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Structure problems for cylindric algebras

J. DONALD MONK

Abstract. Much of the work that has been done in algebraic logic concerns the relationships between kinds of algebras, or global questions about varieties of algebras. Thus the relationships between relation algebras, cylindric algebras, and polyadic algebras is fairly well understood. The representation theory has been carefully studied, and the relationships between various types of representable algebras such as cylindric set algebras, generalized cylindric set algebras, etc., has been thoroughly investigated. This article is not concerned with any of these kinds of problems. Instead, we are interested here in intrinsic questions about cylindric algebras in general and the most important class of such algebras, the regular locally finite dimensional cylindric set algebras. In the selection of these problems we have been guided by two principles: to not go over old ground covered in the books HMT[71], HMT[85] and HMTAN[81], and to see what problem areas in the theory of Boolean algebras naturally transfer to cylindric algebras. We shall take the theory of Boolean algebras as known, although there are many questions open about them, too, of course. All the results about BA's which we need can be found by browsing through the BA handbook BA[88].

We survey some recent new results, with a few proofs, and mention some open problems. We will not mention related algebraic structures like relation algebras, except in passing. We use the notation of HMT[71], HMT[85] without recalling it (some of it is recalled in Maddux's introductory paper in the present volume, and in §2 of the "Open problems" paper).

1. Groups

Groups appear in cylindric algebras in at least two natural ways: as automorphism groups, and via a kind of Galois theory. First we consider the automorphism groups. The central result here is the following theorem of

Andréka and Németi (unpublished): for any $\alpha \geq 2$ and any group G, there is a $\mathsf{Cs}_\alpha^{\mathrm{reg}} \cap \mathsf{Lf}_\alpha$ $\mathfrak{A}$ such that $G \cong \mathrm{Aut}\mathfrak{A}$. Their proof is based on an idea of Maddux [85], who did the same thing for (in general non-representable) relation algebras. (They also proved a similar result for relation set algebras, and after hearing of their theorem, Maddux obtained this result by a different proof.)

There is still a small natural question remaining. What is the situation for CA_1's, or, more generally, for monadic-generated CA_α's? We give a few results for this case, but there are still open questions; the results are due to Andréka, Monk, and Németi. It is convenient to introduce some notation first, connected with the treatment of monadic-generated CA_α's in section 2.2 of HMT[71]. We adjoin to the language for CA_α's unary operation symbols a_κ for all $\kappa < (\alpha+1) \cap \omega$. In a CA_α, $\mathsf{a}_\kappa x$ denotes the element

$$\mathsf{c}_{(\kappa)}\left[\bar{\mathsf{d}}(\kappa \times \kappa) \cdot \prod_{\lambda<\kappa} \mathsf{s}_\lambda^0 x\right].$$

This element is 0-dimensional if $\Delta x \subseteq \{0\}$. A *special term* is a Boolean term built up from the following atomic parts: $\mathsf{d}_{\kappa\lambda}$, $\mathsf{s}_\lambda^0 v_i$, and $\mathsf{a}_\kappa u$, where u is a formal product of various terms v_i and $-v_i$. The *relevant indices* of a special term are the subscripts κ, λ occurring in the parts $\mathsf{d}_{\kappa\lambda}$ or $\mathsf{s}_\kappa^0 v_i$. A convenient consequence of Lemma 2.2.22 of HMT[71] in this terminology is as follows:

(*) If τ is a special term with variables among $v_0, \ldots, v_{m-1}$, and if $\kappa < \alpha$, then there is a special term σ with the following properties:

The variables of σ are also among $v_0, \ldots, v_{m-1}$;

The relevant indices of σ are those of τ except for κ;

For any CA_α $\mathfrak{A}$ and any elements $x_0, \ldots, x_{m-1}$ of A such that $\Delta x_i \subseteq 1$ for all $i < m$ we have $\mathsf{c}_\kappa \tau(x_0, \ldots, x_{m-1}) = \sigma(x_0, \ldots, x_{m-1})$.

We also need the following notation. $\mathrm{Aut}[\mathbf{K}]$ is the class of all automorphism groups of members of $\mathbf{K}$; thus $\mathsf{I}\mathrm{Aut}[\mathbf{K}]$ is the class of all groups isomorphic to automorphism groups of members of $\mathbf{K}$. BA is the class of all Boolean algebras, and G is the class of all groups. For any class $\mathbf{K}$ of cylindric algebras, $\mathsf{Mg}\mathbf{K}$ is the class of all monadic-generated members of $\mathbf{K}$. If $\mathfrak{A}$ is monadic-generated, then $\mathfrak{M}\mathfrak{A}$ is the BA of elements $a \in A$ such that $\Delta a \subseteq \{0\}$.

The following lemma will be useful:

Lemma 1.1. *Suppose that $\alpha > 0$, $\mathfrak{A}$ and $\mathfrak{B}$ are monadic-generated CA_α's, and g is an isomorphism of $\mathfrak{M}\mathfrak{A}$ with $\mathfrak{M}\mathfrak{B}$. Then g extends to an isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$ iff $g\mathsf{a}_\kappa x = \mathsf{a}_\kappa gx$ whenever $\kappa < (\alpha+1)\cap\omega$ and $x \in M\mathfrak{A}$.*

Proof. $\Rightarrow$ is trivial, so we only consider $\Leftarrow$. First note that if τ is a special term with empty relevant index set, then $\tau(x_0, \ldots, x_{m-1}) = 0$ in $\mathfrak{A}$ iff $\tau(gx_0, \ldots, gx_{m-1}) = 0$ in $\mathfrak{B}$, for $x_0, \ldots, x_{m-1} \in M\mathfrak{A}$. It then follows that a product of elements of the set

$$\begin{gathered}\{\mathsf{d}_{\kappa\lambda} : \kappa, \lambda < \alpha\} \cup \{-\mathsf{d}_{\kappa\lambda} : \kappa, \lambda < \alpha\} \cup \\ \{\mathsf{s}^0_\xi x : x \in M\mathfrak{A}, \xi < \alpha\} \cup \{\mathsf{a}_\kappa x : x \in M\mathfrak{A},\ \kappa < (\alpha+1)\cap\omega\}\end{gathered}$$

is 0 iff the corresponding product with each x replaced by gx is 0. This is easily proved by induction on the number i of relevant indices in a product given by a special term, using $(*)$ and the hypothesis of the lemma. In fact, suppose that $\tau(x_0, \ldots, x_{m-1}) = 0$ is such a product. Our initial note takes care of the case $i = 0$, so suppose that $i \neq 0$. Let κ be any relevant index of τ. Choose σ by $(*)$. Then

$$\begin{aligned}\tau(x_0, \ldots, x_{m-1}) = 0 &\text{ iff } \mathsf{c}_\kappa\tau(x_0, \ldots, x_{m-1}) = 0 \\ &\text{ iff } \sigma(x_0, \ldots, x_{m-1}) = 0 \\ &\text{ iff } \sigma(gx_0, \ldots, gx_{m-1}) = 0 \\ &\text{ iff } \mathsf{c}_\kappa\tau(gx_0, \ldots, gx_{m-1}) = 0 \\ &\text{ iff } \tau(gx_0, \ldots, gx_{m-1}) = 0.\end{aligned}$$

