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Cylindric Algebras

PART II

L. HENKIN, J.D. MONK and A. TARSKI †

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PART II

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INTRODUCTION

The present volume continues and completes *Cylindric Algebras, Part I.* Substantial portions of this volume, including much of the opening chapter, can be read independently of its predecessor, and large portions of Part I are unnecessary for following the present material. For this reason we present here a brief, independent introduction to this book.

The concept of cylindric algebras was created to permit the use of algebraic methods in treating two related parts of mathematics. One of these is a very general kind of geometry associated with basic set–theoretic notions, and the other is the theory of deductive systems of mathematical logic. The two domains are connected because models of deductive systems give rise in a natural way to structures within the set–theoretical "spaces". The notion of a cylindric algebra can be considered as a common, algebraic abstraction from its two sources.

First we shall describe the set–theoretical source which gives rise to Chapter 3 (the first chapter in this book), and then we shall describe the logical source and its relation to the remainder of the book.

Given any ordinal α and set U, there are several elementary operations on subsets of the space ${}^{\alpha}U$ of α–termed sequences of elements of U which play a fundamental role in our theory. These are the familiar Boolean operations of union, intersection, and complementation with respect to ${}^{\alpha}U$, and the less familiar operations C_κ of *cylindrification* for each $\kappa<\alpha$: if $X\subseteq{}^{\alpha}U$, then

$$\mathsf{C}_\kappa X=\{u\in{}^{\alpha}U:\text{there is a } v\in X \text{ such that } u_\lambda=v_\lambda \text{ for all } \lambda\in\alpha \text{ with } \lambda\neq\kappa\}.$$

($\mathsf{C}_\kappa X$ is the cylinder generated by translating X parallel to the κth axis of the space ${}^{\alpha}U$.) In addition, there are the important *diagonal hyperplanes* $\mathsf{D}_{\kappa\lambda}$ for $\kappa,\lambda<\alpha$:

$$\mathsf{D}_{\kappa\lambda}=\{u\in{}^{\alpha}U: u_\kappa=u_\lambda\}.$$

A *cylindric set algebra* is a collection of subsets of ${}^{\alpha}U$ closed under all of the Boolean and cylindric operations, having all diagonal hyperplanes as members. The abstract notion of a cylindric algebra is defined by equational axioms which hold in all cylindric set algebras. Part I was devoted to a comprehensive study of these arbitrary cylindric algebras.

In Chapter 3 (the first chapter) of the present work we study cylindric set algebras and related structures in their own right. The second part of this chapter is devoted to theorems on relationships between arbitrary cylindric algebras and the various kinds of set algebras – representation theorems, and construction methods for non–representable algebras. In the first two sections of Chapter 4, the

metamathematics of cylindric algebras is discussed: model theory in section 4.1, and decision problems in section 4.2.

The second source for the concept of cylindric algebras is first–order logic. The basic operations $+$, $\cdot$, $-$, c_κ, $\mathsf{d}_{\kappa\lambda}$ in cylindric algebras correspond to logical notions $\vee$, $\wedge$, $\neg$, $\exists v_\kappa$, $v_\kappa = v_\lambda$. In fact, if Λ is a first–order language and Σ a set of sentences of Λ, define $\varphi \equiv \psi$ iff φ and ψ are formulas of Λ and $\varphi \leftrightarrow \psi$ is derivable from Σ. Then $\equiv$ is an equivalence relation on the set F of all formulas of Λ, and there are operations $+$, $\cdot$, $-$, c_κ, $\mathsf{d}_{\kappa\lambda}$ on the set S of all equivalence classes of formulas which determine a cylindric algebra having S as its set of elements, with $(\varphi/\equiv)+(\psi/\equiv) = (\varphi\vee\psi)/\equiv$, $(\varphi/\equiv)\cdot(\psi/\equiv) = (\varphi\wedge\psi)/\equiv$, $-(\varphi/\equiv) = (\neg\varphi)/\equiv$, $\mathsf{c}_\kappa(\varphi/\equiv) = (\exists v_\kappa\varphi)/\equiv$, and $\mathsf{d}_{\kappa\lambda} = (v_\kappa = v_\lambda)/\equiv$ for all formulas φ,ψ and all $\kappa,\lambda<\omega$.

Section 4.3 is devoted to a careful explication of this connection between cylindric algebras and first–order logic.

There are other ways to algebraize ordinary first–order logic, and there are other logics to algebraize. A survey of some other algebraic logics is given in Chapter 5.

Throughout Part I various "promises" were made about material which would be found in Part II. These are located in this volume at the appropriate places, with the following exceptions, which mainly concern results whose proofs could not be reconstructed. (1) Cf. Part I, p. 258: we do not know whether the equational theory of Mn_α's is decidable. See 4.2.1–4.2.6. (2) Cf. Part I, p. 311: the connection between the class L'_α and infinitary languages is not discussed in this volume. (3) Cf. Part I, p. 339: the result mentioned in Remark 2.5.8 could not be reconstructed in its full form; see 4.3.32. (4) Cf. Part I, p. 343: we do not know whether, for $3\leq\alpha<\omega$ there are identities holding in all finite CA_α's but not in all CA_α's. (5) Cf. Part I, p. 426: we do not know whether, if $\omega\leq\alpha<\beta$, there is a CA_α $\mathfrak{A}$ and a CA_β $\mathfrak{B}$ such that $\mathfrak{A}$ is a generating subreduct of $\mathfrak{B}$ different from $\mathfrak{Rt}_\alpha\mathfrak{B}$. Items (3)–(5) refer to results of Tarski whose proofs could not be reconstructed.

* * *

The first draft of this book was about two–thirds complete when the senior author and originator of the theory of cylindric algebras died. We hope that this volume will be found to conform to his high standards of scholarship and creativity.

Since the appearance of Part I, the theory of cylindric algebras has been advanced primarily by work of H. Andréka and I. Németi, whose many results appear throughout this volume; by R. Maddux, who greatly simplified the solution of a key decision problem and also settled the remaining open equational question (for CA_3's); by D. Resek, who established a strong representation theorem for relativized cylindric algebras; and by M. Rubin, who proved the undecidability of the elementary theory of CA_1's.

For extensive help in the actual writing of the book, the authors express their deep gratitude to Hajnal Andréka and István Németi. The typing of the manuscript was done by Andrea and Steven Monk. For help with corrections we are indebted to Balázs Biró, Roger Maddux, Ildikó Sain, György Serény, and Andréka and Németi. The book was produced from camera–ready copy prepared on an IBM PC with an Epson FX–100 printer, using the Fancy Font typesetting software from Softcraft, Inc.

Notational changes

We have kept the notation from Part I (with additions itemized at the end of the book), except for the following changes or simplifications. Parentheses in superscripts or subscripts are eliminated when no confusion is likely: thus, e.g., we write $Sg^{\mathfrak{A}}X$, $\Delta^{\mathfrak{A}}X$, $\mathfrak{Rb}^{\rho}\mathfrak{B}$ in place of $Sg^{(\mathfrak{A})}X$, $\Delta^{(\mathfrak{A})}X$, $\mathfrak{Rb}^{(\rho)}\mathfrak{B}$. We use the notation $\varphi^{\mathfrak{A}}$ for the α–ary relation defined by φ in $\mathfrak{A}$, rather than the earlier notation $\tilde{\varphi}^{(\mathfrak{A})}$. Because of some possibility of confusion, we retain the notation $c_{(\Gamma)}$ for generalized cylindrification. To avoid second–level subscripts (subscripts on subscripts, etc.), we usually convert notations like f_i or p_{ij} to fi or pij when used at a subscript or superscript level.

Throughout the book, proofs which are trivial are simply omitted.

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CONTENTS

CHAPTER 3

REPRESENTABLE CYLINDRIC ALGEBRAS

3. REPRESENTABLE CYLINDRIC ALGEBRAS

In this chapter we discuss in detail the set–theoretically defined cylindric algebras and the relationships between them and the abstract cylindric algebras which were the focus of attention in Part I. The main notion in section 3.1 is that of a cylindric set algebra, already introduced in 1.1.5 and discussed in passing in later places in Part I. There are several other related notions, most of which were also introduced in Part I. Section 3.1 has an extensive discussion of the relationships between these notions and between the results of applying algebraic operators like H, S, and P to classes of them. Aside from the naturalness of these investigations, they also prepare the ground for section 3.2. In section 3.2 several representation theorems are proved: theorems that certain abstractly defined cylindric algebras are isomorphic to set algebras. Section 3.2 also has a lengthy discussion of several kinds of non–representable cylindric algebras.

3.1. CYLINDRIC SET ALGEBRAS

The discussion of set algebras which we now begin represents a somewhat condensed version of the book Henkin, Monk, Tarski, Andréka, Németi [81']. We introduce most of the basic notions found there, prove the main positive results, and state without proof the main negative results. That book has two parts: Henkin, Monk, Tarski [81'] and Andréka, Németi [81']; for brevity we refer to these parts in this section by [HMT81] and [AN81] respectively.

In addition to cylindric set algebras, introduced in 1.1.5, we have considered several other kinds of set algebras; cf. 1.1.13, 2.2.11, and 2.3.15. We shall repeat here the definitions of these. To unify the treatment we introduce a very general kind of algebra, the cylindric–relativized set algebras. After the basic definitions, we give simple relationships between the classes, and then proceed to discuss various algebraic notions in their applications to our set algebras.

We begin with the definition of the main notions with which we will be concerned in all of Part II: cylindric–relativized set algebras, cylindric set algebras, generalized cylindric set algebras, representable cylindric algebras, and regular cylindric set algebras.

DEFINITION 3.1.1. (i) *If $f \in {}^{\alpha}U$, $\kappa<\alpha$, and $u \in U$, then f^{κ}_{u} is the member of ${}^{\alpha}U$ such that $(f^{\kappa}_{u})\lambda = f\lambda$ if $\lambda \neq \kappa$, while $(f^{\kappa}_{u})\kappa = u$. For typographical reasons we sometimes write $f(\kappa/u)$ in place of f^{κ}_{u}.*

(ii) *Let α be an ordinal and V a set of α–termed sequences. The* base *of V is $\bigcup_{x\in V} Rgx$, which we denote by U in what follows. For all $\kappa,\lambda<\alpha$ we set*

$$\mathsf{D}^{[V]}_{\kappa\lambda} = \{y \in V : y_{\kappa} = y_{\lambda}\},$$

and we let $\mathsf{C}_\kappa^{[V]}$ *be the mapping of* $Sb\,V$ *into* $Sb\,V$ *such that, for every* $X \subseteq V$,

$$\mathsf{C}_\kappa^{[V]} X = \{y \in V : y_u^\kappa \in X \text{ for some } u \in U\}.$$

(*When* V *is implicitly understood we shall write simply* $\mathsf{D}_{\kappa\lambda}$ *or* C_κ.)

(iii) *A is an* α**–dimensional cylindric–relativized field of sets** *iff there is a set* V *of* α*–termed sequences such that* A *is a non–empty family of subsets of* V *closed under all the operations* u, n, ${}_V\sim$ *and* $\mathsf{C}_\kappa^{[V]}$ (*for each* $\kappa<\alpha$), *and containing as elements the subsets* $\mathsf{D}_{\kappa\lambda}^{[V]}$ (*for all* $\kappa,\lambda<\alpha$).

(iv) $\mathfrak{X}$ *is a* **cylindric–relativized set algebra of dimension** α *iff there is a set* V *of* α*–termed sequences such that*

$$\mathfrak{X} = \langle A, \mathsf{u}, \mathsf{n}, {}_V\sim, 0, V, \mathsf{C}_\kappa^{[V]}, \mathsf{D}_{\kappa\lambda}^{[V]} \rangle_{\kappa,\lambda<\alpha},$$

where A *is an* α*–dimensional cylindric–relativized field of sets with unit element* V. Crs_α *is the class of all cylindric–relativized set algebras of dimension* α.

For the remaining parts of this definition we assume that A *is a cylindric–relativized field of sets and* $\mathfrak{X}$, *with universe* A, *is a cylindric–relativized set algebra, both with dimension* α *and unit element* V. *The base of* V *is also called the* **base** *of* A *and* $\mathfrak{X}$. *We let it be* U *in what follows.*

(v) A, *respectively* $\mathfrak{X}$, *is called an* α**–dimensional cylindric field of sets**, *respectively* **set algebra**, *if* $V = {}^\alpha U$. *The class of all cylindric set algebras of dimension* α *is denoted by* Cs_α.

(vi) A, *respectively* $\mathfrak{X}$, *is called an* α**–dimensional generalized cylindric field of sets**, *respectively* **set algebra**, *if* V *has the form* $\bigcup_{i\in I} {}^\alpha Y_i$, *where* $Y_i \neq 0$ *for each* $i \in I$, *and* $Y_i \mathsf{n} Y_j = 0$ *for any two distinct* $i,j \in I$. *The symbol* Gs_α *denotes the class of all generalized cylindric set algebras of dimension* α.

(vii) A *cylindric algebra* $\mathfrak{B}$ *is called* **representable** *if it is isomorphic to a generalized cylindric set algebra. Thus* IGs_α *is the class of all* α*–dimensional representable cylindric algebras.*

(viii) *An element* $x \in A$ *is* **regular** *provided that* $g \in x$ *whenever* $f \in x$, $(\Delta x \mathsf{u} 1) \mathsf{l} f \subseteq g$, *and* $g \in V$. A *and* $\mathfrak{X}$ *are called* **regular** *if each* $x \in A$ *is regular.* (*We assume here that* $\alpha>0$; *if* $\alpha=0$, *all elements, as well as* A *and* $\mathfrak{X}$, *are called regular.*) K *being any class included in* Crs_α, *we denote by* K^{reg} *the class of all regular algebras which belong to* K.

In 1.1.13 we gave extensive motivation for some of the concepts in 3.1.1. Thus cylindric set algebras form the simplest concrete kind of cylindric algebras. For $\alpha \geqq 2$, IGs_α coincides with the class of subdirect products of cylindric set algebras (by 3.1.77 below) and is thus the most natural abstract class of cylindric algebras corresponding to the concrete ones; much of our work in this part will be concerned with this class of representable algebras. (For $\alpha \leqq 1$, a relatively less interesting case, IGs_α does not correspond to the intuitive notion of representable algebra; see 3.1.17, 3.1.70.) Regular locally finite set algebras were implicitly discussed in 2.3.15. They are the most well–behaved algebras from the point of view of representation theory. It turns out that any representable CA is isomorphic to a subdirect product of regular cylindric set algebras (by 3.1.107). As we shall see, the class $\mathsf{Cs}_\alpha^{reg} \mathsf{n} \mathsf{Lf}_\alpha$ of regular locally finite cylindric set algebras plays a very important role in

representation theory and in the relationship of set algebras to logic.

Now we define some subsidiary notions which will be important in establishing the main results about the above algebras and are also interesting in their own right.

DEFINITION 3.1.2. *Let $\mathfrak{A}$ be a cylindric–relativized set algebra with universe A, dimension α, base U, and unit element V.*

(i) *If $A = Sb\,V$, then we denote $\mathfrak{A}$ by $\mathfrak{Sb}\,V$; A and $\mathfrak{A}$ are called respectively a* **full cylindric–relativized field of sets** *and a* **full cylindric–relativized set algebra**.

(ii) *Let W be a set and β an ordinal. ${}^{\beta}W$ is then called the* **Cartesian space with base** W **and dimension** β. *Moreover, for every $p \in {}^{\beta}W$ we set*

$$ {}^{\beta}W^{(p)} = \{x \in {}^{\beta}W : \{\xi < \beta : x_\xi \neq p_\xi\} \text{ is finite}\}, $$

and we call ${}^{\beta}W^{(p)}$ the **weak Cartesian space with base** W **and dimension** β **determined by** p.

(iii) *A, respectively $\mathfrak{A}$, is called an* α**–dimensional weak cylindric field of sets,** *respectively* **set algebra**, *if there is a $p \in {}^{\alpha}U$ such that $V = {}^{\alpha}U^{(p)}$. The class of all α–dimensional weak cylindric set algebras is denoted by Ws_α.*

(iv) *A, respectively $\mathfrak{A}$, is called an* α**–dimensional generalized weak cylindric field of sets,** *respectively* **set algebra**, *if V has the form $\bigcup_{i \in I} {}^{\alpha}Y_i^{(pi)}$, where $pi \in {}^{\alpha}Y_i$ for each $i \in I$ and ${}^{\alpha}Y_i^{(pi)} \cap {}^{\alpha}Y_j^{(pj)} = 0$ for any two distinct $i,j \in I$. We use Gws_α for the class of all generalized weak cylindric set algebras of dimension α.*

In order to finish defining our basic notions, we need the following lemma.

LEMMA 3.1.3. *Suppose $\alpha \geqq 2$ and*

$$ \bigcup_{i \in I} {}^{\alpha}Y_i^{(pi)} = \bigcup_{j \in J} {}^{\alpha}Z_j^{(qj)}, $$

with ${}^{\alpha}Y_i^{(pi)} \cap {}^{\alpha}Y_k^{(pk)} = 0$ for distinct $i,k \in I$, and ${}^{\alpha}Z_j^{(qj)} \cap {}^{\alpha}Z_l^{(ql)} = 0$ for distinct $j,l \in J$.

Then $\{{}^{\alpha}Y_i^{(pi)} : i \in I\} = \{{}^{\alpha}Z_j^{(qj)} : j \in J\}$.

PROOF. By symmetry it suffices to take any $i \in I$ and find $j \in J$ such that ${}^{\alpha}Y_i^{(pi)} \subseteq {}^{\alpha}Z_j^{(qj)}$. Choose $j \in J$ so that $pi \in {}^{\alpha}Z_j^{(qj)}$. We claim

(1) $\quad Y_i \subseteq Z_j$.

For, let $y \in Y_i$. Then $(pi)^0_y \in {}^{\alpha}Y_i^{(pi)}$, so choose $k \in J$ with $(pi)^0_y \in {}^{\alpha}Z_k^{(qk)}$. Thus $(pi)_1 \in Z_j \cap Z_k$. Then $(pi)^0_{(pi)1} \in {}^{\alpha}Z_j^{(qj)} \cap {}^{\alpha}Z_k^{(qk)}$, so $j = k$. Thus $y \in Z_j$, as desired to prove (1).

Now let $r \in {}^{\alpha}Y_i^{(pi)}$ be arbitrary. Thus by (1), $r \in {}^{\alpha}Z_j$. Clearly then $r \in {}^{\alpha}Z_j^{(qj)}$, as desired.

REMARK 3.1.4. Lemma 3.1.3 does not extend to $\alpha = 1$, since clearly $\bigcup_{i \in I} {}^{\alpha}Y_i = {}^{\alpha}(\bigcup_{i \in I} Y_i)$ in this case, but it is trivially true for $\alpha = 0$.

DEFINITION 3.1.5. *Let $\mathfrak{A}$, with universe A, be a Gws_α with unit element $\bigcup_{i \in I} {}^{\alpha}Y_i^{(pi)}$, where ${}^{\alpha}Y_i^{(pi)} \cap {}^{\alpha}Y_j^{(pj)} = 0$ for distinct $i,j \in I$. Assume $\alpha \geqq 2$. The sets Y_i are called the*

subbases *of* A, *or* $\mathfrak{A}$ (*this is justified by* 3.1.3). *We call* A, *or* $\mathfrak{A}$, **normal** *if* $Y_i = Y_j$ *or* $Y_i \cap Y_j = 0$ *for all* $i,j \in I$; **widely distributed** *if* $Y_i \cap Y_j = 0$ *for all distinct* $i,j \in I$; **compressed** *if* $Y_i = Y_j$ *for all* $i,j \in I$. *The class of all normal, widely distributed, or compressed* Gws_α*'s is denoted respectively by* Gws_α^{nm}, Gws_α^{wd} *or* Gws_α^{cm}. *If* $\mathsf{K} \subseteq \mathsf{Gws}_\alpha$ *and* κ *is a cardinal, we denote by* ${}_\kappa\mathsf{K}$ *the class of all* $\mathfrak{A} \in \mathsf{K}$ *all of whose subbases have power* κ, *by* ${}_{<\kappa}\mathsf{K}$ *the class of all those whose subbases have power* $<\kappa$, *and by* ${}_\infty\mathsf{K}$ *all* $\mathfrak{A} \in \mathsf{K}$ *with all subbases infinite.*

DEFINITION 3.1.6. *If* K *is a class of cylindric algebras, we let* $\mathsf{RIK} = \{\mathfrak{R}\mathfrak{l}_b\mathfrak{B} : \mathfrak{B} \in \mathsf{K}, b \in B\}$ (*cf.* 2.2.1).

REMARKS 3.1.7. Having defined our basic notions, we summarize the main results established in this section. The inclusions holding among the various classes of set algebras are indicated in Figure 3.1.8 for $\alpha > 0$; if we consider the classes of isomorphic images of the various set algebras, then the diagram collapses as indicated in Figure 3.1.9. The inclusions in each case are in general proper inclusions. In case $2 \leqq \alpha < \omega$ the classes Ws_α and Cs_α coincide, and so do Gws_α and Gs_α; furthermore, under this assumption each member of any of these classes is regular. IGs_α coincides with the class of isomorphs of subdirect products of Cs_α's for $\alpha \geqq 2$. IGws_α and Ws_α are similarly related. Every Gs_α is isomorphic to a regular Gs_α and to a subdirect product of Cs_α^{reg}'s. The main result of the section is that $\mathsf{IGs}_\alpha = \mathsf{HSP}(\mathsf{Cs}_\alpha^{reg} \cap \mathsf{Lf}_\alpha)$ for $\alpha \geqq \omega$ (3.1.123). The situation with respect to algebraic operators on our classes is summarized in Figures 3.1.10 – 3.1.13.

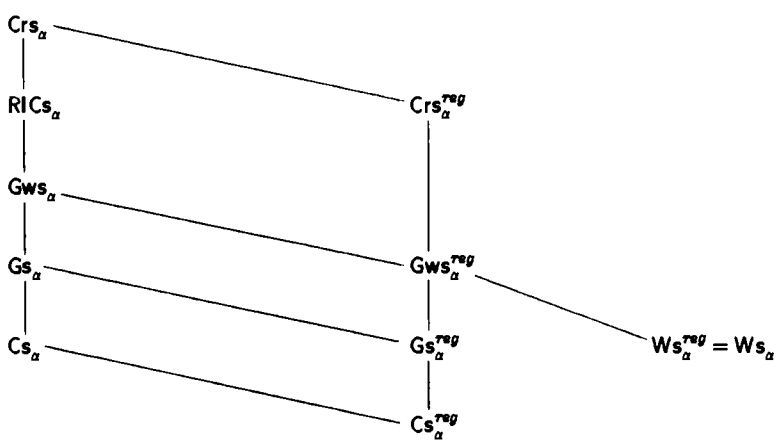

FIGURE 3.1.8
($\alpha > 0$)

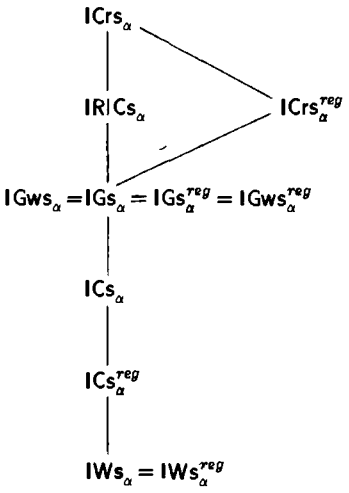

FIGURE 3.1.9
($\alpha>0$)

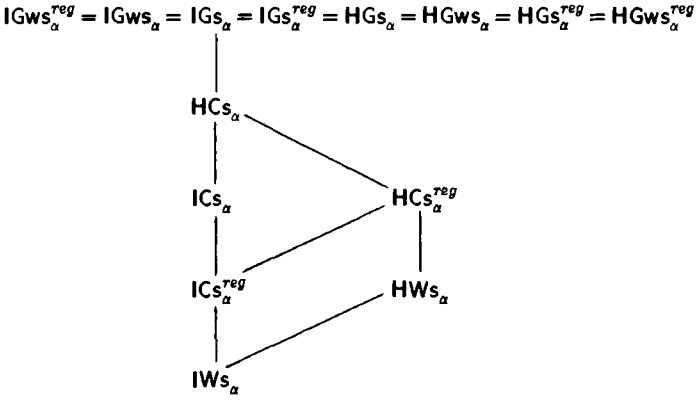

FIGURE 3.1.10
($\alpha \geqq \omega$)

$$\begin{tikzcd}
\mathsf{H}_\infty\mathsf{Cs}_\alpha = \mathsf{I}_\infty\mathsf{Cs}_\alpha = \mathsf{H}_\infty\mathsf{Cs}_\alpha^{reg} = \mathsf{H}_\infty\mathsf{Ws}_\alpha \arrow[d, dash] \\
\mathsf{I}_\infty\mathsf{Cs}_\alpha^{reg} \arrow[d, dash] \\
\mathsf{I}_\infty\mathsf{Ws}_\alpha
\end{tikzcd}$$

FIGURE 3.1.11
$(\alpha \geqq \omega)$

$$\begin{tikzcd}
 & \mathsf{IGws}_\alpha = \mathsf{HSPGws}_\alpha = \mathsf{IGs}_\alpha = \mathsf{HSPGs}_\alpha = \mathsf{SPCs}_\alpha = \mathsf{SPWs}_\alpha & \\
 & \mathsf{HPCs}_\alpha \arrow[u, dash] \arrow[dl, dash] \arrow[dr, dash] & \\
\mathsf{HCs}_\alpha \arrow[d, dash] \arrow[dr, dash] & \mathsf{HPCs}_\alpha^{reg} \arrow[d, dash] \arrow[dl, dash] \arrow[dr, dash] & \mathsf{PCs}_\alpha \arrow[d, dash] \arrow[dl, dash] \\
 & \mathsf{ICs}_\alpha \arrow[d, dash] & \\
\mathsf{HCs}_\alpha^{reg} \arrow[d, dash] \arrow[dr, dash] & \mathsf{HPWs}_\alpha \arrow[dr, dash] & \mathsf{PCs}_\alpha^{reg} \arrow[d, dash] \arrow[dl, dash] \\
 & \mathsf{ICs}_\alpha^{reg} \arrow[d, dash] & \\
\mathsf{HWs}_\alpha \arrow[dr, dash] & & \mathsf{PWs}_\alpha \arrow[dl, dash] \\
 & \mathsf{IWs}_\alpha &
\end{tikzcd}$$

FIGURE 3.1.12
$(\alpha \geqq 2)$

$$\begin{tikzcd}[column sep=small]
 & \mathsf{H}_\infty\mathsf{Ws}_\alpha = \mathsf{H}_\infty\mathsf{Cs}_\alpha = \mathsf{H}_\infty\mathsf{Cs}_\alpha^{reg} = \mathsf{I}_\infty\mathsf{Cs}_\alpha = \mathsf{HP}_\infty\mathsf{Ws}_\alpha = \mathsf{HP}_\infty\mathsf{Cs}_\alpha^{reg} = \mathsf{HSP}_\infty\mathsf{Gs}_\alpha \arrow[d, dash] \\
 & \mathsf{P}_\infty\mathsf{Cs}_\alpha^{reg} \arrow[dl, dash] \arrow[dr, dash] \\
\mathsf{I}_\infty\mathsf{Cs}_\alpha^{reg} \arrow[dr, dash] & & \mathsf{P}_\infty\mathsf{Ws}_\alpha \arrow[dl, dash] \\
 & \mathsf{I}_\infty\mathsf{Ws}_\alpha
\end{tikzcd}$$

FIGURE 3.1.13
($\alpha \geqq \omega$)

Now we begin the proper mathematical discussion in this section by giving, in 3.1.14 – 3.1.29, degenerate cases and simple properties of the notions introduced. Recall that we omit proofs which are trivial.

COROLLARY 3.1.14. *Let* $\mathfrak{A}$ *be a* Crs_α *with base* U *and unit element* V.
(i) $V=0$ *iff* $|A|=1$.
(ii) *If* $V=\{0\}$, *then* $|A|=2$.
(iii) *If* $\alpha=0$, *then* $V\subseteq\{0\}$.
(iv) *If* $\alpha>0$ *and* $U=0$, *then* $V=0$.

Because of this corollary we will frequently make such assumptions as $\alpha>0$, $U\neq 0$, or $V\neq 0$.

COROLLARY 3.1.15. *If* $\alpha<\omega$, U *is any set, and* $p\in {}^\alpha U$, *then* ${}^\alpha U^{(p)}={}^\alpha U$. *Hence for* $\alpha<\omega$ *we have* $\mathsf{Gs}_\alpha=\mathsf{Gws}_\alpha$, *if* $0<\alpha<\omega$ *we have* $\mathsf{Cs}_\alpha=\mathsf{Ws}_\alpha\cup\{\mathfrak{A}_\alpha\}$, *where* $\mathfrak{A}_\alpha$ *is the unique* Cs_α *with universe* 1, *and finally* $\mathsf{Cs}_0=\mathsf{Ws}_0$.

COROLLARY 3.1.16. *Let* $\alpha\geqq 2$. *If* $\mathfrak{A}$ *is a* Gws_α *with every subbase having only one element, then* $\mathfrak{A}$ *is a discrete* Gs_α.

COROLLARY 3.1.17. $\mathsf{Crs}_1=\mathsf{Gws}_1=\mathsf{Gs}_1=\mathsf{Cs}_1=\mathsf{Ws}_1\cup\{\mathfrak{B}\}$, *where* $\mathfrak{B}$ *is the* Crs_1 *with universe* 1; $\mathsf{Crs}_0=\mathsf{Gws}_0=\mathsf{Gs}_0=\{\mathfrak{A}_1,\mathfrak{A}_2\}$ *and* $\mathsf{Cs}_0=\mathsf{Ws}_0=\{\mathfrak{A}_2\}$, *where* $\mathfrak{A}_1$ *and* $\mathfrak{A}_2$ *are the unique* Crs_0*'s with universes* 1 *and* 2 *respectively. Further, every* Crs_0 *is full, and the base of any* Crs_0 *is* 0.

COROLLARY 3.1.18. *If* $\mathfrak{A}$ *is a* Cs_1 *or* Ws_1, $x\in A$, *and* $x\neq 0$, *then* $\mathsf{c}_0x=1$.

By 2.3.14, Corollary 3.1.18 simply expresses that any Cs_1 or Ws_1 with more than one

element is simple. In fact, this is true for any finite dimension. The proof is trivial; the result is stated formally in 3.1.70 below.

COROLLARY 3.1.19. (i) *For any* α, $\mathsf{Cs}_\alpha \cup \mathsf{Gs}_\alpha \cup \mathsf{Ws}_\alpha \cup \mathsf{Gws}_\alpha \subseteq \mathsf{CA}_\alpha$.
(ii) *For* $\alpha \leqq 1$, $\mathsf{Crs}_\alpha \subseteq \mathsf{CA}_\alpha$.
(iii) *For* $\alpha \geqq 2$, $\mathsf{Crs}_\alpha \not\subseteq \mathsf{CA}_\alpha$.

PROOF. Both (i) and (ii) are trivial. To establish (iii), we construct $\mathfrak{A} \in \mathsf{Crs}_\alpha \sim \mathsf{CA}_\alpha$ for $\alpha \geqq 2$ by choosing any set U with $|U|>1$, taking $V = {}^\alpha U \sim \mathsf{D}_{01}^{[W]}$, where $W = {}^\alpha U$, and letting $\mathfrak{A}$ be the full Crs_α with unit element V (3.1.2(i)). We have $\mathsf{D}_{01}^{[V]} = 0$, so that $V = \mathsf{D}_{00}^{[V]} \neq \mathsf{C}_1^{[V]}(\mathsf{D}_{01}^{[V]} \cap \mathsf{D}_{10}^{[V]}) = 0$. Thus $\mathfrak{A}$ fails to satisfy axiom (C_6), whence $\mathfrak{A} \notin \mathsf{CA}_\alpha$ and the proof is complete.

COROLLARY 3.1.20. *If* $\alpha \geqq 2$, *then* $\mathsf{Cs}_\alpha \subset \mathsf{Gs}_\alpha$ *and* $\mathsf{RICs}_\alpha \subseteq \mathsf{Crs}_\alpha = \mathsf{SRICs}_\alpha$; *furthermore,* $\mathsf{Cs}_\alpha \subset \mathsf{Gws}_\alpha^{cm}$ *if* $\alpha \geqq \omega$, *and* $\mathsf{Cs}_\alpha = \mathsf{Gws}_\alpha^{cm}$ *if* $\alpha < \omega$.

It is also known that $\mathsf{Gws}_\alpha \subset \mathsf{RICs}_\alpha$ for $\alpha \geqq 2$; see [HMT81] 2.9–2.14. Also, it is known that although $\mathsf{RICs}_0 = \mathsf{Crs}_0$ and $\mathsf{RICs}_1 = \mathsf{Cs}_1$, we have $\mathsf{RICs}_\alpha \subset \mathsf{Crs}_\alpha$ for $\alpha > 1$; see Henkin, Resek [75'] and [AN81]. See also 3.1.33 and 5.5.7.

COROLLARY 3.1.21. *If* $\alpha \geqq \omega$, *then* $\mathsf{Ws}_\alpha \subset \mathsf{Gws}_\alpha \subset \mathsf{Crs}_\alpha$ *and* $\mathsf{Cs}_\alpha \subset \mathsf{Gs}_\alpha \subset \mathsf{Gws}_\alpha$.

THEOREM 3.1.22. *Let* $\mathfrak{A}$ *be a* Gws_α *with unit element* $\bigcup_{i \in I} {}^\alpha U_i^{(pi)}$, *where* ${}^\alpha U_i^{(pi)} \cap {}^\alpha U_j^{(pj)} = 0$ *for all distinct* $i,j \in I$. *Assume that* ${}^\alpha U_i^{(pi)} \in A$ *for all* $i \in I$, *and* $\alpha \neq 1$. *Then for any* $X \in A$ *the following conditions are equivalent:*
(i) $X = {}^\alpha U_i^{(pi)}$ *for some* $i \in I$;
(ii) X *is a minimal element of* $\mathfrak{A}$ (*under* $\subseteq$) *such that* $X \neq 0$ *and* $\Delta X = 0$.

PROOF. (i)$\Rightarrow$(ii). Clearly for any $i \in I$ we have $0 \neq {}^\alpha U_i^{(pi)}$ and $\Delta {}^\alpha U_i^{(pi)} = 0$. Now suppose that $0 \neq Y \subseteq {}^\alpha U_i^{(pi)}$ and $\Delta Y = 0$. Fix $y \in Y$, and let $x \in {}^\alpha U_i^{(pi)}$ be arbitrary. There is a finite $\Gamma \subseteq \alpha$ such that $(\alpha \sim \Gamma) 1 y \subseteq x$. Thus $x \in \mathsf{c}_{(\Gamma)} Y = Y$, and hence $Y = {}^\alpha U_i^{(pi)}$.

(ii)$\Rightarrow$(i). Assume (ii), and choose $i \in I$ such that $X \cap {}^\alpha U_i^{(pi)} \neq 0$. Since $\Delta {}^\alpha U_i^{(pi)} = 0$, we have $\Delta(X \cap {}^\alpha U_i^{(pi)}) = 0$ by 1.6.6. Hence by (ii), $X = X \cap {}^\alpha U_i^{(pi)}$, while by the implication (i)$\Rightarrow$(ii) already established, ${}^\alpha U_i^{(pi)} = X \cap {}^\alpha U_i^{(pi)}$. Thus (i) holds.

Now we discuss regular set algebras. For the classes Cs_α, Ws_α and Gs_α the definition assumes a simpler form.

COROLLARY 3.1.23. *Let* $\mathfrak{A}$ *be a* Cs_α *with base* U, *and let* $X \in A$. *Then the following conditions are equivalent:*
(i) X *is regular;*
(ii) *For all* $f \in X$ *and all* $g \in {}^\alpha U$, *if* $\Delta X 1 f \subseteq g$ *then* $g \in X$.

PROOF. (ii)$\Rightarrow$(i): trivial. (i)$\Rightarrow$(ii). Assume (i) and the hypotheses of (ii). If $0 \in \Delta X$, the desired conclusion $g \in X$ obvious. Suppose therefore $0 \notin \Delta X$. Then $f_{g0}^0 \in X$ since $0 \notin \Delta X$. Since $(\Delta X \cup 1) 1 f_{g0}^0 \subseteq g$, by the definition of regularity we get $g \in X$.

The next two corollaries are proved in the same way as 3.1.23.

COROLLARY 3.1.24. *Let* $\mathfrak{A}$ *be a* Ws_α *with unit element* ${}^\alpha U^{(p)}$, *and let* $X \in A$. *Then the following conditions are equivalent*:

(i) X *is regular*;

(ii) *For all* $f \in X$ *and all* $g \in {}^\alpha U^{(p)}$, *if* $\Delta X 1 f \subseteq g$ *then* $g \in X$.

See also 3.1.26.

COROLLARY 3.1.25. *Let* $\mathfrak{A}$ *be a* Gs_α *with unit element* $\bigcup_{i \in I} {}^\alpha Y_i$, *where* $Y_i \mathbf{n} Y_j = 0$ *for* $i \neq j$, *and let* $X \in A$. *Then the following conditions are equivalent*:

(i) X *is regular*;

(ii) *For all* $i \in I$, *all* $f \in X \mathbf{n} {}^\alpha Y_i$, *and all* $g \in {}^\alpha Y_i$, *if* $\Delta X 1 f \subseteq g$ *then* $g \in X$.

No analogous simplification of the notion of regularity for arbitrary Gws_α's is known. Weak cylindric set algebras are always regular:

COROLLARY 3.1.26. $\mathsf{Ws}_\alpha^{reg} = \mathsf{Ws}_\alpha$. *Thus if* $\mathfrak{A}$ *is a* Ws_α *with unit element* ${}^\alpha U^{(p)}$, $f \in X \in A$, $g \in {}^\alpha U^{(p)}$, *and* $\Delta X 1 f \subseteq g$, *then* $g \in X$.

PROOF. It suffices by 3.1.24 to prove the second statement; assume its hypothesis. Since $f, g \in {}^\alpha U^{(p)}$, there is a finite $\Gamma \subseteq \alpha$ with $(\alpha \sim \Gamma) 1 f \subseteq g$. Hence $[\alpha \sim (\Gamma \sim \Delta X)] 1 f \subseteq g$, so that $g \in \mathsf{c}_{(\Gamma \sim \Delta X)} X = X$.

COROLLARY 3.1.27. *If* $\alpha < \omega$, *then* $\mathsf{Cs}_\alpha = \mathsf{Cs}_\alpha^{reg}$, $\mathsf{Ws}_\alpha = \mathsf{Ws}_\alpha^{reg}$, $\mathsf{Gs}_\alpha = \mathsf{Gs}_\alpha^{reg}$, *and* $\mathsf{Gws}_\alpha = \mathsf{Gws}_\alpha^{reg}$.

COROLLARY 3.1.28. *If* $\alpha \geqq \omega$, *then* $\mathsf{Ws}_\alpha^{reg} \subset \mathsf{Gws}_\alpha^{reg} \subset \mathsf{Crs}_\alpha^{reg}$ *and* $\mathsf{Cs}_\alpha^{reg} \subset \mathsf{Gs}_\alpha^{reg} \subset \mathsf{Gws}_\alpha^{reg}$.

PROOF. To produce a member of $\mathsf{Gws}_\alpha^{reg} \sim \mathsf{Ws}_\alpha^{reg}$, let U and V be two disjoint sets each with at least two elements, and let $p \in {}^\alpha U$, $q \in {}^\alpha V$ be arbitrary. Let $\mathfrak{A}$ be the full Gws_α with unit element ${}^\alpha U^{(p)} \mathbf{u} {}^\alpha V^{(q)}$. Then it is easily checked, along the lines of the proof of 3.1.26, that $\mathfrak{A} \in \mathsf{Gws}_\alpha^{reg}$, while clearly $\mathfrak{A} \notin \mathsf{Ws}_\alpha$. It is also clear that $\mathfrak{A} \notin \mathsf{Gs}_\alpha$. If we let $\mathfrak{B}$ be the two-element Crs_α with unit element $\{p\}$, we see that $\mathfrak{B} \in \mathsf{Crs}_\alpha^{reg} \sim \mathsf{Gws}_\alpha$ (provided that p has more than one element in its range). Finally, if U and V are as above and $\mathfrak{C}$ is the minimal subalgebra of the full Gs_α with unit element ${}^\alpha U \mathbf{u} {}^\alpha V$, then $\mathfrak{C} \in \mathsf{Gs}_\alpha^{reg} \sim \mathsf{Cs}_\alpha$.

COROLLARY 3.1.29. *For* $\alpha \geqq \omega$ *we have* $\mathsf{Cs}_\alpha^{reg} \subset \mathsf{Cs}_\alpha$, $\mathsf{Gs}_\alpha^{reg} \subset \mathsf{Gs}_\alpha$, *and* $\mathsf{Gws}_\alpha^{reg} \subset \mathsf{Gws}_\alpha$.

PROOF. By 3.1.21 it suffices to exhibit a Cs_α which is not regular. Let $\mathfrak{A}$ be the full Cs_α with base 2, and let $p = \langle 0 : \kappa < \alpha \rangle$. Then ${}^\alpha 2^{(p)} \in A$ and $\Delta {}^\alpha 2^{(p)} = 0$. Hence ${}^\alpha 2^{(p)}$ is not regular, since in a regular Cs_α the only elements X such that $\Delta X = 0$ are the zero element and the unit element.

REMARK 3.1.30. Analyzing the proof just given further, we see that $\mathfrak{A}$ is a Cs_α

which is not even isomorphic to a Cs_α^{reg}. From 3.1.107 it follows that any simple Cs_α is isomorphic to a Cs_α^{reg}; see also 3.1.70. In the other direction, Andréka and Németi have shown that for $\alpha \geqq \omega$, every non-minimal locally finite dimensional regular Cs_α is isomorphic to a non-regular Cs_α; see [HMT81] 3.10.

We noted in 3.1.20 that every Crs_α is a subalgebra of a relativization of a Cs_α. Thus the unit element of any Crs_α is an element of a Cs_α and vice versa (by 3.1.20). The next theorem gives an elementary characterization of unit elements of Gws_α's, Gs_α's, and Cs_α's, as elements of a larger Cs_α for certain α; this theorem was mentioned in 2.2.11.

THEOREM 3.1.31. *Let $\mathfrak{A}$ be a Cs_α with base U, let $V \in A$, and let*

$$F = \{X \in A\colon \mathsf{s}_\lambda^\kappa X \mathbf{n} \mathsf{s}_\kappa^\lambda X = X \text{ for all } \kappa,\lambda < \alpha\}.$$

Then we have:

(i) *For $\alpha \leqq 1$, $F = A$.*

(ii) *For $\alpha = 2$, V is a Cartesian space (i.e., the unit element of a Cs_2) iff $V \in F$; V is equal to ${}^\alpha U$ iff $V \in F$ and $\mathsf{c}_1 V = 1$.*

(iii) *For $3 \leqq \alpha < \omega$, V is a unit element of a Gs_α iff $V \in F$; V is a unit element of a Gs_α with base U iff $V \in F$ and $\mathsf{c}_{(\alpha \sim 1)} V = 1$.*

(iv) *For $3 \leqq \alpha$, V is a unit element of a Gws_α iff $V \in F$.*

PROOF. Condition (i) is obvious. Now we show that if V is a unit element of a Gws_α, then $V \in F$, provided that $\alpha \geqq 3$. This will establish parts of (iii) and (iv). Say $V = \bigcup_{i \in I} {}^\alpha W_i^{(pi)}$, where ${}^\alpha W_i^{(pi)} \mathbf{n} {}^\alpha W_j^{(pj)} = 0$ for any $i,j \in I$ with $i \neq j$. Given any $\kappa,\lambda < \alpha$, we clearly have $V \subseteq \mathsf{s}_\lambda^\kappa V \mathbf{n} \mathsf{s}_\kappa^\lambda V$. Now suppose $f \in \mathsf{s}_\lambda^\kappa V \mathbf{n} \mathsf{s}_\kappa^\lambda V$; thus, say, $f_{f\lambda}^\kappa \in {}^\alpha W_i^{(pi)}$ and $f_{f\kappa}^\lambda \in {}^\alpha W_j^{(pj)}$. Pick $\mu \in \alpha \sim \{\kappa,\lambda\}$ and notice that $f\mu \in W_i \mathbf{n} W_j$. There is a finite $\Gamma \subseteq \alpha$ such that $(\alpha \sim \Gamma) 1 f = (\alpha \sim \Gamma) 1 pi = (\alpha \sim \Gamma) 1 pj$. If therefore we define g by stipulating that $g\nu = f\mu$ for all $\nu \in \Gamma$, and $g\nu = f\nu$ for every $\nu \in \alpha \sim \Gamma$, we get $g \in {}^\alpha W_i^{(pi)} \mathbf{n} {}^\alpha W_j^{(pj)}$. Consequently $i = j$ and $f \in {}^\alpha W_i^{(pi)} \subseteq V$. Hence $V = \mathsf{s}_\lambda^\kappa V \mathbf{n} \mathsf{s}_\kappa^\lambda V$, and we conclude that $V \in F$, as desired.

Next we show that for $\alpha \geqq 2$, every $V \in F$ is the unit element of a Gws_α. (This portion of the proof is due to Andréka and Németi.) This will finish the proof of (iv) and prove parts of (ii) and (iii). For each $f \in V$ let

$$Y_f = \{u \in U\colon f_u^0 \in V\}.$$

The following facts about Y_f show that V is the unit element of a Gws_α.

(1) If $f \in V$, then $f \in {}^\alpha Y_f$.

For, if $\kappa < \alpha$, then $f \in V \subseteq \mathsf{s}_\kappa^0 V$, so $f_{f\kappa}^0 \in V$ and hence $f\kappa \in Y_f$. Thus $f \in {}^\alpha Y_f$ and (1) holds. Next, clearly:

(2) If $f \in V$ and $u \in Y_f$ then $Y_f = Y_{f(0/u)}$.

(3) If $f \in V$, $u \in Y_f$, and $\kappa < \alpha$, then $f_u^\kappa \in V$.

For, we may assume that $\kappa \neq 0$. Then $f^0_u \in V \subseteq \mathsf{s}^\kappa_0 V$, so $f^{0\kappa}_{uu} \in V$. Also, $f \in V \subseteq \mathsf{s}^\kappa_0 V$, whence $f^\kappa_{f0} \in V$. Hence $f^\kappa_u \in \mathsf{s}^0_\kappa V \mathbf{n} \mathsf{s}^\kappa_0 V = V$, as desired.

(4) If $f \in V$, $u \in Y_f$ and $\kappa < \alpha$, then $Y_f \subseteq Y_{f(\kappa/u)}$.

For, by (2) we may assume $\kappa \neq 0$. Assume the hypothesis of (4), and consider any $v \in Y_f$. We have $f^0_u, f^0_v \in V \subseteq \mathsf{s}^\kappa_0 V$, so $f^{0\kappa}_{uu}$, $f^{0\kappa}_{vv} \in V$. Hence $f^{0\kappa}_{vu} \in \mathsf{s}^\kappa_0 V \mathbf{n} \mathsf{s}^0_\kappa V = V$. Thus $v \in Y_{f(\kappa/u)}$, as desired.

Now we claim:

(5) If $f \in V$ and $g \in {}^\alpha Y^{(f)}_f$ then $g \in V$ and $Y_f \subseteq Y_g$.

For, let $f \in V$. For each $g \in {}^\alpha Y^{(f)}_f$ let $n_g = |\{\kappa < \alpha; f\kappa \neq g\kappa\}|$. We check the conclusion of (5) by induction on n_g. If $n_g = 0$, then $f = g$ and the conclusion is trivial. Assume inductively that $n_g > 0$. Fix $\kappa < \alpha$ such that $f\kappa \neq g\kappa$. Applying the inductive hypothesis to $g^\kappa_{f\kappa}$ we infer that $g^\kappa_{f\kappa} \in V$ and $Y_f \subseteq Y_{g(\kappa/f\kappa)}$. Now $g\kappa \in Y_f \subseteq Y_{g(\kappa/f\kappa)}$ and $g^\kappa_{f\kappa} \in V$, so by (3), $g \in V$. Also $g^\kappa_{f\kappa} \in V$ and $g\kappa \in Y_{g(\kappa/f\kappa)}$ yield by (4) $Y_f \subseteq Y_{g(\kappa/f\kappa)} \subseteq Y_g$. This finishes the inductive proof of (5).

Next,

(6) If $f, g \in V$ and ${}^\alpha Y^{(f)}_f \subseteq {}^\alpha Y^{(g)}_g$, then ${}^\alpha Y^{(f)}_f = {}^\alpha Y^{(g)}_g$.

In fact, $f \in {}^\alpha Y^{(f)}_f \subseteq {}^\alpha Y^{(g)}_g$ so by (5), $Y_g \subseteq Y_f$ whence ${}^\alpha Y^{(f)}_f = {}^\alpha Y^{(g)}_g$.

(7) If $f, g \in V$ and ${}^\alpha Y^{(f)}_f \mathbf{n} {}^\alpha Y^{(g)}_g \neq 0$, then ${}^\alpha Y^{(f)}_f = {}^\alpha Y^{(g)}_g$.

Indeed, let $h \in {}^\alpha Y^{(f)}_f \mathbf{n} {}^\alpha Y^{(g)}_g$. Then by (5), $h \in V$ and $Y_f \mathbf{u} Y_g \subseteq Y_h$, hence ${}^\alpha Y^{(f)}_f \mathbf{u} {}^\alpha Y^{(g)}_g \subseteq {}^\alpha Y^{(h)}_h$. Therefore by (6), ${}^\alpha Y^{(f)}_f = {}^\alpha Y^{(g)}_g$.

Now (1), (5) and (7) show that V is the unit element of a Gws_α, as desired.

Next note that for $2 \leqq \alpha < \omega$, $\mathsf{c}_{\langle \alpha \sim 1 \rangle} V = 1$ is equivalent to $\bigcup_{p \in V} Rgp = U$ if $V \in F$. This finishes the proof of (iii) and proves another part of (ii).

It remains only to take $\alpha = 2$ and prove the first part of (ii). The direction $\Rightarrow$ is obvious. Now suppose $V \in F$. Then (1)–(7) above still apply. Note that ${}^\alpha Y^{(f)}_f = {}^\alpha Y_f$ for all $f \in V$. Furthermore, if $f, g \in V$ then $f^0_{g1} \in \mathsf{s}^0_1 V \mathbf{n} \mathsf{s}^1_0 V = V$, and consequently $g1 \in Y_f$. Thus by (1), $Y_f \mathbf{n} Y_g \neq 0$ and hence ${}^\alpha Y_f \mathbf{n} {}^\alpha Y_g \neq 0$. Therefore ${}^\alpha Y_f = {}^\alpha Y_g$ by (7). Hence V is a Cartesian space, finishing the proof.

REMARKS 3.1.32. It can be shown that there is no similar abstract characterization of those elements of a Cs_α which are unit elements of a (1) Gs_α if $\alpha = 2$, (2) Gs_α if $\alpha \geqq \omega$, (3) Ws_α if $\alpha \geqq \omega$, (4) Cs_α if $\alpha \geqq \omega$. For (1) see [HMT81] 2.3. For (2),(3) and (4) see [AN81] 0.6 and 0.7. There is still no such characterization in cases (1),(2),(4) even if we consider elements of a full Cs_α, but there is one for (3): For $\alpha \geqq \omega$, an element V of a full Cs_α $\mathfrak{A}$ is a weak Cartesian space iff $V \neq 0$, $\mathsf{s}^\kappa_\lambda V \mathbf{n} \mathsf{s}^\lambda_\kappa V = V$ for all $\kappa, \lambda < \alpha$, and there is a $Y \in At\mathfrak{Zb}\mathfrak{A}$ such that $V \subseteq Y$. There *is* an abstract characterization of the unit elements X of Cs_α's which are elements of a

given Cs_α, for $\alpha<\omega$:

$$X=\bigcap_{\kappa<\alpha}\mathsf{C}_{(\alpha\sim\{\kappa\})}(\mathsf{d}_\alpha\mathsf{n}X).$$

This is easily checked; it was noticed by Andréka and Németi.
See also 3.1.34 and [AN81] 0.8.

