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Pseudo-Trees and Boolean Algebras

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Abstract. We consider Boolean algebras constructed from pseudo-trees in various ways and make comments about related classes of Boolean algebras.

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0. Introduction

A *pseudo-tree* is a partially ordered set $(T, \leqslant)$ such that for each $t \in T$, the initial segment $\{x \in T: x \leqslant t\}$ of T is linearly ordered, under $\leqslant$; in case each of these initial segments is well-ordered, we get a *tree*. Thus pseudo-trees are a generalization of both trees and linear orders. We will discuss Boolean algebras obtained from pseudo-trees in different ways some of which have been considered in the literature for trees. We also make comments about related classes of Boolean algebras.

For $(T, \leqslant)$ a pseudo-tree and t in T, we let $T \uparrow t = \{x \in T: t \leqslant x\}$ and we define Treealg T, the *pseudo-tree algebra* of T, to be the algebra of subsets of T generated by $\{T \uparrow t: t \in T\}$. Thus for T a tree, it coincides with the tree algebra of T as defined by G. Brenner in [4]; see also [5] and [6]. If T happens to be a linear order, Treealg T coincides with the interval algebra Intalg T. For a basic exposition of these notions, see Section 16, resp. 15 in [17]. Let us point out that a special question on pseudo-tree algebras (which pseudo-tree algebras are isomorphic to interval algebras?) has been considered before in [10].

The paper is organized as follows. In Section 1, we review those results on tree algebras which carry over to pseudo-tree algebras. In Section 2, we apply methods from the theory of semigroup algebras (see [11]) to pseudo-tree algebras, and in Section 3, we study the close connection between pseudo-tree algebras and interval algebras. Sections 4 to 6 present further constructions of Boolean algebras from pseudo-trees: regular open algebras, restricted regular open algebras, pseudo-tree-generated Boolean algebras. In Section 7, we consider the class of *tail algebras* Tailalg P; here Tailalg P is generated from an arbitrary partial order $(P, \leqslant)$ in the same way as Treealg T is generated from a pseudo-tree $(T, \leqslant)$. The final Section 8 compares these and a few related classes of Boolean algebras.

For all unexplained notation and results in set theory, see [13]; in Boolean algebras, see [17]. In particular, we denote the Boolean operations by $+, \cdot, -, 0, 1$ and the Boolean partial ordering by $\leqslant$, and we assume familiarity with Section 16, on tree algebras, of that book.

We thank L. Heindorf for showing us the semigroup-algebraic proofs of Theorem 2.3 and 2.4 which are included here with his kind permission.

1. Facts on Tree Algebras which Generalize to Pseudo-Tree Algebras

We ask here how much of the elementary theory of tree algebras extends to pseudo-tree algebras. The numbers 16.1 to 16.20 below refer to Section 16 in [17].

16.3, 16.6, the normal form lemmas. These carry over to pseudo-tree algebras. 16.6 needs a new proof in part, since the well-foundedness of the tree under consideration was used in a minor way. Instead of picking w of minimal height in $\varepsilon \backslash e_t$, pick it minimal among the finitely many elements which are relevant to the argument, or let it be an arbitrary member of $\varepsilon \backslash e_t$, if there are no relevant points in $\varepsilon \backslash e_t$.

16.4, extension to homomorphisms, carries over because of 16.3.

16.7, each tree algebra Treealg T is isomorphic to a tree algebra Treealg T^* where T^* has a single root. This carries over as follows. If the set of minimal elements (roots) of T is finite and every element of T has a root below it, proceed as in the first part of the proof of 16.7, otherwise as in the second part of that proof (note that pseudo-trees do not necessarily have roots – for example, take a linear ordering with no first element). We give another proof of 16.7 in Theorem 2.3.

16.8, Treealg $(T \uparrow t) \cong (\text{Treealg } T) \restriction (T \uparrow t)$, extends to pseudo-tree algebras.

16.9, Treealg T' embeds into Treealg T for $T' \subseteq T$, carries over with some change in the proof. Namely, if T' has only finitely many roots and every element of T' lies above some root of T', proceed as in Case 2 of 16.9; otherwise proceed as in Case 1.

16.10, description of atoms of Treealg T, remains the same for pseudo-tree algebras.

16.11, ultrafilters of Treealg T correspond to initial chains of T; here a subset C of T is called an initial chain if it is a chain in T, $x \leqslant y$ and $y \in C$ implies $x \in C$, and, if T has only finitely many roots and each element of T has a root below it, then some root belongs to C. This carries over, too; we give an additional piece of information not stated in 16.11. Namely, by identifying each subset C of T with its characteristic function $\chi_C: T \to 2$, we obtain the *natural topology* on the power set $\mathscr{P}T$ of T; $\mathscr{P}T$ is homeomorphic to the product space ${}^T2$, hence Boolean. The set X of initial chains of T is a closed subspace of $\mathscr{P}T$, and the function mapping an ultrafilter p of Treealg T to the initial chain $\{t \in T: T \uparrow t \in p\}$ is a homeomorphism from Ult(Treealg T) onto X, as is easily checked.

16.17, 16.18, the class of tree algebras is closed under finite products and under homomorphic images. This extends to pseudo-tree algebras; we will give in 2.4 a less computational proof of 16.18 than the one contained in [17].

16.20, chains in T and in Treealg T. The theorem carries over to pseudo-tree algebras in the following form. Suppose C is a chain in the pseudo-tree algebra of T and the cardinality κ of C is regular and uncountable. Then there are subchains D of C and E of T of size κ which are isomorphic or conversely isomorphic. This does not follow in an obvious way from the proof of 16.20. But the above result is established in [12] for semigroup algebras, and we show in 2.1(b) that every pseudo-tree algebra is a semigroup algebra.

Concerning 16.12 and 16.19, the relation between tree algebras and algebras embeddable into interval algebras, we refer to Section 3; let us just mention here that 16.12 carries over to pseudo-tree algebras: every pseudo-tree algebra embeds into an interval algebra (Theorem 3.1).