By Sikorski's extension theorem it follows that g extends to an isomorphism of the Boolean part of $\mathfrak{A}$ onto the Boolean part of $\mathfrak{B}$, preserving diagonals and taking $\mathsf{s}^0_\xi x$ to $\mathsf{s}^0_\xi gx$ for all $x \in M\mathfrak{A}$. An application of $(*)$ again shows that g is a cylindric isomorphism. ∎

Theorem 1.2. *Assume that $\alpha > 0$. For any BA $\mathfrak{A}$ there is a $\mathsf{Mg}(\mathsf{Cs}^{\mathrm{reg}}_\alpha \cap \mathsf{Lf}_\alpha)$ $\mathfrak{B}$ such that $\mathfrak{A}$ is isomorphic to $\mathfrak{M}\mathfrak{B}$, and also $\mathrm{Aut}\mathfrak{A} \cong \mathrm{Aut}\mathfrak{B}$.*

Proof. We may assume that $\mathfrak{A}$ is an algebra of subsets of a set U, and that each non-zero member of A is infinite. Let C consist of all elements of $\mathcal{P}({}^\alpha U)$ having the form $\{a \in {}^\alpha U : a_0 \in x\}$ for some $x \in A$, and let $\mathfrak{B}$ be the Cs_α of subsets of ${}^\alpha U$ generated by C. Thus $\mathfrak{B}$ is monadic-generated. Note that C is the universe of $\mathfrak{M}\mathfrak{B}$; this follows from Theorem 2.2.24 of HMT[71]. So we let $\mathfrak{C} = \mathfrak{M}\mathfrak{B}$. Now $\mathfrak{A}$ is isomorphic to $\mathfrak{M}\mathfrak{B}$; a desired isomorphism is $a \mapsto \{x \in {}^\alpha U : x_0 \in a\}$.

Thus we can finish the proof by showing that $\mathrm{Aut}\mathfrak{B}$ is isomorphic to $\mathrm{Aut}\mathfrak{C}$. Given an automorphism f of $\mathfrak{B}$, the restriction of f to C is an automorphism of $\mathfrak{C}$; and that restriction has only f as an extension to an automorphism of $\mathfrak{B}$. That every automorphism of $\mathfrak{C}$ extends to an automorphism of $\mathfrak{B}$ is an easy consequence of Lemma 1.1. In fact, let g be an automorphism of $\mathfrak{C}$. Note that our assumption that all non-zero elements of A are infinite implies that an element $\mathsf{a}_\kappa x$ is 0 iff the element x is 0 ($x \in C$); and this is true iff the element gx is 0. Hence, since each element $\mathsf{a}_\kappa x$ is 0-dimensional, it follows that $g\mathsf{a}_\kappa x = \mathsf{a}_\kappa gx$. So Lemma 1.1 applies to show that g extends to an automorphism of $\mathfrak{B}$. ∎

Lemma 1.3. *Suppose $0 < \alpha < \omega$ and $\mathfrak{A}$ is a monadic-generated $\mathsf{Mg}(\mathsf{Cs}^{\mathrm{reg}}_\alpha \cap \mathsf{Lf}_\alpha)$. Then $\mathfrak{A}$ is isomorphic to a $\mathsf{Mg}(\mathsf{Cs}^{\mathrm{reg}}_\alpha \cap \mathsf{Lf}_\alpha)$ $\mathfrak{B}$ such that for every atom x of $\mathfrak{M}\mathfrak{B}$, if $\mathsf{a}_\alpha x = 1$, then $\{u_0 : u \in x\}$ is infinite.*

Proof. Say that $\mathfrak{A}$ has base U. For each $y \in M\mathfrak{A}$ let $V_y = \{u_0 : u \in y\}$. Note that if y is atomless in $\mathfrak{M}\mathfrak{A}$, then V_y is infinite; but if y is an atom, then V_y can have any cardinality. Now we associate with each atom $y \in M\mathfrak{A}$ such that $\alpha \leq |V_y| < \omega$ an infinite superset V'_y of V_y such that $(V'_y \cap U) \setminus V_y = 0$ and $V'_y \cap V'_z = 0$ for distinct y, z. Then let $U' = U \cup \bigcup\{V'_y : y \text{ is an atom of } \mathfrak{M}\mathfrak{A} \text{ with } \alpha \leq |V_y| < \omega\}$. For each $a \in M\mathfrak{A}$ let $W_a = \{u_0 : u \in a\} \cup \bigcup\{V'_y : y \text{ is an atom of } \mathfrak{M}\mathfrak{A},\ \alpha \leq |V_y| < \omega, \text{ and } y \leq a\}$. Then let $fa = \{u \in {}^\alpha U' : u_0 \in W_a\}$. It is easily seen that f extends to the desired isomorphism, using Lemma 1.1. ∎

Now we are ready for our main positive result about automorphism groups of monadic-generated cylindric algebras.

Theorem 1.4. (i) $\mathbf{I}\mathrm{Aut}[\mathsf{BA}] \subseteq \mathbf{I}\mathrm{Aut}[\mathsf{Mg}(\mathsf{Cs}^{\mathrm{reg}}_\alpha \cap \mathsf{Lf}_\alpha)] \subseteq \mathbf{I}\mathrm{Aut}[\mathsf{Mg}(\mathsf{CA}_\alpha)] \subseteq \mathsf{G}$.