REMARKS 3.1.33. Recall from 3.1.20 that for $\alpha\geqq 2$, $\mathsf{Crs}_\alpha=\mathsf{SRICs}_\alpha$. It is known that $\mathsf{Crs}_\alpha\neq\mathsf{RICs}_\alpha$ for every $\alpha\geqq 1$; see [AN81] 2.3 for the case $\alpha=2$; a simple modification works for other α — see 5.5.7. In [HMT81] 2.9–2.14 it is shown that $\mathsf{Gws}_\alpha\subseteq\mathsf{RICs}_\alpha$ for $\alpha\geqq 2$, by a lengthy argument.

The following result of [AN81] 0.3 shows that Gws's play a distinguished role among all Crs_α's.

THEOREM 3.1.34. *Let V be a set of α–termed sequences. Then the following conditions are equivalent:*
(i) *V is the unit element of a* Gws_α;
(ii) *the full* Crs_α *with unit element V is a* CA_α;
(iii) *every* Crs_α *with unit element V is a* CA_α.

PROOF. Clearly (i)$\Rightarrow$(iii)$\Rightarrow$(ii). Assume (ii), let $\mathfrak{A}$ be the full Crs_α with unit element V, and let U be the base of V. Let $\mathfrak{B}$ be the full Cs_α with base U. Thus $V\in B$.

(1) If $\kappa,\lambda<\alpha$, then $V\subseteq\mathsf{s}_\lambda^\kappa V$ in $\mathfrak{B}$.

This is clear from the definition of the operations in $\mathfrak{A}$, since $\mathsf{c}_\kappa\mathsf{d}_{\kappa\lambda}=1$ in $\mathfrak{A}$ because $\mathfrak{A}$ is a CA_α. Now we consider two cases.

Case 1. $\alpha\geqq 3$. Then by 3.1.31(iv) and (1), to show (i) it suffices to show that $\mathsf{s}_\lambda^\kappa V\mathsf{n}\mathsf{s}_\kappa^\lambda V\subseteq V$ for all distinct $\kappa,\lambda<\alpha$. So, suppose $\kappa,\lambda<\alpha$, $\kappa\neq\lambda$, and $q\in\mathsf{s}_\lambda^\kappa V\mathsf{n}\mathsf{s}_\kappa^\lambda V$. Thus $q_{q\lambda}^\kappa\in V$ and $q_{q\kappa}^\lambda\in V$. Now choose $\mu\in\alpha\sim\{\kappa,\lambda\}$. Since $q_{q\lambda}^\kappa\in V\subseteq\mathsf{s}_\mu^\kappa V$ by (1), we have $q_{q\mu}^\kappa\in V$. Similarly $(q_{q\kappa}^\lambda)_{q\mu}^\kappa\in V$. Thus $q_{q\kappa}^\lambda\in\mathsf{c}_\kappa^{\mathfrak{A}}\{(q_{q\kappa}^\lambda)_{q\mu}^\kappa\}$, $(q_{q\kappa}^\lambda)_{q\mu}^\kappa\in\mathsf{c}_\lambda^{\mathfrak{A}}\{q_{q\mu}^\kappa\}$, and $q_{q\mu}^\kappa\in\mathsf{c}_\kappa^{\mathfrak{A}}\{q_{q\lambda}^\kappa\}$, so $q_{q\kappa}^\lambda\in\mathsf{c}_\kappa^{\mathfrak{A}}\mathsf{c}_\lambda^{\mathfrak{A}}\mathsf{c}_\kappa^{\mathfrak{A}}\{q_{q\lambda}^\kappa\}=\mathsf{c}_\lambda^{\mathfrak{A}}\mathsf{c}_\kappa^{\mathfrak{A}}\{q_{q\lambda}^\kappa\}$ by (ii). Say $p\in V$ and $(\alpha\sim\{\lambda\})\upharpoonright q_{q\kappa}^\lambda\subseteq p$ while $(\alpha\sim\{\kappa\})\upharpoonright q_{q\lambda}^\kappa\subseteq p$. Clearly then $q=p\in V$, as desired.

Case 2. $\alpha\leqq 2$. If $\alpha\leqq 1$, then $\mathsf{Crs}_\alpha=\mathsf{Gws}_\alpha$ by 3.1.17. So assume that $\alpha=2$. It suffices to show that V is an equivalence relation. Suppose $\langle u,v\rangle\in V$; we show that $\langle v,u\rangle\in V$. By (1) we have $\langle u,u\rangle\in V$ and $\langle v,v\rangle\in V$. Hence $\langle u,u\rangle\in\mathsf{c}_1^{\mathfrak{A}}\mathsf{c}_0^{\mathfrak{A}}\{\langle v,v\rangle\}$, so $\langle u,u\rangle\in\mathsf{c}_0^{\mathfrak{A}}\mathsf{c}_1^{\mathfrak{A}}\{\langle v,v\rangle\}$. It follows that $\langle v,u\rangle\in V$. Finally, suppose that $\langle u,v\rangle\in V$ and $\langle v,w\rangle\in V$. We have $\langle w,v\rangle\in V$ by what was just proved, so $\langle u,v\rangle\in\mathsf{c}_0^{\mathfrak{A}}\mathsf{c}_1^{\mathfrak{A}}\{\langle w,w\rangle\}$, where $\langle w,w\rangle\in V$ by (1) again. Hence $\langle u,v\rangle\in\mathsf{c}_1^{\mathfrak{A}}\mathsf{c}_0^{\mathfrak{A}}\{\langle w,w\rangle\}$, so $\langle u,w\rangle\in V$, as desired.

We give one more result on relativization, from [AN81] 2.2(i).

THEOREM 3.1.35. *Suppose* $\mathfrak{A} \in \mathsf{Crs}_\alpha^{reg}$ *and* $a \in Zd\mathfrak{A}$. *Then* $\mathfrak{R}\mathfrak{l}_a\mathfrak{A}$ *is regular.* (*Cf.* 2.2.1)

PROOF. Assume the hypothesis, and let $\mathfrak{B} = \mathfrak{R}\mathfrak{l}_a\mathfrak{A}$ for brevity. Let $b \in B$; we are to show that b is regular. It is easy to see that $\Delta^{\mathfrak{A}} b = \Delta^{\mathfrak{B}} b$; cf. the proof of 2.2.12. Suppose $p \in b$, $q \in a$, and $(1 \mathbf{u} \Delta b) \upharpoonright p \subseteq q$. By the regularity of b in $\mathfrak{A}$ we get $q \in b$, as desired.

Change of Base

Now in 3.1.36–3.1.54 we consider changing the base of a set algebra: Given a set algebra $\mathfrak{A}$ with base U and given a set W, when is $\mathfrak{A}$ isomorphic to a set algebra with base W? Some results along this line will also be established later in this section after more machinery, in particular the ultraproduct construction, is available; see 3.1.105 and 3.1.112–3.1.115.

The simplest way of changing a base is to extend a one–one function from U onto W. This gives algebraic versions of the trivial logical theorem that isomorphism implies elementary equivalence. The algebraic theorem is also trivial:

THEOREM 3.1.36. *Let* $\mathfrak{A}$ *be a* Crs_α *with base* U *and unit element* V. *Suppose* f *is a one–one function from* U *onto a set* W. *For any* $X \in A$ *let* $FX = \{y \in {}^\alpha W : f^{-1} \circ y \in X\}$. *Then* F *is an isomorphism from* $\mathfrak{A}$ *onto a* Crs_α $\mathfrak{B}$ *with base* W *and unit element* FV. *Furthermore:*

(i) *If* $\mathfrak{A}$ *is a* Cs_α *then so is* $\mathfrak{B}$.

(ii) *If* $\mathfrak{A}$ *is a* Ws_α *with unit element* ${}^\alpha U^{(p)}$, *then* $\mathfrak{B}$ *is a* Ws_α *with unit* ${}^\alpha W^{(f \circ p)}$.

(iii) *If* $\mathfrak{A}$ *is a* Gs_α *with unit element* $\bigcup_{i \in I} {}^\alpha S_i$, *where* $S_i \mathbf{n} S_j = 0$ *for* $i \neq j$, *then* $\mathfrak{B}$ *is a* Gs_α *with unit element* $\bigcup_{i \in I} {}^\alpha f^* S_i$ *where* $f^* S_i \mathbf{n} f^* S_j = 0$ *for* $i \neq j$.

(iv) *If* $\mathfrak{A}$ *is a* Gws_α *with unit element* $\bigcup_{i \in I} {}^\alpha S_i^{(p_i)}$ *where* ${}^\alpha S_i^{(p_i)} \mathbf{n} {}^\alpha S_j^{(p_j)} = 0$ *for* $i \neq j$, *then* $\mathfrak{B}$ *is a* Gws_α *with unit element* $\bigcup_{i \in I} {}^\alpha (f^* S_i)^{(f \circ p_i)}$, *where* ${}^\alpha (f^* S_i)^{(f \circ p_i)} \mathbf{n} {}^\alpha (f^* S_j)^{(f \circ p_j)} = 0$ *for* $i \neq j$.

Theorem 3.1.36 leads to the following definition.

DEFINITION 3.1.37. (i) *Suppose* f *is a one–one function from a set* U *onto a set* W, *and* α *is an ordinal. Then for any* $X \subseteq {}^\alpha U$ *we set*

$$(\tilde{f}^\alpha) X = \tilde{f} X = \{y \in {}^\alpha W : f^{-1} \circ y \in X\}.$$

(ii) *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be* Crs_α*'s with bases* U *and* W *respectively. We say that* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are* **base–isomorphic** *if there is a one–one function* f *mapping* U *onto* W *such that* $A \upharpoonright \tilde{f}$ *is an isomorphism from* $\mathfrak{A}$ *onto* $\mathfrak{B}$.

REMARKS 3.1.38. For each $\alpha \geq 1$ there are isomorphic $\mathsf{Cs}_\alpha^{reg} \mathbf{n} \mathsf{Lf}_\alpha$'s which are not base–isomorphic. In fact, one can take two infinite sets U, W with $|U| \neq |W|$ and let $\mathfrak{A}$ and $\mathfrak{B}$ be the minimal subalgebras of $\mathfrak{Gb}^\alpha U$ and $\mathfrak{Gb}^\alpha W$ respectively. However, isomorphism and base isomorphism are closely connected if one of the bases is

finite; we discuss this case in these remarks. First we note the following algebraic version of the logical result according to which any two elementarily equivalent finite structures are isomorphic; it is due to Monk and is found in [HMT81] 3.6:

(1) If $\mathfrak{A},\mathfrak{B} \in \mathsf{Cs}_\alpha^{reg} \cap \mathsf{Lf}_\alpha$, $\mathfrak{A} \cong \mathfrak{B}$, and the base of either $\mathfrak{A}$ or $\mathfrak{B}$ has power $< \alpha \cap \omega$, then $\mathfrak{A}$ and $\mathfrak{B}$ are base–isomorphic.

For $\alpha \geqq \omega$, in (1) the regularity assumption cannot be dropped, Lf_α cannot be replaced by Dc_α, Cs_α cannot be replaced by Gs_α, Ws_α or Gws_α, and the condition that one of the bases is finite cannot be removed. These facts are due to Andréka and Németi; for the proofs, see [HMT81] 3.7, 3.10, 3.11 and [AN81] 3.5.

Next we show that for each $\alpha \geqq 3$ there is a Cs_α with an infinite base not isomorphic to a Cs_α with a finite base. First suppose that $3 \leqq \alpha < \omega$. Then, we claim, the following equation ε holds identically in every Cs_α with a finite base, but fails in some *finite* Cs_α with an infinite base (this equation is due to Andréka and Németi, and replaces a longer one originally used for this purpose):

$$\mathsf{c}_{(\alpha)}[-\mathsf{c}_{(\alpha\sim 1)}x + \mathsf{c}_{(\alpha\sim 2)}x \cdot \mathsf{d}_{01} + \mathsf{c}_{(\alpha\sim 2)}x \cdot \mathsf{s}_1^0\mathsf{s}_2^1\mathsf{c}_{(\alpha\sim 2)}x \cdot -\mathsf{s}_2^1\mathsf{c}_{(\alpha\sim 2)}x] = 1.$$

To show this, let $\mathfrak{A}$ be a Cs_α with base U in which ε does not hold identically; we show that U is infinite; this will establish that ε holds in every Cs_α with a finite base. Thus there is an $X \in A$ such that the following conditions hold:

(2) $\mathsf{C}_{(\alpha\sim 1)}X = {}^\alpha U$;
(3) $\mathsf{C}_{(\alpha\sim 2)}X \subseteq \sim\mathsf{D}_{01}$;
(4) $\mathsf{C}_{(\alpha\sim 2)}X \cap \mathsf{s}_1^0\mathsf{s}_2^1\mathsf{C}_{(\alpha\sim 2)}X \subseteq \mathsf{s}_2^1\mathsf{C}_{(\alpha\sim 2)}X$.

Now let $R = \{2 \upharpoonright u : u \in X\}$. Then R is a binary relation on U satisfying the following conditions:

(5) For all $u \in U$ there is a $v \in U$ with uRv;
(6) For all $u \in U$, not(uRu);
(7) R is transitive.

It follows that U is infinite, as desired.

Now we construct a finite Cs_α $\mathfrak{A}$ with an infinite base such that ε fails to hold in $\mathfrak{A}$. Let $\mathfrak{B} = \langle B, < \rangle$, where B is the set of rational numbers and $<$ is the usual ordering on B. Let Λ be a discourse language for $\mathfrak{B}$, with a sequence $\langle v_\xi : \xi < \alpha \rangle$ of variables. Then $\{\varphi^{\mathfrak{B}} : \varphi$ a formula of $\Lambda\}$ is the universe of a Cs_α $\mathfrak{A}$ with infinite base B. From the usual decision procedure for sentences holding in $\mathfrak{B}$ we see that $\mathfrak{A}$ is finite. Letting $X = \varphi^{\mathfrak{B}}$, φ the formula $v_0 < v_1$, we see that ε fails in $\mathfrak{A}$.

For $\alpha \geqq \omega$, let $\mathfrak{A}$ be any Cs_α with an infinite base. Then $\mathfrak{A}$ has characteristic zero, while any Cs_α with a finite base has characteristic greater than zero. So $\mathfrak{A}$ is not isomorphic to a Cs_α with a finite base.

Concerning these examples we make some remarks on the case $\alpha \leqq 2$. Any Crs_0 has base 0. Any finite Cs_1 is isomorphic to a Cs_1 with a finite base. In fact, let $\mathfrak{A}$ be a

finite Cs_1, say with base U. For each non-zero $a \in A$ choose $u_a \in U$ so that $\langle u_a \rangle \in a$. Let $U' = \{u_a : a \in A\}$, and for any $a \in A$ let $fa = a \cap {}^1U'$. Then f is an isomorphism of $\mathfrak{A}$ onto a Cs_1 with finite base U'. In 3.2.66 we show that any finite Gs_2 (resp. Cs_2) is isomorphic to a Gs_2 (resp. Cs_2) with a finite base.

Next we note:

(8) If $\mathfrak{A}$ and $\mathfrak{B}$ are Gws_α's, $\mathfrak{A} \geqslant \mathfrak{B}$, and $|W| < \alpha \cap \omega$ for some subbase W of $\mathfrak{B}$, then $|W| = |W'|$ for some subbase W' of $\mathfrak{A}$. (Here $\alpha \geqq 2$.)

For, let $\kappa = |W|$. Then, in $\mathfrak{B}$,

$$\bigcap_{\lambda<\mu<\kappa} {\sim}\mathsf{D}_{\lambda\mu} \cap \mathsf{C}_\kappa^\partial \bigcup_{\lambda<\kappa} \mathsf{D}_{\lambda\kappa} \neq 0. \tag{9}$$

In fact, ${}^\alpha W^{(p)}$ is included in the unit element of $\mathfrak{B}$ for some $p \in {}^\alpha W$, and any $q \in {}^\alpha W^{(p)}$ such that $\kappa \restriction q$ maps κ one-one onto W will be a member of the left side of (9). It follows that (9) also holds in $\mathfrak{A}$, and this gives the desired set W'.

One cannot modify (8) by assuming that W is a subbase of $\mathfrak{A}$ and finding W' a subbase of $\mathfrak{B}$, with all other conditions unchanged; see [HMT81] 3.4.

Finally we note that there are Cs_α's $\mathfrak{A}$ and $\mathfrak{B}$ with bases U, W respectively, with $\mathfrak{A} \cong \mathfrak{B}$, $\alpha \leqq |U| < \omega$, and $|U| \neq |W|$. Namely, let $\mathfrak{A}$ and $\mathfrak{B}$ be minimal Cs_α's with bases U and W, respectively, subject only to the condition $\alpha \leqq |U|, |W|$; then $\mathfrak{A} \cong \mathfrak{B}$ by 2.5.30.

We now turn away from the case of finite bases. We consider the possibility of decreasing the base, and first we note that this is not always possible:

THEOREM 3.1.39. *For any ordinal α and any non-empty set U there is a Cs_α, resp. Ws_α, $\mathfrak{A}$ with base U such that if $\mathfrak{B}$ is a Crs_α with base W and $\mathfrak{A} \cong \mathfrak{B}$, then $|U| \leqq |W|$.*

PROOF. First we construct a Cs_α $\mathfrak{A}$ with the indicated property. For each $u \in U$ let $x_u = \{\langle u : \kappa < \alpha \rangle\}$, and let $\mathfrak{A}$ be the Cs_α with base U generated by $\{x_u : u \in U\}$. Suppose $\mathfrak{B}$ is a Crs_α with base W and $f \in Is(\mathfrak{A}, \mathfrak{B})$. Now $\langle x_u : u \in U \rangle$ is a system of pairwise disjoint elements of A, and $x_u \subseteq \mathsf{D}_{\kappa\lambda}$ whenever $u \in U$ and $\kappa, \lambda < \alpha$. Hence for every $u \in U$ there is a $w_u \in W$ such that $\langle w_u : \kappa < \alpha \rangle \in f x_u$; so $w_u \neq w_v$ if $u \neq v$. Hence $|U| \leqq |W|$.

For the Ws_α part, we may assume that $\alpha \geqq \omega$. Fix $u_0 \in U$, set $p = \langle u_0 : \kappa < \alpha \rangle$, and let $\mathfrak{A}$ be the Ws_α with unit element ${}^\alpha U^{(p)}$ generated by $\{\{p_u^0\} : u \in U\}$. Suppose $\mathfrak{B}$ is a Crs_α with base W and $f \in Is(\mathfrak{A}, \mathfrak{B})$. If $|U| < \omega$, say $|U| = \kappa$. Then $\bigcap\{{\sim}\mathsf{D}_{\lambda\mu} : \lambda, \mu < \kappa,\ \lambda \neq \mu\} \neq 0$ in $\mathfrak{A}$, hence in $\mathfrak{B}$, so that there is a $g \in {}^\alpha W$ with $\kappa \restriction g$ one-one, whence $|U| \leqq |W|$. Assume that $|U| \geqq \omega$. Note that $\{p_u^0\} \subseteq \mathsf{D}_{\kappa\lambda}$ for all $\kappa, \lambda \in \alpha \sim \{0\}$. Hence for every $u \in U$, there exist $w, v \in W$ such that $\langle w, v, v, \ldots \rangle \in f\{p_u^0\}$. So if we set $gu = \{\langle w, v \rangle : \langle w, v, v, \ldots \rangle \in f\{p_u^0\}\}$ for all $u \in U$, we have: $0 \neq gu \subseteq W \times W$, and $gu \cap gu' = 0$ for $u \neq u'$. Hence $|U| \leqq |W|$.

REMARK 3.1.40. By 3.1.65, the Cs_α constructed in 3.1.39 is regular.

We now prove a theorem giving important special cases in which it is possible to decrease the base; the theorem is due to Andréka, Monk, and Németi. It is an algebraic version of the downward Löwenheim–Skolem theorem, and is proved as that theorem is proved in Tarski, Vaught [57*]. It generalizes Lemma 5 in Henkin,

Monk [74']. Before giving the theorem, we need a definition and some lemmas.

DEFINITION 3.1.41. (i) *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be* Crs_α*'s with unit elements* V_0 *and* V_1 *and bases* U_0 *and* U_1 *respectively, and let* $F \in Is(\mathfrak{B},\mathfrak{A})$. *We call* F *a* **(strong) ext–isomorphism** *if* $F = \langle X \cap V_0 : X \in B \rangle$ *(resp.* $F = \langle X \cap {}^\alpha W : X \in B \rangle$ *for some* $W \subseteq U_1$*); we then also call* F^{-1} *a* **(strong) sub–isomorphism** *of* $\mathfrak{A}$ *onto* $\mathfrak{B}$.

(ii) *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be* Crs_α*'s and* $F \in Is(\mathfrak{B},\mathfrak{A})$. *Then we call* F *a* **(strong) ext–base–isomorphism** *if* $F = g \circ h$ *for some base–isomorphism* h *and some (strong) ext–isomorphism* g*; then also* F^{-1} *is called a* **(strong) sub–base–isomorphism**.

This definition gives algebraic versions of the notions of elementary substructures, elementary extensions, and elementary embeddings. For more details, see section 4.3, especially towards its end. Note that a strong ext–isomorphism *is* an ext–isomorphism: if $F = \langle X \cap {}^\alpha W : X \in B \rangle$ is a strong ext–isomorphism of $\mathfrak{B}$ onto $\mathfrak{A}$, as in 3.1.41(i), then $V_0 = V_1 \cap {}^\alpha W$ and $F = \langle X \cap V_0 : X \in B \rangle$. The following two trivial lemmas help to clarify these notions.

LEMMA 3.1.42. *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be* Crs_α*'s with unit elements* V_0 *and* V_1 *respectively.*

(i) *If* $\mathfrak{A}$ *is sub–isomorphic to* $\mathfrak{B}$ *and* $X \in B$ *is regular in* $\mathfrak{B}$, *then* $X \cap V_0$ *is regular in* $\mathfrak{A}$.

(ii) *If* $F \in Is(\mathfrak{A},\mathfrak{B})$, *then* F *is a (strong) sub–base–isomorphism iff* $F = g \circ h$ *for some base–isomorphism* h *and some (strong) sub–isomorphism* g.

PROOF. Condition (i) follows immediately from the definitions. We prove only the (non–strong) direction $\Rightarrow$ in (ii), since $\Leftarrow$ is similar. So assume given a base–isomorphism h of $\mathfrak{B}$ onto some Crs_α $\mathfrak{C}$ and a (strong) ext–isomorphism g of $\mathfrak{C}$ onto $\mathfrak{A}$, with $F = h^{-1} \circ g^{-1}$. Then $A \upharpoonright h^{-1}$ is a base–isomorphism of $\mathfrak{A}$ onto some Crs_α $\mathfrak{D}$ with unit element $V_2 \subseteq V_1$. Let $k = \langle X \cap V_2 : X \in B \rangle$. Then it is easily checked that $k = (A \upharpoonright h^{-1}) \circ g \circ h$. Hence k is an ext–isomorphism, and $F = k^{-1} \circ (A \upharpoonright h^{-1})$, as desired.

LEMMA 3.1.43. *Suppose* $\alpha \geqq 1$. *Let* $\mathfrak{B}$ *be a* Crs_α *with base* U, *suppose* $W \subseteq U$, *and suppose that* $\langle X \cap {}^\alpha W : X \in B \rangle$ *is a strong ext–isomorphism of* $\mathfrak{B}$ *onto a* Crs_α $\mathfrak{A}$ *with* $|A| > 1$. *Then the following conditions hold:*

(i) *If* $\mathfrak{B}$ *is a* Cs_α, *then* $\mathfrak{A}$ *is a* Cs_α *with base* W.

(ii) *If* $\mathfrak{B}$ *is a* Ws_α *with unit element* ${}^\alpha U^{(p)}$, *then* $\mathfrak{A}$ *is a* Ws_α *with unit element* ${}^\alpha W^{(q)}$ *for some* $q \in {}^\alpha U^{(p)} \cap {}^\alpha W$.

(iii) *If* $\mathfrak{B}$ *is a* Gs_α *with unit element* $\bigcup_{i \in I} {}^\alpha S_i$, *where* $S_i \cap S_j = 0$ *for* $i \neq j$, *then* $\mathfrak{A}$ *is a* Gs_α *with unit element* $\bigcup_{i \in J} {}^\alpha(S_i \cap W)$, *where* $J = \{i \in I : S_i \cap W \neq 0\}$.

(iv) *If* $\mathfrak{B}$ *is a* Gws_α *with unit element* $\bigcup_{i \in I} {}^\alpha S_i^{(p_i)}$, *where* ${}^\alpha S_i^{(p_i)} \cap {}^\alpha S_j^{(p_j)} = 0$ *for* $i \neq j$, *then* $\mathfrak{A}$ *is a* Gws_α *with unit element of the form* $\bigcup_{i \in J} {}^\alpha(S_i \cap W)^{(q_i)}$, *where* $J \subseteq I$ *and* $q_i \in {}^\alpha S_i^{(p_i)} \cap {}^\alpha W$ *for all* $i \in J$.

REMARK 3.1.44. For each $\alpha \geqq \omega$ there is a regular Cs_α strongly sub–isomorphic to a non–regular Cs_α. Thus 3.1.42(i) holds only in the direction indicated. This example, due to Andréka and Németi, can be found in [HMT81] 3.17.

Now we are ready for the promised decreasing base theorem.

THEOREM 3.1.45. *Let $\mathfrak{A}$ be a* Crs_α *with unit element V and base U. Let κ be an infinite cardinal such that $|A|\leqq\kappa\leqq|U|$. Assume $S\subseteq U$ and $|S|\leqq\kappa$. Then there is a W with $S\subseteq W\subseteq U$ such that $|W|=\kappa$ and:*

(i) *Each of the following conditions* (a)–(c) *implies that $\langle X\mathbf{n}^\alpha W: X\in A\rangle$ is a strong ext–isomorphism of $\mathfrak{A}$ onto a* Crs_α $\mathfrak{B}$:

(a) *$\mathfrak{A}$ is a* Ws_α *with unit element ${}^\alpha U^{(p)}$; then we may assume that $Rgp\subseteq W$ and $\mathfrak{B}$ is a* Ws_α *with unit element ${}^\alpha W^{(p)}$;*

(b) *$\kappa=\kappa^{|\alpha|}$; then if $\mathfrak{A}$ is a* Cs_α *it follows that $\mathfrak{B}$ is a* Cs_α *with base W;*

(c) *$\mathfrak{A}$ is a regular* Gs_α, *and $\kappa=\Sigma_{\mu<\lambda}\kappa^\mu$, where λ is the least infinite cardinal such that $|\Delta X|<\lambda$ for all $X\in A$; then $\mathfrak{B}$ is a regular* Gs_α *with base W, and is a* Cs_α *if $\mathfrak{A}$ is a* Cs_α.

(ii) *If $\mathfrak{A}$ is a* Gws_α, *then $\mathfrak{A}$ is ext–isomorphic to a* Gws_α *with base W.*

(iii) *If $|\alpha|\leqq\kappa$, then $\mathfrak{A}$ is ext–isomorphic to a* Crs_α *with base W.*

PROOF. We assume as given well–orderings of U and V. (i)(a): First note that $|\alpha|\leqq|A|$. There is a subset T_0 of U such that $|T_0|=\kappa$, $S\mathbf{u}Rgp\subseteq T_0$, and $X\mathbf{n}^\alpha T_0\neq 0$ whenever $0\neq X\in A$. Now suppose that $0<\beta<\kappa$ and T_γ has been defined for all $\gamma<\beta$. Let $M=\bigcup_{\gamma<\beta}T_\gamma$ and let

$$T_\beta=M\mathbf{u}\{a\in U:\text{there exist } X\in A,\ \mu<\alpha, \text{ and } u\in{}^\alpha M^{(p)} \text{ such that } a \text{ is the first element of } U \text{ with the property that } u^\mu_a\in X\}.$$

Let $W=T_\kappa=\bigcup_{\gamma<\kappa}T_\gamma$. By induction it is easily seen that $|T_\beta|=\kappa$ for all $\beta\leqq\kappa$; in particular, $|W|=\kappa$. Using 3.1.43, it is clear that to finish the proof of (i)(a) we only need to check that

(1) $\mathbf{C}^{[V]}_\mu X\mathbf{n}^\alpha W\subseteq\mathbf{C}^{[V']}_\mu(X\mathbf{n}^\alpha W)$ for any $\mu<\alpha$, and arbitrary $X\in A$, where $V'=V\mathbf{n}^\alpha W$;

here $V'={}^\alpha W^{(p)}$. Suppose $u\in\mathbf{C}^{[V]}_\mu X\mathbf{n}^\alpha W$. Thus $u\in V={}^\alpha U^{(p)}$, $u\in{}^\alpha W$, and $u^\mu_a\in X$ for some $a\in U$. Since $\{\lambda: u\lambda\neq p\lambda\}$ is finite, $Rgp\subseteq T_0$, and $u\in{}^\alpha W$, we infer that there is a $\beta<\kappa$ such that $u\in{}^\alpha(\bigcup_{\gamma<\beta}T_\gamma)^{(p)}$. Thus there is an $a'\in T_\beta$ such that $u^\mu_{a'}\in X$. Thus $u^\mu_{a'}\in X\mathbf{n}^\alpha W$. Since clearly $u\in V'$, it follows that $u\in\mathbf{C}^{[V']}_\mu(X\mathbf{n}^\alpha W)$.

(i)(b): We make a very similar construction, starting with $T_0\subseteq U$ such that $|T_0|=\kappa$, $S\subseteq T_0$, and $X\mathbf{n}^\alpha T_0\neq 0$ whenever $0\neq X\in A$. To construct T_β we replace ${}^\alpha M^{(p)}$ above by ${}^\alpha M\mathbf{n}V$. The condition $\kappa^{|\alpha|}=\kappa$ is used to check that $|T_\beta|=\kappa$ for all $\beta\leqq\kappa$. To check the crucial condition (1) it is enough to note that $\alpha<cf\kappa$ because $\kappa^{|\alpha|}=\kappa$, and hence any $u\in{}^\alpha W$ is in ${}^\alpha T_\beta$ for some $\beta<\kappa$.

(i)(c): The conclusion is clear if $\mathfrak{A}$ is discrete, so suppose $\mathfrak{A}$ is non–discrete. Then $|\alpha|\leqq|A|$. Furthermore, we may assume that $\alpha\geqq 2$, since the case $\alpha\leqq 1$ is treated by (i)(a) (cf. 3.1.17). We define T_0 as in (i)(b). Now T_β is defined as follows (with $\beta<\kappa$ and M as above):

$$T_\beta=M\mathbf{u}\{a\in U:\text{there exist } X\in A,\ \mu\in\Delta X,\ \nu\in\alpha\sim\{\mu\} \text{ and } u\in{}^{(\Delta X\sim\{\mu\})\cup\{\nu\}}M \text{ such}$$

that a is the first element of U with the property that $v \in X$ for some $v \in V$ with $((\Delta X \sim \{\mu\}) \cup \{\nu\}) \upharpoonright v = u$ and $v\mu = a\}$.

The condition in (i)(c) is used to check that $|T_\beta| = \kappa$ for all $\beta \leq \kappa$, and that $u \in {}^\Gamma W$ with $|\Gamma| < \lambda$ implies that $u \in {}^\Gamma T_\beta$ for some $\beta < \kappa$. Again only the condition (1) remains to be checked. If $\mu \notin \Delta^{\mathfrak{A}} X$, then (1) is obvious, so assume that $\mu \in \Delta^{\mathfrak{A}} X$. Suppose $p \in \mathsf{C}_\mu^{[V]} X \cap {}^\alpha W$. Thus $p \in V$, and $p_a^\mu \in X$ for some $a \in U$; moreover, $p \in {}^\alpha W$. Choose $\nu \in \alpha \sim \{\mu\}$ and let $u = ((\Delta X \sim \{\mu\}) \cup \{\nu\}) \upharpoonright p$. Then $u \in {}^{(\Delta X \sim \{\mu\}) \cup \{\nu\}} \bigcup_{\gamma < \beta} T_\gamma$ for some $\beta < \kappa$, so by the definition of T_β, $v \in X$ for some v such that $((\Delta X \sim \{\mu\}) \cup \{\nu\}) \upharpoonright v = u$ and $v\mu \in T_\beta \subseteq W$. Then $v\nu = u\nu = p\nu$ so it follows since $\mathfrak{A} \in \mathsf{Gs}_\alpha$ and $\alpha \geq 2$ that $p_{v\mu}^\mu \in V$. Clearly $\Delta X \upharpoonright v = \Delta X \upharpoonright p_{v\mu}^\mu$, so by the regularity of X in $\mathfrak{A}$, $p_{v\mu}^\mu \in X$. Hence $p_{v\mu}^\mu \in X \cap {}^\alpha W$; so $p \in \mathsf{C}_\mu^{[V]}(X \cap {}^\alpha W)$, as desired in (1).

(ii): By the proof of (i)(c) we may assume that $\alpha \geq 2$. Let the unit element of $\mathfrak{A}$ be $\bigcup_{i \in I} {}^\alpha T_i^{(pi)}$, where ${}^\alpha T_i^{(pi)} \cap {}^\alpha T_j^{(pj)} = 0$ for $i \neq j$. Choose $J \subseteq I$ with $|J| \leq \kappa$ such that $S \subseteq \bigcup_{j \in J} T_j$, $|\bigcup_{j \in J} T_j| \geq \kappa$, and $X \cap \bigcup_{j \in J} {}^\alpha T_j^{(pj)} \neq 0$ for all non-zero $X \in A$. Now take any $j \in J$. The set $\{X \cap {}^\alpha T_j^{(pj)} : X \in A\}$ is the universe of a Ws_α, as is easily checked; we denote this Ws_α by $\mathfrak{C}_j$. Now if $|T_j| < \kappa$ let $X_j = T_j$. If $|T_j| \geq \kappa$, by (i)(a) there is a set X_j such that $Rgp_j \cup (S \cap T_j) \subseteq X_j \subseteq T_j$, $|X_j| = \kappa$, and $\langle Y \cap {}^\alpha X_j : Y \in C_j \rangle$ is a strong ext-isomorphism of $\mathfrak{C}_j$ onto a Ws_α with unit element ${}^\alpha X_j^{(pj)}$. Let $V' = \bigcup_{j \in J} {}^\alpha X_j^{(pj)}$. Then it is easily checked that $\langle Y \cap V' : Y \in A \rangle$ is an ext-isomorphism of $\mathfrak{A}$ onto a Gws_α with base $W = \bigcup_{j \in J} X_j$ such that $S \subseteq W \subseteq U$ and $|W| = \kappa$, as desired.

(iii): Let Z_0 be a subset of V such that $|Z_0| \leq \kappa$, $S \subseteq \bigcup_{z \in Z_0} Rgz$, $|\bigcup_{z \in Z_0} Rgz| = \kappa$, and $X \cap Z_0 \neq 0$ whenever $0 \neq X \in A$. If $n \in \omega$ and Z_n has been defined, let

$Z_{n+1} = Z_n \cup \{z \in V$: there exist $q \in Z_n$, $X \in A$, and $\mu < \alpha$ such that z is the least element of V with $z \in X$ and $(\alpha \sim \{\mu\}) \upharpoonright z \subseteq q\}$.

Let $V' = \bigcup_{n \in \omega} Z_n$ and $W = \bigcup_{z \in V'} Rgz$. It is easily checked that $\langle X \cap V' : X \in A \rangle$ is the desired ext-isomorphism.

REMARKS 3.1.46. None of the various conditions in the hypothesis of 3.1.45 can simply be omitted. The examples to show this, mainly due to Andréka and Németi, are found in [HMT81] 3.19, in [AN81] 3.7,3.8 and in Sain [82a'],[84']. However, consider the condition

(*) A is generated by a set of power $\leq \kappa$.

If $|\alpha| \leq \kappa$, (*) is equivalent to $|A| \leq \kappa$, and so trivially (*) can replace $|A| \leq \kappa$ in the hypothesis of 3.1.45, as far as the conditions (i)(b) and (iii) are concerned. Sain [82a'],[84'] has shown that the same replacement can be made in condition (i)(c), but that $|A| \leq \kappa$ cannot be weakened to (*) in the case of (i)(a) and (ii).

A question related to the changing base question is this: when is a Ws_α with unit element ${}^\alpha W^{(p)}$ isomorphic to one with unit element ${}^\alpha W^{(q)}$? The following theorem along these lines is a generalization of Lemma 6 of Henkin, Monk [74'] due to Andréka and Németi.

THEOREM 3.1.47. *Let* $\mathfrak{A}$ (*resp.* $\mathfrak{A}'$) *and* $\mathfrak{B}$ (*resp.* $\mathfrak{B}'$) *be* (*the full*) Ws_α'*s with unit elements* V_0 *and* V_1, *and bases* U_0 *and* U_1, *respectively. Consider the following conditions*:

(i) $\mathfrak{A}$ *and* $\mathfrak{B}$ *are base–isomorphic*;

(ii) *there exist* $p' \in V_0$ *and* $q' \in V_1$ *such that* $p'|p'^{-1} = q'|q'^{-1}$ *and* $|U_0 \sim Rgp'| = |U_1 \sim Rgq'|$;

(iii) $\mathfrak{A}' \cong \mathfrak{B}'$;

(iv) $\mathfrak{A}'$ *is base–isomorphic to* $\mathfrak{B}'$.

Then (i) *implies* (ii), *while* (ii), (iii) *and* (iv) *are mutually equivalent.*

PROOF. Say $V_0 = {}^\alpha U_0^{(p)}$ and $V_1 = {}^\alpha U_1^{(q)}$. Now (i)$\Rightarrow$(iv) and (iv)$\Rightarrow$(iii) are trivial. To prove (iii)$\Rightarrow$(ii) let f be an isomorphism from $\mathfrak{A}'$ onto $\mathfrak{B}'$. Choose $q' \in V_1$ such that $f\{p\} = \{q'\}$. If $p\kappa = p\lambda$, then $\{p\} \subseteq \mathsf{D}_{\kappa\lambda}$, so $\{q'\} \subseteq \mathsf{D}_{\kappa\lambda}$ and $q'\kappa = q'\lambda$. By symmetry, $p|p^{-1} = q'|q'^{-1}$. Also, it is easy to check that

$$\begin{aligned} |U_0 \sim Rgp| &= |\{d \in A' : d \text{ is an atom} \leqq \mathsf{C}_0\{p\} \cap \textstyle\bigcap_{0<\kappa<\alpha} \sim \mathsf{D}_{0\kappa} \sim \{p\}\}| \\ &= |\{d \in B' : d \text{ is an atom} \leqq \mathsf{C}_0\{q\} \cap \textstyle\bigcap_{0<\kappa<\alpha} \sim \mathsf{D}_{0\kappa} \sim \{q'\}\}| \\ &= |U_1 \sim Rgq'|. \end{aligned}$$

Thus (ii) holds. It remains only to show that (ii)$\Rightarrow$(iv). Assume (ii), and let $f = \{\langle p'\kappa, q'\kappa\rangle : \kappa < \alpha\}$. Then f is a one–one function from a subset of U_0 into U_1 which can be extended to a one–one function f' from U_0 onto U_1. Thus by 3.1.36 and 3.1.37, $\tilde{f}'$ is an isomorphism from $\mathfrak{A}'$ onto the full Ws_α $\mathfrak{B}''$ with unit element ${}^\alpha U_1^{(f'\circ p)}$. If Γ and Δ are finite subsets of α such that $(\alpha \sim \Gamma) \upharpoonright p \subseteq p'$ and $(\alpha \sim \Delta) \upharpoonright q \subseteq q'$, clearly $(\alpha \sim (\Gamma \cup \Delta)) \upharpoonright (f' \circ p) \subseteq q$. Thus ${}^\alpha U_1^{(f'\circ p)} = {}^\alpha U_1^{(q)}$, as desired.

REMARK 3.1.48. It is easy to see that in 3.1.47, (ii) does not imply (i) in general. From the following interesting theorem of Andréka and Németi it follows that the condition of base–isomorphism in (i) cannot be replaced by isomorphism.

THEOREM 3.1.49. *If* $\mathfrak{A} \in \mathsf{Lf}_\alpha \cap \mathsf{Ws}_\alpha$, $\mathfrak{A}$ *has base* U, *and* $q \in {}^\alpha U$, *then* $\mathfrak{A}$ *is isomorphic to a* Ws_α *with unit element* ${}^\alpha U^{(q)}$.

PROOF. Let $\mathfrak{A}$ have unit element ${}^\alpha U^{(p)}$. For any $X \in A$ let

$$fX = \{u \in {}^\alpha U^{(q)} : \text{there is a } v \in X \text{ with } \Delta X \upharpoonright u \subseteq v\}.$$

Using the regularity of $\mathfrak{A}$ (3.1.26), it is easy to see that f is an isomorphism from $\mathfrak{B}\upharpoonright\mathfrak{A}$ into $\mathfrak{B}\upharpoonright\mathfrak{B}$, where $\mathfrak{B}$ is the full Ws_α with unit element ${}^\alpha U^{(q)}$. Now let $\kappa < \alpha$ and $X \in A$. If $u \in f\mathsf{C}_\kappa X$, choose $v \in \mathsf{C}_\kappa X$ so that $\Delta\mathsf{C}_\kappa X \upharpoonright u \subseteq v$. Define $w \in {}^\alpha U$ by setting, for any $\lambda < \alpha$,

$$w\lambda = \begin{cases} u\lambda & \text{if } \lambda \in \Delta X, \\ v\lambda & \text{if } \lambda \notin \Delta X. \end{cases}$$

Since ΔX is finite, we have $w \in {}^\alpha U^{(p)}$. Now $\Delta\mathsf{C}_\kappa X \subseteq \Delta X$, so $\Delta\mathsf{C}_\kappa X \upharpoonright v \subseteq w$. Hence by

regularity $w \in \mathsf{C}_\kappa X$. Choose $a \in U$ such that $w_a^\kappa \in X$. Now $\Delta X \upharpoonright u_a^\kappa \subseteq w_a^\kappa$, so $u_a^\kappa \in fX$. Thus $u \in \mathsf{C}_\kappa fX$. This proves that $f\mathsf{C}_\kappa X \subseteq \mathsf{C}_\kappa fX$. The converse is straightforward.

A result related to 3.1.47 and 3.1.49 is found in 3.1.113.

The notion of base isomorphism discussed above can be generalized to base homomorphisms. The generalization has some logical applications (see section 4.3). The notion, and most of the results below, are due to Andréka and Németi.

DEFINITION 3.1.50. *Let $\mathfrak{A},\mathfrak{B} \in \mathsf{Crs}_\alpha$ with bases U, U' and unit elements V, W respectively. Suppose that f maps U' into U. We define $\hat{f} = \hat{f}_{\mathfrak{A}W}$, a function with domain A, as follows. For any $x \in A$,*

$$\hat{f}x = \{g \in W : f \circ g \in x\}.$$

In case $\hat{f} \in Hom(\mathfrak{A},\mathfrak{B})$ we call $\hat{f}$ a **base–homomorphism from $\mathfrak{A}$ into $\mathfrak{B}$.**

Note that a base–isomorphism function $\tilde{f}$ depends only on f and α, while a base–homomorphism function $\hat{f}$ depends on f, $\mathfrak{A}$, and W. Nevertheless, base–isomorphisms are special cases of base–homomorphisms, by the following result.

THEOREM 3.1.51. *Let f be a one–one function from U onto U', $\mathfrak{A}$ a Crs_α with base U, and $\tilde{f}$ a base–isomorphism from $\mathfrak{A}$ onto a Crs_α $\mathfrak{B}$ with unit element W. Let $g = f^{-1}$. Then $\tilde{f} = \hat{g}_{\mathfrak{A}W}$.*

The definition of base–homomorphism, while simple, leaves open the question concerning when the function $\hat{f}$ really is a base–homomorphism. This is cleared up by the next result, for Gs's, the case of most interest for us.

THEOREM 3.1.52. *Let $\alpha \geqq 2$. Let $\mathfrak{A},\mathfrak{B} \in \mathsf{Gs}_\alpha$ with bases U, U' and unit elements V, W respectively, where $V = \bigcup_{\iota \in I} {}^\alpha S_\iota$, $W = \bigcup_{j \in J} {}^\alpha T_j$, $S_\iota \cap S_k = 0$ for distinct $i,k \in I$, and $T_j \cap T_l = 0$ for distinct $j,l \in J$. Let f map U' into U. Then the following conditions are equivalent:*

(i) $\hat{f} \in Hom(\mathfrak{A},\mathfrak{B})$.

(ii) *For every $j \in J$ there is an $i \in I$ such that $T_j \upharpoonright f$ is a one–one function with range included in S_ι, and such that for all $x \in A$, $\kappa < \alpha$ and $y \in {}^\alpha T_j$, if $f \circ y \in \mathsf{C}_\kappa x$ then there is an $a \in f^* T_j$ such that $(f \circ y)_a^\kappa \in x$.*

In case $\mathfrak{A} = \mathfrak{Gb}\, V$, the last part of (ii) *can be replaced by the condition $f^* T_j = S_\iota$.*

PROOF. (i)$\Rightarrow$(ii). Suppose that $j \in J$, $i,k \in I$, $i \neq k$, $a,b \in T_j$, $fa \in S_\iota$, $fb \in S_k$. Set $g = \langle a,b,b,\ldots\rangle$. Then $g \in W$ but $f \circ g \notin V$, contradicting $\hat{f}V = W$. Now fix $j \in J$. Choose $i \in I$ so that $f^* T_j \subseteq S_\iota$. Suppose that $a,b \in T_j$, $a \neq b$, and $fa = fb$. Let g be as above. Then $f \circ g \in \mathsf{D}_{01}$ but $g \notin D_{01}$, contradicting $\hat{f}\mathsf{D}_{01} = \mathsf{D}_{01}$. Thus $T_j \upharpoonright f$ is one–one. Next, suppose that $x \in A$, $\kappa < \alpha$, $y \in {}^\alpha T_j$, and $f \circ y \in \mathsf{C}_\kappa x$. Thus $y \in \hat{f}\mathsf{C}_\kappa x = \mathsf{C}_\kappa \hat{f}x$, so choose $b \in T_j$ so that $y_b^\kappa \in \hat{f}x$. Thus $(f \circ y)_{fb}^\kappa = f \circ y_b^\kappa \in x$, as desired. Note that $f^* T_j = S_\iota$ implies that this last part of (ii) holds. Now assume that $\mathfrak{A} = \mathfrak{Gb}\, V$. Let $a \in S_\iota$ and choose $b \in T_j$. Let $y = \langle b,b,\ldots\rangle$, $z = \langle a,fb,fb,\ldots\rangle$. Then $y \in \hat{f}\mathsf{C}_0\{z\} = \mathsf{C}_0\hat{f}\{z\}$, so choose $c \in T_j$ such that $y_c^0 \in \hat{f}\{z\}$. Thus $f \circ y_c^0 = z$, hence $fc = a$, as desired.

(ii)$\Rightarrow$(i). Clearly $\hat{f}$ preserves the Boolean operations. Now suppose $g \in \hat{f}\mathsf{D}_{\kappa\lambda}$. Then $f \circ g \in \mathsf{D}_{\kappa\lambda}$. Say $g \in {}^{\alpha}T_j$, and choose i as in (ii). Since $T_j \restriction f$ is one–one, we get $g\kappa = g\lambda$, hence $g \in \mathsf{D}_{\kappa\lambda}$. Thus $\hat{f}\mathsf{D}_{\kappa\lambda} \subseteq \mathsf{D}_{\kappa\lambda}$, and the converse is obvious.

Now let $x \in A$, $\kappa < \alpha$. Suppose that $g \in \hat{f}\mathsf{C}_{\kappa}x$. Thus $f \circ g \in \mathsf{C}_{\kappa}x$. Say $g \in {}^{\alpha}T_j$. By (ii), choose $b \in T_j$ so that $(f \circ g)^{\kappa}_{fb} \in x$. Thus $f \circ g^{\kappa}_{b} \in x$ and $g^{\kappa}_{b} \in \hat{f}x$, so $g \in \mathsf{C}_{\kappa}\hat{f}x$. The converse is similar.

LEMMA 3.1.53. *Let $\alpha \geqq 2$. Let $\mathfrak{A}, \mathfrak{B} \in \mathsf{Gs}_{\alpha}$, with bases U, U' and unit elements V, W respectively, f a function from U' into U, and $\hat{f}$ a base–homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$. Suppose $a \in A$, $|\Delta a| < \omega$, and a is regular. Then $\hat{f}a$ is regular.*

PROOF. Suppose that $g \in \hat{f}a$, $h \in W$, and $(\Delta\hat{f}a \mathbf{u} 1) \restriction g \subseteq h$. Say $g, h \in {}^{\alpha}T_j$, and choose i by 3.1.52. Recall from 2.3.2 that $\Delta\hat{f}a \subseteq \Delta a$. Let $\Gamma = \Delta a \sim \Delta\hat{f}a$. Define $k \in {}^{\alpha}S_i$ by setting $(\Delta a \mathbf{u} 1) \restriction k \subseteq f \circ g$, $(\alpha \sim (\Delta a \mathbf{u} 1)) \restriction k \subseteq f \circ h$. Now $f \circ g \in a$, so by the regularity of a we get $k \in a$. Since $(\alpha \sim \Gamma) \restriction k \subseteq f \circ h$, it follows that $f \circ h \in \mathsf{c}_{(\Gamma)}a$. Therefore, $h \in \hat{f}\mathsf{c}_{(\Gamma)}a = \mathsf{c}_{(\Gamma)}\hat{f}a = \hat{f}a$, as desired.

The condition $|\Delta a| < \omega$ in Lemma 3.1.53 *is* needed; see [AN81], p. 223.

We use this lemma to prove the next theorem, which shows that base–homomorphisms arise naturally when considering the set–function associated with a point–function.

THEOREM 3.1.54. (Ferenczi) *Let $\alpha \geqq 2$. Let $\mathfrak{A}, \mathfrak{B} \in \mathsf{Gs}_{\alpha}$, with bases U, U' and unit elements V, W respectively. Suppose that f maps W into V. Then the following conditions are equivalent:*

(i) *$f^{-1*} \in Hom(\mathfrak{Cb}\, V, \mathfrak{Cb}\, W)$, and $f^{-1*}x$ is regular for any regular $x \subseteq V$ with $|\Delta x| < \omega$.*

(ii) *There is a g mapping U' into U such that $f = \langle g \circ y : y \in W \rangle$ and f^{-1*} is the base–homomorphism $\hat{g} \in Hom(\mathfrak{Cb}\, V, \mathfrak{Cb}\, W)$.*

PROOF. (i)$\Rightarrow$(ii). For any w let $k_w = \langle w : \kappa < \alpha \rangle$. We claim that for each $w \in U'$, fk_w is a constant function. Assume, on the contrary, that $\kappa, \lambda < \alpha$ and $(fk_w)\kappa \neq (fk_w)\lambda$. Thus $\{fk_w\} \subseteq -\mathsf{D}_{\kappa\lambda}$, so $k_w \in f^{-1*}\{fk_w\} \subseteq -\mathsf{D}_{\kappa\lambda}$, a contradiction.

Hence we may define g mapping U' into U by the condition $fk_w = k_{gw}$ for any $w \in U'$.

Now suppose that $y \in W$ and $\kappa < \alpha$; we show that $(fy)\kappa = g(y\kappa)$. Let $a = \{v \in V : v\kappa = (fy)\kappa\}$. Thus $\Delta a \subseteq \{\kappa\}$ and a is regular. Hence $f^{-1*}a$ is regular. Now let $W = \bigcup_{\iota \in I} {}^{\alpha}S_{\iota}$ with $S_{\iota} \cap S_k = 0$ for $i \neq k$, and choose i so that $y, k_{y\kappa} \in {}^{\alpha}S_{\iota}$. Then $(\Delta f^{-1*}a \mathbf{u} 1) \restriction y \subseteq (k_{y\kappa})^0_{y0}$ and $y \in f^{-1*}a$, so $(k_{y\kappa})^0_{y0} \in f^{-1*}a$, hence $k_{y\kappa} \in f^{-1*}a$. Therefore $k_{gy\kappa} = fk_{y\kappa} \in a$, so $(fy)\kappa = g(y\kappa)$. The rest of (ii) is now clear.

(ii)$\Rightarrow$(i). Obvious, using Lemma 3.1.53 for the regularity assertion.

We prove our main algebraic result concerning base homomorphisms in 3.1.111, after we have more properties of Gs_{α}'s available.

Subalgebras

Now we discuss subalgebras of set algebras. The following theorem is trivial.

