilon} < x & \text{iff} \quad x \in B; \\
g_{G0} < g_{G1}; & \\
g_{G\varepsilon} < g_{H\delta} & \text{iff} \quad A \subset C.
\end{array}$$

(1) If $x < y$ in J, and the ordered pair (x, y) is not of the form (g_{G0}, g_{G1}), then there is an $f \in I$ such that $x < f < y$.

This is clear if $x, y \in I$. Suppose that $x = g_{G\varepsilon}$, with $G = (A, B)$, and $y \in I$. Then $y \in B$. Choose $z \in B$ with $z < y$. Then $x < z < y$, as desired. Suppose that $y = g_{G\varepsilon}$ and $x \in I$, $G = (A, B)$. Then $x \in A$ and the conclusion is clear. Finally, suppose that $x = g_{G\varepsilon}$ and $y = g_{H\delta}$, with $G = (A, B)$ and $H = (C, D)$. Then $A \subset C$. Choose $z \in C \setminus A$. Then $x < z < y$, as desired. So (1) holds.

(2) There is no strictly increasing sequence of elements of I of type ω_2.

For, suppose that $\langle f^\alpha : \alpha < \omega_2\rangle$ is such a sequence. If $\xi < \eta < \omega_2$, then there is a $\chi_{\xi\eta} < \omega_1$ such that $f^\xi \restriction \chi_{\xi\eta} = f^\eta \restriction \chi_{\xi\eta}$, and $f^\xi(\chi_{\xi\eta}) = 0$, $f^\eta(\chi_{\xi\eta}) = 1$.

(2a) If $\xi < \eta < \varrho < \omega_2$, then $\chi_{\xi\varrho} \leq \chi_{\xi\eta}$.

In fact, suppose that $\chi_{\xi\eta} < \chi_{\xi\varrho}$. Then we have $f^\xi \restriction \chi_{\xi\eta} = f^\eta \restriction \chi_{\xi\eta} = f^\varrho \restriction \chi_{\xi\eta}$ and $f^\varrho(\chi_{\xi\eta}) = f^\xi(\chi_{\xi\eta}) = 0$, $f^\eta(\chi_{\xi\eta}) = 1$, so $f^\varrho < f^\eta$, a contradiction. So (2a) holds.

By (2a), for every $\xi < \omega_2$ there is an $\alpha_\xi < \omega_1$ such that for all $\beta \in [\alpha_\xi, \omega_2)$ we have $\chi_{\xi\alpha_\xi} = \chi_{\xi\beta}$. Then there is a $\theta < \omega_1$ such that $|\{\xi < \omega_2 : \chi_{\xi\alpha_\xi} = \theta\}| = \omega_2$. Pick $\xi < \omega_2$ such that $\chi_{\xi\alpha_\xi} = \theta$. Then pick $\eta < \omega_2$ such that $\xi, \alpha_\xi < \eta$ and $\chi_{\eta\alpha_\eta} = \theta$. Pick $\beta < \omega_2$ with $\eta, \alpha_\nu < \beta$ and $\chi_{\eta\beta} = \theta$. Then $f^\xi(\theta) = 0$ and $f^\eta(\theta) = 1$. But also $f^\eta(\theta) = 0$ and $f^\beta(\theta) = 1$, a contradiction. So (2) holds.

Similarly,

(3) There is no strictly decreasing sequence of elements of I of type ω_2.

(4) In J there is no chain of type ω_2 or ω_2^*.

For, suppose that $\langle x_\alpha : \alpha < \omega_2 \rangle$ is a strictly increasing sequence of elements of J. By (1), for each even α choose $y_\alpha \in I$ such that $x_\alpha < y_\alpha < x_{\alpha+2}$. These y_α's are strictly increasing, contradicting (2). Similarly, there is no sequence of type ω_2^*.

The rest of the proof is to take an infinite simple partition P and show that $|P| \leq \omega_1$ leads to a contradiction. We define $c \sim d$ iff $c, d \in L_P$ and one of the following holds:

(a) $c = d$;

(b) $c < d$, and there is a finite sequence $c = e_0, e_1, \ldots, e_m = d$ such that $e_{i+1} = e_i^+$ for all $i < m$, each $e_i \in L_P$;

(c) Like (b), with c and d interchanged.

Clearly "$\sim$" is an equivalence relation on L_P, and each equivalence class is countable.

(5) If α and β are equivalence classes such that $\alpha < \beta$, then there is an equivalence class γ such that $\alpha < \gamma < \beta$.

To prove (5) we consider four cases.

Case 1. α has a largest element, c, and β has a smallest element, d. Applying $(*)$ to $[c, d)$, we get $e \in L_P$ such that $e < d$ and $c < e^+$. Thus $c \leq e$.

Subcase 1.1. $c = e$. Then $e^+ < d$, since otherwise $e^+ = d$ and $\alpha = \beta$. Applying $(*)$ to $[e^+, d)$, we get $v \in L_P$ such that $v < d$ and $e^+ < v^+$. So $c = e < v < d$ and $\alpha < [v] < \beta$, as desired.