(ii) *For $\alpha < \beta$ we have $\mathbf{I}\mathrm{Aut}[\mathsf{Mg}(\mathsf{Cs}^{\mathrm{reg}}_\alpha \cap \mathsf{Lf}_\alpha)] \subseteq \mathbf{I}\mathrm{Aut}[\mathsf{Mg}(\mathsf{Cs}^{\mathrm{reg}}_\beta \cap \mathsf{Lf}_\beta)]$.*

(iii) *For $1 \leq \alpha$ we have $\mathbf{I}\mathrm{Aut}[\mathsf{CA}_1] \subseteq \mathbf{I}\mathrm{Aut}[\mathsf{Mg}(\mathsf{CA}_\alpha)]$.*

(iv) *For $\alpha, \beta \geq \omega$ we have equality in* (ii), *and also* $\mathbf{I}\mathrm{Aut}[\mathsf{Mg}(\mathsf{CA}_\alpha)] = \mathbf{I}\mathrm{Aut}[\mathsf{Mg}(\mathsf{CA}_\beta)]$.

Proof. (i) is an immediate consequence of Theorem 1.2. For (ii), suppose that $\mathfrak{A}$ is a $\mathsf{Mg}(\mathsf{Cs}^{\mathrm{reg}}_\alpha \cap \mathsf{Lf}_\alpha)$; we want to construct a $\mathsf{Mg}(\mathsf{Cs}^{\mathrm{reg}}_\beta \cap \mathsf{Lf}_\beta)$ $\mathfrak{B}$ with the same automorphism group as $\mathfrak{A}$. If $\alpha < \omega$ we may assume that $\mathfrak{A}$ has the property described in Lemma 1.3. For each $a \in M\mathfrak{A}$ let $ga = \{u \in {}^\beta U : u \restriction \alpha \in a\}$, and let $\mathfrak{B}$ be the Cs_β generated by $g[A]$. Clearly g is an isomorphism from $\mathfrak{M}\mathfrak{A}$ onto $\mathfrak{M}\mathfrak{B}$. Let f be an automorphism of $\mathfrak{M}\mathfrak{A}$. It suffices to show that the two conditions

$$f \text{ extends to an automorphism of } \mathfrak{A}$$
$$g \circ f \circ g^{-1} \text{ extends to an automorphism of } \mathfrak{B}$$

are equivalent. First suppose that f extends to an automorphism of $\mathfrak{A}$, and $\mathsf{a}_\kappa ga = 0$ for some $\kappa < (\beta+1) \cap \omega$ and $a \in M\mathfrak{A}$; we want to show that $\mathsf{a}_\kappa gfa = 0$. Since $\mathsf{a}_\kappa ga = 0$, it follows that there are atoms $x_0, \ldots, x_{m-1}$ of $\mathfrak{M}\mathfrak{A}$ such that $a = x_0 + \cdots + x_{m-1}$, and if $\alpha < \omega$ the special property given by Lemma 1.3 implies that $\mathsf{a}_\alpha x_i = 0$ for all $i < m$. An easy argument then shows in any case that $\mathsf{a}_\kappa gfa = 0$, as desired. Conversely, if $\mathsf{a}_\kappa gfa = 0$, then $\mathsf{a}_\kappa ga = 0$ by applying the argument just given to f^{-1} and $g \circ f^{-1} \circ g^{-1}$.

On the other hand, suppose that $g \circ f \circ g^{-1}$ extends to an automorphism of $\mathfrak{B}$, and $\kappa < (\alpha+1) \cap \omega$ with $\mathsf{a}_\kappa a = 0$. Clearly then $\mathsf{a}_\kappa ga = 0$, and so $\mathsf{a}_\kappa gfa = 0$. Thus $\mathsf{a}_\kappa fa = 0$, as desired. For the converse one proceeds as before.

For (iii), let $\mathfrak{A}$ be any CA_1. We may assume that $\mathfrak{A}$ is a subdirect product of $\mathfrak{B}_i$, $i \in I$, where each $\mathfrak{B}_i$ is a Cs_1 with base U_i, $U_i \cap U_j = 0$ for $i \neq j$, and each non-zero element of $\mathfrak{B}_i$ is infinite. For each $a \in A$ let $ga = \bigcup_{i \in I} \{u \in {}^\alpha U_i : u \restriction 1 \in a_i\}$, and let $\mathfrak{C}$ be the Gs_α generated by $g[A]$. So $\mathfrak{C}$ is a $\mathsf{Mg}(\mathsf{CA}_\alpha)$. Clearly g is an isomorphism from $\mathfrak{M}\mathfrak{A}$ onto $\mathfrak{M}\mathfrak{C}$. Next, we claim

$$\mathsf{a}_\kappa ga = g\mathsf{c}_0 a \text{ for any } 0 < \kappa < (\alpha+1) \cap \omega \text{ and any } a \in A.$$

In fact,

$$\begin{aligned}
\mathsf{a}_\kappa ga \cap {}^\alpha U_i = {}^\alpha U_i \ &\text{iff } |\{u_0 : u \in {}^\alpha U_i \cap ga\}| \geq \kappa \\
&\text{iff } |\{u_0 : u \in {}^\alpha U_i,\ u \restriction 1 \in a_i\}| \geq \kappa \\
&\text{iff } a_i \neq 0 \\
&\text{iff } (\mathsf{c}_0 a)_i = {}^1 U_i \\
&\text{iff } g\mathsf{c}_0 \cap {}^\alpha U_i = {}^\alpha U_i,
\end{aligned}$$

as desired.

Now let f be an automorphism of $\mathfrak{M}\mathfrak{A}$. Then f is an automorphism of $\mathfrak{A}$ iff $g \circ f \circ g^{-1}$ extends to an automorphism of $\mathfrak{C}$. In fact, for $\Rightarrow$ we have

$$gfg^{-1}\mathsf{a}_\kappa ga = gf\mathsf{c}_0 a = g\mathsf{c}_0 fa = \mathsf{a}_\kappa gfa,$$

and for $\Leftarrow$ we have

$$f\mathsf{c}_0a = g^{-1}gfg^{-1}g\mathsf{c}_0a = g^{-1}gfg^{-1}\mathsf{a}_1ga = g^{-1}\mathsf{a}_1gfa = \mathsf{c}_0fa.$$

Next we consider the first part of (iv). Say $\omega \leq \alpha < \beta$. Given $\mathfrak{B} \in \mathsf{Mg}(\mathsf{Cs}_\beta^{\mathrm{reg}} \cap \mathsf{Lf}_\beta)$, say with base U, let for $b \in M\mathfrak{B}$ $gb = \{u \in {}^\alpha U : u_0 = v_0$ for some $v \in b\}$, and let $\mathfrak{A}$ be the Cs_α generated by $g[M\mathfrak{B}]$. It is easy to check that g is an isomorphism of $\mathfrak{M}\mathfrak{B}$ with $\mathfrak{M}\mathfrak{A}$, and $g\mathsf{a}_\kappa b = \mathsf{a}_\kappa gb$ for all $\kappa < \omega$ and all $b \in M\mathfrak{B}$. Hence an easy argument shows that an automorphism f of $\mathfrak{M}\mathfrak{B}$ extends to an automorphism of $\mathfrak{B}$ iff $g \circ f \circ g^{-1}$ extends to an automorphism of $\mathfrak{A}$, as desired.