THEOREM 3.1.55. *If* $\mathsf{K} \in \{\mathsf{Crs}_\alpha, \mathsf{Cs}_\alpha, \mathsf{Gs}_\alpha, \mathsf{Crs}_\alpha^{reg}, \mathsf{Ws}_\alpha, \mathsf{Gws}_\alpha, \mathsf{Gws}_\alpha^{nm}, \mathsf{Gws}_\alpha^{wd}, \mathsf{Gws}_\alpha^{cm}, {}_\infty\mathsf{Gws}_\alpha\}$, *then* $\mathsf{SK} = \mathsf{K}$.

Now we want to show the extent of classes of regular set algebras by proving several theorems to the effect that regular elements generate regular set algebras. (This is not true in full generality, however; see 3.1.66.) These results, except for 3.1.64, are in [AN81] section 1; for 3.1.64, see [HMT81] 4.1. We begin with a definition.

DEFINITION 3.1.56. *Let* $\mathfrak{A} \in \mathsf{CA}_\alpha$. (i) *An element* $x \in A$ *is* **small** *iff for every infinite* $\Gamma \subseteq \Delta x$ *and every finite* $\Delta \subseteq \alpha$ *there is a finite* $\Theta \subseteq \Gamma$ *such that* $\mathsf{c}^\partial_{(\Theta)}\mathsf{c}_{(\Delta)}x = 0$. *We denote by* $Sm^{\mathfrak{A}}$ *the set of all small elements of* $\mathfrak{A}$.

(ii) *Assume also* $\Gamma \subseteq \alpha$. *We set*

$$I_\Gamma^{\mathfrak{A}} = \{x \in A : \textit{for every finite } \Delta \subseteq \alpha \textit{ there is a finite } \Theta \subseteq \alpha \sim \Gamma \textit{ such that } \mathsf{c}^\partial_{(\Theta)}\mathsf{c}_{(\Delta)}x = 0\};$$
$$Dm_\Gamma^{\mathfrak{A}} = \{x \in A : |\Delta x \sim \Gamma| < \omega\}.$$

Finally, we denote by $\mathfrak{Dm}_\Gamma^{\mathfrak{A}}$ *the subalgebra of* $\mathfrak{A}$ *with universe* $Dm_\Gamma^{\mathfrak{A}}$.

The last part of this definition is justified by 2.1.4. Note that if $x \in A$ and Δx is finite, then x is small. As usual, we omit the superscripts $^{\mathfrak{A}}$ in $Sm^{\mathfrak{A}}$, etc., when no confusion is likely.

LEMMA 3.1.57. *Let* $\mathfrak{A} \in \mathsf{CA}_\alpha$ *and* $\Gamma \subseteq \alpha$. *Then:*

(i) I_Γ *is an ideal of* $\mathfrak{A}$.

(ii) $I_\Gamma \cap Dm_\Gamma = \{0\}$.

(iii) $Sm \sim Dm_\Gamma \subseteq I_\Gamma$.

PROOF. (i) Clearly $y \leq x \in I_\Gamma$ implies $y \in I_\Gamma$, and $x \in I_\Gamma$, $\kappa < \alpha$ imply $\mathsf{c}_\kappa x \in I_\Gamma$. Now suppose $x, y \in I_\Gamma$. Let Δ be a finite subset of α. Then choose finite $\Theta_1, \Theta_2 \subseteq \alpha \sim \Gamma$ such that $\mathsf{c}^\partial_{(\Theta_1)}\mathsf{c}_{(\Delta)}x = 0$ and $\mathsf{c}^\partial_{(\Theta_2)}\mathsf{c}_{(\Delta \cup \Theta_1)}y = 0$, using the assumption $x, y \in I_\Gamma$. Now we use the general law

$$(*) \quad \mathsf{c}^\partial_{(\Xi)}(u+v) \leq \mathsf{c}^\partial_{(\Xi)}u + \mathsf{c}_{(\Xi)}v,$$

valid in any CA_α for all u, v and all finite $\Xi \subseteq \alpha$, to get

$$\mathsf{c}^\partial_{(\Theta_1 \cup \Theta_2)}\mathsf{c}_{(\Delta)}(x+y) = \mathsf{c}^\partial_{(\Theta_2)}\mathsf{c}^\partial_{(\Theta_1)}(\mathsf{c}_{(\Delta)}x + \mathsf{c}_{(\Delta)}y) \leq \mathsf{c}^\partial_{(\Theta_2)}(\mathsf{c}^\partial_{(\Theta_1)}\mathsf{c}_{(\Delta)}x + \mathsf{c}_{(\Theta_1)}\mathsf{c}_{(\Delta)}y) = \mathsf{c}^\partial_{(\Theta_2)}\mathsf{c}_{(\Theta_1 \cup \Delta)}y = 0,$$

as desired. Thus I_Γ is an ideal.

(ii) Let $y \in I_\Gamma \cap Dm_\Gamma$; we show that $y = 0$. Set $\Delta = \Delta y \sim \Gamma$; *then* $|\Delta| < \omega$ since $y \in Dm_\Gamma$

Because $y \in I_\Gamma$ we can find a finite $\Theta \subseteq \alpha \sim \Gamma$ such that $c^{\partial}_{(\Theta)} c_{(\Delta)} y = 0$. But $\Delta c_{(\Delta)} y \subseteq \Gamma$, so $c^{\partial}_{(\Theta)} c_{(\Delta)} y = c_{(\Delta)} y$. Thus $y \leq c_{(\Delta)} y = 0$, as desired.

(iii) Let $x \in Sm \sim Dm_\Gamma$ be arbitrary. Since $x \notin Dm_\Gamma$, we have $|\Delta x \sim \Gamma| \geq \omega$. In order to show that $x \in I_\Gamma$, let Δ be an arbitrary finite subset of α. Then because x is small, choose a finite subset Θ of $\Delta x \sim \Gamma$ such that $c^{\partial}_{(\Theta)} c_{(\Delta)} x = 0$. This shows that $x \in I_\Gamma$.

The following general fact from the algebraic theory of CA_α's will also be needed:

LEMMA 3.1.58. *Let $\mathfrak{A}$ be a CA_α generated by a set X. Suppose B is a subuniverse of $\mathfrak{A}$ and I is an ideal of $\mathfrak{A}$ such that $B \cap I = \{0\}$. Then $B \subseteq Sg(X \sim I)$.*

PROOF. $\mathfrak{A}/I$ is generated by $\{b/I : b \in X \sim I\}$, and $\langle b/I : b \in B \rangle$ is one–one, so the lemma is clear.

LEMMA 3.1.59. *Let $\mathfrak{A} \in \mathsf{CA}_\alpha$ be generated by $X \subseteq Sm^{\mathfrak{A}}$, and suppose that $\Gamma \subseteq \alpha$. Then $Dm_\Gamma = Sg(X \cap Dm_\Gamma)$.*

PROOF. Assume the hypothesis. Since $Dm_\Gamma \in Su\mathfrak{A}$, we clearly have $Sg(X \cap Dm_\Gamma) \subseteq Dm_\Gamma$. On the other hand, by 3.1.57 and 3.1.58 we have $Dm_\Gamma \subseteq Sg(X \sim I_\Gamma) \subseteq Sg(X \cap Dm_\Gamma)$, as desired.

To proceed further, we need to generalize the notion of regularity.

DEFINITION 3.1.60. *Let $\mathfrak{A}$ be a Crs_α with unit element V, and suppose that $x \in A$ and $\Gamma \subseteq \alpha$. We say that x is* **Γ–regular in $\mathfrak{A}$** *if for all $q \in x$ and all $p \in V$, if $(\Gamma \cup \Delta x) \upharpoonright q \subseteq p$ then $p \in x$.*

Thus x is regular in $\mathfrak{A}$ iff it is 1–regular in $\mathfrak{A}$. Also note that if $\Gamma \subseteq \Delta \subseteq \alpha$ and x is Γ–regular in $\mathfrak{A}$, then x is also Δ–regular in $\mathfrak{A}$.

LEMMA 3.1.61. *Suppose $\kappa \in \Gamma \subseteq \alpha$, $|\Gamma| > 1$, $\mathfrak{A}$ is a Gws_α, and $x \in A$ is Γ–regular in $\mathfrak{A}$. Then x is also $(\Gamma \sim \{\kappa\})$–regular in $\mathfrak{A}$.*

PROOF. Let V be the unit element of $\mathfrak{A}$. Clearly we may assume that $\kappa \notin \Delta x$. Suppose $q \in x$, $p \in V$, and $((\Gamma \sim \{\kappa\}) \cup \Delta x) \upharpoonright q \subseteq p$. Since $|\Gamma| > 1$, there is a $b \in Rgq \cap Rgp$. Now $q^\kappa_b \in \mathsf{C}^{[V]}_\kappa x = x$, and $(\Gamma \cup \Delta x) \upharpoonright q^\kappa_b \subseteq p^\kappa_b \in V$. Hence $p^\kappa_b \in x$ by the Γ–regularity of x. Thus $p \in \mathsf{C}^{[V]}_\kappa x = x$, as desired.

By Lemma 3.1.61 and the comment before it, if x is regular in a Gws_α $\mathfrak{A}$, then it is Γ–regular for every non–empty Γ.

LEMMA 3.1.62. Suppose $\mathfrak{A} \in \mathsf{Gws}^{nm}_\alpha$ *and* $0 \neq \Gamma \subseteq \alpha$. *Then Γ–regular elements generate Γ–regular elements in $\mathfrak{Dm}_\Gamma$.*

PROOF. Assume the hypothesis. Obviously if x is Γ–regular then so is $-x$. Now suppose that $x, y \in Dm_\Gamma$ are both Γ–regular. Set $\Delta = \Gamma \cup \Delta x \cup \Delta y$. Since $|\Delta \sim \Gamma| < \omega$ because $x, y \in Dm_\Gamma$, it suffices by 3.1.61 to show that $x + y$ is Δ–regular; this, however, is

obvious. Clearly $\mathsf{D}_{\kappa\lambda}$ is Γ–regular for all $\kappa,\lambda<\alpha$.

Finally, suppose $\kappa<\alpha$, $x\in Dm_\Gamma$, and x is Γ–regular. Set $\Delta=\Gamma\mathbf{u}\mathbf{\Delta}x$; again it is enough to show that $\mathsf{C}_\kappa x$ is Δ–regular. Suppose $q\in\mathsf{C}_\kappa x$, $p\in V$, and $(\Delta\mathbf{u}\mathbf{\Delta}\mathsf{C}_\kappa x)\upharpoonright q\subseteq p$, where $V=\bigcup_{i\in I}{}^\alpha W_i^{(ri)}$ is the unit element of $\mathfrak{A}$, with ${}^\alpha W_i^{(ri)}\mathbf{n}{}^\alpha W_j^{(rj)}=0$ for $i\neq j$, $W_i=W_j$ or $W_i\mathbf{n}W_j=0$ for all i,j. Say $q\in{}^\alpha W_i^{(ri)}$, $p\in{}^\alpha W_j^{(rj)}$, $q_b^\kappa\in x$. Since we may assume that $\alpha\geqq 2$, normality gives $W_i=W_j$ and $q_b^\kappa\in{}^\alpha W_i^{(ri)}$. Hence $p_b^\kappa\in{}^\alpha W_j^{(rj)}$. Since $(\Gamma\mathbf{u}\mathbf{\Delta}x)\upharpoonright q_b^\kappa\subseteq p_b^\kappa$, we get $p_b^\kappa\in x$. Hence $p\in\mathsf{C}_\kappa x$, as desired.

We can now state the main theorem giving many regular set algebras.

THEOREM 3.1.63. *Let $\mathfrak{A}\in\mathsf{Gws}_\alpha^{nm}$ be generated by a set $X\subseteq Sm^{\mathfrak{A}}$ of regular elements. Then $\mathfrak{A}$ is regular.*

PROOF. Assume the hypothesis. Let $a\in A$ and set $\Gamma=1\mathbf{u}\mathbf{\Delta}a$. Obviously $a\in Dm_\Gamma$, and by 3.1.59, $a\in Sg(X\mathbf{n}Dm_\Gamma)$. By the remark following 3.1.61, every element of $X\mathbf{n}Dm_\Gamma$ is Γ–regular. Hence a is Γ–regular by 3.1.62. Therefore a is regular.

COROLLARY 3.1.64. *If $\mathfrak{A}$ is a Cs_α generated by a set of regular elements with finite dimension sets, then $\mathfrak{A}$ is regular.*

COROLLARY 3.1.65. *Every Gws_α^{nm} generated by a set of atoms is regular.*

PROOF. To apply 3.1.63 to $\mathfrak{A}\in\mathsf{Gws}_\alpha^{nm}$ we want to show that every atom a of $\mathfrak{A}$ is small and regular. Suppose Γ is an infinite subset of $\mathbf{\Delta}a$ and Δ is a finite subset of α. Then $\mathbf{\Delta}a=\alpha$ by 1.10.5(ii), and $a\leqq\mathsf{c}_0-\mathsf{d}_{01}$. Hence $\mathsf{c}_{(\Delta)}a\leqq\mathsf{c}_0-\mathsf{d}_{01}$ and so by 1.10.5(i), $\mathbf{\Delta}\mathsf{c}_{(\Delta)}a=\alpha\sim\Delta$, using also 1.10.3(i). Choose $\kappa\in\Gamma\sim\Delta$. Then $\mathsf{c}_\Delta a>\mathsf{c}_\kappa^\partial\mathsf{c}_{(\Delta)}a\in Cl_{(\Delta)}\mathfrak{A}$, so $\mathsf{c}_\kappa^\partial\mathsf{c}_{(\Delta)}a=0$, as desired.

To show that a is regular, we may assume that $\alpha\geqq\omega$. By 1.10.5(ii), $\mathbf{\Delta}a=\alpha$ or $\mathbf{\Delta}a=0$. *If* $\mathbf{\Delta}a=\alpha$, trivially a is regular. Suppose $\mathbf{\Delta}a=0$; thus by 1.10.5(ii), $a\leqq\mathsf{c}_0^\partial\mathsf{d}_{01}$. Then a is a union of sets ${}^\alpha W$ with W a singleton, hence a is regular by normality of $\mathfrak{A}$.

REMARK 3.1.66. Generalizations of the above results can be found in [AN81] section 4. We note that regular elements do not always generate regular algebras. For example, let $\alpha\geqq\omega$ and let $\mathfrak{B}$ be the full Cs_α with base ω. Set $X=\{x\in{}^\alpha\omega$:for every odd $\kappa<\alpha$, $x\kappa\leqq x0\}$. Clearly $\mathbf{\Delta}X=\{\kappa<\alpha:\kappa$ is odd$\}\mathbf{u}1$, and X is regular. Let $\mathfrak{A}=\mathfrak{Sg}^{\mathfrak{B}}\{X\}$. Now $\mathsf{C}_0X=\{x\in{}^\alpha\omega$:there is a $\lambda<\omega$ such that $x\kappa\leqq\lambda$ for every odd $\kappa<\alpha\}$. Thus $\mathbf{\Delta}\mathsf{C}_0X=0$, while $0\neq\mathsf{C}_0X\neq 1$. Hence C_0X is not regular. This example is extensively generalized in the above reference. In particular, the following result relevant to 3.1.64 is established: If $\mathfrak{C}$ is a full Gws_α, then $\mathfrak{C}$ is normal iff every subalgebra of $\mathfrak{C}$ generated by a set of locally finite dimensional regular elements is regular.

In the example just given we used the following obvious result:

THEOREM 3.1.67. *If $\mathfrak{A}$ is a regular Cs_α, then $Z\mathfrak{A}=\{0,1\}$.*

REMARKS 3.1.68. We state without proof some results about generators in set

algebras. First we consider generation by a single element, a topic which was discussed several places in Part I: 2.1.11, 2.3.22, 2.3.23, and 2.6.25. In particular, following 2.1.11 the following result was stated, the proof being easily obtained from the proof of 2.1.11:

(1) If $2 \leqq \alpha < \omega$ and $\kappa < \omega$, then the full Cs_α with base κ is generated by a single element.

Andréka and Neméti [79a'] solved Problem 2.3 of Part I by showing:

(2) For each $\alpha > 0$ there is a simple finitely generated Cs_α not generated by a single element.

Comer, Andréka, and Németi proved the following (see [HMT81], Comer [83a']):

(3) If $\alpha < \omega$ and $\mathfrak{A}$ is a Cs_α with base of cardinality $\leqq \alpha+1$, then $\mathfrak{A}$ is generated by a single element.

It is not known, in general, if the bound $\alpha+1$ in (3) is best possible. If $\mathfrak{A}$ has base of cardinality $> 1.5\alpha$, then $\mathfrak{A}$ is not generated by a single element (see (8)–(10) below). J. Larson [83'] and [83a'] has further shown the wide extent of Cs_α's with single generators by showing:

(4) For each $\alpha \geqq 2$ there are at least continuum many pairwise non–isomorphic one–generated $\mathsf{Cs}_\alpha^{reg} \cap \mathsf{Lf}_\alpha$'s.

By 2.6.72 this result gives the maximum number of Lf_α's with at most continuum many generators, for every α. Also in Larson [83a'] the following result is given, by an easy argument:

(5) For each $\alpha \geqq 2$ there are $2^{|\alpha| \cup \omega}$ pairwise non–isomorphic one–generated Cs_α's.

Concerning set algebras other than Cs_α's, Andréka and Németi [81a'] established the following strong result. For $2 \leqq \alpha < \omega$, let

> $f\alpha$ = the smallest β such that there is a system $\langle U_\gamma : \gamma < \beta \rangle$ of pairwise disjoint sets with $\omega > |U_\gamma| > 1$ for all $\gamma < \beta$ such that the full Gs_α with unit element $\bigcup_{\gamma<\beta}{}^{\alpha} U_\gamma$ is not generated by a single element.

Then their result is:

(6) For all α with $2 \leqq \alpha < \omega$, $f\alpha = 2^b - 2^c + 1$, where $b = 2^\alpha - 1$ and $c = 2^{\alpha-1} - 1$.

Now we turn to questions about the number of generators in a set algebra. Henkin proved the following (see [HMT81] 4.7):

(7) If $1 \leqq \alpha, \beta < \omega$, then there is a Cs_α $\mathfrak{A}$ with base of size $\alpha \cdot \beta$ which cannot be generated by $< \log_2 \beta$ elements.

For Cs_2's, similar results were obtained by G. Bergman [82'] using an interesting notion of rank. These results suggest the following definition. For $2 \leqq \alpha, \beta < \omega$ let

$q(\alpha,\beta)$ = the smallest $\gamma < \omega$ such that every Cs_α with base β can be generated by γ elements.

Thus the results (3) and (7) can be succinctly formulated as saying $q(\alpha,\beta) = 1$ for $\beta \leqq \alpha+1$ and $q(\alpha,\alpha\cdot\beta) \geqq log_2\beta$. Andréka and Németi, in unpublished work, have shown:

(8) $q(\alpha,\beta) \geqq 2$ if $\beta > 1.5\alpha$.
(9) $q(2,\beta)$ = least integer $\geqq log_2(\beta-1)$ for $\beta > 2$,
(10) $q(\alpha,\alpha\cdot\beta)$ = least integer $\geqq log_2(\beta+\beta/(\alpha-1))$.

Cf. Németi [83a'], Pálfy [80'].
Finally, Erdos, Faber and Larson [81'] showed:

(11) There is a countable Cs_2 not embeddable in any finitely generated Cs_2.

REMARKS 3.1.69. In 3.1.101 we show that the following classes are closed under directed unions: IGs_α and IGws_α (for arbitrary α), ICs_α and IWs_α (for $\alpha < \omega$) and $\mathsf{ICs}_\alpha^{reg} \cap \mathsf{Lf}_\alpha$ (for arbitrary α). In [AN81] 4.14 the authors show that for $\alpha \geqq \omega$, $\mathsf{ICs}_\alpha \cap \mathsf{Lf}_\alpha$ and $\mathsf{ICs}_\alpha^{reg} \cap \mathsf{Dc}_\alpha$ are not closed under directed unions; in Andréka, Németi [84'] this negative result is extended to the class IWs_α.

Homomorphisms

We now give four elementary results about simple set algebras.

THEOREM 3.1.70. (i) *Any* $\mathsf{Cs}_\alpha^{reg} \cap \mathsf{Lf}_\alpha$ *with non-empty base is simple.*
(ii) *Any simple* $\mathsf{Cs}_\alpha \cap \mathsf{Lf}_\alpha$ *is isomorphic to a* Cs_α^{reg}.
(iii) *Any* $\mathsf{Ws}_\alpha \cap \mathsf{Lf}_\alpha$ *is simple.*
(iv) *For* $\alpha < \omega$, *any* Cs_α *with non-empty base is simple.*

PROOF. Only (ii) needs a proof. Let $\mathfrak{A}$ be a simple $\mathsf{Cs}_\alpha \cap \mathsf{Lf}_\alpha$, say with base U. Fix any $p \in {}^\alpha U$. For any $X \in A$ and $q \in {}^\alpha U$ we set $g(q,X) = \langle q\kappa : \kappa \in \Delta X\rangle \cup \langle p\kappa : \kappa \in \alpha \sim \Delta X\rangle$. Then we set, for any $X \in A$, $fX = \{q \in {}^\alpha U : g(q,X) \in X\}$. To show that f is a homomorphism, first suppose that $X \subseteq Y$ and $q \in fX$. Set $\Gamma = \Delta X \oplus \Delta Y$. Then $g(q,Y) \in \mathsf{c}_{(\Gamma)}X$ and $\mathsf{c}_{(\Gamma)}X = \mathsf{c}_{(\Delta X \sim \Delta Y)}X \subseteq \mathsf{c}_{(\Delta X \sim \Delta Y)}Y = Y$. Thus $q \in fY$. Next, suppose that $q \in f(X \cup Y)$. Let $r = (\Delta X \cup \Delta Y) \restriction q \cup (\alpha \sim (\Delta X \cup \Delta Y)) \restriction p$. Then $r \in X \cup Y$; say $r \in W$. Since $((\Delta X \cup \Delta Y) \sim \Delta X) \restriction r \subseteq g(q,X)$, we get $q \in FX$. So f preserves $\cup$. Clearly it preserves $-$ and diagonal elements. Now suppose that $\kappa < \alpha$ and $X \in A$; we show that $f\mathsf{C}_\kappa X = \mathsf{C}_\kappa fX$. If $\kappa \notin \Delta X$, clearly $\mathsf{C}_\kappa fX = fX$ and the result follows. Assume that $\kappa \in \Delta X$. First let $q \in f\mathsf{C}_\kappa X$. Thus $g(q,\mathsf{C}_\kappa X) \in \mathsf{C}_\kappa X$. Let $\Gamma = \Delta X \sim \Delta \mathsf{C}_\kappa X$. Thus Γ is finite. Set

$r=(\Gamma \upharpoonright q)\mathbf{u}(\alpha\sim\Gamma)\upharpoonright g(q,\mathbf{C}_\kappa X)$. Then $r\in\mathbf{C}_\kappa X$ also. Say $r^\kappa_u\in X$. It is easily checked that $r^\kappa_u=g(q^\kappa_u,X)$, and so $q^\kappa_u\in fX$, hence $q\in\mathbf{C}_\kappa fX$. Conversely, suppose that $q\in\mathbf{C}_\kappa fX$: say $q^\kappa_u\in fX$, so that $g(q^\kappa_u,X)\in X$. Now

$$[\Delta\mathbf{C}_\kappa X\mathbf{u}(\alpha\sim\Delta X)]\upharpoonright g(q^\kappa_u,X)\subseteq g(q,\mathbf{C}_\kappa X),$$

so $g(q,\mathbf{C}_\kappa X)\in\mathsf{c}_{(\Gamma)}X\subseteq\mathsf{c}_{(\Gamma)}\mathbf{C}_\kappa X=\mathbf{C}_\kappa X$ and hence $q\in f\mathbf{C}_\kappa X$.

By the simplicity of $\mathfrak{A}$, f is an isomorphism. Clearly $f^*\mathfrak{A}$ is regular.

COROLLARY 3.1.71. *Let $\mathfrak{A}\in\mathsf{Cs}_\alpha\mathbf{u}\mathsf{Ws}_\alpha$ with base U, $|U|>0$. Then the minimal subalgebra of $\mathfrak{A}$ is simple. The characteristic of $\mathfrak{A}$ is $|U|$ if $|U|<\alpha\mathbf{n}\omega$, while it is 0 if $|U|\geqq\alpha\mathbf{n}\omega$.*

COROLLARY 3.1.72. *Let $\mathfrak{A}$ be a Gws_α with subbases $\langle U_i:i\in I\rangle$, each $U_i\neq 0$, $I\neq 0$, and $\alpha\geqq 1$. Then the minimal subalgebra of $\mathfrak{A}$ is simple iff either (1) for all $i,j\in I$ we have $|U_i|=|U_j|<\alpha\mathbf{n}\omega$, in which case $\mathfrak{A}$ has characteristic $|U_i|$ (for any $i\in I$) or (2) for all $i\in I$ we have $|U_i|\geqq\alpha\mathbf{n}\omega$, in which case $\mathfrak{A}$ has characteristic 0.*

REMARKS 3.1.73. Theorem 3.1.70(i) does not extend to arbitrary $\mathsf{Cs}_\alpha\mathbf{n}\mathsf{Lf}_\alpha$'s, or to $\mathsf{Cs}^{reg}_\alpha\mathbf{n}\mathsf{Dc}_\alpha$'s; 3.1.70(iii) does not extend to $\mathsf{Ws}_\alpha\mathbf{n}\mathsf{Dc}_\alpha$'s; 3.1.70(iv) does not extend to Gs_α's and Gws_α's; and 3.1.71 does not extend to Gs_α's and Gws_α's for $\alpha\geqq 2$. For these counterexamples, some due to Andréka and Németi, see [HMT81] 5.5 and [AN81] section 5. The proof of Theorem 3.1.70(ii) is due to Andréka and Németi.

REMARKS 3.1.74. We summarize the situation concerning closure of our classes under homomorphisms; see 3.1.10 and 3.1.11. The positive results are indicated in those two diagrams; the non-trivial ones are found in the following theorems: $\mathsf{Ws}_\alpha\subseteq\mathsf{ICs}^{reg}_\alpha$, 3.1.102; $\mathsf{IGs}_\alpha=\mathsf{IGs}^{reg}_\alpha=\mathsf{IGws}^{reg}_\alpha=\mathsf{IGws}_\alpha$, 3.1.107; $\mathsf{HGws}_\alpha\subseteq\mathsf{IGs}_\alpha$, 3.1.103; for $\alpha\geqq\omega$, ${}_\infty\mathsf{Gws}_\alpha\subseteq\mathsf{I}_\infty\mathsf{Cs}_\alpha$, 3.1.106; ${}_\infty\mathsf{Cs}_\alpha\subseteq\mathsf{I}_\infty\mathsf{Ws}_\alpha$, 3.1.139. Now we mention several counterexamples, assuming always $\alpha\geqq\omega$:

(1) $\mathsf{H}_\infty\mathsf{Cs}^{reg}_\alpha\not\subseteq\mathsf{ICs}^{reg}_\alpha$ (due to Andréka and Németi; see [HMT81] 5.6(4)).
(2) $\mathsf{H}_\infty\mathsf{Ws}_\alpha\not\subseteq\mathsf{IWs}_\alpha\mathbf{u}\{1\}$, where 1 is the one-element Cs_α (due to Monk; see [HMT81] 6.16(2)).
(3) $\mathsf{HCs}^{reg}_\alpha\not\subseteq\mathsf{ICs}_\alpha$ (due to Andréka, Monk, and Németi; see [HMT81] I.5.6(5) and [AN81] 5.3).
(4) $\mathsf{HWs}_\alpha\not\subseteq\mathsf{ICs}_\alpha$ (due to Monk; see [HMT81] 5.6(5)).
(5) $\mathsf{ICs}^{reg}_\alpha\not\subseteq\mathsf{HWs}_\alpha$ (see [AN81] 5.4(v)).
(6) $\mathsf{Gs}_\alpha\not\subseteq\mathsf{HCs}_\alpha$ (also if $2\leqq\alpha<\omega$).

This statement (6) is clear: the minimal subalgebra of any $\mathfrak{A}\in\mathsf{HCs}_\alpha$ is simple or of power 1 by 3.1.70(i), while it is easy to construct a Gs_α for which this is not true.

(7) $\mathsf{H}(\mathsf{Cs}_\alpha\mathbf{n}\mathsf{Lf}_\alpha)\not\subseteq\mathsf{ICs}_\alpha$ (due to Andréka and Németi; see [AN81] 5.7).

It remains open whether $\mathsf{HCs}_\alpha=\mathsf{HCs}^{reg}_\alpha$.

Products

Now in 3.1.75 through 3.1.82 we discuss products of set algebras. We first make a definition extending part of 2.2.1.

DEFINITION 3.1.75. *Let $\mathfrak{A}$ be a* Crs_α *with unit element V and let $W \subseteq V$. Then $rl^{\mathfrak{A}}_W$ is the function with domain A such that for any $a \in A$, $rl^{\mathfrak{A}}_W a = W \cap a$.*

Using this notation, we can conveniently formulate a general result which specializes to give the most important connection between Gs_α's and Cs_α's, and between Gws_α's and Ws_α's.

THEOREM 3.1.76. *Let $\mathfrak{B}$ be a full* Gws_α *with unit element $\bigcup_{\iota \in I} V_\iota$, where $V_\iota \cap V_j = 0$ for $i,j \in I$ and $i \neq j$, and $\Delta V_\iota = 0$ for all $i \in I$. For each $i \in I$ let $\mathfrak{A}_\iota$ be the full* Crs_α *with unit element V_ι. Then $\mathfrak{B} \cong \mathsf{P}_{\iota \in I} \mathfrak{A}_\iota$. In fact, there is a unique $f \in Is(\mathfrak{B}, \mathsf{P}_{\iota \in I} \mathfrak{A}_\iota)$ such that $rl^{\mathfrak{B}}_{V_\iota} = pj_\iota \circ f$ for each $i \in I$.*

PROOF. Clearly there is a unique f mapping B into $\mathsf{P}_{\iota \in I} A_\iota$ and satisfying the final condition. Clearly f is one–one and onto. By 2.3.26, $rl^{\mathfrak{B}}_{V_\iota} \in Ho(\mathfrak{B}, \mathfrak{A}_\iota)$ for each $i \in I$, so $f \in Hom(\mathfrak{B}, \mathsf{P}_{\iota \in I} \mathfrak{A}_\iota)$ by 0.3.6(ii).

COROLLARY 3.1.77. *For $\alpha \geqq 2$ we have* $\mathsf{IGs}_\alpha = \mathsf{SPCs}_\alpha$ *and* $\mathsf{SPCs}^{reg}_\alpha \subseteq \mathsf{IGs}^{reg}_\alpha$.

PROOF. Suppose $\mathfrak{C} \in \mathsf{Gs}_\alpha$; say $\mathfrak{C}$ has unit element $\bigcup_{\iota \in I} {}^\alpha U_\iota$, where $U_\iota \cap U_j = 0$ for distinct $i,j \in I$ and $U_\iota \neq 0$ for all $i \in I$. Let $V_\iota = {}^\alpha U_\iota$ for each $i \in I$, and let $\mathfrak{A}, \mathfrak{B}, f$ be as in 3.1.76. Clearly $\Delta^{\mathfrak{B}} V_\iota = 0$ for all $i \in I$ since $\alpha \geqq 2$. Hence $C1f$ is an isomorphism of $\mathfrak{C}$ onto a subdirect product of Cs_α's, as desired.

Conversely, suppose $\mathfrak{C} \subseteq \mathsf{P}_{\iota \in I} \mathfrak{D}_\iota$, each $\mathfrak{D}_\iota$ a Cs_α with base $U_\iota \neq 0$. We may assume that $U_\iota \cap U_j = 0$ for distinct $i,j \in I$. Let $V_\iota = {}^\alpha U_\iota$ for each $i \in I$, and again let $\mathfrak{B}, \mathfrak{A}$ and f be as in 3.1.76. Clearly $C1f^{-1}$ is an isomorphism of $\mathfrak{C}$ onto a Gs_α.

For the last part of the theorem we assume the notation of the preceding paragraph, with each $\mathfrak{D}_\iota$ regular. To show that $f^{-1*}\mathfrak{C}$ is a regular Gs_α, we use 3.1.25. Hence suppose $c \in C$, $i \in I$, $h \in f^{-1}c \cap {}^\alpha U_\iota$, $g \in {}^\alpha U_\iota$ and $\Delta(f^{-1}c)1h \subseteq g$. Note that $f^{-1}c = \bigcup_{j \in I} c_j$. Clearly $\Delta^{\mathfrak{A}_\iota} c_\iota \subseteq \Delta(\bigcup_{j \in I} c_j)$. Hence, by regularity of $\mathfrak{A}_\iota$ we get $g \in f^{-1}c$, as desired.

REMARK 3.1.78. If $\mathfrak{C} \in \mathsf{Gs}^{reg}_\alpha$, the first part of the proof of 3.1.77 does not necessarily give $C1f \in Ism(\mathfrak{C}, \mathsf{P}_{\iota \in I} \mathfrak{B}_\iota)$, $\mathfrak{B}$ a system of regular Cs_α's. In fact, there is a Gs^{reg}_α $\mathfrak{C}$ with unit element as in the proof above such that for all $i \in I$, $rl_{U_\iota}{}^*\mathfrak{C} \notin \mathsf{ICs}^{reg}_\alpha$; see [AN81] 6.1. Nevertheless, it is true that $\mathsf{SPCs}^{reg}_\alpha = \mathsf{IGs}^{reg}_\alpha$; in fact, also $\mathsf{IGs} = \mathsf{IGs}^{reg}_\alpha$; see 3.1.107. Analogously to 3.1.77 we have:

COROLLARY 3.1.79. *For* $\alpha \geqq 2$ *we have* $\mathsf{IGws}_\alpha = \mathsf{SPWs}_\alpha$.

COROLLARY 3.1.80. *For* $\alpha \geqq 2$ *we have* $\mathsf{PGs}_\alpha = \mathsf{IGs}_\alpha$ *and* $\mathsf{PGws}_\alpha = \mathsf{IGws}_\alpha$.

COROLLARY 3.1.81. *Let* $\mathfrak{A} \in \mathsf{IGs}_\alpha$, $|A|>1$ *and* $\alpha<\omega$. *Then the following conditions are equivalent:*

(i) $\mathfrak{A} \in \mathsf{ICs}_\alpha$;

(ii) $\mathfrak{A}$ *is simple.*

PROOF. By 3.1.77 and 3.1.70(iv) (treating $\alpha<2$ separately).

REMARKS 3.1.82. We now discuss closure of our classes of set algebras under **H**,**S** and **P**; the facts are summarized in Figures 3.1.12 and 3.1.13 for $\alpha \geqq 2$. The non-trivial positive relationships in these figures are proved in the following places below: $\mathsf{IGs}_\alpha = \mathsf{IGws}_\alpha$ in 3.1.107; $\mathsf{IGs}_\alpha = \mathsf{HSPGs}_\alpha = \mathsf{HSPGws}_\alpha$ in 3.1.108; $\mathsf{Ws}_\alpha \subseteq \mathsf{ICs}_\alpha^{reg}$ in 3.1.102; ${}_\infty\mathsf{Gws}_\alpha \subseteq \mathsf{I}_\infty\mathsf{Cs}_\alpha$ for $\alpha \geqq \omega$ in 3.1.106; ${}_\infty\mathsf{Cs}_\alpha \subseteq \mathsf{I}_\infty\mathsf{Ws}_\alpha$ in 3.1.139. Before considering various counterexamples, we indicate the situation for $\alpha \leqq 1$:

(1) For $\alpha \leqq 1$ the classes are just five in number, increasing under inclusion in the following order:

(a) $\mathsf{IWs}_\alpha = \{\mathfrak{A} \in \mathsf{CA}_\alpha : \mathfrak{A}$ is simple$\}$;

(b) $\mathsf{ICs}_\alpha = \{\mathfrak{A} \in \mathsf{CA}_\alpha : \mathfrak{A}$ is simple or $|A| = 1\}$;

(c) $\mathsf{PCs}_\alpha = \{\mathfrak{A} \in \mathsf{CA}_\alpha : \mathfrak{A}$ is a product of simple CA_α's$\}$;

(d) HPCs_α;

(e) $\mathsf{HSPCs}_\alpha = \mathsf{CA}_\alpha = \mathsf{SPWs}_\alpha$.

Now we mention various counterexamples associated with 3.1.12 and 3.1.13, and indicate where they can be found.

(2) $\mathsf{HWs}_\alpha \not\subseteq \mathsf{PCs}_\alpha$ for $\alpha \geqq \omega$; see [HMT81] 6.8(2) and 5.6(5).

(3) $\mathsf{Cs}_\alpha^{reg} \not\subseteq \mathsf{PWs}_\alpha$ and $\mathsf{Cs}_\alpha \not\subseteq \mathsf{PCs}_\alpha^{reg}$ for $\alpha \geqq \omega$ (see [AN81] 6.2(ii),(i)).

(4) $\mathsf{PWs}_\alpha \not\subseteq \mathsf{HCs}_\alpha$ for $\alpha \geqq \omega$; see [HMT81] 6.8(4).

(5) $\mathsf{SPCs}_\alpha \not\subseteq \mathsf{HPCs}_\alpha$; see [HMT81] 6.8(5); the counterexample is due to Andréka, Monk and Németi.

(6) $\mathsf{Cs}_\alpha^{reg} \not\subseteq \mathsf{SPDc}_\alpha$ and $\mathsf{Ws}_\alpha \not\subseteq \mathsf{SPDc}_\alpha$ for all $\alpha>0$; see [HMT81] 6.8(7) for the counterexample due to Andréka and Németi.

In connection with (6) the following general fact about Lf_α's and Dc_α's should be noted: $\mathsf{Dc}_\alpha \not\subseteq \mathsf{SPLf}_\alpha$ (hence $\mathsf{SPDc}_\alpha \not\subseteq \mathsf{SPLf}_\alpha$) for all $\alpha \geqq \omega$; see [HMT81] 6.8(8).

(7) $\mathsf{PWs}_\alpha \not\subseteq \mathsf{IWs}_\alpha$ (an immediate consequence of 3.1.85).

Among the questions about 3.1.12 left open by the above counterexamples, the most important seems to be whether $\mathsf{ICs}_\alpha^{reg} \subseteq \mathsf{HPWs}_\alpha$. The answer to this question (obtained by Andréka and Németi) is, consistently, no; see Sain [82a'].

Concerning 3.1.13, note:

(8) $\mathsf{H}_\infty\mathsf{Ws}_\alpha \not\subseteq \mathsf{P}_\infty\mathsf{Cs}_\alpha^{reg}$; see [AN81] 6.2(vi).

Now we discuss direct indecomposability and related notions.

THEOREM 3.1.83. *Every full* Ws_α *is subdirectly indecomposable.*

PROOF. Let $\mathfrak{A}$ be a full Ws_α, with unit element ${}^{\alpha}U^{(p)}$. We apply 2.4.44. Given $0 \neq y \in A$, choose $f \in y$. Then there is a finite $\Gamma \subseteq \alpha$ such that $(\alpha \sim \Gamma) \upharpoonright f \subseteq p$. Thus $\{p\} \subseteq \mathsf{c}_{(\Gamma)} y$. Hence by 2.4.44, $\mathfrak{A}$ is subdirectly indecomposable.

COROLLARY 3.1.84. *Any subdirectly indecomposable* Cs_α *is isomorphic to a* Ws_α.

PROOF. By 3.1.17, 3.1.20, and 3.1.79.

COROLLARY 3.1.85. *Every* Ws_α *is weakly subdirectly indecomposable.*

PROOF. By 0.3.58(ii), 2.4.47(i), and 3.1.83.

COROLLARY 3.1.86. *Let* $\mathfrak{A} \in \mathsf{IGws}_\alpha$. *Then the following two conditions are equivalent:*
(i) $\mathfrak{A} \in \mathsf{IWs}_\alpha$;
(ii) $\mathfrak{A} \subseteq \mathfrak{B}$ *for some subdirectly indecomposable* $\mathfrak{B} \in \mathsf{IGws}_\alpha$.

PROOF. (i) implies (ii) by 3.1.83. (ii) implies (i) by 3.1.79.

COROLLARY 3.1.87. *Any regular* Cs_α *with non-empty base is directly indecomposable.*

PROOF. By 3.1.67 and 2.4.14.

REMARKS 3.1.88. Throughout these remarks let $\alpha \geqq \omega$.
(1) Examples (I) and (II) in 2.4.50 are Ws_α's which are respectively subdirectly indecomposable but not simple, and weakly subdirectly indecomposable but not subdirectly indecomposable.
(2) We show in 3.1.110 that any direct factor of a Cs_α is isomorphic to a Cs_α; this contrasts with the situation for homomorphic images (see 3.1.74(3)).
(3) The full Cs_α with base of any cardinality $\geqq 2$ is directly decomposable.
(4) A Cs_α^{reg} which is directly indecomposable by 3.1.87 but which is not weakly subdirectly indecomposable can be obtained by modifying Example (III) of 2.4.50; see [HMT81] 6.16(5).
(5) Example (I) in 2.4.50 can be similarly modified to yield a Cs_α^{reg} $\mathfrak{A}$ which is subdirectly indecomposable but not simple; see [HMT81] 6.16(6).
(6) As noted by Andréka and Németi, example (II) in 2.4.50 can be modified to yield a Cs_α^{reg} $\mathfrak{A}$ which is weakly subdirectly indecomposable but not subdirectly indecomposable; see [HMT81] 6.16(7) and [AN81] 6.3.
(7) Andréka and Németi have shown that the converse of Theorem 3.1.87 fails: there is a directly indecomposable Cs_α not isomorphic to a Cs_α^{reg}; see [HMT81] 6.16(8).

(8) It follows from 3.1.107 that any subdirectly indecomposable Cs_α is isomorphic to a regular Cs_α.
(9) Andréka and Németi have shown that there is a $\mathsf{Cs}_\alpha \mathsf{nLf}_\alpha$ which is not isomorphic to a product of directly indecomposable CA_α's; see [AN81] 6.2(i).
(10) In Andréka, Németi [84'] it is shown that there is a weakly subdirectly indecomposable Cs_α^{reg} which is not isomorphic to a Ws_α.

It is open whether every weakly subdirectly indecomposable Cs_α is isomorphic to a regular Cs_α.

Ultraproducts

Now we turn to ultraproducts. After some technical lemmas we will be able to establish with the aid of ultraproducts the main properties of our classes of set algebras. The first very general lemma is due to Monk. For it and the other lemmas we need some notation.

DEFINITION 3.1.89. *Let F be an ultrafilter on a set I, $U=\langle U_i : i\in I\rangle$ a system of sets, and α an ordinal.*

(i) *By an (F,U,α)-**choice function** we mean a function c mapping $\alpha\times(\mathsf{P}_{i\in I}U_i/\bar{F})$ into $\mathsf{P}_{i\in I}U_i$ such that for all $\kappa<\alpha$ and all $y\in\mathsf{P}_{i\in I}U_i/\bar{F}$ we have $c(\kappa,y)\in y$.*

(ii) *If c is an (F,U,α)-choice function, then we define c^+ mapping ${}^\alpha(\mathsf{P}_{i\in I}U_i/\bar{F})$ into $\mathsf{P}_{i\in I}{}^\alpha U_i$ by setting, for all $q\in{}^\alpha(\mathsf{P}_{i\in I}U_i/\bar{F})$ and all $i\in I$*

$$(c^+q)_i=\langle c(\kappa,q_\kappa)_i:\kappa<\alpha\rangle.$$

(iii) *Let $A=\langle A_i : i\in I\rangle$ be a system of sets such that $A_i\subseteq Sb({}^\alpha U_i)$ for all $i\in I$, and let c be an (F,U,α)-choice function. Then there is a unique function $Rep(F,U,\alpha,A,c)$ (usually abbreviated by omitting one or more of the five arguments) mapping $\mathsf{P}_{i\in I}A_i/\bar{F}$ into $Sb^\alpha(\mathsf{P}_{i\in I}U_i/\bar{F})$ such that, for any $a\in\mathsf{P}_{i\in I}A_i$,*

$$Rep(a/\bar{F})=\{q\in{}^\alpha(\mathsf{P}_{i\in I}U_i/\bar{F}) : \{i\in I : (c^+q)_i\in a_i\}\in F\}.$$

LEMMA 3.1.90. *Let F be an ultrafilter on a set I, $U=\langle U_i : i\in I\rangle$ a system of non-empty sets, and α an ordinal. Let c be an (F,U,α)-choice function. Further, let $\mathfrak{A}\in{}^I\mathsf{Crs}_\alpha$, where each $\mathfrak{A}_i$ has base U_i and unit element V_i, and set $V=\langle V_i : i\in I\rangle$.*

Then $Rep(c)$ is a homomorphism from $\mathsf{P}_{i\in I}\mathfrak{A}_i/\bar{F}$ into a Crs_α. Furthermore, for every $0\neq a/\bar{F}\in\mathsf{P}_{i\in I}A_i/\bar{F}$, there is a choice function c such that $Rep(c)(a/\bar{F})\neq 0$. Namely, if $Z\in F$, $s\in\mathsf{P}_{i\in I}V_i$, $s_i\in a_i$ for all $i\in Z$, and $w=\langle\langle s_i\kappa : i\in I\rangle:\kappa<\alpha\rangle$, it follows that if c' is any (F,U,α)-choice function such that $c'(\kappa,w\kappa/\bar{F})=w\kappa$ for all $\kappa<\alpha$, then $Rep(c')(a/\bar{F})\neq 0$.

PROOF. Let $f=Rep(c)$, $X=\mathsf{P}_{i\in I}U_i/\bar{F}$, $T=f(V/\bar{F})$. Clearly f preserves $+$. Now

let $\kappa,\lambda<\alpha$; we show that f preserves $\mathsf{d}_{\kappa\lambda}$. Note that $f\mathsf{d}_{\kappa\lambda}\subseteq T$ since f preserves $+$. Now let $q\in T$. Then $\{i\in I:(c^+q)_i\in V_i\}\in F$, and so

$$q\in f\mathsf{d}_{\kappa\lambda} \quad\text{iff}\quad \{i\in I:(c^+q)_i\in \mathsf{D}_{\kappa\lambda}^{[V_i]}\}\in F \quad\text{iff}\quad \{i\in I:(c^+q)_i\kappa = (c^+q)_i\lambda\}\in F \quad\text{iff}$$
$$\{i\in I:c(\kappa,q\kappa)_i = c(\lambda,q\lambda)_i\in F \text{ iff } q_\kappa = q_\lambda \text{ iff } q\in \mathsf{D}_{\kappa\lambda}^{[T]}.$$

Now let $a\in \mathsf{P}_{i\in I}A_i$. We show that f preserves $-$. Clearly $f(-a/\bar{F})\subseteq T\sim f(a/\bar{F})$. Now let $q\in T\sim f(a/\bar{F})$. Thus $\{i\in I:(c^+q)_i\in V_i\}\in F$ and $\{i\in I:(c^+q)_i\in a_i\}\notin F$, i.e., $\{i\in I:(c^+q)_i\in V_i\sim a_i\}\in F$. Therefore $q\in f(-a/\bar{F})$.

Next, let also $\kappa<\alpha$; we show that f preserves c_κ. First suppose that $q\in f(\mathsf{c}_\kappa a/\bar{F})$. Let $M=\{i\in I:(c^+q)_i\in \mathsf{C}_\kappa^{[V_i]}a_i\}$; thus $M\in F$. Then there is an $s\in \mathsf{P}_{i\in I}U_i$ such that $[(c^+q)_i]^\kappa_{si}\in a_i$ for all $i\in M$. We show that $q^\kappa_u\in f(a/\bar{F})$, where $u=s/\bar{F}$. Let $Z=\{i\in I:si=c(\kappa,s/\bar{F})_i\}$. Then $Z\in F$ since $c(\kappa,s/\bar{F})\in s/\bar{F}$. For any $i\in Z\cap M$ we have

$$(c^+q^\kappa_u)_i=\langle c(\lambda,(q^\kappa_u)\lambda)_i:\lambda<\alpha\rangle = [(c^+q)_i]^\kappa_{c(\kappa,u)i} = [(c^+q)_i]^\kappa_{si}\in a_i.$$

Thus, indeed, $q^\kappa_u\in f(a/\bar{F})$. Hence $q\in \mathsf{C}_\kappa^{[T]}f(a/\bar{F})$. Second, suppose that $q\in \mathsf{C}_\kappa^{[T]}f(a/\bar{F})$. Thus $q\in T$ and $(\alpha\sim\{\kappa\})1q=(\alpha\sim\{\kappa\})1p$ for some $p\in f(a/\bar{F})$. Let $M=\{i\in I:(c^+p)_i\in a_i\}$; thus $M\in F$. Also since $q\in T$ the set $Z=\{i\in I:(c^+q)_i\in V_i\}$ is in F. Now let $i\in M\cap Z$. Then $(c^+q)_i\in V_i$ and $(\alpha\sim\{\kappa\})1(c^+q)_i\subseteq(c^+p)_i\in a_i$, proving that $(c^+q)_i\in \mathsf{C}_\kappa^{[V_i]}a_i$. Thus $q\in f(\mathsf{c}_\kappa a/\bar{F})$, since $M\cap Z\in F$.

We have now verified that f is a homomorphism. For the second part of the conclusion of the lemma, assume its additional hypotheses. Let $q=\langle w\kappa/\bar{F}:\kappa<\alpha\rangle$ and $f'=Rep(c')$. We show that $q\in f'(a/\bar{F})$ (hence $f'(a/\bar{F})\neq 0$, as desired). In fact, for any $i\in Z$ we have

$$(c'^+q)_i=\langle c'(\kappa,q\kappa)_i:\kappa<\alpha\rangle = \langle c'(\kappa,w\kappa/\bar{F})_i:\kappa<\alpha\rangle=\langle (w\kappa)_i:\kappa<\alpha\rangle = s_i\in a_i.$$

So $q\in f'(a/\bar{F})$, as desired.

Now we give some specialized versions of Lemma 3.1.90; they are due to Andréka and Németi. The first one does not put any restrictions on the choice function c or the ultrafilter F involved.

LEMMA 3.1.91. *Assume the hypotheses of* 3.1.90, *and let* $\mathfrak{B}=Rep(c)^*(\mathsf{P}_{i\in I}\mathfrak{A}_i/\bar{F})$. *Then:*

(i) *If* $\mathfrak{A}\in {}^I\mathsf{Gws}_\alpha$ *then* $\mathfrak{B}\in\mathsf{Gws}_\alpha$; *moreover, for* $\alpha\geqq 2$, *every subbase of* $\mathfrak{B}$ *has the form* $\mathsf{P}_{i\in I}Y_i/\bar{F}$ *for some system* $\langle Y_i:i\in I\rangle$ *such that* Y_i *is a subbase of* $\mathfrak{A}_i$ *for each* $i\in I$.

(ii) *If* κ *is a cardinal,* $\alpha\geqq 2$, $\lambda=|{}^I\kappa/\bar{F}|$, *and* $\mathfrak{A}\in {}^I({}_\kappa\mathsf{Gws}_\alpha)$, *then* $\mathfrak{B}\in{}_\lambda\mathsf{Gws}_\alpha$.

(iii) *If* $\alpha\geqq 2$ *and* $\mathfrak{A}\in {}^I\mathsf{Gws}_\alpha^{cm}$ *then* $\mathfrak{B}\in\mathsf{Gws}_\alpha^{cm}$, *and the base of* $\mathfrak{B}$ *is either* $\mathsf{P}_{i\in I}U_i/\bar{F}$ *or* 0.

(iv) *If* $\alpha\geqq 2$ *and* $\mathfrak{A}\in {}^I\mathsf{Gws}_\alpha^{nm}$ *then* $\mathfrak{B}\in\mathsf{Gws}_\alpha^{nm}$.

(v) *If* $\mathfrak{A}\in {}^I\mathsf{Cs}_\alpha$ *then* $\mathfrak{B}\in\mathsf{Cs}_\alpha$ *and* $\mathfrak{B}$ *has base* $\mathsf{P}_{i\in I}U_i/\bar{F}$.