Subcase 1.2. $c < e$. Then $\alpha < [e] < \beta$, as desired.

Case 2. α has a largest element c, but β has no smallest element. Then we can choose a sequence $\langle d_n : n \in \omega \rangle$ of elements of β which is strictly decreasing. For each even $n \in \omega$, by (1) there is an $f_n \in I$ such that $d_{n+2} < f_n < d_n$. Then there exist $g, h \in I$ such that $c < g < h < f_n$ for each n. Applying $(*)$ to $[g, h)$, we get $e \in L_P$ such that $e < h$ and $g < e^+$. Thus $c \leq e$.

Subcase 2.1. $c = e$. Now $e^+ < f_n$ for each n. It follows that there is a $k \in I$ such that $e^+ < k < f_n$ for each n. Applying $(*)$ to $[e^+, k)$, we get $u \in L_P$ such that $u < k$ and $e^+ < u^+$. So $c < u$ and $\alpha < [u] < \beta$.

Subcase 2.2. $c < e$. Then $\alpha < [e] < \beta$.

Case 3. α has no largest element, but β has a smallest element, c. This is similar to Case 2.

Case 4. α has no largest element, and β has no smallest element. Then we can easily find $h < k$ such that $\alpha < h$ and $k < \beta$, $h, k \in I$, by the arguments above. Applying $(*)$ to $[h, k)$, we get the desired result.

This proves (5).

Now suppose that $|P| = \aleph_0$. Fix $c_0 \in L_P$. Define

$$c_{n+1} = \begin{cases} c_n^+ & \text{if } c_n^+ \in L_P, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Case 1. There is an n such that c_{n+1} is undefined; we take the first such n. So $c_n^+ \notin L_P$.

Subcase 1.1. $c_n^+ < \infty$.

Subcase 1.1.1. $c_n^+ = g_{G0}$ for some gap G. Applying $(*)$ to $[g_{G0}, g_{G1})$, we get $e \in L_P$ such that $e < g_{G1}$ and $g_{G0} < e^+$. So $e = c_n^+$, a contradiction.

Subcase 1.1.2. Otherwise, there is a sequence $\langle f_\alpha : \alpha < \omega_1 \rangle$ strictly decreasing with limit c_n^+. We now define $d_\alpha \in L_P$. Suppose that d_β has been defined for all $\beta < \alpha$ such that $c_n^+ < d_\beta$. Choose $f_\gamma < d_\beta$ for each $\beta < \alpha$. Applying $(*)$ to $[f_{\gamma+1}, f_\gamma)$, we get $d_\alpha \in L_P$ such that $d_\alpha < f_\gamma$ and $f_{\gamma+1} < d_\alpha^+$. So $c_n^+ < d_\alpha$. This construction gives ω_1 elements of L_P, a contradiction.

Subcase 1.2. $c_n^+ = \infty$. We define $d_0 = c_0$ and

$$d_{n+1} = \begin{cases} \text{the } a \in L_P \text{ such that } a^+ = d_n & \text{if there is such } a, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Subcase 1.2.1. There is an n such that d_{n+1} is undefined; we take the least such n. Since L_P is infinite, $d_n \neq 0$.

Subcase 1.2.1.1. $d_n = g_{G1}$ for some gap G. Applying $(*)$ to $[g_{G0}, g_{G1})$, we get $e \in L_P$ such that $e < g_{G1}$ and $g_{G0} < e^+$. So $e^+ = d_n$, a contradiction.

Subcase 1.2.1.2. Otherwise there is a sequence $\langle f_\alpha : \alpha < \omega_1 \rangle$ of members of I strictly increasing with limit d_n; we can then proceed as in 1.1.2 to get a contradiction.

Subcase 1.2.2. d_n is defined for all n. For each even n choose $f_n \in I$ so that $d_{n+2} < f_n < d_n$. Then choose $g, h \in I$ so that $g < h < f_n$ for all n. Applying $(*)$ to $[g, h)$, we get $e \in L_P$ such that $e < h$ and $g < e^+$. So $[e] < [d_0]$. Now (5) shows that there are infinitely many equivalence classes, densely ordered, and therefore isomorphic to the rationals. Take a gap (A, B) in the ordered set of equivalence classes of character (ω, ω^*). By the above ideas, we can then find $g, h \in I$ such that $g < h$ and $A < g < h < B$. Choose $r \in L_P$ so that $r < h$ and $g < r^+$. Then $A < [r] < B$, a contradiction.

Case 2. c_n is defined for all n. This case can be treated like 1.2.2.

