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On endomorphism bases

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In the article Baldwin, Berman, Glass, Hodges [1], with which we assume acquaintance, the authors prove a general theorem about endomorphism bases and apply it in several situations. We give here several comments. First, their proof of the theorem gives a slightly stronger result (Theorem 1 below). Second, this stronger result does not extend to singular cardinals (Example 1). Third, there is a weaker result that does hold for singular cardinals of cofinality $>\omega$ (Theorem 2). Fourth, this result does not hold for singular cardinals of cofinality ω (Example 2). In Example 2 we also apply these theorems to Boolean algebras.

The proof of Theorem 1 in [1] clearly gives the following result:

THEOREM 1. *In addition to the conclusion of Theorem* 1 *in* [1] *we have that for any mapping f from Y' into Y', the endomorphism f^* of A corresponding to f maps X into X.*

EXAMPLE 1. Let κ be singular; say $\langle \mu_\alpha : \alpha < cf\kappa \rangle$ is a strictly increasing sequence of infinite cardinals with supremum κ. Let A be a free Boolean algebra with free generators $\langle a_\alpha : \alpha < cf\kappa \rangle$ and $\langle b_{\alpha\beta} : \alpha < cf\kappa,\ \beta < \mu_\alpha \rangle$. Let $Y = \{a_\alpha \cdot b_{\alpha\beta} : \alpha < cf\kappa,\ \beta < \mu_\alpha\}$. Then $X = \{a_\alpha : \alpha < cf\kappa\} \cup \{b_{\alpha\beta} : \alpha < cf\kappa,\ \beta < \mu_\alpha\}$ is an endomorphism base for A. We claim, however, that the conclusion of Theorem 1 fails. For, suppose that $Y' \in [Y]^\kappa$ is an endomorphism base, with the additional property mentioned. Then there exist distinct $\alpha, \delta < cf\kappa$ and distinct $\beta, \gamma < \mu_\alpha$, $\varepsilon < \mu_\delta$ such that $a_\alpha \cdot b_{\alpha\beta}$, $a_\alpha \cdot b_{\alpha\gamma}$, $a_\delta \cdot b_{\delta\varepsilon} \in Y'$. Take a mapping f of Y' into Y' such that $f(a_\alpha \cdot b_{\alpha\beta}) = \alpha_\alpha \cdot b_{\alpha\beta}$ and $f(a_\alpha \cdot b_{\alpha\gamma}) = a_\delta \cdot b_{\delta\varepsilon}$. If f^* is the extension of f to an endomorphism of A, clearly f^* does not map X into X.

Now we formulate a notion which enables us to extend Theorem 1 in a weaker form to singular cardinals. We call a subset X of A a *singular endomorphism base* if $\kappa = |X|$ is singular, and there is a partition $\langle Y_\alpha : \alpha < cf\kappa \rangle$ of X such that for all $\tau : cf\kappa \to cf\kappa$ and all $f : X \to X$, if $f[Y_\alpha] \subseteq Y_{\tau\alpha}$ for all $\alpha < cf\kappa$, then f extends to an endomorphism f^* of A in a functorial way.

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THEOREM 2. *Let A be a structure of countable type generated by an endomorphism base X. Suppose that κ is a singular cardinal with $cf\kappa > \omega$, and $Y \in [A]^\kappa$. Then there is a singular endomorphism base $Z \in [Y]^\kappa$ such that the indicated extensions map X into X.*

Proof. We proceed as in the proof of Theorem 1 of [1], using the well-known double Δ-system lemma instead of the usual Δ-system lemma (see, e.g., Monk [2], Theorem 10.6). We then obtain $Z \in [Y]^\kappa$ and a partition $\langle W_\alpha : \alpha < cf\kappa \rangle$ of Z such that for some term t, every element $b \in W_\alpha$, $\alpha < cf\kappa$, can be written as $b = t^A(\bar{c}, \bar{d}_\alpha, \bar{e}_b)$, where the $\bar{c}$, $\bar{d}$, $\bar{e}_b$ are finite sequences of elements of X, all disjoint for various α and b. Given $\tau : cf\kappa \to cf\kappa$ and $f : Z \to Z$ such that $f[W_\alpha] \subseteq W_{\tau\alpha}$ for all $\alpha < cf\kappa$, we define $g : X \to X$ by

$$gx = \begin{cases} i\text{-th term of } \bar{e}_{fb}, \text{ if } x \text{ is the} \\ \quad i\text{-th term of } \bar{e}_b, \text{ for some } b \in Z, \\ i\text{-th term of } \bar{d}_{\tau\alpha}, \text{ if } x \text{ is the} \\ \quad i\text{-th term of } \bar{d}_\alpha, \text{ for some } \alpha < cf\kappa \\ x \qquad \text{otherwise.} \end{cases}$$

The rest of the proof is as for Theorem 1 of [1].

EXAMPLE 2. We apply both theorems to Boolean algebras, and show in particular that Theorem 2 does not hold for singular κ of cofinality ω. In [2], Corollary 10.9, it is shown that if $cf\kappa > \omega$, then any set of κ elements of a free BA contains an independent subset of size κ. The above results generalize this, according to the following

FACT. If X is a singular endomorphism base of a BA A, or an infinite endomorphism base of A, then X is independent.

Proof. We take only the case of a singular endomorphism base X, with notation as above. Suppose that Y and Z are disjoint finite subsets of X; we show that $\prod_{y \in Y} y \cdot \prod_{z \in Z} -z \neq 0$. Choose $\alpha < cf\kappa$ such that $|Y_\alpha| > 1$. Let $z : cf\kappa \to cf\kappa$ be such that $\tau\beta = \alpha$ for all $\beta < cf\kappa$. Choose $u, v \in Y_\alpha$ with $u \nleq v$. Then let $f : X \to Y_\alpha$ be such that $f[Y] \subseteq \{u\}$ and $f[Z] \subseteq \{v\}$. The extension of f to an endomorphism of A shows that $\prod_{y \in Y} y \cdot \prod_{z \in Z} -z \neq 0$.

If A is a free BA and $\kappa \leq |A|$ is singular of cofinality ω, it is well-known that there is a subset $X \subseteq A$ of power κ with no independent subset of power κ. By the FACT, this shows that the condition $cf\kappa > \omega$ in Theorem 2 cannot be removed. To construct such a subset, let $\langle a_n : n \in \omega \rangle$ be a partition of A, and for each $n \in \omega$

let Y_n be a subset of the set of free generators not appearing in the canonical forms of any a_m, of power μ_n, where the μ_n are strictly increasing with supremum κ and the Y_n are pairwise disjoint. Then, as is easily seen,

$$X = \{a_n \cdot y : n < \omega, y \in Y_n\}$$

is the desired set.

REFERENCES

[1] J. T. BALDWIN, J. BERMAN, A. M. W. GLASS and W. A. HODGES, *A combinatorial fact about free algebras*, Alg. Univ. *15* (1982), 145–152.
[2] J. D. MONK, *Independence in Boolean algebras*, Per. Math. Hungar. *14* (1983), 269–308.

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