PROOF. Let $f=Rep(c)$. First suppose $\mathfrak{A}\in {}^I\mathsf{Cs}_\alpha$. Thus $V_i={}^\alpha U_i$ for each $i\in I$. We

need to prove that $f(V/\bar{F}) = {}^{\alpha}X$, where $X = \mathsf{P}_{i\in I} U_i/\bar{F}$. Let $q \in {}^{\alpha}X$. Then $(c^+q)_i \in {}^{\alpha}U_i = V_i$ for all $i \in I$, so $q \in f(V/\bar{F})$. The converse being obvious, we thus have $\mathfrak{B} \in \mathsf{Cs}_\alpha$ with base X. Thus (v) holds.

Next, suppose that $\mathfrak{A} \in {}^I\mathsf{Gws}_\alpha$. Then for each $i \in I$ we can write

$$V_i = \bigcup\{{}^{\alpha}Y_{ij}^{(pij)} : j \in J_i\},$$

where ${}^{\alpha}Y_{ij}^{(pij)} \cap {}^{\alpha}Y_{ik}^{(pik)} = 0$ whenever $j \neq k$. Let $S_{ij} = {}^{\alpha}Y_{ij}^{(pij)}$. Now for each $j \in \mathsf{P}_{i\in I} J_i$ we set

$$W_j = \{q \in {}^{\alpha}X : \{i \in I : (c^+q)_i \in S_{i,ji}\} \in F\},$$
$$Q_j = \mathsf{P}_{i\in I}\, Y_{i,ji}/\bar{F}^{(U)}.$$

Now we claim

(1) $f(V/\bar{F}) = \bigcup\{W_j : j \in \mathsf{P}_{i\in I} J_i\}$.

For, let $q \in f(V/\bar{F})$. Let $M = \{i \in I : (c^+q)_i \in V_i\}$, so that $M \in F$. Choose $j \in \mathsf{P}_{i\in I} J_i$ so that $(c^+q)_i \in S_{i,ji}$ for all $i \in M$. Thus $q \in W_j$, as desired. Clearly each $W_j \subseteq f(V/\bar{F})$, so (1) holds.

(2) If $j,k \in \mathsf{P}_{i\in I} J_i$ and $j/\bar{F} \neq k/\bar{F}$, then $W_j \cap W_k = 0$.

For, assume the hypothesis of (2) and let $q \in W_j$. Let $Z = \{i \in I : (c^+q)_i \in S_{i,ji}\}$ and let $H = \{i \in I : (c^+q)_i \in S_{i,ki}\}$. Then $Z \cap H \subseteq \{i \in I : ji = ki\} \notin F$, while $Z \in F$, so $H \notin F$. Thus $q \notin W_k$, as desired.

(3) For any $j \in \mathsf{P}_{i\in I} J_i$ we have $W_j = \bigcup_{q\in W_j} {}^{\alpha}Q_j^{(q)}$.

For, first let $q \in W_j$. Let $\kappa < \alpha$. Now $\{i \in I : (c^+q)_i \in S_{i,ji}\} \in F$ and hence $\{i \in I : c(\kappa, q\kappa)_i \in Y_{i,ji}\} \in F$, and further, since $c(\kappa, q\kappa) \in q\kappa$, $q\kappa \in Q_j$. Thus $\subseteq$ in (3) holds. For the other direction, it suffices to take $q \in W_j$, $\kappa < \alpha$, $u \in Q_j$, and show that $q_u^\kappa \in W_j$. Let $M = \{i \in I : (c^+q)_i \in S_{i,ji}\}$ and $Z = \{i \in I : c(\kappa, u)_i \in Y_{i,ji}\}$. Then $M \in F$ since $q \in W_j$ and $Z \in F$ since $u \in Q_j$ and $c(\kappa, u) \in u$. So $M \cap Z \in F$. Let $i \in M \cap Z$. Then

$$(c^+q_u^\kappa)_i = \langle c(\lambda, q_u^\kappa\lambda)_i : \lambda < \alpha\rangle = \langle c(\lambda, q\lambda)_i : \lambda < \alpha\rangle^\kappa_{c(\kappa,u)i} = [(c^+q)_i]^\kappa_{c(\kappa,u)i} \in S_{i,ji}.$$

Thus $q_u^\kappa \in W_j$, as desired.

Now (1), (2), (3) immediately yield that $\mathfrak{B} \in \mathsf{Gws}_\alpha$, upon noting that $W_j = W_k$ if $j/\bar{F} = k/\bar{F}$; they also give all other parts of 3.1.91 except for the following two points.

First, the final assertion of (iii): assume the hypothesis of (iii), and suppose the base of $\mathfrak{B}$ is non-empty. By (1), choose $j \in \mathsf{P}_{i\in I} J_i$ and $q \in W_j$. The base of $\mathfrak{B}$ is clearly contained in X; now assume $x \in X$. Then clearly $q_x^0 \in W_j$ and so x is in the base of $\mathfrak{B}$, as desired.

Second, we prove (iv); so we assume $\alpha \geqq 2$ and $\mathfrak{A} \in {}^I\mathsf{Gws}_\alpha^{nm}$. By (2) and (3) it is

enough to assume that $j,k \in \mathsf{P}_{i\in I} J_i$ with $j/\bar{F} \neq k/\bar{F}$ and show that $Q_j \cap Q_k = 0$ or $Q_j = Q_k$. The assumption yields $\{i \in I: ji \neq ki\} \in F$. Now $\mathfrak{A} \in {}^I\mathsf{Gws}_\alpha^{nm}$ implies that $\{i \in I: ji \neq ki\} \subseteq \{i \in I: Y_{i,ji} = Y_{i,ki}\} \cup \{i \in I: Y_{i,ji} \cap Y_{i,ki} = 0\}$, so the desired conclusion is clear. This finishes the proof of 3.1.91.

LEMMA 3.1.92. *Assume* $\alpha \geq 2$. *Let* $\mathsf{K} \in \{\mathsf{Gws}_\alpha^{nm}, \mathsf{Gws}_\alpha^{cm}, \mathsf{Gs}_\alpha, \mathsf{Cs}_\alpha, \mathsf{Ws}_\alpha\}$ *and let* $\mathfrak{A} \in {}^I\mathsf{K}$. *Let* F *be an ultrafilter on* I, *and* $U_i \neq 0$ *the base of* $\mathfrak{A}_i$ *for each* $i \in I$.

Then for every non-zero $x \in \mathsf{P}_{i\in I} A_i/\bar{F}$ *there is an* (F,U,α)*-choice function* c *such that, with* $\mathfrak{B} = Rep(c)^*(P_{i\in I}\mathfrak{A}_i/F)$, *we have:* $\mathfrak{B} \in \mathsf{K}$, $Rep(c)(x) \neq 0$, *and* $\mathfrak{B}$ *has base* $\mathsf{P}_{i\in I} U_i/\bar{F}$.

PROOF. The cases $\mathsf{K} = \mathsf{Gws}_\alpha^{cm}$ and $\mathsf{K} = \mathsf{Cs}_\alpha$ are given by 3.1.90 and 3.1.91. Throughout the rest of the proof we assume that V_i is the unit element of $\mathfrak{A}_i$ for each $i \in I$, $X = \mathsf{P}_{i\in I} U_i/\bar{F}$, and Y, J, p, S, W and Q are as in the proof of 3.1.91. Say $x = a/F$.

First we take the case $\mathsf{K} = \mathsf{Gws}_\alpha^{nm}$. We may assume that there is an $r \in \mathsf{P}_{i\in I} J_i$ such that $p_{i,ri} \in a_i$ for all $i \in I$. Now we define an equivalence relation $\equiv$ on $\mathsf{P}_{i\in I} J_i$ by setting, for all $j,k \in \mathsf{P}_{i\in I} J_i$,

$$j \equiv k \text{ iff } \{i \in I: Y_{i,ji} = Y_{i,ki}\} \in F.$$

Let l be a function mapping $\mathsf{P}_{i\in I} J_i/\equiv$ into $\mathsf{P}_{i\in I} J_i$ such that $lz \in z$ for all $z \in \mathsf{P}_{i\in I} J_i/\equiv$ and $l(r/\equiv) = r$. Set $l' = \langle\langle p_{i,(lz)i} : i \in I\rangle : z \in \mathsf{P}_{i\in I} J_i/\equiv\rangle$. Now we claim (continuing the enumeration in the proof of 3.1.91):

(4) If $z,w \in \mathsf{P}_{i\in I} J_i/\equiv$, $z \neq w$ and $\kappa < \alpha$ then $(pj_\kappa \circ l'z)/\bar{F} \neq (pj_\kappa \circ l'w)/\bar{F}$.

In fact, then the set $Z = \{i \in I: Y_{i,(lz)i} \neq Y_{i,(lw)i}\}$ is in F and by normality, $Y_{i,(lz)i} \cap Y_{i,(lw)i} = 0$ for all $i \in Z$. Hence $(l'z)_i\kappa \neq (l'w)_i\kappa$ for all $i \in Z$ and all $\kappa < \alpha$. Thus the conclusion of (4) holds.

By (4), there is an (F,U,α)-choice function c such that:

(5) $c(\kappa, (pj_\kappa \circ l'z)/\bar{F}) = pj_\kappa \circ l'z$ for all $\kappa < \alpha$ and $z \in \mathsf{P}_{i\in I} J_i/\equiv$.

Now let $f = Rep(c)$. By 3.1.91(iv), $\mathfrak{B} \in \mathsf{Gws}_\alpha^{nm}$. Note that if $w = \langle\langle p_{i,ri}\kappa : i \in I\rangle : \kappa < \alpha\rangle$, then for each $\kappa < \alpha$, $w\kappa = pj_\kappa \circ l'(r/\equiv)$ since $l(r/\equiv) = r$. Hence by (5) the hypotheses in the second part of 3.1.90 are fulfilled, and we infer that $fx \neq 0$. It remains only to show that the base of $\mathfrak{B}$ is X. It is obviously a subset of X, so suppose $t \in \mathsf{P}_{i\in I} U_i$. Choose $v \in \mathsf{P}_{i\in I} J_i$ such that $t_i \in Y_{i,vi}$ for all $i \in I$. Set $y = t/\bar{F}$ and let

$$q = \langle\langle (pj_\kappa \circ l'(v/\equiv))/\bar{F} : \kappa < \alpha\rangle_y^C.$$

We claim that $q \in f(\langle V_i : i \in I\rangle/\bar{F})$, hence y is in the base of $\mathfrak{B}$. To prove this, let $l(v/\equiv) = w$. Thus $w \in v/\equiv$, so the set $M = \{i \in I: Y_{i,vi} = Y_{i,wi}\}$ is in F. Also, $c(0,y) \in y$, so the set $N = \{i \in I: c(0,y)_i = t_i\}$ is in F. Now for any $i \in M \cap N$ we have:

$$(c^+q)_i = \langle c(\kappa,q_\kappa)_i:\kappa<\alpha\rangle = \langle (pj_\kappa\circ l'(v/\equiv))_i:\kappa<\alpha\rangle^0_{c(0,y)i} \text{ by } (5)$$
$$= \langle (l'(v/\equiv))_i\kappa:\kappa<\alpha\rangle^0_{li} = \langle p_{i,wi}\kappa:\kappa<\alpha\rangle^0_{li} \in {}^\alpha Y_{i,wi}^{(p(i,wi))} \subseteq V_i .$$

Thus $q\in f(\langle V_i:i\in I\rangle/\bar{F})$, as desired. This completes the case $\mathsf{K}=\mathsf{Gws}^{nm}_\alpha$.

Next, suppose $\mathsf{K}=\mathsf{Gs}_\alpha$. We retain the notation above. Note that if $z\in \mathsf{P}_{i\in I}J_i/\equiv$, $\kappa<\alpha$, and $i\in I$, then $(pj_\kappa\circ l'z)_i=(l'z)_i\kappa=p_{i,(lz)i}\kappa\in Y_{i,(lz)i}$. Thus $pj_\kappa\circ l'z\in \mathsf{P}_{i\in I}Y_{i,(lz)i}$. Therefore $(pj_\kappa\circ l'z)/\bar{F}\in Q_{lz}$. Also observe that if $Q_j\cap Q_k\neq 0$ with $j,k\in \mathsf{P}_{i\in I}J_i$, then $j\equiv k$. Furthermore, $lz=l(lz/\equiv)$ for all $z\in \mathsf{P}_{i\in I}J_i/\equiv$, and of course $l(j/\equiv)\equiv j$. It follows that there is an (F,U,α)–choice function c satisfying not only (5) but also the following condition:

(6) If $\kappa<\alpha$, $j\in \mathsf{P}_{i\in I}J_i$ and $q\in Q_j$, then $c(\kappa,q)\in \mathsf{P}_{i\in I}Y_{i,l(j/\equiv)i}$.

Thus by the first part of this proof it remains only to show that $\mathfrak{B}$ is a Gs_α. Now we claim

(7) If $j,k\in \mathsf{P}_{i\in I}J_i$ and $j\not\equiv k$, then $Q_j\cap Q_k=0$.

In fact, assume $y\in \mathsf{P}_{i\in I}Y_{i,ji}$. Let $H=\{i\in I: y_i\in Y_{i,ki}\}$. Clearly $H\subseteq\{i\in I: Y_{i,ji}=Y_{i,ki}\}\notin F$, so $H\notin F$ and hence $y\notin Q_k$. So (7) holds. Hence by (1) and (3) it suffices to show that ${}^\alpha Q_j\subseteq f(\langle V_i:i\in I\rangle/\bar{F})$ for each $j\in \mathsf{P}_{i\in I}J_i$. Let $q\in {}^\alpha Q_j$. Then for any $\kappa<\alpha$ and $i\in I$ we have by (6) $(c^+q)_i\kappa=c(\kappa,q_\kappa)_i\in Y_{i,l(j/\equiv)i}$. So $(c^+q)_i\in {}^\alpha Y_{i,l(j/\equiv)i}\subseteq V_i$. Hence $q\in f(\langle V_i:i\in I\rangle\bar{F})$, as desired. This finishes the case $\mathsf{K}=\mathsf{Gs}_\alpha$.

Only the case $\mathsf{K}=\mathsf{Ws}_\alpha$ remains. Since $\mathsf{Ws}_\alpha\subseteq\mathsf{Gws}^{cm}_\alpha$, by 3.1.91(iii) the conclusion that the base of $\mathfrak{B}$ should be $\mathsf{P}_{i\in I}U_i/\bar{F}$ will follow from the other conditions. Let $N=\{i\in I:|U_i|>1\}$. Let s and w be as in the last part of 3.1.90. Then it is easily seen that there is an (F,U,α)–choice function c satisfying the following two conditions:

(8) $c(\kappa,w\kappa/\bar{F})=w\kappa$ for all $\kappa<\alpha$;
(9) $c(\kappa,y)_i\neq w_\kappa i$ whenever $\kappa<\alpha$, $y\in X$, $y\neq w_\kappa/\bar{F}$, and $i\in N$.

Again let $f=Rep(c)$. By 3.1.90, $fx\neq 0$. Now let $q=\langle w_\kappa/\bar{F}:\kappa<\alpha\rangle$. We shall show that $f(V/\bar{F})={}^\alpha X^{(q)}$. Note that $q\in {}^\alpha X$.

If $N\notin F$, then $|X|=1$ and hence $0\neq f(V/\bar{F})\subseteq {}^\alpha X=\{q\}={}^\alpha X^{(q)}$, so $f(V/\bar{F})={}^\alpha X^{(q)}$.

Assume that $N\in F$. Let $p\in {}^\alpha X$, $i\in N$, and $\kappa<\alpha$. Then $(c^+p)_i\kappa=s_i\kappa$ iff $p_\kappa=w_\kappa/\bar{F}$, and hence $(c^+p)_i\in V_i$ iff $p\in {}^\alpha X^{(q)}$. Since $N\in F$ it follows that $p\in f(V/\bar{F})$ iff $p\in {}^\alpha X^{(q)}$, as desired. This finishes the proof of 3.1.92.

Our next version of 3.1.90 concerns regularity.

LEMMA 3.1.93. *Assume the hypotheses of* 3.1.90. *Let* $f=Rep(c)$. *Also suppose* $a\in \mathsf{P}_{i\in I}A_i$ *and for each* $i\in I$, a_i *is regular in* $\mathfrak{A}_i$ *and* $\Delta a_i\subseteq\Delta f(a/\bar{F})$. *Then* $f(a/\bar{F})$ *is regular.*

PROOF. We assume all the hypotheses. Let $\Gamma=1\cup\Delta f(a/\bar{F})$ and assume that $p\in f(a/\bar{F})$, $q\in f(V/\bar{F})$, and $\Gamma\upharpoonright p\subseteq q$. We want to show that $q\in f(a/\bar{F})$. Let

$$H = \{i \in I : (c^+p)_i \in a_i \text{ and } (c^+q)_i \in V_i\};$$

thus $H \in F$. By the definition of c^+, for all $i \in I$ we have $\Gamma 1(c^+p)_i \subseteq (c^+q)_i$. Thus for any $i \in H$, the regularity of a_i implies that $(c^+q)_i \in a_i$. Since $H \in F$, it follows that $q \in f(a/\bar{F})$, as desired.

Two more extensions of 3.1.90 give algebraic versions of the upward Lowenheim, Skolem theorem: they are due to Andréka and Németi (see [HMT81] 7.12 and 7.23), generalizing results of Henkin, Monk [74'].

LEMMA 3.1.94. *Let $\mathfrak{A}$ be a* Crs_α *with base U and unit element V, and let F be an ultrafilter on a set I. Let c be an $(F,\langle U:i \in I\rangle,\alpha)$–choice function, and let $f = Rep(F,\langle U:i \in I\rangle,\alpha,\langle \mathfrak{A}:i \in I\rangle,c)$. Define $\delta \in {}^A({}^I A/\bar{F})$ and $\varepsilon \in {}^U({}^I U/\bar{F})$ by*

$$\delta = \langle\langle a:i \in I\rangle/\bar{F} : a \in A\rangle,$$
$$\varepsilon = \langle\langle u:i \in I\rangle/\bar{F} : u \in U\rangle.$$

We assume also that for every $r \in V$ there is a $Z \in F$ such that, for all $\kappa<\alpha$, $c(\kappa,\varepsilon(r\kappa)) \supseteq \langle r\kappa : i \in Z\rangle$. Finally, let $g = f \circ \delta$ and $\mathfrak{B} = g^*\mathfrak{A}$. Then:

(i) *ε is one–one, and $\bar{\varepsilon}$ is a base–isomorphism from $\mathfrak{A}$ onto some* Crs_α.

(ii) *$rl^{\mathfrak{B}}_{\bar{\varepsilon}V}$ is an ext–isomorphism of $\mathfrak{B}$ onto $\bar{\varepsilon}^*\mathfrak{A}$.*

(iii) *δ is an isomorphism of $\mathfrak{A}$ into ${}^I\mathfrak{A}/\bar{F}$.*

(iv) *g is a sub–base–isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$.*

(v) *$\bar{\varepsilon} = rl^{\mathfrak{B}}_{\bar{\varepsilon}V} \circ g$.*

PROOF. (iii) is a special case of 0.3.67, and (i) is immediate from 3.1.36 and 3.1.37. By 3.1.90, f is a homomorphism. Hence g is a homomorphism. Thus it is now clear that (ii) and (iv) follow from (v). To establish (v), let $a \in A$; we want to show that

(1) $ga \cap \bar{\varepsilon}V = \{\varepsilon \circ s : s \in a\}$.

First suppose $q \in ga \cap \bar{\varepsilon}V$. Since $q \in \bar{\varepsilon}V$, there is an $s \in V$ such that $q = \varepsilon \circ s$. By the hypothesis of the lemma choose $Z \in F$ such that $c(\kappa,\varepsilon(s\kappa)) \supseteq \langle s\kappa : i \in Z\rangle$ for all $\kappa<\alpha$. Let $H = \{i \in I : (c^+q)_i \in a\}$; thus $H \in F$. Since $H \cap Z \in F$, we can choose $i \in H \cap Z$. Then

$$s = \langle s\kappa : \kappa<\alpha\rangle = \langle c(\kappa,\varepsilon(s\kappa))_i : \kappa<\alpha\rangle = \langle c(\kappa,q\kappa)_i : \kappa<\alpha\rangle = (c^+q)_i \in a.$$

Thus $q = \varepsilon \circ s$ implies that q is in the right side of (1).

Second suppose $q = \varepsilon \circ s$ with $s \in a$. Since $a \subseteq V$, we have $q \in \bar{\varepsilon}V$. Again by the hypothesis of the lemma let $Z \in F$ be such that $c(\kappa,\varepsilon s\kappa) \supseteq \langle s\kappa : i \in Z\rangle$ for all $\kappa<\alpha$. Then $(c^+q)_i = s \in a$ for all $i \in Z$, so $q \in ga \cap \bar{\varepsilon}V$.

LEMMA 3.1.95. *Assume $\alpha>0$. Let $\mathfrak{A}$ be a* Gws_α *with base U and unit element V, and let F be an $|\alpha|$–regular ultrafilter on some set I. Then there is an $(F,\langle U:i \in I\rangle,\alpha)$–choice function c such that, if we let f,δ,ε,g and $\mathfrak{B}$ be as in 3.1.94 then in addition to (i)–(v) there we have the following conditions:*

(vi) *g is a strong sub–base–isomorphism;*
(vii) $\bar{\varepsilon} V = {}^{\alpha}(\varepsilon^* U) \cap f\delta V$;
(viii) *the base of* $\mathfrak{B}$ *is* ${}^I U/\bar{F}$;
(ix) $\mathfrak{A} \in \mathsf{K}$ *implies* $\mathfrak{B} \in \mathsf{K}$ *for all* $\mathsf{K} \in \{\mathsf{Ws}_\alpha, \mathsf{Cs}_\alpha, \mathsf{Gs}_\alpha, \mathsf{Gws}_\alpha, \mathsf{Gws}_\alpha^{reg}\}$.

PROOF. Assume the hypothesis. Say $V = \bigcup_{j \in J} {}^{\alpha} Y_j^{(pj)}$, where $0 = {}^{\alpha} Y_j^{(pj)} \cap {}^{\alpha} Y_k^{(pk)}$ for distinct $j,k \in J$. Let $Y = \langle Y_j : j \in J \rangle$ and $X = {}^I U/\bar{F}$ Let $R = \{\langle j,k \rangle \in {}^2 J : Y_j = Y_k\}$. Thus R is an equivalence relation on J. Let K be a subset of J having exactly one element in common with each equivalence class under R. Let $L \subseteq {}^I K$ have exactly one element in common with each equivalence class under $\bar{F}^{\langle (K : \iota \in I) \rangle}$, with $\langle j : i \in I \rangle \in L$ for each $j \in K$. Now we define functions $w \in {}^X({}^I U)$ and $v \in {}^X({}^I J)$. Let $y \in X$. Choose $k_y = k \in {}^I K$ such that $y \cap \mathsf{P}_{\iota \in I} Y_{k\iota} \neq 0$, and k constant if $y = \varepsilon u$ for some $u \in U$. Let vy be the unique element of $L \cap (k/\bar{F})$. Thus $y \cap \mathsf{P}_{\iota \in I} Y_{(vy)\iota} \neq 0$, so we can pick $wy \in y \cap \mathsf{P}_{\iota \in I} Y_{(vy)\iota}$ with $wy = \langle u : i \in I \rangle$ if $y = \varepsilon u$. Now w and v have the following properties:

(1) $wy \in y$ for all $y \in X$;
(2) $w\varepsilon u = \langle u : i \in I \rangle$ for all $u \in U$;
(3) for all $y \in X$ and $i \in I$ we have $(wy)i \in Y_{(vy)i}$;
(4) if $k,q \in Rgv$ and $\{i \in I : Y_{k\iota} = Y_{q\iota}\} \in F$, then $k = q$.

Since F is $|\alpha|$–regular, choose $h \in {}^I\{\Gamma \subseteq \alpha : |\Gamma| < \omega\}$ such that $\{i \in I : \kappa \in hi\} \in F$ for all $\kappa < \alpha$. Now let c be the $(F, \langle U : i \in I \rangle, \alpha)$–choice function such that for all $\kappa < \alpha$, $y \in X$, and $i \in I$,

$$c(\kappa, y)_i = \begin{cases} p_{(vy)i}\kappa & \text{if } \kappa \notin hi \text{ and } y \notin \varepsilon^* U, \\ (wy)i & \text{otherwise.} \end{cases}$$

Note that $c(\kappa, y) \in \mathsf{P}_{i \in I} Y_{(vy)i}$ for all $\kappa < \alpha$ and $y \in X$. Let $f, \delta, \varepsilon, g$ and $\mathfrak{B}$ be as in 3.1.94. Then (vii) is easily checked, using (2), and (vi) clearly follows from (vii). To prove (viii), let Z be the base of $\mathfrak{B}$; we are to show that $Z = X$, and it is obvious that $Z \subseteq X$. Suppose $y \in \varepsilon^* U$; say $y = \varepsilon u$ with $u \in U$; in fact say $u \in Y_j$ with $j \in J$. Now let $q = \langle \varepsilon p_j \kappa : \kappa < \alpha \rangle_y^0$ Thus $q \in {}^{\alpha} X$, and for any $i \in I$ we have $(c^+ q)_\iota = (p_j)_u^0 \in {}^{\alpha} Y_j^{(pj)} \subseteq V$ using (2) and the definition of c. It follows that $q \in f\delta V$, and hence $y \in Z$. Now let $y \in X \sim \varepsilon^* U$. Let $q = \langle y : \kappa < \alpha \rangle$. Then for any $i \in I$,

$$(c^+ q)_\iota = \langle c(\kappa, y)_\iota : \kappa < \alpha \rangle = (\alpha \sim hi) \upharpoonright p_{(vy)h} u \langle (wy)i : \kappa \in hi \rangle \in {}^{\alpha} Y_{(vy)h}^{(p(vy)i)} \subseteq V.$$

Hence again $q \in f\delta V$ and $y \in Z$. So (viii) holds.

Now we turn to the parts of (ix). The cases $\mathsf{K} = \mathsf{Gws}_\alpha$ and $\mathsf{K} = \mathsf{Cs}_\alpha$ are taken care of by 3.1.91. If $\mathfrak{A}$ is regular, then so is $\mathfrak{B}$ by 3.1.93 since g is an isomorphism. Next suppose $\mathsf{K} = \mathsf{Gs}_\alpha$. Thus for all $j,k \in J$ we have $Y_j = Y_k$ or $Y_j \cap Y_k = 0$. For each $r \in Rgv$ we set $Q_r = \mathsf{P}_{\iota \in I} Y_{r\iota}/\bar{F}^U$. Now

(5) if $r, r' \in Rgv$ and $r \neq r'$, then $Q_r \cap Q_{r'} = 0$.

For, suppose $y \in \mathsf{P}_{i \in I} Y_{r\iota}$, $z \in \mathsf{P}_{\iota \in I} Y_{r'\iota}$, and $\{i \in I : y_i = z_i\} \in F$. Then $\{i \in I : Y_{ri} = Y_{r'\iota}\} \in F$, so by (4) $r = r'$.

(6) $gV \subseteq \bigcup_{r \in Rgv} {}^{\alpha}Q_r$.

For, let $q \in gV$. Thus the set $M = \{i \in I: (c^+q)_i \in V\}$ is in F. We claim that $q \in {}^{\alpha}Q_{vq0}$. For each $i \in M$ choose $ji \in J$ so that $(c^+q)_i \in {}^{\alpha}Y_{ji}$. Now for any $i \in M$ and $\kappa < \alpha$ we have $(c^+q)_i\kappa \in Y_{ji}$ and also $(c^+q)_i\kappa = c(\kappa, q\kappa)_i \in Y_{(vq\kappa)i}$ by the note following the definition of c, so $Y_{ji} = Y_{(vq\kappa)i}$. So for any $i \in M$ and $\kappa < \alpha$ we have $c(\kappa, q\kappa)_i \in Y_{(vq\kappa)i} = Y_{ji} = Y_{(vq0)i}$. Thus $q\kappa \in Q_{vq0}$ for any $\kappa < \alpha$, as desired.

(7) If $r \in Rgv$, $\kappa < \alpha$, *and* $y \in Q_r$, then $c(\kappa, y) \in \mathsf{P}_{i \in I} Y_{ri}$.

For, choose $z \in y \cap \mathsf{P}_{i \in I} Y_{ri}$. By the remark after the definition of c we also have $c(\kappa, y) \in y \cap \mathsf{P}_{i \in I} Y_{(vy)i}$. Thus the set $M = \{i \in I: z_i = c(\kappa, y)_i\}$ is in F. Since $M \subseteq \{i \in I: Y_{ri} = Y_{(vy)i}\}$, we infer from (4) that $vy = r$, so (7) follows.

(8) For all $r \in Rgv$ we have ${}^{\alpha}Q_r \subseteq gV$.

In fact, let $q \in {}^{\alpha}Q_r$. By (7) we have, for all $\kappa < \alpha$, $c(\kappa, q\kappa) \in \mathsf{P}_{i \in I} Y_{ri}$. Hence for all $i \in I$, $(c^+q)_i = \langle c(\kappa, q\kappa)_i : \kappa < \alpha \rangle \in {}^{\alpha}Y_{ri} \subseteq V$. Hence $q \in gV$.

By (5), (6), (8) we have $\mathfrak{B} \in \mathsf{Gs}_\alpha$.

In the case $\mathsf{K} = \mathsf{Ws}_\alpha$ we redefine c. Let $\mathfrak{A} \in \mathsf{Ws}_\alpha$, say $V = {}^{\alpha}U^{(p)}$. We may assume that $|U| > 1$. For each $u \in U$ and $\kappa < \alpha$ let $c(\kappa, \varepsilon u) = \langle u : i \in I \rangle$. Now suppose $y \in X \sim \varepsilon^* U$ and $\kappa < \alpha$. Choose $k_y \in y$ and let $Z_y = \{i \in I: k_y i \neq p\kappa\}$. Thus $Z_y \in F$ since $y \neq \varepsilon p\kappa$. Let $c(\kappa, y) = Z_y \upharpoonright k_y \mathbf{u} \langle u_y : i \in \alpha \sim Z_y \rangle$, where $u_y \in U \sim \{p\kappa\}$. Thus c is an $(F, \langle U: i \in I \rangle, \alpha)$–choice function, with the additional property

(9) for all $\kappa < \alpha$ and all $y \in X \sim \{\varepsilon p\kappa\}$ we have $p\kappa \notin Rgc(\kappa, y)$.

Now to establish (vi)–(viii) as well as $\mathfrak{B} \in \mathsf{Ws}_\alpha$ it suffices to show that $gV = {}^{\alpha}X^{(\varepsilon^* p)}$. First suppose $q \in {}^{\alpha}X^{(\varepsilon^* p)}$. Let $i \in I$. Then $(c^+q)_i \in {}^{\alpha}U$, obviously. Let $\Gamma = \{\kappa < \alpha: q\kappa \neq \varepsilon p\kappa\}$. Then Γ is finite, and for $\kappa \in \alpha \sim \Gamma$ we have $(c^+q)_i\kappa = c(\kappa, q\kappa)_i = c(\kappa, \varepsilon p\kappa)_i = p\kappa$. Thus $(c^+q)_i \in {}^{\alpha}U^{(p)} = V$. Hence $q \in gV$.

Conversely, suppose $q \in {}^{\alpha}X \sim {}^{\alpha}X^{(\varepsilon^* p)}$; we show that $q \notin gV$. Let $\Gamma = \{\kappa < \alpha: q\kappa \neq \varepsilon p\kappa\}$; thus Γ is infinite. For any $\kappa \in \Gamma$ and $i \in I$ we have $c(\kappa, q\kappa)_i \neq p\kappa$ by (9). Therefore $(c^+q)_i \notin {}^{\alpha}U^{(p)} = V$ for all $i \in I$, and consequently $q \notin gV$.

This completes the proof of 3.1.95.

REMARKS 3.1.96. Lemmas 3.1.90–3.1.93 are algebraic forms of the Łoś lemma for ultraproducts. In [AN81] it is shown that various hypotheses in 3.1.90–3.1.95 are essential. Andréka and Németi also showed that these lemmas do not generalize to arbitrary reduced products.

Now we use the above lemmas to prove various closure properties of our classes of set algebras. First we consider closure under ultraproducts.

THEOREM 3.1.97. $\mathsf{UpK} = \mathsf{IK}$ *for* $\mathsf{K} \in \{\mathsf{Gws}_\alpha, \mathsf{Gs}_\alpha, \mathsf{Gws}_\alpha^{nm}\}$, *if* $\alpha \geq 2$.

PROOF. By Lemmas 3.1.91 and 3.1.92 we have $\mathsf{UpK} \subseteq \mathsf{SPK}$ for each class K considered. We have $\mathsf{IK} \subseteq \mathsf{UpK}$ in general, by 0.3.69, while $\mathsf{SPK} \subseteq \mathsf{IK}$ for $\mathsf{K} \in \{\mathsf{Gws}_\alpha, \mathsf{Gs}_\alpha\}$ by 3.1.77 and 3.1.79. One can show that $\mathsf{SPGws}_\alpha^{nm} \subseteq \mathsf{IGws}_\alpha^{nm}$ by using 3.1.76.

THEOREM 3.1.98. $\mathsf{UpCs}_\alpha = \mathsf{ICs}_\alpha$ *for* $\alpha < \omega$.

PROOF. This is true by 3.1.91(v) and 3.1.70(iv), since simplicity is an elementary property for $\alpha < \omega$ (by 2.3.14).

THEOREM 3.1.99. *For* $\alpha \leqq 1$ *we have* $\mathsf{UpGs}_\alpha = \mathsf{IGs}_\alpha$.

PROOF. By 3.1.91(i), 3.1.17, and simplicity again.

REMARK 3.1.100. For $\alpha \geqq \omega$, ICs_α and IWs_α are not closed under ultraproducts (but see 3.1.109 below); this was first noticed by Monk [65]. In fact, let $\mathfrak{A}$ be *any* Cs_α whose base U satisfies $2 \leqq |U| < \omega$. Let $I = 2^a$, where $a = 2^{|\alpha|}$, and choose an ultrafilter F over I such that $|{}^I A/\bar{F}| = 2^{|I|}$ (see, e.g., Chang, Keisler [73**]). Now $\mathfrak{A}$ has characteristic $|U|$, and hence so does ${}^I\mathfrak{A}/\bar{F}$. But any Cs_α of characteristic $|U|$ has base of cardinality $|U|$, and hence has at most $|I|$ elements. Thus ${}^I\mathfrak{A}/\bar{F} \notin \mathsf{ICs}_\alpha$. The same construction works for Ws_α.

We can use the above results to show that some of the classes of isomorphs of our set algebras are closed under directed unions; see [HMT81] 7.10 and 7.11, and Andréka, Németi [75'].

THEOREM 3.1.101. $\bigcup \mathsf{L} \in \mathsf{K}$ *whenever* L *is a non-empty subset of* K *directed by* $\subseteq$, *for* $\mathsf{K} = \mathsf{ICs}_\alpha$ *if* $\alpha < \omega$, *and for arbitrary* α, *for* $\mathsf{K} \in \{\mathsf{IGws}_\alpha, \mathsf{IGs}_\alpha, \mathsf{IGs}_\alpha^{nm}, \mathsf{IWs}_\alpha \cap \mathsf{Lf}_\alpha, \mathsf{ICs}_\alpha^{reg} \cap \mathsf{Lf}_\alpha\}$.

PROOF. This is immediate from 0.3.71 and 3.1.97–3.1.99 since $\mathsf{SK} = \mathsf{K}$, for all but the last two choices of K. Now we take the case $\mathsf{K} = \mathsf{IWs}_\alpha \cap \mathsf{Lf}_\alpha$. If L is as indicated, then $\bigcup \mathsf{L} \in \mathsf{IGws}_\alpha$ by what has already been said, while $\bigcup \mathsf{L}$ is simple by 3.1.70(ii) and 0.2.35(ii). Hence $\bigcup \mathsf{L} \in \mathsf{IWs}_\alpha \cap \mathsf{Lf}_\alpha$ by 3.1.79 and 0.3.58(i).

Now assume $\mathsf{K} = \mathsf{ICs}_\alpha^{reg} \cap \mathsf{Lf}_\alpha$. By the above we may assume that $\alpha \geqq \omega$. Also we may assume that $|B| > 1$ for each $\mathfrak{B} \in \mathsf{L}$. For each $\mathfrak{B} \in \mathsf{L}$ let $f_{\mathfrak{B}}$ be an isomorphism onto a regular Cs_α $\mathfrak{A}_{\mathfrak{B}}$ with non-empty base $U_{\mathfrak{B}}$; further, let $M_{\mathfrak{B}} = \{\mathfrak{C} \in \mathsf{L} : \mathfrak{B} \subseteq \mathfrak{C}\}$. Let F be an ultrafilter on L such that $M_{\mathfrak{B}} \in F$ for each $\mathfrak{B} \in L$. There is an isomorphism g of $\mathsf{P}_{\mathfrak{B} \in L} \mathfrak{B}/\bar{F}$ onto $\mathsf{P}_{\mathfrak{B} \in L} \mathfrak{A}_{\mathfrak{B}}/\bar{F}$ such that $g(b/\bar{F}) = \langle f_{\mathfrak{B}} b_{\mathfrak{B}} : \mathfrak{B} \in \mathsf{L} \rangle / \bar{F}$ for all $b \in \mathsf{P}_{\mathfrak{B} \in L} B$. Let c be an (F, U, α)-choice function, where $U = \langle U_{\mathfrak{B}} : \mathfrak{B} \in \mathsf{L} \rangle$, and let $f = Rep(c)$. By Lemmas 3.1.90 and 3.1.91(v), f is a homomorphism from $\mathsf{P}_{\mathfrak{B} \in L} \mathfrak{A}_{\mathfrak{B}}/\bar{F}$ onto a Cs_α. Now for each $b \in \bigcup_{\mathfrak{B} \in L} B$ let

$$h'b = \langle b : b \in B, \mathfrak{B} \in \mathsf{L} \rangle \cup \langle 0^{(\mathfrak{B})} : b \notin B, \mathfrak{B} \in L \rangle,$$

and let $hb = h'b/\bar{F}$. Then h is an isomorphism of $\bigcup \mathsf{L}$ into $\mathsf{P}_{\mathfrak{B} \in L} \mathfrak{B}/\bar{F}$ (see the proof of

0.3.71.) Now let $\mathfrak{C} = (f \circ g \circ h)^* \bigcup \mathsf{L}$. We claim that $f \circ g \circ h$ is an isomorphism, and $\mathfrak{C} \in \mathsf{Cs}_\alpha^{reg} \cap \mathsf{Lf}_\alpha$.

First note that each member of L is simple, by 3.1.70(i); hence $\bigcup \mathsf{L}$ is simple, by 0.2.35(ii). Thus $f \circ g \circ h$ is an isomorphism. By 2.1.13, $\mathfrak{C} \in \mathsf{Lf}_\alpha$. To show that $\mathfrak{C}$ is regular we apply 3.1.93. Let $b \in \bigcup_{\mathfrak{B} \in L} B$. Then for each $\mathfrak{B} \in \mathsf{L}$, $f_{\mathfrak{B}}(h'b)_{\mathfrak{B}}$ is regular in $\mathfrak{A}_{\mathfrak{B}}$, and

$$\Delta f_{\mathfrak{B}}(h'b)_{\mathfrak{B}} = \Delta(h'b)_{\mathfrak{B}} \subseteq \Delta b = \Delta(fghb)$$

since $f \circ g \circ h$ is an isomorphism. Therefore by 3.1.93, $fghb$ is regular, as desired. This finishes the proof of 3.1.101.

In 3.1.69 we discussed possible extensions of 3.1.101.

Now we turn to four major facts about our set algebras: $\mathsf{Ws}_\alpha \subseteq \mathsf{ICs}_\alpha^{reg}$, $\mathsf{HGws}_\alpha \subseteq \mathsf{IGs}_\alpha$, $\mathsf{ICs}_\alpha = \mathsf{IGws}_\alpha^{cm}$ for $\alpha \geqq \omega$, and $\mathsf{I}_\infty\mathsf{Gws}_\alpha = \mathsf{I}_\infty\mathsf{Cs}_\alpha$ for $\alpha \geqq \omega$. These results have many important corollaries which we state and discuss afterwards. We begin with $\mathsf{Ws}_\alpha \subseteq \mathsf{ICs}_\alpha^{reg}$, a result of Andréka and Németi, generalizing the result $\mathsf{Ws}_\alpha \subseteq \mathsf{ICs}_\alpha$ of Henkin, Monk [74']; see [HMT81] 7.13.

THEOREM 3.1.102. $\mathsf{Ws}_\alpha \subseteq \mathsf{ICs}_\alpha^{reg}$.

PROOF. Let $\mathfrak{A}$ be a Ws_α with unit element $V = {}^\alpha U^{(p)}$. We shall modify the following proof in order to prove 3.1.105 below; for that reason we keep α arbitrary (although for $\alpha < \omega$ Theorem 3.1.102 is trivial), and do things a little more generally than is needed here. Let $|I| \geqq |\alpha| \cup \omega$, and let F be a $|I|$-regular ultrafilter over I (for the notion of κ-regular ultrafilter see Chang, Keisler [73**]). Then, as is easily seen, there is a function $h \in {}^I\{\Gamma \subseteq \alpha : |\Gamma| < \omega\}$ such that $\{i \in I : \kappa \in hi\} \in F$ for all $\kappa < \alpha$. Now let δ and ε be as in 3.1.94 and set $X = {}^I U/\bar{F}$. Let c be an $(F, \langle U : i \in I \rangle, \alpha)$-choice function satisfying the following condition:

(1) For all $\kappa < \alpha$, $i \in I$, and all $y \in X$, if $\kappa \notin hi$ then $c(\kappa, y)_i = p\kappa$; if $\kappa \in hi$ and $y = \varepsilon u$ with $u \in U$ then $c(\kappa, y)_i = u$.

Let $f = Rep(c)$. We shall show that $f \circ \delta$ is the desired isomorphism. By 3.1.90 $f \circ \delta$ is a homomorphism onto a Crs_α $\mathfrak{B}$. Now we show that $f\delta V = {}^\alpha X$, so that $\mathfrak{B}$ is a Cs_α. Since $f\delta V \subseteq {}^\alpha X$ trivially, we show the other inclusion. Let $q \in {}^\alpha X$. It suffices to show that $(c^+q)_i \in V$ for *all* $i \in I$. So, let $i \in I$. Note that $(c^+q)_i \in {}^\alpha U$. *If* $\kappa \notin hi$, then by (1) we have $(c^+q)_i\kappa = c(\kappa, q\kappa)_i = p\kappa$. Since hi is finite, it follows that $(c^+q)_i \in {}^\alpha U^{(p)}$, as desired.

To show that $f \circ \delta$ is an isomorphism we apply 3.1.94; we have only one hypothesis of 3.1.94 left to check. Let $r \in V$. Then there is a finite $\Gamma \subseteq \alpha$ such that $(\alpha \sim \Gamma) \upharpoonright r = (\alpha \sim \Gamma) \upharpoonright p$. Let $Z = \{i \in I : \Gamma \subseteq hi\}$. By the choice of h we have $Z \in F$. Let $\kappa < \alpha$ and $i \in Z$; we show that $c(\kappa, \varepsilon r\kappa)_i = r\kappa$ (as desired). If $\kappa \in \Gamma$, then $\kappa \in hi$ and so $c(\kappa, \varepsilon r\kappa)_i = r\kappa$ by (1). If $\kappa \notin \Gamma$ then $r\kappa = p\kappa$ and $c(\kappa, \varepsilon r\kappa)_i = r\kappa$ by either clause of (1). Thus the hypotheses of 3.1.94 hold, and hence $f \circ \delta$ is an isomorphism.

It remains to show that $\mathfrak{B}$ is regular; but since $f \circ \delta$ is an isomorphism, this is immediate from 3.1.93 and 3.1.26.

Our next result, $\mathsf{HGws}_\alpha \subseteq \mathsf{IGs}_\alpha$, is one of the earliest results about set algebras, and is due to Henkin and Tarski. The original proof was circuitous, however, using the representation theory of section 3.2. The present more direct proof is due to Andréka and Németi; see [HMT81] 7.15.

THEOREM 3.1.103. $\mathsf{HGws}_\alpha \subseteq \mathsf{IGs}_\alpha$.

PROOF. The case $\alpha<2$ is trivial, so assume $\alpha \geqq 2$. Let $\mathfrak{A} \in \mathsf{Gws}_\alpha$, and let L be an ideal of $\mathfrak{A}$. We want to show that $\mathfrak{A}/L \in \mathsf{IGs}_\alpha$.

Case 1. $\alpha<\omega$. For each $z \in A$ let $kz = -\mathsf{c}_{(\alpha)}z$. Note that kz is zero–dimensional and $\mathfrak{Rl}_{kz}\mathfrak{A} \in \mathsf{Gs}_\alpha$. Let $\mathfrak{B} = \langle \mathfrak{Rl}_{kz}\mathfrak{A} : z \in L \rangle$. Let F be an ultrafilter over L such that $\{v \in L : v \geqq z\} \in F$ for all $z \in L$. Now define h mapping A into $\mathsf{P}_{z\in L}B_z/\bar{F}$ by setting, for any $a \in A$,

$$ha = \langle a \cdot kz : z \in L \rangle / \bar{F}.$$

Now $\langle a \cdot kz : a \in A \rangle \in Ho(\mathfrak{A}, \mathfrak{B}_z)$ for each $z \in L$ by 2.3.26, so $\langle \langle a \cdot kz : z \in L \rangle : a \in A \rangle \in Hom(\mathfrak{A}, \mathsf{P}_{z\in L}\mathfrak{B}_z)$ by 0.3.6(ii). Hence by 0.3.61 we infer that $h \in Hom(\mathfrak{A}, \mathsf{P}_{z\in L}\mathfrak{B}_z/\bar{F})$. Now we claim that $h^*\mathfrak{A} \cong \mathfrak{A}/L$; to show this it suffices to show that $h^{-1*}0 = L$. If $a \in L$, then $a \cdot -\mathsf{c}_{(\alpha)}v = 0$ for all $v \geqq a$, so $\{v \in L : v \geqq a\} \subseteq \{v \in L : a \cdot kv = 0\}$; since $\{v \in L : v \geqq a\} \in F$, it follows that $ha = 0$. On the other hand, if $a \in A \sim L$ and $z \in L$ then $a \not\leqq \mathsf{c}_{(\alpha)}z$ and hence $a \cdot kz \neq 0$; thus $ha \neq 0$, as desired.

Thus $h^*\mathfrak{A} \cong \mathfrak{A}/L$, so $\mathfrak{A}/L \in \mathsf{SUpGs}_\alpha$. By 3.1.97, $\mathfrak{A}/L \in \mathsf{IGs}_\alpha$.

Case 2. $\alpha \geqq \omega$. Using 3.1.77 it is enough to take any $a \in A \sim L$ and find a homomorphism h of $\mathfrak{A}$ onto a Cs_α such that $ha \neq 0$ and $h^*L = \{0\}$. Let

$$I = L \times \{\Gamma \subseteq \alpha : |\Gamma| < \omega\},$$

and let F be an ultrafilter on I such that $\{\langle v, \Delta \rangle \in I : v \geqq z,\ \Delta \supseteq \Gamma\} \in F$ for all $\langle z, \Gamma \rangle \in I$. Let $\mathfrak{A}$ have base U and unit element

$$V = \bigcup \{ {}^\alpha Y_j^{(pj)} : j \in J \},$$

where ${}^\alpha Y_i^{(pi)} \cap {}^\alpha Y_j^{(pj)} = 0$ for distinct $i, j \in J$. Let $X = {}^I U/\bar{F}$. Since $a \notin L$, there is a function $r \in {}^I V$ such that $r(z, \Gamma) \in a \cdot -\mathsf{c}_{(\Gamma)}z$ for all $\langle z, \Gamma \rangle \in I$. Then there is a function $j \in {}^I J$ such that $ri \in {}^\alpha Y_{ji}^{(pji)}$ for all $i \in I$. Next we let

$$Q = \{k/\bar{F} : k \in \mathsf{P}_{i\in I} Y_{ji}\},\ \ w = \langle \langle (ri)_\kappa : i \in I \rangle : \kappa < \alpha \rangle.$$

Let c be an $(F, \langle U : i \in I \rangle, \alpha)$–choice function such that the following conditions hold.

(1) If $\kappa<\alpha$ and $y \in Q$, then $c(\kappa, y) \in \mathsf{P}_{i\in I} Y_{ji}$.
(2) If $\kappa<\alpha$, then $c(\kappa, w\kappa/\bar{F}) = w\kappa$.
(3) If $\kappa<\alpha$, $y \in X$, $(z, \Gamma) \in I$, and $\kappa \notin \Gamma$, then $c(\kappa, y)_{z\Gamma} = r(z, \Gamma)_\kappa$.

Let $f = Rep(F, \langle U : i \in I \rangle, \alpha, \langle A : i \in I \rangle, c)$. Also let δ be as in 3.1.94. Then we claim that $f \circ \delta$ is the desired homomorphism. First we note that by the second part of 3.1.90 and by (2), we have $f\delta a \neq 0$. Next we show that $(f \circ \delta)^*L = \{0\}$. Let $z \in L$. Set

$Z=\{\langle v,\Gamma\rangle\in I: v\geqq z\}$; thus $Z\in F$. Let $q\in{}^{\alpha}X$. We show, for any $i\in Z$, that $(c^{+}q)_{\iota}\notin z$ (thus $q\notin f\delta z$, as desired). Let $i\in Z$, say $i=\langle v,\Gamma\rangle$, where $v\geqq z$. By (3) we have $c(\kappa,q\kappa)_{\iota}=(ri)_{\kappa}$ for all $\kappa\in\alpha\sim\Gamma$; thus $(\alpha\sim\Gamma)\upharpoonright(c^{+}q)_{\iota}=(\alpha\sim\Gamma)\upharpoonright ri$. Since $ri\notin c_{(\Gamma)}v$ it follows that $(c^{+}q)_{\iota}\notin c_{(\Gamma)}v$ and, since $v\geqq z$, $(c^{+}q)_{\iota}\notin z$.

It remains only to show that $(f\circ\delta)^{*}\mathfrak{A}$ is a Cs_{α}; we show that $f\delta V={}^{\alpha}Q$.

(4) For any $q\in{}^{\alpha}X$ and any $i\in I$, $(c^{+}q)_{\iota}\in V$ iff $(c^{+}q)_{\iota}\in{}^{\alpha}Y_{j\iota}$.

For, if $q\in{}^{\alpha}X$ and $i\in I$, then with $i=(z,\Gamma)$ we have by (3) $(\alpha\sim\Gamma)\upharpoonright(c^{+}q)_{\iota}=(\alpha\sim\Gamma)\upharpoonright ri$. An easy argument yields (4).

Now suppose $q\in{}^{\alpha}Q$. By (1), $(c^{+}q)_{\iota}\in{}^{\alpha}Y_{j\iota}$ for all $i\in I$, so by (4) $(c^{+}q)_{\iota}\in V$ for all $i\in I$ and hence $q\in f\delta V$. On the other hand, let $q\in f\delta V$. Then the set $Z=\{i\in I:(c^{+}q)_{\iota}\in V\}$ is in F. By (4), $(c^{+}q)_{\iota}\in{}^{\alpha}Y_{j\iota}$ for each $i\in Z$, i.e., $c(\kappa,q\kappa)_{\iota}\in Y_{j\iota}$ for all $\kappa<\alpha$ and all $i\in Z$. Thus $q\in{}^{\alpha}Q$. This completes the proof.

The third of our major applications of ultraproducts, that $\mathsf{ICs}_{\alpha}=\mathsf{IGws}_{\alpha}^{cm}$ for $\alpha\geqq 0$ is due to Andréka and Németi; see [HMT81], proof of 7.17; it generalizes the result, due to Monk, that any direct factor of a Cs_{α} is isomorphic to a Cs_{α} (Corollary 3.1.110 below). Note that $\mathsf{ICs}_{\alpha}=\mathsf{IGws}_{\alpha}^{cm}$ trivially if $\alpha<\omega$.

THEOREM 3.1.104 $\mathsf{ICs}_{\alpha}=\mathsf{IGws}_{\alpha}^{cm}$. *Moreover, any compressed* Gws_{α} *is sub–isomorphic to a* Cs_{α}.

