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REPRESENTABLE CYLINDRIC ALGEBRAS

Leon HENKIN
Department of Mathematics, University of California, Berkeley, CA 94720, USA

J. Donald MONK
Department of Mathematics, University of Colorado, Boulder, CO 80309, USA

Alfred TARSKI†

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A cylindric algebra consists of a Boolean algebra endowed with an additional structure consisting of distinguished elements and operations, satisfying a certain system of equational axioms. The introduction and study of these algebras has its motivation in two parts of mathematics: the deductive systems of first-order logic, and a portion of elementary set theory dealing with spaces of various dimensions. This paper investigates the relationships between the abstract notion of a cylindric algebra and its source in elementary set-theoretic geometry.

The precise set-theoretic or geometric notion giving rise to the abstractly defined notion of cylindric algebras is that of a cylindric set algebra. To see what this amounts to, we start with any set U and ordinal α and form the α-dimensional Cartesian space ${}^{\alpha}U$ over U. The points of ${}^{\alpha}U$ are the sequences of length α whose components are in U, i.e., the functions u mapping α into U. Among the subsets of the space ${}^{\alpha}U$ we distinguish the *diagonal sets* $D_{\kappa\lambda}$ for each $\kappa, \lambda < \alpha$, where

$$D_{\kappa\lambda} = \{u \in {}^{\alpha}U : u\kappa = u\lambda\}.$$

And among the operations mapping subsets of ${}^{\alpha}U$ to other such subsets we distinguish the *cylindrifications* C_κ for each $\kappa < \alpha$, where for any $X \subseteq {}^{\alpha}U$,

$$C_\kappa X = \{u \in {}^{\alpha}U : \text{for some } x \in X \text{ we have } u\lambda = x\lambda \text{ for every } \lambda \neq \kappa,\ \lambda < \alpha\}.$$

The set $C_\kappa X$ is the cylinder generated by X in the κth direction.

Using the diagonal sets $D_{\kappa\lambda}$ and cylindrifications C_κ, we now define a *cylindric*

† Alfred Tarski died in Berkeley on October 27, 1983, after the manuscript for this paper had been completed. He was 82. The ideas of his which are incorporated in this paper were developed over the years since 1948, or possibly even earlier. He introduced both of the other authors to these ideas, thus stimulating their own interest in the subject.

set algebra of dimension α with base U as a system of the form

$$\mathfrak{A} = \langle A, \cup, \cap, \sim, 0, {}^{\alpha}U, C_\kappa, D_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha}$$

such that $\langle A, \cup, \cap, \sim, 0, {}^{\alpha}U\rangle$ is a Boolean set algebra of subsets of ${}^{\alpha}U$, $D_{\kappa\lambda} \in A$ whenever $\kappa, \lambda < \alpha$, and A is closed under each operation C_κ, $\kappa < \alpha$. We use the notation Cs_α to refer to the class of all cylindric set algebras of dimension α.

There are many equations that hold identically in every cylindric set algebra of a given dimension; in addition to all of the Boolean identities, we may mention

$$C_\kappa 0 = 0, \qquad X \cap C_\kappa X = X, \qquad C_\kappa(X \cap C_\kappa Y) = C_\kappa X \cap C_\kappa Y$$

as simple examples. Certain of these identities have been selected to serve as axioms for the abstractly defined notion of cylindric algebras. Thus, a *cylindric algebra of dimension* α (CA_α for short) can be described as any system $\langle B, +, \cdot, -, 0, 1, c_\kappa, d_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha}$ similar to Cs_α's which satisfies a certain set of prescribed equational identities. The identities which serve as axioms for defining CA_α's have been selected from all identities holding in every Cs_α because they are simple, and because for an important class of cylindric algebras (for α infinite), the class of *locally finite-dimensional* CA_α's (Lf_α's for short), an equation holds identically in all Lf_α's if and only if it holds in all Cs_α's. The importance of Lf_α's, in particular Lf_ω's, derives from the study of first-order logic. Thus if T is any mathematical theory formulated in a first-order language, then there is a natural way to associate an Lf_ω with T. An Lf_α is any CA_α such that, for each of its elements x, we have $c_\kappa x = x$ for all but finitely many $\kappa < \alpha$.

The book [5] is devoted to the abstract theory of cylindric algebras, and [6] to the theory of cylindric set algebras and related structures. A cylindric algebra of dimension α is *representable* if it is isomorphic to a subdirect product of cylindric set algebras of dimension α. In this article we give various sufficient conditions for representability, some of which are also necessary, and we describe some constructions of non-representable cylindric algebras.

Although our work is based on [5] and [6], we have made efforts to keep our discussion understandable for readers who have not made a thorough study of those works, by repeating some definitions and giving extensive references to results from [5] and [6] which we use.

We use Rp_α for the class of all representable CA_α's, i.e., the class of all CA_α's isomorphic to a subdirect product of Cs_α's.

For readers who are familiar with cylindric algebras we now describe the contents of this article in more detail. We begin with a technical result, II.5, which has two important corollaries. The first is that every Lf_α with $\alpha \geqslant \omega$ is representable (this result already has several proofs in the literature). It has many important immediate consequences; e.g., $\mathrm{Dc}_\alpha \subseteq \mathrm{Rp}_\alpha$ for $\alpha \geqslant \omega$, $\mathrm{Rp}_\alpha = \mathrm{SNr}_\alpha\mathrm{CA}_{\alpha+\omega}$ for any α, $\mathrm{Mn}_\alpha \subseteq \mathrm{Rp}_\alpha$. The second corollary is a characterization of representability in terms of embeddability in algebras with 'thin' elements (II.13).

an atomic algebra with all atoms rectangular (II.16). Third, we show that every CA_α of positive characteristic is representable (II.53). Our last major positive representation result is that any CA_2 satisfying two additional simple equations is representable (II.65). We conclude the article with the description of three methods for constructing non-representable CA_α's (the existence of non-representable CA_α's is, however, known from the literature; see [5] and [10]).

This article is the second in a series which is to form a part of the second volume of [5]. (The first is contained in [6]). For this reason we number definitions, theorems, etc. by II.1, II.2, etc.

Our first major result is a very strong sufficient condition for representability; it is an algebraic version of the completeness theorem for first-order logic. Our algebraic proof will be a version of Henkin's proof of the completeness theorem. Recall that his proof first starts by adjoining constants, which are used to eliminate quantifiers (in a certain sense). So we start by discussing an algebraic version of constants. There are at least three such versions. One can algebraically express properties of the formula $v_0 = \mathbf{c}$, $\mathbf{c}$ a constant; this is the method we actually use. Or, one can concentrate on the operation of substituting a constant for a variable in a formula; the corresponding algebraic notion of a special kind of endomorphism is used extensively in Halmos' related theory of polyadic algebras, and occurs as a derived notion in our development — see II.3. Lastly, one can think of a constant as a variable which one is not allowed to quantify; see the proof of II.7 and also Remarks II.9.

For the following definition, recall that CA_α is the class of all cylindric algebras of dimension α; $s^\kappa_\lambda x = c_\kappa(d_{\kappa\lambda} \cdot x)$ for all distinct $\kappa, \lambda < \alpha$, and $s^\kappa_\kappa x = x$; and $\Delta y = \{\kappa < \alpha : c_\kappa y \neq y\}$.

Definition II.1. Let $\alpha \geqslant 2$ and let $\mathfrak{A} \in CA_\alpha$.

(i) For $\kappa < \alpha$, an element x of A is *κ-thin* if $\Delta x \subseteq \{\kappa\}$, $x \cdot s^\kappa_\lambda x \leqslant d_{\kappa\lambda}$ for some $\lambda \in \alpha \sim \{\kappa\}$ and $c_\kappa x = 1$.

(ii) $\mathfrak{A}$ is *rich* if for every $y \in A$ such that $\Delta y \subseteq 1$ and $y \neq 0$ there is a 0-thin element x such that $x \cdot c_0 y \leqslant y$.

Remarks II.2. It will be shown shortly that if x is κ-thin, then $x \cdot s^\kappa_\lambda x \leqslant d_{\kappa\lambda}$ for *every* $\lambda \in \alpha \sim \{\kappa\}$. If $\mathfrak{A}$ is a Cs_α with base U and $u \in U$, then $x = \{t \in {}^\alpha U : t0 = u\}$ is 0-thin. (Cs_α is the class of all cylindric set algebras of dimension α.) Thin elements are an algebraic version of individual constants. Thus let Λ be a language with an individual constant $\mathbf{c}$. Let Σ be a consistent set of sentences in Λ. Then $(v_0 = \mathbf{c})/\equiv_\Sigma$ is a 0-thin element in the CA_α $\mathfrak{Fm}^{(\Lambda)}/\equiv_\Sigma$ (cf. [5, 1.1.9, 1.1.10]).

We need two lemmas concerning thin elements. In the proof of these lemmas, and throughout the rest of this article, we use some elementary arithmetic of

and their derivation from the axioms for CA_α's is usually easy (see [5, Chapter 1, exclusive of 1.8–1.11]).

Lemma II.3. *Suppose $\alpha \geq 2$, $\kappa < \alpha$, $\mathfrak{A}$ is a CA_α, and x is a κ-thin element of $\mathfrak{A}$. Then:*

(i) $x \cdot s^\kappa_\lambda x \leq d_{\kappa\lambda}$ *for every* $\lambda < \alpha$.

(ii) *If* $\lambda < \alpha$, *then* $s^\kappa_\lambda x$ *is* λ*-thin.*

(iii) *If* $\lambda \neq \kappa$, *then* $c_\lambda - c_\kappa(x \cdot -d_{\kappa\lambda}) = 1$.

(iv) *If* $\lambda \neq \kappa$, $y \in A$, *and* $c_\lambda[x \cdot y \cdot c_\kappa(x \cdot -y)] \leq c_\kappa(x \cdot -d_{\kappa\lambda})$, *then* $c_\kappa(x \cdot -y) = -c_\kappa(x \cdot y)$.

(v) *Under the assumptions of* (iv) *we have* $x \cdot c_\kappa(x \cdot y) = x \cdot y$.

Proof. (i) By II.1 choose $\mu \in \sim\{\kappa\}$ such that $x \cdot s^\kappa_\mu x \leq d_{\kappa\mu}$. Applying s^μ_λ to both sides of this inequality we get $x \cdot s^\kappa_\lambda x \leq d_{\kappa\lambda}$, as desired.

(ii) We have $s^\kappa_\lambda x \cdot s^\lambda_\kappa s^\kappa_\lambda x = s^\kappa_\lambda x \cdot x \leq d_{\kappa\lambda}$, $\Delta s^\kappa_\lambda x \subseteq \{\lambda\}$, and $c_\lambda s^\kappa_\lambda x = c_\kappa x = 1$, so $s^\kappa_\lambda x$ is λ-thin.

(iii) By (i), we have $x \cdot -d_{\kappa\lambda} \cdot s^\kappa_\lambda x = 0$, so $c_\kappa(x \cdot -d_{\kappa\lambda}) \cdot s^\kappa_\lambda x = 0$, hence $s^\kappa_\lambda x \leq -c_\kappa(x \cdot -d_{\kappa\lambda})$. Therefore

$$1 = c_\kappa x = c_\lambda s^\kappa_\lambda x = c_\lambda - c_\kappa(x \cdot -d_{\kappa\lambda}),$$

as desired.

(iv) The assumption yields $c_\lambda[x \cdot y \cdot c_\kappa(x \cdot -y)] \cdot -c_\kappa(x \cdot -d_{\kappa\lambda}) = 0$, hence $c_\kappa(x \cdot y) \cdot c_\kappa(x \cdot -y) \cdot c_\lambda - c_\kappa(x \cdot -d_{\kappa\lambda}) = 0$. Hence the desired result follows from (iii), since $c_\kappa(x \cdot y) + c_\kappa(x \cdot -y) = c_\kappa x = 1$.

(v) We have

$$\begin{aligned} x \cdot c_\kappa(x \cdot y) &= x \cdot -c_\kappa(x \cdot -y) = x \cdot c^\partial_\kappa(-x + y) \\ &\leq x \cdot (-x + y) = x \cdot y \leq x \cdot c_\kappa(x \cdot y). \end{aligned}$$

For the next lemma, recall that if Γ is a finite subset of α, $\Gamma = \{\gamma 0, \ldots, \gamma(\kappa - 1)\}$, then $c_{(\Gamma)}x = c_{\gamma 0} \cdots c_{\gamma(\kappa-1)}x$.

Lemma II.4. *Suppose $2 \leq \alpha$, Γ is a finite subset of α, and $\mathfrak{A}$ is a CA_α. Suppose x is a function with domain Γ such that x_κ is a κ-thin element of $\mathfrak{A}$ for every $\kappa \in \Gamma$. Furthermore, assume that the equality*

$$c_\lambda[y \cdot z \cdot c_\kappa(y \cdot -z)] \cdot c_\kappa(c_\lambda y \cdot -d_{\kappa\lambda}) = 0$$

holds for all distinct κ, $\lambda < \alpha$ and all $y, z \in A$. Then for any $y \in A$ we have

$$c_{(\Gamma)}\Big(\prod_{\kappa\in\Gamma} x_\kappa \cdot -y\Big) = -c_{(\Gamma)}\Big(\prod_{\kappa\in\Gamma} x_\kappa \cdot y\Big),$$

$$\prod_{\kappa\in\Gamma} x_\kappa \cdot c_{(\Gamma)}\Big(\prod_{\kappa\in\Gamma} x_\kappa \cdot y\Big) = \prod_{\kappa\in\Gamma} x_\kappa \cdot y.$$

From the following theorem we will be able to derive the main representation theorem using two additional easy lemmas. Recall here that a CA_α $\mathfrak{A}$ is *simple* if it has exactly two ideals; $\mathfrak{A} \in \mathrm{Lf}_\alpha$ if Δa is finite for all $a \in A$. A Cs_α $\mathfrak{A}$ as in the introduction is *regular* if for all $a \in A$ and all $u, v \in {}^\alpha U$, if $u \in a$ and $\Delta a \upharpoonright u = \Delta a \upharpoonright v$, then $v \in a$; $\mathrm{Cs}_\alpha^{\mathrm{reg}}$ is the class of all regular Cs_α's. For $\alpha \geqslant \omega$, a $\mathrm{Cs}_\alpha^{\mathrm{reg}} \cap \mathrm{Lf}_\alpha$ corresponds exactly to a model in some language; see [6, pp. 3, 57].

Theorem II.5. *Suppose $2 \leqslant \alpha$, $\mathfrak{A}$ is a simple rich Lf_α, and for all distinct $\kappa, \lambda < \alpha$ and all $x, y \in A$ the equality $c_\lambda[x \cdot y \cdot c_\kappa(x \cdot -y)] \cdot -c_\kappa(c_\lambda x \cdot -d_{\kappa\lambda}) = 0$ holds. Then $\mathfrak{A}$ is isomorphic to a $\mathrm{Cs}_\alpha^{\mathrm{reg}}$.*

Proof. Let U be the set of all 0-thin elements of $\mathfrak{A}$. We shall define an isomorphism f of $\mathfrak{A}$ onto a regular Cs_α with base U. Because $\mathfrak{A}$ is simple, we will only need to check the homomorphism conditions for f. For any $a \in A$ let

$$fa = \left\{ u \in {}^\alpha U : \prod_{\kappa \in \Delta a} s_\kappa^0 u_\kappa \leqslant a \right\}.$$

To check that f preserves $-$, first suppose that $u \in fa \cap f(-a)$. Since $\Delta a = \Delta(-a)$, this implies that $\prod_{\kappa \in \Delta a} s_\kappa^0 u_\kappa = 0$. But $c_{(\Delta a)} \prod_{\kappa \in \Delta a} s_\kappa^0 u_\kappa = \prod_{\kappa \in \Delta a} c_\kappa s_\kappa^0 u_\kappa = 1$, contradiction. Second, we show that $fa \cup f(-a) = {}^\alpha U$. Suppose $u \in {}^\alpha U \sim fa$. Thus $\prod_{\kappa \in \Delta a} s_\kappa^0 u_\kappa \cdot -a \neq 0$, so by simplicity, $c_{(\Delta a)}(\prod_{k \in \Delta a} s_\kappa^0 u_\kappa \cdot -a) = 1$. Then by II.4 we have $c_{(\Delta a)}(\prod_{\kappa \in \Delta a} s_\kappa^0 u_\kappa \cdot a) = 0$, so $\prod_{\kappa \in \Delta a} s_\kappa^0 u_\kappa \leqslant -a$ and hence $u \in f(-a)$, as desired.

Next we show that f preserves $\cdot$. Suppose first that $u \in f(a \cdot b)$. Thus $\prod_{\kappa \in \Delta(a \cdot b)} s_\kappa^0 u_\kappa \leqslant a \cdot b$. Let $\Gamma = \Delta(a \cdot b) \cup \Delta a$. Then $\prod_{\kappa \in \Gamma} s_\kappa^0 u_\kappa \leqslant a$, so

$$\prod_{\kappa \in \Delta a} s_\kappa^0 u_\kappa = c_{(\Gamma \sim \Delta a)} \prod_{\kappa \in \Gamma} s_\kappa^0 u_\kappa \leqslant c_{(\Gamma \sim \Delta a)} a = a,$$

and $u \in fa$. Similarly, $u \in fb$. Second, suppose that $u \in fa \cap fb$. Then $\prod_{\kappa \in \Delta a} s_\kappa^0 u_\kappa \leqslant a$ and $\prod_{\kappa \in \Delta b} s_\kappa^0 u_\kappa \leqslant b$, so $\prod_{\kappa \in \Delta a \cup \Delta b} s_\kappa^0 u_\kappa \leqslant a \cdot b$. As above one argues to show that $\prod_{\kappa \in \Delta(a \cdot b)} s_\kappa^0 u_\kappa \leqslant a \cdot b$.

For cylindrifications, suppose that $\lambda < \alpha$, and first suppose that $u \in fc_\lambda a$. We may assume that $\lambda \in \Delta a$. Thus with $b = c_\lambda a$,

$$\prod_{\kappa \in \Delta b} s_\kappa^0 u_\kappa \leqslant c_\lambda a.$$

We want to find a 0-thin element v such that $u_v^\lambda \in fa$. Note that

$$\begin{aligned} & c_\lambda c_{(\Delta a \sim \{\lambda\})}\left[\left(\prod_{\kappa \in \Delta a \sim \{\lambda\}} s_\kappa^0 u_\kappa\right) \cdot a\right] \\ &= c_{(\Delta a \sim \{\lambda\})}\left(\prod_{\kappa \in \Delta a \sim \{\lambda\} \sim \Delta b} s_\kappa^0 u_\kappa\right) \cdot c_\lambda\left[\left(\prod_{\kappa \in \Delta b} s_\kappa^0 u_\kappa\right) \cdot a\right] \\ &= c_{(\Delta a \sim \{\lambda\})}\left(\prod s_\kappa^0 u_\kappa\right) = 1; \end{aligned}$$

it follows that the element $x = c_{(\Delta a \sim \{\lambda\})}[(\prod_{\kappa \in \Delta a \sim \{\lambda\}} s^0_\kappa u_\kappa) \cdot a]$ is non-zero and has dimension set $\subseteq \{\lambda\}$. Hence let v be a 0-thin element $\leqslant s^\lambda_0 x$. Set $u' = u^\lambda_v$. Then

$$\begin{aligned}\prod_{\kappa \in \Delta a} s^0_\kappa u'_\kappa &= \prod_{\kappa \in \Delta a \sim \{\lambda\}} s^0_\kappa u_\kappa \cdot s^0_\lambda v \\ &\leqslant \prod_{\kappa \in \Delta a \sim \{\lambda\}} s^0_\kappa u_\kappa \cdot c_{(\Delta a \sim \{\lambda\})}\left[\left(\prod_{\kappa \in \Delta a \sim \{\lambda\}} s^0_\kappa u_\kappa\right) \cdot a\right] \\ &\leqslant a \qquad \text{by II.4,}\end{aligned}$$

as desired. Second, suppose that $u \in c_\lambda fa$. Let v be a 0-thin element such that $u^\lambda_v \in fa$. Thus

$$(1) \qquad \prod_{\kappa \in \Delta a \sim \{\lambda\}} s^0_\kappa u_\kappa \cdot s^0_\lambda v \leqslant a.$$

Let $\Gamma = (\Delta a \sim \Delta c_\lambda a) \cup \{\lambda\}$. Applying $c_{(\Gamma)}$ to both sides of (1) we get

$$\prod_{\kappa \in \Delta b} s^0_\kappa u_\kappa \leqslant c_\lambda a,$$

hence $u \in fc_\lambda a$, as desired.

Now for diagonal elements, suppose $\lambda, \mu < \alpha$; we may assume that $\lambda \neq \mu$. First suppose that $u \in fd_{\lambda\mu}$. We may assume that $\mathfrak{A}$ is non-discrete, and hence $\Delta d_{\lambda\mu} = \{\lambda, \mu\}$. Thus $s^0_\lambda u_\lambda \cdot s^0_\mu u_\mu \leqslant d_{\lambda\mu}$. Hence

$$s^0_\mu u_\mu = c_\lambda(s^0_\lambda u_\lambda \cdot s^0_\mu u_\mu) \leqslant c_\lambda(d_{\lambda\mu} \cdot s^0_\lambda u_\lambda) = s^0_\mu u_\lambda$$

and so $u_\mu \leqslant u_\lambda$. By symmetry $u_\mu = u_\lambda$, so $u \in D_{\lambda\mu}$. Second, suppose that $u \in D_{\lambda\mu}$, so that $u_\mu = u_\lambda$. Then

$$s^0_\lambda u_\lambda \cdot s^0_\mu u_\mu = s^0_\lambda u_\lambda \cdot s^\lambda_\mu s^0_\lambda u_\lambda \leqslant d_{\lambda\mu}$$

since $s^0_\lambda u_\lambda$ is λ-thin by II.3(ii). Therefore $u \in fd_{\lambda\mu}$.

It is obvious that fa is regular for each $a \in A$, so the proof is complete.

Lemma II.6. *Suppose $2 \leqslant \alpha$, $\mathfrak{A}$ is a rich Lf_α, and I is an ideal of $\mathfrak{A}$. Then $\mathfrak{A}/I$ is rich.*

Proof. Suppose $a \in A$, $\Delta(a/I) \subseteq I$, and $a/I \neq 0$. Then $\Delta c_{(\Delta a \sim 1)} a \subseteq 1$ and $c_{(\Delta a \sim 1)} a \neq 0$, so let x be a 0-thin element $\leqslant c_{(\Delta a \sim 1)} a$. Clearly x/I is 0-thin and $x/I \leqslant a/I$.