Thus $|L_P| = \aleph_1$. Then we get a gap (A, B) in the ordered set of equivalence classes of character (ω_1, ω_1^*), since there is always a symmetric gap by an argument of Hausdorff, and (ω, ω^*) is ruled out by the above argument. But this clearly gives rise to an (ω_1, ω_1^*) gap $G = (C, D)$ in I, so that for every $\alpha \in A$ there is an $f \in C$ such that $\alpha < f$, and for every $\beta \in B$ there is a $g \in D$ such that $g < \beta$. Thus $A < g_{G0} < g_{G1} < B$. Choose $e \in L_P$ such that $e < g_{G1}$ and $g_{G0} < e^+$. Then $C < [e] < D$, a contradiction. □

A rigid interval algebra shows that $|\mathrm{Aut}(A)| < \mathfrak{c}_{\mathrm{mm}}(A)$ is possible. This completes the picture for the place of $\mathfrak{c}_{\mathrm{mm}}$ among the functions of Monk [4]: only $\mathfrak{c}_{\mathrm{mm}} \leq \mathfrak{c}$ and relations implied by it.

We now give a construction of a rigid interval algebra with $\mathfrak{c}_{\mathrm{mm}}$ arbitrarily large; this is a modification of a construction of Todorčević [6]. Note that the rigid tree algebra above is not isomorphic to an interval algebra. For the following results, let λ be an infinite regular cardinal, and set $\kappa = (2^\lambda)^+$. Let U be the set of all $\alpha < \kappa$ such that $\mathrm{cf}\, \alpha = \lambda$ and α cannot be written as $\tau + \lambda$. Clearly U is stationary in κ. By SOLOVAY's theorem we can write $U = \bigcup_{\alpha<\kappa} V_\alpha$, where $\langle V_\alpha : \alpha < \kappa\rangle$ is a system of pairwise disjoint stationary subsets of κ. For $f, g \in {}^\lambda\kappa$, $f \prec g$ denotes the lexicographic order; thus this relation holds if there is an ordinal $\chi_{fg} < \lambda$ such that $f\restriction\chi_{fg} = g\restriction\chi_{fg}$ and $f(\chi_{fg}) < g(\chi_{fg})$. Now we define $\langle f_\alpha : \alpha \in U\rangle$ such that f_α is a strictly increasing continuous function from λ into α with supremum α; the definition goes by induction. Let $\langle(\Delta_\alpha, \Theta_\alpha) : \alpha < \kappa\rangle$ enumerate all pairs (Δ, Θ) of disjoint subsets of U each of power $< \lambda$. Now for each $\alpha \in U$, let $(\Delta'_\alpha, \Theta'_\alpha) = (\Delta_\beta, \Theta_\beta)$, where $\beta < \kappa$ is such that $\alpha \in V_\beta$. Thus $\langle(\Delta'_\alpha, \Theta'_\alpha) : \alpha \in U\rangle$ enumerates all pairs (Δ, Θ) with Δ and Θ subsets of U of power $< \lambda$, and each pair is enumerated repeatedly by one of the stationary sets V_β. Suppose that f_β has been defined for all $\beta < \alpha$ with $\beta, \alpha \in S$.

Case 1. For all $\beta \in \Delta'_\alpha$ and all $\gamma \in \Theta'_\alpha$ we have $\beta < \alpha$, $\gamma < \alpha$, $f_\beta \prec f_\gamma$, and for any $\delta < \chi_{f_\beta f_\gamma}$, $f_\beta(\delta) < \alpha$ and $f_\gamma(\delta) < \alpha$. Now we define $g \in {}^\lambda\kappa$ by recursion:

$$g(0) = \sup\{f_\beta(0) : \beta \in \Delta'_\alpha\};$$

for δ limit,

$$g(\delta) = \sup_{\varepsilon<\delta} g(\varepsilon);$$

finally,

$$g(\delta+1) = \max(\sup\{f_\beta(\delta+1) : f_\beta\restriction(\delta+1) = g\restriction(\delta+1)\}, g(\delta)+1).$$

(We count the sup of the empty set as 0.) Thus g is strictly increasing and continuous.

(1) $f_\beta \preceq g$ for all $\beta \in \Delta'_\alpha$.

For, suppose that $\beta \in \Delta'_\alpha$ and $g \prec f_\beta$. Clearly $\chi_{f_\beta g}$ is a successor ordinal $\delta+1$; and this is clearly impossible.

(2) $g \preceq f_\gamma$ for all $\gamma \in \Theta'_\alpha$.

For, suppose that $\gamma \in \Theta'_\alpha$ and $f_\gamma \prec g$. Clearly $\chi_{g f_\gamma}$ is a successor ordinal $\delta+1$. Thus $f_\gamma(\delta+1) < g(\delta+1)$ and $f_\gamma\restriction(\delta+1) = g\restriction(\delta+1)$. Now $f_\gamma(\delta) < f_\gamma(\delta+1)$, so $g(\delta)+1 = f_\gamma(\delta)+1 \le f_\gamma(\delta+1)$. Hence there is a $\beta \in \Delta_\alpha$ such that $f_\beta\restriction(\delta+1) = g\restriction(\delta+1)$ and $f_\gamma(\delta+1) < f_\beta(\delta+1)$, a contradiction.

(3) $g \neq f_\gamma$ for all $\gamma \in \Theta'_\alpha$.