PROOF. Obviously $\mathsf{Cs}_{\alpha}\subseteq\mathsf{Gws}_{\alpha}^{cm}$. Now let $\mathfrak{A}\in\mathsf{Gws}_{\alpha}^{cm}$, $\alpha\geqq\omega$. Say the unit element of $\mathfrak{A}$ is V and its base is U. Let F be a $|\alpha|$–regular ultrafilter on α. Let ε and δ be as in 3.1.94. Let $X={}^{\alpha}U/\bar{F}$. Let c be an $(F,\langle U:\lambda<\alpha\rangle,\alpha)$–choice function such that

(1) for all $\kappa<\alpha$ and all $u\in U$, $c(\kappa,\varepsilon u)=\langle u:\lambda<\alpha\rangle$.

Let $f=Rep(F,\langle U:\lambda<\alpha\rangle,\alpha,\langle A:\lambda<\alpha\rangle,c)$. Then the hypotheses of 3.1.94 are met, and hence $f\circ\delta$ is an isomorphism. Now by 3.1.91(iii) $(f\circ\delta)^{*}\mathfrak{A}\in\mathsf{Gws}_{\alpha}^{cm}$, and the base of $(f\circ\delta)^{*}\mathfrak{A}$ is X. Let $W={}^{\alpha}X\sim f\delta V$. If $W=0$ we are finished, so assume that $W\neq 0$. Note that W itself is the unit element of some Gws_{α}. Now we claim

(2) $\mathfrak{A}\geqslant\mathfrak{B}$ for some Gws_{α} $\mathfrak{B}$ with unit element W.

In fact, applying 2.3.26(ii) to the full Gws_{α} with unit element V and then restricting it to $\mathfrak{A}$ we find that $\mathfrak{A}\geqslant\mathfrak{C}$ for some Ws_{α} $\mathfrak{C}$ with base U. By 3.1.102 $\mathfrak{C}$ is isomorphic to a Cs_{α} $\mathfrak{D}$; looking at the proof of 3.1.102 we see that we may assume that the base of $\mathfrak{D}$ is X. Since $W\subseteq{}^{\alpha}X$ is zero–dimensional, we get a Gws_{α} $\mathfrak{B}$ with unit element W such that $\mathfrak{D}\geqslant\mathfrak{B}$, as desired in (2); let g be a homomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$.

By 0.3.6(ii), $\mathfrak{A}$ is isomorphic to a subalgebra of $\mathfrak{B}\times(f\circ\delta)^{*}\mathfrak{A}$, and by 3.1.76 $\mathfrak{B}\times(f\circ\delta)^{*}\mathfrak{A}$ is isomorphic to a Cs_{α} $\mathfrak{A}'$; in fact, the function $h=\langle gau\,f\delta a:a\in A\rangle$ is an isomorphism from $\mathfrak{A}$ into $\mathfrak{A}'$. It is easily checked that $ha\cap\tilde{\varepsilon}V=\tilde{\varepsilon}a$ for all $a\in A$, using 3.1.94(v), so the second statement of the theorem follows.

For the next result, ${}_\infty\mathsf{Gws}_\alpha \subseteq \mathsf{I}_\infty\mathsf{Cs}_\alpha$ for $\alpha \geqq \omega$, we need a lemma about increasing bases. The lemma is interesting in its own right; we discuss increasing bases further below. For this result, due to Andréka and Németi, see [HMT81] 7.19; it is a sharper form of part of 3.1.102.

THEOREM 3.1.105 *Assume $\alpha \geqq 2$. Let $\mathfrak{A}$ be a Ws_α with infinite base U. Let γ be a cardinal such that $|A|\cdot|U| \leqq \gamma \leqq \Sigma_{\mu<\lambda}\gamma^\mu$, where λ is the least infinite cardinal such that $|\Delta x|<\lambda$ for all $x \in A$. Then $\mathfrak{A}$ is sub–isomorphic to a Cs_α^{reg} with base of power γ.*

PROOF. Let $\mathfrak{A}$ have unit element ${}^\alpha U^{(p)}$. Let $I = max(\alpha,\gamma)$, and let F be a $|I|$–regular ultrafilter on I. Introducing the notation in the proof of 3.1.102 we see from that proof that $f\circ\delta$ is an isomorphism of $\mathfrak{A}$ onto a Cs_α^{reg} $\mathfrak{B}$, and $\bar{\varepsilon}^*\mathfrak{A}$ is sub–isomorphic to $\mathfrak{B}$. Also note that $\mathfrak{B}$ has base X. By Proposition 4.3.7 of Chang, Keisler [73**] we have $|X| = |{}^I U| > \gamma$. Hence we may apply 3.1.45(i)(c) to get a subset W of X such that $\varepsilon^* U \subseteq W$, $|W| = \gamma$, and such that $\mathfrak{B}$ is strongly ext–isomorphic to a Cs_α^{reg} $\mathfrak{C}$ with base W. Thus $\bar{\varepsilon}^*\mathfrak{A}$ is sub–isomorphic to $\mathfrak{C}$. The desired conclusion follows.

Our final major result on set algebras follows; it is due to Henkin and Monk [74'].

THEOREM 3.1.106. For $\alpha \geqq \omega$ *we have* ${}_\infty\mathsf{Gws}_\alpha \subseteq \mathsf{I}_\infty\mathsf{Cs}_\alpha$.

PROOF. Let $\mathfrak{A}$ be a ${}_\infty\mathsf{Gws}_\alpha$, say with unit element $\bigcup_{i\in I} V_i$, where $V_i = {}^\alpha U_i^{(p_i)}$ and U_i is infinite for all $i \in I$, and $V_i \cap V_j = 0$ for distinct $i,j \in I$. Choose $i \in {}^A I$ so that $a \cap V_{ia} \neq 0$ for all $a \in A\sim\{0\}$. For each $a \in A$ let $\mathfrak{B}_a$ be the full Ws_α with unit element V_{ia}, and set $h_a = \mathsf{rl}^{\mathfrak{A}}_{V_{ia}}$ (recall 3.1.75). Thus $h_a \in Hom(\mathfrak{A},\mathfrak{B}_a)$ for all $a \in A$, and $h_a a \neq 0$ if $a \neq 0$. Let $\kappa = \bigcup_{i\in I}|U_i|\mathbf{u}|A|\mathbf{u}|\alpha|$. Now let $\langle W_a : a \in A\rangle$ be such that $2^\kappa = \bigcup_{a\in A} W_a$, $|W_a| = 2^\kappa$ for each $a \in A$, and $W_a \cap W_{a'} = 0$ for distinct $a,a' \in A$. By 3.1.105 $\mathfrak{B}_a$ is isomorphic to a Cs_α $\mathfrak{C}_a$ with base W_a, for each $a \in A$; let $j_a \in Is(\mathfrak{B}_a,\mathfrak{C}_a)$ for each $a \in A$. Choose $z \in {}^A({}^\alpha(2^\kappa))$ such that $z_a \in j_a h_a a$ for each $a \in A\sim\{0\}$. For each $a \in A\sim\{0\}$ let $X_a = {}^\alpha(2^\kappa)^{(za)}$, and let $X_0 = {}^\alpha(2^\kappa)\sim\bigcup\{X_a : a \in A\sim\{0\}\}$. Since $\alpha \leqq \kappa$, it is easy to see that for every $a \in A\sim\{0\}$ there is a base isomorphism $\tilde{f}_a$ of $\mathfrak{C}_a$ onto a Cs_α with base 2^κ such that $Rg z_a \upharpoonright f_a \subseteq Id$. Thus $z_a \in f_a j_a h_a$ for each $a \in A\sim\{0\}$, and $X_a \cap X_b = 0$ if $a \neq b$. For every $a \in A$ let $\mathfrak{D}_a$ be the full Gws_α with unit element X_a and set $g_a = f_a \circ \mathsf{rl}_{X_a} \circ j_a \circ h_a$. Thus $g_a \in Hom(\mathfrak{A},\mathfrak{D}_a)$ for all $a \in A$, and $g_a a \neq 0$ for $a \neq 0$. Hence $\mathfrak{A} \cong |\subseteq \mathsf{P}_{a\in A}\mathfrak{D}_a$. By 3.1.76 we have $\mathsf{P}_{a\in A}\mathfrak{D}_a \in \mathsf{ICs}_\alpha$, as desired.

Now we give corollaries of the above results. First we have various characterizations of the class IGs_α of representable CA_α's:

COROLLARY 3.1.107. $\mathsf{IGs}_\alpha = \mathsf{IGs}_\alpha^{reg} = \mathsf{IGws}_\alpha = \mathsf{IGws}_\alpha^{reg} = \mathsf{SPWs}_\alpha = \mathsf{SPCs}_\alpha = \mathsf{SPCs}_\alpha^{reg}$, *for* $\alpha \geqq 2$.

PROOF. We have

$$\mathsf{I}\mathsf{Gws}_\alpha \subseteq \mathsf{SPWs}_\alpha \subseteq \mathsf{SPCs}_\alpha^{reg} \subseteq \mathsf{IGs}_\alpha^{reg} \subseteq \mathsf{IGws}_\alpha^{reg} \subseteq \mathsf{IGws}_\alpha$$

and

$$\mathsf{SPCs}_\alpha^{reg} \subseteq \mathsf{SPCs}_\alpha \subseteq \mathsf{IGs}_\alpha \subseteq \mathsf{IGws}_\alpha,$$

using 3.1.79, 3.1.102 and 3.1.77.

Another important corollary is that IGs_α is algebraically closed (cf. 3.1.123).

COROLLARY 3.1.108. *For $\alpha \geqq 2$ we have* $\mathsf{IGs}_\alpha = \mathsf{HSPGs}_\alpha = \mathsf{HSPGws}_\alpha = \mathsf{HSPWs}_\alpha = \mathsf{HSPCs}_\alpha = \mathsf{HSPCs}_\alpha^{reg}$.

PROOF. We have

$$\mathsf{IGs}_\alpha \subseteq \mathsf{HSPWs}_\alpha = \mathsf{HSPCs}_\alpha = \mathsf{HSPCs}_\alpha^{reg} \subseteq \mathsf{HSPGs}_\alpha = \mathsf{HGs}_\alpha \subseteq \mathsf{HGws}_\alpha \subseteq \mathsf{IGs}_\alpha$$

using 3.1.107, 3.1.77, and 3.1.103.

Another algebraically closed class is the important class $\mathsf{I}_\infty\mathsf{Cs}_\alpha$, for $\alpha \geqq \omega$ (see also the end of the section).

COROLLARY 3.1.109. *For $\alpha \geqq \omega$ we have* $\mathsf{I}_\infty\mathsf{Gs}_\alpha = \mathsf{HSP}_\infty\mathsf{Gs}_\alpha = \mathsf{HSP}_\infty\mathsf{Gws}_\alpha = \mathsf{I}_\infty\mathsf{Gws}_\alpha = \mathsf{HSP}_\infty\mathsf{Ws}_\alpha = \mathsf{I}_\infty\mathsf{Cs}_\alpha = \mathsf{HSP}_\infty\mathsf{Cs}_\alpha^{reg}$.

PROOF. As for 3.1.108, using also 3.1.106.

Finally, we have a corollary relevant to our discussion of homomorphic images of set algebras (see 3.1.74):

COROLLARY 3.1.110. *Any direct factor of a* Cs_α *is isomorphic to a* Cs_α.

PROOF For $\alpha < \omega$ this is trivial, by 3.1.70(iv). For $\alpha \geqq \omega$ we can use 3.1.104 since for any CA_α $\mathfrak{A}$ a direct factor of $\mathfrak{A}$ has the form $\mathfrak{R}\mathfrak{l}_b\mathfrak{A}$ for some $b \in A$, by 2.4.8, and any zero-dimensional element in a Gws_α^{cm} is the unit element of a Gws_α^{cm}.

We are now in a position to prove the main result on base homomorphisms (due to Andréka and Németi). For a similar theorem see Lemma 7 in Németi [83b']. See 3.1.50 – 3.1.54.

THEOREM 3.1.111. *Let $\alpha \geqq 2$. Let $\mathfrak{A} \in \mathsf{IGs}_\alpha$ and let κ be any cardinal.*

(i) *There is a $\mathfrak{B} \in \mathsf{Gs}_\alpha^{reg} \cap \mathsf{I}\mathfrak{A}$ such that for every $\mathfrak{C} \in_{<\kappa} \mathsf{Gs}_\alpha^{reg}$ and every $h \in Hom(\mathfrak{B},\mathfrak{C})$, h is a base-homomorphism.*

(ii) *We may omit "reg" in (i).*

(iii) *In both (i) and (ii), if $2^{|A|\cup|\alpha|\cup\omega} < \kappa$ we may assume that $\mathfrak{B} \in_{<\kappa} \mathsf{Gs}_\alpha$.*

PROOF (i). We may assume that $2^{|A|\cup|\alpha|\cup\omega} < \kappa$. Then we shall construct $\mathfrak{B} \in_{<\kappa} \mathsf{Gs}_\alpha$ satisfying (i). Clearly we can find $\langle \mathfrak{C}_i : i \in I \rangle$ and $\langle f_i : i \in I \rangle$ satisfying the following

conditions:

(1) $\mathfrak{C}_i \in {}_{<\kappa}\mathsf{Cs}_\alpha^{reg}$ and $f_i \in Ho(\mathfrak{A},\mathfrak{C}_i)$ for each $i \in I$.
(2) If $i,j \in I$, $i \neq j$, and U_i, U_j are the bases of $\mathfrak{C}_i$ and $\mathfrak{C}_j$ respectively, then $U_i \cap U_j = 0$.
(3) If $\mathfrak{D} \in {}_{<\kappa}\mathsf{Cs}_\alpha^{reg}$ and $g \in Ho(\mathfrak{A},\mathfrak{D})$, then there is an $i \in I$ and a base–isomorphism $h \in Is(\mathfrak{C}_i,\mathfrak{D})$ such that $g = h \circ f_i$.

Now for each $a \in A$, let $ga = \bigcup_{i \in I} f_i a$. Then $g \in Ho(\mathfrak{A},\mathfrak{B})$ for some $\mathfrak{B} \in {}_{<\kappa}\mathsf{Gs}_\alpha^{reg}$, as is easily checked. Actually, g is an isomorphism. For, suppose $0 \neq a \in A$. By 3.1.107, there is a Cs_α^{reg} $\mathfrak{D}$ and an $h \in Ho(\mathfrak{A},\mathfrak{D})$ with $ha \neq 0$. By Theorem 3.1.45(i)(c) we may assume that the base of $\mathfrak{D}$ has power $\leq 2^{|A| \cup \omega \cup |\alpha|}$, which we know has size $<\kappa$. Hence by (3) choose $i \in I$ and a base–isomorphism $k \in Is(\mathfrak{C}_i,\mathfrak{D})$ such that $h = k \circ f_i$. Thus $f_i a \neq 0$, as desired.

Now suppose that $\mathfrak{D} \in {}_{<\kappa}\mathsf{Gs}_\alpha^{reg}$ and $h \in Hom(\mathfrak{B},\mathfrak{D})$. Say $\mathfrak{D}$ has unit element $V = \bigcup_{j \in J} {}^\alpha U_j$, where $U_j \cap U_k$ for $j \neq k$. Now fix $j \in J$. Let $k = r l_W^{\mathfrak{D}}$, where $W = {}^\alpha U_j$ (see Definition 3.1.75). Let $l = k \circ h \circ g$ and $\mathfrak{E} = l^* \mathfrak{A}$. Thus $l \in Ho(\mathfrak{A},\mathfrak{E})$, and $\mathfrak{E} \in {}_{<\kappa}\mathsf{Cs}_\alpha^{reg}$. Hence by (3) there is an $ij \in I$ and a base–isomorphism $t \in Is(\mathfrak{C}_{ij},\mathfrak{E})$ such that $l = t \circ f_{ij}$. Say $t = \tilde{u}_j$, where u_j is a one–one function from the base of $\mathfrak{C}_{ij}$ onto U_j. Now set $s = \bigcup_{j \in J} u_j^{-1}$. Thus s maps the base of $\mathfrak{D}$ into the base of $\mathfrak{B}$. We claim that $h = \hat{s}$, finishing the proof. Let $b \in B$ and $v \in V$. Say $ga = b$, $j \in J$, and $v \in {}^\alpha U_j$. Let S be the base of $\mathfrak{C}_{ij}$. Then

$$v \in \hat{s}b \text{ iff } s \circ v \in b \text{ iff } u_j^{-1} \circ v \in b \text{ iff } u_j^{-1} \circ v \in b \cap {}^\alpha S \text{ iff } v \in \tilde{u}_j f_j a \text{ iff } v \in t f_j a \text{ iff } v \in la \text{ iff } v \in hb,$$

as desired.

This finishes the proof of (i). The proof of (ii) is obtained by obvious changes, using 3.1.45(i)(b) instead of 3.1.45(i)(c).

Now we prove some results about increasing a base of a set algebra. The first result is due to Andréka and Németi, in [HMT81] 7.25, with the parts dealing with Ws_α and Cs_α^{reg} due to Henkin, Monk [74'].

THEOREM 3.1.112. *Assume $\alpha \geq 1$. Let $\mathfrak{A}$ be a Gws_α with an infinite base U, and suppose that $|A| \cup |U| \leq \kappa$. Then $\mathfrak{A}$ is strongly sub–isomorphic to a Gws_α $\mathfrak{B}$ with base of cardinality κ.*

Moreover:

(i) *If $\mathfrak{A} \in \mathsf{K} \in \{\mathsf{Ws}_\alpha, \mathsf{Gws}_\alpha, \mathsf{Gs}_\alpha, \mathsf{Gws}_\alpha^{reg}, \mathsf{Gs}_\alpha^{reg}\}$, then $\mathfrak{B} \in \mathsf{K}$;*

(ii) *If $\mathfrak{A} \in \mathsf{Cs}_\alpha$ (resp. $\mathfrak{A} \in \mathsf{Cs}_\alpha^{reg}$) and $\kappa = \kappa^{|\alpha|}$, then $\mathfrak{B} \in \mathsf{Cs}_\alpha$ (resp. $\mathfrak{B} \in \mathsf{Cs}_\alpha^{reg}$).*

PROOF. The parts concerning Ws_α, Gws_α, Cs_α and Cs_α^{reg} are immediate from 3.1.42, 3.1.45, and 3.1.95. For the other parts, dealing with Gs_α, $\mathsf{Gws}_\alpha^{reg}$ and Gs_α^{reg}, we use a direct construction, not involving ultraproducts (which actually works for some other classes). Suppose $\mathfrak{A}$ is a Gws_α, say with base U and unit element $V = \bigcup_{j \in J} {}^\alpha Y_j^{(pj)}$, where ${}^\alpha Y_j^{(pj)} \cap {}^\alpha Y_k^{(pk)} = 0$ for distinct $j,k \in J$. Let $\langle Z_\beta : \beta < \kappa \rangle$ be a system of pairwise disjoint sets, with $|Z_\beta| = |U|$ for all $\beta < \kappa$, and let f_β be a one–one function mapping U

onto Z_β for each $\beta<\kappa$; further, we assume that $Z_0 = U$ and $f_0 = U \upharpoonright Id$. Then for distinct $\langle\beta,j\rangle,\langle\gamma,k\rangle \in \kappa\times J$ we have ${}^\alpha(f_\beta^* Y_j)^{(f_\beta\circ pj)} \cap {}^\alpha(f_\gamma^* Y_k)^{(f_\gamma\circ pk)} = 0$. Let $\mathfrak{B}$ be the full Gws_α with unit element $\bigcup_{\beta<\kappa}\bigcup_{j\in J}{}^\alpha(f_\beta^* Y_j)^{(f_\beta\circ pj)}$. For each $x\in A$ let $gx = \bigcup_{\beta<\kappa}\{y\in {}^\alpha Z_\beta : f_\beta^{-1}\circ y \in x\}$. It is easily checked that g is an isomorphism of $\mathfrak{A}$ onto a subalgebra $\mathfrak{C}$ of $\mathfrak{B}$, and in fact $x = {}^\alpha U \cap gx$ for all $x\in A$, i.e., g is a strong sub-isomorphism of $\mathfrak{A}$ onto $\mathfrak{C}$. In case $\mathfrak{A}\in \mathsf{Gs}_\alpha$, it is clear that $\mathfrak{C}\in \mathsf{Gs}_\alpha$; and also $\mathfrak{A}\in \mathsf{Gws}_\alpha$ implies $\mathfrak{C}\in \mathsf{Gws}_\alpha$.

Finally, suppose $\mathfrak{A}\in \mathsf{Gws}_\alpha^{reg}$. Suppose $x\in A$, $y\in gx$, $z\in gV$, and $(\Delta gx \cup 1)\upharpoonright y \subseteq z$. Say $y\in {}^\alpha Z_\beta$ and $f_\beta^{-1}\circ y\in x$, while $z\in {}^\alpha Z_\gamma$ and $f_\gamma^{-1}\circ z\in V$. Since $y0 = z0$, we have $\beta=\gamma$. Thus since $\Delta x = \Delta gx$, because g is an isomorphism, we get $f_\beta^{-1}\circ z\in x$ by the regularity of x, and hence $z\in gx$.

We wish to strengthen the part of 3.1.112 dealing with Cs_α. To this end we need the following result concerning changing the function p in the unit element ${}^\alpha U^{(p)}$ in a Ws_α. This result of Andréka and Németi, found in [HMT81] 7.27, generalizes a result of Henkin, Monk [74'].

THEOREM 3.1.113. *Suppose $\alpha\geqq\omega$. Let $\mathfrak{A}$ be a Ws_α with unit element ${}^\alpha U^{(p)}$, and let $q\in {}^\alpha U$. Then $\mathfrak{A}$ is homomorphic to a Ws_α with unit element ${}^\alpha Y^{(q)}$ with $U\subseteq Y$, where if $|U|<\omega$ or $|U\sim Rgq| = |U|\geqq|A|$ then one may take $Y = U$.*

PROOF. We may assume that $|U|>1$. Let $I = \{\Gamma\subseteq\alpha : |\Gamma|<\omega\}$, and let F be an ultrafilter on I such that $\{\Gamma\in I : \Delta\subseteq\Gamma\}\in F$ for all $\Delta\in I$. Let $X = {}^I U/\bar{F}$. Choose $k\in {}^\alpha U$ so that $k\kappa \neq p\kappa$ for all $\kappa<\alpha$. Let ε be as in 3.1.94. Then there is an $(F,\langle U : i\in I\rangle,\alpha)$-choice function c such that for all $y\in X$, $\Gamma\in I$, and $\kappa\in\alpha\sim\Gamma$,

$$c(\kappa,y)_\Gamma = \begin{cases} p\kappa & \text{if } y = \varepsilon q\kappa, \\ k\kappa & \text{otherwise.}\end{cases}$$

Now let $f = Rep(F,\langle U : i\in I\rangle,\alpha,\langle A : i\in I\rangle,c)$. Thus by 3.1.90 f is a homomorphism from ${}^I\mathfrak{A}/\bar{F}$ onto some Crs_α $\mathfrak{B}$. Since the function δ of 3.1.94 is an isomorphism of $\mathfrak{A}$ into ${}^I\mathfrak{A}/\bar{F}$, $f\circ\delta\in Hom(\mathfrak{A},\mathfrak{B})$. Now if $h\in {}^\alpha X$ and $\Gamma\in I$, then $\{\kappa\in\alpha\sim\Gamma : (c^+h)_\Gamma\kappa\neq p\kappa\} = \{\kappa\in\alpha\sim\Gamma : c(\kappa,h\kappa)_\Gamma\neq p\kappa\} = \{\kappa\in\alpha\sim\Gamma : h\kappa\neq\varepsilon q\kappa\}$, so $(c^+h)_\Gamma\in {}^\alpha U^{(p)}$ iff $h\in {}^\alpha X^{(\varepsilon\circ q)}$. Thus $\mathfrak{B}$ is a Ws_α with unit element ${}^\alpha X^{(\varepsilon\circ q)}$. Choose $Y\supseteq U$ together with a function l mapping Y one-one onto X such that $\varepsilon\subseteq l$. By 3.1.36, l^{-1} induces an isomorphism t from $\mathfrak{B}$ onto a Ws_α with unit element ${}^\alpha Y^{(q)}$. Thus $t\circ f\circ\delta$ is the desired homomorphism.

If $|U|<\omega$, then $|U| = |X|$ and hence $Y = U$. Assume now $|U\sim Rgq| = |U|\geqq|A|$. By 3.1.45(i)(a) $t^*f^*\delta^*\mathfrak{A}$ is ext-isomorphic to a Ws_α with unit element ${}^\alpha W^{(q)}$ for some W with $|U| = |W|$, $U\subseteq W$. Thus there is a one-one function s mapping W onto U such that $s\circ q = q$. By 3.1.36, s induces an isomorphism u from $t^*f^*\delta^*\mathfrak{A}$ onto a Ws_α with unit element ${}^\alpha U^{(q)}$, and hence $u\circ t\circ f\circ\delta$ is the desired homomorphism for the last part of the theorem.

The following theorem and corollary are also due to Andréka and Németi; see [AN81] 7.12 and 7.12.1.

THEOREM 3.1.114. *Assume $\alpha \geqq \omega$ and $\mathfrak{A}$ is a* Gws_α *with base U and unit element $V = \bigcup_{i \in I} {}^\alpha Y_i^{(pi)}$, $I \neq 0$, where ${}^\alpha Y_i^{(pi)} \cap {}^\alpha Y_j^{(pj)} = 0$ for $i \neq j$, each Y_i infinite. Suppose $\kappa \geqq |A| \cap (2^{|\alpha|} + \Sigma_{i \in I} 2^{|Y_i|})$, $\kappa > \alpha$, $\kappa \geqq |U|$. Then $\mathfrak{A}$ is subisomorphic to some $\mathfrak{C} \in_\kappa \mathsf{Cs}_\alpha$. If $\mathfrak{A}$ is compressed, then we may assume that the sub–isomorphism is strong.*

PROOF. By 3.1.79 and its proof, from 3.1.76 we can write $\mathfrak{A} \cong | \subseteq_d \mathsf{P}_{i \in I} \mathfrak{B}_i$, each $\mathfrak{B}_i$ a Ws_α with unit element ${}^\alpha Y_i^{(pi)}$. We use also the specific form of this isomorphism; $x \in A$ goes to $\langle x \cap {}^\alpha Y_i^{(pi)} : i \in I \rangle$. Note that $|B_i| \leqq |A|$ and $|B_i| \leqq 2^{|\alpha \cup |Y_i|}$. By 3.1.45 and 3.1.95, $\mathfrak{B}_i$ is strongly sub–isomorphic to a Ws_α $\mathfrak{D}_i$ with unit element ${}^\alpha T_i^{(pi)}$, where $\kappa = |T_i| = |T_i \sim Rg(pi)| = |T_i \sim Y_i|$. Set $r_i = rl_Z$ with $Z = {}^\alpha Y_i$; thus $r_i \in Is(\mathfrak{D}_i, \mathfrak{B}_i)$. Now let S be any set such that $U \subseteq S$ and $|S \sim U| = \kappa$. Then for each $i \in I$ there is a one–one function k_i from T_i onto S such that $Y_i \restriction k_i \subseteq Id$. Let $z_i = \tilde{k}_i \circ r_i^{-1}$. Thus z_i is a strong sub–base–isomorphism of $\mathfrak{B}_i$ onto a Ws_α $\mathfrak{C}_i$ with unit element ${}^\alpha S^{(pi)}$. We now set $W = \bigcup_{i \in I} {}^\alpha S^{(pi)}$ and $Q = {}^\alpha S \sim W$. We can write

$$Q = \bigcup_{j \in J} {}^\alpha S^{(qj)}$$

with ${}^\alpha S^{(qj)} \cap {}^\alpha S^{(qk)} = 0$ for distinct $j, k \in J$. Now fix $e \in I$. Note that $\mathfrak{C}_e \in \mathsf{H}\mathfrak{A}$, $\mathfrak{C}_e$ has unit element ${}^\alpha S^{(pe)}$, and $\kappa = |S| \geqq |C_e|$ (since $|C_e| = |B_e|$). Now for any $j \in J$ we have $|S \sim Rg(qj)| = \kappa$ since $\kappa > \alpha$, so by 3.1.113 we have $\mathfrak{C}_e \geqslant \mathfrak{R}_j$ for some Ws_α $\mathfrak{R}_j$ with unit element ${}^\alpha S^{(qj)}$. Let $h_j \in Ho(\mathfrak{A}, \mathfrak{R}_j)$ for every $j \in J$. Finally, for any $x \in A$ set

$$kx = \bigcup_{i \in I} z_i(x \cap {}^\alpha Y_i^{(pi)}) \cup \bigcup_{j \in J} h_j x$$

It is clear that k is an isomorphism from $\mathfrak{A}$ onto a Cs_α with base S. To show that k is a sub–isomorphism it suffices to take any $x \in A$ and show that $V \cap kx = x$. First note that $V \cap kx = V \cap \bigcup_{i \in I} z_i(x \cap {}^\alpha Y_i^{(pi)})$ and, since $z_i(x \cap {}^\alpha Y_i^{(pi)}) \subseteq {}^\alpha S^{(pi)}$ for all $i \in I$, we further get

(1) $V \cap kx = \bigcup_{i \in I} {}^\alpha Y_i \cap z_i(x \cap {}^\alpha Y_i^{(pi)})$

(Here we use the fact that ${}^\alpha Y_j^{(pj)} \cap {}^\alpha S^{(pi)} = 0$ for $i \neq j$, which is easily checked.) Now for any $i \in I$ we have

$${}^\alpha Y_i \cap z_i(x \cap {}^\alpha Y_i^{(pi)}) = {}^\alpha Y_i \cap \tilde{k}_i r_i^{-1}(x \cap {}^\alpha Y_i^{(pi)}) = {}^\alpha Y_i \cap r_i^{-1}(x \cap {}^\alpha Y_i^{(pi)}) = r_i r_i^{-1}(x \cap {}^\alpha Y_i^{(pi)}) = x \cap {}^\alpha Y_i^{(pi)}.$$

Thus from (1) we get $V \cap kx = x$, as desired.

Finally, suppose that $\mathfrak{A}$ is compressed. Thus $Y_i = U$ for all $i \in I$. By the above reasoning, if $x \in A$ and $i \in I$ then $V \cap z_i(x \cap {}^\alpha Y_i^{(pi)}) = {}^\alpha U \cap z_i(x \cap {}^\alpha Y_i^{(pi)}) = x \cap {}^\alpha Y_i^{(pi)}$; hence ${}^\alpha U \cap kx = {}^\alpha U \cap \bigcup_{i \in I} z_i(x \cap {}^\alpha Y_i^{(pi)}) = x$; so k is strong. This completes the proof of 3.1.114.

COROLLARY 3.1.115. *Suppose $\alpha \geqq \omega$, and κ and λ are infinite cardinals. Then:*
(i) *If $\kappa \geqq 2^{|\alpha \cup \lambda|}$, then* ${}_\lambda\mathsf{Gws}_\alpha \subseteq \mathsf{I}_\kappa \mathsf{Cs}_\alpha$;

(ii) *If* $\mathfrak{A}$ *is a* ${}_\lambda\mathsf{Cs}_\alpha$, *and* $\kappa \geqq 2^{|\alpha \cup \lambda|}$, *then* $\mathfrak{A}$ *is strongly sub–isomorphic to some* ${}_\kappa\mathsf{Cs}_\alpha$.

PROOF. (i) Assume the hypothesis, and let $\mathfrak{A} \in {}_\lambda\mathsf{Gws}_\alpha$. If the unit element V of $\mathfrak{A}$ is as in 3.1.114, we may assume that $|I| \leqq |A|$, by picking $J \subseteq I$, $|J| \leqq |A|$, with $a \cap \bigcup_{\iota \in J} {}^\alpha Y_\iota^{(p\iota)} \neq 0$ for all $a \in A \sim \{0\}$, then relativizing to $\bigcup_{\iota \in J} {}^\alpha Y_\iota^{(p\iota)}$. Hence the hypotheses of 3.1.114 apply, and the desired result follows. (ii) is treated similarly.

REMARKS 3.1.116. We make some remarks about further results concerning ultraproducts of set algebras, in particular about various counterexamples.
(1) The special hypotheses in 3.1.105 are necessary; see [HMT81] 7.30(a).
(2) In 3.1.113 one cannot replace *homomorphic* by *isomorphic*, and the condition $|U \sim Rgq| = |U|$ is necessary; see [HMT81] 7.30(b),(c). On the other hand, it is not known if the condition $|U| \geqq |A|$ is needed.
(3) Various results are known concerning increasing the sizes of subbases in a Gws_α in addition to 3.1.114 and 3.1.115; see [HMT81] 7.26, 7.28.
(4) In [AN81] 7.2 an example is given of a directed union of members of ICs_α $(\alpha \geqq \omega)$ which is not only not isomorphic to a Cs_α, but is not even isomorphically embeddable in an ultraproduct of Cs_α's.
(5) In [AN81] 7.4 an algebra $\mathfrak{A} \in \mathsf{Cs}_\alpha^{reg}$ is constructed such that ${}^2\mathfrak{A} \notin \mathsf{UpCs}_\alpha^{reg}$, and an algebra $\mathfrak{B} \in \mathsf{Ws}_\alpha$ is constructed such that ${}^2\mathfrak{A} \notin \mathsf{Up}(\mathsf{Gws}_\alpha^{cm})^{reg}$.
(6) Using ultraproducts, one can easily establish the following unpublished result of Richard Thompson, characterizing IWs_α's among IGs_α's: for any $\mathfrak{A} \in \mathsf{IGs}_\alpha$, we have $\mathfrak{A} \in \mathsf{IWs}_\alpha$ iff the following condition holds: there is a function Γ mapping A into the collection of finite subsets of α such that for every finite subset X of $A \sim \{0\}$ we have $\prod_{x \in X} \mathsf{c}_{(\Gamma x)} x \neq 0$. This has as a corollary that for any countable $\mathfrak{A} \in \mathsf{IGs}_\alpha$, $\mathfrak{A} \in \mathsf{IWs}_\alpha$ iff $\mathfrak{A}$ is weakly subdirectly indecomposable. For uncountable $\mathfrak{A}$ this is no longer true (cf. 3.1.88 (10)).

Reducts

Now we turn to our last main topic of this section, reducts and neat embeddings for set algebras. We give three methods for going from set algebras of one dimension to ones of another. The first method is given in the following lemma.

LEMMA 3.1.117. *Let* $\mathfrak{A}$ *be a* Crs_β *with base* U *and unit element* V. *Let* α *be an ordinal and let* $\rho \in {}^\alpha\beta$ *be one–one. Fix* $x \in X \in A$. *For each* $y \in {}^\alpha U$ *set*

$$y^+ = ((\beta \sim Rg\rho) \upharpoonright x) \mathsf{u} (y \circ \rho^{-1});$$

thus $y^+ \in {}^\beta U$. *For all* $Y \in A$ *let* $fY = \{y \in {}^\alpha U : y^+ \in Y\}$.

Then f *is a homomorphism from* $\mathfrak{Rb}^\rho\mathfrak{A}$ *onto a* Crs_α, *and* $fX \neq 0$.

PROOF. Let $W = fV$. Clearly f preserves $+$ and $-$. Since $(x \circ \rho)^+ = x$, we have

$x \circ \rho \in fX$; thus $fX \neq 0$. It is routine to check that f preserves $\mathsf{d}_{\kappa\lambda}$ for all $\kappa,\lambda<\alpha$. Now suppose that $Y \in A$, $\kappa<\alpha$, and $y \in W$. For brevity set $\mathfrak{B} = \mathfrak{Rb}^{\rho}\mathfrak{A}$. To prove that $fc_{\kappa}^{\mathfrak{B}} Y \subseteq \mathsf{C}_{\kappa}^{[W]} fY$, let $y \in fc_{\kappa}^{\mathfrak{B}} Y$. Thus $y \in f\mathsf{C}_{\rho\kappa}^{[V]} Y$, that is, $y^{+} \in \mathsf{C}_{\rho\kappa}^{[V]} Y$. Thus $y^{+} \in V$ and $(y^{+})_{u}^{\rho\kappa} \in Y$ for some $u \in U$. It is easily checked that $(y_{u}^{\kappa})^{+} = (y^{+})_{u}^{\rho\kappa}$; hence $(y_{u}^{\kappa})^{+} \in Y$, so $y_{u}^{\kappa} \in fY$ and therefore $y \in \mathsf{C}_{\kappa}^{[W]} fY$. The other inclusion $\supseteq$ is established similarly.

THEOREM 3.1.118. *If α and β are ordinals with $\alpha \geq 2$, and $\rho \in {}^{\alpha}\beta$ is one–one, then* $\mathsf{Rd}_{\alpha}^{\rho}\mathsf{Gws}_{\beta} \subseteq \mathsf{IGws}_{\alpha}$ *and* $\mathsf{Rd}_{\alpha}^{\rho}\mathsf{Gs}_{\beta} \subseteq \mathsf{IGs}_{\alpha}$.

PROOF. By easy arguments, it suffices to show that $\mathsf{Rd}_{\alpha}^{\rho}\mathsf{Ws}_{\beta} \subseteq \mathsf{IGws}_{\alpha}$ and $\mathsf{Rd}_{\alpha}^{\rho}\mathsf{Cs}_{\beta} \subseteq \mathsf{IGs}_{\alpha}$. Take any $\mathfrak{A} \in \mathsf{Ws}_{\beta}$; to show that $\mathfrak{Rb}_{\alpha}^{\rho}\mathfrak{A} \in \mathsf{IGws}_{\alpha}$ it suffices by 2.4.39 and 3.1.79 to take any non–zero $X \in A$ and find a homomorphism f of $\mathfrak{Rb}_{\alpha}^{\rho}\mathfrak{A}$ into some Ws_{α} such that $fX \neq 0$. Say $\mathfrak{A}$ has unit element ${}^{\beta}U^{(p)}$; and choose $x \in X$. For each $y \in {}^{\alpha}U$ let y^{+} be as in 3.1.117, and let f be defined as in 3.1.117. Thus by 3.1.117, f is a homomorphism from $\mathfrak{Rb}_{\alpha}^{\rho}\mathfrak{A}$ onto a Crs_{α} $\mathfrak{B}$, and $fX \neq 0$. For each $y \in {}^{\alpha}U$ let $\Gamma_{y} = \{\kappa \in Rg\rho : y\rho^{-1}\kappa \neq p\kappa\}$. The following three statements are then easily checked:

(1) $f({}^{\beta}U^{(p)}) = \{y \in {}^{\alpha}U : |\Gamma_{y}| < \omega\}$;
(2) $|\Gamma_{y}| < \omega$ iff $|\rho^{-1*}\Gamma_{y}| < \omega$, for any $y \in {}^{\alpha}U$;
(3) $\rho^{-1*}\Gamma_{y} = \{\kappa < \alpha : y\kappa \neq p\rho\kappa\}$.

From (1)–(3) it follows that $f({}^{\beta}U^{(p)}) = {}^{\alpha}U^{(p\circ\rho)}$, as desired. The case Cs is even easier.

In case $\rho \in {}^{\alpha}\beta$ is onto β in 3.1.117, we have $y^{+} = y \circ \rho^{-1}$ and f is an isomorphism. This leads to the following obvious result.

THEOREM 3.1.119. *Let α and β be ordinals and let $\rho \in {}^{\alpha}\beta$ be one–one and onto. Then* $\mathsf{I}Rd_{\alpha}^{\rho}\mathsf{K}_{\beta} = \mathsf{IK}_{\alpha}$ *for all* $\mathsf{K} \in \{\mathsf{Crs}, \mathsf{Gws}, \mathsf{Gws}^{nm}, \mathsf{Gws}^{wd}, \mathsf{Gws}^{cm}, \mathsf{Gws}^{reg}, \mathsf{Gs}, \mathsf{Cs}, \mathsf{Ws}\}$.

Our second method for changing dimension goes from smaller ordinals to larger ones in general; it is formulated in the following lemma.

LEMMA 3.1.120. *Let $\mathfrak{A}$ be a Crs_{α} with base U and unit element V. Let β be an ordinal, and let $\rho \in {}^{\alpha}\beta$ be one–to–one. Suppose $W \subseteq {}^{\beta}U$ and the following conditions hold:*
(i) $V = \{x \in {}^{\alpha}U : x = y \circ \rho$ *for some* $y \in W\}$;
(ii) *for all $y \in W$, $\kappa<\alpha$, and $u \in U$, if $(y\circ\rho)_{u}^{\kappa} \in V$, then $y_{u}^{\rho\kappa} \in W$.*
Let $\mathfrak{B}$ be the full Crs_{β} with unit element W, and let $fX = \{y \in W : y\circ\rho \in X\}$.
Then $f \in Ism(\mathfrak{A}, \mathfrak{Rb}_{\alpha}^{\rho}\mathfrak{B})$, and $c_{\kappa}^{\mathfrak{B}} fX = fX$ for all $X \in A$ and all $\kappa \in \beta \sim Rg\rho$.

PROOF. Clearly f preserves $+$. Now let $y \in W$ and $X \in A$. By (i) we have $y\circ\rho \in V$. Hence $y \in f(V \sim X)$ iff $y\circ\rho \notin X$ iff $y \in W \sim fX$. Thus f preserves $-$. If $0 \neq X \in A$, choose $x \in X$. By (i) there is a $y \in W$ such that $x = y\circ\rho$. Hence $y \in fX$. This shows that f is one–one. Clearly f preserves $\mathsf{d}_{\kappa\lambda}$ for all $\kappa,\lambda<\alpha$. Now suppose that $X \in A$, $\kappa<\alpha$, and $y \in f\mathsf{C}_{\kappa}^{[V]} X$; we want to show that $y \in \mathsf{C}_{\rho\kappa}^{[W]} fX$. By the definition of f we have $y \in W$ and $y\circ\rho \in \mathsf{C}_{\kappa}^{[V]} X$. Hence $y\circ\rho \in V$ and $(y\circ\rho)_{u}^{\kappa} \in X$ for some u. Since $(y\circ\rho)_{u}^{\kappa} \in V$, our assumption (ii) yields $y_{u}^{\rho\kappa} \in W$. Since $(y\circ\rho)_{u}^{\kappa} = y_{u}^{\rho\kappa}\circ\rho$, this shows that $y_{u}^{\rho\kappa} \in fX$, and

hence $y \in \mathsf{C}_{\rho\kappa}^{[W]} fX$. The converse is similar. Thus $f \in Ism(\mathfrak{A}, \mathfrak{Rb}^{\rho}\mathfrak{B})$.

Now suppose $X \in A$, $\kappa \in \beta \sim Rg\rho$, and $y \in \mathsf{C}_{\kappa}^{[W]} fX$. Thus $y \in W$ and $y_u^{\kappa} \in fX$ for some u. Hence $y_u^{\kappa} \in W$ and $y_u^{\kappa} \circ \rho \in X$. Since $\kappa \notin Rg\rho$, we have $y_u^{\kappa} \circ \rho = y \circ \rho$. Hence $y \in fX$. This shows that $\mathsf{C}_{\kappa}^{[W]} fX = fX$, as desired.

THEOREM 3.1.121. *Let α and β be ordinals, $2 \leq \alpha$, and let $\rho \in {}^{\alpha}\beta$ be one-one. Suppose* $\mathsf{K} \in \{\mathsf{Gws}, \mathsf{Gws}^{nm}, \mathsf{Gws}^{wd}, \mathsf{Gws}^{cm}, \mathsf{Gws}^{nmreg}, \mathsf{Gs}, \mathsf{Cs}, \mathsf{Ws}, \mathsf{Gs}^{reg}, \mathsf{Cs}^{reg}\}$. *Then* $\mathsf{K}_{\alpha} \subseteq \mathsf{ISRd}_{\alpha}^{\rho}\mathsf{K}_{\beta}$.

PROOF. First suppose $\mathsf{K} = \mathsf{Gws}$. Let $\mathfrak{A} \in \mathsf{Gws}_{\alpha}$. Say $\mathfrak{A}$ has base U and unit element $\bigcup_{i \in I} {}^{\alpha}Y_i^{(pi)}$, where ${}^{\alpha}Y_i^{(pi)} \cap {}^{\alpha}Y_j^{(pj)} = 0$ for $i \neq j$. For each $i \in I$ choose $qi \in {}^{\beta}Y_i$ such that $qi \circ \rho = pi$, and set $W = \bigcup_{i \in I} {}^{\beta}Y_i^{(qi)}$. Clearly ${}^{\beta}Y_i^{(qi)} \cap {}^{\beta}Y_j^{(qj)} = 0$ for $i \neq j$. The conditions (i) and (ii) in 3.1.120 are easily verified. Thus 3.1.120 yields $\mathfrak{A} \in \mathsf{ISRd}_{\alpha}^{\rho}\mathsf{Gws}_{\beta}$, as desired. This argument also takes care of $\mathsf{K} \in \{\mathsf{Gws}^{nm}, \mathsf{Gws}^{wd}, \mathsf{Gws}^{cm}, \mathsf{Ws}\}$. If $\mathfrak{A} \in \mathsf{Gs}_{\alpha}$, say $\mathfrak{A}$ has unit element $\bigcup_{i \in I} {}^{\alpha}Y_i$, where $Y_i \cap Y_j = 0$ for distinct $i, j \in I$. We set $W = \bigcup_{i \in I} {}^{\beta}Y_i$ and use 3.1.120 again. $\mathsf{K} = \mathsf{Cs}_{\alpha}$ follows in the same way. For regularity, with $\mathfrak{B}$ as in 3.1.120 we take $\mathfrak{C} = \mathfrak{Cg}^{\mathfrak{B}} f^* A$ and use Lemma 3.1.62 with $\Gamma = Rg\rho$.

Specializing 3.1.121 and its proof to the case $\alpha \leq \beta$ and $\rho = \alpha \restriction Id$ we get:

THEOREM 3.1.122. *Suppose $2 \leq \alpha \leq \beta$ and* $\mathsf{K} \in \{\mathsf{Gws}, \mathsf{Gws}^{nm}, \mathsf{Gws}^{wd}, \mathsf{Gws}^{cm}, \mathsf{Gws}^{reg}, \mathsf{Gs}, \mathsf{Cs}, \mathsf{Ws}, \mathsf{Gs}^{reg}, \mathsf{Cs}^{reg}\}$. *Then* $\mathsf{K}_{\alpha} \subseteq \mathsf{ISNr}_{\alpha}\mathsf{K}_{\beta}$.

Now we can combine ideas from the above two constructions to prove another of the basic results about set algebras, due to Henkin and Tarski; the proof is due to Monk:

THEOREM 3.1.123. *For $\alpha \geq \omega$ we have* $\mathsf{IGs}_{\alpha} = \mathsf{HSP}(\mathsf{Cs}_{\alpha}^{reg} \cap \mathsf{Lf}_{\alpha})$.

PROOF. The direction $\supseteq$ follows from 3.1.108. To prove $\subseteq$ we shall use 2.6.4 in the following way. Let $\mathsf{K} = \mathsf{SP}(\mathsf{Cs}_{\alpha}^{reg} \cap \mathsf{Lf}_{\alpha})$. We take any $\mathfrak{A} \in \mathsf{Cs}_{\alpha}$, any $X \in A \sim \{0\}$, any $\gamma < \omega$, and any one-one $\rho \in {}^{\gamma}\alpha$ and find $\mathfrak{C} \in \mathsf{Cs}_{\alpha}^{reg} \cap \mathsf{Lf}_{\alpha}$ and $f \in Hom(\mathfrak{Rb}_{\gamma}^{\rho}\mathfrak{A}, \mathfrak{Rb}_{\gamma}^{\rho}\mathfrak{C})$ such that $fX \neq 0$. Then $\mathfrak{Rb}_{\gamma}^{\rho}\mathfrak{A} \in \mathsf{Rd}^{\rho}K$, and so by 2.6.4(i) we get $\mathfrak{A} \in \mathsf{SUpK} \subseteq \mathsf{HSP}(\mathsf{Cs}_{\alpha}^{reg} \cap \mathsf{Lf}_{\alpha})$. Then 3.1.77 finishes the proof.

So, suppose $\mathfrak{A}, X, \gamma$ and ρ are as indicated. Let U be the base of $\mathfrak{A}$, and fix $x \in X$. For any $y \in {}^{\alpha}U$ we let $y^* = ((\alpha \sim Rg\rho) \restriction x) \mathsf{u} (Rgp \restriction y)$. Then for any $Y \in A$ let $fY = \{y \in {}^{\alpha}U : y^* \in Y\}$. Let $\mathfrak{B}$ be the full Cs_{α} with base U. Clearly f preserves u and $\sim$ relative to ${}^{\alpha}U$. It is easily checked that f takes $\mathsf{d}_{\kappa\lambda}$ of $\mathfrak{Rb}_{\gamma}^{\rho}\mathfrak{A}$ to $\mathsf{d}_{\kappa\lambda}$ of $\mathfrak{Rb}_{\gamma}^{\rho}\mathfrak{B}$. Let $\mathfrak{A}' = \mathfrak{Rb}_{\gamma}^{\rho}\mathfrak{A}$ and $\mathfrak{B}' = \mathfrak{Rb}_{\gamma}^{\rho}\mathfrak{B}$. Take $Y \in A$ and $\kappa < \gamma$. We show that $f\mathsf{c}_{\kappa}^{\mathfrak{A}'} Y = \mathsf{c}_{\kappa}^{\mathfrak{B}'} fY$. Let $y \in f\mathsf{c}_{\kappa}^{\mathfrak{A}'} Y$. Thus $y^* \in \mathsf{C}_{\rho\kappa} Y$. Say $(y^*)_u^{\rho\kappa} \in Y$. Clearly $(y^*)_u^{\rho\kappa} = (y_u^{\rho\kappa})^*$. Thus $(y_u^{\rho\kappa})^* \in Y$, so $y_u^{\rho\kappa} \in fY$, hence $y \in \mathsf{C}_{\rho\kappa} fY = \mathsf{c}_{\kappa}^{\mathfrak{B}'} fY$. This proves that $f\mathsf{c}_{\kappa}^{\mathfrak{A}'} Y \subseteq \mathsf{c}_{\kappa}^{\mathfrak{B}'} fY$. The converse is similar. Thus $f \in Hom(\mathfrak{Rb}_{\gamma}^{\rho}\mathfrak{A}, \mathfrak{Rb}_{\gamma}^{\rho}\mathfrak{B})$. Since $x^* = x$, we have $x \in fX$; thus $fX \neq 0$.

Next we claim that for any $Y \in A$ we have $\mathsf{\Delta}^{\mathfrak{B}} fY \subseteq Rg\rho$. For, suppose $\kappa \in \alpha \sim Rg\rho$ and $y \in \mathsf{C}_{\kappa} fY$. Thus $y_u^{\kappa} \in fY$ for some u. Hence $(y_u^{\kappa})^* \in Y$. But $(y_u^{\kappa})^* = y^*$, so $y \in fY$. This

proves the claim.

Also, fY is regular in $\mathfrak{B}$ for any $Y \in A$. For, suppose $y \in fY$, $z \in {}^{\alpha}U$, and $\Delta fY \upharpoonright y \subseteq z$. Let $\Gamma = Rg\rho \sim \Delta fY$. Now $y^* \in Y$, and $(\alpha \sim \Gamma) \upharpoonright y^* \subseteq z^*$. Hence $z^* \in \mathsf{C}_{(\Gamma)}Y = \mathsf{c}^{\mathfrak{A}'}_{(\Theta)}Y$, so $z \in f\mathsf{c}^{\mathfrak{A}'}_{(\Theta)}Y = \mathsf{c}^{\mathfrak{B}'}_{(\Theta)}fY = \mathsf{C}_{(\Gamma)}fY = fY$, where $\Theta = \rho^{-1*}\Gamma$, as desired.

Now let $\mathfrak{C} = \mathfrak{Cg}^{\mathfrak{B}}\{fY : Y \in A\}$. By the above and 3.1.64, $\mathfrak{C} \in \mathsf{Cs}^{reg}_{\alpha} \cap \mathsf{Lf}_{\alpha}$. Clearly $f \in Hom(\mathfrak{Rb}^{\rho}_{\gamma}\mathfrak{A}, \mathfrak{Rb}^{\rho}_{\gamma}\mathfrak{C})$, as desired.

Our third, and last, construction method for changing dimension in set algebras is due to Andréka and Németi, and the results using it are also due to them; see [AN81] 4.7.1.1, 4.7.1.2, 8.1.