Finally, we take the second part of (iv). Suppose that $\omega \leq \alpha < \beta$. First suppose that $\mathfrak{A}$ is a $\mathsf{Mg}(\mathsf{CA}_\alpha)$; say $\mathfrak{A}$ is a Gs_α with unit element $\bigcup_{i\in I} {}^\alpha U_i$, $U_i \cap U_j = 0$ for $i \neq j$, the relativization of $\mathfrak{A}$ to ${}^\alpha U_i$ being regular for each $i \in I$. Let $ga = \{u \in \bigcup_{i\in I} {}^\beta U_i : u \restriction \alpha \in a\}$ for each $a \in M\mathfrak{A}$; then proceed as above. Second, suppose that $\mathfrak{B}$ is a Gs_β with unit element $\bigcup_{i\in I} {}^\beta U_i$, $U_i \cap U_j = 0$ for $i \neq j$, the relativization of $\mathfrak{B}$ to ${}^\beta U_i$ being regular for each $i \in I$. Let $gb = \{u \restriction \alpha : u \in b\}$ for each $b \in M\mathfrak{B}$; then proceed as above. ∎

Now we consider possible improvements of Theorem 1.4. Obviously $\mathbf{I}\mathrm{Aut}[\mathsf{BA}] = \mathbf{I}\mathrm{Aut}[\mathsf{Mg}(\mathsf{Cs}_1^{\mathrm{reg}} \cap \mathsf{Lf}_1)]$. For $\alpha > 1$, equality no longer holds:

Proposition 1.5. *For $\alpha > 1$ we have $\mathbf{I}\mathrm{Aut}[\mathsf{BA}] \subset \mathbf{I}\mathrm{Aut}[\mathsf{Mg}(\mathsf{Cs}_\alpha^{\mathrm{reg}} \cap \mathsf{Lf}_\alpha)]$.*

Proof. By Boolean-algebraic results it is enough to show that there is a monadic-generated $\mathsf{Cs}_\alpha^{\mathrm{reg}} \cap \mathsf{Lf}_\alpha$ with automorphism group of size 4. This is easy: let U be an infinite set, and pick two elements of U, u and w, and let the remaining elements be divided into two equal-sized infinite parts U' and U''. Let $x = \{a \in {}^\alpha U : a_0 = u\}$, and let y be defined similarly using w. And let $z_0 = \{a \in {}^\alpha U : a_0 \in U'\}$, and define z_1 similarly using U''. Take the CA_α $\mathfrak{A}$ generated by $\{x, y, z_0, z_1\}$. It is easy to check that $\mathfrak{A}$ is as desired, using base automorphisms. ∎

Also, not every group is isomorphic to the automorphism group of a monadic-generated cylindric algebra:

Proposition 1.6. $\mathbf{I}\mathrm{Aut}[\mathsf{Mg}(\mathsf{CA}_\alpha)] \neq \mathsf{G}$.

Proof. We show that, like BA's, every non-trivial automorphism group of a monadic-generated CA_α has a member of order 2. To prove this, let $\mathfrak{A}$ be a monadic-generated CA_α with non-trivial automorphism group. We consider two cases.

Case 1. There is a zero-dimensional element moved by some automorphism. Then by arguing in the easy way one does for BA's, there exist disjoint non-zero zero-dimensional elements a and b such that $fa = b$ for some automorphism f. The following automorphism g is then non-trivial and of order 2: $x \mapsto f(x \cdot a) + f^{-1}(x \cdot b) + x \cdot -a \cdot -b$.

Case 2. All zero-dimensional elements are fixed by all automorphisms. Now there exist an automorphism f and non-zero disjoint elements a and b such that $\Delta a = 1$, $\Delta b = 1$, and $fa = b$. We define the function g on $M\mathfrak{A}$ just as in case 1; g is at least an automorphism of $\mathfrak{M}A$ of order 2. We want to show that it extends to an automorphism h of $\mathfrak{A}$. This is seen similarly to the above; the new facts to observe are the following, valid for any $x \in M\mathfrak{A}$:

$$a_\kappa x = fa_\kappa x = a_\kappa fx; \;\; a_\kappa x = a_\kappa gx;$$

The first part is clear. The second part follows upon noticing that if u and v are disjoint members of $Nr_1\mathfrak{A}$ then

$$\mathsf{c}_{(\kappa)}\left(\overline{\mathsf{d}}(\kappa \times \kappa) \cdot \prod_{\lambda<\kappa} \mathsf{s}^0_\lambda(u+v)\right) = \sum_{\mu+\nu=\kappa}\left[\mathsf{c}_{(\mu)}\left(\overline{\mathsf{d}}(\mu \times \mu) \cdot \prod_{\lambda<\mu} \mathsf{s}^0_\lambda u\right) \cdot \mathsf{c}_{(\nu)}\left(\overline{\mathsf{d}}(\nu \times \nu) \cdot \prod_{\lambda<\nu} \mathsf{s}^0_\lambda v\right)\right];$$

and this is easily checked by verifying it in set algebras.

So h exists by Lemma 1.1. ∎

Now we make some observations about monadic-generated set algebras, leading up to the next proposition. Let $\mathfrak{A}$ be a monadic-generated Cs_α with base U, and let $\mathfrak{B} = \mathfrak{M}\mathfrak{A}$. Then with each $b \in B$ we can associate a subset tb of U such that $b = \{u \in {}^\alpha U : u_0 \in tb\}$. Note that if tb is finite, then there are atoms $c_0, \ldots, c_{m-1}$ of $\mathfrak{B}$ such that $b = c_0 + \cdots + c_{m-1}$ and $\langle tc_i : i < m\rangle$ is a partition of tb.