For, suppose the contrary, $g = f_\gamma$, $\gamma \in \Theta'_\alpha$. Then $f_\beta \prec g$ for all $\beta \in \Delta'_\alpha$. Let $\delta = \sup_{\beta\in\Delta'_\alpha}(\chi_{f_\beta g}+1)$. Then $g(\varepsilon+1) = g(\varepsilon)+1$ for all $\varepsilon \in [\delta, \lambda)$. Thus γ has the form $\tau + \lambda$, contradicting the definition of S. So (3) holds.

Now let $\delta = \sup\{\chi_{g f_\gamma} + 1 : \gamma \in \Theta'_\alpha\}$. Let $f_\alpha\restriction(\delta+1) = g\restriction(\delta+1)$, $f_\alpha(\delta+1) = g(\delta+1)+1$, and let f_α be strictly increasing and continuous with supremum α. This finishes the definition is Case 1.

Case 2. Case 1 fails. Then we let f_α be any strictly increasing continuous function with supremum α.

Lemma 5. *Suppose that C is club in κ. Suppose that Δ and Θ are disjoint subsets of $U \cap C$ of size $< \lambda$, and $f_\beta \prec f_\gamma$ for all $\beta \in \Delta$ and $\gamma \in \Theta$. Then there is an $\alpha \in U \cap C$ such that $f_\beta \prec f_\alpha \prec f_\gamma$ for all $\beta \in \Delta$ and $\gamma \in \Theta$.*

Proof. Choose $\delta \in U$ so that

$$(\forall\beta \in \Delta)(\forall\gamma \in \Theta)(\beta < \delta,\ \gamma < \delta \text{ and both } f_\beta\restriction\chi_{f_\beta f_\gamma} \text{ and } f_\gamma\restriction\chi_{f_\beta f_\gamma} \text{map into } \delta).$$

Say $(\Delta,\Theta) = (\Delta_\xi,\Theta_\xi)$, and $\xi \in V_\eta$. The set $V_\eta \cap C \cap [\delta,\kappa)$ is nonempty; let α be any member of it. Note that $(\Delta'_\alpha,\Theta'_\alpha) = (\Delta,\Theta)$. Hence Case 1 above applies, and the desired conclusion follows. □

According to Lemma 5, $\mathrm{c_{mm}}(\mathrm{Intalg}(L(U \cap C)) \geq \lambda$ for any club C in κ.

Lemma 6. *If S' is a stationary subset of U, then in $L(S')$ there is no set of size λ^+ which is anti-well-ordered.*

Proof. Suppose that $\langle g_\alpha : \alpha < \lambda^+\rangle$ is a strictly decreasing sequence of elements of $L(S')$.

(1) If $\xi < \eta < \varrho < \lambda^+$, then $\chi_{g_\xi g_\varrho} \leq \chi_{g_\xi g_\eta}$.

In fact, suppose that $\tau := \chi_{g_\xi g_\eta} < \chi_{g_\xi g_\varrho}$. Then we have $g_\eta \restriction \tau = g_\xi \restriction \tau = g_\varrho \restriction \tau$, and $g_\varrho(\tau) = g_\xi(\tau) > g_\eta(\tau)$, so $g_\eta \prec g_\varrho$, a contradiction. So (1) holds.

It follows that $(\forall \xi < \lambda^+)(\exists \alpha_\xi < \lambda^+)(\forall \beta \in [\alpha_\xi,\lambda^+))\,(\chi_{g_\xi g_{\alpha_\xi}} = \chi_{g_\xi g_\beta})$. Now each $\chi_{g_\xi g_{\alpha_\xi}}$ is less than λ, so there is a $\theta < \lambda$ such that $|\{\xi < \lambda^+ : \chi_{g_\xi g_{\alpha_\xi}} = \theta\}| = \lambda^+$. Hence we can define $\xi_0, \xi_1, \ldots$ so that $\chi_{g_{\xi_i} g_{\alpha_{\xi_i}}} = \theta$ and $\xi_i, \alpha_{\xi_i} < \xi_{i+1}$ for all $i < \omega$. Now $\xi_{i+1} \in [\xi_{\xi_i},\lambda^+)$, so $\chi_{g_{\xi_i} g_{\xi_{i+1}}} = \theta$. So $g_{\xi_i}(\theta) > g_{\xi_{i+1}}(\theta)$ for all $i < \omega$, a contradiction. □

Lemma 7. *Suppose that $S \subseteq U$. Let*

$$S' = \{\alpha \in S : (\exists \xi_\alpha < \lambda)\,[\{\beta \in S : f_\alpha \prec f_\beta \text{ and } f_\alpha \restriction \xi_\alpha \subseteq f_\beta\} \text{ is non-stationary}\}]\}.$$

Then S' is non-stationary.