2. Connection with Semigroup Algebras

We show in this section how to derive many of the preceding results from Section 16 of [17] by using the theory of semigroup algebras. The presentation of this material follows mostly an outline by L. Heindorf.

Call a subset H of a Boolean algebra B *disjunctive* if, for h and $h_1, \ldots, h_n \in H$ satisfying $h \leqslant h_1 + \cdots + h_n$, there is $i \in \{1, \ldots, n\}$ such that $h \leqslant h_i$. If H is disjunctive then $0 \notin H$ – otherwise, $0 \leqslant$ the sum of the empty subset of H, contradicting disjunctiveness. B is a *semigroup algebra* if it is generated by a subset H such that $0, 1 \in H$, H is closed under multiplication (thus a subsemigroup of $(B, \cdot)$) and $H \backslash \{0\}$ is disjunctive. Finally, call $H \subseteq B$ a *ramification set* if for all $s, t \in H$ either $s \cdot t = 0$ or $s \leqslant t$ or $t \leqslant s$; H is then a pseudo-tree under the inverse of the Boolean partial order.

REMARK 2.1 (a) If $R \subseteq B$ is a ramification set, then so is every subset of R. Moreover, $R \cup \{0\}$ is closed under multiplication.

(b) Let T be a pseudo-tree. Then the set $R = \{T \uparrow t; t \in T\}$ of canonical generators of Treealg T is a ramification set; it is also disjunctive, as shown in (c). If T has a single root r, then $1 = T \uparrow r \in R$. Thus $H = R \cup \{\emptyset\}$ shows that Treealg T is a semigroup algebra.

(c) More generally, let $(P, \leqslant)$ be a partial order. In the power set algebra of P, the set $R = \{P \uparrow p : p \in P\}$ (where $P \uparrow p = \{q \in P : p \leqslant q\}$) is disjunctive (this situation will be studied more closely in Section 7). For let $p, p_1, \ldots, p_n \in P$ and $P \uparrow p \subseteq P \uparrow p_1 \cup \cdots \cup P \uparrow p_n$. Now $p \in P \uparrow p$, so $p \in P \uparrow p_i$ for some i and $P \uparrow p \subseteq P \uparrow p_i$.

PROPOSITION 2.2. *(a) $H \subseteq B$ is disjunctive iff, for each $M \subseteq H$, there is a (unique) homomorphism f_M from $\langle H \rangle$ (the subalgebra of B generated by H), to $\mathscr{P}M$ (the power set algebra of M), sending $h \in H$ to $M \downarrow h = \{m \in M : m \leqslant h\}$.*

(*b*) *Let* $T \subseteq B^+ = B\backslash\{0\}$ *be a disjunctive ramification set and* X, Y *finite subsets of* T. *Then* $\Pi X \leqslant \Sigma Y$ *iff either* $(X = \emptyset$ *and* $\Sigma Y = 1)$ *or* $(x \cdot x' = 0$, *for some* $x, x' \in X)$ *or* $(x \leqslant y$, *for some* $x \in X$ *and* $y \in Y)$.

(*c*) *Let* $T \subseteq B^+$ *be a ramification set and* $T^* \subseteq T$ *maximally disjunctive. Then* $\langle T^*\rangle = \langle T\rangle$.

Proof. For the non-trivial direction of (b), suppose $\Pi X \leqslant \Sigma Y$ and none of the first two cases applies. Then X is non-empty, $x = \Pi X$ is an element of T, and by disjunctiveness, Case 3 applies.

(a) First assume that H is disjunctive and that $M \subseteq H$. In order to apply Sikorski's criterion on the extension of mappings to homomorphisms (see Theorem 12.2 in [21] resp. 5.5, 5.9 in [17]), assume that $h_1, \ldots, h_n, k_1, \ldots, k_m \in H$ and $h_1 \cdot \ldots \cdot h_n \leqslant k_1 + \cdots + k_m$; we want to show that $(M \downarrow h_1) \cap \cdots \cap (M \downarrow h_n) \subseteq (M \downarrow k_1) \cup \cdots \cup (M \downarrow k_m)$. Hence suppose that $x \in (M \downarrow h_1) \cap \cdots \cap (M \downarrow h_n)$. So $x \leqslant h_1 \cdot \ldots \cdot h_n$ and hence $x \leqslant k_1 + \cdots + k_m$. Since H is disjunctive, $x \leqslant k_i$ for some i, and hence $x \in (M \downarrow k_i)$, as desired.

Conversely assume the indicated condition and supoose that $h, h_1, \ldots, h_n \in H$ and $h \leqslant h_1 + \cdots + h_n$. Consider $M = \{h\}$ and $f_M : \langle H\rangle \to \mathscr{P}M$ as in the statement of (a). Then

$$h \in M \downarrow h = f_M(h) \subseteq f_M(h_1) \cup \cdots \cup f_M(h_n) = (M \downarrow h_1) \cup \cdots \cup (M \downarrow h_n),$$

hence $h \leqslant h_i$ for some i.

(c) Let $t \in T\backslash T^*$ with the aim of showing $t \in \langle T^*\rangle$. Since $t \notin T^*$, $T^* \cup \{t\}$ is not disjunctive.

Case 1. There are $t_i \in T^*$ such that $t \leqslant t_1 + \cdots + t_n$ but $t \nleqslant t_i$ for all i. Let n be minimal for this situation. It follows that $t \cdot t_i \neq 0$, hence $t_i \leqslant t$ and $t = t_1 + \cdots + t_n \in \langle T^*\rangle$. Note that, by minimality of n, the t_i are pairwise disjoint.

