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THE CONTRIBUTIONS OF ALFRED TARSKI TO ALGEBRAIC LOGIC

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One of the most extensive parts of Tarski's contributions to logic is his work on the algebraization of the subject. His work here involves Boolean algebras, relation algebras, cylindric algebras, Boolean algebras with operators, Brouwerian algebras, and closure algebras. The last two are less developed in his work, although his contributions are basic to other work in those subjects. At any rate, not being conversant with the latest developments in those fields, we shall concentrate on an exposition of Tarski's work in the first four areas, trying to put them in the perspective of present-day developments.

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Boolean algebras. Tarski's main papers concerning Boolean algebras can be divided into these categories:

1. *Fundamentals.* Several of Tarski's papers in the 1930's were concerned with the foundations of the theory of Boolean algebras, a fact that was recognized by Birkhoff in the first edition of his book, ***Lattice theory***, in which he described M. H. Stone and Tarski jointly as the creators of the modern theory of BA's. Clearly in the modern spirit of the subject was Tarski [35]. This important paper may be considered to have two main parts. In the first, he introduces the general infinite distributive law and shows that every complete atomic BA is isomorphic to a power set algebra. This may be considered as the beginning of the study of infinite distributive laws; see below. In the second part of the paper, he proves the fundamental theorem that any logic give rise to a Boolean algebra, in the familiar way. This is the first appearence in the literature of the well-known and important Lindenbaum-Tarski algebras. There is some historical controversy as to the inventor of these algebras, although to one not directly involved it seems obvious that Tarski was the sole inventor. See Henkin, Monk and Tarski [71^{m}, p. 169, footnote 2] and Surma [1982].

2. *Logical questions.* In Tarski [49^{a}] the important result is stated that the elementary theory of BA's is decidable. His proof never appeared in print in detail,

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but it was presented in seminars in Berkeley. It is a classical, but somewhat involved, elimination of quantifiers. The theorem has been considerably strengthened: the elementary theory of BA's with an ω-sequence of ideals is decidable (Rabin [1969]). On the other hand, the elementary theory of a BA with a subalgebra is undecidable (Rubin [1976]; independently shown by McKenzie). There are several other known results concerning decidability or undecidability of the theory of BA's in various languages. A survey is to appear in Weese [198–], [198–a].

3. In a series of abstracts, Tarski discussed the relative strengths of prime ideal theorems for various kinds of BA's and the axiom of choice. This work set the stage for the later work of Halpern and others.

4. The deeper algebraic theory of BA's was treated by Tarski in several papers. In Mostowski and Tarski [39a] the notion of an interval algebra, or BA with an ordered basis, was introduced, and their most important properties were established. This turns out to be a very interesting class of BA's, and it has been further investigated by many people; for example, see Mayer and Pierce [1960], Rotman [1972], and Rubin [1983]. See also Henkin, Monk and Tarski [71$^{\text{m}}$]. In Tarski [37] and Tarski [39], [45], notions of κ-complete ideals and related topics were introduced, starting an active area which might be called set-theoretical ideal theory; for some recent developments, see, for example, Baumgartner, Taylor and Wagon [1982]. Erdös and Tarski [43] proved the very useful result that $\min\{\kappa$: every set of pairwise disjoint elements in $\mathfrak{A}$ has power $<\kappa\}$ is always a regular cardinal. Smith and Tarski [57] made a thorough study of distributive laws in BA's; this paper gives the central core of results known about such laws at the present time. For a survey of recent developments, see Jech [198–]. In Erdös and Tarski [43], [61b] and Keisler and Tarski [64], properties of BA's associated with large cardinals were discussed.

5. *Measure algebras.* Horn and Tarski [48c] is an important early paper in this area.

Brouwerian algebras and closure algebras. Tarski's work here was done in collaboration with J. C. C. McKinsey, in the papers McKinsey and Tarski [44], [46], [48]. The first of these papers studies closure algebras from the universal-algebraic point of view. The second is mainly about Brouwerian algebras, but also indicates their connections with closure algebras. The third paper applies the first two in the study of intuitionistic and modal logic.