Proof. Suppose that S' is stationary. Choose $\xi < \lambda$ and a stationary $S'' \subseteq S'$ such that $\xi_\alpha = \xi$ for all $\alpha \in S''$. Next, for $\alpha \in S''$ let $h(\alpha) = \sup\{f_\alpha(\eta) + 1 : \eta < \xi\}$. So h is regressive, and so there is an $\eta < \kappa$ and a stationary $S''' \subseteq S''$ such that $h(\alpha) = \eta$ for all $\alpha \in S'''$. Next, note that $|{}^\xi\eta| \leq 2^\lambda < \kappa$. It follows that there is a stationary $S^{\mathrm{iv}} \subseteq S'''$ such that $f_\alpha \restriction \xi = f_\beta \restriction \xi$ for all $\alpha,\beta \in S^{\mathrm{iv}}$. Because of this, and the definition of S', for all $\alpha \in S^{\mathrm{iv}}$ the set $\{\beta \in S^{\mathrm{iv}} : f_\alpha \prec f_\beta\}$ is non-stationary; say that C_α is a club disjoint from this set. Now define by recursion

$$\beta_\alpha \in S^{\mathrm{iv}} \cap \textstyle\bigcap_{\xi<\alpha} C_{\beta_\xi} \setminus \{\beta_\xi : \xi < \alpha\}$$

for all $\alpha < \kappa$. If $\xi < \alpha < \kappa$, then $\beta_\alpha \in C_{\beta_\xi}$, and so by the choice of the C_ν's, $f_{\beta_\alpha} \prec f_{\beta_\xi}$. This contradicts Lemma 6. □

Lemma 8. *Suppose that $S \subseteq U$ and S is stationary in κ. Then $L(S)$ is not the union of fewer than κ well-ordered subsets.*

Proof. Suppose it is such a union. Then there is a stationary subset S' of S such that $L(S')$ is well-ordered.

(1) If $\alpha < \beta < \gamma$, all in S', then $\chi_{f_\alpha f_\gamma} \leq \chi_{f_\alpha f_\beta}$.

To prove this, suppose that $\tau := \chi_{f_\alpha f_\beta} < \chi_{f_\alpha f_\gamma}$. Then we have $f_\beta \restriction \tau = f_\alpha \restriction \tau = f_\gamma \restriction \tau$, and $f_\gamma(\tau) = f_\alpha(\tau) < f_\beta(\tau)$, so $f_\gamma \prec f_\beta$, a contradiction.

By (1), for every $\xi \in S'$ there is an $\alpha_\xi \in S'$ such that for all $\beta \in S' \cap [\alpha_\xi,\kappa)$ we have $\chi_{f_\xi f_{\alpha_\xi}} = \chi_{f_\xi f_\beta}$. Since $\chi_{f_\xi f_{\alpha_\xi}} < \lambda < \kappa$, it follows that there is a $\theta < \lambda$ and a stationary subset S'' of S' such that for all $\xi \in S''$ we have $\chi_{f_\xi f_{\alpha_\xi}} = \theta$. By Lemma 7, the set

$$S^{\mathrm{iv}} := \{\alpha \in S''' : (\forall \mu < \lambda)\,(\{\beta \in S''' : f_\alpha \prec f_\beta \text{ and } f_\alpha \restriction \mu \subseteq f_\beta\} \text{ is stationary}\}$$

is also stationary. Pick any $\alpha \in S^{\mathrm{iv}}$, and take a $\beta \in S'''$ such that $f_\alpha \prec f_\beta$ and $f_\alpha \restriction (\theta+1) \subseteq f_\beta$. This contradicts the above. □

Lemma 9. *If S is a stationary subset of U, then $L(S)$ has a well-ordered subset of size λ^+.*

Proof. This is clear from Lemma 6 and the partition relation $(2^\lambda)^+ \to (\lambda^+)^2_\lambda$. □

Lemma 10. *If S and S' are subsets of U and $L(S)$ is order-isomorphic to a subset of $L(S')$, then $S \setminus S'$ is non-stationary.*

Proof. Suppose that $S_0 = S \setminus S'$ is stationary. Let H be the given isomorphism, and write $H(f_\alpha) = f_{h(\alpha)}$ for all $\alpha \in S$. Thus h is one–one, so there is a stationary subset S_1 of S_0 on which h is strictly increasing. For $\tau \leq \lambda$ let $A_\tau = \{f_{h(\alpha)} \restriction \tau : \alpha \in S_1\}$. Thus $|A_\lambda| = \kappa$. Choose τ minimum such that $|A_\tau| = \kappa$.

(1) τ is not a limit ordinal.

For, suppose that it is. For each $\sigma \leq \lambda$ define

$$\alpha \equiv_\sigma \gamma \quad \text{iff} \quad \alpha, \beta \in S_1 \text{ and } f_{h(\alpha)} \restriction \sigma = f_{h(\beta)} \restriction \sigma.$$

Thus "$\equiv_\sigma$" is an equivalence relation on S_1, and $|S_1/\equiv_\sigma| = |A_\sigma|$, as is seen by the map $[\alpha]_{\equiv_\sigma} \mapsto f_{h(\alpha)} \restriction \sigma$.