Case 2. There are $t_0, \ldots, t_n \in T^*$ such that $t_0 \leqslant t + t_1 + \cdots + t_n$ but $t_0 \nleqslant t$ and $t_0 \nleqslant t_i$ for $1 \leqslant i \leqslant n$, and again, n is minimal for this situation. As in Case 1, t_0 is the disjoint sum of t and $t_1, \ldots, t_n$. So $t = t_0 \cdot -(t_1 + \cdots + t_n) \in \langle T^*\rangle$. □

THEOREM 2.3. *For every Boolean algebra* B, *the following are equivalent*:

(*a*) B *is isomorphic to Treealg* T, *for some pseudo-tree* T *with a single root*,

(*b*) B *is isomorphic to Treealg* T, *for some pseudo-tree* T,

(*c*) B *is generated by a ramification set* $S \subseteq B^+$,

(*d*) B *is generated by a ramification set* $R \subseteq B^+$ *such that* $1 \in R$ *and* R *is disjunctive.*

Proof. Trivially, (a) implies (b) and (b) implies (c) – see 2.1(b).

(c) implies (d): if $S \subseteq B^+$ is a ramification set generating B, then so is $S \cup \{1\}$. Pick $R \subseteq S \cup \{1\}$ such that $1 \in R$ and R is maximally disjunctive. R verifies (d), by 2.2.(c).

(d) implies (a): let B be generated by a ramification set $T \subseteq B^+$ such that $1 \in T$ and T is disjunctive. So T is a pseudo-tree under the converse $\leqslant_T$ of the Boolean partial order $\leqslant$ of B. By 2.2.(a), consider the homomorphism $f_T: B \to \mathscr{P}T$ mapping

$t \in T$ to $T \uparrow t = \{x \in T: t \leqslant_T x\}$. Clearly f_T maps B onto Treealg T. It is also one-one, for let $t_1 \cdot \ldots \cdot t_n \nleqslant s_1 + \cdots + s_m$, where t_i, $s_j \in T$. By 2.2.(b), $t = t_1 \cdot \ldots \cdot t_n \in T$ and $t \nleqslant s_j$ for all j. Thus

$$t \in \bigcap_i f_T(t_i) \setminus \bigcup_j f_T(s_j),$$

as desired. □

COROLLARY 2.4. (*a*) *Homomorphic images of pseudo-tree algebras are pseudo-tree algebras.*

(*b*) *Every pseudo-tree algebra is retractive.*

Proof. Let B be a pseudo-tree algebra, hence (by 2.3) generated by a disjunctive ramification set $T \subseteq B^+$ where $1 \in T$, and let $f: B \to A$ be an epimorphism of Boolean algebras. Then $R = f[T]$ is a ramification set generating A, hence A is (up to isomorphism) a pseudo-tree algebra, by 2.3.

We find an embedding $g: A \to B$ satisfying $f \circ g = id_A$ as follows. Pick $R^* \subseteq R \setminus \{0\}$ such that $1 \in R^*$ and R^* is maximally disjunctive; R^* generates A, by 2.2.(c). For $r \in R^*$, pick a preimage $g(r) \in T$ of r under f such that $g(1) = 1$. The map $g: R^* \to B$ extends to a homomorphism by 2.2.(b) and since, for $r, r' \in R^*$, $r \cdot r' = 0$ implies $g(r) \cdot g(r') = 0$ and $r \leqslant r'$ implies $g(r) \leqslant g(r')$. □

2.4.(b) follows also from 3.1 below and Rubin's theorem that subalgebras of interval algebras are retractive (see 15.18 in [17]). We note that the proofs of 2.3 and 2.4 are still valid when the notion of a pseudo-tree is replaced everywhere by that of a tree, thus giving some well-known facts on tree algebras.

3. Connection with Interval Algebras

We prove in Theorem 3.1 the analogue of 16.12 in [17] and then discuss the question whether the converse holds. Theorem 3.1 has first been shown by M. Bekkali, by a proof different from ours.

THEOREM 3.1. *Every pseudo-tree algebra embeds into an interval algebra.*

Proof. Let T be a pseudo-tree; we may assume by 16.7 that T has a single root min T. Consider, in the first order language $\{<\} \cup \{c_t : t \in T, t \neq \min T\} \cup \{d_t : t \in T, t \neq \min T\}$ (where c_t and d_t are distinct constant symbols), the theory Σ expressing that, for every model $\mathscr{A} = (L, <, a_t, b_t)_{t \in T, t \neq \min T}$ of Σ, the following hold.

L is a dense linear order with a first element min L

$\min L < a_t < b_t$, for t in T, $t \neq \min T$

$a_t < a_s$ and $b_s < b_t$, for $t < s$ in T, $t \neq \min T$

$b_t < a_s$ or $b_s < a_t$, if $t, s \in T \setminus \{\min T\}$ are incomparable.

Clearly, if $\mathscr{A} = (L, <, \dots)$ is a model of Σ, then there is an embedding from Treealg T into Intalg L mapping $T \uparrow \min T$ to L and $T \uparrow t$, $t \neq \min T$, to the interval $[a_t, b_t)$.

In fact, Σ has a model by the compactness theorem of first order predicate logic: for if $T_0 \subseteq T$ is finite, then the finite subset of Σ which consists of those sentences referring only to points in T_0 has a model because T_0 is a (well-founded) tree and by, 16.12 in [17], its tree algebra embeds into an interval algebra. □

We do not know whether the converse of Theorem 3.1 holds:

PROBLEM 1. Is every subalgebra of an interval algebra isomorphic to a pseudo-tree algebra?

In view of Theorem 3.1, a positive answer would imply that the class of pseudo-tree algebras is closed under taking subalgebras. Note that Example 16.19 in [17] does not solve the question since it is not hard to check that the algebra B in this example is isomorphic to Treealg T where T is the disjoint union of κ linear orders, each of them inversely well-ordered of type ω_1^*. In fact, there are two results giving a positive answer to Problem 1 for special situations. Let us note that both results show once more that the algebra in Example 16.19 of [17] is a pseudo-tree algebra and that a special case of 3.2(a) has earlier been proved by van Douwen (cf. [8]): every subalgebra of the interval algebra of ω_1 is a pseudo-tree algebra. Also Problem 1 above is stated in [8].