For the proof of the next result, we use the follow notions. If $\alpha \leqslant \beta$ and $\mathfrak{A}$ is a CA_β, then $\mathfrak{Rd}_\alpha$ is the CA_α obtained from $\mathfrak{A}$ by deleting c_κ for $\alpha \leqslant \kappa < \beta$ and $d_{\kappa\lambda}$ if $\alpha \leqslant \kappa < \beta$ or $\alpha \leqslant \lambda < \beta$; $\mathfrak{Nr}_\alpha$ is the subalgebra of $\mathfrak{Rd}_\alpha$ with universe $\{a \in A : \Delta a \subseteq \alpha\}$.

Lemma II.7. *Suppose $\omega \leqslant \alpha$ and $\mathfrak{A} \in \mathrm{Lf}_\alpha$. Then $\mathfrak{A}$ can be isomorphically embedded in a rich Lf_α.*

Proof. By a simple transfinite argument it suffices to show that if $0 \neq a \in A$ and $\Delta a \subseteq 1$, then there is an extension $\mathfrak{B} \in \mathrm{Lf}_\alpha$ *of* $\mathfrak{A}$ such that $x \cdot c_0 a \leqslant a$ for some 0-thin element x of B. To this end, let $\mathfrak{C}$ be an $\mathrm{Lf}_{\alpha+1}$ such that $\mathfrak{A} \subseteq \mathfrak{Nr}_\alpha \mathfrak{C}$, by [5, 2.6.49]. Let $\mathfrak{D} = \mathfrak{Rd}_\alpha \mathfrak{C}$, and let $I = Ig^{(\mathfrak{D})}\{-c_0((a + -c_0 a) \cdot d_{0\alpha})\}$. Then $A \cap I = \{0\}$. In fact, suppose $y \in A \cap I$. Thus $y \cdot c_0((a + -c_0 a) \cdot d_{0\alpha}) = 0$, so $y \cdot -c_0 a = 0$ and $y \cdot s^0_\alpha a = 0$. Since $\Delta a \subseteq \alpha$ and $\Delta y \subseteq \alpha$ we have $y \cdot c_0 a = 0$. Hence $y = 0$, as desired. Clearly also $(a + -c_0 a) \cdot d_{0\alpha} \notin I$. Let $x = (a + -c_0 a) \cdot d_{0\alpha}$. Then in $\mathfrak{D}/I$ we have that x/I is 0-thin and $(x/I) \cdot c_0(a/I) \leqslant a/I$. This finishes the proof.

The main representation theorem now follows:

Theorem II.8. *For* $\alpha \geqslant \omega$ *we have* $\mathrm{Lf}_\alpha \subseteq \mathrm{SPCs}^{\mathrm{reg}}_\alpha$.

Proof. By II.7 we may assume that our given $\mathfrak{A} \in \mathrm{Lf}_\alpha$ is rich, and by [5, 2.4.52] and II.6 that $\mathfrak{A}$ is simple By [5, 1.11.7], the equations indicated in II.5 hold in $\mathfrak{A}$. Hence the conclusion follows by II.5. (The proof of 1.11.7 is easy.)

Remarks II.9. The result just established is due to Tarski. The proof is due to Henkin. Tarski's original proof can be sketched as follows, using the apparatus developed in [5]. We start with a simple Lf_α $\mathfrak{A}$, $\alpha \geqslant \omega$. First we neatly embed $\mathfrak{A}$ in a simple Lf_β $\mathfrak{C}$ such that $|B| = \beta$; it suffices to show that $\mathfrak{C} \in \mathrm{ICs}^{\mathrm{reg}}_\beta$. Using this cardinality condition it is easy to construct an ultrafilter F on $\mathfrak{Bl}\,\mathfrak{B}$ satisfying the following condition.

(*) For all $\kappa < \beta$ and all $x \in \beta$, if $c_\kappa x \in F$, then $s^\kappa_\lambda x \in F$ for some $\lambda \in \beta \sim \Delta x$.

Now we can define an equivalence relation $\equiv$ on β by setting $\kappa \equiv \lambda$ iff $d_{\kappa\lambda} \in F$. Let U be the set of all $\equiv$-classes, and let $\phi \in {}^U\beta$ be a choice function: $\phi u \in u$ for all $u \in U$. Then the desired isomorphism f is defined by

$$fb = \{x \in {}^\beta U : s^+_{\phi \circ x} b \in F\}$$

for any $b \in B$, where s^+ is the substitution function introduced in [5, 1.11.13].

Proofs of II.8 have appeared in the literature. Except for a different argument avoiding neat embedding, the proof just sketched appears in [1]. A proof similar to the one of II.5, using thin elements, is given in [13] and a proof *using* the completeness theorem is carried out in [12]. Via the correspondence between Lf_α's and polyadic equality algebras for infinite α (see [3]), representation theorems for polyadic algebra yield II.8 again; see [4].

Now we give the most important corollaries of II.8. Here we use various notions introduced in [5]. These corollaries will not be used in the rest of his article. The first one gives several characterizations of representability.

$\mathrm{SNr}_\alpha \mathrm{CA}_{\alpha+\beta}$ *for each* $\beta \geqslant \omega$. *Moreover,* $\mathfrak{A} \in \mathrm{CA}_\alpha$ *is representable iff every finite reduct of* $\mathfrak{A}$ *is representable.*

Proof. The second two equalities are found in [5, 2.6.34 and 2.6.35]. For the first equality, assume first $\alpha < \omega$. Then by [5, 2.6.48], II.8, [6, I.7.4, I.8.7],

$$\begin{aligned}\mathrm{SNr}_\alpha \mathrm{CA}_{\alpha+\omega} &= \mathrm{SNr}_\alpha \mathrm{Lf}_{\alpha+\omega} \subseteq \mathrm{SNr}_\alpha \mathrm{SPCs}^{\mathrm{reg}}_{\alpha+\omega} = \mathrm{SNr}_\alpha \mathrm{IGs}_{\alpha+\omega} \\ &= \mathrm{IGs}_\alpha \subseteq \mathrm{SNr}_\alpha \mathrm{CA}_{\alpha+\omega},\end{aligned}$$

giving the desired result. If $\alpha \geqslant \omega$, then by [5, 2.6.52], II.8, [6, I.7.16, I.8.7],

$$\mathrm{SNr}_\alpha \mathrm{CA}_{\alpha+\omega} = \mathrm{SUpLf}_\alpha \subseteq \mathrm{SUpIGs}_\alpha = \mathrm{IGs}_\alpha \subseteq \mathrm{SNr}_\alpha \mathrm{CA}_{\alpha+\omega},$$

again giving the desired result. The last statement follows easily from [5, 2.6.47] and the above.

Theorem II.10 extends to $\alpha = 0$ and $\alpha = 1$; the arguments here are rather trivial, and will be given separately in II.54, II.55. Now we give some additional sufficient conditions for representability which follow from II.8.

Theorem II.11. *Let* $\alpha \geqslant \omega$ *and* $\mathfrak{A} \in \mathrm{CA}_\alpha$. *Then each of the following conditions is sufficient for* $\mathfrak{A}$ *to be representable:*

(i) $\mathfrak{A} \in \mathrm{Lf}_\alpha$.

(ii) $\mathfrak{A} \in \mathrm{Dc}_\alpha$.

(iii) $\mathfrak{A} \in \mathrm{Ss}_\alpha$.

(iv) *For every finite* $\Gamma \subseteq \alpha$ *and every non-zero* $x \in A$ *there exist distinct* $\kappa, \lambda \in \alpha \sim \Gamma$ *such that* $x \cdot d_{\kappa\lambda} \neq 0$.

(v) *For every finite sequence* ρ *without repeating terms and with range included in* α, *and for every non-zero* $x \in A$ *there exist a function* h *and* $\kappa < \alpha$ *such that* h *is an endomorphism of* $\mathfrak{Rd}^{(\rho)}\mathfrak{A}$, $\kappa \in \alpha \sim \mathrm{Rg}\,\rho$, $c_\kappa^{(\mathfrak{A})} \circ h = h$, *and* $hx \neq 0$.

(vi) $\mathfrak{A}$ *is of characteristic* $\kappa > 0$.

(vii) *For every* $\kappa < \alpha$ *and every* $x \in A$, $c_\kappa x = \sum_{\lambda<\alpha} s_\lambda^\kappa x$.

Proof. By [5, 2.6.49, 2.6.50, 2.6.54], and II.10.

Condition II.11(vi), that $\mathfrak{A}$ has non-zero characteristic, remains a sufficient condition for representability when $\alpha < \omega$; see II.51.

Theorem II.12. *Every monadic-generated* CA_α *is representable*; *hence every minimal* CA_α *is representable.*

Proof. By [5, 2.6.56] and II.10.

Now we return to the basic Theorem II.5. Its formulation does not have the

assumption $\alpha \geqslant \omega$ which is present in II.8, needed there because of Lemma II.7. This leads to the following characterization of representability for $2 \leqslant \alpha < \omega$, due to Henkin and Tarski, announced in [7]. For the notion of Gs_α, see [6, I.1.1 and I.6.3]; IGs_α coincides with Rp_α for $\alpha \geqslant 2$.

Theorem II.13. *Suppose $2 \leqslant \alpha < \omega$ and $\mathfrak{A}$ is a CA_α. Then $\mathfrak{A}$ is representable iff $\mathfrak{A}$ can be embedded in a rich CA_α $\mathfrak{B}$ such that in $\mathfrak{B}$ all of the equations*

$$c_\lambda(x \cdot y \cdot c_\kappa(x \cdot -y)) \cdot -c_\kappa(c_\lambda x \cdot -d_{\kappa\lambda}) = 0$$

hold, for all distinct $\kappa, \lambda < \alpha$ and all $x, y \in B$.

Proof. The direction $\Leftarrow$ is immediate from 2.4.52, II.6, and II.5. For $\Rightarrow$, suppose that $\mathfrak{A}$ is representable. Say $\mathfrak{A} \cong \mathfrak{C}$, where $\mathfrak{C}$ is a Gs_α with unit element V. Let $\mathfrak{B}$ be the full Gs_α with unit element V. It suffices to show that $\mathfrak{B}$ is rich and the indicated equations hold in $\mathfrak{B}$. Say $V = \bigcup_{i \in I} {}^\alpha U_i$, where $U_i \cap U_j = 0 \neq U_i$ for distinct $i, j \in I$. Suppose $0 \neq b \in B$ and $\Delta b \subseteq 1$. Choose $t_i \in {}^\alpha U_i$ for all $i \in I$ so that $t_i \in b$ if $b \cap {}^\alpha U_i \neq 0$. Let

$$x = \bigcup_{i \in I} \{s \in {}^\alpha U_i : s0 = t_i 0\}$$

It is easily checked that x is 0-thin and $x \cdot c_0 b \leqslant b$. Thus $\mathfrak{B}$ is rich. That the given equations hold in $\mathfrak{B}$ is immediate from [6, I.8.6. and I.11.7], but this can also be checked directly. This finishes the proof.

Now we turn to another characterization of representability, involving rectangular atoms. It depends on the following theorem which is of independent interest. Theorems II.14 and II.16 are due to Henkin and Tarski, announced in [7]. For the notion of a rectangular element, see [5, 1.10.6]; Gws_α is defined in [6, I.1.1], and in [6, I.7.14] it is shown that $\mathrm{IGws}_\alpha = \mathrm{Rp}_\alpha$ for $\alpha \geqslant 2$. We also need the following notions; see [6]. Given an ordinal α, a set U, and a subset $V \subseteq {}^\alpha U$ we define

$$C_\kappa^{[V]} X = \{x \in V : y \in X \text{ for some } y \text{ with } (\alpha \sim \{\kappa\}) \upharpoonright x \subseteq y\},$$
$$D_{\kappa\lambda} = \{x \in V : x\kappa = x\lambda\}$$

for any $X \subseteq V$ and $\kappa, \lambda < \alpha$. A Crs_α is an algebra $\mathfrak{A} = \langle A, \cup, \cap, {}_V\!\sim, 0, V, C_\kappa^{[V]}, D_{\kappa\lambda} \rangle_{\kappa,\lambda<\alpha}$ closed under the indicated operations. If A consists of all subsets of V, we call $\mathfrak{A}$ *full*.

Theorem II.14. *Let $\mathfrak{A}$ be any atomic CA_α, $\alpha \geqslant 2$. Then the following conditions are equivalent:*

(i) *Every atom of $\mathfrak{A}$ is rectangular.*

(ii) *There is an isomorphism of $\mathfrak{A}$ onto a $\mathrm{Gws}_\alpha \mathfrak{B}$ that carries each atom of $\mathfrak{A}$ to*

Discussion II.15. We shall only prove, below, that (i) implies (ii), as the converse is very easy to check.

To see the idea of our proof, consider first the case where $\alpha < \omega$, and where $\mathfrak{A}$ is simple. We wish to represent $\mathfrak{A}$ isomorphically as a Cs_α with some base U. Thus the unit set of $\mathfrak{A}$ is a hypercube, ${}^\alpha U$, and our isomorphism will be determined by assigning to each atom x of $\mathfrak{A}$ a single point $\langle a_0, \ldots, a_{\alpha-1}\rangle \in {}^\alpha U$.

Our idea is to take for U the set of atoms on the 'principal diagonal' d of $\mathfrak{A}$, $d = \prod_{\kappa,\lambda<\alpha} d_{\kappa\lambda}$. Then for any atom x of $\mathfrak{A}$ and any $\kappa < \alpha$, we will get the component $a_\kappa \in U$ by forming the hyperplane $c_{(\alpha\sim\{\kappa\})}x$ and intersecting it with d. This mapping of atoms x of $\mathfrak{A}$ to points $\langle a_0, \ldots, a_{\alpha-1}\rangle$ of ${}^\alpha U$ is one-one and onto, because any $\langle a_0, \ldots, a_{\alpha-1}\rangle \in {}^\alpha U$ will be the coordinates of a unique atom x of $\mathfrak{A}$, obtained by intersecting all of the hyperplanes $c_{(\alpha\sim\{\kappa\})}a_\kappa$.

Of course we cannot use such a construction in case $\alpha \geqslant \omega$, since in that case $\mathfrak{A}$ has no principal diagonal d and no hyperplanes $c_{(\alpha\sim\{\kappa\})}x$. Furthermore, we are not restricting ourselves to the case where $\mathfrak{A}$ is simple. Nevertheless, we can incorporate the geometric ideas of the preceding paragraph in the desired proof, as follows.

Proof of II.14, (i) *implies* (ii). We define a binary relation E on $\mathrm{At}\,\mathfrak{A} \times \alpha$ by the following rule, where $\mathrm{At}\,\mathfrak{A}$ is the set of all atoms of $\mathfrak{A}$. For any $a, b \in \mathrm{At}\,\mathfrak{A}$ and $\xi, \eta \in \alpha$, $\langle a, \xi\rangle E\langle b, \eta\rangle$ iff there is some finite $\Gamma \subseteq \alpha$ with $\xi, \eta \in \Gamma$ such that $c_{(\Gamma\sim\{\xi\})}a \cdot d_\Gamma = c_{(\Gamma\sim\{\eta\})}b \cdot d_\Gamma$. (Recall that $d_\Gamma = \prod_{\kappa,\lambda\in\Gamma} d_{\kappa\lambda}$.) It is clear that E is reflexive and symmetric, but it is also transitive. For if $\langle a, \xi\rangle E\langle b, \eta\rangle E\langle c, \zeta\rangle$, then we have finite $\Gamma, \Delta \subseteq \alpha$ with $\xi, \eta \in \Gamma$, $\eta, \zeta \in \Delta$,

$$c_{(\Gamma\sim\{\xi\})}a \cdot d_\Gamma = c_{(\Gamma\sim\{\eta\})}b \cdot d_\Gamma, \qquad c_{(\Delta\sim\{\eta\})}b \cdot d_\Delta = c_{(\Delta\sim\{\zeta\})}c \cdot d_\Delta.$$

Putting $\Phi = \Gamma \cup \Delta$, so that $\xi, \eta, \zeta \in \Phi$, if we apply $c_{(\Phi\sim\Gamma)}$ to both sides of the first equation we get $c_{(\Phi\sim\{\xi\})}a \cdot d_\Gamma = c_{(\Phi\sim\{\eta\})}b \cdot d_\Gamma$ hence $c_{(\Phi\sim\{\xi\})}a \cdot d_\Phi = c_{(\Phi\sim\{\eta\})}b \cdot d_\Phi$. Similarly $c_{(\Phi\sim\{\eta\})}b \cdot d_\Phi = c_{(\Phi\sim\{\zeta\})}c \cdot d_\Phi$. Thus $\langle a, \xi\rangle E\langle c, \zeta\rangle$, as desired.

Now let U be the set of all equivalence classes of $\mathrm{At}\,\mathfrak{A} \times \alpha$ under E. Define a map h of $\mathrm{At}\,\mathfrak{A}$ into ${}^\alpha U$ by setting $(ha)\xi = \langle a, \xi\rangle/E$ for each $a \in \mathrm{At}\,\mathfrak{A}$ and $\xi \in \alpha$. Let V be the range of h, and let $\mathfrak{C}$ be the full Crs_α with unit element V. For each $x \in A$ let $jx = \{ha : a \text{ an atom} \leqslant x\}$. To complete our proof it suffices to show that $\mathfrak{C}$ is a Gws_α and j is an isomorphism of $\mathfrak{A}$ into $\mathfrak{C}$. (The proof below could be simplified slightly by working with the atom structure of $\mathfrak{A}$, defined in [5, 2.7.32]. But we do not want to assume acquaintance with that material.)

Before proceeding we note the following fact about our relation E, which is easily established:

(1) *If $a, b \in \mathrm{At}\,\mathfrak{A}$ and $\kappa < \alpha$, then $\langle a, \kappa\rangle E\langle b, \kappa\rangle$ iff there is a finite $\Gamma \subseteq \alpha$ such that $\kappa \notin \Gamma$ and $c_{(\Gamma)}a = c_{(\Gamma)}b$.*

$\langle a, 0\rangle E\langle b, 0\rangle$, so $c_{(\Gamma)}a = c_{(\Gamma)}b$ for some Γ; we choose such a Γ of smallest cardinality. If $\Gamma = 0$, then $a = b$, as desired. Suppose $\Gamma \neq 0$, and choose $\xi \in \Gamma$. Since $\langle a, \xi\rangle E\langle b, \xi\rangle$, choose a finite $\Delta \subseteq \alpha$ such that $\xi \notin \Delta$ and $c_{(\Delta)}a = c_{(\Delta)}b$. Then, using the fact that a and b are rectangular,

$$c_{(\Gamma\cap\Delta)}a = c_{(\Gamma)}a \cdot c_{(\Delta)}a = c_{(\Gamma)}b \cdot c_{(\Delta)}b = c_{(\Gamma\cap\Delta)}b.$$

But $\xi \in \Gamma \sim (\Gamma \cap \Delta)$, so $|\Gamma \cap \Delta| < |\Gamma|$, a contradiction. So $a = b$, and h is one–one.

It follows easily that j is an isomorphism of $\mathfrak{BI}\,\mathfrak{A}$ into $\mathfrak{BI}\,\mathfrak{C}$. ($\mathfrak{BI}\,\mathfrak{D}$ is the Boolean part of $\mathfrak{D}$ for any CA_α $\mathfrak{D}$.) Next, suppose that $\kappa, \lambda < \alpha$. To show that $jd_{\kappa\lambda} = D^{[V]}_{\kappa\lambda}$, it suffices to assume that $\kappa \neq \lambda$. Suppose that $a \in \mathrm{At}\,\mathfrak{A}$, and first suppose that $a \leqslant d_{\kappa\lambda}$ (so that $ha \in jd_{\kappa\lambda}$). Then $c_\kappa a \cdot d_{\kappa\lambda} = c_\lambda a \cdot d_{\kappa\lambda} = a \cdot d_{\kappa\lambda}$, so $\langle a, \kappa\rangle E\langle a, \lambda\rangle$; hence $(ha)\kappa = (ha)\lambda$ and $ha \in D^{[V]}_{\kappa\lambda}$. Second suppose that $ha \in D^{[V]}_{\kappa\lambda}$. This $\langle a, \kappa\rangle E\langle a, \lambda\rangle$, so choose a finite $\Gamma \subseteq \alpha$ with $\kappa, \lambda \in \Gamma$ and $c_{(\Gamma\sim\{\kappa\})}a \cdot d_\Gamma = c_{(\Gamma\sim\{\lambda\})}a \cdot d_\Gamma$. Then $c_{(\Gamma\sim\{\kappa,\lambda\})}a \cdot d_\Gamma = c_{(\Gamma\sim\{\kappa\})}a \cdot c_{(\Gamma\sim\{\lambda\})}a \cdot d_\Gamma$ (since a is rectangular) $= c_{(\Gamma\sim\{\kappa\})}a \cdot d_\Gamma \neq 0$ (by [5, 1.10.13]). Hence $0 \neq a \cdot c_{(\Gamma\sim\{\kappa,\lambda\})}d_\Gamma = a \cdot d_{\kappa\lambda}$ using [5, 1.8.6]. Thus $a \leqslant d_{\kappa\lambda}$, as desired.

Next we show that j preserves c_κ for any $\kappa < \alpha$. Let $a \in \mathrm{At}\,\mathfrak{A}$, $x \in A$. First suppose that $a \leqslant c_\kappa x$, so that $ha \in jc_\kappa x$. Choose $b \in \mathrm{At}\,\mathfrak{A}$ with $b \leqslant x$ and $a \leqslant c_\kappa b$. Then by [5, 1.10.3(i)], $c_\kappa a = c_\kappa b$. Hence by (1), $(ha)\lambda = (hb)\lambda$ for all $\lambda \neq \kappa$. So $ha \in C^{[V]}_\kappa jx$, as desired. Second, suppose that $ha \in C^{[V]}_\kappa jx$. Say $b \in \mathrm{At}\,\mathfrak{A}$, $b \leqslant x$, $(\alpha \sim \{\kappa\}) \upharpoonright ha \subseteq hb$. Since $\alpha \geqslant 2$, there is a $\lambda \in \alpha \sim \{\kappa\}$, hence $(ha)\lambda = (hb)\lambda$, hence $c_{(\Gamma)}a = c_{(\Gamma)}b$ for some finite $\Gamma \subseteq \alpha$. We choose such a Γ with $|\Gamma|$ minimum. Suppose there is a $\mu \in \Gamma \sim \{\kappa\}$. Then $(ha)\mu = (hb)\mu$, so by (1) there is a finite $\Delta \subseteq \alpha$ with $c_{(\Delta)}a = c_{(\Delta)}b$. Since a and b are rectangular we easily get $c_{(\Gamma\cap\Delta)}a = c_{(\Gamma\cap\Delta)}b$. Since $|\Gamma \cap \Delta| < |\Gamma|$, this is a contradiction. Thus $\Gamma \subseteq \{\kappa\}$, so $c_\kappa a = c_\kappa b$. Hence $a \leqslant c_\kappa x$ and $ha \in jc_\kappa x$, as desired.