(2) $(S_1/\equiv_\tau) = \{\bigcap_{\sigma<\tau} u_\sigma : u \in \prod_{\sigma<\tau}(S_1/\equiv_\sigma)\} \setminus \{\emptyset\}$.

This condition is clear, since $[\alpha]_{\equiv_\tau} = \bigcap_{\sigma<\tau}[\alpha]_{\equiv_\sigma}$ for all $\alpha \in S_1$.

Now $|S_1/\equiv_\sigma| < \kappa$ for all $\sigma < \tau$, so $|\prod_{\sigma<\tau}(S_1/\equiv_\sigma)| \leq (2^\lambda)^\tau < \kappa$, a contradiction. This proves (1).

Say $\tau = \sigma + 1$. Now

$$S_1 = \bigcup_{g \in A_\sigma} \{\alpha \in S_1 : f_{h(\alpha)} \restriction \sigma = g\},$$

and $|A_\sigma| < \kappa$, so there is a stationary $S_2 \subseteq S_1$ such that $f_{h(\alpha)} \restriction \sigma = f_{h(\beta)} \restriction \sigma$ for all $\alpha, \beta \in S_2$. Then the function $f_\alpha \mapsto f_{h(\alpha)}(\sigma)$ is an order preserving embedding of $L(S_2)$ into κ, contradicting Lemma 8. □

Lemma 11. *Suppose that $S, S' \subseteq U$, neither $L(S)$ nor $L(S')$ have a smallest element, and there is an isomorphism of* Intalg$(L(S))$ *into* Intalg$(L(S'))$. *Then $S \setminus S'$ is non-stationary.*

Proof. Suppose that $S_0 = S \setminus S'$ is stationary. Recall from [3] that in the interval algebras a 0 is adjoined, by our assumption. Let $b_\alpha = [0, f_\alpha)$ for all $\alpha \in S_0$. Let H be the given isomorphism into, and for each $\alpha \in S_0$ write

$$H(b_\alpha) = [x^0_\alpha, y^0_\alpha) \cup [x^1_\alpha, y^1_\alpha) \cup \ldots \cup [x^{n(\alpha)-1}_\alpha, y^{n(\alpha)-1}_\alpha),$$

where $x^0_\alpha < y^0_\alpha < x^1_\alpha < y^1_\alpha < \cdots < y^{n(\alpha)-1}_\alpha \leq \infty$. There is an $n \in \omega$ and a stationary $S_1 \subseteq S_0$ such that $n(\alpha) = n$ for all $\alpha \in S_1$. Now we claim

(1) There is a stationary $S_2 \subseteq S_1$ such that $x^0_\alpha = x^0_\beta$ for all $\alpha, \beta \in S_2$.

For, there is a stationary $S'_1 \subseteq S_1$ such that either $x^0_\alpha = 0$ for all $\alpha \in S'_1$, or $x^0_\alpha \neq 0$ for all $\alpha \in S'_1$. The first case gives (1), so suppose that the second case holds. For all $\alpha \in S'_1$ write $x^0_\alpha = f_{h(\alpha)}$. Thus h maps S'_1 into S', and $S'_1 \cap S' = \emptyset$, so $h(\alpha) \neq \alpha$ for all $\alpha \in S'_1$. If h is constant on a stationary subset of S'_1, this gives (1), so assume that h is one–one on some stationary subset S''_1 of S'_1. If $\alpha, \beta \in S''_1$ and $f_\alpha \prec f_\beta$, then $b_\alpha < b_\beta$, hence $H(b_\alpha) < H(b_\beta)$, and so $x^0_\beta \leq x^0_\alpha$; since h is one–one, $x^0_\beta < x^0_\alpha$. By Lemma 9 there is a well-ordered subset of $L(S''_1)$ of size λ^+, so this gives an anti-well-ordered subset of $L(S')$ of size λ^+, contradicting Lemma 6. Hence (1) holds.

Thus we can write, for any $\alpha \in S_2$,

$$H(b_\alpha) = [x^0, y^0_\alpha) \cup [x^1_\alpha, y^1_\alpha) \cup \ldots \cup [x^{n-1}_\alpha, y^{n-1}_\alpha).$$

We now claim

(2) There is a stationary subset S_3 of S_2 such that $y^0_\alpha = y^0_\beta$ for all $\alpha, \beta \in S_3$.

To prove this, first note that it cannot be true that $n = 1$ and there is a stationary subset S_2 such that $y^0_\alpha = \infty$ for all α in that subset. So we may assume that $y^0_\alpha \neq \infty$ for all $\alpha \in S_2$. For each $\alpha \in S_2$ write $y^0_a = f_{l(\alpha)}$. If l is constant on some stationary subset of S_2, this gives (2). So assume that l is one–one on a stationary subset S'_2 of S_2. If $\alpha, \beta \in S'_2$ and $f_\alpha \prec f_\beta$, then $b_\alpha < b_\beta$, and hence $f_{l(\alpha)} \prec f_{l(\beta)}$. This shows that there is an embedding of $L(S'_2)$ into $L(S')$. This contradicts Lemma 10, since $S'_2 \setminus S' = S'_2$. So (2) holds.