Proposition 1.7. *Assume the above notation, and suppose that f maps B into B. Then f extends to an automorphism of $\mathfrak{A}$ iff f is an automorphism of $\mathfrak{B}$ and for all $0 < \kappa < \alpha \cap \omega$, f permutes $\{b : b$ is an atom of $\mathfrak{B}$ and $|tb| = \kappa\}$.*

Proof. For all such κ, $|tb| = \kappa$ iff

$$\mathsf{c}_{(\kappa)}\left(\bar{d}(\kappa \times \kappa) \cdot \prod_{\lambda<\kappa} \mathsf{s}^0_\lambda b\right) \cdot -\mathsf{c}_{(\kappa+1)}\left(\bar{d}((\kappa+1) \times (\kappa+1)) \cdot \prod_{\lambda<\kappa+1} \mathsf{s}^0_\lambda b\right) = 1.$$

From this observation, $\Rightarrow$ in the theorem is clear. For $\Leftarrow$, we apply Lemma 1.1. To do this, it suffices to show that $\mathsf{a}_\kappa b = 0$ implies that $\mathsf{a}_\kappa fb = 0$. By the hypothesis and the above observation, there are atoms $c_0, \ldots, c_{m-1}$ of $\mathfrak{B}$ such that $b = c_0 + \cdots + c_{m-1}$. And the hypothesis for our direction then implies easily that $\mathsf{a}_\kappa fb = 0$. ∎

Corollary 1.8. *Suppose that $0 < \alpha < \omega$ and G is a finite group. Then G is isomorphic to the automorphism group of some $\mathsf{Mg}(\mathsf{Cs}_\alpha)$ iff G is isomorphic to the direct product of α many symmetric groups.*

Proof. $\Rightarrow$. Suppose G is the automorphism group of $\mathfrak{A}$, with the above notation ($\mathfrak{B}$, etc.) For each $0 < \kappa < \alpha$ let $X_\kappa = \{b : b \text{ is an atom of } \mathfrak{B} \text{ and } |tb| = \kappa\}$. Thus by Proposition 1.7, the set X_κ is finite. Let $b = -(\sum_{\kappa<\alpha} \sum X_\kappa)$. Then Proposition 1.7 says that G is isomorphic to the direct product of all the symmetric groups on X_κ with the automorphism group of $\mathfrak{B} \upharpoonright \mathfrak{B} \upharpoonright b$. The result now follows from Boolean algebraic facts.

$\Leftarrow$ is easily seen by a direct construction, using Proposition 1.7. ∎

Corollary 1.9. *For $\omega > \alpha < \beta$ we have*

$$\mathbf{I}\mathrm{Aut}[\mathsf{Mg}(\mathsf{Cs}_\alpha^{\mathrm{reg}} \cap \mathsf{Lf}_\alpha)] \neq \mathbf{I}\mathrm{Aut}[\mathsf{Mg}(\mathsf{Cs}_\beta^{\mathrm{reg}} \cap \mathsf{Lf}_\beta)].$$

Proof. According to Corollary 1.8, $\mathbf{I}\mathrm{Aut}[\mathsf{Mg}(\mathsf{Cs}_\beta^{\mathrm{reg}} \cap \mathsf{Lf}_\beta)]$ contains a group of size $2^{\alpha+1}$ but there is no such group in $\mathbf{I}\mathrm{Aut}[\mathsf{Mg}(\mathsf{Cs}_\alpha^{\mathrm{reg}} \cap \mathsf{Lf}_\alpha)]$. ∎

Proposition 1.10. *For $0 < \alpha < \omega$ and $0 < \beta$ we have $\mathbf{I}\mathrm{Aut}[\mathsf{Mg}(\mathsf{Cs}_\alpha^{\mathrm{reg}} \cap \mathsf{Lf}_\alpha)] \not\subseteq \mathbf{I}\mathrm{Aut}[\mathsf{Mg}(\mathsf{CA}_\beta)]$.*

Proof. Let $\langle \mathfrak{A}_\gamma : \gamma < \alpha + 2 \rangle$ be a system of pairwise non-isomorphic rigid infinite BA's, and let $\mathfrak{B}$ be a four-element BA. By Theorem 1.2, for each $\gamma < \alpha + 2$ let $\mathfrak{C}_\gamma$ be a $\mathsf{Mg}(\mathsf{Cs}_\beta)$ with the properties mentioned there relative to the BA $\mathfrak{A}_\gamma \times \mathfrak{B}$. Then $\prod_{\gamma<\alpha+2} \mathfrak{C}_\gamma$ is the algebra desired. ∎

As mentioned at the beginning, there are still some open problems here.

Problem 1. Give an abstract characterization of the automorphism groups of monadic-generated CA_α's, at least relative to the (abstractly unknown) class of automorphism groups of BA's.

Problem 2.
For $0 < \alpha < \omega$ and $\alpha < \beta$, is $\mathbf{I}\mathrm{Aut}[\mathsf{Mg}(\mathsf{CA}_\alpha)] \subseteq \mathbf{I}\mathrm{Aut}[\mathsf{Mg}(\mathsf{CA}_\beta)]$?
For $\alpha \geq \omega$, is $\mathbf{I}\mathrm{Aut}[\mathsf{Mg}(\mathsf{Cs}_\alpha^{\mathrm{reg}} \cap \mathsf{Lf}_\alpha)] = \mathbf{I}\mathrm{Aut}[\mathsf{Mg}(\mathsf{CA}_\alpha)]$?

The other place in the theory of cylindric algebras where groups naturally appear is in a kind of Galois theory. We do not have any definite problems to mention here, but we would like to call the reader's attention to this little area of algebraic logic, which has not received a systematic treatment in any of the books. So we will indicate the basic notions involved, and state without proof two of the main theorems involving them.

Let α be an ordinal and U a non-empty set. We denote by Sym U the group of all permutations of U, and by $\mathfrak{P}U$ the CA_α of all subsets of ${}^\alpha U$. With each $g \in$ Sym U we can associate the base automorphism $\tilde{g}$ of $\mathfrak{P}U$ defined by $\tilde{g}X = \{y \in {}^\alpha U : g^{-1} \circ y \in X\}$. Now if G is a subgroup of Sym U, we define $Fx^\alpha_{lf}G$ to be the set of all $X \subseteq {}^\alpha U$ such that ΔX is finite and $\tilde{g}X = X$ for all $g \in G$. It is easy to see that $Fx^\alpha_{lf}G$ is the universe of a $\mathsf{Cs}^{\mathrm{reg}}_\alpha \cap \mathsf{Lf}_\alpha$, which we shall denote by $\mathfrak{Fr}^\alpha_{lf}G$. Here is the first theorem we want to mention:

Theorem. (Daigneault [64], Comer [84]) *Suppose that U is a finite non-empty set and $|U| \leq \alpha+1$. Then $\mathfrak{Fr}^\alpha_{lf}$ is an antiisomorphism from the lattice of all subgroups of* Sym U *onto the lattice of all locally finite regular set algebras with base U*

This theorem is similar to a much earlier theorem of Krasner [38] which can be formulated in terms of (non locally-finite) polyadic algebras. The condition $|U| \leq \alpha + 1$ is really necessary; see Driessel [68] and Comer [84].