Before proceeding, we note the following facts:

(2) *If Γ is a finite subset of α, $\Gamma \neq \alpha$, and $x \in A$, then $C^{[V]}_{(\Gamma)}jx = \{ha : a$ is an atom and there is an atom $b \leqslant x$ such that $(\alpha \sim \Gamma) \upharpoonright ha \subseteq hb\}$.*

For, $C^{[V]}_{(\Gamma)}jx = jc_{(\Gamma)}x = \{ha : a$ is an atom $\leqslant c_{(\Gamma)}x\}$. Hence if $ha \in C^{[V]}_{(\Gamma)}jx$, let b be an atom $\leqslant x$ such that $a \leqslant c_{(\Gamma)}b$. Then $c_{(\Gamma)}a = c_{(\Gamma)}b$, so by (1) we have $(\alpha \sim \Gamma) \upharpoonright ha \subseteq hb$. Conversely, suppose a and b are atoms, $b \leqslant x$, and $(\alpha \sim \Gamma) \upharpoonright ha \subseteq hb$. Choose $\kappa \in \alpha \sim \Gamma$. Then $(ha)\kappa = (hb)\kappa$, so by (1) there is a finite $\Delta \subseteq \alpha$ such that $c_{(\Delta)}a = c_{(\Delta)}b$; using rectangularity we easily find that $c_{(\Gamma)}a = c_{(\Gamma)}b$. Thus $a \leqslant c_{(\Gamma)}x$, so $ha \in C^{[V]}_{(\Gamma)}jx$, as desired.

(3) *Suppose $a \in \mathrm{At}\,\mathfrak{A}$, $\kappa < \alpha$, x is a function with domain α, $(\alpha \sim \{\kappa\}) \upharpoonright ha \subseteq x$, b is an atom, $b \leqslant c_{(\Gamma)}a$ for some finite $\Gamma \subseteq \alpha$, $\lambda < \alpha$, and $x\kappa = (hb)\lambda$. Then $x \in V$.*

To prove this, we may assume that $\kappa, \lambda \in \Gamma$. First suppose $\kappa = \lambda$. We have

we have $(\alpha \sim \{\kappa\}) \upharpoonright ha \subseteq hc$ and $(hc)\kappa = (hb)\kappa = x\kappa$. Thus $hc = x$, as desired. Now suppose that $\kappa \neq \lambda$. Then there is an atom $c \leqslant c_\kappa a \cdot c_{(\Gamma \sim \{\kappa\})}(d_{\kappa\lambda} \cdot c_{(\Gamma \sim \{\lambda\})}b)$; say d is an atom $\leqslant d_{\kappa\lambda} \cdot c_{(\Gamma \sim \{\lambda\})}b$ and $c \leqslant c_{(\Gamma \sim \{\kappa\})}d$. Then $(\alpha \sim \{\kappa\}) \upharpoonright ha \subseteq hc$, and $(hc)\kappa = (hd)\kappa = (hd)\lambda = (hb)\lambda = x\kappa$, using (1). So again $hc = x$, as desired.

Now it remains only to show that V is a Gws_α unit element. First suppose that $\alpha < \omega$. For each α-atom k, let $U_k = \{(ha)\kappa : a \text{ is an atom } \leqslant k,\ \kappa < \alpha\}$.

(4) *If k and l are distinct α-atoms, then $U_k \cap U_l = 0$.*

For, suppose $x \in U_k \cap U_l$. Say $x = (ha)\kappa = (hb)\lambda$ with a and b atoms $\leqslant k$ and l respectively, and $\kappa, \lambda < \alpha$. Thus $\langle a, \kappa\rangle E \langle b, \lambda\rangle$, so there is a finite $\Gamma \subseteq \alpha$ with $\kappa, \lambda \in \Gamma$ and $c_{(\Gamma \sim \{\kappa\})}a \cdot d_\Gamma = c_{(\Gamma \sim \{\lambda\})}b \cdot d_\Gamma$. Hence $c_{(\Gamma \sim \{\kappa\})}a \cdot c_{(\Gamma \sim \{\lambda\})}b \neq 0$ and so $k = l$.

(5) $V = \bigcup \{{}^\alpha U_k : k \text{ an } \alpha\text{-atom}\}$.

For, if a is any atom, obviously $ha \in {}^\alpha U_k$, with $k = c_{(\alpha)}a$. Thus $\subseteq$ holds. The direction $\supseteq$ follows from the following statement (with $\kappa = \alpha$):

(6) *Suppose that k is an α-atom, a is an atom $\leqslant k$, $\kappa \leqslant \alpha$, $x \in {}^\alpha U_k$, and $x\lambda = (ha)\lambda$ for all $\lambda \in \alpha \sim \kappa$. Then $x \in V$.*

We prove (6) by induction on κ. The case $\kappa = 0$ is trivial. Assume that it is true for $\kappa - 1$ ($\kappa > 0$), and assume its hypotheses. Let $(\alpha \sim \{\kappa - 1\}) \upharpoonright x \subseteq y$, $y(\kappa - 1) = (ha)(\kappa - 1)$, $y \in {}^\alpha U_k$. Thus $y\lambda = (ha)\lambda$ for all $\lambda \in \alpha \sim (\kappa - 1)$, so $y \in V$ by the induction hypothesis. Then (3) yields $x \in V$, as desired.

Second, we suppose that $\alpha \geqslant \omega$. We define an equivalence relation $\equiv$ on V by setting $x \equiv y$ iff $x, y \in V$ and $|\{\kappa < \alpha : x\kappa \neq y\kappa\}| < \omega$. For each $\equiv$-class k, let $Y_k = \{x\kappa : x \in k,\ \kappa < \alpha\}$. Thus ${}^\alpha Y^{(x)}_{x/\equiv} \cap V \subseteq x/\equiv$ for any $x \in V$, and hence it suffices to show that ${}^\alpha Y^{(x)}_{x/\equiv} \subseteq V$ for any $x \in V$. To do this it suffices to show:

(7) *If Γ is a finite subset of α, $x \in V$, $y \in {}^\alpha Y_{x/\equiv}$, and $(\alpha \sim \Gamma) \upharpoonright x \subseteq y$, then $y \in V$.*

We prove (7) by induction on $|\Gamma|$. The case $\Gamma = 0$ is trivial. Assume inductively that $\Gamma \neq 0$, and fix $\kappa \in \Gamma$. Let z be the element of ${}^\alpha Y_{x/\equiv}$ such that $(\alpha \sim \{\kappa\}) \upharpoonright y \subseteq z$ and $z\kappa = x\kappa$. Then $z \in V$ by the inductive hypothesis; say $z = hb$ with $b \in \mathrm{At}\,\mathfrak{A}$. Since $y\kappa \in Y_{x/\equiv}$, there exist $w \in V$, a finite $\Delta \subseteq \alpha$, and a $\lambda < \alpha$ such that $(\alpha \sim \Delta) \upharpoonright x \subseteq w$ and $y\kappa = w\lambda$. Let v be such that $(\alpha \sim \{\kappa\}) \upharpoonright w \subseteq v$ and $v\kappa = w\lambda$. Then by (3), $v \in V$, say $v = hd$. Let $\Omega = \Gamma \cup \Delta \cup \{\kappa\}$. Then for any $\mu \in \alpha \sim \Omega$ we have $y\mu = x\mu = z\mu = w\mu = v\mu$. Hence $C^{[V]}_{(\Omega)}\{x\} \subseteq C^{[V]}_{(\Omega)}\{hb\} \cap C^{[V]}_{(\Omega)}\{hd\}$ by (2) so $c_{(\Omega)}b \cdot c_{(\Omega)}d \neq 0$, hence $c_{(\Omega \sim \{\kappa\})}d \cdot c_\kappa b \neq 0$. Let e be an atom $\leqslant c_{(\Omega \sim \{\kappa\})}d \cdot c_\kappa b$. Thus $c_\kappa e = c_\kappa b$, so $(\alpha \sim \{\kappa\}) \upharpoonright he = (\alpha \sim \{\kappa\}) \upharpoonright hb \subseteq y$. Also, $c_{(\Omega \sim \{\kappa\})}e = c_{(\Omega \sim \{\kappa\})}d$, so by (1) $(he)\kappa = (hd)\kappa = w\lambda = y\kappa$. Thus $y = he \in V$, as desired. This finishes the proof of II.14.

Theorem II.16. *For $\alpha \geqslant 2$, a CA_α is representable iff it can be embedded in an*

The proof is immediate from II.14.

Our next representation result, due to Henkin, is that every CA_α, $\alpha < \omega$, of positive characteristic is representable; we have already noted in II.11 that this applies for $\alpha \geqslant \omega$. Recall from [5, 2.4.61] that a CA_α has positive characteristic provided that its minimal subalgebra is simple and there is a $\lambda < \alpha \cap \omega$ such that $c_{(\lambda+1)}(\prod_{\kappa,\mu<\lambda,\kappa\neq\mu} - d_{\kappa\mu}) = 0$.

Remark II.17. To establish this representation we need an auxiliary result, due independently to Comer and Henkin, that for $\alpha < \omega$ a CA_α of positive characteristic can be provided with a substitution operator s satisfying the conditions in [5, 1.11.12]. Since we do not need all the conditions of [5, 1.11.12], we only check those parts actually needed. To establish this result it is convenient to use the main theorem in Jonsson [8], which we repeat here for the reader's convenience. Suppose that I is any set. An *elementary transformation* of I is a mapping of I into itself of the form $[x/y]$ or $[x/y, y/x]$ with $x, y \in I$. Recall that $[x/y]$ is that mapping of I into I which sends x to y and z to z for all $z \in I \sim \{x\}$; $[x/y, y/x]$ sends x to y, y to x, and z to z for all $z \in I \sim \{x, y\}$. For brevity we denote $[x/y, y, x]$ by $[x, y]$.

Jónsson's theorem is as follows. Let $\langle S, \cdot, e\rangle$ be a semigroup with identity e. Suppose that we are given a mapping s from the elementary transformations of I into S. Then the following conditions are equivalent:

(A) s extends to a mapping s^+ from the set of all finite transformation of I into S such that $s^+(\sigma \circ \tau) = s^+\sigma \circ s^+\tau$ for any two such transformations, and $s^+(I \restriction \mathrm{Id}) = e$. (A transformation of I is just a mapping of I into I. A transformation σ of I is *finite* if $\{x \in I : \sigma x \neq x\}$ is finite.)

(B) If x, y, z are distinct elements of I and y, u, z are distinct elements of I then the following conditions hold:

(I) $s[x, y] = s[y, x]$,
(II) $s[x, y] \cdot s[y, x] = e$,
(III) $s[x, y] \cdot s[x, z] = s[y, z] \cdot s[x, y]$,
(IV) $s[x, y] \cdot s[z/x] = s[z/y] \cdot s[x, y]$,
(V) $s[x, y] \cdot s[y/x] = s[x/y]$,
(VI) $s[y/x] \cdot s[z/u] = s[z/u] \cdot s[y/x]$,
(VII) $s[y/x] \cdot s[y/u] = s[y/u]$.

It is a lengthy proces to establish the auxiliary result of II.17, and we break the proof into a series of lemmas. For all the lemmas we assume that $\alpha < \omega$, and we use $\kappa, \lambda, \mu, \ldots$ for ordinals $<\alpha$. Some of the lemmas are true for arbitrary CA_α's, $<\omega$, but for many we indicate by (A) the additional assumption that the

from [5, Section 1.5] will be used without explicit citation. The proof is due to Henkin and Monk.

We begin with a result in the spirit of [5, Section 1.5].

Lemma II.18. *If* $\kappa \neq \lambda$, $\{\kappa, \lambda\} \cap \{\mu, \nu, \rho, \sigma\} = 0$ $|\{\mu, \nu, \rho\}| = |\{\mu, \nu, \sigma\}| = 3$, $\{\kappa, \lambda\} \cap \{\gamma, \delta, \varepsilon, \xi\} = 0$, $|\{\gamma, \delta, \varepsilon\}| = |\{\gamma, \delta, \xi\}| = 3$, *and* $e = -d_{\kappa\lambda} \cdot d_{\mu\rho} \cdot d_{\nu\sigma} \cdot d_{\gamma\varepsilon} \cdot d_{\delta\xi}$, *then*

$$e \cdot {}_\mu s(\kappa, \lambda) c_\mu c_\nu (x \cdot d_{\mu\rho} \cdot d_{\nu\sigma}) = e \cdot {}_\gamma s(\kappa, \lambda) c_\gamma c_\delta (x \cdot d_{\gamma\varepsilon} \cdot d_{\delta\xi}).$$

Proof. Clearly one of the following conditions holds: $\mu \neq \gamma, \varepsilon$ or $\mu \neq \delta, \xi$ or $\nu \neq \gamma, \varepsilon$ or $\nu \neq \delta, \xi$. By symmetry, say $\mu \neq \gamma, \varepsilon$. Then

$$\begin{aligned} e \cdot {}_\mu s(\kappa, \lambda) c_\mu c_\nu (x \cdot d_{\mu\rho} \cdot d_{\nu\sigma}) &= e \cdot {}_\mu s(\kappa, \lambda) c_\mu c_\gamma (x \cdot d_{\mu\rho} \cdot d_{\gamma\varepsilon}) \\ &= e \cdot {}_\gamma s(\kappa, \lambda) c_\mu c_\gamma (x \cdot d_{\mu\rho} \cdot d_{\gamma\varepsilon}); \end{aligned}$$

if $\rho \neq \gamma$, this is equal to $e \cdot {}_\gamma s(\kappa, \lambda) c_\gamma (x \cdot d_{\gamma\epsilon})$ which is equal to $e \cdot {}_\gamma s(\kappa, \lambda) c_\gamma c_\delta (x \cdot d_{\gamma\varepsilon} \cdot d_{\delta\xi})$. If $\rho = \gamma$, then it is equal to

$$\begin{aligned} e \cdot {}_\gamma s(\kappa, \lambda) c_\mu c_\gamma (x \cdot d_{\mu\varepsilon} \cdot d_{\gamma\varepsilon}) &= e \cdot {}_\gamma s(\kappa, \lambda) c_\gamma (x \cdot d_{\gamma\varepsilon}) \\ &= e \cdot {}_\gamma s(\kappa, \lambda) c_\gamma c_\delta (x \cdot d_{\gamma\varepsilon} \cdot d_{\delta\xi}). \end{aligned}$$

This finishes the proof.

Definition II.19. We set

$$g_{\rho\sigma} = \prod \{-d_{\mu\nu} : \mu, \nu < \alpha, \mu \neq \nu, \{\mu, \nu\} \cap \{\rho, \sigma\} = 0\},$$

$$e_\rho = g_{\rho\rho},$$

$$f_{\rho\sigma} = d_{\rho\sigma} \cdot e_\rho.$$

We give some properties of these notions which will be used later.

Lemma II.20. (1) $f_{\rho\sigma} = f_{\sigma\rho}$.

(ii) *If* $|\{\kappa, \lambda, \mu\}| = 3$, *then* $f_{\mu\lambda} \cdot s^\mu_\kappa x = f_{\mu\lambda} \cdot s^\mu_\kappa (f_{\mu\kappa} \cdot x)$.

(iii) *If* $|\{\kappa, \lambda, \mu\}| = 3$, *then* $f_{\mu\lambda} \cdot s^\mu_\kappa x = 0$ *iff* $f_{\mu\kappa} \cdot x = 0$.

(iv) (A) *If* $\rho \neq \sigma$, *then* $e_\rho \cdot e_\sigma \leqslant d_{\rho\sigma}$.

Lemma II.21 (A). *If* $\rho \neq \sigma$, *then* $f_{\rho\sigma} \cdot c_\rho c_\sigma (f_{\rho\sigma} \cdot x) = f_{\rho\sigma} \cdot x$.

Proof. We only need to show $\leqslant$. Now $f_{\rho\sigma} \cdot c_\rho c_\sigma (f_{\rho\sigma} \cdot x) \cdot -x = 0$ iff $c_\rho (f_{\rho\sigma} \cdot -x) \cdot c_\rho (f_{\rho\sigma} \cdot x) = 0$. Since $c_\rho (f_{\rho\sigma} \cdot -x) \cdot c_\sigma (f_{\rho\sigma} \cdot x) = s^\rho_\sigma - x \cdot e_\rho \cdot s^\sigma_\rho x \cdot e_\sigma$ and $e_\rho \cdot e_\sigma \leqslant d_{\rho\sigma}$, the desired conclusion follows.

Proof. Again we only need to show $\leqslant$. We have

$$\begin{aligned} f_{\rho\sigma} \cdot c_\mu(f_{\rho\sigma} \cdot x) &\leqslant d_{\rho\sigma} \cdot e_\rho \cdot c_\mu c_\rho(f_{\rho\sigma} \cdot x) \\ &= d_{\rho\sigma} \cdot e_\rho \cdot c_\mu(e_\rho \cdot s^\rho_\sigma x) = d_{\rho\sigma} \cdot c_\rho(f_{\mu\rho} \cdot c_\mu(e_\rho \cdot s^\rho_\sigma x)) \\ &= d_{\rho\sigma} \cdot c_\rho(f_{\mu\rho} \cdot c_\mu c_\rho(f_{\mu\rho} \cdot s^\rho_\sigma x)) = d_{\rho\sigma} \cdot c_\rho(f_{\mu\rho} \cdot s^\rho_\sigma x) \leqslant x \qquad \text{(By II.20)}, \end{aligned}$$

as desired.

Lemma II.23 (A). *If* $|\{\kappa, \lambda, \rho, \sigma\}| = 4$, *then*

$$f_{\rho\sigma} \cdot {}_\rho s(\kappa, \lambda) c_\rho c_\sigma(x \cdot f_{\rho\sigma}) = f_{\rho\sigma} \cdot {}_\rho s(\kappa, \lambda) s^\rho_\sigma x.$$

Proof. We have

$$\begin{aligned} f_{\rho\sigma} \cdot {}_\rho s(\kappa, \lambda) c_\rho c_\sigma(x \cdot f_{\rho\sigma}) &= f_{\rho\sigma} \cdot {}_\rho s(\kappa, \lambda)(f_{\lambda\rho} \cdot c_\sigma(s^\rho_\sigma x \cdot e_\rho)) \quad \text{by II.20(ii))} \\ &= f_{\rho\sigma} \cdot {}_\rho s(\kappa, \lambda)(f_{\lambda\rho} \cdot c_\sigma(s^\rho_\sigma x \cdot f_{\lambda\rho})) \\ &= f_{\rho\sigma} \cdot {}_\rho s(\kappa, \lambda)(f_{\lambda\rho} \cdot s^\rho_\sigma x) \qquad \text{(by II.22)} \\ &= f_{\rho\sigma} \cdot {}_\rho s(\kappa, \lambda) s^\rho_\sigma x \qquad \text{(by II.20(ii))}. \end{aligned}$$

Now we are ready to define $s[\kappa, \lambda]$, to be used as indicated in II.17.

Definition II.24 For $\kappa \neq \lambda$ we set

$$\begin{aligned} s[\kappa, \lambda]x = d_{\kappa\lambda} \cdot x &+ \sum_{\rho \neq \kappa, \lambda} - d_{\kappa\lambda} \cdot d_{\kappa\rho} \cdot s^\kappa_\lambda s^\lambda_\rho x \\ &+ \sum_{\rho \neq \kappa, \lambda} - d_{\kappa\lambda} \cdot d_{\lambda\rho} \cdot s^\lambda_\kappa s^\kappa_\rho x \\ &+ \sum \{-d_{\kappa\lambda} \cdot d_{\mu\rho} \cdot d_{\nu\sigma} \cdot {}_\mu s(\kappa, \lambda) c_\mu c_\nu(d_{\mu\rho} \cdot d_{\nu\sigma} \cdot x): \\ &\qquad \{\kappa, \lambda\} \cap \{\mu, \nu, \rho, \sigma\} = 0, |\{\mu, \nu, \rho\}| = 3 = |\{\mu, \nu, \sigma\}|\} \\ &+ \sum_{|\{\kappa, \lambda, \rho, \sigma\}| = 4} f_{\rho\sigma} \cdot {}_\rho s(\kappa, \lambda) s^\rho_\sigma x. \end{aligned}$$

We list some immediate properties of $s[\kappa, \lambda]$:

Lemma II.25. *Assume* $\kappa \neq \lambda$.

(i) $s[\kappa, \lambda]$ *preverves* $+$.
(ii) $s[\kappa, \lambda]0 = 0$.
(iii) (A) $s[\kappa, \lambda]1 = 1$.
(iv) $s[\kappa, \lambda] = s[\lambda, \kappa]$.
(v) $s[\kappa, \lambda]d_{\kappa\lambda} = d_{\kappa\lambda}$.
(vi) *If* $\rho \neq \kappa, \lambda$, *then* $s[\kappa, \lambda]d_{\kappa\rho} = d_{\lambda\rho}$ *and* $s[\kappa, \lambda]d_{\lambda\rho} = d_{\kappa\rho}$.

Next we give some lemmas which will enable us to break further arguments concerning $s[\kappa, \lambda]$ into cases.

Lemma II.26. *If* $\kappa \neq \lambda$, *then* $d_{\kappa\lambda} \cdot s[\kappa, \lambda]x = d_{\kappa\lambda} \cdot x$.

Lemma II.27. *If* $|\{\kappa, \lambda, \rho\}| = 3$, *then* $-d_{\kappa\lambda} \cdot d_{\kappa\rho} \cdot s[\kappa, \lambda]x = -d_{\kappa\lambda} \cdot d_{\kappa\rho} \cdot s^{\kappa}_{\lambda} s^{\lambda}_{\rho} x$.