Now it is clear that we can continue this argument, finally reaching a stationary set on which $H(b_\alpha)$ is constant, a contradiction. □

Theorem 12 *There is a mono-rigid interval algebra A with* $\mathrm{c_{mm}}(A) \geq \lambda$.

(Recall our assumptions on λ and κ.)

Proof. We apply Lemma 7 to U in place of S and get S' as indicated. Let $S = U \setminus S'$. Thus

(1) For all $\alpha \in S$ and all $\xi < \lambda$ the set $\{\beta \in S : f_\alpha \prec f_\beta \text{ and } f_\alpha \restriction \xi \subseteq f_\beta\}$ is stationary.

Hence

(2) If $f, g \in L(S)$ and $f \prec g$, then $\{\beta \in S : f \prec f_\beta \prec g\}$ is stationary.

For, let $f = f_\alpha$. By (1), the set $\{\beta \in S : f_\alpha \prec f_\beta \text{ and } f_\alpha \restriction (\chi_{f_\alpha g} + 1) \subseteq f_\beta\}$ is stationary. For β in this set we have $f \prec f_\beta \prec g$, as desired.

Now suppose that H is a one-one embedding of $\mathrm{Intalg}(L(S))$ into $\mathrm{Intalg}(L(S))$, and it is not the identity. Then

(3) There is a nonzero element $a \in \mathrm{Intalg}(L(S))$ such that $a \cdot H(a) = 0$.

For, choose x so that $x \neq H(x)$. If $x \not\leq H(x)$, we can take $a = x \cdot -H(x)$. If $H(x) \not\leq x$, we can take $a = H(x) \cdot -x$. So (3) holds.

Let $S' = \{\alpha \in S : f_\alpha \in a\}$ and $S'' = \{\alpha \in S : f_\alpha \in H(a)\}$. By (2), these are stationary subsets of κ. They are disjoint, and there is an embedding of $\mathrm{Intalg}(L(S'))$ into $\mathrm{Intalg}(L(S''))$. This contradicts Lemma 11. □

3 Tree algebras

If T has only one root, and has an element a of level ω such that there is no other element of level ω with the same predecessors as a, then $\mathrm{c_{mm}}(\mathrm{Treealg}\, T) = \omega$. For, let $b_0, b_1, \ldots$ be the predecessors of a in increasing order, with b_0 the root. The desired partition is $(T \uparrow b_0) \setminus (T \uparrow b_1), (T \uparrow b_1) \setminus (T \uparrow b_2), \ldots, T \uparrow a$.

By an earlier remark, it is possible to have $\mathrm{c_{mm}}(A) < \pi\chi(A)$ for a tree algebra A. One can have $\mathrm{Depth}(A) < \mathrm{c_{mm}}(A)$: Finco κ.

Proposition 13. *For any tree algebra A, $\mathrm{c_{mm}}(A) \le \chi(A)$.*

Proof. We use Theorem 14.12 of Monk [4]. Say $A = \mathrm{Treealg}\, T$, where T has only one root r. We are assuming that A is infinite, so T is infinite.

Case 1. T has finite height. Then some element $t \in T$ has an infinite set X of immediate successors. Then the following is a partition in A: $(T \uparrow r) \setminus (T \uparrow t)$ (omitted if $t = r$), and $T \uparrow x$ for $x \in X$. By [4, Theorem 14.12], this proves that $\mathrm{c_{mm}}(A) \le \chi(A)$.

Case 2. T has infinite height. If T has no infinite chain, then some level is infinite, and the lowest such level gives at one lower level an element with infinitely many immediate successors, to which the argument of Case 1 applies. So we may assume that T has an infinite chain. Let C be an initial chain of order type ω, and let X be the set of all immediate successors of C. Let $C = \{b_0, b_1, \ldots\}$, in increasing order. Then a partition is furnished by $(T \uparrow b_0) \setminus (T \uparrow b_1)$, $(T \uparrow b_1) \setminus (T \uparrow b_2), \ldots$, along with all elements $T \uparrow x$ for $x \in X$. Again, [4, Theorem 14.12] yields that $\mathrm{c_{mm}}(A) \le \chi(A)$. □

Note that $\mathrm{c_{mm}}(A) < |\mathrm{Aut}(A)|$ is possible for a tree algebra: $\mathrm{Finco}\,\omega \times \mathrm{Finco}\,\kappa$. A rigid tree algebra shows that $|\mathrm{Aut}(A)| < \mathrm{c_{mm}}(A)$ is possible, and Theorem 2 shows that one can have $\mathrm{c_{mm}}$ arbitrarily large. All of these remarks fix for tree algebras the place of $\mathrm{c_{mm}}$ among the 21 functions of Monk [4]: one has $\mathrm{c_{mm}}(A) < \chi(A)$, and no other relations except those implied by this.