Of course it is natural to consider also the case in which U is infinite. There are then some obvious restrictions on both the set algebras and the groups involved in the theorem. Call a set algebra $\mathfrak{A}$ *locally complete* if for every finite J, the BA of elements x with $\Delta x \subseteq J$ is closed under arbitrary unions. It is obvious that for any subgroup G of Sym U the algebra $\mathfrak{Fr}^\alpha_{lf}G$ is locally complete. Call a subgroup G of Sym U *closed* if for every $f \in$ Sym U, if for every finite $F \subseteq U$ there is a $g \in G$ such that $f \restriction F = g \restriction F$, then $f \in G$. This really is a closure operation (the intersection of arbitrarily many closed sets is closed). And it is easy to check that if H is the closure of G, then $\mathfrak{Fr}^\alpha_{lf}G = \mathfrak{Fr}^\alpha_{lf}H$. Thus it is natural in trying to extend the above theorem to restrict to closed groups.

Theorem. (Carter, Driessel) *If $\aleph_0 = |U| \leq \alpha$, then $\mathfrak{Fr}^\alpha_{lf}$ is an antiisomorphism from the lattice of all closed subgroups of* Sym U *onto the lattice of all locally complete locally finite regular set algebras with base U.*

For this theorem, see Driessel [68], where also an example showing that the condition $|U| \leq \alpha$ is necessary can be found.

For uncountable U there are some results of Reyes [70].

Also, the Galois theory has been investigated for relation algebras. See Jónsson [84] and Wielandt [69].

2. Endomorphisms

For any cylindric algebra $\mathfrak{A}$ we let $\mathrm{End}\mathfrak{A}$ be the endomorphism monoid of $\mathfrak{A}$. Note, first of all, that $\mathrm{Aut}\mathfrak{A}$ is first-order definable in $\mathrm{End}\mathfrak{A}$—as the set of all invertible elements.

First we consider regular locally finite set algebras. They are all simple, and so the endomorphism monoid has the special property that all elements are left-cancellative ($ab = ac$ implies that $b = c$). So the following question is natural.

Problem 3. For $\alpha > 1$, is $\mathbf{I}\mathrm{End}[\mathsf{Cs}_\alpha^{\mathrm{reg}} \cap \mathsf{Lf}_\alpha]$ the class of all left-cancellative monoids?

Of course this is not the case for $\alpha = 1$, because of the above group results.

Looking for a representation of more general monoids, it is natural to consider the whole class of CA_α's:

Problem 4. For $\alpha > 1$, is $\mathbf{I}\mathrm{End}[\mathsf{CA}_\alpha]$ the class of all monoids? This might be interesting also for the class of representable CA_α's.

Optimists — who think the answer to Problem 4 is yes — might want to try for the stronger result that the category of CA_α's is alg-universal (see Pultr, Trnková [80]).

There is one more thing about endomorphisms which is worth mentioning. We have the notion of rigidity for Boolean algebras — it means that the BA has only the identity automorphism. There is the general algebraic notion of *endo-rigidity* — meaning that the algebra has only the identity endomorphism. For BA's, this notion is vacuous — there are no such things which are non-trivial since, for example, every BA has lots of two-valued endomorphisms. (So, Boolean algebraists have used the term *endo-rigid* for a weaker notion.) Perhaps it is a surprise that for CA_α's they exist (this theorem and its corollary are essentially due to Andréka and Németi, unpublished):

Theorem 2.1. *For each $\alpha > 0$ there is a non-trivial endo-rigid $\mathsf{Cs}_\alpha^{\mathrm{reg}} \cap \mathsf{Lf}_\alpha$.*

Proof. Let $\mathfrak{A}$ be an infinite mono-rigid BA. We may assume that $\mathfrak{A}$ is a set algebra of subsets of a set U. For each $a \in A$ let

$$x_a = \{u \in {}^\alpha U : u_0 \in a\}.$$

Let $\mathfrak{B}$ be the subalgebra of the Cs_α of all subsets of ${}^\alpha U$ generated by all of these sets x_a for $a \in A$. If f is an endomorphism of $\mathfrak{B}$ it is one-one, and $f \restriction M\mathfrak{B}$ is an endomorphism of $M\mathfrak{B}$. But $\mathfrak{M}\mathfrak{B}$ is isomorphic to $\mathfrak{A}$, so $f \restriction M\mathfrak{B}$ must be the identity, and so the same is true of f itself. ∎

The following obvious corollary of this theorem shows another respect in which the behaviour of endomorphisms is different for CA's as opposed to BA's:

Corollary 2.2. *Let $\alpha > 0$. There are non-isomorphic non-trivial $\mathsf{Cs}_\alpha^{\mathrm{reg}} \cap \mathsf{Lf}_\alpha$'s with isomorphic endomorphism monoids.* ∎

3. Subalgebras

The first question that occurs about subalgebras is

Problem 5. Characterize the lattices of subalgebras of CA_α's and $\mathsf{Cs}_\alpha^{\mathrm{reg}} \cap \mathsf{Lf}_\alpha$'s in lattice-theoretic terms, for $\alpha > 0$ and $\alpha > 1$ respectively.

Recall in this connection that the lattice of subalgebras of BA's has a fairly simple characterization, which holds also, of course, for $\mathsf{Cs}_1^{\mathrm{reg}} \cap \mathsf{Lf}_1$'s. See Monk [88] or Grätzer, Koh, Makkai [72].

Now we turn to three related concepts involving subalgebras: cofinality, Jónsson algebras, and the descending chain condition for subalgebras.

The cofinality of a CA_α $\mathfrak{A}$, denoted by cf $\mathfrak{A}$, is the smallest infinite cardinal κ such that $\mathfrak{A}$ has a strictly ascending sequence of subalgebras with union $\mathfrak{A}$, or ∞ if no such cardinal exists. In fact, cf $\mathfrak{A} < \infty$ iff $\mathfrak{A}$ is not finitely generated. For a general reference concerning this notion see Gould, Morel, Tsinakis [86], and for the cofinality of BA's, see Koppelberg [77], van Douwen [89], and Just [88]. It is obvious that $\mathrm{cf}(\mathfrak{B}\mathfrak{l}\,\mathfrak{A}) \leq \mathrm{cf}\,\mathfrak{A}$ for any CA_α $\mathfrak{A}$.