Boolean algebras with operators. Tarski developed this subject in joint work with Bjarni Jónsson; see Jónsson and Tarski [51a] and [52]. The subject is still alive, and there have been later papers on this subject by Henkin, Monk, and others. The natural algebraic notion of a BA with operators can be considered to be a general algebraic framework for many extensions of classical sentential logic. The main, very useful, result in the two papers is the extension of certain operators on a BA to operators on the power set of its Stone space, and the preservation of certain kinds of sentences by this extension. In connection with this main theorem, the notion of a complex algebra is introduced; the theorem can be formulated as saying that every BA with certain kinds of operators can be embedded in the complex algebra of a certain relational structure. Below we explain exactly what this means for the complex algebras of groups, which is essentially a familiar construction to mathematicians. Applications of the main theorem are discussed for closure algebras, cylindric algebras, and relation algebras. It is likely that recent work on dynamic algebras and on general algebraic logics could be simplified and helped by the use of results about BA's with operators; see Kozen [1979] and Andréka, Gergely and Németi [1977].

Relation algebras. The theory of binary relations, introduced by Peirce, was first extensively developed developed on the arithmetic level by E. Schröder [1890]. The purely algebraic theory originates with Tarski [41]. A *relation algebra*, for brevity an RA, is by definition an algebraic structure $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, ;, {}^{\cup}, 1' \rangle$ such that $\langle A, +, \cdot, -, 0, 1 \rangle$ is a BA, ; is a binary, and $^{\cup}$ a unary, operation on A, and $1' \in A$, subject to certain equational postulates:

$$(x; y); z = x; (y; z), \tag{1}$$

$$(x + y); z = x; z + y; z, \tag{2}$$

$$x; 1' = x, \tag{3}$$

$$x^{\cup\cup} = x, \tag{4}$$

$$(x + y)^{\cup} = x^{\cup} + y^{\cup}, \tag{5}$$

$$(x; y)^{\cup} = y^{\cup}; x^{\cup}, \tag{6}$$

$$x^{\cup}; [-(x; y)] \cdot y = 0. \tag{7}$$

This is an abstraction from the following concrete case: A is a collection of binary relations on some set U, closed under the Boolean operations $\cup$, $\cap$, $\setminus$ (complementation relative to $U \times U$), with $\{(u, u): u \in U\} \in A$, and closed under the relation-theoretic operations $|$ (relative product) and $^{-1}$ (conversion):

$$R \,|\, S = \{(u, w): \exists v[(u, v) \in R \text{ and } (v, w) \in S]\},$$
$$R^{-1} = \{(u, v): (v, u) \in R\}.$$

Of course, $|$, $^{-1}$ and $\{(u, u): u \in U\}$ correspond to ;, $^{\cup}$ and 1', and the postulates (1)–(7) are easily seen to hold in this concrete case. An RA is *representable* if it is isomorphic to a subdirect product of concrete ones. The arithmetic of relation algebras is very rich. Its most extensive development is found in Chin and Tarski [51]. Probably all of the concrete relation algebraic identities found in the books of Schröder can be proved to hold in any RA. The algebraic theory of relation algebras is treated in Jónsson and Tarski [51a], [52]. In particular, they prove some representation theorems (relations algebras are isomorphic to some almost concrete ones), and discuss complex algebras for relation algebras. For clarity, take the group notion first. Given a group G, we can form a relation algebra $\mathfrak{A}_G = \langle G, \cup, \cap, \setminus, 0, G, ;, {}^{\cup}, 1' \rangle$ in which $X; Y = X \cdot Y$(complex product), $X^{\cup} = X^{-1}$, and $1' = \{e\}$ (e the identity of G). This relation algebra has the special property that $x; y = 0$ implies that $x = 0$ or $y = 0$; an RA with this property is called *integral*. Jónsson and Tarski left open the question whether every representable integral RA is embeddable in the complex algebra of a group. McKenzie [1970] answered this question negatively: the class of RA's embeddable in the complex algebra of groups is not even finitely axiomatizable over the class of integral representable relation algebras. The main result in Jónsson and Tarski [52] about complex algebras of RA's is that an RA is representable iff it is isomorphically embeddable in the complex algebra of a generalized Brandt groupoid. The notion of a Brandt groupoid comes up in finite combinatorics. Thus here, for perhaps the first time, we see the interplay of algebraic logic and finite combinatorics—a recurrent theme in this area (see below).

One of the deepest results concerning RA's was obtained by Tarski quite early, but never published: the equational theories of RA's and of representable RA's are undecidable—see Tarski [41], [53[a]b]. His proof directly translated set theory into RA's (essentially), and forms a basis for his later work with Givant (see below). A much simpler proof was recently obtained by Maddux [1978].