Proof. $\geqslant$ is clear. Now let $r = -d_{\kappa\lambda} \cdot d_{\kappa\rho} \cdot s^{\kappa}_{\lambda} s^{\lambda}_{\rho} x$. For $\sigma \neq \kappa, \kappa, \rho$ let $t = -d_{\kappa\lambda} \cdot d_{\kappa\rho} \cdot d_{\lambda\sigma}$. Now $-d_{\kappa\lambda} \cdot d_{\kappa\rho} \cdot d_{\kappa\sigma} \cdot s^{\kappa}_{\lambda} s^{\lambda}_{\sigma} x \leqslant r$; $-d_{\kappa\lambda} \cdot d_{\kappa\rho} \cdot d_{\lambda\rho} \cdot s^{\lambda}_{\kappa} s^{\kappa}_{\rho} x = 0$; and

$$t \cdot s^{\lambda}_{\kappa} s^{\kappa}_{\sigma} x = t \cdot s^{\lambda}_{\rho} s^{\kappa}_{\sigma} x = t \cdot s^{\kappa}_{\sigma} s^{\lambda}_{\rho} x \leqslant r.$$

Next, suppose that $\{\kappa, \lambda\} \cap \{\rho, \mu, \nu, \sigma\} = 0$. Suppose $|\{\rho, \mu, \nu\}| = 3 = |\{\rho, \mu, \sigma\}|$. Let $t = -d_{\kappa\lambda} \cdot d_{\kappa\rho} \cdot d_{\rho\nu} \cdot d_{\mu\sigma}$. Then

$$\begin{aligned} t \cdot {}_{\rho}s(\kappa, \lambda) c_{\rho} c_{\mu} (d_{\rho\nu} \cdot d_{\mu\sigma} \cdot x) &= t \cdot s^{\kappa}_{\lambda} s^{\lambda}_{\rho} c_{\rho} (d_{\rho\nu} \cdot d_{\mu\sigma} \cdot x) \\ &= t \cdot s^{\kappa}_{\lambda} s^{\lambda}_{\rho} (d_{\rho\nu} \cdot d_{\mu\sigma} \cdot x) \leqslant r. \end{aligned}$$

If on the other hand $|\{\mu, \nu, \rho\}| = 3 = |\{\mu, \nu, \sigma\}|$, then with $t = -d_{\kappa\lambda} \cdot d_{\kappa\rho} \cdot d_{\mu\rho} \cdot d_{\nu\sigma}$,

$$t \cdot {}_{\mu}s(\kappa, \lambda) c_{\mu} c_{\nu} (d_{\mu\rho} \cdot d_{\nu\sigma} \cdot x) = t \cdot s^{\kappa}_{\lambda} s^{\lambda}_{\mu} (d_{\mu\rho} \cdot d_{\nu\sigma} \cdot x) \leqslant r.$$

Finally, suppose that $\{\kappa, \lambda, \rho\} \cap \{\mu, \nu, \sigma, \tau\} = 0$ and $|\{\mu, \nu, \sigma\}| = 3 = |\{\mu, \nu, \tau\}|$. Let $t = -d_{\kappa\lambda} \cdot d_{\kappa\rho} \cdot d_{\mu\sigma} \cdot d_{\nu\tau}$. Then

$$\begin{aligned} t \cdot {}_{\mu}s(\kappa, \lambda) c_{\mu} c_{\nu} (d_{\mu\sigma} \cdot d_{\nu\tau} \cdot x) &= t \cdot s_{\lambda} s^{\mu}_{\kappa} s^{\lambda}_{\mu} c^{\kappa}_{\mu} (d_{\mu\sigma} \cdot d_{\nu\tau} \cdot x) \\ = t \cdot s^{\mu}_{\lambda} s^{\kappa}_{\mu} s^{\lambda}_{\rho} c_{\mu} (d_{\mu\sigma} \cdot d_{\nu\tau} \cdot x) &= t \cdot s^{\kappa}_{\lambda} s^{\lambda}_{\rho} c_{\mu} (d_{\mu\sigma} \cdot d_{\nu\tau} \cdot x) \leqslant r. \end{aligned}$$

By the cases in the definition of $s[\kappa, \lambda]x$ the lemma follows: note that $-d_{\kappa\lambda} \cdot d_{\kappa\rho}$ has 0 intersection with the last sum.

By symmetry we obtain:

Lemma II.28. *If* $|\{\kappa, \lambda, \rho\}| = 3$, *then* $-d_{\kappa\lambda} \cdot d_{\lambda\rho} \cdot s[\kappa, \lambda]x = -d_{\kappa\lambda} \cdot d_{\lambda\rho} \cdot s^{\lambda}_{\kappa} s^{\kappa}_{\rho} x$.

Lemma II.29. *Suppose* $\{\kappa, \lambda\} \cap \{\mu, \nu, \rho, \sigma\} = 0$ *and* $|\{\mu, \nu, \rho\}| = 3 = |\{\mu, \nu, \sigma\}|$. *Then*

$$-d_{\kappa\lambda} \cdot d_{\mu\rho} \cdot d_{\nu\sigma} \cdot s[\kappa, \lambda]x = -d_{\kappa\lambda} \cdot d_{\mu\rho} \cdot d_{\nu\sigma} \cdot {}_{\mu}s(\kappa, \lambda) c_{\mu} c_{\nu} (x \cdot d_{\mu\rho} \cdot d_{\nu\sigma}).$$

Proof. $\geqslant$ is clear. Now let r be the right-side of the indicated equation. If $t = -d_{\kappa\lambda} \cdot d_{\mu\rho} \cdot d_{\nu\sigma} \cdot d_{\kappa\mu}$, then $t \cdot s^{\kappa}_{\lambda} s^{\lambda}_{\mu} x = t \cdot s^{\kappa}_{\lambda} s^{\lambda}_{\mu} (x \cdot d_{\mu\rho} \cdot d_{\nu\sigma}) \leqslant t \cdot r$. Now suppose $\eta \neq \kappa, \lambda, \mu, \nu, \sigma, \rho$, and let $t = -d_{\kappa\lambda} \cdot d_{\mu\rho} \cdot d_{\nu\sigma} \cdot d_{\kappa\eta}$. Then

$$t \cdot s^{\kappa}_{\lambda} s^{\lambda}_{\eta} x = t \cdot s^{\kappa}_{\lambda} s^{\lambda}_{\eta} s^{\mu}_{\rho} x = t \cdot s^{\kappa}_{\lambda} s^{\mu}_{\eta} s^{\lambda}_{\eta} s^{\mu}_{\rho} x$$

Finally, suppose also $\{\kappa, \lambda\} \cap \{\gamma, \delta, \varepsilon, \xi\} = 0$ and $|\{\gamma, \delta, \varepsilon\}| = 3 = |\{\gamma, \delta, \xi\}|$. Then

$$-d_{\kappa\lambda} \cdot d_{\mu\rho} \cdot d_{\nu\sigma} \cdot d_{\gamma\varepsilon} \cdot d_{\delta\xi} \cdot {}_{\gamma}s(\kappa, \lambda)c_{\gamma}c_{\delta}(x \cdot d_{\gamma\varepsilon} \cdot d_{\delta\xi}) \leq r$$

by II.18, reasoning as in the proof of II.27.

Lemma II.30. *If* $|\{\kappa, \lambda, \rho, \sigma\}| = 4$, *then* $f_{\rho\sigma} \cdot s[\kappa, \lambda]x = f_{\rho\sigma} \cdot {}_{\rho}s(\kappa, \lambda)s^{\rho}_{\sigma}x$.

Now we shall verify the conditions mentioned in II.17. We assume (A). First note by II.26–II.30 that $s[\kappa, \lambda]x \cdot s[\kappa, \lambda] - x = 0$ if $\kappa \neq \lambda$. Thus by II.25(i)–(iii) we have:

Lemma II.31 (A). *If* $\kappa \neq \lambda$, *then* $s[\kappa, \lambda]$ *is a Boolean endomorphism.*

Lemma II.32 (A). *If* $\kappa \neq \lambda$, *then* $s[\kappa, \lambda]c_{\kappa}x = s^{\lambda}_{\kappa}c_{\kappa}x$.

Proof. By II.26–II.30 there are two non-trivial cases. First, suppose that $\{\kappa, \lambda\} \cap \{\mu, \nu, \rho, \sigma\} = 0$ and $|\{\mu, \nu, \rho\}| = 3 = |\{\mu, \nu, \sigma\}|$. Let $t = -d_{\kappa\lambda} \cdot d_{\mu\rho} \cdot d_{\nu\sigma}$. Then

$$\begin{aligned} t \cdot s[\kappa, \lambda]c_{\kappa}x &= t \cdot {}_{\mu}s(\kappa, \lambda)c_{\mu}c_{\nu}(c_{\kappa}x \cdot d_{\mu\rho} \cdot d_{\nu\sigma}) \\ &= t \cdot s^{\mu}_{\lambda}s^{\lambda}_{\nu}c_{\mu}(d_{\mu\rho} \cdot c_{\kappa}x) = t \cdot s^{\lambda}_{\kappa}c_{\kappa}x. \end{aligned}$$

Second, suppose that $|\{\kappa, \lambda, \rho, \sigma\}| = 4$. Then

$$\begin{aligned} f_{\rho\sigma} \cdot s[\kappa, \lambda]c_{\kappa}x &= f_{\rho\sigma} \cdot {}_{\rho}s(\kappa, \lambda)s^{\rho}_{\sigma}c_{\kappa}x \\ &= f_{\rho\sigma} \cdot s^{\rho}_{\kappa}s^{\lambda}_{\rho}s^{\rho}_{\sigma}c_{\kappa}x = f_{\rho\sigma} \cdot s^{\lambda}_{\kappa}s^{\rho}_{\sigma}c_{\kappa}x = f_{\rho\sigma} \cdot s^{\lambda}_{\kappa}c_{\kappa}x. \end{aligned}$$

Lemma II.33 (A). *If* $\kappa \neq \lambda$, *then* $s[\kappa, \lambda] \circ s^{\kappa}_{\lambda} = s^{\lambda}_{\kappa}$.

Lemma II.34 (A). *If* $\kappa \neq \lambda$, *then* $s[\kappa, \lambda]s[\kappa, \lambda]x = x$.

Proof. We proceed by cases according to the definition of $s[\kappa, \lambda]x$. Clearly $s[\kappa, \lambda](d_{\kappa\lambda} \cdot x) = d_{\kappa\lambda} \cdot s[\kappa, \lambda]x = d_{\kappa\lambda} \cdot x$, using II.25(v) and II.26. If $\rho \neq \kappa, \lambda$, then, using II.33,

$$s[\kappa, \lambda](-d_{\kappa\lambda} \cdot d_{\kappa\rho} \cdot s^{\kappa}_{\lambda}s^{\lambda}_{\rho}x) = -d_{\kappa\lambda} \cdot d_{\lambda\rho} \cdot s^{\lambda}_{\rho}x = -d_{\kappa\lambda} \cdot d_{\lambda\rho} \cdot x.$$

By symmetry, $s[\kappa, \lambda](-d_{\kappa\lambda} \cdot d_{\lambda\rho} \cdot s^{\lambda}_{\kappa}s^{\kappa}_{\rho}x) = -d_{\kappa\lambda} \cdot d_{\kappa\rho} \cdot x$. Now suppose $\{\kappa, \lambda\} \cap \{\mu, \nu, \rho, \sigma\} = 0$ and $|\{\mu, \nu, \rho\}| = 3 = |\{\mu, \nu, \sigma\}|$. Let $t = -d_{\kappa\lambda} \cdot d_{\mu\rho} \cdot d_{\nu\sigma}$. Then

$$\begin{aligned} &s[\kappa, \lambda](t \cdot {}_{\mu}s(\kappa, \lambda)c_{\mu}c_{\nu}(x \cdot d_{\mu\rho} \cdot d_{\nu\sigma})) \\ &\quad = t \cdot {}_{\mu}s(\kappa, \lambda)c_{\mu}c_{\nu}({}_{\mu}s(\kappa, \lambda)c_{\mu}c_{\nu}(x \cdot d_{\mu\rho} \cdot d_{\nu\sigma}) \cdot d_{\mu\rho} \cdot d_{\nu\sigma}) \qquad \text{(by II.29)} \end{aligned}$$

Finally, suppose that $|\{\kappa, \lambda, \rho, \sigma\}| = 4$. Then

$$\begin{aligned}
&s[\kappa, \lambda](f_{\rho\sigma} \cdot {}_\rho s(\kappa, \lambda) s^\rho_\sigma x) \\
&\quad = f_{\rho\sigma} \cdot {}_\rho s(\kappa, \lambda) s^\rho_{\sigma\rho} s(\kappa, \lambda) c_\rho c_\sigma (x \cdot f_\rho \sigma) \qquad \text{(by II.23)} \\
&\quad = f_{\rho\sigma} \cdot {}_\rho s(\kappa, \lambda)_\rho s(\kappa, \lambda) c_\rho c_\sigma (x \cdot f_{\rho\sigma}) = f_{\rho\sigma} \cdot c_\rho c_\sigma (x \cdot f_{\rho\sigma}) \\
&\quad = x \cdot f_{\rho\sigma} \qquad \text{(by II.21)}.
\end{aligned}$$

In the next few lemmas we aim toward the result that $c_\mu s[\kappa, \lambda]x = s[\kappa, \lambda]c_\mu x$ for $|\{\kappa, \lambda, \mu\}| = 3$, one of the hardest results to establish.

Lemma II.35. *If* $|\{\kappa, \kappa, \mu\}| = 3$, *then* $d_{\kappa\lambda} \cdot c_\mu s[\kappa, \lambda]x \leqslant s[\kappa, \lambda]c_\mu x$.

Lemma II.36. *If* $|\{\kappa, \lambda, \mu, \rho\}| = 4$, *then* $-d_{\kappa\lambda} \cdot d_{\kappa\rho} \cdot c_\mu s[\kappa, \lambda]x \leqslant s[\kappa, \lambda]c_\mu x$.

Lemma II.37. *If* $|\{\kappa, \lambda, \mu, \rho, \sigma, \xi\}| = 6 = |\{\kappa, \lambda, \mu, \rho, \sigma, \eta\}|$, *then*

$$-d_{\kappa\lambda} \cdot d_{\rho\xi} \cdot d_{\sigma\eta} \cdot c_\mu s[\kappa, \lambda]x \leqslant s[\kappa, \lambda]c_\mu x.$$

Lemma II.38 (A). *If* $|\{\kappa, \lambda, \mu, \rho, \sigma\}| = 5$, *then*

$$-d_{\kappa\lambda} \cdot d_{\kappa\mu} \cdot d_{\rho\sigma} \cdot g_{\mu\rho} \cdot c_\mu s[\kappa, \lambda]x \leqslant s[\kappa, \lambda]c_\mu x.$$

Proof. Let $y = -d_{\kappa\lambda} \cdot d_{\kappa\mu} \cdot d_{\rho\sigma} \cdot g_{\mu\rho}$. First note that $y \cdot s[\kappa, \lambda]c_\mu x = y \cdot s^\kappa_\lambda s^\lambda_\mu c_\mu x$. Hence

$$\begin{aligned}
&y \cdot c_\mu(d_{\mu\nu} \cdot s[\kappa, \lambda]x) = y \cdot s[\kappa, \lambda]x \leqslant s[\kappa, \lambda]c_\mu x; \\
&y \cdot c_\mu(d_{\mu\lambda} \cdot s[\kappa, \lambda]x) = y \cdot c_\mu(d_{\mu\lambda} \cdot s^\lambda_\kappa s^\kappa_\mu x) \\
&\quad \leqslant y \cdot {}_\mu s(\lambda, \kappa) c_\mu c_\rho (x \cdot d_{\rho\sigma}) = y \cdot {}_\mu s(\kappa, \lambda) c_\mu x \leqslant s[\kappa, \lambda]c_\mu x; \\
&y \cdot c_\mu(d_{\mu\rho} \cdot s[\kappa, \lambda]x) = y \cdot c_\mu(d_{\mu\rho} \cdot {}_\mu s(\kappa, \lambda) c_\mu c_\rho (x \cdot d_{\mu\rho} \cdot d_{\rho\sigma})) \\
&\qquad\qquad = y \cdot {}_\mu s(\kappa, \lambda) c_\mu (x \cdot d_{\mu\rho}) \leqslant s[\kappa, \lambda]c_\mu x;
\end{aligned}$$

if $\nu \neq \kappa, \lambda, \mu, \rho, \sigma$, then $y \cdot c_\mu(d_{\mu\nu} \cdot s[\kappa, \lambda]x) \leqslant s[\kappa, \lambda]c_\mu x$ similarly. Finally,

$$\begin{aligned}
y \cdot c_\mu(f_{\rho\sigma} \cdot s[\kappa, \lambda]x) &= y \cdot c_\mu(f_{\rho\sigma} \cdot {}_\rho s(\kappa, \lambda) s^\rho_\sigma x) \\
&\leqslant y \cdot {}_\rho s(k, \lambda) s^\rho_\sigma c_\mu x = y \cdot s^\rho_\mu s^\kappa_\lambda s^\rho_\sigma c_\mu x \\
&= y \cdot s^\kappa_\lambda s^\lambda_\mu s^\rho_\sigma c_\mu x = y \cdot s^\kappa_\lambda s^\lambda_\mu c_\mu x.
\end{aligned}$$

Lemma II.39 (A). *If* $|\{\kappa, \lambda, \mu\}| = 3$, *then* $f_{\kappa\mu} \cdot c_\mu s[\kappa, \lambda]x \leqslant s[\kappa, \lambda]c_\mu x$.

Proof. By II.27 we have $f_{\kappa\mu} \cdot s[\kappa, \lambda]c_\mu x = f_{\kappa\mu} \cdot s^\kappa_\lambda s^\lambda_\mu c_\mu x$. Now

$$f_{\kappa\mu} \cdot c_\mu(d_{\kappa\mu} \cdot s[\kappa, \lambda]x) = f_{\kappa\mu} \cdot s[\kappa, \lambda]x \leqslant s[\kappa, \lambda]c_\mu x.$$

Next,

$$\begin{aligned} f_{\kappa\mu} \cdot c_\mu(d_{\mu\lambda} \cdot s[\kappa, \lambda]x) &= f_{\kappa\mu} \cdot c_\mu(d_{\mu\lambda} \cdot s^\lambda_\kappa s^\kappa_\mu x) \\ &= f_{\kappa\mu} \cdot {}_\mu s(\lambda, \kappa)x = f_{\kappa\mu} \cdot {}_\mu s(\lambda, \kappa)(f_{\kappa\mu} \cdot x) \qquad \text{(by II.20(ii))} \\ &= f_{\kappa\mu} \cdot {}_\mu s(\lambda, \kappa)(f_{\kappa\mu} \cdot c_\lambda(f_{\kappa\mu} \cdot x)) \qquad \text{(by II.22)} \\ &= f_{\kappa\mu} \cdot s^\mu_\lambda s^\kappa_\mu c_\lambda(f_{\kappa\mu} \cdot x) = f_{\kappa\mu} \cdot s^\kappa_\lambda s^\mu_\lambda c_\lambda(f_{\kappa\mu} \cdot x). \end{aligned}$$

Now

$$\begin{aligned} f_{\kappa\lambda} \cdot s^\mu_\lambda c_\lambda(f_{\kappa\mu} \cdot x) &= f_{\kappa\lambda} \cdot s^\mu_\kappa c_\lambda(f_{\kappa\mu} \cdot x) \\ &= f_{\kappa\lambda} \cdot c_\lambda(e_\mu \cdot s^\mu_\kappa x) = f_{\kappa\lambda} \cdot c_\lambda\Big(e_\mu \cdot \sum_{\nu \neq \mu} d_{\mu\nu} \cdot s^\mu_\kappa x\Big) \\ &= f_{\kappa\lambda} \cdot c_\lambda(f_{\mu\lambda} \cdot s^\mu_\kappa x) \leqslant s^\lambda_\mu c_\mu x \end{aligned}$$

and hence $f_{\kappa\mu} \cdot c_\mu(d_{\mu\lambda} \cdot s[\kappa, \lambda]x) \leqslant s^\kappa_\lambda s^\lambda_\mu c_\mu x$ by II.20(ii). Finally, if $\rho \neq \kappa, \lambda, \mu$, then

$$f_{\kappa\mu} \cdot c_\mu(f_{\mu\rho} \cdot s[\kappa, \lambda]x = f_{\kappa\mu} \cdot c_\mu(f_{\mu\rho} \cdot {}_\mu s(\kappa, \lambda)s^\mu_\rho x) \leqslant s^\kappa_\lambda s^\lambda_\mu c_\mu x,$$

and the proof is complete.

Lemma II.40 (A). *If* $|\{\kappa, \lambda, \mu, \rho, \sigma\}| = 5$, *then*

$$-d_{\kappa\lambda} \cdot d_{\mu\rho} \cdot d_{\mu\sigma} \cdot g_{\rho\sigma} \cdot c_\mu s[\kappa, \lambda]x \leqslant s[\kappa, \lambda]c_\mu x.$$

Proof. Let $t = -d_{\kappa\lambda} \cdot d_{\mu\rho} \cdot d_{\mu\sigma} \cdot g_{\rho\sigma}$. Then

$$\begin{aligned} t \cdot s[\kappa, \lambda]c_\mu x &= t \cdot {}_\mu s(\kappa, \lambda)c_\mu c_\rho(c_\mu x \cdot d_{\mu\rho} \cdot d_{\mu\sigma}) \\ &= t \cdot {}_\mu s(\kappa, \lambda)s^\mu_\rho c_\mu x \\ &= t \cdot {}_\rho s(\kappa, \lambda)s^\rho_\sigma c_\mu x. \end{aligned}$$

Now $t \cdot c_\mu(d_{\mu\rho} \cdot s[\kappa, \lambda]x) = t \cdot s[\kappa, \lambda]x \leqslant s[\kappa, \lambda]c_\mu x$. Similarly, $t \cdot c_\mu(d_{\mu\sigma} \cdot s[\kappa, \lambda]x) \leqslant s[\kappa, \lambda]c_\mu x$. Suppose $\nu \neq \kappa, \lambda, \mu, \rho, \sigma$.

$$\begin{aligned} t \cdot c_\mu(d_{\mu\nu} \cdot s[\kappa, \lambda]x) &= t \cdot c_\mu(d_{\mu\nu} \cdot {}_\mu s(\kappa, \lambda)c_\mu c_\rho(x \cdot d_{\mu\nu} \cdot d_{\rho\sigma})) \\ &= t \cdot {}_\rho s(\kappa, \lambda)s^\rho_\sigma s^\mu_\nu x \leqslant s[\kappa, \lambda]c_\mu x. \end{aligned}$$

Finally,

$$t \cdot c_\mu(f_{\rho\sigma} \cdot s[\kappa, \lambda]x) = t \cdot c_\mu(f_{\rho\sigma} \cdot {}_\rho s(\kappa, \lambda)s^\rho_\sigma x) \leqslant s[\kappa, \lambda]c_\mu x.$$

The following lemma is established in a very similar manner:

Lemma II.41 (A). *If* $|\{\kappa, \lambda, \mu, \rho, \sigma, \nu\}| = 6$ *then*

$$-d_{\kappa\lambda} \cdot d_{\mu\rho} \cdot d_{\sigma\nu} \cdot g_{\mu\sigma} \cdot c_\mu s[\kappa, \lambda]x \leqslant s[\kappa, \lambda]c_\mu x.$$

Lemma II.42 (A). *If* $|\{\kappa, \lambda, \mu, \rho\}| = 4$, *then* $f_{\mu\rho} \cdot c_\mu s[\kappa, \lambda]x \leqslant s[\kappa, \lambda]c_\mu x$.