THEOREM 3.2. (*a*) (*Bonnet*; *cf.* (*i*) $\Rightarrow$ (*iv*) *in the main theorem of* [2]). *Every superatomic subalgebra of an interval algebra is a pseudo-tree algebra.*

(*b*) *Let L be a linear order, J a set of half-open intervals in L. Then the subalgebra of Intalg L generated by J is isomorphic to a pseudo-tree algebra.*

Proof of (*b*). Let $I = \{[a, b) : a < b \text{ in } L\}$ be the set of all half-open intervals of L; for $x, y \in I$, write

$$x \mid y \text{ iff } (x \subseteq y \text{ or } y \subseteq x \text{ or } x \cap y = \emptyset);$$

otherwise write $x \nmid y$. I.e., $x \mid y$ means that x, y don't overlap. Thus $T \subseteq I$ is a ramification set, as defined in Section 2, iff $x \mid y$ holds for all $x, y \in T$.

For $r \in L$, call $x \in I$ *r-centered* if r is one of the end points of x. The importance of this notion is that the family of all r-centered intervals is a ramification set in Intalg L.

CLAIM. Let $x \in I$ be r-centered and let $y \in I$ be such that $x \nmid y$. Then there are $s, t \in I$, both r-centered, such that s and t are generated by x and y, and y is generated by s and t.

The proof splits into four easy cases.

Case 1. $x = [r, b)$ for some $b > r$, and $y = [u, v)$ where $r < u < b < v$. Then put $s = [r, u)$, $t = [r, v)$; clearly $s = x \setminus y$, $t = x \cup y$, and $y = t \setminus s$.

Case 2. $x = [r, b)$ and $y = [u, v)$ where $u < r < v < b$. Put $s = [u, r)$, $t = [r, v)$; then $s = y \setminus x$, $t = x \cap y$, and $y = s \cup t$.

Case 3. $x = [a, r)$ and $y = [u, v)$ where $u < a < v < r$. Similar to Case 1.

Case 4. $x = [a, r)$ and $y = [u, v)$ where $a < u < r < v$. Similar to Case 2.

This finishes the Claim.

To prove the theorem, let B be the algebra generated by J and let $I_B = I \cap B$. Thus I_B generates B. We shall construct a ramification set $T \subseteq I_B$ generating B; by Theorem 2.3, this completes the proof.

Without loss of generality, L is infinite; let $\{r_\alpha : \alpha < \kappa\}$ be an enumeration of L where $\kappa = |L|$. We define, by induction on $\alpha < \kappa$, a subalgebra B_α of B and a ramification set $T_\alpha \subseteq B$:

$$B_\alpha = \left\langle \bigcup_{\nu < \alpha} T_\nu \right\rangle$$

$$T_\alpha = \{x \in I_B : x \text{ is } r_\alpha\text{-centered}\} \setminus B_\alpha.$$

Clearly:

(1) if $x \in I_B$ is r_α-centered, then $x \in B_\alpha \cup T_\alpha \subseteq B_{\alpha+1}$

(2) if $x \in T_\alpha$, $y \in I_B$ and $x \nmid y$ then $y \in B_{\alpha+1}$.

This follows by applying the Claim to $r = r_\alpha$: we get $s, t \in \langle x, y \rangle \subseteq B$ which are r_α-centered such that $y \in \langle s, t \rangle$. Now $s, t \in B_{\alpha+1}$ by (1) and hence $y \in B_{\alpha+1}$.

Let $T = \bigcup_{\alpha < \kappa} T_\alpha$. T generates B, since B is generated by I_B and for $x \in I_B$ r_α-centered, (1) implies that $x \in B_{\alpha+1} \subseteq \langle T \rangle$.

We finally show that T is a ramification set; so let $\alpha < \beta < \kappa$, $x \in T_\alpha$, and $y \in T_\beta$ with the aim of showing that $x \mid y$. But if $x \nmid y$ then (2) implies that $y \in B_{\alpha+1} \subseteq B_\beta$, contradicting $T_\beta \cap B_\beta = \emptyset$. □

4. Restricted Regular Open Algebras of Pseudo-Trees

In this section we are interested in a subalgebra $\mathrm{RRO}(T^{-1})$, the restricted regular open algebra, of the regular open algebra $\mathrm{RO}(T^{-1})$ of a pseudo-tree T; we'll prove that restricted regular open algebras of pseudo-trees are pseudo-tree algebras and the converse holds for trees but not, in general, for pseudo-trees.

Let us briefly recall the construction of the complete Boolean algebra $\mathrm{RO}(P)$ for P a partially ordered set (see, e.g., [17, Section 4.2], for details). For $p \in P$, we define $u_p = P \downarrow p = \{q \in P : q \leqslant p\}$. The set $\{u_p : p \in P\}$ is the base of the *partial order topology* on P, and $\mathrm{RO}(P)$ is the regular open algebra of P with this topology. For $p \in P$, write $e(p) = \operatorname{int} \operatorname{cl}(u_p)$. We then have, for arbitrary p and q in P,

(3) if $q \leqslant p$, then $e(q) \leqslant e(p)$

(4) if p and q are incomparable, then $e(p) \cdot e(q) = 0$

(5) if P is separative (i.e., if for all $p, q \in P$ satisfying $q \nleqslant p$ there is $r \leqslant q$ incompatible with p) then $e(p) = u_p$.

Moreover,

(6) $D = \{e(p): p \in P\}$ is a dense subset of RO(P) (i.e. for every $x \in \mathrm{RO}(P)\setminus\{\emptyset\}$ there is $p \in P$ such that $e(p) \leqslant x$)

(7) if $P = T^{-1} = (T, \geqslant_T)$ where $(T, \leqslant_T)$ is a pseudo-tree, then

$$e(p) = \{q \in T\text{: every } t \geqslant_T q \text{ is comparable with } p\}.$$

The *restricted regular open algebra* RRO(P) of P is then defined to be the subalgebra of RO(P) generated by the elements $e(p)$, $p \in P$; a dense subalgebra of RO(P). Note that every Boolean algebra A is isomorphic to RRO(P), for some partial order P, since $A \cong \mathrm{RRO}(A^+)$.