It turned out, surprisingly, that not every RA is representable. This was shown by Lyndon [1950] (with a correction in Lyndon [1956]). Elegant constructions of nonrepresentable RA's were given in Jónsson [1959] and Lyndon [1961]. Lyndon's construction associates an RA $\mathfrak{A}_G$ with a projective geometry G, with $\mathfrak{A}_G$ representable iff G cannot be embedded in a geometry of higher dimension. This construction led immediately to a proof that the class of representable RA's is not finitely axiomatizable over RA (Monk [1964]). Here the Bruck-Ryser theorem on the nonexistence of projective planes of certain orders was used.

One of the biproducts of Tarski's fundamental investigations in model theory was the important result (in Tarski [55]) that the class of representable relation algebras is a variety.

Early in the development of the theory of relation algebras, Tarski realized that the ideas could be used to achieve a philosophically interesting basis for set theory in which variables are not used and only equations occur (see Tarski [53[a]c]). He returned to these ideas in the 70's, and spent the last years of his life developing with S. Givant a lengthy monograph on this subject (Givant and Tarski [8–[m]]).

The ideas in the calculus of relations have turned out to have relevance in theoretical computer science. In particular, the dynamic algebras of Kozen and Pratt (see Kozen [1979]) embody a variation of RA's in which there is an additional operation of transitive closure. (See also the above discussion of BA's with operators.) Cf. here also Ng and Tarski [77[a]].

For an up-to-date account of RA's, see Jónsson [19].

Cylindric algebras. Although relation algebras may be considered to be algebraic versions of a (rather limited) portion of predicate logic, this connection is not immediately obvious. RA's are more directly motivated set-theoretically. Cylindric algebras, on the other hand, are most convincingly motivated via full first-order logic.

A *cylindric algebra of dimension* α (α an arbitrary ordinal)—a CA_α—is by definition an algebraic structure $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, c_\kappa, d_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha}$ such that $\langle A, +, \cdot, -, 0, 1\rangle$ is a BA, each c_κ is a unary operation on A, each $d_{\kappa\lambda} \in A$, and the following equational identities hold:

(1) $$c_\kappa 0 = 0,$$

(2) $$x \cdot c_\kappa x = x,$$

(3) $$c_\kappa(x \cdot c_\kappa y) = c_\kappa x \cdot c_\kappa y,$$

(4) $$c_\kappa c_\lambda x = c_\lambda c_\kappa x,$$

(5) $$d_{\kappa\kappa} = 1,$$

(6) $$\text{if } \kappa \neq \lambda, \mu, \text{ then } d_{\lambda\mu} = c_\kappa(d_{\lambda\kappa} \cdot d_{\kappa\mu}),$$

(7) $$\text{if } \kappa \neq \lambda \text{ then } c_\kappa(d_{\kappa\lambda} \cdot x) \cdot c_\kappa(d_{\kappa\lambda} \cdot -x) = 0.$$

The logical motivation is this: given a theory Γ in a first-order language L, define $\varphi \equiv \psi$ iff φ and ψ are formulas such that $\Gamma \vdash \varphi \leftrightarrow \psi$. This is an equivalence relation on the set of all formulas, and the collection of equivalence classes forms a cylindric algebra under operations such that $[\varphi] + [\psi] = [\varphi \vee \psi]$, $-[\varphi] = [\neg \varphi]$, $c_\kappa[\varphi] = [\exists v_\kappa \varphi]$ and $d_{\kappa\lambda} = [v_\kappa = v_\lambda]$. In analogy with Boolean algebras, this algebra could be called the *Lindenbaum-Tarski algebra* of the theory Γ. (The author suggests that at least for this construction, Tarski's name be associated with Lindenbaum's—see above.) CA_α's can also be motivated set-theoretically: one considers a BA A of subsets of ${}^\alpha U$ for some set U, closed additionally under the following operations:

$$\mathbf{C}_\kappa X = \{u \in {}^\alpha U: \text{for some } v \in X, u_\lambda = v_\lambda \text{ for all } \lambda \neq \kappa\} \quad (\textit{cylindrification}),$$
$$\mathbf{D}_{\kappa\lambda} = \{u \in {}^\alpha U: u_\kappa = u_\lambda\} \quad (\textit{diagonal set}).$$

CA_α's isomorphic to a subdirect product of such set algebras are called *representable*. These concrete CA's also arise in a natural, obvious fashion from models of theories.