Proof. We have

$$\begin{aligned} f_{\mu\rho} \cdot s[\kappa, \lambda]c_\mu x &= f_\mu\rho \cdot {}_\mu s(\kappa, \lambda)c_\mu x \\ &= f_{\mu\rho} \cdot s[\lambda, \kappa]c_\mu x \\ &= f_{\mu\rho} \cdot {}_\mu s(\lambda, \kappa)c_\mu x. \end{aligned}$$

Now note that $f_{\mu\rho} \cdot c_\mu s[\kappa, \lambda]x = \sum_{\nu\neq\mu} f_{\mu\rho} \cdot c_\mu(d_{\mu\nu} \cdot s[\kappa, \lambda]x)$. We have

$$\begin{aligned} f_{\mu\rho} \cdot c_\mu(d_{\mu\rho} \cdot s[\kappa, \lambda]x) &= f_{\mu\rho} \cdot s[\kappa, \lambda]x \leqslant s[\kappa, \lambda]c_\mu x; \\ f_{\mu\rho} \cdot c_\mu(d_{\mu\kappa} \cdot s[\kappa, \lambda]x) &= f_{\mu\rho} \cdot c_\mu(d_{\mu\kappa} \cdot s^\kappa_\lambda s^\lambda_\mu x) \\ &\leqslant s[\kappa, \lambda]c_\mu x; \end{aligned}$$

$f_{\mu\rho} \cdot c_\mu(d_{\mu\lambda} \cdot s[\kappa, \lambda]x) \leqslant s[\kappa, \lambda]c_\mu x$ similarly. Finally, if $\nu \neq \kappa, \lambda, \mu, \rho$, then

$$\begin{aligned} f_{\mu\rho} \cdot c_\mu(d_{\mu\nu} \cdot s[\kappa, \lambda]x) &= f_{\mu\rho} \cdot c_\mu(f_{\mu\nu} \cdot s[\kappa, \lambda]x) \\ &= f_{\mu\rho} \cdot c_\mu(f_{\mu\nu} \cdot {}_\mu s(\kappa, \lambda)s^\mu_\nu x) \leqslant s[\kappa, \lambda]c_\mu x. \end{aligned}$$

Lemma II.43 (A). *If* $|\{\kappa, \lambda, \rho, \sigma\}| = 4$, *then* $f_{\rho\sigma} \cdot c_\mu s[\kappa, \lambda]x \leqslant s[\kappa, \lambda]c_\mu x$.

Proof. We have

$$\begin{aligned} f_{\rho\sigma} \cdot s[\kappa, \lambda]c_\mu x &= f_{\rho\sigma} \cdot {}_\rho s(\kappa, \lambda)s^\rho_\sigma c_\mu x = f_{\rho\sigma} \cdot {}_\mu s(\kappa, \lambda)s^\rho_\sigma c_\mu x \\ &= f_{\rho\sigma} \cdot {}_\rho s(\lambda, \kappa)s^\rho_\sigma c_\mu x. \end{aligned}$$

Hence

$$f_{\rho\sigma} \cdot c_\mu(d_{\mu\nu} \cdot s[\kappa, \lambda]x) = f_{\rho\sigma} \cdot {}_\mu s(\kappa, \lambda)x \leqslant s[\kappa, \lambda]c_\mu x;$$

$f_{\rho\sigma} \cdot c_\mu(d_{\mu\lambda} \cdot s[\kappa, \kappa]x) \leqslant s[\kappa, \lambda]c_\mu x$ similarly. Further,

$$\begin{aligned} f_{\rho\sigma} \cdot c_\mu(d_{\mu\rho} \cdot s[\kappa, \lambda]x) &= f_{\rho\sigma} \cdot c_\mu(d_{\mu\rho} \cdot {}_\mu s(\kappa, \lambda)c_\mu c_\rho(x \cdot d_{\mu\rho} \cdot d_{\mu\sigma})) \\ &\leqslant s[\kappa, \lambda]c_\mu x. \end{aligned}$$

If $\nu \neq \kappa, \lambda, \rho, \sigma$, $f_{\rho\sigma} \cdot c_\mu(d_{\mu\nu} \cdot s[\kappa, \lambda]x) \leqslant s[\kappa, \lambda]c_\mu x$ similarly. Finally, $f_{\rho\sigma} \cdot c_\mu(f_{\rho\sigma} \cdot s[\kappa, \lambda]x) = f_{\rho\sigma} \cdot s[\kappa, \lambda]x \leqslant s[\kappa, \lambda]c_\mu x$ by II.22.

Now by II.35–II.43 we have:

Lemma II.44 (A). *If* $|\{\kappa, \lambda, \mu\}| = 3$, *then* $c_\mu s[\kappa, \lambda]x \leqslant s[\kappa, \lambda]c_\mu x$.

Lemma II.45 (A). *If* $|\{\kappa, \lambda, \mu\}| = 3$, *then* $c_\mu s[\kappa, \lambda]x = s[\kappa, \lambda]c_\mu x$.

Proof. Applying II.44 to $s[\kappa, \lambda]x$ in place of x, and using II.34, we obtain $c_\mu x \leqslant s[\kappa, \lambda]c_\mu s[\kappa, \lambda]x$. Applying $s[\kappa, \lambda]$ to both sides and using II.34 we get $s[\kappa, \lambda]c_\mu x \leqslant c_\mu s[\kappa, \lambda]x$. Combining this with II.44 gives the desired result.

Proof. $s[\kappa, \lambda]s^{\mu}_{\kappa}x = c_{\mu}s[\kappa, \lambda](d_{\mu\nu} \cdot x) = c_{\mu}(d_{\mu\lambda} \cdot s[\kappa, \lambda]x) = s^{\mu}_{\lambda}s[\kappa, \lambda]x$, using II.45.

Lemma II.47. *If $\kappa \neq \lambda$, then $c_{\kappa}s[\kappa, \lambda]x \leqslant s^{\kappa}_{\lambda}c_{\lambda}x$.*

Proof. We have $c_{\kappa}(d_{\kappa\lambda} \cdot s[\kappa, \lambda]x) = c_{\kappa}(d_{\kappa\lambda} \cdot x) \leqslant s^{\kappa}_{\lambda}c_{\lambda}x$. If $\rho \neq \kappa, \lambda$, then

$$c_{\kappa}(-d_{\kappa\lambda} \cdot d_{\kappa\rho} \cdot s[\kappa, \lambda]x) = c_{\kappa}(-d_{\kappa\lambda} \cdot d_{\kappa\rho} \cdot s^{\kappa}_{\lambda}s^{\lambda}_{\rho}x) \leqslant s^{\kappa}_{\lambda}c_{\lambda}x;$$

$$\begin{aligned} c_{\kappa}(-d_{\kappa\lambda} \cdot d_{\lambda\rho} \cdot s[\kappa, \lambda]x) &= c_{\kappa}(-d_{\kappa\lambda} \cdot d_{\lambda\rho} \cdot s^{\lambda}_{\kappa}s^{\kappa}_{\rho}x) \\ &\leqslant c_{\kappa}(-d_{\kappa\lambda} \cdot d_{\lambda\rho} \cdot s^{\kappa}_{\rho}c_{\lambda}x) \leqslant s^{\kappa}_{\lambda}c_{\lambda}x. \end{aligned}$$

If $\{\kappa, \lambda\} \cap \{\mu, \nu, \rho, \sigma\} = 0$ and $|\{\mu, \nu, \rho\}| = 3 = |\{\mu, \nu, \sigma\}|$, then, with $t = -d_{\kappa\lambda} \cdot d_{\mu\rho} \cdot d_{\nu\sigma}$,

$$\begin{aligned} c_{\kappa}(t \cdot s[\kappa, \lambda]x) &= c_{\kappa}(t \cdot {}_{\mu}s(\kappa, \lambda)c_{\mu}c_{\nu}(d_{\mu\rho} \cdot d_{\nu\sigma} \cdot x)) \\ &\leqslant c_{\kappa}(t \cdot s^{\kappa}_{\lambda}c_{\mu}(c_{\lambda}x \cdot d_{\mu\rho})) \leqslant s^{\kappa}_{\lambda}c_{\lambda}x. \end{aligned}$$

Finally, if $|\{\kappa, \lambda, \rho, \sigma\}| = 4$, then

$$\begin{aligned} c_{\kappa}(f_{\rho\sigma} \cdot s[\kappa, \lambda]x) &= c_{\kappa}(f_{\rho\sigma} \cdot {}_{\rho}s(\kappa, \lambda)s^{\rho}_{\sigma}x) \\ &\leqslant c_{\kappa}(f_{\rho\sigma} \cdot s^{\rho}_{\kappa}s^{\kappa}_{\lambda}s^{\rho}_{\sigma}c_{\lambda}x) \leqslant s^{\kappa}_{\lambda}c_{\lambda}x. \end{aligned}$$

Lemma II.48 (A). *If $k \neq \lambda$, then $c_{\kappa}s[\kappa, \lambda]x = s^{\kappa}_{\lambda}c_{\lambda}x$.*

Proof. Applying II.47 with κ and λ exchanged to $s[\kappa, \lambda]x$, we get $c_{\lambda}x \leqslant s^{\lambda}_{\kappa}c_{\kappa}s[\kappa, \lambda]x$, using II.25(iv), II.34. Hence $s^{\kappa}_{\lambda}c_{\lambda}x \leqslant c_{\kappa}s[\kappa, \lambda]x$. Then II.47 yields the desired result.

Lemma II.49 (A). *If $\kappa \neq \lambda$, then $c_{\kappa}s[\kappa, \lambda]x = s[\kappa, \lambda]c_{\lambda}x$.*

Proof. By II.32 and II.48.

The last lemma needed in order to apply Jónsson's theorem in II.17 requires a lengthy proof:

Lemma II.50 (A). *If $|\{\kappa, \lambda, \mu\}| = 3$, then $s[\kappa, \lambda]s[\kappa, \mu]x = s[\lambda, \mu]s[\kappa, \lambda]x$.*

Proof. We have

$$s[\kappa, \lambda]s[\kappa, \mu](x \cdot d_{\kappa\lambda} \cdot d_{\kappa\mu}) = x \cdot d_{\kappa\lambda} \cdot d_{\kappa\mu} = s[\lambda, \mu]s[\kappa, \lambda]x \cdot d_{\kappa\lambda} \cdot d_{\kappa\mu};$$

$$\begin{aligned} (1) \quad s[\kappa, \lambda]s[\kappa, \mu](x \cdot d_{\kappa\lambda} \cdot -d_{\kappa\mu}) &= s[\kappa, \lambda](d_{\mu\lambda} \cdot -d_{\kappa\mu} \cdot s^{\mu}_{\kappa}s^{\kappa}_{\lambda}x) \\ &= d_{\mu\kappa} \cdot -d_{\lambda\mu} \cdot s^{\mu}_{\lambda}s^{\lambda}_{\kappa}x; \end{aligned}$$

$$\begin{aligned} (2) \quad s[\lambda, \mu]s[\kappa, \lambda](x \cdot d_{\kappa\lambda} \cdot -d_{\kappa\mu}) &= s[\lambda, \mu](d_{\kappa}\lambda \cdot -d_{\lambda\mu} \cdot x) \\ &= d_{\kappa\lambda} \cdot -d_{\lambda\mu} \cdot s^{\mu}_{\lambda}s^{\lambda}_{\kappa}x; \end{aligned}$$

(3) $s[\lambda, \mu]s[\kappa, \lambda](x \cdot d_{\kappa\mu} \cdot -d_{\kappa\lambda}) = s[\lambda, \mu](d_{\lambda\mu} \cdot -d_{\kappa\lambda} \cdot s^{\lambda}_{\kappa}s^{\kappa}_{\mu}x)$
$= d_{\lambda\mu} \cdot -d_{\kappa\mu} \cdot s^{\lambda}_{\kappa}s^{\kappa}_{\mu}x;$

$s[\kappa, \lambda]s[\kappa, \mu](x \cdot d_{\lambda\mu} \cdot -d_{\kappa\lambda}) = d_{\kappa\lambda} \cdot -d_{\mu\lambda} \cdot s^{\kappa}_{\mu}s^{\mu}_{\lambda}x$ by symmetry from (3);

$s[\lambda, \mu]s[\kappa, \lambda](x \cdot d_{\lambda\mu} \cdot -d_{\kappa\lambda}) = d_{\kappa\lambda} \cdot -d_{\mu\kappa} \cdot s^{\kappa}_{\mu}s^{\mu}_{\lambda}x$ by symmetry from (1).

Now, let $t = -d_{\kappa\lambda} \cdot -d_{\kappa\lambda} \cdot -d_{\kappa\mu} \cdot -d_{\lambda\mu}$. Suppose $\rho \neq \kappa, \lambda, \mu$. Then

(4) $s[\kappa, \lambda]s[\kappa, \mu](x \cdot t \cdot d_{\kappa\rho}) = s[\kappa, \lambda](t \cdot d_{\mu\rho} \cdot s^{\mu}_{\kappa}s^{\kappa}_{\rho}x)$
$= t \cdot d_{\mu\rho} \cdot s^{\mu}_{\lambda}s^{\lambda}_{\kappa}s^{\kappa}_{\rho}x;$

(5) $s[\lambda, \mu]s[\kappa, \lambda](x \cdot t \cdot d_{\kappa\rho}) = s[\lambda, \mu](t \cdot d_{\lambda}\rho \cdot s^{\lambda}_{\kappa}s^{\kappa}_{\rho}x)$
$= t \cdot d_{\mu\rho} \cdot s^{\mu}_{\lambda}s^{\lambda}_{\kappa}s^{\kappa}_{\rho}x;$

(6) $s[\kappa, \lambda]s[\kappa, \mu](x \cdot t \cdot d_{\lambda\rho}) = s[\kappa, \lambda](t \cdot d_{\lambda\rho} \cdot s[\kappa, \mu]x)$
$= t \cdot d_{\kappa\rho} \cdot s^{\kappa}_{\lambda}s^{\lambda}_{\rho}s[\kappa, \mu]x) = t \cdot d_{\kappa\rho} \cdot c_{\kappa}s[\kappa, \mu](d_{\mu\lambda} \cdot s^{\lambda}_{\rho}x)$
$= t \cdot d_{\kappa\rho} \cdot s^{\kappa}_{\mu}s^{\mu}_{\lambda}s^{\lambda}_{\rho}x$ (by II.48)
$= s[\lambda, \mu]s[\kappa, \lambda](x \cdot t \cdot d_{\lambda\rho})$ by symmetry from (4);

$s[\kappa, \lambda]s[\kappa, \mu](x \cdot t \cdot d_{\mu\rho}) = t \cdot d_{\lambda\rho} \cdot s^{\lambda}_{\kappa}s^{\kappa}_{\mu}s^{\mu}_{\rho}x$
$= s[\lambda, \mu]s[\kappa, \lambda](x \cdot t \cdot d_{\mu\rho})$ by symmetry from (5), (6).

Now suppose that $\{\kappa, \lambda, \mu\} \cap \{\nu, \rho, \sigma, \tau\} = 0$ and $|\{\nu, \rho, \sigma\}| = 3 = |\{\nu, \rho, \tau\}|$. Let $y = t \cdot d_{\nu\sigma} \cdot d_{\rho\tau}$. Then

$$\begin{aligned}
s[\kappa, \lambda]s[\kappa, \mu](x \cdot y) &= s[\kappa, \lambda](y \cdot {}_{\nu}s(\kappa, \mu)c_{\nu}c_{\rho}(x \cdot d_{\nu\sigma} \cdot d_{\rho\tau})) \\
&= y \cdot {}_{\nu}s(\kappa, \lambda)c_{\nu}c_{\rho}({}_{\nu}s(\kappa, \mu)c_{\nu}c_{\rho}(x \cdot d_{\nu\sigma} \cdot d_{\rho\tau}) \cdot d_{\nu\sigma} \cdot d_{\rho\tau}) \\
&= y \cdot {}_{\nu}s(\kappa, \lambda){}_{\nu}s(\kappa, \mu)c_{\nu}c_{\rho}(x \cdot d_{\nu\sigma} \cdot d_{\rho\tau}) \\
&= y \cdot {}_{\nu}s(\lambda, \mu){}_{\nu}s(\kappa, \lambda)c_{\nu}c_{\rho}(x \cdot d_{\nu\sigma} \cdot d_{\rho\tau}) \qquad \text{by [5, 1.5.18]} \\
&= s[\lambda, \mu]s[\kappa, \lambda](x \cdot y) \qquad \text{by symmetry.}
\end{aligned}$$

Finally, suppose that $\{\kappa, \lambda, \mu\} \cap \{\rho, \sigma\} = 0$ and $\rho \neq \sigma$. Then, using II.23,

$$\begin{aligned}
s[\kappa, \lambda]s[\kappa, \mu](x \cdot f_{\rho\sigma}) &= s[\kappa, \lambda](f_{\rho\sigma} \cdot {}_{\rho}s(\kappa, \mu)c_{\rho}c_{\sigma}(x \cdot f_{\rho\sigma})) \\
&= f_{\rho\sigma} \cdot {}_{\rho}s(\kappa, \lambda)s^{\rho}_{\sigma\rho}s(\kappa, \mu)c_{\rho}c_{\sigma}(x \cdot f_{\rho\sigma}) \\
&= f_{\rho\sigma} \cdot {}_{\rho}s(\lambda, \mu){}_{\rho}s(\kappa, \lambda)c_{\rho}c_{\sigma}(x \cdot f_{\rho\sigma}) \\
&= s[\lambda, \mu]s[\kappa, \lambda](x \cdot y) \qquad \text{by symmetry.}
\end{aligned}$$

Finally we are ready for the auxiliary result of Comer and Henkin:

Theorem II.51. *Suppose $\alpha < \omega$, and $\mathfrak{A}$ is a CA_{α} of positive characteristic. Then there is a function s assigning to every $\tau \in {}^{\alpha}\alpha$ an endomorphism s_{τ} of $\mathfrak{Bl}\,\mathfrak{A}$ such that the following conditions hold for any $\sigma, \tau \in {}^{\alpha}\alpha$ and $\kappa, \lambda \in \alpha$:*

(ii) $s_\sigma = A \upharpoonright \mathrm{Id}$ *if* $\sigma = \alpha \upharpoonright \mathrm{Id}$.
(iii) $s_{[\kappa/\lambda]} = s^\kappa_\lambda$.
(iv) $s_\tau d_{\kappa\lambda} = d_{\tau\kappa,\tau\lambda}$.
(v) *If* $(\alpha \sim \{\kappa\}) \upharpoonright \sigma \subseteq \tau$, *then* $s_\sigma c_\kappa x = s_\tau c_\kappa x$.
(vi) *If* $\kappa \notin \mathrm{Rg}\,\sigma$, *then* $c_\kappa s_\sigma x = s_\sigma x$.
(vii) *If* $\sigma^{-1*}\{\kappa\} = \{\lambda\}$, *then* $c_\kappa s_\sigma x = s_\sigma c_\lambda x$.

Proof. By II.18–II.50 and Jónsson's theorem in II.17 we obtain a function s assigning to every $\tau \in {}^\alpha\alpha$ an endomorphism s_τ of $\mathfrak{Bl}\,\mathfrak{A}$ such that (i)–(iii) holds and $s_{[\kappa,\lambda]} = s[\kappa, \lambda]$ for any distinct $\kappa, \lambda < \alpha$. Now (iv) follows from II.25(v)–(vii). For (v), we may assume that $\alpha \geqslant 2$. Choose $\lambda \in \alpha \sim \{\kappa\}$. Then $\sigma \circ [\kappa/\lambda] = \tau \circ [\kappa/\lambda]$, and hence

$$s_\sigma c_\kappa x = s_\sigma s^\kappa_\lambda c_\kappa x = s_{\sigma\circ[\kappa/\lambda]} c_\kappa x = s_\rho c_\kappa x.$$

For (iv), choose $\lambda \in \alpha \sim \{\kappa\}$. Then $[\kappa/\lambda] \circ \sigma = \sigma$, so

$$c_\kappa s_\sigma x = c_\kappa s_{[\kappa/\lambda]\circ\sigma} x = c_\kappa s^\kappa_\lambda s_\sigma x = s^\kappa_\lambda s_\sigma x = s_\sigma x.$$

Finally, for (vii) note that $([\kappa, \lambda] \circ \sigma)^{-1*}\{\lambda\} = \{\lambda\}$. Hence $[\kappa, \lambda] \circ \sigma$ can be writeen as a composition of replacements and transpositions none of which involve λ. Hence by II.45 we have $x_\lambda s_{[\kappa,\lambda]\circ\sigma} x = s_{[\kappa,\lambda]\circ\sigma} c_\lambda x$, hence $c_\lambda s[\kappa, \lambda] s_\sigma x = s[\kappa, \lambda] s_\sigma c_\lambda x$. Hence by II.49, $c_\kappa s_\sigma x = s_\sigma c_\lambda x$, as desired.

The following result is closely related to II.51, but will not be needed for the positive characteristic representation theorem.