DEFINITION 3.1.124. *Let α and β be ordinals, and let $\rho \in {}^{\alpha}\beta$ be one–one. We define a function rb^{ρ} with domain the class of all β–termed sequences such that, for any β–termed sequence f,*

$$rb^{\rho}f = \langle\langle f_{\rho\kappa}, (\beta \sim Rg\rho) \upharpoonright f\rangle : \kappa < \alpha\rangle.$$

Then rd^{ρ} is the function whose domain is the class of all sets of β–termed sequences such that, for any such class X,

$$rd^{\rho}X = (rb^{\rho})^*X.$$

LEMMA 3.1.125. *Let α and β be ordinals with $\alpha \geq 2$ and let $\rho \in {}^{\alpha}\beta$ be one–one. Suppose $\mathfrak{A}$ is the full Gws_{β} with unit element $V = \bigcup_{i \in I} {}^{\beta}Y_i^{(pi)}$, where ${}^{\beta}Y_i^{(pi)} \cap {}^{\beta}Y_j^{(pj)} = 0$ for distinct $i,j \in I$. Then*

(i) *$V \upharpoonright rb^{\rho}$ is one–to–one;*

(ii) *$rd^{\rho} \in Is(\mathfrak{Rb}^{\rho}_{\alpha}\mathfrak{A}, \mathfrak{Cb}\, rd^{\rho}V)$;*

(iii) *$\mathfrak{Cb}rd^{\rho}V$ is a Gws_{α} with unit element $rd^{\rho}V = \bigcup_{j \in J} {}^{\alpha}W_j^{\langle qj\rangle}$, where ${}^{\alpha}W_j^{\langle qj\rangle} \cap {}^{\alpha}W_k^{\langle qk\rangle} = 0$ for distinct $j,k \in J$, $J = \{\langle i,h\rangle : i \in I,\ h \in {}^{\beta \sim Rg\rho}Y_i,\ |\{\kappa \in \beta \sim Rg\rho : h\kappa \neq (pi)\kappa\}| < \omega\}$, and for any $\langle i,h\rangle \in J$, $W_{ih} = Y_i \times \{h\}$ and $qih = \langle\langle (pi)\rho\kappa, h\rangle : \kappa < \alpha\rangle$;*

(iv) *If $\mathsf{K} \in \{\mathsf{Gws}^{nm}, \mathsf{Gws}^{wd}, \mathsf{Gs}\}$ and $\mathfrak{A} \in \mathsf{K}_{\beta}$, then $\mathfrak{Cb}rd^{\rho}V \in \mathsf{K}_{\alpha}$.*

(v) *If $X \in A$ is regular, then $rd^{\rho}X$ is regular.*

PROOF. Clearly rb^{ρ} is a one–one function, so (i) holds. Hence rd^{ρ} is an isomorphism from $\mathfrak{Bl}\mathfrak{Rb}^{\rho}_{\alpha}\mathfrak{A}$ onto $\mathfrak{Bl}\mathfrak{Cb}rd^{\rho}V$. Clearly rd^{ρ} preserves $d_{\kappa\lambda}$ for all $\kappa,\lambda < \alpha$. Next, let $\kappa < \alpha$ and $X \in A$. To show that $rd^{\rho}\mathsf{c}_{\kappa}X = \mathsf{C}_{\kappa}rd^{\rho}X$, let $g \in rd^{\rho}\mathsf{c}_{\kappa}X$. Say $g = rb^{\rho}f$ with $f \in \mathsf{C}_{\rho\kappa}X$. Then $f^{\rho\kappa}_u \in X$ for some u. Note that $rb^{\rho}f^{\rho\kappa}_u = g^{\kappa}_v$, where $v = \langle u, (\beta \sim Rg\rho) \upharpoonright f\rangle$. Hence $g^{\kappa}_v \in rd^{\rho}X$ and so $g \in \mathsf{C}_{\kappa}rd^{\rho}X$. Thus $rd^{\rho}\mathsf{c}_{\kappa}X \subseteq \mathsf{C}_{\kappa}rd^{\rho}X$. The converse is similar. Thus (ii) holds.

Now we turn to (iii). Let $g \in rd^{\rho}V$. Say $g = rb^{\rho}f$, $f \in {}^{\alpha}Y_i^{(pi)}$. Set $h = (\beta \sim Rg\rho) \upharpoonright f$. Clearly then $g \in {}^{\alpha}W_{ih}$. Let $\Gamma = \{\kappa < \beta : f\kappa \neq (pi)\kappa\}$; so, Γ is finite. For $\kappa \in \alpha \sim \rho^{-1*}\Gamma$ we have $g\kappa = \langle f_{\rho\kappa}, h\rangle = \langle (pi)\rho\kappa, h\rangle = (qih)\kappa$. Hence $g \in {}^{\alpha}W_{ih}^{\langle qih\rangle}$. Conversely, suppose $g \in {}^{\alpha}W_{ih}^{\langle qih\rangle}$. Then there is a function f with domain β such that $h \subseteq f$ and $g\kappa = \langle f\rho\kappa, h\rangle$ for every $\kappa < \alpha$. Hence $f \in {}^{\beta}Y_i$. Let $\Gamma = \{\kappa < \alpha : g\kappa \neq (qih)\kappa\}$ and $\Delta = \{\kappa \in \beta \sim Rg\rho : h\kappa \neq (pi)\kappa\}$. Thus $\rho^*\Gamma \cup \Delta$ is finite. For any $\kappa \in \beta \sim (\rho^*\Gamma \cup \Delta)$ we have $f\kappa = h\kappa = (pi)\kappa$ if $\kappa \notin Rg\rho$, and $f\kappa = (pi)\kappa$

if $\kappa \in Rg\rho$, since $\langle f\kappa, h\rangle = g\rho^{-1}\kappa = (qih)\rho^{-1}\kappa = \langle (pi)\kappa, h\rangle$. So $f \in {}^{\beta}Y_i^{\langle pi\rangle}$ and hence $g = rb^{\rho}f \in rd^{\rho}V$. Hence we have checked that $rd^{\rho}V = \bigcup_{j\in J}{}^{\alpha}W_j^{\langle qj\rangle}$. Now suppose that $\langle i,h\rangle$, $\langle j,k\rangle$ are distinct members of J, and $g \in {}^{\alpha}W_{ih}^{\langle qih\rangle} \cap {}^{\alpha}W_{jk}^{\langle qjk\rangle}$. Then since $g0 \in W_{ih} \cap W_{jk}$ we have $h = k$. Now there is a function f with domain β such that $g\kappa = \langle f\rho\kappa, h\rangle$ for all $\kappa < \alpha$ with $h \subseteq f$. By the arguments above, $f \in {}^{\beta}Y_i^{\langle pi\rangle} \cap {}^{\beta}Y_j^{\langle pj\rangle}$, so $i = j$. Hence (iii) holds.

(iv) is easy. For (v), let $X \in A$ be regular. Suppose $g \in rd^{\rho}X$, $h \in rd^{\rho}V$, $(1 \cup \Delta rd^{\rho}X) \upharpoonright g \subseteq h$; we are to show that $h \in rd^{\rho}X$. Clearly $\Delta rd^{\rho}X = \rho^{-1*}\Delta X$. Say $g = rb^{\rho}f$, $f \in X$, and $h = rb^{\rho}k$, $k \in V$. Now $g0 = h0$, so $(\beta \sim Rg\rho) \upharpoonright f = (\beta \sim Rg\rho) \upharpoonright k$. It follows that $(\{\rho 0\} \cup \Delta X) \upharpoonright f \subseteq k$. Now by 3.1.61 and the remark after 3.1.60, X is $\{\rho 0\}$–regular in $\mathfrak{A}$, so it follows that $k \in X$. Hence $h \in rd^{\rho}X$, as desired. This completes the proof of 3.1.125.

THEOREM 3.1.126. *Let α and β be ordinals with $\alpha \geqq 2$ and let $\rho \in {}^{\alpha}\beta$ be one–one. Let* $\mathsf{K} \in \{\mathsf{Gws}, \mathsf{Gws}^{wd}, \mathsf{Gws}^{nm}, \mathsf{Gws}^{reg}, \mathsf{Gs}, \mathsf{Gs}^{reg}\}$. *Then* $\mathsf{IK}_{\alpha} = \mathsf{SRd}_{\alpha}^{\rho}\mathsf{IK}_{\beta}$.

PROOF: By 3.1.121 and 3.1.125.

THEOREM 3.1.127. *Let α and β be ordinals with $2 \leqq \alpha \leqq \beta$ and let* $\mathsf{K} \in \{\mathsf{Gws}, \mathsf{Gws}^{wd}, \mathsf{Gws}^{nm}, \mathsf{Gws}^{reg}, \mathsf{Gs}, \mathsf{Gs}^{reg}\}$. *Then* $\mathsf{IK}_{\alpha} = \mathsf{SNr}_{\alpha}\mathsf{IK}_{\beta}$.

PROOF: By 3.1.122 and 3.1.125.

Finally we note the following easy consequence of 3.1.120 for Cs_{α}'s:

THEOREM 3.1.128. *If $1 \leqq \beta$ then* $\mathsf{Cs}_1 = \mathsf{SINr}_1\mathsf{Cs}_{\beta}$.

REMARKS 3.1.129. We make some comments about possible improvements of these results on changing dimension.

(1) Various extensions of the important Theorem 3.1.123 are found in [AN81] 7.1 and 7.2. For example, $\mathsf{I}_{\infty}\mathsf{Cs}_{\alpha} = \mathsf{SUp}(\mathsf{Cs}_{\alpha}^{reg} \cap \mathsf{Lf}_{\alpha})$ for $\alpha \geqq \omega$.

(2) A reduct of a Cs_{α} is not necessarily isomorphic to a Cs_{α} (cf. 3.1.125 and 3.1.126). For example, suppose $3 \leqq \beta \leqq \alpha$ and $2^{a} > 2^{b}$, where $a = 2^{|\alpha|}$ and $b = 2^{|\beta|}$. Let $\mathfrak{A}$ be the full Cs_{α} with base 2. Then $\mathfrak{Rb}_{\beta}\mathfrak{A}$ cannot be isomorphic to a Cs_{β}. For more detailed results see [AN81] 8.10, where it is shown, for example, that if $1 < \alpha < \beta$ then $\mathsf{Rd}_{\alpha}\mathsf{Cs}_{\beta}^{reg} \not\subseteq \mathsf{ICs}_{\alpha}$.

(3) In connection with (2) we should mention the following interesting results of Andréka and Németi [AN81] 8.12: for $\alpha \leqq \beta$ we have: (a) $\mathsf{ICs}_{\alpha} \cap \mathsf{Lf}_{\alpha} = \mathsf{SRd}_{\alpha}(\mathsf{ICs}_{\beta} \cap \mathsf{Lf}_{\beta})$ iff $2^{|\alpha|} = 2^{|\beta|}$; (b) $\mathsf{ICs}_{\alpha} \cap \mathsf{Lf}_{\alpha} = \mathsf{SNr}_{\alpha}(\mathsf{ICs}_{\beta} \cap \mathsf{Lf}_{\beta})$ iff either $\alpha, \beta < \omega$ or $2^{|\alpha|} = 2^{|\beta|}$. Also in [AN81] 8.18 they show that if $1 < \alpha < \beta$ then (c) $\mathsf{IWs}_{\alpha} = \mathsf{SNr}_{\alpha}\mathsf{IWs}_{\beta}$ and (d) $\mathsf{ICs}_{\alpha} = \mathsf{SNr}_{\alpha}\mathsf{ICs}_{\beta}$ iff $\beta < \alpha + \omega$.

The classes ${}_\kappa\mathsf{Gs}_\alpha$

We now want to strengthen 3.1.106 and obtain similar but weaker results for the classes ${}_\kappa\mathsf{Gs}_\alpha$; these results are due to Andréka and Németi [AN81] 7.1, 3.14.

DEFINITION 3.1.130. *For κ a cardinal ≥ 1 and α an ordinal $\geq\omega$, let $\mathfrak{R}f(\kappa,\alpha)$ be the subalgebra of $\mathfrak{Cb}({}^\alpha\kappa)$ with universe $\{X{:}X\subseteq{}^\alpha\kappa,\ |\Delta X|<\omega,\ X$ regular$\}$.*

This definition is justified by 3.1.64. The first lemma is of some independent interest. It is a generalization, due to Monk, of Andréka, Németi [81] Lemma 7.1.1.

LEMMA 3.1.131. *Suppose $\alpha\geq\omega$ and $\mathfrak{A}$ is a simple non-discrete Lf_α. Then ${}^J\mathfrak{A}\in\mathsf{SUp}\{\mathfrak{A}\}$ for any $J\neq 0$.*

PROOF By 0.3.9(vi) and 0.3.72(i) we have ${}^J\mathfrak{A}\in\mathsf{SUp}({}^\alpha\mathfrak{A})$. Hence by 0.3.70(i) it suffices to show that ${}^\alpha\mathfrak{A}\in\mathsf{SUp}\{\mathfrak{A}\}$. Let $I=\{\Gamma{:}\Gamma\subseteq\alpha,\ 0<|\Gamma|<\omega\}$ and let F be an ultrafilter on I such that $\{\Delta\in I{:}\Gamma\subseteq\Delta\}\in F$ for every $\Gamma\in I$.

Now we define a function $z\in{}^\alpha({}^IA)$. Let $\Gamma\in I$. Choose $\langle z_{\lambda\Gamma}{:}\lambda\in\alpha\rangle$ so that the following conditions hold:

(1) $z_{\lambda\Gamma}\neq 0$ for all $\lambda\in\Gamma$, and $z_{\lambda\Gamma}=0$ if $\lambda\notin\Gamma$.
(2) $\Sigma_{\lambda\in\Gamma}z_{\lambda\Gamma}=1$;
(3) $z_{\lambda\Gamma}\cdot z_{\mu\Gamma}=0$ if $\lambda,\mu\in\Gamma$ and $\lambda\neq\mu$;
(4) $\Delta z_{\lambda\Gamma}\subseteq\alpha\sim\Gamma$ for all $\lambda\in\Gamma$.

It is easy to find $\langle z_{\lambda\Gamma}{:}\lambda\in\alpha\rangle$ with these properties, starting from elements $\mathsf{d}_{\mu\nu}$, $\mu,\nu\in\alpha\sim\Gamma$, and using the facts that $\Sigma\{\mathsf{d}_{\mu\nu}{:}\nu\in\Delta\}=1$ if Δ is infinite (since $\mathfrak{A}\in\mathsf{Lf}_\alpha$), and that $\mathfrak{A}$ is non-discrete. Now for any $\lambda<\alpha$ we have

(5) $z_\lambda/\bar{F}\neq 0$;
(6) $\Delta(z_\lambda/\bar{F})=0$.

Condition (5) holds since $\{\Gamma{:}\lambda\in\Gamma\}\subseteq\{\Gamma{:}z_{\lambda\Gamma}\neq 0\}$ and $\{\Gamma{:}\lambda\in\Gamma\}\in F$. For (6), $\{\Gamma{:}\{\lambda,\kappa\}\subseteq\Gamma\}\subseteq\{\Gamma{:}\mathsf{c}_\kappa z_{\lambda\gamma}=z_{\lambda\Gamma}\}$, and $\{\Gamma{:}\{\lambda,\kappa\}\subseteq\Gamma\}\in F$, yielding (6).

Now for any $y\in{}^\alpha A$ let ey be the member of ${}^IA$ such that, for any $\Gamma\in I$,

$$(ey)\Gamma=\Sigma_{\lambda\in\Gamma}z_{\lambda\Gamma}\cdot y_\lambda$$

Finally, set $fy=ey/\bar{F}$ for any $y\in{}^\alpha A$. We claim that f is the desired isomorphism of ${}^\alpha\mathfrak{A}$ into ${}^I\mathfrak{A}/\bar{F}$. Clearly e preserves $+$, and so f does also. By (2) we have $(e1)\Gamma=1$ for any $\Gamma\in I$, and so $f1=1$. Also, clearly $f0=0$. If $y\cdot y'=0$, by (3) it is clear that $ey\cdot ey'=0$ hence $fy\cdot fy'=0$. Thus f preserves $-$.

To show that f preserves c_λ, where $\lambda<\alpha$, let $y\in{}^\alpha A$. Then for any $\Gamma\in I$ with $\lambda\in\Gamma$ we have

$$\mathsf{c}_\lambda((ey)\Gamma)=\mathsf{c}_\lambda\Sigma_{\mu\in\Gamma}z_{\mu\Gamma}\cdot y_\mu=\Sigma_{\mu\in\Gamma}z_{\mu\Gamma}\cdot\mathsf{c}_\lambda y_\mu\ \text{(by (4))}\ =(e\mathsf{c}_\lambda y)\Gamma.$$

Hence $c_\lambda fy = fc_\lambda y$ since $\{\Gamma \in I : \lambda \in \Gamma\} \in I$.

If $\lambda, \mu < \alpha$ and $\Gamma \in I$, clearly $(ed_{\lambda\mu})\Gamma = d_{\lambda\mu}$, using (2). Thus f preserves $d_{\lambda\mu}$.

Finally, suppose $y \in {}^\alpha A$ and $y \neq 0$. Say $y_\lambda \neq 0$. Let $\Omega = \Delta y_\lambda$. Then $c_{(\Omega)} y_\lambda = 1$. Now for any $\Gamma \in I$ with $\lambda \in \Gamma$ we have, by (3), $(z_\lambda \cdot ey)\Gamma = z_{\lambda\Gamma} \cdot y_\lambda$. It follows that $z_\lambda/\bar{F} \cdot fy = z_\lambda/\bar{F} \cdot \langle y_\lambda : \Gamma \in I\rangle/\bar{F}$. Hence, using (5) and (6),

$$c_{(\Omega)}(z_\lambda/\bar{F} \cdot fy) = c_{(\Omega)}(z_\lambda/\bar{F} \cdot \langle y_\lambda : \Gamma \in I\rangle/\bar{F}) = z_\lambda/\bar{F} \cdot c_{(\Omega)}(\langle y_\lambda : \Gamma \in I\rangle/\bar{F}) = z_\lambda/\bar{F} \neq 0.$$

Hence $fy \neq 0$, as desired.

LEMMA 3.1.132. *If $\kappa \geqq 2$ is a cardinal and $\alpha \geqq \omega$, then* ${}_\kappa\mathsf{Gs}_\alpha \subseteq \mathsf{SUp}({}_\kappa\mathsf{Gs}_\alpha \mathsf{n} \mathsf{Lf}_\alpha)$.

PROOF. Let $\mathfrak{A} \in {}_\kappa\mathsf{Gs}_\alpha$. By 3.1.122 and its proof there is an isomorphism f from $\mathfrak{A}$ into $\mathfrak{Nr}_\alpha\mathfrak{B}$ for some ${}_\kappa\mathsf{Gs}_{\alpha+\alpha}\ \mathfrak{B}$. We may assume that f^*A generates $\mathfrak{B}$. Let $I = \{\Gamma : \Gamma \subseteq \alpha, |\Gamma| < \omega\}$. For each $\Gamma \in I$ let ρ_Γ be a one–one function mapping α into $\alpha+\alpha$ such that $\Gamma \upharpoonright Id \subseteq \rho_\Gamma$ and $\rho_\Gamma^*(\alpha \sim \Gamma) \subseteq (\alpha+\alpha) \sim \alpha$. Let $\mathfrak{C}_\Gamma$ be the ρ_Γ–reduct of $\mathfrak{B}$ for each $\Gamma \in I$. Since f^*A generates $\mathfrak{B}$ it is clear that $\mathfrak{C}_\Gamma \in \mathsf{Lf}_\alpha$ for any $\Gamma \in I$; see the proof of 2.6.64. By 3.1.118 and its proof we have $\mathfrak{C}_\Gamma \in \mathsf{I}_\kappa\mathsf{Gs}_\alpha$ for any $\Gamma \in I$. Finally, it is clear that there is an isomorphism from $\mathfrak{A}$ into $\mathsf{P}_{\Gamma \in I}\mathfrak{C}_\Gamma/\bar{F}$, where F is any ultrafilter on I such that $\{\Delta : \Gamma \subseteq \Delta \in I\} \in F$ for every $\Gamma \in I$.

LEMMA 3.1.133. *If κ and λ are cardinals with $\omega \leqq \lambda \leqq \kappa$, and if α is an ordinal $\geqq \omega$, then* ${}_\kappa\mathsf{Gs}_\alpha \subseteq \mathsf{SUp}_\lambda\mathsf{Gs}_\alpha$.

PROOF. In this proof we use natural extensions of some of our notions to sets Γ of ordinals; thus we speak of CA_Γ's, $\mathfrak{Rd}_\Gamma$, etc. Let $\mathfrak{A} \in {}_\kappa\mathsf{Gs}_\alpha$. Note that $\mathfrak{Rd}_\Gamma\mathfrak{A} \in \mathsf{I}_\infty\mathsf{Gs}_\Gamma$ for every $\Gamma \subseteq \alpha$ by 3.1.118 and its proof. Let $I = \{\langle\Gamma, \mathfrak{B}\rangle : \Gamma \subseteq \alpha, |\Gamma| < \omega$, and $\mathfrak{B}$ is a finitely generated subalgebra of $\mathfrak{Rd}_\Gamma\mathfrak{B}\}$. Then if $\langle\Gamma, \mathfrak{B}\rangle \in I$ we have $|B| \leqq \omega$, and hence by 3.1.45(i)(c) it follows that $\mathfrak{B} \in \mathsf{I}_\lambda\mathsf{Gs}_\Gamma$; by 3.1.121 and its proof, ${}_\lambda\mathsf{Gs}_\Gamma \subseteq \mathsf{ISRd}_\Gamma({}_\lambda\mathsf{Gs}_\alpha)$, so we may choose $\mathfrak{C}_{\Gamma\mathfrak{B}} \in {}_\lambda\mathsf{Gs}_\alpha$ such that $\mathfrak{B} \cong \mathsf{I} \subseteq \mathfrak{Rd}_\Gamma\mathfrak{C}_{\Gamma\mathfrak{B}}$. Let F be any ultrafilter on I such that $\{\langle\Gamma, \mathfrak{B}\rangle : \Gamma' \subseteq \Gamma, \mathfrak{B}' \subseteq \mathfrak{B}, \langle\Gamma, \mathfrak{B}\rangle \in I\} \in F$ *for* any $\langle\Gamma', \mathfrak{B}'\rangle \in I$. Then it is easily seen that $\mathfrak{B}$ can be isomorphically embedded in $\mathsf{P}_{\langle\Gamma, \mathfrak{B}\rangle \in I}\mathfrak{C}_{\Gamma\mathfrak{B}}/\bar{F}$, as desired.

THEOREM 3.1.134. *For any cardinal $\kappa \geqq 2$ and any ordinal $\alpha \geqq \omega$ we have* ${}_\kappa\mathsf{Ws}_\alpha \mathsf{n} \mathsf{Lf}_\alpha \subseteq \mathsf{I}({}_\kappa\mathsf{Cs}_\alpha^{reg} \mathsf{n} \mathsf{Lf}_\alpha)$.

PROOF. For $\kappa < \omega$ this result follows from the proof of 3.1.102. The present proof works also for $\kappa < \omega$, however. Let $I = \{\Gamma : \Gamma \subseteq \alpha, |\Gamma| < \omega\}$, and let F be an ultrafilter on I such that $\{\Gamma : \Gamma \in I, \Delta \subseteq \Gamma\} \in F$ for all $\Delta \in I$. Let $\mathfrak{A} \in {}_\kappa\mathsf{Ws}_\alpha \mathsf{n} \mathsf{Lf}_\alpha$; say the unit element of $\mathfrak{A}$ is ${}^\alpha\kappa^{(p)}$. We now introduce some special notation. If $\Gamma \in I$ and $q \in {}^\alpha\kappa$, we denote by $p[\Gamma/q]$ the function $(\alpha \sim \Gamma) \upharpoonright p \cup (\Gamma \upharpoonright q)$. Now for any $x \in A$ we set

$$fx = x \cup \{q \in {}^\alpha\kappa \sim ({}^\alpha\kappa^{(p)}) : \{\Gamma \in I : p[\Gamma/q] \in x\} \in F\}.$$

Clearly f preserves the Boolean operations, and is one–one. Now assume that $\xi < \alpha$ and $q \in fc_\xi x$ with $x \in A$. If $q \in {}^\alpha\kappa^{(p)}$, it is clear that $q \in c_\xi fx$. Assume that $q \notin {}^\alpha\kappa^{(p)}$. Let

$M=\{\Gamma\in I: p[\Gamma/q]\in \mathsf{c}_\xi x\}$. Thus $M\in F$. We may assume that $\Delta x\cup\{\xi\}\subseteq\Gamma$ for all $\Gamma\in M$. Fix $\Gamma\in M$, and choose $u\in\kappa$ such that $(p[\Gamma/q])^\xi_u\in x$. We claim that $q^\xi_u\in fx$, hence $q\in \mathsf{c}_\xi fx$, as desired. In fact, let $N=M\cap\{\Omega\in I:\Gamma\subseteq\Omega\}$. Then $N\in F$, and for any $\Omega\in N$ we have

$$\Delta x\upharpoonright(p[\Omega/q^\xi_u])=\Delta x\upharpoonright q^\xi_u=\Delta x\upharpoonright(p[\Gamma/q])^\xi_u,$$

so by 3.1.24, 3.1.26 it follows that $p[\Omega/q^\xi_u]\in x$. This shows that $q^\xi_u\in fx$. Thus we have proved that $f\mathsf{c}_\xi x\subseteq \mathsf{c}_\xi fx$. The other inclusion is easily established, and it is also easy to prove that f preserves diagonal elements. The proof is complete.

LEMMA 3.1.135. *For any cardinal $\kappa\geqq\omega$ and any ordinal $\alpha\geqq\omega$ we have* $\mathsf{I}_\infty\mathsf{Cs}_\alpha=\mathsf{HSP}_\kappa\mathsf{Cs}_\alpha$.

PROOF. By 3.1.109 it suffices to prove the inclusion $\subseteq$. Let $\mathfrak{A}\in{}_\infty\mathsf{Cs}_\alpha$. By 3.1.112 we may assume that the base of $\mathfrak{A}$ has power $\geqq\kappa$. Then by 3.1.133 we have

$$\begin{aligned}\mathfrak{A}\in\mathsf{SUp}_\kappa\mathsf{Gs}_\alpha&=\mathsf{SUpSP}_\kappa\mathsf{Cs}_\alpha \text{ by 3.1.77 and its proof}\\ &\subseteq\mathsf{HSP}_\kappa\mathsf{Cs}_\alpha.\end{aligned}$$

THEOREM 3.1.136. *For any cardinal $\kappa\geqq 2$ and any ordinal $\alpha\geqq\omega$ we have:*

$$\begin{aligned}\mathsf{SUp}_\kappa\mathsf{Gs}_\alpha&=\mathsf{SUp}\{\mathfrak{R}\mathsf{f}(\kappa,\alpha)\}=\mathsf{SUp}({}_\kappa\mathsf{Cs}^{reg}_\alpha\cap\mathsf{Lf}_\alpha)=\mathsf{HSP}_\kappa\mathsf{Gs}_\alpha\\ &=\mathsf{HSP}\{\mathfrak{R}\mathsf{f}(\kappa,\alpha)\}=\mathsf{HSP}({}_\kappa\mathsf{Cs}^{reg}_\alpha\cap\mathsf{Lf}_\alpha).\end{aligned}$$

For $\kappa\geqq\omega$ these classes are all equal to $\mathsf{I}_\infty\mathsf{Cs}_\alpha$. For $\kappa<\omega$ they are all equal to $\mathsf{I}_\kappa\mathsf{Gs}_\alpha$.

PROOF. We have

$$\begin{aligned}\mathsf{SUp}_\kappa\mathsf{Gs}_\alpha&\subseteq\mathsf{SUp}({}_\kappa\mathsf{Gs}_\alpha\cap\mathsf{Lf}_\alpha)\ (3.1.132)\\ &\subseteq\mathsf{SUpSP}({}_\kappa\mathsf{Ws}_\alpha\cap\mathsf{Lf}_\alpha)\\ &\subseteq\mathsf{SUpSP}({}_\kappa\mathsf{Cs}^{reg}_\alpha\cap\mathsf{Lf}_\alpha)\ (3.1.134)\\ &\subseteq\mathsf{SUp}\{\mathfrak{R}\mathsf{f}(\kappa,\alpha)\}\ (3.1.131)\\ &\subseteq\mathsf{SUp}({}_\kappa\mathsf{Cs}^{reg}_\alpha\cap\mathsf{Lf}_\alpha)\\ &\subseteq\mathsf{SUp}_\kappa\mathsf{Gs}_\alpha.\end{aligned}$$

If $\kappa<\omega$, then by 3.1.103 and its proof we have

$$\mathsf{SUp}_\kappa\mathsf{Gs}_\alpha\subseteq\mathsf{HSP}_\kappa\mathsf{Gs}_\alpha=\mathsf{H}_\kappa\mathsf{Gs}_\alpha\subseteq\mathsf{I}_\kappa\mathsf{Gs}_\alpha$$

and the desired conclusions follow easily. If $\kappa\geqq\omega$, then $\mathsf{HSP}_\kappa\mathsf{Gs}_\alpha=\mathsf{H}_\kappa\mathsf{Gs}_\alpha\subseteq{}_\lambda\mathsf{Cs}_\alpha$ for some $\lambda\geqq\kappa$ by 3.1.103 and its proof, and ${}_\lambda\mathsf{Cs}_\alpha\subseteq\mathsf{SUp}_\kappa\mathsf{Gs}_\alpha$ by 3.1.133. Again the desired conclusions follow.

COROLLARY 3.1.137. For any $\alpha\geqq\omega$ we have $\mathsf{IGs}_\alpha=\mathsf{HSP}\{\mathfrak{R}\mathsf{f}(\kappa,\alpha):\kappa\leqq\omega\}$.

PROOF. By 3.1.123, 3.1.133, and 3.1.136, since

$$\begin{aligned}\mathbf{P}(\mathsf{Cs}_\alpha^{rcg}\mathbf{n}\mathsf{Lf}_\alpha) &\subseteq \mathbf{P}(\{\mathfrak{R}\mathfrak{f}(\kappa,\alpha):\kappa<\omega\}\,\mathbf{u}\,\{\mathfrak{R}\mathfrak{f}(\kappa,\alpha):\kappa\geqq\omega\})\\ &\subseteq \mathbf{P}(\{\mathfrak{R}\mathfrak{f}(\kappa,\alpha):\kappa<\omega\}\,\mathbf{u}\,\mathbf{S}\mathsf{Up}_\omega\mathsf{Gs}_\alpha)\\ &\subseteq \mathbf{P}(\{\mathfrak{R}\mathfrak{f}(\kappa,\alpha):\kappa<\omega\}\,\mathbf{u}\,\mathbf{HSP}\{\mathfrak{R}\mathfrak{f}(\omega,\alpha)\})\\ &\subseteq \mathbf{HSP}\{\mathfrak{R}\mathfrak{f}(\kappa,\alpha):\kappa\leqq\omega\}.\end{aligned}$$

REMARK 3.1.138. On the basis of the results above on reducts, it is a trivial matter to verify the statement on p.343 of Part I to the effect that if $\alpha\geqq\omega$ and H_α is the class of all hereditarily non-discrete CA_α's, then $\mathfrak{Sr}_\beta\mathsf{H}_\alpha$ satisfies the conclusions of 0.4.54. In fact, by 0.5.17 it suffices to show that for each finite $J\subseteq\alpha$ the class $\mathsf{Rd}_J\mathsf{H}_\alpha$ satisfies the conditions of 0.4.54; by 0.4.36 it is enough to check that $\mathbf{S}\mathsf{Rd}_J\mathsf{H}_\alpha$ has a finite member with more than one element, and this is clear from the proof of 3.1.117.

The following theorem is due to Andréka, Monk, and Németi.

THEOREM 3.1.139. ${}_\infty\mathsf{Cs}_\alpha\subseteq\mathbf{H}_\infty\mathsf{Ws}_\alpha$.

PROOF. We may assume that α is a limit ordinal. Using 0.2.16 and 3.1.112 it is enough to show that $\mathfrak{C}\in\mathbf{H}_\infty\mathsf{Ws}_\alpha$, with $\mathfrak{C}=\mathfrak{Gb}({}^\alpha\kappa)$, and $\kappa^{|\alpha|}=\kappa\geqq\omega$. Let $p=(0:\xi<\alpha)$, $\mathfrak{B}=\mathfrak{Gb}({}^\alpha\kappa^{(p)})$, and for $s\in{}^\alpha\kappa$ let $inds=\{\xi<\alpha: s\xi\neq 0\}$. Let $\mathcal{F}=\{\Gamma:\Gamma\subseteq\alpha,\ |\Gamma|<\omega\}$. There is a function T mapping ${}^\alpha\kappa^{(p)}$ into $\mathcal{F}$ such that (a) $\forall\Delta\in\mathcal{F}\ \forall t\in{}^\Delta\kappa\ \forall\mu\in\kappa\sim\{0\}$ $\exists s\in{}^\alpha\kappa^{(p)}\sim\{p\}$ $(Ts=\Delta,\ t\subseteq s,\ s(maxinds)=\mu)$, (b) if $s\in{}^\alpha\kappa^{(p)}$, $\xi\in Ts$, $\mu\in\kappa$, then $Ts=T(s^\xi_\mu)$, (c) $\forall s\in{}^\alpha\kappa^{(p)}\sim\{p\}(maxinds\notin Ts)$. To construct T, for each $\beta\in\alpha$ let l_β be a one-one function from κ onto $\{(\Delta,\mu):\Delta\in\mathcal{F}\sim\{0\},\ max\Delta=\beta,\ \mu<\kappa\}$. Then for any $s\in{}^\alpha\kappa^{(p)}$ let $Ts=\Delta$ if $s\neq p$ and for some $\beta\in\alpha$, $maxinds=\beta+2$, $max\Delta=\beta$, and $l_\beta s(\beta+1)=\langle\Delta,s(\beta+2)\rangle$, $Ts=0$ otherwise. Then (a)–(c) are clear. Now for each $\Gamma\in\mathcal{F}$ let $e_\Gamma=\{s\in{}^\alpha\kappa^{(p)}:\Gamma\notin Ts\}$. Set $I=Ig^{\mathfrak{B}}\{e_\Gamma:\Gamma\in\mathcal{F}\}$. Let f be a one-one function from ${}^\alpha\kappa$ onto $\kappa\sim 1$. Define $Fp=0$, and for $s\in{}^\alpha\kappa^{(p)}\sim\{p\}$ let $Fs=(Ts\restriction s)\,\mathbf{u}\,(\alpha\sim Ts)\restriction f^{-1}(s(maxinds))$. For any $a\in C$ set $g'a=F^{-1*}a$, $ga=g'a/I$. Clearly $g\in Hom(\mathfrak{Bl}\mathfrak{C},\mathfrak{Bl}(\mathfrak{B}/I))$. To show that g is one-one, suppose that $a\neq 0$ but $g'a\in I$. Say $s\in a$. Now if $\Gamma\in\mathcal{F}$ and $\xi<\alpha$ then $\mathsf{c}^{\mathfrak{C}}_\xi e_\Gamma\subseteq e_{\Gamma\cup\{\xi\}}$, as is easily checked, using (b). Hence there exist $m\in\omega$ and $\Gamma_1,\ldots,\Gamma_m\in\mathcal{F}$ such that $g'a\subseteq e_{\Gamma_1}\mathbf{u}\ldots\mathbf{u}e_{\Gamma_m}$. Let $\Delta=\Gamma_1\mathbf{u}\ldots\mathbf{u}\Gamma_m$. Apply (a) to get $t\in{}^\alpha\kappa^{(p)}\sim\{p\}$ such that $Tt=\Delta$, $\Delta\restriction s\subseteq t$, and $t(maxindt)=fs$. Then $Ft=s$, so $t\in g'a\sim(e_{\Gamma_1}\mathbf{u}\ldots\mathbf{u}e_{\Gamma_m})$, contradiction. For $\xi,\eta<\alpha$ we have $g'\mathsf{d}^{\mathfrak{C}}_{\xi\eta}\sim e_{\{\xi,\eta\}}=\mathsf{d}^{\mathfrak{B}}_{\xi\eta}\sim e_{\{\xi\eta\}}$ and $g'\mathsf{c}^{\mathfrak{C}}_\xi a\sim e_{\{\xi\}}=\mathsf{c}^{\mathfrak{B}}_\xi a\sim e_{\{\xi\}}$, as is easily checked. (For the last equality, note that if $s\in{}^\alpha\kappa^{(p)}\sim e_{\{\xi\}}$ then $\xi\in Ts$ and $Fs^\xi_\mu=(Fs)^\xi_\mu$ for any $\mu<\kappa$, using (a)–(c).) Hence $g\in Ism(\mathfrak{C},\mathfrak{B}/I)$, as desired.

3.2. REPRESENTATION THEORY

Recall from Definition 3.1.1 (vii) that a cylindric algebra $\mathfrak{A}$ is *representable* if it is isomorphic to a Gs_α. For $\alpha \geqq 2$ this is equivalent to saying that $\mathfrak{A}$ is isomorphic to a subdirect product of Cs_α's (by Theorem 3.1.77). We have emphasized in Part I the importance of this notion, which is the main topic of study in this volume. In this section we give the main representation theorems for cylindric algebras: sufficient conditions for representability, and abstract characterizations of IGs_α. We also describe some constructions of non-representable CA_α's and briefly survey some other kinds of representability. Most of the results in this section will be found in Henkin, Monk, Tarski [84'].

In more detail, the contents of this section are as follows. We begin with the main sufficient condition for representability: for $\alpha \geqq \omega$, every Lf_α is isomorphic to a subdirect product of Cs_α^{reg}'s. This result has many important corollaries, giving other sufficient conditions for representability and some characterizations of it. For example, it follows from this result and theorems of Part I that for $\alpha \geqq \omega$ every dimension-complemented CA_α is representable, while for any α, a CA_α is representable iff it can be neatly embedded in a $\mathsf{CA}_{\alpha+\omega}$. After this main result we give another equivalent condition for representability, involving rectangular atoms. Then we prove that a finite dimensional CA_α of positive characteristic is always representable; this is also true for α infinite but then it is a trivial corollary of our main result and 2.6.54 of Part I. The last main positive representation result applies to any CA_2 satisfying two special equations. Then we give several methods for constructing non-representable cylindric algebras. We close the section with a brief discussion of other possible ways of representing CA_α's, namely by Crs_α's or by sheaves. Those comments extend our discussion at the end of Part I.

Main Theorems

The theorem that every Lf_α for $\alpha \geqq \omega$ is representable is an algebraic version of the completeness theorem for first-order logic. See section 4.3 for a more extensive treatment of this connection. Our algebraic proof will be a version of Henkin's proof of the completeness theorem. Recall that his proof starts by adjoining constants, which are used to eliminate quantifiers (in a certain sense). So we start by discussing an algebraic version of constants. There are at least three such versions. One can algebraically express properties of the formula $v_0 = c$, c a constant; this is the method we actually use. Or, one can concentrate on the operation of substituting a constant for a variable in a formula; the corresponding algebraic notion of a special kind of endomorphism is used extensively in Halmos' related theory of

polyadic algebras, and occurs as a derived notion in our development – see 3.2.4. Lastly, one can think of a constant as a variable which one is not allowed to quantify; see the proof of 3.2.7, and also Remark 3.2.9.

DEFINITION 3.2.1. *Let* $\alpha \geqq 2$ *and let* $\mathfrak{A} \in \mathsf{CA}_\alpha$. (i) *For* $\kappa < \alpha$, *an element* x *of* A *is* κ**–thin** *if* $\Delta x \subseteq \{\kappa\}$, $x \cdot \mathsf{s}^\kappa_\lambda x \leqq \mathsf{d}_{\kappa\lambda}$ *for some* $\lambda \in \alpha \sim \{\kappa\}$, *and* $\mathsf{c}_\kappa x = 1$.
(ii) $\mathfrak{A}$ *is* **rich** *if for every* $y \in A$ *such that* $\Delta y \subseteq 1$ *and* $y \neq 0$ *there is a* 0*–thin element* x *such that* $x \cdot \mathsf{c}_0 y \leqq y$.

REMARKS 3.2.2. It will be shown shortly that if x is κ–thin then $x \cdot \mathsf{s}^\kappa_\lambda x \leqq \mathsf{d}_{\kappa\lambda}$ for *every* $\lambda \in \alpha \sim \{\kappa\}$. If $\mathfrak{A}$ is a Cs_α with base U, and $u \in U$, then $x = \{t \in {}^\alpha U : t0 = u\}$ is 0–thin. Conversely, if x is 0–thin and regular, then $x = \{t \in {}^\alpha U : t0 = u\}$ for some $u \in U$. Thin elements are an algebraic version of individual constants. Thus let Λ be a language with an individual constant $\mathbf{c}$. Let Σ be a consistent set of sentences in Λ. Then $(\mathsf{v}_0 = \mathbf{c})/\equiv_\Sigma$ is a 0–thin element in the CA_α $\mathfrak{Fm}^\Lambda/\equiv_\Sigma$ (cf. 1.1.9, 1.1.10).

We need two lemmas concerning thin elements.

LEMMA 3.2.3. *Suppose* $\alpha \geqq 2$, $\kappa < \alpha$, $\mathfrak{A}$ *is a* CA_α, *and* x *is a* κ*–thin element of* $\mathfrak{A}$. *Then*:
(i) $x \cdot \mathsf{s}^\kappa_\lambda x \leqq \mathsf{d}_{\kappa\lambda}$ *for every* $\lambda < \alpha$;
(ii) *if* $\lambda < \alpha$, *then* $\mathsf{s}^\kappa_\lambda x$ *is* λ*–thin*;
(iii) *if* $\lambda \neq \kappa$, *then* $\mathsf{c}_\lambda - \mathsf{c}_\kappa(x \cdot - \mathsf{d}_{\kappa\lambda}) = 1$;
(iv) *if* $\lambda \neq \kappa$, $y \in A$, *and* $\mathsf{c}_\lambda[x \cdot y \cdot \mathsf{c}_\kappa(x \cdot - y)] \leqq \mathsf{c}_\kappa(x \cdot - \mathsf{d}_{\kappa\lambda})$, *then* $\mathsf{c}_\kappa(x \cdot - y) = -\mathsf{c}_\kappa(x \cdot y)$;
(v) *under the assumptions of* (iv) *we have* $x \cdot \mathsf{c}_\kappa(x \cdot y) = x \cdot y$.

PROOF. (i) By 3.2.1 choose $\mu \in \alpha \sim \{\kappa\}$ such that $x \cdot \mathsf{s}^\kappa_\mu x \leqq \mathsf{d}_{\kappa\mu}$; thus $\mathsf{c}_\mu x = x$. Applying s^μ_λ to both sides of the inequality we get $x \cdot \mathsf{s}^\kappa_\lambda x \leqq \mathsf{d}_{\kappa\lambda}$, as desired.
(ii). We have $\mathsf{s}^\kappa_\lambda x \cdot \mathsf{s}^\lambda_\kappa \mathsf{s}^\kappa_\lambda x = \mathsf{s}^\kappa_\lambda x \cdot x \leqq \mathsf{d}_{\kappa\lambda}$, $\Delta \mathsf{s}^\kappa_\lambda x \subseteq \{\lambda\}$, *and* $\mathsf{c}_\lambda \mathsf{s}^\kappa_\lambda x = \mathsf{c}_\kappa x = 1$, so $\mathsf{s}^\kappa_\lambda x$ is λ–thin.
(iii). By (i) we have $x \cdot - \mathsf{d}_{\kappa\lambda} \cdot \mathsf{s}^\kappa_\lambda x = 0$, so $\mathsf{c}_\kappa(x \cdot - \mathsf{d}_{\kappa\lambda}) \cdot \mathsf{s}^\kappa_\lambda x = 0$, hence $\mathsf{s}^\kappa_\lambda x \leqq -\mathsf{c}_\kappa(x \cdot - \mathsf{d}_{\kappa\lambda})$. Therefore

$$1 = \mathsf{c}_\kappa x = \mathsf{c}_\lambda \mathsf{s}^\kappa_\lambda x = \mathsf{c}_\lambda - \mathsf{c}_\kappa(x \cdot - \mathsf{d}_{\kappa\lambda}),$$

as desired.
(iv). The assumption yields $\mathsf{c}_\lambda[x \cdot y \cdot \mathsf{c}_\kappa(x \cdot - y)] \cdot - \mathsf{c}_\kappa(x \cdot - \mathsf{d}_{\kappa\lambda}) = 0$, hence $\mathsf{c}_\kappa(x \cdot y) \cdot \mathsf{c}_\kappa(x \cdot - y) \cdot \mathsf{c}_\lambda - \mathsf{c}_\kappa(x \cdot - \mathsf{d}_{\kappa\lambda}) = 0$. Hence the desired result follows from (iii), since $\mathsf{c}_\kappa(x \cdot y) + \mathsf{c}_\kappa(x \cdot - y) = \mathsf{c}_\kappa x = 1$.
(v). We have

$$x \cdot \mathsf{c}_\kappa(x \cdot y) = x \cdot - \mathsf{c}_\kappa(x \cdot - y) = x \cdot \mathsf{c}^\partial_\kappa(-x + y) \leqq x \cdot (-x + y) = x \cdot y \leqq x \cdot \mathsf{c}_\kappa(x \cdot y).$$

LEMMA 3.2.4. *Suppose* $2 \leqq \alpha$, Γ *is a finite subset of* α, *and* $\mathfrak{A}$ *is a* CA_α. *Suppose* x *is a function with domain* Γ *such that* x_κ *is a* κ*–thin element of* $\mathfrak{A}$ *for every* $\kappa \in \Gamma$. *Furthermore, assume that the equality*

$$\mathsf{c}_\lambda[y \cdot z \cdot \mathsf{c}_\kappa(y \cdot - z)) \cdot - \mathsf{c}_\kappa(\mathsf{c}_\lambda y \cdot - \mathsf{d}_{\kappa\lambda})] = 0$$

holds for all distinct $\kappa,\lambda<\alpha$ *and all* $\mathbf{y},z\in A$. *Then for any* $\mathbf{y}\in A$ *we have*

$$\mathsf{c}_{(\Gamma)}(\Pi_{\kappa\in\Gamma}\boldsymbol{x}_\kappa\cdot-\mathbf{y})=-\mathsf{c}_{(\Gamma)}(\Pi_{\kappa\in\Gamma}\boldsymbol{x}_\kappa\cdot\mathbf{y});\ \Pi_{\kappa\in\Gamma}\boldsymbol{x}_\kappa\cdot\mathsf{c}_{(\Gamma)}(\Pi_{\kappa\in\Gamma}\boldsymbol{x}_\kappa\cdot\mathbf{y})=\Pi_{\kappa\in\Gamma}\boldsymbol{x}_\kappa\cdot\mathbf{y}.$$

PROOF. This lemma is an obvious generalization of 3.2.3 (iv), (v).

From the following theorem we will be able to derive the main representation theorem using two additional easy lemmas.

THEOREM 3.2.5. *Suppose* $2\leqq\alpha$, $\mathfrak{A}$ *is a simple rich* Lf_α, *and for all distinct* $\kappa,\lambda<\alpha$ *and all* $\boldsymbol{x},\mathbf{y}\in A$ *the equality* $\mathsf{c}_\lambda[\boldsymbol{x}\cdot\mathbf{y}\cdot\mathsf{c}_\kappa(\boldsymbol{x}\cdot-\mathbf{y})]\cdot-\mathsf{c}_\kappa(\mathsf{c}_\lambda\boldsymbol{x}\cdot-\mathsf{d}_{\kappa\lambda})=0$ *holds. Then* $\mathfrak{A}$ *is isomorphic to a* Cs_α^{reg}.

PROOF. Let U be the set of all 0–thin elements of $\mathfrak{A}$. We shall define an isomorphism f of $\mathfrak{A}$ onto a regular Cs_α with base U. Because $\mathfrak{A}$ is simple, we will only need to check the homomorphism conditions for f. For any $a\in A$ let

$$fa=\{\boldsymbol{u}\in{}^\alpha U:\Pi_{\kappa\in\Delta a}\mathsf{s}_\kappa^0\boldsymbol{u}_\kappa\leqq a\}$$

To check that f preserves $-$, first suppose that $\boldsymbol{u}\in fa\cap f(-a)$. Since $\Delta a=\Delta(-a)$, this implies that $\Pi_{\kappa\in\Delta a}\mathsf{s}_\kappa^0\boldsymbol{u}_\kappa=0$. But $\mathsf{c}_{(\Delta a)}\Pi_{\kappa\in\Delta a}\mathsf{s}_\kappa^0\boldsymbol{u}_\kappa=\Pi_{\kappa\in\Delta a}\mathsf{c}_\kappa\mathsf{s}_\kappa^0\boldsymbol{u}_\kappa=1$, contradiction. Second, we show that $fa\cup f(-a)={}^\alpha U$. Suppose $\boldsymbol{u}\in{}^\alpha U\sim fa$. Thus $\Pi_{\kappa\in\Delta a}\mathsf{s}_\kappa^0\boldsymbol{u}_\kappa\cdot-a\neq0$, so by simplicity, $\mathsf{c}_{(\Delta a)}(\Pi_{\kappa\in\Delta a}\mathsf{s}_\kappa^0\boldsymbol{u}_\kappa\cdot-a)=1$. Then by 3.2.4 we have $\mathsf{c}_{(\Delta a)}(\Pi_{\kappa\in\Delta a}\mathsf{s}_\kappa^0\boldsymbol{u}_\kappa\cdot a)=0$, so $\Pi_{\kappa\in\Delta a}\mathsf{s}_\kappa^0\boldsymbol{u}_\kappa\leqq-a$ and hence $\boldsymbol{u}\in f(-a)$, as desired.

Next we show that f preserves $\cdot$. Suppose first that $\boldsymbol{u}\in f(a\cdot b)$. Thus $\Pi_{\kappa\in\Delta(a\cdot b)}\mathsf{s}_\kappa^0\boldsymbol{u}_\kappa\leqq a\cdot b$. Let $\Gamma=\Delta(a\cdot b)\cup\Delta a$. Then $\Pi_{\kappa\in\Gamma}\mathsf{s}_\kappa^0\boldsymbol{u}_\kappa\leqq a$, so

$$\Pi_{\kappa\in\Delta a}\mathsf{s}_\kappa^0\boldsymbol{u}_\kappa=\mathsf{c}_{(\Gamma\sim\Delta a)}\Pi_{\kappa\in\Gamma}\mathsf{s}_\kappa^0\boldsymbol{u}_\kappa\leqq\mathsf{c}_{(\Gamma\sim\Delta a)}a=a,$$

and $\boldsymbol{u}\in fa$. Similarly, $\boldsymbol{u}\in fb$. Second, suppose that $\boldsymbol{u}\in fa\cap fb$. Then $\Pi_{\kappa\in\Delta a}\mathsf{s}_\kappa^0\boldsymbol{u}_\kappa\leqq a$ and $\Pi_{\kappa\in\Delta b}\mathsf{s}_\kappa^0\boldsymbol{u}_\kappa\leqq b$, so $\Pi_{\kappa\in\Delta a\cup\Delta b}\mathsf{s}_\kappa^0\boldsymbol{u}_\kappa\leqq a\cdot b$. As above one argues to show that $\Pi_{\kappa\in\Delta(a\cdot b)}\mathsf{s}_\kappa^0\boldsymbol{u}_\kappa\leqq a\cdot b$.