Theorem 3.1. *If $\mathfrak{A}$ is an infinite CA_1, then* $\operatorname{cf}\mathfrak{A} \leq 2^\omega$.

Proof. *Case 1.* $\mathfrak{A}$ has an infinite simple homomorphic image $\mathfrak{B}$. From the simple form of simple CA_1's and from Boolean algebraic results it follows that $\operatorname{cf}\mathfrak{B} \leq 2^\omega$, and so the same is true for $\mathfrak{A}$.

Case 2. All simple homomorphic images of $\mathfrak{A}$ are finite. It follows that $\mathsf{c}_0[A]$ is infinite, and hence there is a system $\langle x_i : i < \omega\rangle$ of disjoint nonzero c_0-closed elements. For each $i < \omega$ let I_i be a maximal ideal such that $-x_i \in I_i$. Since $-x_i + -x_j = 1$ for distinct $i, j < \omega$, it follows that for distinct $i, j < \omega$ we have $x_i \in I_j$, and hence the natural homomorphism f from $\mathfrak{A}$ into $\prod_{i<\omega}\mathfrak{A}/I_i$ has infinite image. This shows that $\mathfrak{A}$ has a homomorphic image of power $\leq 2^\omega$, and the theorem follows again. ∎

A *Jónsson algebra* is an infinite algebra $\mathfrak{A}$ with more elements than fundamental operations which has no proper subalgebra of size $|A|$. This differs slightly from the usual notion. Note that a Jónsson algebra has cofinality $\operatorname{cf}|A|$.

Theorem 3.2. *Suppose that $\alpha \geq 3$, κ is a regular cardinal $> |\alpha| \cup \omega$, and there is a Jónsson groupoid of power κ. Then there is a $\mathsf{Cs}_\alpha^{\mathrm{reg}} \cap \mathsf{Lf}_\alpha$ with cofinality κ.*

Proof. Let $\langle U, \cdot\rangle$ be a Jónsson groupoid with $|U| = \kappa$. For each $u \in U$ let

$$a_u = \{x \in {}^\alpha U : x_0 = u\},\ b = \{x \in {}^\alpha U : x_2 = x_0 \cdot x_1\}.$$

Let $\mathfrak{A}$ be the Cs_α of subsets of ${}^\alpha U$ generated by all of these elements. Clearly $\operatorname{cf}\mathfrak{A} \leq \kappa$. Suppose that $\langle \mathfrak{B}_\xi : \xi < \lambda\rangle$ is an increasing sequence of subalgebras of $\mathfrak{A}$ with union $\mathfrak{A}$, where $\lambda < \kappa$. Choose $\xi < \lambda$ such that $C \stackrel{\text{def}}{=} \{u \in U : a_u \in B_\xi\}$ has κ elements, and $b \in B_\xi$. If $u, v \in C$, then $a_{u\cdot v} = \mathsf{s}_0^2\mathsf{c}_0\mathsf{c}_1(a_u \cdot \mathsf{s}_1^0 a_v \cdot b)$, and hence $u \cdot v \in C$. It follows that $C = U$, and hence $\mathfrak{B}_\xi = \mathfrak{A}$, contradiction. ∎

This theorem implies that there are many CA_α's with high cofinality, by the known results on the existence of Jónsson groupoids (see, e.g., Jónsson [72] and Shelah [88]). We note in passing that Theorem 3.2 extends to relation algebras. Namely, let G be a Jónsson group of infinite cardinality κ. Let $\mathfrak{A}$ be the relation algebra which consists of the finite and cofinite subsets of G, with the complex algebra operations. It is easily checked that $\mathfrak{A}$ has cofinality κ.

The main question left open by Theorem 3.2 is as follows.

Problem 6. What is the cofinality of CA_2's?

We turn to the problem of existence of Jónsson cylindric algebras. It may be enlightening to go over the proof that there are no infinite Jónsson BA's. Let A be an infinite BA. Choose $a \in A$ such that $A \restriction a$ and $A \restriction -a$ both have more than two elements. Say $|A \restriction a| = |A|$. Then $\langle (A \restriction a) \cup \{-a\} \rangle$ is a proper subalgebra of A with the same number of elements as A. This argument generalizes to show the following theorem, for which we need a lemma.

Lemma 3.3. *Let A be an infinite* BA*, X a subalgebra of A, and $|X| < |A|$. Then A has a proper subalgebra B such that $|B| = |A|$ and $X \subseteq B$.*

Proof. There are two possibilities. *Case 1.* There is an element $a \in A$ such that $|A \restriction a| = |A|$ and $|A \restriction -a| > |X|$. Then the subalgebra of $\mathfrak{A}$ generated by $(A \restriction a) \cup \{x \cdot -a : x \in X\}$ is as desired.

Case 2. The ideal $I \stackrel{\text{def}}{=} \{a \in A : |A \restriction a| < |A|\}$ is prime, and $|A \restriction a| \leq |X|$ for all $a \in I$. Let J be the ideal in $\mathfrak{A}$ generated by $X \cap I$. Thus $|J| < |A|$, and hence $|A/J| = |A|$. Let $\mathfrak{C}$ be a proper subalgebra of A/J with $|A|$ elements. Then $B \stackrel{\text{def}}{=} \bigcup C$ is as desired. ∎

Theorem 3.4. *There is no Jónsson* CA_1.

Proof. Let $\mathfrak{A}$ be an infinite CA_1. If $|\mathsf{c}_0[A]| = |A|$, we can take a proper subalgebra B of $\mathsf{c}_0[A]$ as a BA with $|A|$ elements, and take the subalgebra of $\mathfrak{A}$ which it generates; this works by Theorem 2.2.24 of HMT[71]. On the other hand, if $\mathsf{c}_0[A]$ has fewer elements than A, then the lemma yields a proper Boolean subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ with $|A|$ elements containing $\mathsf{c}_0[A]$, and clearly B is actually a subuniverse of $\mathfrak{A}$, as desired. ∎

This argument can be extended to CA_2's, in the following way. First a lemma.

Lemma 3.5. *Let $\mathfrak{A}$ be a CA_α, a a zero-dimensional element of A, and $\mathfrak{B}$ a subalgebra of $\mathfrak{A} \restriction a$. Let $\mathfrak{C} = \mathfrak{Sg}(B)$. Then $C \restriction a = B$.*

Proof. Let $Y = \{y \in A : y \cdot a \in B\}$. Then $B \subseteq Y$, and Y is closed under all of the Boolean and cylindric operations. Hence $Y = C$. The lemma follows. ∎

Theorem 3.6. *There is no Jónsson* CA_2.