Theorem II.52. *Let* $\mathfrak{C}$ *be a* $\mathrm{CA}_{\alpha+2}$ *and* $\mathfrak{A} = \mathfrak{Nr}_\alpha\,\mathfrak{C}$. *Then there is a function* s *assigning to every finite transformation* σ *of* α *an endomorphism* s_σ *of* $\mathfrak{Bl}\,\mathfrak{A}$ *such that the conditions of* II.51 *hold for all finite transformations,* σ, τ *of* α *and all* $\kappa, \lambda < \alpha$.

Proof. This time for distinct $\kappa, \lambda < \alpha$ we let $s[\kappa, \lambda]x = {}_\alpha s(\kappa, \lambda)x$, for all $x \in A$. The conditions in II.17 and II.51 follow easily using results in [5, Section 1.5].

Both II.51 and II.52 express relationships between cylindric and polyadic algebras. In this connection, see also [3]. Also, substitutions are definable in any Gs_α with all subbases of size $\leqslant \alpha + 1$, for $\alpha < \omega$; this is a result of Andréka and Németi.

Now we turn to the promised representation theorem of Henkin.

Theorem II.53. *Let* $\alpha < \omega$. *If* $\mathfrak{A}$ *is a subdirect product of* CA_α*'s of positive characteristic, then* $\mathfrak{A}$ *is representable.*

available a substitution operation s satisfying (i)–(vii) there. Let κ be the characteristic of $\mathfrak{A}$. Thus $\bar{d}(\kappa \times \kappa) \neq 0$, and so there is an ultrafilter F on $\mathfrak{Bl}\,\mathfrak{A}$ such that $\bar{d}(\kappa \times \kappa) \in F$. (Recall that $\bar{d}(\kappa \times \kappa) = \prod_{\xi,\eta\langle\kappa,\xi\neq\eta} - d_{\xi\eta}$.) Now for any $x \in A$ let

$$fx = \{\tau \in {}^{\alpha}\kappa : s_\tau x \in F\}.$$

From this definition it is clear that f preserves $+$ and $-$. Since $\mathfrak{A}$ is simple, it suffices to show that f preserves $d_{\xi\eta}$ and c_ξ for all $\xi, \eta < \alpha$. We have $\tau \in fd_{\xi\eta}$ iff $s_\tau d_{\xi\eta} \in F$ iff $d_{\tau\xi,\tau\eta} \in F$ iff $\tau\xi = \tau\eta$, since $-d_{\mu\nu} \in F$ for all distinct $\mu, \nu < \kappa$. Thus f preserves $d_{\xi\eta}$.

To show that f preserves c_ξ, we need several steps.

(1) *If $\kappa \leqslant \xi < \alpha$ and $c_\xi x \in F$, then $s^\xi_\eta x \in F$ for some $\eta < \kappa$.*

For,

$$c_\xi x = c_\xi\Big(x \cdot \sum_{\mu,\nu<\kappa,\mu\neq\nu} d_{\mu\nu} + x \cdot \sum_{\mu<\kappa} d_{\mu\xi}\Big) = \sum_{\mu,\nu<\kappa,\mu\neq\nu} d_{\mu\nu} \cdot c_\xi x + \sum_{\mu<\kappa} s^\xi_\mu x.$$

Since $d_{\mu\nu} \notin F$ for all distinct $\mu, \nu < \kappa$, the conclusion of (1) follows. Let $V = {}^{\alpha}\kappa$.

(2) *If $\xi < \kappa$ and $\tau \in fc_\xi x$, then $\tau \in C^{[V]}_\xi fx$.*

For, under the hypothesis of (2) we have $s_\tau c_\xi x \in F$. Let $\sigma = \tau^\xi_\kappa$. Then $\sigma^{-1*}\{\kappa\} = \{\xi\}$, $(\alpha \sim \{\xi\}) \restriction \tau \subseteq \sigma$, so by II.51(v) and (vi) we get $s_\tau c_\tau x = s_\sigma c_\xi x = c_\kappa s_\sigma x$. Hence by (1) choose $\eta < \kappa$ with $s^\kappa_\eta s_\sigma x \in F$. Thus $[\kappa/\eta] \circ \sigma \in fx$ and $(\alpha \sim \{\xi\}) \restriction \tau \subseteq [\kappa/\eta] \circ \sigma$, so $\tau \in C_\xi fx$.

(3) *If $\kappa \leqslant \xi < \alpha$ and $\tau \in fc_\xi x$, then $\tau \in C^{[V]}_\xi fx$.*

The proof is similar to that of (2), using $\sigma = \tau^\xi_\xi$.

(4) *If $\xi < \alpha$ and $\tau \in C^{[V]}_\xi fx$, then $\tau \in fc_\xi x$.*

For, any $\tau^\xi_\eta \in fx$, with $\eta < \kappa$. Let $\sigma = \tau^\xi_\eta$. Then $s_\sigma x \in F$, so $s_\sigma c_\xi \in F$. But $s_\sigma c_\xi x = s_\tau c_\xi x$ by II.51(v), so $s_\rho c_\xi x \in F$, hence $\tau \in Fc_\xi x$. This completes the proof of II.53.

Theorem II.53 has been generalized by Andréka and Németi: they show that if $\alpha < \omega$ and $c_\kappa(x \cdot d(\alpha \times \alpha)) \cdot d(\alpha \times \alpha) = x \cdot d(\alpha + \alpha)$ for all $\kappa < \alpha$ and all x, then $\mathfrak{A}$ is representable (the hypothesis of II.53 is equivalent to $d(\alpha \times \alpha) = 0$).

Next we consider representability for CA_0's and CA_1's, both cases being rather trivial.

Theorem II.54. *Every CA_0 is representable.*

Proof. By [5, 2.4.52] it suffices to take the case of a simple CA_0 $\mathfrak{A}$. Thus $|A| = 2$,

Theorem II.55. *Every* CA_1 *is representable.*

Proof. Again it suffices to take a simple $\mathrm{CA}_1\,\mathfrak{A}$. Thus by [5, 2.13.14] the cylindrification c_0 is given by: $c_0 0 = 0$, $c_0 a = 1$ if $a \neq 0$. By the Boolean representation theorem, say f is an isomorphism from $\mathfrak{Bl}\,\mathfrak{A}$ onto a field of subsets of some set U. Define $ga = \{\langle u\rangle : u \in fa\}$ for all $a \in A$. Clearly g is an isomorphism from $\mathfrak{A}$ onto a Cs_1.

Corollary II.56. *For* $\alpha \leqslant 1$ *we have* $\mathrm{CA}_\alpha = \mathrm{SPCs}_\alpha = \mathrm{SNr}_\alpha \mathrm{CA}_{\alpha+\omega}$.

Proof. By [6, I.8.6], II.54, and II.55.

Our last two positive representation results in this section concern CA_2's. The first result is that any CA_2 is isomorphic to a Crs_2. To formulate the other result, recall from [5, 2.6.42] that for each $\alpha \geqslant 2$ there is a CA_α in which the equation

$$c_1(x \cdot y \cdot c_0(x \cdot -y)) \cdot -c_0(c_1 x \cdot -d_{01}) = 0$$

fails to hold identically; on the other hand, by [5, 2.6.41] this equation holds for every representable CA_α. Of course the same two statements hold if we replace 0 and 1 by arbitrary distinct $\kappa, \lambda < \alpha$. The second result is that if the above equation and its symmetric form with 0 and 1 interchanged hold in a $\mathrm{CA}_2\,\mathfrak{A}$, then $\mathfrak{A}$ is representable. These two results are due to Henkin and Tarski, announced in [7]. The underlying idea goes gack to Evertt and Ulam [2]. We shall derive both results from a rather technical lemma, II.59. For it we need some preparation.

Lemma II.57. *For any non-empty set* U *there is a partition* $\mathcal{P}$ *of* ${}^2U$ *such that* $|\mathcal{P}| = |U|$, $D_{01}^{[V]} \in \mathcal{P}$, *where* $V = {}^2U$, *and* $C_\kappa^{[V]} X = {}^2U$ *for each* $X \in \mathcal{P}$ *and each* $\kappa < 2$.

Proof. Let $\circ$ be a group operation on U. For each $u \in U$ let $X_u = \{\langle v, u \circ v\rangle : v \in U\}$. If $\langle v, w\rangle \in X_u \cap X_u$, then $x = u \circ v = u' \circ v$, hence $u = u'$. Given any $\langle v, w\rangle \in {}^2U$ we have $\langle v, w\rangle \in X_u$, where $u = w \circ v^{-1}$. Clearly $X_u \neq 0$ for all $u \in U$. Thus $\{X_u : u \in U\}$ is a partition of ${}^2U$, and $|\{X_u : u \in U\}| = |U|$. If e is the identity of the group $\langle u, \circ\rangle$, then $X_e = D_{01}^{[V]}$. Finally, let $u \in U$; we show that $C_0 X_u = C_1 X_u = V$. Let $\langle v, w\rangle \in V$ be arbitrary. Then $\langle u^{-1} \circ w, w\rangle \in X_u$ and $\langle v, u \circ v\rangle \in X_u$ so $\langle v, w\rangle \in C_0 X_u \cap C_1 X_u$, as desired.

Lemma II.58. *Suppose* $|U| = |U'| \geqslant \kappa > 0$, κ *a cardinal, and* $V = U \times U'$. *Then there is a partition* $\mathcal{P}$ *of* V *such that* $|\mathcal{P}| = \kappa$, $D_{01}^{[V]} \in \mathcal{P}$ *if* $U = U'$ *and* $\kappa > 1$, *and* $C_0^{[V]} X = C_1^{[V]} X = V$ *for all* $X \in \mathcal{P}$.

Proof. Set $W = {}^\alpha U$. Let $\mathcal{P}$ be as in II.57. Choose $\mathcal{P}' \subseteq \mathcal{P}$ with $|\mathcal{P}'| = \kappa$ and

let $\mathcal{P}'' = (\mathcal{P}' \sim \{X\}) \cup \{Y\}$. Let F be a one–one function from U' onto U, with $f = U \restriction \mathrm{Id}$ if $U = U'$. For each X let $X^* = \{\langle u, v\rangle : u \in U,\ v \in U,$ and $\langle u, fv\rangle \in X\}$. Finally, let $\mathcal{P} = \{X^* : X \in \mathcal{P}''\}$. Clearly $\mathcal{P}$ is as desired.

Lemma II.59. *Let $\mathfrak{A}$ be a simple complete atomic* CA_2. *An atom $a \in \mathrm{At}\,\mathfrak{A}$ is said to be* defective *if the following condition holds*:

(*) $c_0 a \cdot c_1 a \leqslant d_{01}$, *and there exist $\kappa, \lambda < 2$ and $x, y, z \in A$ such that $\{\kappa, \lambda\} = 2$, $x \leqslant c_\kappa a$, and $c_\lambda(x \cdot y) \cdot c_\lambda(x \cdot z) \cdot -c_\lambda(x \cdot y \cdot x) \neq 0$.*

Let I be the set of all defective atoms of $\mathfrak{A}$. Then $\mathfrak{A}$ is isomorphic to a Crs_2 *whose unit element has the form ${}^2U \sim \bigcup_{i \in I}({}^2X_i \sim D_{01}^{[W]})$, where U is some set, $W = {}^2U$, $X_i \cap X_j = 0$ for $i \neq j$ and $X_i \subseteq U$ for all $i \in I$.*

Proof. Let $\mathrm{Dat} = \{a \in \mathrm{At}\,\mathfrak{A} : a \leqslant d_{01}\}$. An element $a \in \mathrm{Dat}$ is *small* if $c_0 a \cdot c_1 a \leqslant d_{01}$; otherwise it is *big*. Note that if a is small, then $a = c_0 a \cdot c_1 a$, since $c_0 a \cdot c_1 a = c_0(a \cdot d_{01}) \cdot c_1(a \cdot d_{01}) \cdot d_{01} = a$. For $a, b \in \mathrm{Dat}$ we set $A_{ab} = \{z \in \mathrm{At}\,\mathfrak{A} : z \leqslant c_1 a \cdot c_0 b\}$. In Fig. II.60 we illustrate these notions. Now we establish some properties of A_{ab}:

(1) $A_{ab} \neq 0$ *for all $a, b \in \mathrm{Dat}$.*

For, $c_2 c_1(c_1 a \cdot c_0 b) = c_0(c_1 a \cdot c_1 c_0 b) = c_0 c_1 a \cdot c_1 c_0 b = 1$ by simplicity.

(2) $A_{aa} = \{a\}$ *if a is small.*

For, if $b \in A_{aa}$, then $b \leqslant d_{01}$ and so $b = b \cdot d_{01} = b \cdot c_0 a \cdot c_1 a \cdot d_{01} = b \cdot a$, hence $a = b$.

(3) *For all $a, b \in \mathrm{Dat}$ and $z \in \mathrm{At}\,\mathfrak{A}$, $z \in A_{ab}$ if $c_1 z = c_1 a$ and $c_0 z = c_0 b$.*

For, $\Leftarrow$ is clear. $\Rightarrow$: $z \leqslant c_1 a$, so $c_1 z \leqslant c_1 a$. But $c_1 a$ is a $\{1\}$-atom by [5, 1.10.3], so $c_1 z = c_1 a$. Similarly, $c_0 z = c_0 b$.

(4) *For all $a, b, c, d \in \mathrm{Dat}$, if $A_{ab} \cap A_{cd} \neq 0$, then $a = c$ and $b = d$.*

For, say $z \in A_{ab} \cap A_{cd}$. Then, using (3),

$$\begin{aligned} a &= d_{01} \cdot c_1(a \cdot d_{01}) \qquad (\text{since } a \leqslant d_{01}) \\ &= d_{01} \cdot c_1 a = d_{01} \cdot c_1 z = d_{01} \cdot c_1 c = c. \end{aligned}$$

Similarly, $b = d$.

(5) $\mathrm{At}\,\mathfrak{A} = \bigcup_{a, b \in \mathrm{Dat}} A_{ab}$.

For, let $z \in \mathrm{At}\,\mathfrak{A}$. By [5, 1.10.3], $a = c_1 z \cdot d_{01}$ and $b = c_0 z \cdot d_{01}$ are atoms; thus $a, b \in \mathrm{Dat}$. Clearly $c_1 a = c_1 z$ and $c_0 b = c_0 z$, so $z \in A_{ab}$ by (3).

Now for each $a \in \mathrm{Dat}$ we let

$$X_a = \begin{cases} \{\langle a, x\rangle : x \in \mathrm{At}\,\mathfrak{A}\} & \text{if } a \text{ is big, or small and defective,} \\ \{\langle a, x\rangle : \end{cases}$$

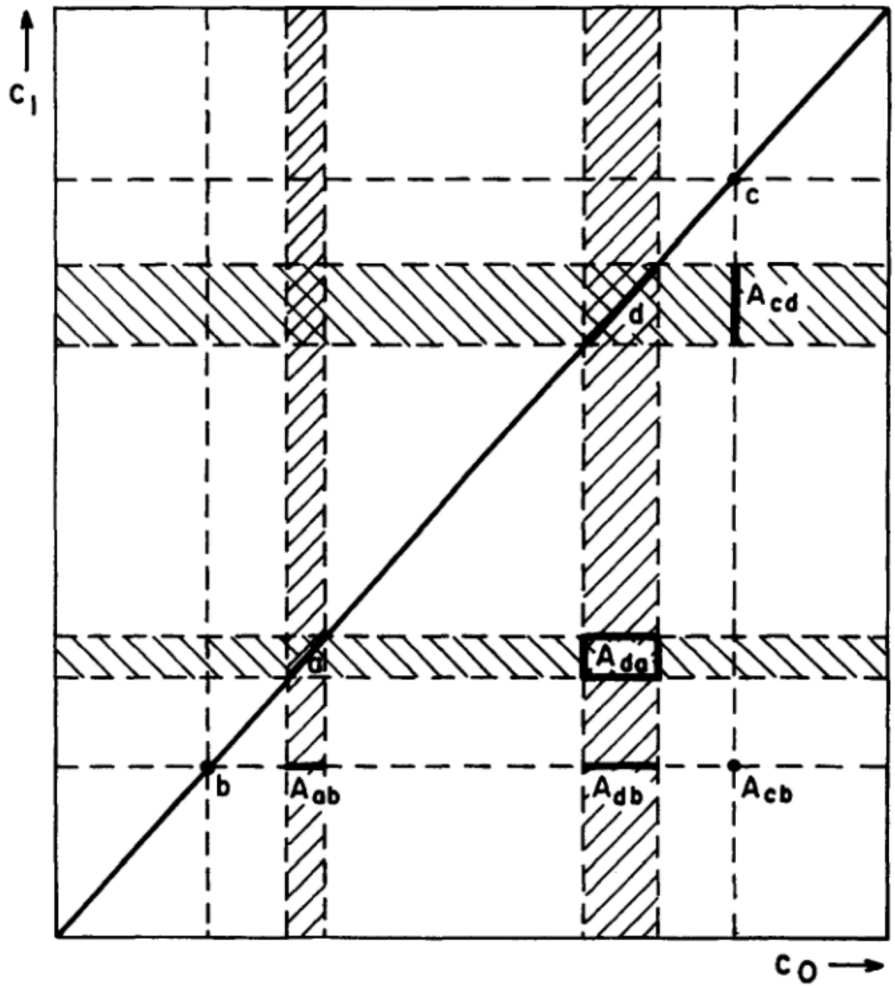

Fig. II.60. a and d are big atoms, b and c are small.

Set $U = \bigcup_{a \in \mathrm{Dat}} X_a$, $W = {}^2U$. With I as in the statement of the lemma, let $V = {}^2U \sim \bigcup_{a \in I} ({}^2X_a \sim D_{01}^{[W]})$. Thus

(6) $V = \bigcup_{a,b \in \mathrm{Dat}, a \neq b} X_a \times X_b \cup \bigcup_{a \in \mathrm{Dat} \sim I} X_a \times X_b \cup \bigcup_{a \in I} ((X_a \times X_a) \cap D_{01}^{[W]})$;

(7) $(X_a \times X_b) \cap (X_c \times X_d) \neq 0$ *implies* $a = c$ *and* $b = d$.

Now we shall define ϕ mapping At $\mathfrak{A}$ into Sb V, by defining for all $a, b \in \mathrm{Dat}$ its restriction $A_{ab} \upharpoonright \phi$. We shall do this so that the following conditions hold:

(8) *If* $x, y \in A_{ab}$ *and* $x \neq y$, *then* $\phi x \cap \phi y = 0 \neq \phi x$.

(9) *If* $a, b \in \mathrm{Dat}$ *and* $a \neq b$, *then* $\bigcup_{z \in A_{ab}} \phi z = X_a \times X_b$.

(10) *If* $a \in \mathrm{Dat} \sim I$, *then* $\bigcup_{z \in A_{aa}} \phi z = X_a \times X_a$.

(11) *If* $a \in I$, *then* $\bigcup_{z \in A_{aa}} \phi z = X_a \times X_a \cap D_{01}^{[W]}$.

(12) $C_0^{[V]} \phi z = V \cap (U \times X_b)$ *and* $C_1^{[V]} \gamma z = V \cap (X_a \times U)$ *for all* $z \in A_{ab}$.

Case 1. a and b are large, and $a \neq b$. Thus $|X_a| = |X_b| = |\mathrm{At}\,\mathfrak{A}| \geqslant |A_{ab}| > 0$. By II.58, let $A_{ab} \upharpoonright \phi$ be a one–one function from A_{ab} onto a partition of $X_a \times X_b$ such that $C_0^{[Y]}\phi z = C_1^{[Y]}\phi z = X_a \times X_b$ for all $z \in A_{ab}$, where $Y = X_a \times X_b$. Clearly (8)–(12) hold.

Case 2. a is big, $a = b$. Thus $c_1 a \cdot c_0 a \leqslant d_{01}$, so $|A_{aa}| > 1$. Hence by II.58 we can carry through the construction in Case 1 with $\phi a = D_{01}^{[W]} \cap (X_a \times X_a)$.

Case 3. a defective, $a = b$. Then $A_{aa} = \{a\}$, and we set $\phi a = (X_a \times X_b) \cap D_{01}^{[W]}$. Clearly (8)–(12) hold.

Case 4. a small non-defective; or b small non-defective. Say a small non-defective. We shall show that $c_1 a \cdot c_0 b$ is an atom. Let, $x = c_1 a \cdot c_0 b$ and suppose that x is not an atom. Say $x = y + z$ where $0 \neq y, z$ and $y \cdot z = 0$. Since a is small and non-defective we have $c_0 y \cdot c_0 z \leqslant c_0(y \cdot z) = 0$. But $y \leqslant c_0 b$ so $c_0 y = c_0 b$, and similarly $c_0 z = c_0 b$, so $c_0 b = 0$, contradiction.

Thus $A_{ab} = \{c_1 a \cdot c_0 b\}$. We set $\phi(c_1 a \cdot c_0 b) = X_a \times X_b$. Clearly (8)–(12) hold.

This finishes the definition of ϕ. By (1)–(12) ϕ is a one–one function from $\mathrm{At}\,\mathfrak{A}$ onto a partition of V. Hence if we let

$$fx = \bigcup \{\phi a : a \leqslant x,\ a \in \mathrm{At}\,\mathfrak{A}\}$$

for all $x \in A$, we obtain an isomorphism of $\mathfrak{Bl}\,\mathfrak{A}$ into the BA of all subsets of V. It remains to check that f preserves cylindrifications and diagonal elements.

By symmetry we prove only that f preserves c_0. First suppose $\langle u, v\rangle \in fc_0x$. Say $\langle u, v\rangle \in \phi a$ with $a \leqslant c_0 x$, $a \in \mathrm{At}\,\mathfrak{A}$. Then there is a $b \in \mathrm{At}\,\mathfrak{A}$ with $b \leqslant x$, and $a \leqslant c_0 b$. By (5), say $a \leqslant c_1 c \cdot c_0 e$ with $c, e \in \mathrm{Dat}$, and $b \leqslant c_1 s \cdot c_0 t$ with $s, t \in \mathrm{Dat}$. Then $c_0 e = c_0 a = c_0 b = c_0 t$, so $e = t$. Since $\langle u, v\rangle \in \phi a$, by (8)–(12) we have $v \in X_e$, hence $\langle u, v\rangle \in V \cap (U \times X_e)$ and so by (12),

$$\langle u, v\rangle \in C_0^{[V]}\phi b \subseteq C_0^{[V]} fx,$$

as desired. Conversely, suppose $\langle u, v\rangle \in C_0^{[V]} fx$. Say $\langle w, v\rangle \in fx$, $\langle w, v\rangle \in \phi a$ with $a \leqslant x$, $a \in \mathrm{At}\,\mathfrak{A}$, $a \in A_{bc}$ with $b, c \in \mathrm{Dat}$. Say $\langle u, v\rangle \in \phi e$, $e \in A_{st}$. Then $v \in X_c$, $v \in X_t$, so $c = t$. Since $c_0 t = c_0 a \leqslant c_0 x$, we have $e \leqslant c_0 x$, hence $\langle u, v\rangle \in fc_0x$, as desired.