For cylindrifications, suppose that $\lambda<\alpha$ and first suppose that $\boldsymbol{u}\in f\mathsf{c}_\lambda a$. We may assume that $\lambda\in\Delta a$. Thus

$$\Pi_{\kappa\in\Delta b}\mathsf{s}_\kappa^0\boldsymbol{u}_\kappa\leqq\mathsf{c}_\lambda a,\ \text{with}\ b=\mathsf{c}_\lambda a.$$

We want to find a 0–thin element v such that $\boldsymbol{u}_v^\lambda\in fa$. Note that

$$\begin{aligned}&\mathsf{c}_\lambda\mathsf{c}_{(\Delta a\sim\{\lambda\})}[(\Pi_{\kappa\in\Delta a\sim\{\lambda\}}\mathsf{s}_\kappa^0\boldsymbol{u}_\kappa)\cdot a]\\&=\mathsf{c}_{(\Delta a\sim\{\lambda\})}((\Pi_{\kappa\in\Delta a\sim\{\lambda\}\sim\Delta b}\mathsf{s}_\kappa^0\boldsymbol{u}_\kappa)\cdot\mathsf{c}_\lambda[(\Pi_{\kappa\in\Delta b}\mathsf{s}_\kappa^0\boldsymbol{u}_\kappa)\cdot a])\\&=\mathsf{c}_{(\Delta a\sim\{\lambda\})}(\Pi_{\kappa\in\Delta a\sim\{\lambda\}}\mathsf{s}_\kappa^0\boldsymbol{u}_\kappa)=1\ \ (\text{with}\ b=\mathsf{c}_\lambda a);\end{aligned}$$

it follows that the element $\boldsymbol{x}=\mathsf{c}_{(\Delta a\sim\{\lambda\})}[(\Pi_{\kappa\in\Delta a\sim\{\lambda\}}\mathsf{s}_\kappa^0\boldsymbol{u}_\kappa)\cdot a]$ is non–zero and has dimension set $\subseteq\{\lambda\}$. Hence let v be a 0–thin element $\leqq\mathsf{s}_0^\lambda\boldsymbol{x}$. Set $\boldsymbol{u}'=\boldsymbol{u}_v^\lambda$. Then

$$\begin{aligned}&\Pi_{\kappa\in\Delta a}s^0_\kappa u'_\kappa=\Pi_{\kappa\in\Delta a\sim\{\lambda\}}s^0_\kappa u_\kappa\cdot s^0_\lambda v\\&\leq\Pi_{\kappa\in\Delta a\sim\{\lambda\}}s^0_\kappa u_\kappa\cdot c_{(\Delta a\sim\{\lambda\})}[(\Pi_{\kappa\in\Delta a\sim\{\lambda\}}s^0_\kappa u_\kappa)\cdot a]\\&\leq a \text{ by } 3.2.4,\end{aligned}$$

as desired. Second, suppose that $u\in c_\lambda fa$. Let v be a 0—thin element such that $u^\lambda_v\in fa$. Thus

(1) $\Pi_{\kappa\in\Delta a\sim\{\lambda\}}s^0_\kappa u_\kappa\cdot s^0_\lambda v\leq a.$

Let $\Gamma=(\Delta a\sim\Delta c_\lambda a)\cup\{\lambda\}$. Applying $c_{(\Gamma)}$ to both sides of (1) we get

$$\Pi_{\kappa\in\Delta b}s^0_\kappa u_\kappa\leq c_\lambda a, \text{ with } b=c_\lambda a,$$

hence $u\in fc_\lambda a$, as desired.

Now for diagonal elements, suppose $\lambda,\mu<\alpha$; we may assume that $\lambda\neq\mu$. First suppose that $u\in fd_{\lambda\mu}$. We may assume that $\mathfrak{A}$ is non—discrete, and hence $\Delta d_{\lambda\mu}=\{\lambda,\mu\}$. Thus $s^0_\lambda u_\lambda\cdot s^0_\mu u_\mu\leq d_{\lambda\mu}$. Hence

$$s^0_\mu u_\mu=c_\lambda(s^0_\lambda u_\lambda\cdot s^0_\mu u_\mu)\leq c_\lambda(d_{\lambda\mu}\cdot s^0_\lambda u_\lambda)=s^0_\mu u_\lambda$$

and so $u_\mu\leq u_\lambda$ by applying s^μ_0. By symmetry $u_\mu=u_\lambda$, so $u\in D_{\lambda\mu}$. Second, suppose that $u\in D_{\lambda\mu}$, so that $u_\mu=u_\lambda$. Then

$$s^0_\lambda u_\lambda\cdot s^0_\mu u_\mu=s^0_\lambda u_\lambda\cdot s^\lambda_\mu s^0_\mu u_\lambda\leq d_{\lambda\mu}$$

since $s^0_\lambda u_\lambda$ is λ—thin by 3.2.3(ii). Therefore $u\in fd_{\lambda\mu}$.

It is obvious that fa is regular for each $a\in A$, so the proof is complete.

LEMMA 3.2.6. *Suppose* $2\leq\alpha$, $\mathfrak{A}$ *is a rich* Lf_α, *and* I *is an ideal of* $\mathfrak{A}$. *Then* $\mathfrak{A}/I$ *is rich.*

PROOF. Suppose $a\in A$, $\Delta(a/I)\subseteq 1$, and $a/I\neq 0$. Then $\Delta c_{(\Delta a\sim 1)}a\subseteq 1$ and $c_{(\Delta a\sim 1)}a\neq 0$, so let x be a 0—thin element $\leq c_{(\Delta a\sim 1)}a$. Clearly x/I is 0—thin and $x/I\leq a/I$.

LEMMA 3.2.7. *Suppose* $\omega\leq\alpha$ *and* $\mathfrak{A}\in\mathsf{Lf}_\alpha$. *Then* $\mathfrak{A}$ *can be isomorphically embedded in a rich* Lf_α.

PROOF. By a simple transfinite argument it suffices to show that if $0\neq a\in A$ and $\Delta a\subseteq 1$ then there is an extension $\mathfrak{B}\in\mathsf{Lf}_\alpha$ of $\mathfrak{A}$ such that $x\cdot c_0 a\leq a$ for some 0—thin element x of B. To this end, let $\mathfrak{C}$ be an $\mathsf{Lf}_{\alpha+1}$ such that $\mathfrak{A}\subseteq\mathfrak{Nr}_\alpha\mathfrak{C}$, by 2.6.49. Let $\mathfrak{D}=\mathfrak{Sb}_a\mathfrak{C}$, and let $I=Ig^{\mathfrak{D}}\{-c_0((a+-c_0a)\cdot d_{0\alpha})\}$. Then $A\cap I=\{0\}$. In fact, suppose $y\in A\cap I$. Thus $y\cdot c_0((a+-c_0a)\cdot d_{0\alpha})=0$, so $y\cdot -c_0a=0$ and $y\cdot s^0_\alpha a=0$. Since $\Delta a\subseteq\alpha$ and $\Delta y\subseteq\alpha$ we have $y\cdot c_0a=0$. Hence $y=0$, as desired. Clearly also $(a+-c_0a)\cdot d_{0\alpha}\notin I$. Let $x=(a+-c_0a)\cdot d_{0\alpha}$. Then in $\mathfrak{D}/I$ we have that x/I is 0—thin and $(x/I)\cdot c_0(a/I)\leq a/I$. This finishes the proof.

The main representation theorem now follows:

THEOREM 3.2.8. *For $\alpha \geqq \omega$ we have* $\mathsf{Lf}_\alpha \subseteq \mathsf{SPCs}_\alpha^{reg}$.

PROOF. By 3.2.7 we may assume that our given $\mathfrak{A} \in \mathsf{Lf}_\alpha$ is rich, and by 2.4.52 and 3.2.6 that $\mathfrak{A}$ is simple. By 1.11.7, the equations indicated in 3.2.5 hold in $\mathfrak{A}$. Hence the conclusion follows by 3.2.5.

REMARKS 3.2.9. The result just established is due to Tarski, while the proof is due to Henkin. Tarski's original proof can be sketched as follows, using the apparatus developed in Part I. We start with a simple Lf_α $\mathfrak{A}$, $\alpha \geqq \omega$. First we neatly embed $\mathfrak{A}$ in a simple Lf_β $\mathfrak{B}$ such that $|B| = \beta$; it suffices to show that $\mathfrak{B} \in \mathsf{ICs}_\beta^{reg}$. Using this cardinality condition it is easy to construct an ultrafilter F on $\mathfrak{B}\restriction\mathfrak{B}$ satisfying the following condition:

(*) For all $\kappa < \beta$ and all $x \in B$, if $\mathsf{c}_\kappa x \in F$ then $\mathsf{s}_\lambda^\kappa x \in F$ for some $\lambda \in \beta \sim \Delta x$.

Now we can define an equivalence relation $\equiv$ on β by setting $\kappa \equiv \lambda$ iff $\mathsf{d}_{\kappa\lambda} \in F$. Let U be the set of all $\equiv$-classes, and let $\varphi \in {}^U\beta$ be a choice function: $\varphi u \in u$ for all $u \in U$. Then the desired isomorphism f is defined by

$$fb = \{x \in {}^\beta U : \mathsf{s}^+_{\varphi \circ x} b \in F\}$$

for any $b \in B$, where s^+ is the substitution function introduced in 1.11.13.

Proofs of 3.2.8 have appeared in the literature. The proof just sketched, except for the neat embedding part, essentially appears in Andréka, Németi [75']. A proof similar to the one of 3.2.5, using thin elements, is given in Monk [78'], and a proof *using* the completeness theorem is carried out in Monk [76**]. Via the correspondence between Lf_α's and polyadic equality algebras for infinite α (see Galler [57] and also section 5.4), representation theorems for polyadic algebras yield 3.2.5 again; see Halmos [57a].

Now we give the most important corollaries of 3.2.8. The first one gives several characterizations of representability.

THEOREM 3.2.10. *Let $\alpha \geqq 2$. Then* $\mathsf{IGs}_\alpha = \mathsf{SPCs}_\alpha^{reg} = \mathsf{SNr}_\alpha\mathsf{CA}_{\alpha+\omega} = \bigcap_{\kappa<\omega}\mathsf{SNr}_\alpha\mathsf{CA}_{\alpha+\kappa} = \mathsf{SNr}_\alpha\mathsf{CA}_{\alpha+\beta}$ *for each $\beta \geqq \omega$. Moreover, $\mathfrak{A} \in \mathsf{CA}_\alpha$ is representable iff every finite reduct of $\mathfrak{A}$ is representable.*

PROOF. The last two equalities are in 2.6.34 and 2.6.35, the first in 3.1.107. For the second equality, assume $\alpha < \omega$. Then by 2.6.48, 3.1.107, 3.1.127, and 3.2.8,

$$\mathsf{SNr}_\alpha\mathsf{CA}_\omega = \mathsf{SNr}_\alpha\mathsf{Lf}_\omega \subseteq \mathsf{SNr}_\alpha\mathsf{SPCs}_\omega^{reg} = \mathsf{SNr}_\alpha\mathsf{IGs}_\omega = \mathsf{IGs}_\alpha \subseteq \mathsf{SNr}_\alpha\mathsf{CA}_\omega,$$

giving the desired result. If $\alpha \geqq \omega$, then by 2.6.52, 3.1.108, 3.1.127, and 3.2.8,

$$\mathsf{SNr}_\alpha\mathsf{CA}_{\alpha+\omega} = \mathsf{SUpLf}_\alpha \subseteq \mathsf{SUpIGs}_\alpha = \mathsf{IGs}_\alpha \subseteq \mathsf{SNr}_\alpha\mathsf{CA}_{\alpha+\omega},$$

again as desired. The last statement follows easily from 2.6.47 and the above.

Theorem 3.2.10 does not extend verbatim to $\alpha \leqq 1$; we treat this rather trivial case separately in 3.2.54 and 3.2.55. Now we give some additional sufficient conditions for representability which follow from Theorem 3.2.8.

THEOREM 3.2.11. *Let $\alpha \geqq \omega$ and $\mathfrak{A} \in \mathsf{CA}_\alpha$. Then each of the following conditions is sufficient for $\mathfrak{A}$ to be representable:*

(i) $\mathfrak{A} \in \mathsf{Lf}_\alpha$;

(ii) $\mathfrak{A} \in \mathsf{Dc}_\alpha$;

(iii) $\mathfrak{A} \in \mathsf{Ss}_\alpha$;

(iv) *for every finite $\Gamma \subseteq \alpha$ and every non-zero $x \in A$ there exist distinct $\kappa, \lambda \in \alpha \sim \Gamma$ such that $x \cdot \mathsf{d}_{\kappa\lambda} \neq 0$;*

(v) *for every finite sequence ρ without repeating terms and with range included in α, and for every non-zero $x \in A$ there exist a function h and $\kappa < \alpha$ such that h is an endomorphism of $\mathfrak{Rb}^\rho \mathfrak{A}$, $\kappa \in \alpha \sim Rg\rho$, $\mathsf{c}_\kappa^{\mathfrak{A}} \circ h = h$, and $hx \neq 0$;*

(vi) *$\mathfrak{A}$ is of characteristic $\kappa > 0$;*

(vii) *for every $\kappa < \alpha$ and every $x \in A$, $\mathsf{c}_\kappa x = \Sigma_{\lambda < \alpha} \mathsf{s}_\lambda^\kappa x$.*

PROOF. By 2.6.49, 2.6.50, 2.6.54, and 3.2.10.

Condition 3.2.11(vi), that $\mathfrak{A}$ has non-zero characteristic, remains a sufficient condition for representabiliy when $\alpha < \omega$; see 3.2.53.

THEOREM 3.2.12. *Assume that $\alpha \geqq 2$. Every monadic-generated CA_α is representable; hence every minimal CA_α is representable.*

PROOF. By 2.6.56 and 3.2.10.

Now we return to the basic theorem 3.2.5. Its formulation does not have the assumption $\alpha \geqq \omega$ which is present in 3.2.8, needed there because of Lemma 3.2.7. This leads to the following characterization of representability for $2 \leqq \alpha < \omega$, due to Henkin and Tarski.

THEOREM 3.2.13. *Suppose $2 \leqq \alpha < \omega$ and $\mathfrak{A}$ is a CA_α. Then $\mathfrak{A}$ is representable iff $\mathfrak{A}$ can be embedded in a rich CA_α $\mathfrak{B}$ such that in $\mathfrak{B}$ all of the equations*

$$\mathsf{c}_\lambda(x \cdot y \cdot \mathsf{c}_\kappa(x \cdot -y)) \cdot -\mathsf{c}_\kappa(\mathsf{c}_\lambda x \cdot -\mathsf{d}_{\kappa\lambda}) = 0$$

hold, for all distinct $\kappa, \lambda < \alpha$ and all $x, y \in B$.

PROOF. The direction $\Leftarrow$ is immediate from 2.4.52, 3.2.6, and 3.2.5. For $\Rightarrow$, suppose that $\mathfrak{A}$ is representable. Say $\mathfrak{A} \cong \mathfrak{C}$, where $\mathfrak{C}$ is a Gs_α with unit element V. Let $\mathfrak{B}$ be the full Gs_α with unit element V. It suffices to show that $\mathfrak{B}$ is rich and the indicated equations hold in $\mathfrak{B}$. Say $V = \bigcup_{i \in I} {}^\alpha U_i$, where $U_i \cap U_j = 0 \neq U_i$ for distinct $i, j \in I$. Suppose $0 \neq b \in B$ and $\Delta b \subseteq 1$. Choose $t_i \in {}^\alpha U_i$ for all $i \in I$ so that $t_i \in b$ if $b \cap {}^\alpha U_i \neq 0$. Let

$$x = \bigcup_{i \in I} \{s \in {}^\alpha U_i : s0 = t_i 0\}.$$

It is easily checked that x is 0-thin and $x \cdot \mathsf{c}_0 b \leqq b$. Thus $\mathfrak{B}$ is rich. That the given

equations hold in $\mathfrak{B}$ is immediate from 3.1.122 and 1.11.7, but this can also be checked directly. This finishes the proof.

Now we turn to another characterization of representability, involving rectangular atoms. For the notion of rectangular element, see 1.10.6. The characterization depends on the following theorem which is of independent interest. Theorems 3.2.14 and 3.2.16 are due to Henkin and Tarski.

THEOREM 3.2.14. *Let $\mathfrak{A}$ be any atomic* CA_α, $\alpha \geqq 2$. *Then the following conditions are equivalent*:

(i) *Every atom of $\mathfrak{A}$ is rectangular.*

(ii) *There is an isomorphism of $\mathfrak{A}$ onto a* Gws_α $\mathfrak{B}$ *that carries each atom of $\mathfrak{A}$ to a singleton element of $\mathfrak{B}$.*

DISCUSSION 3.2.15. We shall only prove, below, that (i) implies (ii), as the converse is very easy to check.

To see the idea of our proof, consider first the case where $\alpha<\omega$, and where $\mathfrak{A}$ is simple. We wish to represent $\mathfrak{A}$ isomorphically as a Cs_α $\mathfrak{B}$ with some base U. Thus the unit set of $\mathfrak{B}$ is a hypercube ${}^\alpha U$, and our isomorphism will be determined by assigning to each atom a of $\mathfrak{A}$ a single point $\langle a_0,\ldots,a_{\alpha-1}\rangle \in {}^\alpha U$.

Our idea is to take for U the set of atoms on the "principal diagonal" d of $\mathfrak{A}$, $d=\prod_{\kappa,\lambda<\alpha} \mathsf{d}_{\kappa\lambda}$. Then for any atom a of $\mathfrak{A}$ and any $\kappa<\alpha$, we will get the component $a_\kappa \in U$ by forming the hyperplane $\mathsf{c}_{(\alpha\sim\{\kappa\})}a$ and intersecting it with d. This mapping of atoms a of $\mathfrak{A}$ to points $\langle a_0,\ldots,a_{\alpha-1}\rangle$ of ${}^\alpha U$ is one–one and onto, because any $\langle a_0,\ldots,a_{\alpha-1}\rangle \in {}^\alpha U$ will be the coordinates of a unique atom a of $\mathfrak{A}$, obtained by intersecting all of the hyperplanes $\mathsf{c}_{(\alpha\sim\{\kappa\})}a_\kappa$.

Of course we cannot use such a construction in case $\alpha \geqq \omega$, since in that case $\mathfrak{A}$ has no principal diagonal d and no hyperplanes $\mathsf{c}_{(\alpha\sim\{\kappa\})}a$. Furthermore, we are not restricting ourselves to the case where $\mathfrak{A}$ is simple. Nevertheless, we can incorporate the geometric ideas of the preceding paragraph in the desired proof, as follows.

PROOF of 3.2.14, (i) $\Rightarrow$ (ii). We define a binary relation E on $At\mathfrak{A}\times\alpha$ by the following rule, where $At\mathfrak{A}$ is the set of all atoms of $\mathfrak{A}$. For any $a,b \in At\mathfrak{A}$ and $\xi,\eta \in \alpha$, $\langle a,\xi\rangle E\langle b,\eta\rangle$ iff there is some finite $\Gamma\subseteq\alpha$ with $\xi,\eta\in\Gamma$ such that $\mathsf{c}_{(\Gamma\sim\{\xi\})}a\cdot\mathsf{d}_\Gamma = \mathsf{c}_{(\Gamma\sim\{\eta\})}b\cdot\mathsf{d}_\Gamma$. (Recall that $\mathsf{d}_\Gamma=\prod_{\kappa,\lambda\in\Gamma}\mathsf{d}_{\kappa\lambda}$.) It is clear that E is reflexive and symmetric, but it is also transitive. For if $\langle a,\xi\rangle E\langle b,\eta\rangle E\langle c,\zeta\rangle$, then we have finite $\Gamma,\Delta\subseteq\alpha$ with $\xi,\eta\in\Gamma$, $\eta,\zeta\in\Delta$, $\mathsf{c}_{(\Gamma\sim\{\xi\})}a\cdot\mathsf{d}_\Gamma=\mathsf{c}_{(\Gamma\sim\{\eta\})}b\cdot\mathsf{d}_\Gamma$, $\mathsf{c}_{(\Delta\sim\{\eta\})}b\cdot\mathsf{d}_\Delta=\mathsf{c}_{\Delta\sim\{\zeta\})}c\cdot\mathsf{d}_\Delta$. Putting $\Phi=\Gamma\cup\Delta$, so that $\xi,\eta,\zeta\in\Phi$, if we apply $\mathsf{c}_{(\Phi\sim\Gamma)}$ to both sides of the first equation we get $\mathsf{c}_{(\Phi\sim\{\xi\})}a\cdot\mathsf{d}_\Gamma=\mathsf{c}_{(\Phi\sim\{\eta\})}b\cdot\mathsf{d}_\Gamma$, hence $\mathsf{c}_{(\Phi\sim\{\xi\})}a\cdot\mathsf{d}_\Phi=\mathsf{c}_{(\Phi\sim\{\eta\})}b\cdot\mathsf{d}_\Phi$. Similarly $\mathsf{c}_{(\Phi\sim\{\eta\})}b\cdot\mathsf{d}_\Phi=\mathsf{c}_{(\Phi\sim\{\zeta\})}c\cdot\mathsf{d}_\Phi$. Thus $\langle a,\xi\rangle E\langle c,\zeta\rangle$, as desired.

Now let U be the set of all equivalence classes of $At\mathfrak{A}\times\alpha$ under E. Define a map h of $At\mathfrak{A}$ into ${}^\alpha U$ by setting $(ha)\xi=\langle a,\xi\rangle/E$ for each $a\in At\mathfrak{A}$ and $\xi\in\alpha$. Let V be the range of h, and let $\mathfrak{C}$ be the full Crs_α with unit element V. For each $x\in A$ let $jx=\{ha: a \text{ an atom } \leqq x\}$. To complete our proof it suffices to show that $\mathfrak{C}$ is a Gws_α and j is an isomorphism of $\mathfrak{A}$ into $\mathfrak{C}$. (The proof below could be simplified slightly

by working with the atom structure of $\mathfrak{A}$, defined in 2.7.32. But we do not want to assume acquaintance with that material.)

Before proceeding we note the following fact about our relation E, which is easily established:

(1) If $a,b \in At\mathfrak{A}$ and $\kappa<\alpha$, then $\langle a,\kappa\rangle E\langle b,\kappa\rangle$ iff there is a finite $\Gamma\subseteq\alpha$ such that $\kappa\notin\Gamma$ and $\mathsf{c}_{(\Gamma)}a=\mathsf{c}_{(\Gamma)}b$.

Now we show that h is one–one. Suppose that $ha=hb$. In particular, $\langle a,0\rangle E\langle b,0\rangle$, so $\mathsf{c}_{(\Gamma)}a=\mathsf{c}_{(\Gamma)}b$ for some Γ, by (1); we choose such a Γ of smallest cardinality. If $\Gamma=0$, then $a=b$, as desired. Suppose $\Gamma\neq 0$, and choose $\xi\in\Gamma$. Since $\langle a,\xi\rangle E\langle b,\xi\rangle$, choose a finite $\Delta\subseteq\alpha$ such that $\xi\notin\Delta$ and $\mathsf{c}_{(\Delta)}a=\mathsf{c}_{(\Delta)}b$. Then, using the fact that a and b are rectangular,

$$\mathsf{c}_{(\Gamma\cap\Delta)}a=\mathsf{c}_{(\Gamma)}a\cdot\mathsf{c}_{(\Delta)}a=\mathsf{c}_{(\Gamma)}b\cdot\mathsf{c}_{(\Delta)}b=\mathsf{c}_{(\Gamma\cap\Delta)}b.$$

But $\xi\in\Gamma\sim(\Gamma\cap\Delta)$, so $|\Gamma\cap\Delta|<|\Gamma|$, a contradiction. So $a=b$, and h is one–one.

It follows easily that j is an isomorphism of $\mathfrak{Bl}\mathfrak{A}$ into $\mathfrak{Bl}\mathfrak{C}$. Next suppose that $\kappa,\lambda<\alpha$. To show that $j\mathsf{d}_{\kappa\lambda}=\mathsf{D}^{[V]}_{\kappa\lambda}$, it suffices to assume that $\kappa\neq\lambda$. Suppose that $a\in At\mathfrak{A}$, and first suppose that $a\leqq\mathsf{d}_{\kappa\lambda}$ (so that $ha\in j\mathsf{d}_{\kappa\lambda}$). Then $\mathsf{c}_\kappa a\cdot\mathsf{d}_{\kappa\lambda}=\mathsf{c}_\lambda a\cdot\mathsf{d}_{\kappa\lambda}=a\cdot\mathsf{d}_{\kappa\lambda}$, so $\langle a,\kappa\rangle E\langle a,\lambda\rangle$; hence $(ha)\kappa=(ha)\lambda$ and $ha\in\mathsf{D}^{[V]}_{\kappa\lambda}$. Second suppose that $ha\in\mathsf{D}^{[V]}_{\kappa\lambda}$. Thus $\langle a,\kappa\rangle E\langle a,\lambda\rangle$, so choose a finite $\Gamma\subseteq\alpha$ with $\kappa,\lambda\in\Gamma$ and $\mathsf{c}_{(\Gamma\sim\{\kappa\})}a\cdot\mathsf{d}_\Gamma=\mathsf{c}_{(\Gamma\sim\{\lambda\})}a\cdot\mathsf{d}_\Gamma$. Then $\mathsf{c}_{(\Gamma\sim\{\kappa,\lambda\})}a\cdot\mathsf{d}_\Gamma=\mathsf{c}_{(\Gamma\sim\{\kappa\})}a\cdot\mathsf{c}_{(\Gamma\sim\{\lambda\})}a\cdot\mathsf{d}_\Gamma$ (since a is rectangular) $=\mathsf{c}_{(\Gamma\sim\{\kappa\})}a\cdot\mathsf{d}_\Gamma\neq 0$ (by 1.10.3). Hence $0\neq a\cdot\mathsf{c}_{(\Gamma\sim\{\kappa,\lambda\})}\mathsf{d}_\Gamma=a\cdot\mathsf{d}_{\kappa\lambda}$ using 1.8.6. Thus $a\leqq\mathsf{d}_{\kappa\lambda}$, as desired.

Next we show that j preserves c_κ for any $\kappa<\alpha$. Let $a\in At\mathfrak{A}$, $x\in A$. First suppose that $a\leqq\mathsf{c}_\kappa x$, so that $ha\in j\mathsf{c}_\kappa x$. Choose $b\in At\mathfrak{A}$ with $b\leqq x$ and $a\leqq\mathsf{c}_\kappa b$. Then by 1.10.3($i$), $\mathsf{c}_\kappa a=\mathsf{c}_\kappa b$. Hence by (1), $(ha)\lambda=(hb)\lambda$ for all $\lambda\neq\kappa$. So $ha\in\mathsf{C}^{[V]}_\kappa jx$, as desired. Second, suppose that $ha\in\mathsf{C}^{[V]}_\kappa jx$. Say $b\in At\mathfrak{A}$, $b\leqq x$, $(\alpha\sim\{\kappa\})\upharpoonright ha\subseteq hb$. Since $\alpha\geqq 2$, there is a $\lambda\in\alpha\sim\{\kappa\}$, hence $(ha)\lambda=(hb)\lambda$, hence $\mathsf{c}_{(\Gamma)}a=\mathsf{c}_{(\Gamma)}b$ for finite $\Gamma\subseteq\alpha$. We choose such a Γ with $|\Gamma|$ minimum. Suppose there is a $\mu\in\Gamma\sim\{\kappa\}$. Then $(ha)\mu=(hb)\mu$, so by (1) there is a finite $\Delta\subseteq\alpha$ with $\mathsf{c}_{(\Delta)}a=\mathsf{c}_{(\Delta)}b$, $\mu\notin\Delta$. Since a and b are rectangular we easily get $\mathsf{c}_{(\Gamma\cap\Delta)}a=\mathsf{c}_{(\Gamma\cap\Delta)}b$. Since $|\Gamma\cap\Delta|<|\Gamma|$, this is a contradiction. Thus $\Gamma\subseteq\{\kappa\}$, so $\mathsf{c}_\kappa a=\mathsf{c}_\kappa b$. Hence $a\leqq\mathsf{c}_\kappa x$ and $ha\in j\mathsf{c}_\kappa x$, as desired.

Before proceeding, we note the following facts:

(2) If Γ is a finite subset of α, $\Gamma\neq\alpha$, and $x\in A$, then $\mathsf{C}^{[V]}_{(\Gamma)}jx=\{ha$: a is an atom and there is an atom $b\leqq x$ such that $(\alpha\sim\Gamma)\upharpoonright ha\subseteq hb\}$.

For, $\mathsf{C}^{[V]}_{(\Gamma)}jx=j\mathsf{c}_{(\Gamma)}x=\{ha$: a is an atom $\leqq\mathsf{c}_{(\Gamma)}x\}$. Hence if $ha\in\mathsf{C}^{[V]}_{(\Gamma)}jx$, let b be an atom $\leqq x$ such that $a\leqq\mathsf{c}_{(\Gamma)}b$. Then $\mathsf{c}_{(\Gamma)}a=\mathsf{c}_{(\Gamma)}b$, so by (1) we have $(\alpha\sim\Gamma)\upharpoonright ha\subseteq hb$. Conversely, suppose a and b are atoms, $b\leqq x$, and $(\alpha\sim\Gamma)\upharpoonright ha\subseteq hb$. Choose $\kappa\in\alpha\sim\Gamma$. Then $(ha)\kappa=(hb)\kappa$, so by (1) there is a finite $\Delta\subseteq\alpha$ such that $\mathsf{c}_{(\Delta)}a=\mathsf{c}_{(\Delta)}b$; using

rectangularity we easily find that $c_{(\Gamma)}a = c_{(\Gamma)}b$. Thus $a \leq c_{(\Gamma)}x$, so $ha \in \mathbf{C}^{|V|}_{(\Gamma)}jx$, as desired.

(3) Suppose $a \in At\mathfrak{A}$, $\kappa<\alpha$, x is a function with domain α, $(\alpha\sim\{\kappa\}) \upharpoonright ha \subseteq x$, b is an atom, $b \leq c_{(\Gamma)}a$ for some finite $\Gamma \subseteq \alpha$, $\lambda<\alpha$, and $x\kappa = (hb)\lambda$. Then $x \in V$.

To prove this, we may assume that $\kappa,\lambda \in \Gamma$. First suppose $\kappa = \lambda$. We have $c_\kappa a \cdot c_{(\Gamma\sim\{\kappa\})}b \neq 0$ since $b \leq c_{(\Gamma)}a$; let c be an atom $\leq c_\kappa a \cdot c_{(\Gamma\sim\{\kappa\})}b$. Then by (1) we have $(\alpha\sim\{\kappa\}) \upharpoonright ha \subseteq hc$ and $(hc)\kappa = (hb)\kappa = x\kappa$. Thus $hc = x$, as desired. Now suppose that $\kappa \neq \lambda$. Then there is an atom $c \leq c_\kappa a \cdot c_{(\Gamma\sim\{\kappa\})}(d_{\kappa\lambda} \cdot c_{(\Gamma\sim\{\lambda\})}b)$; say d is an atom $\leq d_{\kappa\lambda} \cdot c_{(\Gamma\sim\{\lambda\})}b$ and $c \leq c_{(\Gamma\sim\{\kappa\})}d$. Then $(\alpha\sim\{\kappa\}) \upharpoonright ha \subseteq hc$, and $(hc)\kappa = (hd)\kappa = (hd)\lambda = (hb)\lambda = x\kappa$, using (1). So again $hc = x$, as desired.

Now it remains only to show that V is a Gws_α unit element. First suppose that $\alpha<\omega$. For each α-atom k, let $U_k = \{(ha)\kappa\colon a \text{ is an atom } \leq k,\ \kappa<\alpha\}$.

(4) If k and l are distinct α-atoms, then $U_k \cap U_l = 0$.

For, suppose $x \in U_k \cap U_l$. Say $x = (ha)\kappa = (hb)\lambda$ with a and b atoms $\leq k$ and l respectively, and $\kappa,\lambda<\alpha$. Thus $\langle a,\kappa\rangle E \langle b,\lambda\rangle$, so there is a finite $\Gamma \subseteq \alpha$ with $\kappa,\lambda \in \Gamma$ and $c_{(\Gamma\sim\{\kappa\})}a \cdot d_\Gamma = c_{(\Gamma\sim\{\lambda\})}b \cdot d_\Gamma$. Hence $c_{(\Gamma\sim\{\kappa\})}a \cdot c_{(\Gamma\sim\{\lambda\})}b \neq 0$ and so $k = l$.

(5) $V = \bigcup\{{}^\alpha U_k\colon k \text{ an } \alpha\text{-atom}\}$.

For, if a is any atom, obviously $ha \in {}^\alpha U_k$, with $k = c_{(a)}a$. Thus $\subseteq$ holds. The direction $\supseteq$ follows from the following statement (with $\kappa = \alpha$):

(6) Suppose that k is an α-atom, a is an atom $\leq k$, $\kappa \leq \alpha$, $x \in {}^\alpha U_k$, and $x\lambda = (ha)\lambda$ for all $\lambda \in \alpha\sim\kappa$. Then $x \in V$.

We prove (6) by induction on κ. The case $\kappa = 0$ is trivial. Assume that it is true for $\kappa-1$ $(\kappa>0)$, and assume its hypotheses. Let $(\alpha\sim\{\kappa-1\}) \upharpoonright x \subseteq y$, $y(\kappa-1) = (ha)(\kappa-1)$, $y \in {}^\alpha U_k$. Thus $y\lambda = (ha)\lambda$ for all $\lambda \in \alpha\sim(\kappa-1)$, so $y \in V$ by the induction hypothesis. Then (3) yields $x \in V$, as desired.

Second, we suppose that $\alpha \geq \omega$. We define an equivalence relation $\equiv$ on V by setting $x \equiv y$ iff $x,y \in V$ and $|\{\kappa<\alpha\colon x\kappa \neq y\kappa\}|<\omega$. For each $\equiv$-class k, let $Y_k = \{x\kappa\colon x \in k,\ \kappa<\alpha\}$. Thus ${}^\alpha Y^{(x)}_{x/\equiv} \cap V \subseteq x/\equiv$ for any $x \in V$, and hence it suffices to show that ${}^\alpha Y^{(x)}_{x/\equiv} \subseteq V$ for any $x \in V$. To do this it suffices to show:

(7) If Γ is a finite subset of α, $x \in V$, $y \in {}^\alpha Y_{x/\equiv}$, and $(\alpha\sim\Gamma) \upharpoonright x \subseteq y$, then $y \in V$.

We prove (7) by induction on $|\Gamma|$. The case $\Gamma = 0$ is trivial. Assume inductively that $\Gamma \neq 0$, and fix $\kappa \in \Gamma$. Let z be the element of ${}^\alpha Y_{x/\equiv}$ such that $(\alpha\sim\{\kappa\}) \upharpoonright y \subseteq z$ and $z\kappa = x\kappa$. Then $z \in V$ by the inductive hypothesis; say $z = hb$ with $b \in At\mathfrak{A}$. Since $y\kappa \in Y_{x/\equiv}$, there exist $w \in V$, a finite $\Delta \subseteq \alpha$, and a $\lambda<\alpha$ such that $(\alpha\sim\Delta) \upharpoonright x \subseteq w$ and $y\kappa = w\lambda$. Let v be such that $(\alpha\sim\{\kappa\}) \upharpoonright w \subseteq v$ and $v\kappa = w\lambda$. Then by (3), $v \in V$, say $v = hd$. Let $\Omega = \Gamma \cup \Delta \cup \{\kappa\}$. Then for any $\mu \in \alpha\sim\Omega$ we have $y\mu = x\mu = z\mu = w\mu = v\mu$. Hence $\mathbf{C}^{|V|}_{(\Omega)}\{x\} \subseteq \mathbf{C}^{|V|}_{(\Omega)}\{hb\} \cap \mathbf{C}^{|V|}_{(\Omega)}\{hd\}$ by (2), so $c_{(\Omega)}b \cdot c_{(\Omega)}d \neq 0$, hence $c_{(\Omega\sim\{\kappa\})}d \cdot c_\kappa b \neq 0$.

Let e be an atom $\leq \mathsf{c}_{(\Omega\sim\{\kappa\})}d\cdot\mathsf{c}_\kappa b$. Thus $\mathsf{c}_\kappa e=\mathsf{c}_\kappa b$, so $(\alpha\sim\{\kappa\})\upharpoonright he=(\alpha\sim\{\kappa\})\upharpoonright hb\subseteq y$. Also, $\mathsf{c}_{(\Omega\sim\{\kappa\})}e=\mathsf{c}_{(\Omega\sim\{\kappa\})}d$, so by (1) $(he)\kappa=(hd)\kappa=w\lambda=y\kappa$. Thus $y=he\in V$, as desired. This finishes the proof of 3.2.14.

THEOREM 3.2.16. *For $\alpha\geq 2$, a CA_α is representable iff it can be embedded in an atomic CA_α in which all atoms are rectangular.*

The proof is immediate from 3.2.14. Theorem 3.2.16 can be given a formulation using atomic structures; see Remarks 2.7.45.

Positive Characteristic

Our next representation result, due to Henkin, is that every CA_α of positive characteristic is representable for $\alpha<\omega$; we have already noted in 3.2.11 that this applies for $\alpha\geq\omega$.

REMARK 3.2.17. To establish this representation result we need an auxiliary result, due independently to Comer and Henkin, that for $\alpha<\omega$ any CA_α of positive characteristic can be provided with a substitution operator s satisfying the conditions in 1.11.12. Actually we do not need all of the conditions of 1.11.12, so we just state those that we do need. To establish this result it is convenient to use the main theorem in Jónsson [62], which we repeat here for the reader's convenience. Suppose that I is any set. An *elementary transformation* of I is a mapping of I into itself of the form $[x/y]$ or $[x/y,y/x]$ with $x,y\in I$. Recall that $[x/y]$ is that transformation of I which sends x to y and z to z for all $z\in I\sim\{x\}$; $[x/y,y/x]$ sends x to y, y to x, and z to z for all $z\in I\sim\{x,y\}$. For brevity we denote $[x/y,y/x]$ by $[x,y]$.

Jónsson's theorem is as follows. Let $\langle S,\cdot,e\rangle$ be a semigroup with identity e. Suppose that we are given a mapping s from the elementary transformations of I into S. Then the following conditions are equivalent:

(A) s extends to a mapping s^+ from the set of all finite transformations of I into S such that $\mathsf{s}^+(\sigma\circ\tau)=\mathsf{s}^+\sigma\cdot\mathsf{s}^+\tau$ for any two such transformations and $\mathsf{s}^+(I\upharpoonright Id)=e$. (A *transformation* of I is just a mapping of I into I. A transformation σ of I is *finite* if $\{x\in I:\sigma x\neq x\}$ is finite.)

(B) If x,y,z are distinct elements of I and y,u,z are distinct elements of I then the following conditions hold:

(I) $\mathsf{s}[x,y]=\mathsf{s}[y,x]$,
(II) $\mathsf{s}[x,y]\cdot\mathsf{s}[y,x]=e$,
(III) $\mathsf{s}[x,y]\cdot\mathsf{s}[x,z]=\mathsf{s}[y,z]\cdot\mathsf{s}[x,y]$,
(IV) $\mathsf{s}[x,y]\cdot\mathsf{s}[z/x]=\mathsf{s}[z/y]\cdot\mathsf{s}[x,y]$,
(V) $\mathsf{s}[x,y]\cdot\mathsf{s}[y/x]=\mathsf{s}[x/y]$,
(VI) $\mathsf{s}[y/x]\cdot\mathsf{s}[z/u]=\mathsf{s}[z/u]\cdot\mathsf{s}[y/x]$,
(VII) $\mathsf{s}[y/x]\cdot\mathsf{s}[y/u]=\mathsf{s}[y/u]$.

It is a lengthy process to establish this auxiliary result, and we break the proof into a series of lemmas. *For all of the lemmas we assume that $\alpha<\omega$, and we use $\kappa,\lambda,\mu,\ldots$ for ordinals $<\alpha$.* Some of the lemmas are true for arbitrary CA_α's, $\alpha<\omega$, but for many we indicate by (A) the additional assumption that the CA_α involved has positive characteristic. To check some of the lemmas, results from section 1.5 will be used without explicit citation. The proof is due to Henkin and Monk.

We begin with a result in the spirit of section 1.5.

LEMMA 3.2.18. *If* $\kappa \neq \lambda$, $\{\kappa,\lambda\} \cap \{\mu,\nu,\rho,\sigma\} = 0$, $|\{\mu,\nu,\rho\}| = |\{\mu,\nu,\sigma\}| = 3$, $\{\kappa,\lambda\} \cap \{\gamma,\delta,\varepsilon,\xi\} = 0$, $|\{\gamma,\delta,\varepsilon\}| = |\{\gamma,\delta,\xi\}| = 3$, *and* $e = -\mathsf{d}_{\kappa\lambda}\cdot\mathsf{d}_{\mu\rho}\cdot\mathsf{d}_{\nu\sigma}\cdot\mathsf{d}_{\gamma\varepsilon}\cdot\mathsf{d}_{\delta\xi}$, *then* $e\cdot{}_\mu\mathsf{s}(\kappa,\lambda)\mathsf{c}_\mu\mathsf{c}_\nu(x\cdot\mathsf{d}_{\mu\rho}\cdot\mathsf{d}_{\nu\sigma}) = e\cdot{}_\gamma\mathsf{s}(\kappa,\lambda)\mathsf{c}_\gamma\mathsf{c}_\delta(x\cdot\mathsf{d}_{\gamma\varepsilon}\cdot\mathsf{d}_{\delta\xi})$.

PROOF. Clearly one of the following conditions holds: $\mu \neq \gamma,\varepsilon$ or $\mu \neq \delta,\xi$ or $\nu \neq \gamma,\varepsilon$ or $\nu \neq \delta,\xi$. By symmetry, say $\mu \neq \gamma,\varepsilon$. Then

$$e\cdot{}_\mu\mathsf{s}(\kappa,\lambda)\mathsf{c}_\mu\mathsf{c}_\gamma(x\cdot\mathsf{d}_{\mu\rho}\cdot\mathsf{d}_{\nu\sigma}) = e\cdot{}_\mu\mathsf{s}(\kappa,\lambda)\mathsf{c}_\mu\mathsf{c}_\gamma(x\cdot\mathsf{d}_{\mu\rho}\cdot\mathsf{d}_{\gamma\varepsilon}) = e\cdot{}_\gamma\mathsf{s}(\kappa,\lambda)\mathsf{c}_\mu\mathsf{c}_\gamma(x\cdot\mathsf{d}_{\mu\rho}\cdot\mathsf{d}_{\gamma\varepsilon});$$

if $\rho \neq \gamma$, this is equal to $e\cdot{}_\gamma\mathsf{s}(\kappa,\lambda)\mathsf{c}_\gamma(x\cdot\mathsf{d}_{\gamma\varepsilon})$ which is equal to $e\cdot{}_\gamma\mathsf{s}(\kappa,\lambda)\mathsf{c}_\gamma\mathsf{c}_\delta(x\cdot\mathsf{d}_{\gamma\varepsilon}\cdot\mathsf{d}_{\delta\xi})$. If $\rho = \gamma$, then it is equal to

$$e\cdot{}_\gamma\mathsf{s}(\kappa,\lambda)\mathsf{c}_\mu\mathsf{c}_\gamma(x\cdot\mathsf{d}_{\mu\varepsilon}\cdot\mathsf{d}_{\gamma\varepsilon}) = e\cdot{}_\gamma\mathsf{s}(\kappa,\lambda)\mathsf{c}_\gamma(x\cdot\mathsf{d}_{\gamma\varepsilon}) = e\cdot{}_\gamma\mathsf{s}(\kappa,\lambda)\mathsf{c}_\gamma\mathsf{c}_\delta(x\cdot\mathsf{d}_{\gamma\varepsilon}\cdot\mathsf{d}_{\delta\xi}).$$

This finishes the proof.

DEFINITION 3.2.19. *We set*

$$\begin{aligned} g_{\rho\sigma} &= \Pi\{-\mathsf{d}_{\mu\nu} : \mu,\nu<\alpha,\ \mu \neq \nu, \{\mu,\nu\} \cap \{\rho,\sigma\} = 0\}; \\ e_\rho &= g_{\rho\rho}; \\ f_{\rho\sigma} &= \mathsf{d}_{\rho\sigma}\cdot e_\rho. \end{aligned}$$

We give some properties of these notions which will be used later.

LEMMA 3.2.20. (i) $f_{\rho\sigma} = f_{\sigma\rho}$.
(ii) *If* $|\{\kappa,\lambda,\mu\}| = 3$, *then* $f_{\mu\lambda}\cdot\mathsf{s}^\mu_\kappa x = f_{\mu\lambda}\cdot\mathsf{s}^\mu_\kappa(f_{\mu\kappa}\cdot x)$.
(iii) *If* $|\{\kappa,\lambda,\mu\}| = 3$, *then* $f_{\mu\lambda}\cdot\mathsf{s}^\mu_\kappa x = 0$ *iff* $f_{\mu\kappa}\cdot x = 0$.
(iv) (A) *If* $\rho \neq \sigma$, *then* $e_\rho\cdot e_\sigma \leqq \mathsf{d}_{\rho\sigma}$.

LEMMA 3.2.21 (A). *If* $\rho \neq \sigma$, *then* $f_{\rho\sigma}\cdot\mathsf{c}_\rho\mathsf{c}_\sigma(f_{\rho\sigma}\cdot x) = f_{\rho\sigma}\cdot x$.

PROOF. We only need to show $\leqq$. Now $f_{\rho\sigma}\cdot\mathsf{c}_\rho\mathsf{c}_\sigma(f_{\rho\sigma}\cdot x)\cdot -x = 0$ iff $\mathsf{c}_\rho(f_{\rho\sigma}\cdot -x)\cdot\mathsf{c}_\sigma(f_{\rho\sigma}\cdot x) = 0$. Since $\mathsf{c}_\rho(f_{\rho\sigma}\cdot -x)\cdot\mathsf{c}_\sigma(f_{\rho\sigma}\cdot x) = \mathsf{s}^\rho_\sigma -x\cdot e_\rho\cdot\mathsf{s}^\sigma_\rho x\cdot e_\sigma$ and $e_\rho\cdot e_\sigma \leqq \mathsf{d}_{\rho\sigma}$, the desired conclusion follows.

LEMMA 3.2.22 (A). *If* $|\{\rho,\sigma,\mu\}| = 3$, *then* $f_{\rho\sigma}\cdot\mathsf{c}_\mu(f_{\rho\sigma}\cdot x) = f_{\rho\sigma}\cdot x$.

PROOF. Again we only need to show $\leqq$. We have

$$\begin{aligned} f_{\rho\sigma}\cdot\mathsf{c}_\mu(f_{\rho\sigma}\cdot x) &\leqq \mathsf{d}_{\rho\sigma}\cdot e_\rho\cdot\mathsf{c}_\mu\mathsf{c}_\rho(f_{\rho\sigma}\cdot x) \\ &= \mathsf{d}_{\rho\sigma}\cdot e_\rho\cdot\mathsf{c}_\mu(e_\rho\cdot\mathsf{s}^\rho_\sigma x) = \mathsf{d}_{\rho\sigma}\cdot\mathsf{c}_\rho(f_{\mu\rho}\cdot\mathsf{c}_\mu(e_\rho\cdot\mathsf{s}^\rho_\sigma x)) \\ &= \mathsf{d}_{\rho\sigma}\cdot\mathsf{c}_\rho(f_{\mu\rho}\cdot\mathsf{c}_\mu\mathsf{c}_\rho(f_{\mu\rho}\cdot\mathsf{s}^\rho_\sigma x)) = \mathsf{d}_{\rho\sigma}\cdot\mathsf{c}_\rho(f_{\mu\rho}\cdot\mathsf{s}^\rho_\sigma x) \leqq x \text{ (by 3.2.21)}, \end{aligned}$$

as desired.

LEMMA 3.2.23 (A). *If* $|\{\kappa,\lambda,\rho,\sigma\}|=4$, *then* $f_{\rho\sigma}\cdot{}_\rho\mathsf{s}(\kappa,\lambda)\mathsf{c}_\rho\mathsf{c}_\sigma(x\cdot f_{\rho\sigma})=f_{\rho\sigma}\cdot{}_\rho\mathsf{s}(\kappa,\lambda)\mathsf{s}^\rho_\sigma x$.

PROOF. We have

$$\begin{aligned}
f_{\rho\sigma}\cdot{}_\rho\mathsf{s}(\kappa,\lambda)\mathsf{c}_\rho\mathsf{c}_\sigma(x\cdot f_{\rho\sigma}) &= f_{\rho\sigma}\cdot{}_\rho\mathsf{s}(\kappa,\lambda)(f_{\lambda\rho}\cdot\mathsf{c}_\sigma(\mathsf{s}^\rho_\sigma x\cdot e_\rho)) \text{ (by 3.2.20(ii))}\\
&= f_{\rho\sigma}\cdot{}_\rho\mathsf{s}(\kappa,\lambda)(f_{\lambda\rho}\cdot\mathsf{c}_\sigma(\mathsf{s}^\rho_\sigma x\cdot f_{\lambda\rho}))\\
&= f_{\rho\sigma}\cdot{}_\rho\mathsf{s}(\kappa,\lambda)(f_{\lambda\rho}\cdot\mathsf{s}^\rho_\sigma x) \text{ (by 3.2.22)}\\
&= f_{\rho\sigma}\cdot{}_\rho\mathsf{s}(\kappa,\lambda)\mathsf{s}^\rho_\sigma x \text{ (by 3.2.20(ii))}.
\end{aligned}$$

Now we are ready to define $\mathsf{s}[\kappa,\lambda]$, to be used as indicated in 3.2.17

DEFINITION 3.2.24. *For* $\kappa\neq\lambda$ *we set*

$$\begin{aligned}
\mathsf{s}[\kappa,\lambda]x &= \mathsf{d}_{\kappa\lambda}\cdot x+\Sigma_{\rho\neq\kappa,\lambda}-\mathsf{d}_{\kappa\lambda}\cdot\mathsf{d}_{\kappa\rho}\cdot\mathsf{s}^\kappa_\lambda\mathsf{s}^\lambda_\rho x\\
&\quad+\Sigma_{\rho\neq\kappa,\lambda}-\mathsf{d}_{\kappa\lambda}\cdot\mathsf{d}_{\lambda\rho}\cdot\mathsf{s}^\lambda_\kappa\mathsf{s}^\kappa_\rho x\\
&\quad+\Sigma\{-\mathsf{d}_{\kappa\lambda}\cdot\mathsf{d}_{\mu\rho}\cdot\mathsf{d}_{\nu\sigma}\cdot{}_\mu\mathsf{s}(\kappa,\lambda)\mathsf{c}_\mu\mathsf{c}_\nu(\mathsf{d}_{\mu\rho}\cdot\mathsf{d}_{\nu\sigma}\cdot x):\{\kappa,\lambda\}\cap\{\mu,\nu,\rho,\sigma\}=0,\ |\{\mu,\nu,\rho\}|=3=|\{\mu,\nu,\sigma\}|\}\\
&\quad+\Sigma_{|\{\kappa,\lambda,\rho,\sigma\}|=4}f_{\rho\sigma}\cdot{}_\rho\mathsf{s}(\kappa,\lambda)\mathsf{s}^\rho_\sigma x.
\end{aligned}$$

We list some immediate properties of $\mathsf{s}[\kappa,\lambda]$; for (v)–(vii) see 3.2.26–3.2.30.

LEMMA 3.2.25. *Assume* $\kappa\neq\lambda$.
(i) $\mathsf{s}[\kappa,\lambda]$ *preserves* $+$.
(ii) $\mathsf{s}[\kappa,\lambda]0=0$.
(iii) (A) $\mathsf{s}[\kappa,\lambda]1=1$.
(iv) (A) $\mathsf{s}[\kappa,\lambda]=\mathsf{s}[\lambda,\kappa]$.
(v) $\mathsf{s}[\kappa,\lambda]\mathsf{d}_{\kappa\lambda}=\mathsf{d}_{\kappa\lambda}$.
(vi) If $\rho\neq\kappa,\lambda$, *then* $\mathsf{s}[\kappa,\lambda]\mathsf{d}_{\kappa\rho}=\mathsf{d}_{\lambda\rho}$ *and* $\mathsf{s}[\kappa,\lambda]\mathsf{d}_{\lambda\rho}=\mathsf{d}_{\kappa\rho}$.
(vii) If $|\{\kappa,\lambda,\rho,\sigma\}|=4$, *then* $\mathsf{s}[\kappa,\lambda]d_{\rho\sigma}=d_{\rho\sigma}$.

Next we give some lemmas which will enable us to break further arguments concerning $\mathsf{s}[\kappa,\lambda]$ into cases.

LEMMA 3.2.26. *If* $\kappa\neq\lambda$, *then* $\mathsf{d}_{\kappa\lambda}\cdot\mathsf{s}[\kappa,\lambda]x=\mathsf{d}_{\kappa\lambda}\cdot x$.

LEMMA 3.2.27. *If* $|\{\kappa,\lambda,\rho\}|=3$, *then* $-\mathsf{d}_{\kappa\lambda}\cdot\mathsf{d}_{\kappa\rho}\cdot\mathsf{s}[\kappa,\lambda]x=-\mathsf{d}_{\kappa\lambda}\cdot\mathsf{d}_{\kappa\rho}\cdot\mathsf{s}^\kappa_\lambda\mathsf{s}^\lambda_\rho x$.