PROPOSITION 4.1. *A Boolean algebra is isomorphic to the restricted regular open algebra of a pseudo-tree iff it is generated by a dense ramification set. In particular, it is then isomorphic to a pseudo-tree algebra, by Theorem* 2.3.

Proof. If $B = \mathrm{RRO}(T^{-1})$ for some pseudo-tree T, then $R = \{e(p): p \in T\}$ is, by (3) and (4) above, a ramification set generating B. R is dense in $\mathrm{RO}(T^{-1})$ and hence in B.

Conversely let B be generated by a dense ramification set $T \subseteq B^+$. Let $f: B \to \mathrm{RO}(T^{-1})$ be the unique embedding mapping $t \in T$ to $e(t)$. Now $f[B]$ is generated by $\{e(t): t \in T\}$, so $B \cong f[B] = \mathrm{RRO}(T^{-1})$. □

The question whether, conversely, every pseudo-tree algebra is isomorphic to $\mathrm{RRO}(T^{-1})$ for some pseudo-tree T, has different answers for trees and pseudo-trees.

PROPOSITION 4.2. *Every tree algebra Treealg S is isomorphic to RRO(T^{-1}), for some tree T. If Treealg S is atomless, i.e., if every element of S has infinitely many successors, then $T \cong S$, without loss of generality.*

Proof. Let $B =$ Treelag S where S has, without loss of generality, a single root. Then

$$R = \{S \uparrow s: s \in S\} \cup \{t \in B: t \text{ an atom of } B\}$$

is a conversely well-founded dense ramification set in B generating B. By 4.1, B is isomorphic to $\mathrm{RRO}(T^{-1})$ where $T = R^{-1}$. If B is atomless then clearly $T \cong S$. □

The analog of Proposition 4.2 for pseudo-trees does not hold.

EXAMPLE 3.3. *Let B be the interval algebra (=pseudo-tree algebra) of the half-open real unit interval* [0, 1). *Then B is not isomorphic to RRO(T^{-1}) for any pseudo-tree T.*

Proof. Suppose that, by 4.1, $T \subseteq B^+$ is a dense ramification set generating B. Since B has a countable dense subalgebra, it follows that T has a countable subset $\{t_n : n \in \omega\}$ which is dense in B.

We not construct, for $n \in \omega$, elements s_n of T and $I_n = [a_n, b_n)$ of $\mathrm{Int}(s_n)$; here $\mathrm{Int}(s_n)$ is the finite set of intervals whose union is s_n. Let $s_0 \in T$ and $I_0 \in \mathrm{Int}(s_0)$ be arbitrary. Having defined s_n and $I_n = [a_n, b_n)$, choose $r, s \in [0, 1)$ such that $a_n < r < s < b_n$ and $s_{n+1} \in T$ such that $s_{n+1} \subseteq [r, s)$ and, if $t_n \cap [r, s) \neq \emptyset$ then s_{n+1} is a proper subinterval of $t_n \cap [r, s)$. Then choose $I_{n+1} \in \mathrm{Int}(s_{n+1})$ arbitrarily. This finishes the construction. Note that $s_{n+1} \subseteq [r, s) \subset I_n \subseteq s_n$, so $s_n <_T s_{n+1}$, where $\leqslant_T$ is the converse of $\leqslant_B$, restricted to T.

Claim. For any $t \in T$, either $t \leqslant_T s_n$ for some n or t is incomparable with s_n for some n.

For, choose n such that $t_n \subseteq t$; thus $t \leqslant_T t_n$. If $t \leqslant_T s_{n+1}$ or t is incomparable with s_{n+1}, the claim is proved. Otherwise, $s_{n+1} <_T t \leqslant_T t_n$ and $t_n \subseteq s_{n+1}$, which contradicts the construction.

Now let, in the interval $[1, 0)$, $a = \sup_{n \in \omega} a_n$. By the claim, for all $t \in T$, either $t \leqslant_T s_n$ for some n, hence $s_n \subseteq t$ and a is an interior point of t, or t and s_n are incomparable for some n, hence $t \cap s_n = \emptyset$ and a is an interior point of s_n. In either case, a is not an end point of some interval of t. Hence $[0, a)$ is not generated by $T \subseteq B$, a contradiction. □

5. Regular Open Algebras of Pseudo-Trees

Our main result in this section is the equivalence of several constructions of complete Boolean algebras, including regular open algebras of pseudo-trees. These algebras seem to have been first considered in [9].

Except for the part involving minimally generated algebras, Theorem 5.1 in a more general topological form is essentially due to [22]; see also [7] who obtained the same results, evidently independently but later. The proofs in both papers are difficult to follow, and our specialization to Boolean algebras is not quite straightforward, so we give a self-contained proof here, which is, moreover, simpler, at least given the development of the theory of minimally generated algebras.

A Boolean algebra B is defined to be *minimally generated* if B can be represented as the union of an increasing continuous chain $(B_\alpha)_{\alpha<\rho}$ for some ordinal ρ such that B_0 is the two-element algebra and, for $\alpha + 1 < \rho$, $B_{\alpha+1}$ is *minimal* over B_α, i.e., there is no subalgebra of $B_{\alpha+1}$ lying properly between B_α and $B_{\alpha+1}$.

A fact we will use in the proof of Theorem 5.1 that is pseudo-tree algebras, i.e. algebras generated by ramification sets (see 2.3) are minimally generated. This follows from Theorem 3.1 and the facts established in [18] that interval algebras are minimally generated and that subalgebras of minimally generated ones are minimally generated. A more direct proof runs like this. Assume R is a ramification set generating B; well-order $R = \{x_\alpha : \alpha < \rho\}$ for some ordinal ρ and let B_α be the subalgebra of B generated by $\{x_\beta : \beta < \alpha\}$. Applying Lemma 1.1 and Proposition 1.3 from [18], it is easily shown that $B_{\alpha+1}$ is minimal over B_α.