To show that f preserves d_{01}, first suppose that $\langle u, v\rangle \in fd_{01}$. Say $\langle u, v\rangle \in \phi a$ with $a \leqslant d_{01}$, $a \in \mathrm{At}\,\mathfrak{A}$. Thus $a \in \mathrm{Dat}$ and $a \in A_{aa}$. *If* a is big, then by Case 2, $u = v$; if a is small and defective, then $u = v$ by Case 3; and if a is small and non-defective, then $\phi a = X_a \times X_a$ by Case 4, and $|X_a| = 1$ by (2), so $u = v$. Thus $\langle u, v\rangle \in D_{01}^{[V]}$. Conversely, suppose that $\langle u, v\rangle \in D_{01}^{[V]}$. Thus $u = v$. Say $\langle u, v\rangle \in \phi a$, where $a \in \mathrm{At}\,\mathfrak{A}$. Say $a \in A_{bc}$. By construction it is clear that $b = c$, and then $a = b$. This completes the proof of II.59.

Theorem II.61. $\mathrm{CA}_2 \subseteq \mathrm{ICrs}_2$.

to a Crs_2. By [5, 2.7.15 and 2.7.17], $\mathfrak{A} \subseteq \mathfrak{C}$ for some simple complete atomic CA_2 $\mathfrak{A}$. The desired result now follows from II.59.

Remark II.62. Theorem II.61 gives a general geometric representation theorem for CA_2's. We know from [5, 2.6.41 and 2.6.42] that not every CA_2 is representable. Thus there is a simple CA_2 $\mathfrak{A}$ which is not isomorphic to a Cs_2. By II.61, $\mathfrak{A}$ is isomprhic to a Crs_2 with unit element V. The set V does not have the form ${}^2U$, but the diagonal element $D_{01}^{[V]}$ consists of elements of the form $\langle u, u\rangle$. It is also possible to represent $\mathfrak{A}$ isomorphically as a field of subsets of some set ${}^2U$ with the cylindrifications $C_\kappa^{[V]}$, $\kappa < 2$, but with an equivalence relationn on U in place of $D_{01}^{[V]}$, where $V = {}^2U$. In fact, in the proof of II.59, Case 3, one simply takes $\phi a = X_a \times X_a$, and otherwise the proof remains the same.

Andréka and Németi have given a direct proof of II.61, and used it to give a somewhat shorter proof of II.65 below.

To establish our other representation theorem concerning CA_2's we need two lemmas.

Lemma II.63. *Suppose $\alpha \geq 2$, $\kappa, \lambda < \alpha$, $\kappa \neq \lambda$, and the equation*

(i) $\quad c_\lambda(x \cdot y \cdot c_\kappa(x \cdot -y)) \cdot -c_\kappa(c_\lambda x \cdot -d_{\kappa\lambda}) = 0$

holds in a CA_α $\mathfrak{A}$. Then the following condition holds for all $x, y, z \in A$:

(ii) *If $c_\lambda x \cdot s_\lambda^\kappa c_\lambda x \leq d_{\kappa\lambda}$, then $c_\kappa(x \cdot y) \cdot c_\kappa(x \cdot z) \leq c_\kappa(x \cdot y \cdot z)$.*

Proof. Assume the hypothesis of the lemma and of (ii). Then $c_\lambda x \cdot -d_{\kappa\lambda} \leq s_\lambda^\kappa - c_\lambda x$, and so for any $w \in A$ we have

$$c_\lambda(x \cdot w \cdot c_\kappa(x \cdot -w)) \leq c_\kappa(c_\lambda x \cdot -d_{\kappa\lambda}) \leq s_\lambda^\kappa - c_\lambda x,$$
$$c_\lambda(x \cdot w \cdot c_\kappa(x \cdot -w)) \cdot c_\kappa(d_{\kappa\lambda} \cdot c_\lambda x) = 0,$$
$$c_\lambda(c_\kappa(x \cdot w) \cdot c_\kappa(x \cdot -w)) \cdot d_{\kappa\lambda} \cdot c_\lambda x = 0,$$
$$c_\kappa(x \cdot w) \cdot c_\kappa(x \cdot -w) \cdot c_\lambda x = 0,$$
$$c_\kappa(c_\kappa(x \cdot w) \cdot c_\kappa(x \cdot -w) \cdot c_\lambda x) = 0,$$
$$c_\kappa(x \cdot w) \cdot c_\kappa(x \cdot -w) \cdot c_\kappa c_\lambda x = 0,$$
$$c_\kappa(x \cdot w) \cdot c_\kappa(x \cdot -w) = 0.$$

Hence, if we let $fw = c_\kappa(x \cdot w)$ for all $w \in A$, we see that f is a homomorphism from $\mathfrak{Bl}\,\mathfrak{A}$ into $\mathfrak{Bl}\,\mathfrak{Rl}_y\,\mathfrak{A}$, with $y = c_\kappa x$. Hence the conclusion of (ii) follows.

Lemma II.64. *Suppose $\alpha \geq 2$, $\kappa, \lambda < \alpha$, and $k \neq \lambda$. Let $\mathfrak{A}$ be a CA_α. Then* II.63(ii) *holds for all $x, y, z \in A$ iff the following inequality holds for all $x, y, z \in A$:*

Proof. Clearly (*) implies II.63(ii). Now assume that II.63(ii) holds for all $x, y, z \in A$, and let $x, y, z \in A$ be arbitrary. Set $u = c_\lambda x \cdot s_\lambda^\kappa c_\lambda x \cdot -d_{\kappa\lambda}$, $x_1 = x \cdot c_\kappa c_\lambda u$, $x_2 = x \cdot -c_\kappa c_\lambda u$. Then

$$c_\lambda x_2 \cdot s_\lambda^\kappa c_\lambda x_2 \cdot -d_{\kappa\lambda} = c_\lambda x \cdot s_\lambda^\kappa c_\lambda x \cdot -d_{\kappa\lambda} \cdot -c_\kappa c_\lambda u = u \cdot -c_\kappa c_\lambda u = 0;$$

hence, by II.63(ii), we have

(1) $\quad c_\kappa(x_2 \cdot y) \cdot c_\kappa(x_2 \cdot z) \leqslant c_\kappa(x_2 \cdot y \cdot z).$

Since $x_1 \leqslant c_\kappa c_\lambda u$, we have $c_\kappa x_1 \leqslant c_\kappa c_\lambda u$ and hence

(2) $\quad c_\kappa(x_1 \cdot y) \cdot c_\kappa(x_1 \cdot z) \leqslant c_\kappa c_\lambda u.$

Now note that $c_\kappa x_1 \cdot c_\kappa x_2 = 0 = c_\lambda x_1 \cdot c_\lambda x_2$. Hence

$$\begin{aligned} c_\kappa(x \cdot y) \cdot c_\kappa(x \cdot z) &= (c_\kappa(x_1 \cdot y) + c_\kappa(x_2 \cdot y)) \cdot (c_\kappa(x_1 \cdot z) + c_\kappa(x_2 \cdot z)) \\ &= c_\kappa(x_1 \cdot y) \cdot c_\kappa(x_1 \cdot z) + c_\kappa(x_1 \cdot y) \cdot c_\kappa(x_2 \cdot z) \\ &\quad + c_\kappa(x_2 \cdot y) \cdot c_\kappa(x_1 \cdot z) + c_\kappa(x_2 \cdot y) \cdot c_\kappa(x_2 \cdot z) \\ &= c_\kappa(x_1 \cdot y) \cdot c_\kappa(x_1 \cdot z) + c_\kappa(x_2 \cdot y) \cdot c_\kappa(x_2 \cdot z) \\ &\leqslant c_\kappa c_\lambda u + c_\kappa(x_2 \cdot y \cdot z) \\ &\leqslant c_\kappa c_\lambda u + c_\kappa(x \cdot y \cdot z), \end{aligned}$$

as desired.

We are now ready for the second representation theorem for CA_2's:

Theorem II.65. *The following are equivalent, for any* $\mathfrak{A} \in CA_2$:

(i) $\mathfrak{A} \in SNr_2CA_3$.

(ii) *For any* $x, y \in A$, *the following two equations hold*:

$$c_1(x \cdot y \cdot c_0(x \cdot -y)) \cdot -c_0(c_1 x \cdot -d_{01}) = 0,$$
$$c_0(x \cdot y \cdot c_1(x \cdot -y)) \cdot -c_1(c_0 x \cdot -d_{01} = 0.$$

(iii) $\mathfrak{A}$ *is representable.*

Proof. (i)$\Rightarrow$(ii) by [5, 2.6.41 and its proof], and (iii)$\Rightarrow$(i) by [6, I.8.6]. To show that (ii)$\Rightarrow$(iii) it suffices to take the case $\mathfrak{A}$ simple, by [5, 2.4.52]. Let $\mathfrak{B} = \mathfrak{Cm}\,\mathfrak{A}$. By [5, 2.7.16, 2.7.17], II.63, and II.64, $\mathfrak{B}$ is a simple complete and atomic CA_2 in which the inequalities II.64(*) hold for $\kappa \neq \lambda$, $\kappa, \lambda \in 2$; hence also the implications II.63(ii) hold. It suffices to show that $\mathfrak{B}$ is isomorphic to a Cs_2. By II.59 it suffices to show that for an arbitrary atom a of $\mathfrak{B}$, a is not defective. So assume that $c_0 a \cdot c_1 a \leqslant d_{01}$, $\{\kappa, \lambda\} = 2$, $x, y, z \in B$, and $0 \neq x < c_\kappa a$. Then $c_\kappa x \cdot s_\kappa c_\kappa x = c_\kappa a \cdot c_\lambda(d_{\kappa\lambda} \cdot c_\kappa a) = c_\kappa a \cdot c_\lambda a \leqslant d_{01}$. Hence by II.63(ii), $c_\lambda(x \cdot y) \cdot c_\lambda(x \cdot z) \leqslant c_\lambda(x \cdot y \cdot z)$, as desired. This finishes the proof of II.65.

Remark II.66. In the rest of this paper we shall describe various methods for

method of splitting elements, pp. 386–394, pp. 407–408. Other methods are found in [9], [10], [11]. Here we shall describe three more methods: permutation models (II.67), dilation (II.68), and twisting (II.70). These methods, due to Henkin, are like those in [5], in that to establish the non-representability we exhibit explicit equations holding in all Rp_α's but not in the ones constructed. In [9], [10], [11], the non-representability is recognized by other means.

Construction II.67 (Permutation models). First we describe the general framework and then we make a specific construction. Let U be a non-empty set and $\mathfrak{A}$ the full Cs_α with base U, where α is arbitrary. Recall from [6, I.3.1, I.3.5] that to every permutation f of U there corresponds a base-automorphism $\bar{f}$ of $\mathfrak{A}$. Throughout this construction, if G is a set of permutations of U we shall denote by $\bar{G}$ the set of all $\bar{f}$ for $f \in G$. If $\mathfrak{B}$ is any CA_α and H is a set of automorphisms of $\mathfrak{B}$ then we let $\mathrm{Fx}_H\,\mathfrak{B} = \{b \in B : fb = b \text{ for all } f \in H\}$. Clearly $\mathrm{Fx}_H\,\mathfrak{B}$ is a subuniverse of $\mathfrak{B}$. We denote by $\mathfrak{Fr}_H\,\mathfrak{B}$ the subalgebra of $\mathfrak{B}$ with universe $\mathrm{Fx}_H\,\mathfrak{B}$. The permutation model method for constructing a CA_α consists in choosing a suitable U, $\mathfrak{A}$, G as above, forming $\mathfrak{C} = \mathfrak{Fr}_K\,\mathfrak{A}$ with $K = \bar{G}$, selecting an appropriate $c \in C$, and finally forming $\mathfrak{Rl}_c\,\mathfrak{C}$.

We shall now give a particular construction of this kind. We assume that $3 \leqslant \alpha$. Let W be a set such that $\alpha \cap W = 0$ and $|\alpha| = |W|$; say that $'$ is a one–one function from α onto W. Thus $\kappa' \in W$ for all $\kappa < \alpha$. For each $\kappa < \alpha$ let $a\kappa = \kappa$ and $a\kappa' = \kappa$. Given a permutation τ of α we define a permutation τ' of W by setting $\tau'\kappa' = (\tau\kappa)'$ for all $\kappa < \alpha$.

Let $U = \alpha \cup W$, and let $\mathfrak{A}$ be the full Cs_α with base U. Set $G = \{\tau \cup \tau' : \tau$ is a permutation of α, and $|\{\kappa < \alpha : \tau\kappa \neq \kappa\}| < \omega\}$. Let $\mathfrak{C} = \mathfrak{Fr}_H\,\mathfrak{A}$, $H = \bar{G}$. For each $s \in {}^\alpha U$ let $s^- = \{g \circ s : g \in G\}$. Clearly $s^- \in C$ for every $s \in {}^\alpha U$. Let

$$V = (\alpha \restriction \mathrm{Id})^- \cup \bigcup \{s^- : a \circ s \text{ is not one–one}\}.$$

Thus $V \in C$. Finally let $\mathfrak{D} = \mathfrak{Rl}_V\,\mathfrak{C}$. We shall show that $\mathfrak{D}$ is a non-representable CA_α.

To show that $\mathfrak{D}$ is a CA_α, it suffices by [5, 2.2.3] to check (C_4) and (C_6). To prove (C_4) it suffices to show the following:

(1) *If $\kappa, \lambda < \alpha$ and $s \in V$, then $c_\kappa^{(\mathfrak{C})} c_\lambda^{(\mathfrak{D})} s^- = \{t \in V :$ there exist $u, v \in U$ such that $t_{uv}^{\kappa\lambda} \in s^-\}$.*

To prove (1), we may ssume that $\kappa \neq \lambda$. The inclusion $\subseteq$ is obvious. For $\supseteq$, suppose $t \in V$, $u, v \in U$, and $t_{uv}^{\kappa\lambda} \in s^-$. If $t_u^\kappa \in V$, the desired conclusion is clear, so assume that $t_u^\kappa \notin V$. This implies that $t_\mu \neq au$ and $t_\mu \neq at\lambda$ for any $\mu \in \alpha \sim \{\kappa, \lambda\}$. Let τ be the transposition $[au/at\lambda, at\lambda/au]$ of α, and let $g = \tau \cup \tau'$. Thus $g \in G$. Let $r = g \circ t_{uv}^{\kappa\lambda}$; so $r \in s^-$. Now, if $\mu \in \alpha \sim \{\kappa, \lambda\}$, then by the remark above, $r\mu = gt\mu = t\mu$. Furthermore, $at_{r\kappa}^\kappa \kappa = ar\kappa = agu = at\lambda = at_{tk}^\kappa \lambda$. Therefore $t_{r\kappa} \in V$ and

Condition (C_6) is obvious, so $\mathfrak{D}$ is a CA_α. To show that $\mathfrak{D}$ is not representable, we consider the following inequality, first discussed in Thompson [14], and mentioned in [5, 1.5.22]:

(1) $$c_\kappa x \cdot c_\lambda y \cdot c_\mu z \leqslant c_\kappa c_\lambda c_\mu (c_\mu(c_\lambda x \cdot c_\kappa y) \cdot c_\lambda(c_\mu x \cdot c_\kappa z) \cdot c_\kappa(c_\mu y \cdot c_\lambda z))$$

Here κ, λ, μ are arbitrary distinct ordinals $<\alpha$. It is easy to check that (1) holds in every representable CA_α. We show that it fails in $\mathfrak{D}$. For simplicity take $\kappa = 0$, $\lambda = 1$, $\mu = 2$. We let

$$x = \langle 1', 1, 2, 3, \ldots \rangle^-,$$
$$y = \langle 0, 2', 2, 3, \ldots \rangle^-,$$
$$z = \langle 0, 1, 0', 3, \ldots \rangle^-,$$
$$f = \alpha \restriction \mathrm{Id}.$$

Clearly $f \in c_0 x \cdot c_1 y \cdot c_2 z$ in $\mathfrak{D}$. Suppose f is in the right side of (1). We then obtain elements $u, v, w, a, b, c, m, n, p, q, r, s \in U$ and $g_0, \ldots, g_5 \in G$ with the following properties:

$$f^{012}_{uvw} \in c_2(c_1 x \cdot c_0 y) \cdot c_1 x \cdot c_2 z) \cdot c_0(c_2 y \cdot c_1 z), \qquad f^{012}_{uva} \in c_1 x \cdot c_0 y,$$
$$f^{012}_{ubw} \in c_2 x \cdot c_0 z, \qquad f^{012}_{cvw} \in c_2 y \cdot c_1 z,$$
$$f^{012}_{uma} = g_0 \circ \langle 1', 1, 2, 3, \ldots \rangle, \qquad f^{012}_{nva} = g_1 \circ \langle 0, 2', 2, 3, \ldots \rangle,$$
$$f^{012}_{ubp} = g_2 \circ \langle 1', 1, 2, 3, \ldots \rangle, \qquad f^{012}_{qbw} = g_3 \circ \langle 0, 1, 0', 3, \ldots \rangle,$$
$$f^{012}_{cvr} = g_4 \circ \langle 0, 2', 2, 3, \ldots \rangle, \qquad f^{012}_{csw} = g_5 \circ \langle 0, 1, 0', 3, \ldots \rangle.$$

Therefore $g_0, \ldots, g_5$ permute $\{0, 1, 2\}$, and $u = g_0 1'$, $v = g_1 2'$, $w = g_3 0'$, so $u, v, w \in W$. Since $f^{012}_{uvw} \in V$, it follows that $|\{u, v, w\}| \leqslant 2$. By symmetry, say $u = v$. Then $g_0 1' = u = v = g_1 2' = (g_1 2)' = a' = (g_0 2)' = g_0 2'$, contradiction.

Construction II.68 (Dilation). The method in II.67, aside from the use permutations, was relativization. Since we relativized an atomic Cs_α, we can say that we *deleted* atoms. Here we want to do the opposite, *add* atoms.

First we explain the general procedure. We start with some Ca_α $\mathfrak{B} = \langle B, T_\kappa, E_{\kappa\lambda} \rangle_{\kappa,\lambda<\alpha}$; recall from [5, 2.7.38] that a Ca_α is a relational structure which is the atom structure of some complete and atomic CA_α. Suppose $a \in {}^\alpha B$, and the following two conditions hold (recall that T_κ is an equivalence relation on B for every $\kappa < \alpha$):

(1) $(a_\kappa / T_\lambda) \cap (a_\lambda / T_\kappa) \neq 0$ for all κ, $\lambda < \alpha$;

(2) $a_\mu \notin E_{\kappa\lambda}$ if κ, λ, μ are distinct ordinals $<\alpha$.

Then we choose some element $n \notin B$ and form a relational structure $\mathfrak{B}' = \langle B', T'_\kappa, E'_{\kappa\lambda} \rangle_{\kappa,\lambda<\alpha}$ as follows. We set $B' = B \cup \{n\}$. For any $\kappa < \alpha$, T'_κ is an equivalence relation on B', $T'_\kappa \cap (B \times B) = T_\kappa$, and for any $b \in B$, $b T'_\kappa n$ iff

$bT_\kappa a_\kappa$. Finally, $E'_{\kappa\lambda} = E_{\kappa\lambda}$ for κ and λ distinct ordinals $<\alpha$, while $E'_{\kappa\kappa} = B'$ for all $\kappa < \alpha$. We claim that $\mathfrak{B}'$ is a Ca_α. To prove this it suffices to check the conditions of [5, 2.7.40]. Of these conditions, (i), (iii) and (v) are obvious. To prove (ii) it suffices to show that $T'_\kappa \mid T'_\lambda \subseteq T'_\lambda \mid T'_\kappa$ for distinct $\kappa, \lambda < \alpha$. Assume that $b(T'_\kappa \mid T'_\lambda)c$, say $bT'_\kappa eT'_\lambda c$. We may assume that $n \in \{b, e, c\}$. By symmetry it suffices to consider the following two cases.

Case 1. $b, e \in B$, $c = n$. Thus $bT_\kappa eT_\lambda a_\lambda$, and by (1) there is a $g \in B$ with $a_\kappa T_\lambda g T_\kappa a_\lambda$, so by (ii) for $\mathfrak{B}$, $bT_\lambda hT_\kappa a_\kappa$ for some $h \in B$. Hence $bT'_\lambda hT'_\kappa n$, as desired.

Case 2. $b, c \in B$, $e = n$. Thus $bT_\kappa a_\kappa$ and $cT_\lambda a_\lambda$. By (1), $a_\kappa T_\lambda g T_\kappa a_\lambda$ for some $g \in B$. Hence by (ii) for $\mathfrak{B}$, $bT_\lambda hT_\kappa c$ for some $h \in B$, as desired.

Now we check [5, 2.7.40(iv)]. Assume $\kappa, \lambda, \mu < \alpha$ and $\mu \neq \kappa, \lambda$. Suppose $bT'_\mu c \in E'_{\kappa\mu} \cap E'_{\mu\lambda}$. If $\kappa = \lambda$, obviously $b \in E'_{\kappa\lambda}$. Assume $\kappa \neq \lambda$. If $b \in B$, obviously $b \in E'_{\kappa\lambda}$. Suppose $b = n$. Thus $a_\mu T_\mu c \in E_{\kappa\mu} \cap E_{\mu\lambda}$, so $a_\mu \in E_{\kappa\lambda}$, contradicting (2). We have now shown that $T'^*_\mu(E'_{\kappa\mu} \cap E'_{\mu\lambda}) \subseteq E'_{\kappa\lambda}$.