4 Complete BA's

For complete BA's A one has $\mathrm{c_{mm}}(A) = \omega$, so the function is trivial.

5 Superatomic BA's

The following possibilities for BA's were mentioned above: $\mathrm{c_{mm}}(A) < \mathrm{Length}(A)$ and $\mathrm{c_{mm}}(A) < \pi\chi(A)$. For superatomic algebras A the inequality $\mathrm{t}(A) < \mathrm{c_{mm}}(A)$ is also possible: $\mathrm{Finco}\,\kappa$. Here $\mathrm{t}(A)$ is the *tightness* of A, i.e. the supremum of depths of homomorphic images of A. Thus the diagram for superatomic BA's is just like for the general case.

6 Atomic BA's

The following possibilities for atomic BA's were mentioned above: $\chi(A) < \mathrm{c_{mm}}(A)$ (with difference necessarily small), $\mathrm{c_{mm}}(A) < \mathrm{Depth}(A)$, and $\mathrm{t}(A) < \mathrm{c_{mm}}(A)$, the difference big. Also, we have $\mathrm{Length}(A) < \mathrm{c_{mm}}(A)$ and $\mathrm{Ind}(A) < \mathrm{c_{mm}}(A)$ for the algebra $A = \mathrm{Finco}\,\kappa$, and $\mathrm{c_{mm}}(A) < \mathrm{Ind}(A)$ for $A = \mathcal{P}\kappa$. Thus the diagram for atomic BA's is like for the general case.

7 Atomless BA's

Proposition 14. *For atomless algebras the inequality $\chi(A) < \mathrm{c_{mm}}(A)$ is possible.*

Proof. In fact, let C be as in 25.192 of Monk [4], with $2^\omega < \kappa$. We claim that $\mathrm{c_{mm}}(C) = 2^\kappa$. To see this, suppose on the contrary that $\langle h(b_\alpha, F_\alpha, a_\alpha) : \alpha < \lambda\rangle$ is a system of nonzero pairwise disjoint elements of C with sum 1, where each $b_\alpha \in B$, F_α is a finite subset of I disjoint from b_α, $a_\alpha \in \prod_{i\in F_\alpha}(A_i \setminus \{0,1\})$, and $\lambda < 2^\kappa$; we want to get a contradiction. Note that $|I| = 2^\kappa$. Then

(1) For all $i \in I$, either there is $\alpha < \lambda$ such that $i \in b_\alpha$, or for all $\alpha < \lambda$, $i \notin b_\alpha$ and $\sum_{\alpha<\lambda, i\in F_\alpha} a_\alpha i = 1$.

For, suppose not: then there is an $i \in I$ such that for all $\alpha < \lambda$, $i \notin b_\alpha$ and

$$c := \sum_{\alpha<\lambda, i\in F_\alpha} a_\alpha i \neq 1.$$

Then $h(0, \{i\}, -c) \cdot h(b_\alpha, F_\alpha, a_\alpha) = 0$ for all $\alpha < \lambda$, a contradiction. So (1) holds.

Let $J = \{i \in I : (\exists\alpha < \lambda)(i \in b_\alpha)\}$. There are now two possibilities:

Case 1. $|J| > \lambda$. For each $i \in J$, choose $\alpha_i < \lambda$ such that $i \in b_{\alpha_i}$. In our case, there are distinct $i, j \in J$ such that $\alpha_i = \alpha_j$, a contradiction.

Case 2. $|J| \leq \lambda$. Then $|I \setminus J| = 2^\kappa$. For all $i \in I \setminus J$ we have $\sum_{\alpha<\lambda, i\in F_\alpha} a_\alpha i = 1$, and so there is an $\alpha_i < \lambda$ such that $i \in F_{\alpha_i}$. It follows that there is a $\beta < \lambda$ and an $L \in [I \setminus J]^{\lambda^+}$ such that $\alpha_i = \beta$ for all $i \in L$. Then F_β is infinite, a contradiction. □

It follows that the diagram for atomless algebras is like the general diagram.

References

[1] Comfort, W. W., and S. Negrepontis, The Theory of Ultrafilters. Springer-Verlag, Berlin-Heidelberg-New York 1982.

[2] van Douwen, E. K., The integers and topology. In: Handbook of Set-theoretic Topology. North-Holland Publ. Comp., Amsterdam 1984, pp. 111 – 167.

[3] Koppelberg, S., General Theory of Boolean algebras. Handbook of Boolean Algebras (J. D. Monk and R. Bonnet, eds.), Vol. 1. North-Holland Publ. Comp., Amsterdam 1989.

[4] Monk, J. D., Cardinal Invariants on Boolean Algebras. Birkhäuser Verlag, Basel-Berlin-Boston 1995.

[5] Rosenstein, J., Linear Orderings. Academic Press, Lodon-New York 1982.

[6] Todorčević, S., Very strongly rigid Boolean algebras. Publ. Inst. Math. (Beograd) 27 (1980), 267 – 277.

(Received: September 28, 1995)