THEOREM 5.1. *The following are equivalent, for a Boolean algebra A.*

(a) *A is isomorphic to $RO(T^{-1})$, for some tree T*

(b) *A is isomorphic to $RO(T^{-1})$, for some pseudo-tree T*

(c) *A is isomorphic to $(\text{Intalg } L)^{cm}$ for some linear order L (here B^{cm} denotes the completion of a Boolean algebra B)*

(d) *A is isomorphic to $(\text{Treealg } T)^{cm}$, for some pseudo-tree T*

(e) *A is isomorphic to $(\text{Treealg } T)^{cm}$, for some tree T*

(f) *A is isomorphic to B^{cm}, for some minimally generated Boolean algebra B.*

Proof. Trivially, (a) implies (b) and each of (c) and (e) implies (d).

(b) implies (c): the algebra $B = \text{RRO}(T^{-1})$ is dense in $\text{RO}(T^{-1})$ and, by 4.1, a pseudo-tree algebra. By Theorem 3.1, we can assume that B is a subalgebra of Intalg K, for some linear order K. Applying Corollary 5.10 of [17], we obtain an algebra C which has B as a dense subalgebra and is a homomorphic image of Intalg K. It follows that $C \cong \text{Intalg } L$, for some linear order L (see e.g. [17, 15.9]) and that $A \cong B^{cm} \cong C^{cm} \cong (\text{Intalg } L)^{cm}$.

(d) implies (f) since pseudo-tree algebras are minimally generated, as shown above.

(f) implies both (a) and (e): let B be minimally generated. By [18, 4.3] (and the remark after 2.4 of this paper), B has a dense subalgebra C which is isomorphic to Treelag S, for some tree S; thus $A \cong B^{cm} \cong (\text{Treealg } S)^{cm}$. Moreover, by 4.2, $C \cong \text{RRO}(T^{-1})$ for some tree T; thus $A \cong B^{cm} \cong C^{cm} \cong \text{RO}(T^{-1})$. □

There are many complete Boolean algebras not isomorphic to the regular open algebra of a pseudo-tree – see, e.g., [9], [14]. In fact, regular open algebras of pseudo-trees have quite special properties, as proved essentially in [14], [18]:

PROPOSITION 5.2. *Let $A = RO(T^{-1})$, T a pseudo-tree. Then*

(a) *$\pi(A)$ (the smallest cardinality of a dense subset of A) equals $c(A)$ (the cellularity of A, defined to be $\sup\{|X| : X$ a disjoint subset of $A\}$) or $c(A)^+$*

(b) *if κ is an infinite cardinal and $\pi(A \upharpoonright a) > \kappa$ for all $a \in A \setminus \{0\}$, then A satisfies the $(\kappa, 2)$-distributive law.*

There is another well-known construction of a complete Boolean algebra from a linear order L: endow L with its order topology having the open intervals $(-\infty, a)$, (a, ∞) (where $a \in L$) and (a, b) (where $a < b$ in L), as a base and consider the regular open algebra $\text{RO}(L)$ of L in the order topology. We have the following additonal part of William's theorem 5.1.

THEOREM 5.3. *The following are equivalent, for a Boolean algebra A.*

(a) *A is isomorphic to $(\text{Intalg } L)^{cm}$, for some linear order L (i.e., condition (c) of 5.1 holds).*

(*b*) *A is isomorphic to $RO(L)$, for some linear order L equipped with the order topology.*

Proof. (a) implies (b): this is obvious if L is a dense linear order, for in this case, the map sending $[a, b) \in \text{Intalg } L$ to $(a, b) \in \text{RO}(L)$ extends to an isomorphism from $(\text{Intalg } L)^{cm}$ onto $\text{RO}(L)$.

Otherwise, fix a partition X of unity in Intalg L such that each $x \in X$ is a half-open interval of L and $(\text{Intalg } L) \upharpoonright x$ is either the two-element algebra or atomless. Pick linear orders $(K_x, <_x)$ such that $(\text{Intalg } L) \upharpoonright x \cong \text{Intalg } K_x$, K_x has a first element and, if $|K_x| > 1$, then K_x is a dense linear order without last element. Let $<$ be a well-ordering of X such that, if $|K_x| = 1$ and $|K_y| > 1$ then $x > y$; $K = \bigcup_{x \in X} K_x$ is totally ordered by

$$s < t \text{ iff} [(s, t \in K_x \text{ and } s <_x t) \text{ or } (s \in K_x, t \in K_y \text{ and } x < y)].$$

Then

$$\begin{aligned}(\text{Intalg } L)^{cm} &\cong \prod_{x \in X} ((\text{Intalg } L) \upharpoonright x)^{cm} \\ &\cong \prod_{x \in X} (\text{Intalg } K_x)^{cm} \cong \prod_{x \in X} \text{RO}(K_x) \\ &\cong \text{RO}(K).\end{aligned}$$

(b) implies (a): if $A = \text{RO}(K)$ for a linear order K, then

$$C = \{\text{int cl}((-\infty, x)) : x \in K\}$$

is a chain in A. The subalgebra of A generated by C is (isomorphic to) an interval algebra and dense in A. Hence $A \cong (\text{Intalg } L)^{cm}$ for some linear order L. □

6. Pseudo-Tree-Generated Algebras

A Boolean algebra A is said to be (*pseudo-*) *tree-generated* if there is a (pseudo-) tree T, under the inverse of the Boolean ordering of A, such that A is generated by T. Note that we do not require here that incomparable elements of T are disjoint.

PROBLEM 2. Can one construct, in ZFC, a Boolean algebra which is not pseudo-tree-generated?

The following remarks shed some light on this question.

1. The supremum of cardinalities of incomparable families in a Boolean algebra A is the same as the supremum of the cardinalities of trees in A; see [19].

2. In [3] an interval algebra of power $\kappa = \text{cf}(2^\omega)$ is constructed which has no incomparable subset of size κ; this is an example of an algebra which is pseudo-tree-generated but not tree-generated.