That $E'_{\kappa\lambda} \subseteq T'^*_\mu(E'_{\kappa\mu} \cap E'_{\mu\lambda})$ is clear if $\kappa \neq \lambda$. To check this for $\kappa = \lambda$ it suffices to show that $n \in T'^*_\mu E'_{\kappa\mu}$. Since $a_\mu \in E_{\kappa\kappa} = T^*_\mu E_{\kappa\mu}$, choose $b \in E_{\kappa\mu}$ with $a_\mu T_\mu b$. Thus $nT'_\mu b$, so $n \in T'^*_\mu E'_{\kappa\mu}$. This finishes the general description.

Now we shall construct a non-representable CA_α using the method of dilation. Assume that $3 \leq \alpha < \omega$. Let W, G and $\mathfrak{C}$ be as in II.67. Let $\mathfrak{B} = \mathfrak{At}\,\mathfrak{C}$, the atom structure of $\mathfrak{C}$. So $\mathfrak{B} \in \mathrm{Ca}_\alpha$; we write $\mathfrak{B} = \langle B, T_\kappa, E_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha}$ as above. For each $\kappa < \alpha - 1$ let s_κ be the member of ${}^\alpha\alpha$ such that for all $\mu < \alpha$,

$$s_\kappa\mu = \begin{cases} \mu & \text{if } \mu \leq \kappa, \\ \mu - 1 & \text{if } \kappa < \mu < \alpha, \end{cases}$$

and let $s_{\alpha-1}$ be such that for all $\mu < \alpha$,

$$s_{\alpha-1}\mu = \begin{cases} \mu & \text{if } \mu < \alpha - 1, \\ 0 & \text{if } \mu = \alpha - 1. \end{cases}$$

Let $a_\kappa = s_\kappa^-$ for each $\kappa < \alpha$. We now check the conditions (1), (2). For (1), it suffices to take distinct $\kappa, \lambda < \alpha$. Say $\kappa < \lambda$. We treat only the case $\kappa + 1 < \lambda < \alpha - 1$ and leave other possibilities to the reader. Let t be the member of ${}^\alpha\alpha$ such that for all $\mu < \alpha$,

$$t\mu = \begin{cases} \mu & \text{if } \mu \leq \kappa, \\ \mu - 1 & \text{if } \kappa < \mu < \lambda, \\ \lambda & \text{if } \mu = \lambda, \\ \mu - 1 & \text{if } \lambda < \mu < \alpha. \end{cases}$$

Clearly $t^- T_\lambda a_\kappa$. Let τ be the cyclic permutation $(\kappa, \lambda - 1, \lambda - 2, \ldots, \kappa + 1)$ of α

and set $g = \tau \cup \tau'$. Then $g \circ s_\lambda \in s_\lambda^-$ and for any $\mu < \alpha$,

$$(g \circ s_\lambda)_\mu = \begin{cases} \mu & \text{if } \mu < \kappa, \\ \lambda - 1 & \text{if } \mu = \kappa, \\ \mu - 1 & \text{if } \kappa < \mu < \lambda, \\ \lambda & \text{if } \mu = \lambda, \\ \mu - 1 & \text{if } \lambda < \mu < \alpha. \end{cases}$$

Thus $t^- T'_\kappa a_\lambda$, as desired. So we take (1) as established. Condition (2) is obvious.

Thus by the general procedure we obtain a $\mathrm{Ca}_\alpha\, \mathfrak{B}' = \langle B', T'_\kappa, E'_{\kappa\lambda} \rangle_{\kappa,\lambda<\alpha}$ by adjoining a new element n. Let $\mathfrak{D} = \mathfrak{Cm}\, \mathfrak{B}'$. Thus $\mathfrak{D}$ is a CA_α by [5, 2.7.39]. We show that $\mathfrak{D}$ is non-representable by considering a new equation, which is an algebraic version of the associativity of relative product of binary relations. To formulate it, let $\mathfrak{E}$ be an arbitrary CA_β, $3 \leqslant \beta$. We define a binary operation ; on E by setting, for any $x, y \in E$,

$$x; y = c_2(s_2^1 c_2 x \cdot s_2^0 c_2 y).$$

Note that if $\mathfrak{F}$ is a Cs_β with base X and $x, y \in F$, then

$$x; y = \{z \in {}^\beta X\colon \text{ there exists } u \in X \text{ with } z_u^1 \in c_2 x \text{ and } z_u^0 \in c_2 y\},$$

which shows the relationship of ; with relative product. This also shows that the equation

(3) $\quad x; (y; z) = (x; y); z$

holds in every representable CA_β. Now we shall show that it does not hold in $\mathfrak{D}$.

To this end, let $x = \langle 0, 0', 1, 2, 3, \ldots \rangle^-$, $y = \langle 0', 1, 1, 2, 3, \ldots \rangle^-$, $z = \langle 1, 0, 1, 2, 3, \ldots \rangle^-$. Note that $s_2^1 c_2 y = c_1(d_{12} \cdot c_2 y) = c_1 y$ and similarly $s_2^0 c_2 z = c_0 z$. Hence

(4) $\quad x; (y; z) = c_2(s_2^1 c_2 x \cdot s_2^0 c_2(c_1 y \cdot c_0 z)),$

(5) $\quad (x; y); z = c_2(s_2^1 c_2(s_2^1 c_2 x \cdot s_2^0 c_2 y) \cdot c_0 z).$

(All of the operations above are in $\mathfrak{D}$. For simplicity we treat n as well as each element s^- for $s \in {}^\alpha U$ as an atom of $\mathfrak{D}$.) Now we claim

(6) *If* $s \in {}^\alpha U \cap (x; (y; z))$, *then* $s_0 = s_1$.

For, $d_{12} \cdot c_2 x = \{0, 0', 0', 2, 3, \ldots\rangle^-$, hence $a_1 \nleqslant s_2^1 c_2 x$ and so $n \nleqslant s_2^1 c_2 x$. It follows that there is a $u \in U$ such that $s_u^2 \in s_2^1 c_2 x \cdot s_2^0 c_2(c_1 y \cdot c_0 z)$. Now $c_1 y \cdot c_0 z = \langle 0', 0, 1, 2, 3, \ldots \rangle^-$, so $d_{02} \cdot c_2(c_1 y \cdot c_0 z) = \langle 0', 0, 0', 2, 3, \ldots \rangle^-$. Therefore $s_2^1 c_2 x \cdot s_2^0 c_2(c_1 y \cdot c_0 z) = \langle 0, 0, 0', 2, 3, \ldots \rangle^-$. Hence $s_0 = s_1$, as desired.

(7) $\quad \langle 0, 1, 0, 2, 3, \ldots \rangle^- \leqslant c_2 n.$

$n \not\leqslant x;(y;z)$. Now we show that $n \leqslant (x;y);z$, so that (3) fails. Clearly $n \leqslant c_0 z$. Now $d_{12} \cdot c_2 x = \langle 0, 0', 0', 2, 3, \ldots \rangle^-$ and $d_{02} \cdot c_2 y = \langle 0', 1, 0', 2, 3, \ldots \rangle^-$. Hence $s_2^1 c_2 x \cdot s_2^0 c_2 y = \langle 0, 1, 0', 2, 3, \ldots \rangle^-$, and hence $d_{12} \cdot c_2(s_2^1 c_2 x \cdot s_2^0 c_2 y) = \langle 0, 1, 1, 2, 3, \ldots \rangle^-$. Thus $n \leqslant s_2^1 c_2(s_2^1 c_2 x \cdot s_2^0 c_2 y)$. So $n \leqslant (x;y);z$, as desired.

Remark II.69. By combining many algebras using ultraproducts we can obtain infinite-dimensional CA_α's in which the equation II.68(3) fails. This is a general method, enabling one always to restrict oneself to the case $\alpha < \omega$ when considering such equations. The method is essentially described in the proof of [5, 2.6.4], but we sketch it here. Suppose $\alpha \geqslant \omega$. Let $I = \{\Gamma : 3 \subseteq \Gamma \subseteq \alpha \text{ and } |\Gamma| < \omega\}$. For each $\Gamma \in I$ let $\beta\Gamma = |\Gamma|$ and let $\rho\Gamma$ be a one–one function from $\beta\Gamma$ onto Γ such that $\beta \restriction \mathrm{Id} \subseteq \rho\Gamma$; moreover, let $M_\Gamma = \{\Delta \in I : \Gamma \subseteq \Delta\}$. Furthermore, let $\mathfrak{A}_\Gamma$ be a $\mathrm{CA}_{\beta\Gamma}$ in which II.68(3) fails. Let $\mathfrak{B}_\Gamma$ be an algebra similar to CA_α's such that $\mathfrak{A}_\Gamma = \mathfrak{Rd}^{(\rho\Gamma)}\mathfrak{B}_\Gamma$, for each $\Gamma \in I$ (extending [5, 2.6.1] in the natural way). Let U be an ultrafilter on I such that $M_\Gamma \in U$ for every $\Gamma \in I$. Then $P_{\Gamma \in I}\mathfrak{B}_\Gamma/\tilde{U}$ is easily seen to be the desired algebra.

Constructing II.70 (Twisting). This method, roughly speaking, consists of starting from a complete atomic CA_α $\mathfrak{A}$, selecting atoms $a, b \in A$ and an ordinal $\kappa < \alpha$, and redefining c_κ on a and b by interchanging the action of c_κ on a in b, *in part* ('twisting').

Specifically, suppose $3 \leqslant \alpha < \omega$. Let $\mathfrak{B}$ be the full Cs_α with base $2\alpha - 2$, and let G be the set of all permutations of $2\alpha - 2$ of the form $[0/1, 1/0] \circ [2\kappa/(2\kappa + 1), (2\kappa + 1)/2\kappa]$ for $0 < \kappa < \alpha - 1$. Set $\mathfrak{A} = \mathfrak{Fr}_H \mathfrak{B}$, $H = \tilde{G}$ (see II.67). Let $\mathfrak{C} = \langle \mathrm{At}\,\mathfrak{A}, T_\kappa, E_{\kappa\lambda} \rangle_{\kappa,\lambda<\alpha}$ be the atom structure of $\mathfrak{A}$. We consider the atoms $\langle 3, 0, 0, 4, 6, 8, \ldots \rangle^-$ and $\langle 3, 1, 1, 4, 6, 8, \ldots \rangle^-$ of $\mathfrak{A}$ and the ordinal $1 < \alpha$. Let $T'_\kappa = T_\kappa$ for all $\kappa \in \alpha \sim \{1\}$, and let $E'_{\kappa\lambda} = E_{\kappa\lambda}$ for all $\kappa, \lambda < \alpha$. Now we define T'_1, a certain equivalence relation on $\mathrm{At}\,\mathfrak{A}$. Let $M = T^\star\{\langle 3, 0, 0, 4, 6, 8, \ldots \rangle^-, \langle 3, 1, 1, 4, 6, 8, \ldots \rangle^-\}$. If $x \in \mathrm{At}\,\mathfrak{A} \sim M$, then the T'_1-class of x is x/T_1. Further, we set

$$\begin{aligned}
\langle 3, 0, 0, 4, 6, 8, \ldots \rangle^-/T' &= \{\langle 3, 0, 0, 4, 6, 8, \ldots \rangle^-, \langle 3, 1, 0, 4, 6, 8, \ldots \rangle^-\} \\
&\quad \cup \{\langle 3, \kappa, 1, 4, 6, 8, \ldots \rangle^- : 2 \leqslant \kappa < 2\alpha - 2\}, \\
\langle 3, 1, 1, 4, 6, 8, \ldots, \rangle^-/T'_1 &= \{\langle 3, 1, 1, 4, 6, 8, \ldots \rangle^-, \langle 3, 0, 1, 4, 6, 8, \ldots \rangle^-\} \\
&\quad \cup \{\langle 3, \kappa, 0, 4, 6, 8, \ldots \rangle^- : 2 \leqslant \kappa < 2\alpha - 2\}.
\end{aligned}$$

Note the symmetry of this definition with respect to interchanging 0 and 1. This gives us a new relational structure $\mathfrak{D} = \langle \mathrm{At}\,\mathfrak{A}, T'_\kappa, E'_{\kappa\lambda} \rangle_{\kappa,\lambda<\alpha}$. We claim that $\mathfrak{D}$ is a Ca_α (see [5, 2.7.38]), and to see this we again want to check the conditions of [5, 2.7.40]. Of these conditions, (i) and (iii) are obviious. For (v) we need to check that if $\lambda \neq 1$, $a, b \in E_{1\lambda}$, and aT'_1b, then $a = b$; this is clear by inspection.

and so $T_1'^*(E_{\kappa 1} \cap E_{1\lambda}) = T_1^*(E_{\kappa 1} \cap E_{1\lambda}) = E_{\kappa\lambda}$. So we only need to check (iv) when $\mu = 1$ and $\kappa = \lambda$; so suppose $\kappa \neq 1$ — we want to show that $\mathrm{At}\,\mathfrak{A} = T_1'^* E_{1\kappa}$. Since $T_1'^*\{\langle 3, 0, 0, 4, 6, 8, \ldots\rangle^-, \langle 3, 1, 1, 4, 6, 8, \ldots\rangle^-\} = T_1^*\{\langle 3, 0, 0, \ldots, 4, 6, 8, \ldots\rangle^-, \langle 3, 1, 1, 4, 6, 8, \ldots\rangle^-\} = M$, this is clear.

It remains only to check (ii). So suppose $\lambda \in \alpha \sim \{1\}$. We need two auxiliary statements.

(1) *If* $a \in \mathrm{At}\,\mathfrak{A}$, *then* $\langle 3, 0, 0, 4, 6, 8, \ldots\rangle^-(T_1^* \mid T_\lambda)a$ *iff* $\langle 3, 1, 1, 4, 6, 7, \ldots\rangle^-(T_1' \mid T_\lambda)a$.

To prove (1), by symmetry it suffices to take the direction $\Rightarrow$. Say $\langle 3, 0, 0, 4, 6, 8, \ldots\rangle^- T_1' b T_\lambda a$. We seek b' such that $\langle 3, 1, 1, 4, 6, 8, \ldots\rangle^- T_1' b' T_\lambda a$. We carry though the proof in full for the case $\lambda = 0$, and leave the other cases to the reader. If $b = \langle 3, 0, 0, 4, 6, 8, \ldots\rangle^-$, let $b' = \langle 2, 0, 0, 4, 6, 8, \ldots\rangle^-$; thus $b' T_0 a$ and $b' = \langle 3, 1, 1, 4, 6, 8, \ldots\rangle^-$ as desired. If $b = \langle 3, 1, 0, 4, 6, 8, \ldots\rangle^-$, let $b' = \langle 2, 0, 1, 4, 6, 8, \ldots\rangle^-$; so $b' T_0 a$ and $b' = \langle 3, 0, 1, 4, 6, 8, \ldots\rangle^- T_1' \langle 3, 1, 1, 4, 6, 8, \ldots\rangle^-$ as desired. If $b = \langle 3, 2, 1, 4, 6, 8, \ldots\rangle^-$ let $b' = \langle 2, 2, 1, 4, 6, 8, \ldots\rangle^- = \langle 3, 3, 0, 4, 6, 8, \ldots\rangle^-$; if $b = \langle 3, 3, 1, 4, 6, 8, \ldots\rangle^-$ let $b^- = \langle 2, 3, 1, 4, 6, 8, \ldots\rangle^- = \langle 3, 2, 0, 4, 6, 8, \ldots\rangle^-$; and if $b = \langle 3, \mu, 1, 4, 6, 8, \ldots\rangle^-$ with $4 \leqslant \mu < 2\alpha - 2$ let $b' = \langle 2, \mu, 1, 4, 6, 8, \ldots\rangle^- = \langle 3, \mu, 0, 4, 6, 8, \ldots\rangle^-$.

(2) *If* $a \in \mathrm{At}\,\mathfrak{A}$, *then* $\langle 3, 0, 0, 4, 6, 8, \ldots\rangle^-(T_\lambda \mid T_1')a$ *if* $\langle 3, 1, 1, 4, 6, 8, \ldots\rangle^-(T_\lambda \mid T_1')a$.

Again it suffices to show the direction $\Rightarrow$. Say $\langle 3, 0, 0, 4, 6, 8, \ldots\rangle^- T_\lambda b T_1' a$. We seek b' such that $\langle 3, 1, 1, 4, 6, 8, \ldots\rangle^- T_\lambda b' T_1' a$. This time we carry out in detail only the case $\lambda \geqslant 3$. Let $s = \langle 3, 0, 0, 4, 6, 8, \ldots\rangle$. Thus $b = (s_\mu^\lambda)^-$ for some $\mu < 2\alpha - 2$. Using the member $[0/1, 1/0] \circ [(2\lambda - 2)/(2\lambda - 1), (2\lambda - 1/(2\lambda - 2)]$ of G we see that $b = (s_{11\nu}^{12\lambda})^-$ for some $\nu < 2\alpha - 2$. Thus we can set $b' = b$. So (2) holds.

We also need the following corollary of (1):

(3) *If* $a \in M$ *and* $b \in \mathrm{At}\,\mathfrak{A}$, *then* $a(T_1' \mid T_\lambda)b$ *iff there is a* $c \in M$ *with* $cT_\lambda b$.

For, the implication $\Rightarrow$ is clear. Suppose conversely that $c \in M$ and $cT_\lambda b$. Without loss of generality, say $\langle 3, 0, 0, 4, 6, 8, \ldots\rangle^- T_1' c$. Thus $\langle 3, 0, 0, 4, 6, 8, \ldots\rangle^-(T_1' \mid T_\lambda)b$, so by (1), $a(T_1' \mid T_\lambda)b$.

Suppose now that $a, b \in \mathrm{At}\,\mathfrak{A}$; we want to show that $a(T_1' \mid T_\lambda)b$ iff $a(T_\lambda \mid T_1')b$. By symmetry it suffices to consider the following three cases.

Case 1. $a, b \notin M$. Then $a(T_1' \mid T_\lambda)b$ iff $a(T_1 \mid T_\lambda)b$ iff $a(T_\lambda \mid T_1)b$ iff $a(T_\lambda \mid T_1' b)$.

Case 2. $a, b \in M$. By (3) we have $a(T_1' \mid T_\lambda)b$ and $a(T_\lambda \mid T_1')b$.

Case 3. $a \in M$, $b \notin M$. Suppose first that $a(T_\lambda \mid T_1')b$. Thus $a(T_\lambda \mid T_1)b$, so $aT_1 \mid T_\lambda b$. Hence by (3) $aT_1' \mid T_\lambda b$. Conversely, suppose that $a(T_1' \mid T_\lambda)b$. Then there is a $c \in M$ such that $cT_\lambda b$. Say without loss of generality $\langle 3, 0, 0, 4,$

$(T_\lambda \mid T_1')b$, so by (2), $\langle 3, 1, 1, 4, 6, 8, \ldots\rangle^-(T_\lambda \mid T_1')b$ and hence $\langle 3, 1, 1, 4, 6, 8, \ldots\rangle^-(T_1 \mid T_\lambda)b$. It follows that $a(T_1 \mid T_\lambda)b$, hence $a(T_\lambda \mid T_1')b$, as desired. We have now shown that $\mathfrak{D}$ is a Ca_α.

Thus $\mathfrak{Cm}\,\mathfrak{D}$ is a CA_α. We now show that it is not representable. To do this we first consider the equation

$$(4) \qquad {}_2s(0, 1)c_2x = {}_2s(1, 0)c_2x.$$

As is easily checked, it holds identically in every representable CA_α. We show that it fails in $\mathfrak{Cm}\,\mathfrak{D}$. (See [5, 1.5.14] and the comments following it.) Take $x = \{0, 3, 0, 4, 6, 8, \ldots\rangle^-$. Then

$$\begin{aligned}
{}_2s(0, 1)c_2x &= s_0^2 s_1^0 s_2^1 c_2 x = s_0^2 s_1^0 c_1\langle 0, 3, 3, 4, 6, 8, \ldots\rangle^- \\
&= s_0^2 c_0\langle 0, 0, 3, 4, 6, 8, \ldots\rangle^- \\
&= c_2\langle 3, 0, 3, 4, 6, 8, \ldots\rangle^-; \\
{}_2s(1, 0)c_2x &= s_1^2 s_0^1 s_2^0 c_2 x = s_1^2 s_0^1 c_0\langle 0, 3, 0, 4, 6, 8, \ldots\rangle^- \\
&= s_1^2 c_1\langle 3, 3, 0, 4, 6, 8, \ldots\rangle^- \\
&= c_2\langle 3, 1, 1, 4, 6, 8, \ldots\rangle^-.
\end{aligned}$$

Since clearly $\langle 3, 1, 1, 4, 6, 8, \ldots\rangle \notin c_2\langle 3, 0, 3, 4, 4, 6, 8, \ldots\rangle^-$, we see that (4) does fail in $\mathfrak{Cm}\,\mathfrak{D}$. $\mathfrak{Cm}\,\mathfrak{D}$ can also be used to show the failure of two further similar equations (5) and (6) which follow.

$$(5) \qquad {}_2s(0, 1){}_2s(0, 1)c_2x = c_2x.$$

Again it is easy to check that (5) holds identically in every representable CA_α. It fails in $\mathfrak{Cm}\,\mathfrak{D}$ with the same element x as above. It is in fact routine to check that

$${}_2s(0, 1){}_2s(0, 1)c_2x = c_2\langle 1, 3, 1, 4, 6, 8, \ldots\rangle^-,$$

while clearly $\langle 0, 3, 0, 4, 6, 8, \ldots\rangle \notin c_2\langle 1, 3, 1, 4, 6, 8\rangle^-$. Finally, assume that $4 \leqslant \alpha$, and consider the equation

$$(6) \qquad {}_2s(0, 1){}_2s(0, 3)c_2y = {}_2s(1, 3){}_2s(0, 1)c_2y.$$

Again it is easy to check that this equation holds in every representable CA_α. Let $y = \langle 4, 3, 0, 0, 6, 8, \ldots\rangle^-$. Then one can check that

$$\begin{aligned}
{}_2s(0, 1){}_2s(0, 3)c_2y &= c_2\langle 3, 0, 3, 4, 6, 8, \ldots\rangle^-, \\
{}_2s(1, 3){}_2s(0, 1)c_2y &= c_2\langle 3, 1, 1, 4, 6, 8, \ldots\rangle^-,
\end{aligned}$$

so (6) does fail in $\mathfrak{Cm}\,\mathfrak{D}$.

In all of the above the indices $0, 1, 2, 3 < \alpha$ played a special role. Of course one can modify the construction of $\mathfrak{D}$ in an obvious way to take care similarly of other

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