Proof. Let $\mathfrak{A}$ be an infinite CA_2. We want to find a proper subalgebra of $\mathfrak{A}$ with just as many elements as A. If $|A \restriction -\mathsf{c}_0(-\mathsf{d}_{01})| = |A|$, Lemma 3.5 plus the non-existence of a Jónsson BA yield the result. So we may assume that $\mathsf{c}_0(-\mathsf{d}_{01}) = 1$.

Second, if $|\mathsf{c}_0[\mathfrak{A}]| = |A|$, we may take a proper subalgebra of $\mathsf{c}_0[\mathfrak{A}]$ as a BA, of power $|A|$, and take the subalgebra it generates; Theorem 2.2.24 of HMT[71] assures that we get a proper subalgebra of $\mathfrak{A}$. So we may assume that $|\mathsf{c}_0[\mathfrak{A}]| < |A|$

Next, note that $|\mathsf{c}_0[\mathfrak{A}]| = |\mathsf{c}_1[\mathfrak{A}]|$. In fact, $x \mapsto \mathsf{c}_1(\mathsf{d}_{01} \cdot x)$ is a one-one function from $\mathsf{c}_0[\mathfrak{A}]$ onto $\mathsf{c}_1[\mathfrak{A}]|$.

It follows that the Boolean subalgebra X of $\mathfrak{A}$ generated by $\mathsf{c}_0[\mathfrak{A}] \cup \mathsf{c}_1[\mathfrak{A}] \cup \{\mathsf{d}_{01}\}$ has fewer than $|A|$ elements. By Lemma 3.3, let $\mathfrak{B}$ be a proper Boolean subalgebra of $\mathfrak{A}$ with $|A|$ elements, and with $X \subseteq B$. Clearly $\mathfrak{B}$ is as desired. ∎

We do not know the situation for $\alpha > 2$:

Problem 7. For $\alpha > 2$ is there a Jónsson CA_α?

Finally, we turn our attention to the descending chain condition for subalgebras (DCCS); recall that an algebra has this condition if every strictly decreasing sequence of subalgebras is finite. Clearly a DCCS algebra of power κ has a Jónsson subalgebra of power κ (with suitable restrictions on the number of operations). So we have the following corollary of Theorems 3.4 and 3.6.

Corollary 3.7. *There is no infinite* CA_1 *or* CA_2 *satisfying DCCS.* ∎

Problem 8. For $\alpha > 2$ is there a CA_α of size $> |\alpha|$ satisfying DCCS?

The last concept involving subalgebras which we will consider is *irredundance.* A subset X of an algebra $\mathfrak{A}$ is *irredundant* if for all $x \in X$, x is not in the subalgebra of $\mathfrak{A}$ generated by $X \backslash \{x\}$. In the theory of BA's the problem of constructing a BA $\mathfrak{A}$ with no irredundant subset of size $|A|$ has received quite a bit of attention. It turns out that the existence of such an uncountable BA with no uncountable irredundant subset is independent of ZFC. Using a general method of transfering Boolean structure to cylindric structure, Andréka and Németi [87] showed that for $\alpha < \omega$ there is a Cs_α of power $\aleph_1$ with no irredundant set of that size, under the same set-theoretic hypotheses as for the BA construction. This gives rise to the following questions.

Problem 9. Can one prove in ZFC that for $0 < \alpha < \omega_1$ there is a Cs_α of size $\aleph_1$ with no irredundant subset of that size?

Problem 10. For $\alpha \geq \omega$, is there a $\mathsf{Cs}^{\mathrm{reg}}_\alpha \cap \mathsf{Lf}_\alpha$ of size $|\alpha|^+$ with no irredundant subset of that size?

4. Independence

Where for Boolean algebras there appears to be only one natural notion of independence, for cylindric algebras we consider three notions; let $\mathfrak{A}$ be any CA_α and X a set of generators of $\mathfrak{A}$:

X is *Marczewski-independent* if X $\{\mathfrak{A}\}$-freely generates $\mathfrak{A}$.

X is CA_α*-independent* if X CA_α-freely generates $\mathfrak{A}$.

X is $\mathbf{HSP}\mathsf{Cs}_\alpha$*-independent* if $\mathfrak{A} \in \mathbf{HSP}\mathsf{Cs}_\alpha$ and X $\mathbf{HSP}\mathsf{Cs}_\alpha$-freely generates $\mathfrak{A}$.

(See HMTI Definition 0.4.23 for the definitions involved.) These independence notions are clearly related to irredundance introduced above. The following obvious implications hold between them, and no other implications are valid for all α:

$$\begin{aligned} \mathsf{CA}_\alpha - \text{independence} &\Rightarrow \text{Marczewski-independence;} \\ \mathbf{HSP}\mathsf{Cs}_\alpha - \text{independence} &\Rightarrow \text{Marczewski-independence;} \\ \text{Marczewski-independence} &\Rightarrow \text{irredundance.} \end{aligned}$$

That these are in general the only implications is seen as follows. Let $\mathbf{K}$ be a proper subvariety of CA_α, and let $\mathfrak{A}$ be a member of $\mathbf{K}$ freely generated by an infinite set X. Then X is Marczewski-independent but not CA_α-independent. Similar arguments work to show that Marczewski-independence does not imply $\mathbf{HSP}\mathsf{Cs}_\alpha$-indepen dence and $\mathbf{HSP}\mathsf{Cs}_\alpha$-independence does not imply CA_α-in dependence (for $\alpha > 1$). Since $\mathsf{CA}_1 = \mathbf{HSP}\mathsf{Cs}_1$, the corresponding two notions of independence coincide for $\alpha = 1$. For $\alpha > 1$ there are non-representable CA_α's, and hence CA_α-independence does not imply $\mathbf{HSP}\mathsf{Cs}_\alpha$-independence then. An example of an irredundant set which is not Marczewski-independent can be obtained easily using the methods of Andréka, Németi [87].

Andréka and Németi [87] showed that there are arbitrarily large CA_α's with no nonempty Marczewski-independent subsets; for $\alpha < \omega$ one can even take the algebra to be complete. The following question is open.

Problem 11. For α infinite, are there (arbitrarily large) complete CA_α's with no nonempty Marczewski-independent subsets?

References

BA[88], HMT[71], HMT[85], and HMTAN[81] refer respectively to the books Monk [88], Henkin, Monk, Tarski [71], Henkin, Monk, Tarski [85], and Henkin, Monk, Tarski, Andréka, Németi [81].

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J. Donald Monk

Department of Mathematics
University of Colorado
Boulder
CO 80309-0426, USA.