3. Rubin's algebra constructed in [20] under ($\diamond$) is not pseudo-tree generated. In fact, any uncountable subset of this algebra has three distinct elements a, b, c such that $c = a \cdot b$; this is clearly not true in an uncountable algebra generated by the inverse of a pseudo-tree.

7. Tail Algebras and Disjunctively Generated Algebras

We consider here a generalization of pseudo-tree algebras. For $(P, \leq)$ a partial order and $p \in P$, define $P \uparrow p = \{q \in P : p \leqslant q\}$ and let Tailalg P, the *tail algebra* of P, be the subalgebra of the power set algebra of P generated by $R = \{P \uparrow p : p \in P\}$. Call a Boolean algebra *disjunctively generated* if it is generated by some disjunctive subset (as defined in Section 2).

Thus we have the following relations: every pseudo-tree algebra is a semigroup algebra (see 2.1(b)); semigroup algebras are isomorphic to tail algebras (see 7.1 below); tail algebras are disjunctively generated (see 2.1(c)). Our main results state that the largest class mentioned here (the class of disjunctively generated algebras) does not contain all Boolean algebras (by 7.3, resp. 7.4) but that, on the other hand, the (possibly smaller) class of tail algebras is quite close to the class of all Boolean algebras by Theorem 7.5.

PROPOSITION 7.1. *Every semigroup algebra is isomorphic to a tail algebra.*

Proof. Let the Boolean algebra A be generated by a subset H such that H is closed under multiplication, $0, 1 \in H$, and $P = H \backslash \{0\}$ is disjunctive. Consider the homomorphism $f = f_P$ from A onto Tailalg (P^{-1}) given by 2.2.(a), i.e., $f(p) = P \downarrow p$ for $p \in P$; we want to show that f is one-one. By Sikorski's criterion, we have to show that $f(p_1) \cap \cdots \cap f(p_n) \subseteq f(q_1) \cup \cdots \cup f(q_m)$ (where $p_i, q_j \in P$) implies $p_1 \cdot \ldots \cdot p_n \leqslant q_1 + \cdots + q_m$. Without loss of generality, $p = p_1 \cdot \ldots \cdot p_n$ is non-zero and hence in P. By definition of $f = f_P$ in 2.2(a), $p \in f(p_1) \cap \cdots \cap f(p_n)$. So $p \in f(q_j)$ for some j, thus $p \leqslant q_j$, and $p \leqslant q_1 + \cdots + q_m$, as desired. □

LEMMA 7.2. *Let P be an infinite partially ordered set. Then: either P has a strictly ascending chain of type ω, or P has a strictly descending chain of type ω, or P is well-founded (with, say, P_α as its α'th level) and there is some $n \in \omega$ such that P_n is infinite.*

Proof. Assume P has no descending chain of type ω (so P is well-founded) and no infinite level $P_n (n \in \omega)$. For each $n \in \omega$, let

$$T_n = \{(p_0, \ldots, p_n) : p_i \in P_i, \text{ for } i \leqslant n, \text{ and } p_0 < \cdots < p_n\}.$$

So $T = \bigcup_{n \in \omega} T_n$ is an infinite tree ordered by end extension of sequences in which every level is finite and non-empty. But then T has an infinite branch, which yields an increasing chain of type ω in P. □

THEOREM 7.3. *Every infinite disjunctively generated algebra has a countably infinite homomorphic image.*

Proof. Say $A = \langle P \rangle$, where P is an infinite disjunctive subset of A. We apply Lemma 7.2 to P^{-1}, and have three cases.

Case 1. There is in P an ascending sequence $(p_n : n \in \omega)$. Let then $M = \{p_n : n \in \omega\}$, and consider the homomorphism f_M given by 2.2(a). Then f_M maps each $p \in P$ to an initial segment of M, and since $f_M(p_n) = \{p_0, \ldots, p_n\}$ and P generates A, it follows that the image of A under f_M is the finite-cofinite algebra on M, a countable algebra.

Case 2. There is in P a descending sequence of type ω. This is similar to Case 1, again considering $M = \{p_n : n \in \omega\}$.

Case 3. P^{-1} is well-founded, and for some (minimal) $n \in \omega$, P_n is infinite. Consider $M = P_n$ and $f = f_M$ as in 2.2(a). Note that

1. $f(p) = \emptyset$ if $p \in P_\alpha, \alpha > n$
2. $f(p) = \{p\}$ for $p \in P_n$
3. $\{f(p) : p \in P_k, k < n\}$ is finite.

It follows that the image of A under f is superatomic, since its quotient under the ideal generated by the atoms is finite. It is well-known, and easy to check, that every superatomic algebra has a countable homomorphic image, giving the desired result. □

A related result is included in [11]: every infinite semigroup algebra has a countably infinite homomorphic image. In fact, more is true as pointed out by Heindorf: by [1, Thm. 3.2], every infinite subalgebra of a semigroup algebra has a countably infinite homomorphic image.

A statement slightly weaker than the following Corollary was proved, but not published, by A. Blass and S. Koppelberg, answering a question by G. Brenner. Recall that a Boolean algebra A is said to have the *countable separation property* if for any countable subsets X and Y of A satisfying $x \cdot y = 0$ for all $x \in X$ and $y \in Y$, there is some $a \in A$ such that $x \leqslant a$ and $y \leqslant -a$ for all $x \in X$ and $y \in Y$.

COROLLARY 7.4. *No infinite Boolean algebra having the countable separation property is disjunctively generated.*

Proof. The countable separation property is inherited by homomorphic images (see [17, 5.27]); moreover, no countably infinite algebra has the countable separation property. □

The following theorem is due to S. Koppelberg and L. Heindorf.

THEOREM 7.5. *Every Boolean algebra is a retract of a tail algebra, in particular, every Boolean algebra is embeddable into a tail algebra.*

Proof. This is trivial for a finite Boolean algebra B with, say, n atoms – just take a tree with n roots and no other points. So let B be an infinite Boolean algebra; we may assume that it is the algebra of clopen subsets of some Boolean space X.