The basic ideas of the theory of cylindric algebras were developed by Tarski in collaboration with his students L. H. Chin and F. B. Thompson, in 1948–1952. In addition to relation algebras, precursors are the *projective algebras* of Everett and Ulam [1946]. Shortly after the invention of CA_α's, L. Henkin made notable contributions to the subject. Later, extensive work was done by J. D. Monk, S. Comer, H. Andréka, and I. Németi. Tarski himself continued to make many technical contributions to the subject until about 1970. The theory of CA_α's is expounded primarily in three substantial monographs: Henkin, Monk and Tarski [71^m + 85^m], and Henkin, Monk, Tarski, Andréka and Németi [81^m]. We indicate some highlights of the development:

(A) The CA_α's defined from logic as above have two pecularities: α is infinite, and the algebras are locally finite-dimensional (for every x, there is a finite $\Gamma \subseteq \alpha$ such that $c_\kappa x = x$ for all $\kappa \in \alpha \backslash \Gamma$). *The* main representation theorem for CA_α's is that every locally finite-dimensional infinite dimensional CA_α is representable. There is a close connection between this result and the completeness theorem for first-order logic. In particular, each is rather easily derivable from the other. The representation result is due to Tarski, but the first published proof of the result is found in Henkin [1956], using the completeness theorem. For Tarski's original proof, in a somewhat simpler form, see Andréka and Németi [1975] (although they did not know Tarski's proof).

(B) For any dimension greater than 1 there are nonrepresentable CA_α's. This was realized by Tarski at an early stage in the development of the subject. In fact, for α

$\geqq 3$ the class of representable CA_α's is not finitely axiomatizable over CA_α (Monk [1969], using a form of the finite Ramsey theorem in a rather complicated way).

(C) The class of representable CA_2's can be characterized by two equations in addition to the CA_2 equations (Henkin and Tarski [61a]). This result and the construction of nonrepresentable cylindric algebras can be conveniently proved using the complex algebras associated with CA's.

(D) The equational theory of CA_α's and of representable CA_α's is undecidable if $3 \leqq \alpha \leqq \omega$ (this is due to Tarski, except for the case of CA_3; that case, and a simpler proof of the whole result, are due to Maddux [1980]).

(E) The equational theories of CA_2's and of representable CA_2's are decidable (Henkin and D. Scott; this appears in Henkin, Monk and Tarski [85^{m}] for the first time).

(F) By extending the notion of a first-order language, one can show that every CA_α is isomorphic to the CA_α of formulas with respect to some language; this was noticed by Henkin and Tarski at an early stage. A proof and a lengthy account of the connections of cylindric algebras with logic are found in Henkin, Monk and Tarski [85^{m}]. In particular, the development of the theory of cylindric algebras has singled out two new kinds of languages: ones with only finitely many variables, corresponding to CA_α's with α finite (see Henkin [1966] for the first extensive treatment of such languages), and what could be called the *finitary logic of infinitary relations*—languages which have infinitely long atomic formulas, but only the usual finitary connectives (see Henkin [1956]). The latter languages correspond exactly to cylindric algebras of infinite dimensions.

(G) As for relation algebras, there are some connections with theoretical computer science which are still in the developmental stage; see Imieliński and Lipski [1984] and Plotkin [1984].

(H) A little-developed aspect of the theory of cylindric algebras concerns the analysis of the Lindenbaum-Tarski algebras of well-known theories. Tarski was especially interested in this topic. He stressed the usefulness of such analyses, suggesting in particular that this should be carried out in detail for the theory of real-closed fields, algebraically expressing his well-known decision method. Essentially the only substantial theorem in this area is the theorem of D. Myers [1976] characterizing the CA_ω of formulas (with no axioms).

(I) It has been shown that relation algebras correspond in a definable fashion to certain three-dimensional CA_α's, in principle reducing the theory of RA's to that of CA_3's. This is a result of Maddux [1978], extending a partial result of Monk [1961].

(J) Very recently, starting with the book Henkin, Monk, Tarski, Andréka and Németi [81^{m}], the study of cylindric algebras has entered a technical phase which is hard to describe in an historical article such as this. Some of the results and problems are rather difficult, and even require some advanced set-theoretical techniques.

(K) There are many other algebraic versions of predicate logic, invented after cylindric algebras were. Many relationships are known concerning these versions; a survey of various kinds of algebraic logics and their relationships with cylindric algebras is found in Chapter 5 of Henkin, Monk and Tarski [85^{m}]. We mention two important cases: the polyadic algebras of Halmos [1962] and diagonal-free CA_α's.

The relational systems associated with the latter by the complex algebra construction are especially interesting: such a relational system is just a nonempty set together with an α-indexed system of commuting equivalence relations on the set. Polyadic algebras form a convenient algebraic framework for discussing the infinitary $L_{\kappa\lambda}$ languages.

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