For each $b \in B$, take two new points p_b, q_b such that the points p_b, q_b $(b \in B)$, are pairwise distinct and not in X. Then put

$$U = \{p_b, q_b : b \in B\}$$

$$P = U \cup X$$

and define a partial order on P by setting $p_b < x$ and $q_b < x$ for all $x \in b$. Thus, for $b \in B$, $P \uparrow p_b = \{p_b\} \cup b$, $P \uparrow q_b = \{q_b\} \cup b$ and $b = P \uparrow p_b \cap P \uparrow q_b \in \text{Tailalg } P$.

We define a map e from B into the power set algebra of P by fixing a non-isolated point x^* of X and putting $e(b) = b$ if $x^* \notin b$ and $e(b) = U \cup b$ if $x^* \in b$. It is easily checked that e embeds B into the power set algebra of P and that $e(b) = -e(-b) \in \text{Tailag } P$ if $x^* \notin b$; hence e is an embedding from B into Tailag P.

Now define

$$I = \{a \in \text{Tailalg } P : a \cap X \text{ consists of finitely many non-isolated points}\},$$

an ideal of Tailalg P. The theorem is a consequence of the following two statements:

(1) $e[B] \cap I = \{0\}$

(2) Tailalg P is generated by $e[B] \cup I$ (cf. [17, 15.21]).

To prove (1), assume that $b \in B$ and $e(b) \in I$. Now $b = e(b) \cap X$ is a clopen subset of X consisting of finitely many non-isolated points (since $e(b) \in I$), and it follows that $b = 0$.

To prove (2), we fix $p \in P$ with the aim of showing that $P \uparrow p$ is generated by $e[B] \cup I$.

Case 1. p is a non-siolated point of X. Then $P \uparrow p = \{p\} \in I$.

Case 2. p is an isolated point of X. Again $P \uparrow p = \{p\}$. Now $\{p\}$ is a clopen subset of X, hence is in B, and $e(\{p\}) = \{p\}$, as desired.

Case 3. $p = p_b$ with $b \in B$ and $x^* \notin b$ (the case $p = q_b$ is similar). Then $P \uparrow p = \{p\} \cup b$, and $b = e(b) \in e[B]$. Now $\{p_b\} = (P \uparrow p_b) \setminus (P \uparrow q_b)$, so $\{p\} \in$ Tailalg P and hence $\{p\} \in I$.

Case 4. $p = p_b$ with $b \in B$ and $x^* \in b$ (the case $p = q_b$ is similar). Then $P \uparrow p = \{p\} \cup b$, and $\{p\} \in I$ as in Case 3. Now $e(b) = U \cup b$, and so $b = e(b) \setminus U$; also $U \in \text{Tailalg } P$ since $U = P \setminus ((P \uparrow p_1) \cap (P \uparrow q_1))$. So $U \in I$. Hence again $P \uparrow p$ is generated by $e[B] \cup I$. □

8. Comparison of Some Properties of Boolean Algebras

By way of a partial summary of the above results, we give a diagram of the properties of Boolean algebras that we have dealt with.

A is a tree algebra $\qquad$ A is an interval algebra

A is a pseudo-tree algebra

A is a semigroup algebra

A is a tail algebra $\qquad$ A is minimally generated

A is disjunctively generated

A has a countable homomorphic image

Implications go from top to bottom, and the non-trivial ones have been proved in 2.1.(b), 7.1, 7.3, and a remark before 5.1. As concerns the reversability of the implications, and possibly some other implications, we make the following remarks. There is an interval algebra not isomorphic to a tree algebra: see [17, 16.21]. There is a tree algebra not isomorphic to an interval algebra: see [17, 16.22]. Every uncountable free algebra is a semigroup algebra but not minimally generated. Superatomic algebras are minimally generated, but not necessarily retractive, hence not necessarily isomorphic to pseudo-tree algebras. Algebras with a countably infinite homomorphic image are not necessarily disjunctively generated, as shown by the following example.

EXAMPLE 8.1. Let $C = A \times B$ where B is infinite and complete and A is countably infinite. We shall use the fact (see e.g. [15]) that B is not the union of a strictly ascending chain $(B_n)_{n \in \omega}$ of subalgebras B_n. C has A as a countably infinite homomorphic image; assume for the contradiction that it is generated by a disjunctive subset P. Write $A = \{a_n : n \in \omega\}$ and define, for $i \in \omega$,

$$P_i = \{p \in P : pr_1(p) \in \{a_0, \ldots, a_i\}\},$$

$$C_i = \text{the subalgebra of } C \text{ generated by } P_i$$

(where pr_1, pr_2 are the projections onto the first respectively second coordinate). Thus P is the union of the chain $(P_i)_{i \in \omega}$, C is the union of the chain $(C_i)_{i \in \omega}$, and B is the union of the chain $(pr_2[C_i])_{i \in \omega}$. By the above choice of B, there is $m \in \omega$ such that $pr_2[C_m] = B$; we can pick m so large that also the element $(1, 0)$ of C is in C_m.

Now C_m is disjunctively generated by $P_m \subseteq P$. On the other hand, $pr_1[C_m]$ is a finite Boolean algebra, $pr_2[C_m] = B$ is complete and, by $(1, 0) \in C_m$,

$$C_m = pr_1[C_m] \times pr_2[C_m],$$

a complete Boolean algebra. This contradicts Corollary 7.4.

Not every tail algebra is a semigroup algebra, as shown by Theorem 7.5, plus Theorem 2 and Proposition 1 of [11]. Finally, under $(\diamond)$, there is a minimally generated Boolean algebra with no countably infinite homomorphic image [16]. The remaining possibilities in the diagram are open, giving rise to the following problems.

PROBLEM 3. Is every disjunctively generated algebra isomorphic to a tail algebra?

PROBLEM 4. Is there in ZFC a minimally generated Boolean algebra with no countably infinite homomorphic image?

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