PROOF. $\geqq$ is clear. Now let $r=-\mathsf{d}_{\kappa\lambda}\cdot\mathsf{d}_{\kappa\rho}\cdot\mathsf{s}^\kappa_\lambda\mathsf{s}^\lambda_\rho x$. For $\sigma\neq\kappa,\lambda,\rho$ let $t=-\mathsf{d}_{\kappa\lambda}\cdot\mathsf{d}_{\kappa\rho}\cdot\mathsf{d}_{\lambda\sigma}$. Now $-\mathsf{d}_{\kappa\lambda}\cdot\mathsf{d}_{\kappa\rho}\cdot\mathsf{d}_{\kappa\sigma}\cdot\mathsf{s}^\kappa_\lambda\mathsf{s}^\lambda_\sigma x\leqq r$; $-\mathsf{d}_{\kappa\lambda}\cdot\mathsf{d}_{\kappa\rho}\cdot\mathsf{d}_{\lambda\rho}\cdot\mathsf{s}^\lambda_\rho\mathsf{s}^\kappa_\rho x=0$; and

$$t\cdot\mathsf{s}^\lambda_\kappa\mathsf{s}^\kappa_\sigma x=t\cdot\mathsf{s}^\lambda_\rho\mathsf{s}^\kappa_\sigma x=t\cdot\mathsf{s}^\kappa_\sigma\mathsf{s}^\lambda_\rho x\leqq r.$$

Next, suppose that $\{\kappa,\lambda\}\cap\{\rho,\mu,\nu,\sigma\}=0$. Suppose $|\{\rho,\mu,\nu\}|=3=|\{\rho,\mu,\sigma\}|$. Let $t=-\mathsf{d}_{\kappa\lambda}\cdot\mathsf{d}_{\kappa\rho}\cdot\mathsf{d}_{\rho\nu}\cdot\mathsf{d}_{\mu\sigma}$. Then

$$t\cdot_\rho \mathsf{s}(\kappa,\lambda)\mathsf{c}_\rho\mathsf{c}_\mu(\mathsf{d}_{\rho\nu}\cdot\mathsf{d}_{\mu\sigma}\cdot x) = t\cdot\mathsf{s}_\lambda^\kappa\mathsf{s}_\rho^\lambda\mathsf{c}_\rho(\mathsf{d}_{\rho\nu}\cdot\mathsf{d}_{\mu\sigma}\cdot x) = t\cdot\mathsf{s}_\lambda^\kappa\mathsf{s}_\rho^\lambda(\mathsf{d}_{\rho\nu}\cdot\mathsf{d}_{\mu\sigma}\cdot x) \leqq r.$$

If on the other hand $|\{\mu,\nu,\rho\}| = 3 = |\{\mu,\nu,\sigma\}|$, then with $t = -\mathsf{d}_{\kappa\lambda}\cdot\mathsf{d}_{\kappa\rho}\cdot\mathsf{d}_{\mu\rho}\cdot\mathsf{d}_{\nu\sigma}$,

$$t\cdot_\mu \mathsf{s}(\kappa,\lambda)\mathsf{c}_\mu\mathsf{c}_\nu(\mathsf{d}_{\mu\rho}\cdot\mathsf{d}_{\nu\sigma}\cdot x) = t\cdot\mathsf{s}_\lambda^\kappa\mathsf{s}_\mu^\lambda(\mathsf{d}_{\mu\rho}\cdot\mathsf{d}_{\nu\sigma}\cdot x) \leqq r.$$

Finally, suppose that $\{\kappa,\lambda,\rho\}\cap\{\mu,\nu,\sigma,\tau\} = 0$ and $|\{\mu,\nu,\sigma\}| = 3 = |\{\mu,\nu,\tau\}|$. Let $t = -\mathsf{d}_{\kappa\lambda}\cdot\mathsf{d}_{\kappa\rho}\cdot\mathsf{d}_{\mu\sigma}\cdot\mathsf{d}_{\nu\tau}$. Then

$$\begin{aligned} t\cdot_\mu \mathsf{s}(\kappa,\lambda)\mathsf{c}_\mu\mathsf{c}_\nu(\mathsf{d}_{\mu\sigma}\cdot\mathsf{d}_{\nu\tau}\cdot x) &= t\cdot\mathsf{s}_\lambda^\mu\mathsf{s}_\kappa^\lambda\mathsf{s}_\mu^\kappa\mathsf{c}_\mu(\mathsf{d}_{\mu\sigma}\cdot\mathsf{d}_{\nu\tau}\cdot x) \\ = t\cdot\mathsf{s}_\lambda^\mu\mathsf{s}_\mu^\kappa\mathsf{s}_\rho^\lambda\mathsf{c}_\mu(\mathsf{d}_{\mu\sigma}\cdot\mathsf{d}_{\nu\tau}\cdot x) &= t\cdot\mathsf{s}_\lambda^\kappa\mathsf{s}_\rho^\lambda\mathsf{c}_\mu(\mathsf{d}_{\mu\sigma}\cdot\mathsf{d}_{\nu\tau}\cdot x) \leqq r. \end{aligned}$$

The lemma follows.

By symmetry we obtain:

LEMMA 3.2.28. *If* $|\{\kappa,\lambda,\rho\}| = 3$, *then* $-\mathsf{d}_{\kappa\lambda}\cdot\mathsf{d}_{\lambda\rho}\cdot\mathsf{s}[\kappa,\lambda]x = -\mathsf{d}_{\kappa\lambda}\cdot\mathsf{d}_{\lambda\rho}\cdot\mathsf{s}_\kappa^\lambda\mathsf{s}_\rho^\kappa x$.

LEMMA 3.2.29. *Suppose* $\{\kappa,\lambda\}\cap\{\mu,\nu,\rho,\sigma\} = 0$ *and* $|\{\mu,\nu,\rho\}| = 3 = |\{\mu,\nu,\sigma\}|$. *Then* $-\mathsf{d}_{\kappa\lambda}\cdot\mathsf{d}_{\mu\rho}\cdot\mathsf{d}_{\nu\sigma}\cdot\mathsf{s}[\kappa,\lambda]x = -\mathsf{d}_{\kappa\lambda}\cdot\mathsf{d}_{\mu\rho}\cdot\mathsf{d}_{\nu\sigma}\cdot_\mu\mathsf{s}(\kappa,\lambda)\mathsf{c}_\mu\mathsf{c}_\nu(x\cdot\mathsf{d}_{\mu\rho}\cdot\mathsf{d}_{\nu\sigma})$.

PROOF. $\geqq$ is clear. Now let r be the right-side of the indicated equation. If $t = -\mathsf{d}_{\kappa\lambda}\cdot\mathsf{d}_{\mu\rho}\cdot\mathsf{d}_{\nu\sigma}\cdot\mathsf{d}_{\kappa\mu}$, then $t\cdot\mathsf{s}_\lambda^\kappa\mathsf{s}_\mu^\lambda(x\cdot\mathsf{d}_{\mu\rho}\cdot\mathsf{d}_{\nu\sigma}) \leqq t\cdot r$. Now suppose $\eta \neq \kappa,\lambda,\mu,\nu,\sigma,\rho$, and let $t = -\mathsf{d}_{\kappa\lambda}\cdot\mathsf{d}_{\mu\rho}\cdot\mathsf{d}_{\nu\sigma}\cdot\mathsf{d}_{\kappa\eta}$. Then

$$t\cdot\mathsf{s}_\lambda^\kappa\mathsf{s}_\eta^\lambda x = t\cdot\mathsf{s}_\lambda^\kappa\mathsf{s}_\eta^\lambda\mathsf{s}_\rho^\mu x = t\cdot\mathsf{s}_\lambda^\kappa\mathsf{s}_\eta^\mu\mathsf{s}_\eta^\lambda\mathsf{s}_\rho^\mu x = t\cdot\mathsf{s}_\eta^\mu\mathsf{s}_\lambda^\kappa\mathsf{s}_\mu^\lambda\mathsf{s}_\rho^\mu x = t\cdot\mathsf{s}_\kappa^\mu\mathsf{s}_\lambda^\kappa\mathsf{s}_\mu^\lambda\mathsf{s}_\rho^\mu x \leqq r.$$

Finally, suppose also $\{\kappa,\lambda\}\cap\{\gamma,\delta,\varepsilon,\xi\} = 0$ and $|\{\gamma,\delta,\varepsilon\}| = 3 = |\{\gamma,\delta,\xi\}|$. Then

$$-\mathsf{d}_{\kappa\lambda}\cdot\mathsf{d}_{\mu\rho}\cdot\mathsf{d}_{\gamma\sigma}\cdot\mathsf{d}_{\gamma\varepsilon}\cdot\mathsf{d}_{\delta\xi}\cdot_\gamma\mathsf{s}(\kappa,\lambda)\mathsf{c}_\gamma\mathsf{c}_\delta(x\cdot\mathsf{d}_{\gamma\varepsilon}\cdot\mathsf{d}_{\delta\xi}) \leqq r$$

by 3.2.18.

LEMMA 3.2.30. *If* $|\{\kappa,\lambda,\rho,\sigma\}| = 4$, *then* $f_{\rho\sigma}\cdot\mathsf{s}[\kappa,\lambda]x = f_{\rho\sigma}\cdot_\rho\mathsf{s}(\kappa,\lambda)\mathsf{s}_\sigma^\rho x$.

Now we shall verify the conditions mentioned in 3.2.17. First note by 3.2.26–3.2.30 that $\mathsf{s}[\kappa,\lambda]x\cdot\mathsf{s}[\kappa,\lambda]-x = 0$ if $\kappa \neq \lambda$. Thus by 3.2.25(i)–(iii) we have:

LEMMA 3.2.31 (A). *If* $\kappa \neq \lambda$, *then* $\mathsf{s}[\kappa,\lambda]$ *is a Boolean endomorphism.*

LEMMA 3.2.32 (A). *If* $\kappa \neq \lambda$, *then* $\mathsf{s}[\kappa,\lambda]\mathsf{c}_\kappa x = \mathsf{s}_\kappa^\lambda\mathsf{c}_\kappa x$.

PROOF. By 3.2.26–3.2.30 there are two non-trivial cases. First, suppose that $\{\kappa,\lambda\}\cap\{\mu,\nu,\rho,\sigma\} = 0$ and $|\{\mu,\nu,\rho\}| = 3 = |\{\mu,\nu,\sigma\}|$. Let $t = -\mathsf{d}_{\kappa\lambda}\cdot\mathsf{d}_{\mu\rho}\cdot\mathsf{d}_{\nu\sigma}$. Then

$$\begin{aligned} t\cdot \mathsf{s}[\kappa,\lambda]\mathsf{c}_\kappa x &= t\cdot_\mu \mathsf{s}(\kappa,\lambda)\mathsf{c}_\mu\mathsf{c}_\nu(\mathsf{c}_\kappa x\cdot \mathsf{d}_{\mu\rho}\cdot\mathsf{d}_{\nu\sigma}) \\ &= t\cdot \mathsf{s}^\nu_\kappa\, \mathsf{s}^\lambda_\nu \mathsf{c}_\nu(\mathsf{d}_{\nu\sigma}\cdot\mathsf{c}_\kappa x) = t\cdot \mathsf{s}^\lambda_\kappa \mathsf{c}_\kappa x. \end{aligned}$$

Second, suppose that $|\{\kappa,\lambda,\rho,\sigma\}| = 4$. Then

$$\begin{aligned} f_{\rho\sigma}\cdot \mathsf{s}[\kappa,\lambda]\mathsf{c}_\kappa x &= f_{\rho\sigma}\cdot_\rho \mathsf{s}(\kappa,\lambda)\mathsf{s}^\rho_\sigma \mathsf{c}_\kappa x \\ &= f_{\rho\sigma}\cdot \mathsf{s}^\rho_\kappa \mathsf{s}^\lambda_\rho \mathsf{s}^\rho_\sigma \mathsf{c}_\kappa x \\ &= f_{\rho\sigma}\cdot \mathsf{s}^\lambda_\kappa \mathsf{c}_\kappa x. \end{aligned}$$

LEMMA 3.2.33 (A). *If* $\kappa \neq \lambda$, *then* $\mathsf{s}[\kappa,\lambda]\circ \mathsf{s}^\kappa_\lambda = \mathsf{s}^\lambda_\kappa$.

LEMMA 3.2.34 (A). *If* $\kappa \neq \lambda$, *then* $\mathsf{s}[\kappa,\lambda]\mathsf{s}[\kappa,\lambda]x = x$.

PROOF. We proceed by cases according to the definition of $\mathsf{s}[\kappa,\lambda]x$. Clearly $\mathsf{s}[\kappa,\lambda](\mathsf{d}_{\kappa\lambda}\cdot x) = \mathsf{d}_{\kappa\lambda}\cdot \mathsf{s}[\kappa,\lambda]x = \mathsf{d}_{\kappa\lambda}\cdot x$, using 3.2.25(v) and 3.2.26. If $\rho \neq \kappa,\lambda$, then, using 3.2.33,

$$\begin{aligned} \mathsf{s}[\kappa,\lambda](-\mathsf{d}_{\kappa\lambda}\cdot \mathsf{d}_{\kappa\rho}\cdot \mathsf{s}^\kappa_\lambda \mathsf{s}^\lambda_\rho x) &= -\mathsf{d}_{\kappa\lambda}\cdot \mathsf{d}_{\lambda\rho}\cdot \mathsf{s}^\lambda_\rho x \\ &= -\mathsf{d}_{\kappa\lambda}\cdot \mathsf{d}_{\lambda\rho}\cdot x. \end{aligned}$$

By symmetry, $\mathsf{s}[\kappa,\lambda](-\mathsf{d}_{\kappa\lambda}\cdot \mathsf{d}_{\lambda\rho}\cdot \mathsf{s}^\lambda_\kappa \mathsf{s}^\kappa_\rho x) = -\mathsf{d}_{\kappa\lambda}\cdot \mathsf{d}_{\kappa\rho}\cdot x$. Now suppose $\{\kappa,\lambda\}\cap\{\mu,\nu,\rho,\sigma\} = 0$ and $|\{\mu,\nu,\rho\}| = 3 = |\{\mu,\nu,\sigma\}|$. Let $t = -\mathsf{d}_{\kappa\lambda}\cdot \mathsf{d}_{\mu\rho}\cdot \mathsf{d}_{\nu\sigma}$. Then

$$\begin{aligned} &\mathsf{s}[\kappa,\lambda](t\cdot_\mu \mathsf{s}(\kappa,\lambda)\mathsf{c}_\mu\mathsf{c}_\nu(x\cdot \mathsf{d}_{\mu\rho}\cdot\mathsf{d}_{\nu\sigma})) = \\ &t\cdot_\mu \mathsf{s}(\kappa,\lambda)\mathsf{c}_\mu\mathsf{c}_\nu({}_\mu\mathsf{s}(\kappa,\lambda)\mathsf{c}_\mu\mathsf{c}_\nu(x\cdot \mathsf{d}_{\mu\rho}\cdot\mathsf{d}_{\nu\sigma})\cdot \mathsf{d}_{\mu\rho}\cdot\mathsf{d}_{\nu\sigma}) \text{ (by 3.2.29)} \\ &= t\cdot_\mu \mathsf{s}(\kappa,\lambda)_\mu \mathsf{s}(\kappa,\lambda)\mathsf{c}_\mu\mathsf{c}_\nu(x\cdot \mathsf{d}_{\mu\rho}\cdot\mathsf{d}_{\nu\sigma}) = t\cdot x. \end{aligned}$$

Finally, suppose that $|\{\kappa,\lambda,\rho,\sigma\}| = 4$. Then

$$\begin{aligned} \mathsf{s}[\kappa,\lambda](f_{\rho\sigma}\cdot_\rho \mathsf{s}(\kappa,\lambda)\mathsf{s}^\rho_\sigma x) &= f_{\rho\sigma}\cdot_\rho \mathsf{s}(\kappa,\lambda)\mathsf{s}^\rho_\sigma{}_\rho \mathsf{s}(\kappa,\lambda)\mathsf{c}_\rho\mathsf{c}_\sigma(x\cdot f_{\rho\sigma}) \text{ (by 3.2.23 and 3.2.20)} \\ &= f_{\rho\sigma}\cdot_\rho \mathsf{s}(\kappa,\lambda)_\rho \mathsf{s}(\kappa,\lambda)\mathsf{c}_\rho\mathsf{c}_\sigma(x\cdot f_{\rho\sigma}) \\ &= f_{\rho\sigma}\cdot \mathsf{c}_\rho\mathsf{c}_\sigma(x\cdot f_{\rho\sigma}) = x\cdot f_{\rho\sigma} \text{ (by 3.2.21)}. \end{aligned}$$

In the next few lemmas we aim toward the result that $\mathsf{c}_\mu \mathsf{s}[\kappa,\lambda]x = \mathsf{s}[\kappa,\lambda]\mathsf{c}_\mu x$ for $|\{\kappa,\lambda,\mu\}| = 3$, one of the hardest results to establish.

LEMMA 3.2.35. *If* $|\{\kappa,\lambda,\mu\}| = 3$, *then* $\mathsf{d}_{\kappa\lambda}\cdot \mathsf{c}_\mu \mathsf{s}[\kappa,\lambda]x \leqq \mathsf{s}[\kappa,\lambda]\mathsf{c}_\mu x$.

LEMMA 3.2.36. *If* $|\{\kappa,\lambda,\mu,\rho\}| = 4$, *then* $-\mathsf{d}_{\kappa\lambda}\cdot \mathsf{d}_{\kappa\rho}\cdot \mathsf{c}_\mu \mathsf{s}[\kappa,\lambda]x \leqq \mathsf{s}[\kappa,\lambda]\mathsf{c}_\mu x$.

LEMMA 3.2.37. *If* $|\{\kappa,\lambda,\mu,\rho,\sigma,\xi\}| = 6 = |\{\kappa,\lambda,\mu,\rho,\sigma,\eta\}|$, *then* $-\mathsf{d}_{\kappa\lambda}\cdot \mathsf{d}_{\rho\xi}\cdot \mathsf{d}_{\sigma\eta}\cdot \mathsf{c}_\mu \mathsf{s}[\kappa,\lambda]x \leqq \mathsf{s}[\kappa,\lambda]\mathsf{c}_\mu x$.

LEMMA 3.2.38 (A). *If* $|\{\kappa,\lambda,\mu,\rho,\sigma\}|=5$, *then* $-\mathsf{d}_{\kappa\lambda}\cdot\mathsf{d}_{\kappa\mu}\cdot\mathsf{d}_{\rho\sigma}\cdot g_{\mu\rho}\cdot\mathsf{c}_\mu\mathsf{s}[\kappa,\lambda]x\leqq\mathsf{s}[\kappa,\lambda]\mathsf{c}_\mu x$.

PROOF. Let $y=-\mathsf{d}_{\kappa\lambda}\cdot\mathsf{d}_{\kappa\mu}\cdot\mathsf{d}_{\rho\sigma}\cdot g_{\mu\rho}$. First note that $y\cdot\mathsf{s}[\kappa,\lambda]\mathsf{c}_\mu x=y\cdot\mathsf{s}_\lambda^\kappa\mathsf{s}_\mu^\lambda\mathsf{c}_\mu x$. Hence

$$y\cdot\mathsf{c}_\mu(\mathsf{d}_{\mu\kappa}\cdot\mathsf{s}[\kappa,\lambda]x)=y\cdot\mathsf{s}[\kappa,\lambda]x\leqq\mathsf{s}[\kappa,\lambda]\mathsf{c}_\mu x;$$
$$y\cdot\mathsf{c}_\mu(\mathsf{d}_{\mu\lambda}\cdot\mathsf{s}[\kappa,\lambda]x)=y\cdot\mathsf{c}_\mu(\mathsf{d}_{\mu\lambda}\cdot\mathsf{s}_\kappa^\lambda\mathsf{s}_\mu^\kappa x)\leqq y\cdot{}_\mu\mathsf{s}(\lambda,\kappa)\mathsf{c}_\mu\mathsf{c}_\rho(x\cdot\mathsf{d}_{\rho\sigma})=y\cdot{}_\mu\mathsf{s}(\kappa,\lambda)\mathsf{c}_\mu x\leqq\mathsf{s}[\kappa,\lambda]\mathsf{c}_\mu x;$$
$$y\cdot\mathsf{c}_\mu(\mathsf{d}_{\mu\rho}\cdot\mathsf{s}[\kappa,\lambda]x)=y\cdot\mathsf{c}_\mu(\mathsf{d}_{\mu\rho}\cdot{}_\mu\mathsf{s}(\kappa,\lambda)\mathsf{c}_\mu\mathsf{c}_\sigma(x\cdot\mathsf{d}_{\mu\rho}\cdot\mathsf{d}_{\rho\sigma}))=y\cdot{}_\mu\mathsf{s}(\kappa,\lambda)\mathsf{c}_\mu(x\cdot\mathsf{d}_{\mu\rho})\leqq\mathsf{s}[\kappa,\lambda]\mathsf{c}_\mu x;$$

if $\nu\neq\kappa,\lambda,\mu,\rho,\sigma$, then $y\cdot\mathsf{c}_\mu(\mathsf{d}_{\mu\nu}\cdot\mathsf{s}[\kappa,\lambda]x)\leqq\mathsf{s}[\kappa,\lambda]\mathsf{c}_\mu x$ similarly. Finally,

$$y\cdot\mathsf{c}_\mu(f_{\rho\sigma}\cdot\mathsf{s}[\kappa,\lambda]x)=y\cdot\mathsf{c}_\mu(f_{\rho\sigma}\cdot{}_\rho\mathsf{s}(\kappa,\lambda)\mathsf{s}_\sigma^\rho x)$$
$$\leqq y\cdot{}_\rho\mathsf{s}(\kappa,\lambda)\mathsf{s}_\sigma^\rho\mathsf{c}_\mu x=y\cdot\mathsf{s}_\mu^\rho\mathsf{s}_\lambda^\kappa\mathsf{s}_\rho^\lambda\mathsf{s}_\sigma^\rho\mathsf{c}_\mu x=y\cdot\mathsf{s}_\lambda^\kappa\mathsf{s}_\mu^\lambda\mathsf{s}_\sigma^\rho\mathsf{c}_\mu x=y\cdot\mathsf{s}_\lambda^\kappa\mathsf{s}_\mu^\lambda\mathsf{c}_\mu x.$$

LEMMA 3.2.39(A). *If* $|\{\kappa,\lambda,\mu\}|=3$, *then* $f_{\kappa\mu}\cdot\mathsf{c}_\mu\mathsf{s}[\kappa,\lambda]x\leqq\mathsf{s}[\kappa,\lambda]\mathsf{c}_\mu x$.

PROOF. By 3.2.27 we have $f_{\kappa\mu}\cdot\mathsf{s}[\kappa,\lambda]\mathsf{c}_\mu x=f_{\kappa\mu}\cdot\mathsf{s}_\lambda^\kappa\mathsf{s}_\mu^\lambda\mathsf{c}_\mu x$. Now $f_{\kappa\mu}\cdot\mathsf{c}_\mu(\mathsf{d}_{\kappa\mu}\cdot\mathsf{s}[\kappa,\lambda]x)=f_{\kappa\mu}\cdot\mathsf{s}[\kappa,\lambda]x\leqq\mathsf{s}[\kappa,\lambda]\mathsf{c}_\mu x$. Next,

$$\begin{aligned}f_{\kappa\mu}\cdot\mathsf{c}_\mu(\mathsf{d}_{\mu\lambda}\cdot\mathsf{s}[\kappa,\lambda]x)&=f_{\kappa\mu}\cdot\mathsf{c}_\mu(\mathsf{d}_{\mu\lambda}\cdot\mathsf{s}_\kappa^\lambda\mathsf{s}_\mu^\kappa x)\\&=f_{\kappa\mu}\cdot{}_\mu\mathsf{s}(\lambda,\kappa)x=f_{\kappa\lambda}\cdot{}_\mu\mathsf{s}(\lambda,\kappa)(f_{\kappa\mu}\cdot x)\ \text{(by 3.2.20(ii))}\\&=f_{\kappa\mu}\cdot{}_\mu\mathsf{s}(\lambda,\kappa)(f_{\kappa\mu}\cdot\mathsf{c}_\lambda(f_{\kappa\mu}\cdot x))\ \text{(by 3.2.22)}\\&=f_{\kappa\mu}\cdot\mathsf{s}_\lambda^\mu\mathsf{s}_\mu^\kappa\mathsf{c}_\lambda(f_{\kappa\mu}\cdot x)=f_{\kappa\mu}\cdot\mathsf{s}_\lambda^\kappa\mathsf{s}_\lambda^\mu\mathsf{c}_\lambda(f_{\kappa\mu}\cdot x).\end{aligned}$$

Now

$$\begin{aligned}f_{\kappa\lambda}\cdot\mathsf{s}_\lambda^\mu\mathsf{c}_\lambda(f_{\kappa\mu}\cdot x)&=f_{\kappa\lambda}\cdot\mathsf{s}_\kappa^\mu\mathsf{c}_\lambda(f_{\kappa\mu}\cdot x)=f_{\kappa\lambda}\cdot\mathsf{c}_\lambda(e_\mu\cdot\mathsf{s}_\kappa^\mu x)=f_{\kappa\lambda}\cdot\mathsf{c}_\lambda(e_\mu\cdot\textstyle\sum_{\nu\neq\mu}\mathsf{d}_{\mu\nu}\cdot\mathsf{s}_\kappa^\mu x)\\&=f_{\kappa\lambda}\cdot\mathsf{c}_\lambda(f_{\mu\lambda}\cdot\mathsf{s}_\kappa^\mu x)\leqq\mathsf{s}_\mu^\lambda\mathsf{c}_\mu x\end{aligned}$$

and hence $f_{\kappa\mu}\cdot\mathsf{c}_\mu(\mathsf{d}_{\mu\lambda}\cdot\mathsf{s}[\kappa,\lambda]x)\leqq\mathsf{s}_\lambda^\kappa\mathsf{s}_\mu^\lambda\mathsf{c}_\mu x$ by 3.2.20(ii). Finally, if $\rho\neq\kappa,\lambda,\mu$, then

$$f_{\kappa\mu}\cdot\mathsf{c}_\mu(f_{\mu\rho}\cdot\mathsf{s}[\kappa,\lambda]x)=f_{\kappa\mu}\cdot\mathsf{c}_\mu(f_{\mu\rho}\cdot{}_\mu\mathsf{s}(\kappa,\lambda)\mathsf{s}_\rho^\mu x)\leqq\mathsf{s}_\lambda^\kappa\mathsf{s}_\mu^\lambda\mathsf{c}_\mu x,$$

and the proof is complete.

LEMMA 3.2.40 (A). *If* $|\{\kappa,\lambda,\mu,\rho,\sigma\}|=5$, *then* $-\mathsf{d}_{\kappa\lambda}\cdot\mathsf{d}_{\mu\rho}\cdot\mathsf{d}_{\mu\sigma}\cdot g_{\rho\sigma}\cdot\mathsf{c}_\mu\mathsf{s}[\kappa,\lambda]x\leqq\mathsf{s}[\kappa,\lambda]\mathsf{c}_\mu x$.

PROOF. Let $t=-\mathsf{d}_{\kappa\lambda}\cdot\mathsf{d}_{\mu\rho}\cdot\mathsf{d}_{\mu\sigma}\cdot g_{\rho\sigma}$. Then

$$t\cdot\mathsf{s}[\kappa,\lambda]\mathsf{c}_\mu x=t\cdot{}_\mu\mathsf{s}(\kappa,\lambda)\mathsf{c}_\mu\mathsf{c}_\rho(\mathsf{c}_\mu x\cdot\mathsf{d}_{\mu\rho}\cdot\mathsf{d}_{\mu\sigma})=t\cdot{}_\mu\mathsf{s}(\kappa,\lambda)\mathsf{s}_\sigma^\rho\mathsf{c}_\mu x=t\cdot{}_\mu\mathsf{s}(\kappa,\lambda)\mathsf{c}_\mu x.$$

Now $t\cdot\mathsf{c}_\mu(\mathsf{d}_{\mu\rho}\cdot\mathsf{s}[\kappa,\lambda]x)=t\cdot\mathsf{s}[\kappa,\lambda]x\leqq\mathsf{s}[\kappa,\lambda]\mathsf{c}_\mu x$. Similarly, $t\cdot\mathsf{c}_\mu(\mathsf{d}_{\mu\sigma}\cdot\mathsf{s}[\kappa,\lambda]x)\leqq\mathsf{s}[\kappa,\lambda]\mathsf{c}_\mu x$. Suppose

$\nu \neq \kappa, \lambda, \mu, \rho, \sigma$. Then

$$\begin{aligned} t\cdot \mathsf{c}_\mu(\mathsf{d}_{\mu\nu}\cdot \mathsf{s}[\kappa,\lambda]x) &= t\cdot \mathsf{c}_\mu(\mathsf{d}_{\mu\nu}\cdot {}_\mu\mathsf{s}(\kappa,\lambda)\mathsf{c}_\mu\mathsf{c}_\rho(x\cdot \mathsf{d}_{\mu\nu}\cdot \mathsf{d}_{\rho\sigma}))\\ &= t\cdot {}_\rho\mathsf{s}(\kappa,\lambda)\,\mathsf{s}^\rho_\sigma\mathsf{s}^\mu_\nu x \leqq \mathsf{s}[\kappa,\lambda]\mathsf{c}_\mu x. \end{aligned}$$

Finally,

$$t\cdot \mathsf{c}_\mu(f_{\rho\sigma}\cdot \mathsf{s}[\kappa,\lambda]x) = t\cdot \mathsf{c}_\mu(f_{\rho\sigma}\cdot {}_\rho\mathsf{s}(\kappa,\lambda)\mathsf{s}^\rho_\sigma x) \leqq \mathsf{s}[\kappa,\lambda]\mathsf{c}_\mu x.$$

The following lemma is established in a very similar manner:

LEMMA 3.2.41 (A). *If* $|\{\kappa,\lambda,\mu,\rho,\sigma,\nu\}| = 6$ *then* $-\mathsf{d}_{\kappa\lambda}\cdot \mathsf{d}_{\mu\rho}\cdot \mathsf{d}_{\sigma\nu}\cdot g_{\mu\sigma}\cdot \mathsf{c}_\mu\mathsf{s}[\kappa,\lambda]x \leqq \mathsf{s}[\kappa,\lambda]\mathsf{c}_\mu x$.

LEMMA 3.2.42 (A). *If* $|\{\kappa,\lambda,\mu,\rho\}| = 4$, *then* $f_{\mu\rho}\cdot \mathsf{c}_\mu\mathsf{s}[\kappa,\lambda]x \leqq \mathsf{s}[\kappa,\lambda]\mathsf{c}_\mu x$.

PROOF. We have

$$f_{\mu\rho}\cdot \mathsf{s}[\kappa,\lambda]\mathsf{c}_\mu x = f_{\mu\rho}\cdot {}_\mu\mathsf{s}(\kappa,\lambda)\mathsf{c}_\mu x = f_{\mu\rho}\cdot \mathsf{s}[\lambda,\kappa]\mathsf{c}_\mu x = f_{\mu\rho}\cdot {}_\mu\mathsf{s}(\lambda,\kappa)\mathsf{c}_\mu x.$$

Now note that $f_{\mu\rho}\cdot \mathsf{c}_\mu\mathsf{s}[\kappa,\lambda]x = \Sigma_{\nu\neq\mu} f_{\mu\rho}\cdot \mathsf{c}_\mu(\mathsf{d}_{\mu\nu}\cdot \mathsf{s}[\kappa,\lambda]x)$. We have

$$\begin{aligned} f_{\mu\rho}\cdot \mathsf{c}_\mu(\mathsf{d}_{\mu\rho}\cdot \mathsf{s}[\kappa,\lambda]x) &= f_{\mu\rho}\cdot \mathsf{s}[\kappa,\lambda]x \leqq \mathsf{s}[\kappa,\lambda]\mathsf{c}_\mu x;\\ f_{\mu\rho}\cdot \mathsf{c}_\mu(\mathsf{d}_{\mu\kappa}\cdot \mathsf{s}[\kappa,\lambda]x) &= f_{\mu\rho}\cdot \mathsf{c}_\mu(\mathsf{d}_{\mu\kappa}\cdot \mathsf{s}^\kappa_\lambda\mathsf{s}^\lambda_\mu x) \leqq \mathsf{s}[\kappa,\lambda]\mathsf{c}_\mu x; \end{aligned}$$

$f_{\mu\rho}\cdot \mathsf{c}_\mu(\mathsf{d}_{\mu\lambda}\cdot \mathsf{s}[\kappa,\lambda]x) \leqq \mathsf{s}[\kappa,\lambda]\mathsf{c}_\mu x$ similarly. Finally, if $\nu \neq \kappa, \lambda, \mu, \rho$, then

$$f_{\mu\rho}\cdot \mathsf{c}_\mu(\mathsf{d}_{\mu\nu}\cdot \mathsf{s}[\kappa,\lambda]x) = f_{\mu\rho}\cdot \mathsf{c}_\mu(f_{\mu\nu}\cdot \mathsf{s}[\kappa,\lambda]x) = f_{\mu\rho}\cdot \mathsf{c}_\mu(f_{\mu\nu}\cdot {}_\mu\mathsf{s}(\kappa,\lambda)\mathsf{s}^\mu_\nu x) \leqq \mathsf{s}[\kappa,\lambda]\mathsf{c}_\mu x.$$

LEMMA 3.2.43 (A). *If* $|\{\kappa,\lambda,\mu,\rho,\sigma\}| = 5$, *then* $f_{\rho\sigma}\cdot \mathsf{c}_\mu\mathsf{s}[\kappa,\lambda]x \leqq \mathsf{s}[\kappa,\lambda]\mathsf{c}_\mu x$.

PROOF. We have $f_{\rho\sigma}\cdot \mathsf{s}[\kappa,\lambda]\mathsf{c}_\mu x = f_{\rho\sigma}\cdot {}_\rho\mathsf{s}(\kappa,\lambda)\mathsf{s}^\rho_\sigma\mathsf{c}_\mu x = f_{\rho\sigma}\cdot {}_\mu\mathsf{s}(\kappa,\lambda)\mathsf{s}^\rho_\sigma\mathsf{c}_\mu x = f_{\rho\sigma}\cdot {}_\rho\mathsf{s}(\lambda,\kappa)\mathsf{s}^\rho_\sigma\mathsf{c}_\mu x$. Hence

$$f_{\rho\sigma}\cdot \mathsf{c}_\mu(\mathsf{d}_{\mu\kappa}\cdot \mathsf{s}[\kappa,\lambda]x) = f_{\rho\sigma}\cdot {}_\mu\mathsf{s}(\kappa,\lambda)x \leqq \mathsf{s}[\kappa,\lambda]\mathsf{c}_\mu x;$$

$f_{\rho\sigma}\cdot \mathsf{c}_\mu(\mathsf{d}_{\mu\lambda}\cdot \mathsf{s}[\kappa,\lambda]x) \leqq \mathsf{s}[\kappa,\lambda]\mathsf{c}_\mu x$ similarly. Further,

$$f_{\rho\sigma}\cdot \mathsf{c}_\mu(\mathsf{d}_{\mu\rho}\cdot \mathsf{s}[\kappa,\lambda]x) = f_{\rho\sigma}\cdot \mathsf{c}_\mu(\mathsf{d}_{\mu\rho}\cdot {}_\mu\mathsf{s}(\kappa,\lambda)\mathsf{c}_\mu\mathsf{c}_\rho(x\cdot \mathsf{d}_{\mu\rho}\cdot \mathsf{d}_{\mu\sigma})) \leqq \mathsf{s}[\kappa,\lambda]\mathsf{c}_\mu x.$$

If $\nu \neq \kappa, \lambda, \rho, \sigma$, then $f_{\rho\sigma}\cdot \mathsf{c}_\mu(\mathsf{d}_{\mu\nu}\cdot \mathsf{s}[\kappa,\lambda]x) \leqq \mathsf{s}[\kappa,\lambda]\mathsf{c}_\mu x$ similarly. Finally, $f_{\rho\sigma}\cdot \mathsf{c}_\mu(f_{\rho\sigma}\cdot \mathsf{s}[\kappa,\lambda]x) = f_{\rho\sigma}\cdot \mathsf{s}[\kappa,\lambda]x \leqq \mathsf{s}[\kappa,\lambda]\mathsf{c}_\mu x$ by 3.2.22.

Now by 3.2.35–3.2.43 we have

LEMMA 3.2.44 (A). *If* $|\{\kappa,\lambda,\mu\}| = 3$, *then* $\mathsf{c}_\mu\mathsf{s}[\kappa,\lambda]x \leqq \mathsf{s}[\kappa,\lambda]\mathsf{c}_\mu x$.

LEMMA 3.2.45 (A). *If* $|\{\kappa,\lambda,\mu\}|=3$, *then* $\mathsf{c}_\mu\mathsf{s}[\kappa,\lambda]x=\mathsf{s}[\kappa,\lambda]\mathsf{c}_\mu x$.

PROOF. Applying 3.2.44 to $\mathsf{s}[\kappa,\lambda]x$ in place of x, and using 3.2.34, we obtain $\mathsf{c}_\mu x\leqq\mathsf{s}[\kappa,\lambda]\mathsf{c}_\mu\mathsf{s}[\kappa,\lambda]x$. Applying $\mathsf{s}[\kappa,\lambda]$ to both sides and using 3.2.34 we get $\mathsf{s}[\kappa,\lambda]\mathsf{c}_\mu x\leqq\mathsf{c}_\mu\mathsf{s}[\kappa,\lambda]x$. Combining this with 3.2.44 gives the desired result.

LEMMA 3.2.46 (A). *If* $|\{\kappa,\lambda,\mu\}|=3$, *then* $\mathsf{s}[\kappa,\lambda]\mathsf{s}^\mu_\kappa x=\mathsf{s}^\mu_\lambda\mathsf{s}[\kappa,\lambda]x$.

PROOF. $\mathsf{s}[\kappa,\lambda]\mathsf{s}^\mu_\kappa x=\mathsf{c}_\mu\mathsf{s}[\kappa,\lambda](\mathsf{d}_{\mu\kappa}\cdot x)=\mathsf{c}_\mu(\mathsf{d}_{\mu\lambda}\cdot\mathsf{s}[\kappa,\lambda]x)=\mathsf{s}^\mu_\lambda\mathsf{s}[\kappa,\lambda]x$, using 3.2.45.

LEMMA 3.2.47. *If* $\kappa\neq\lambda$, *then* $\mathsf{c}_\kappa\mathsf{s}[\kappa,\lambda]x\leqq\mathsf{s}^\kappa_\lambda\mathsf{c}_\lambda x$.

PROOF. We have $\mathsf{c}_\kappa(\mathsf{d}_{\kappa\lambda}\cdot\mathsf{s}[\kappa,\lambda]x)=\mathsf{c}_\kappa(\mathsf{d}_{\kappa\lambda}\cdot x)\leqq\mathsf{s}^\kappa_\lambda\mathsf{c}_\lambda x$. If $\rho\neq\kappa,\lambda$, then

$$\mathsf{c}_\kappa(-\mathsf{d}_{\kappa\lambda}\cdot\mathsf{d}_{\kappa\rho}\cdot\mathsf{s}[\kappa,\lambda]x)=\mathsf{c}_\kappa(-\mathsf{d}_{\kappa\lambda}\cdot\mathsf{d}_{\kappa\rho}\cdot\mathsf{s}^\kappa_\lambda\mathsf{s}^\lambda_\rho x)\leqq\mathsf{s}^\kappa_\lambda\mathsf{c}_\lambda x;$$
$$\mathsf{c}_\kappa(-\mathsf{d}_{\kappa\lambda}\cdot\mathsf{d}_{\lambda\rho}\cdot\mathsf{s}[\kappa,\lambda]x)=\mathsf{c}_\kappa(-\mathsf{d}_{\kappa\lambda}\cdot\mathsf{d}_{\lambda\rho}\cdot\mathsf{s}^\lambda_\kappa\mathsf{s}^\kappa_\rho x)\leqq\mathsf{c}_\kappa(-\mathsf{d}_{\kappa\lambda}\cdot\mathsf{d}_{\lambda\rho}\cdot\mathsf{s}^\kappa_\rho\mathsf{c}_\lambda x)\leqq\mathsf{s}^\kappa_\lambda\mathsf{c}_\lambda x.$$

If $\{\kappa,\lambda\}\cap\{\mu,\nu,\rho,\sigma\}=0$ and $|\{\mu,\nu,\rho\}|=3=|\{\mu,\nu,\sigma\}|$, then, with $t=-\mathsf{d}_{\kappa\lambda}\cdot\mathsf{d}_{\mu\rho}\cdot\mathsf{d}_{\nu\sigma}$,

$$\mathsf{c}_\kappa(t\cdot\mathsf{s}[\kappa,\lambda]x)=\mathsf{c}_\kappa(t\cdot{}_\mu\mathsf{s}(\kappa,\lambda)\mathsf{c}_\mu\mathsf{c}_\nu(\mathsf{d}_{\mu\rho}\cdot\mathsf{d}_{\nu\sigma}\cdot x))\leqq\mathsf{c}_\kappa(t\cdot\mathsf{s}^\kappa_\lambda\mathsf{c}_\mu(\mathsf{c}_\lambda x\cdot\mathsf{d}_{\mu\rho}))\leqq\mathsf{s}^\kappa_\lambda\mathsf{c}_\lambda x.$$

Finally, if $|\{\kappa,\lambda,\rho,\sigma\}|=4$ then

$$\mathsf{c}_\kappa(f_{\rho\sigma}\cdot\mathsf{s}[\kappa,\lambda]x)=\mathsf{c}_\kappa(f_{\rho\sigma}\cdot{}_\rho\mathsf{s}(\kappa,\lambda)\mathsf{s}^\rho_\sigma x)\leqq\mathsf{c}_\kappa(f_{\rho\sigma}\cdot\mathsf{s}^\rho_\kappa\mathsf{s}^\kappa_\lambda\mathsf{s}^\rho_\sigma\mathsf{c}_\lambda x)\leqq\mathsf{s}^\kappa_\lambda\mathsf{c}_\lambda x.$$

LEMMA 3.2.48 (A). *If* $\kappa\neq\lambda$, *then* $\mathsf{c}_\kappa\mathsf{s}[\kappa,\lambda]x=\mathsf{s}^\kappa_\lambda\mathsf{c}_\lambda x$.

PROOF. Applying 3.2.47, with κ and λ exchanged, to $\mathsf{s}[\kappa,\lambda]x$, we get $\mathsf{c}_\lambda x\leqq\mathsf{s}^\lambda_\kappa\mathsf{c}_\kappa\mathsf{s}[\kappa,\lambda]x$, using 3.2.25(iv), 3.2.34. Hence $\mathsf{s}^\kappa_\lambda\mathsf{c}_\lambda x\leqq\mathsf{c}_\kappa\mathsf{s}[\kappa,\lambda]x$. Then 3.2.47 yields the desired result.

LEMMA 3.2.49 (A). *If* $\kappa\neq\lambda$, *then* $\mathsf{c}_\kappa\mathsf{s}[\kappa,\lambda]x=\mathsf{s}[\kappa,\lambda]\mathsf{c}_\lambda x$.

PROOF. By 3.2.32 and 3.2.48.

The last lemma needed in order to apply Jónsson's theorem in 3.2.17 requires a lengthy proof:

LEMMA 3.2.50 (A). *If* $|\{\kappa,\lambda,\mu\}|=3$, *then* $\mathsf{s}[\kappa,\lambda]\mathsf{s}[\kappa,\mu]x=\mathsf{s}[\lambda,\mu]\mathsf{s}[\kappa,\lambda]x$.

PROOF. We have

$$\mathsf{s}[\kappa,\lambda]\mathsf{s}[\kappa,\mu](x\cdot\mathsf{d}_{\kappa\lambda}\cdot\mathsf{d}_{\kappa\mu})=x\cdot\mathsf{d}_{\kappa\lambda}\cdot\mathsf{d}_{\kappa\mu}=\mathsf{s}[\lambda,\mu]\mathsf{s}[\kappa,\lambda]x\cdot\mathsf{d}_{\kappa\lambda}\cdot\mathsf{d}_{\kappa\mu};$$

(1) $\mathsf{s}[\kappa,\lambda]\mathsf{s}[\kappa,\mu](x\cdot\mathsf{d}_{\kappa\lambda}\cdot-\mathsf{d}_{\kappa\mu})=\mathsf{s}[\kappa,\lambda](\mathsf{d}_{\mu\lambda}\cdot-\mathsf{d}_{\kappa\mu}\cdot\mathsf{s}^\mu_\kappa\mathsf{s}^\kappa_\lambda x)=\mathsf{d}_{\mu\kappa}\cdot-\mathsf{d}_{\lambda\mu}\cdot\mathsf{s}^\mu_\lambda\mathsf{s}^\lambda_\kappa x;$

(2) $\mathsf{s}[\lambda,\mu]\mathsf{s}[\kappa,\lambda](x\cdot\mathsf{d}_{\kappa\lambda}\cdot-\mathsf{d}_{\kappa\mu})=\mathsf{s}[\lambda,\mu](\mathsf{d}_{\kappa\lambda}\cdot-\mathsf{d}_{\lambda\mu}\cdot x)=\mathsf{d}_{\kappa\mu}\cdot-\mathsf{d}_{\lambda\mu}\cdot\mathsf{s}^\mu_\lambda\mathsf{s}^\lambda_\kappa x;$

$\mathsf{s}[\kappa,\lambda]\mathsf{s}[\kappa,\mu](x\cdot\mathsf{d}_{\kappa\mu}\cdot-\mathsf{d}_{\kappa\lambda})=\mathsf{d}_{\mu\lambda}\cdot-\mathsf{d}_{\kappa\lambda}\cdot\mathsf{s}^\lambda_\kappa\mathsf{s}^\kappa_\mu x$ by symmetry from (2);

(3) $\mathsf{s}[\lambda,\mu]\mathsf{s}[\kappa,\lambda](x\cdot\mathsf{d}_{\kappa\mu}\cdot-\mathsf{d}_{\kappa\lambda})=\mathsf{s}[\lambda,\mu](\mathsf{d}_{\lambda\mu}\cdot-\mathsf{d}_{\kappa\lambda}\cdot\mathsf{s}^\lambda_\kappa\mathsf{s}^\kappa_\mu x)=\mathsf{d}_{\lambda\mu}\cdot-\mathsf{d}_{\kappa\mu}\cdot\mathsf{s}^\lambda_\kappa\mathsf{s}^\kappa_\mu x;$

$$\mathsf{s}[\kappa,\lambda]\mathsf{s}[\kappa,\mu](x\cdot\mathsf{d}_{\lambda\mu}\cdot-\mathsf{d}_{\kappa\lambda}) = \mathsf{d}_{\kappa\lambda}\cdot-\mathsf{d}_{\mu\lambda}\cdot\mathsf{s}^{\kappa}_{\mu}\mathsf{s}^{\mu}_{\lambda}x \text{ by symmetry from (3);}$$
$$\mathsf{s}[\lambda,\mu]\mathsf{s}[\kappa,\lambda](x\cdot\mathsf{d}_{\lambda\mu}\cdot-\mathsf{d}_{\kappa\lambda}) = \mathsf{d}_{\kappa\lambda}\cdot-\mathsf{d}_{\mu\kappa}\cdot\mathsf{s}^{\kappa}_{\mu}\mathsf{s}^{\mu}_{\lambda}x \text{ by symmetry from (1).}$$

Now let $t = -\mathsf{d}_{\kappa\lambda}\cdot-\mathsf{d}_{\kappa\mu}\cdot-\mathsf{d}_{\lambda\mu}$. Suppose $\rho \neq \kappa,\lambda,\mu$. Then

$$\begin{aligned}
&(4)\ \mathsf{s}[\kappa,\lambda]\mathsf{s}[\kappa,\mu](x\cdot t\cdot\mathsf{d}_{\kappa\rho}) = \mathsf{s}[\kappa,\lambda](t\cdot\mathsf{d}_{\mu\rho}\cdot\mathsf{s}^{\mu}_{\kappa}\mathsf{s}^{\kappa}_{\rho}x) = t\cdot\mathsf{d}_{\mu\rho}\cdot\mathsf{s}^{\mu}_{\lambda}\mathsf{s}^{\lambda}_{\kappa}\mathsf{s}^{\kappa}_{\rho}x;\\
&(5)\ \mathsf{s}[\lambda,\mu]\mathsf{s}[\kappa,\lambda](x\cdot t\cdot\mathsf{d}_{\kappa\rho}) = \mathsf{s}[\lambda,\mu](t\cdot\mathsf{d}_{\lambda\rho}\cdot\mathsf{s}^{\lambda}_{\kappa}\mathsf{s}^{\kappa}_{\rho}x) = t\cdot\mathsf{d}_{\mu\rho}\cdot\mathsf{s}^{\mu}_{\lambda}\mathsf{s}^{\lambda}_{\kappa}\mathsf{s}^{\kappa}_{\rho}x;\\
&(6)\ \mathsf{s}[\kappa,\lambda]\mathsf{s}[\kappa,\mu](x\cdot t\cdot\mathsf{d}_{\lambda\rho}) = \mathsf{s}[\kappa,\lambda](t\cdot\mathsf{d}_{\lambda\rho}\cdot\mathsf{s}[\kappa,\mu]x)\\
&\qquad = t\cdot\mathsf{d}_{\kappa\rho}\cdot\mathsf{s}^{\kappa}_{\lambda}\mathsf{s}^{\lambda}_{\rho}\mathsf{s}[\kappa,\mu]x = t\cdot\mathsf{d}_{\kappa\rho}\cdot\mathsf{c}_{\kappa}\mathsf{s}[\kappa,\mu](\mathsf{d}_{\mu\lambda}\cdot\mathsf{s}^{\lambda}_{\rho}x)\\
&\qquad = t\cdot\mathsf{d}_{\kappa\rho}\cdot\mathsf{s}^{\kappa}_{\mu}\mathsf{s}^{\mu}_{\lambda}\mathsf{s}^{\lambda}_{\rho}x \text{ (by 3.2.48)}\\
&\qquad = \mathsf{s}[\lambda,\mu]\mathsf{s}[\kappa,\lambda](x\cdot t\cdot\mathsf{d}_{\lambda\rho}) \text{ by symmetry from (4);}\\
&\mathsf{s}[\kappa,\lambda]\mathsf{s}[\kappa,\mu](x\cdot t\cdot\mathsf{d}_{\mu\rho}) = t\cdot\mathsf{d}_{\lambda\rho}\cdot\mathsf{s}^{\lambda}_{\kappa}\mathsf{s}^{\kappa}_{\mu}\mathsf{s}^{\mu}_{\rho}x\\
&\qquad = \mathsf{s}[\lambda,\mu]\mathsf{s}[\kappa,\lambda](x\cdot t\cdot\mathsf{d}_{\mu\rho}) \text{ using (6).}
\end{aligned}$$

Now suppose that $\{\kappa,\lambda,\mu\}\cap\{\nu,\rho,\sigma,\tau\} = 0$ and $|\{\nu,\rho,\sigma\}| = 3 = |\{\nu,\rho,\tau\}|$. Let $y = t\cdot\mathsf{d}_{\nu\sigma}\cdot\mathsf{d}_{\rho\tau}$. Then

$$\begin{aligned}
\mathsf{s}[\kappa,\lambda]\mathsf{s}[\kappa,\mu](x\cdot y) &= \mathsf{s}[\kappa,\lambda](y\cdot_{\nu}\mathsf{s}(\kappa,\mu)\mathsf{c}_{\nu}\mathsf{c}_{\rho}(x\cdot\mathsf{d}_{\nu\sigma}\cdot\mathsf{d}_{\rho\tau}))\\
&= y\cdot_{\nu}\mathsf{s}(\kappa,\lambda)\mathsf{c}_{\nu}\mathsf{c}_{\rho}({}_{\nu}\mathsf{s}(\kappa,\mu)\mathsf{c}_{\nu}\mathsf{c}_{\rho}(x\cdot\mathsf{d}_{\nu\sigma}\cdot\mathsf{d}_{\rho\tau})\cdot\mathsf{d}_{\nu\sigma}\cdot\mathsf{d}_{\rho\tau})\\
&= y\cdot_{\nu}\mathsf{s}(\kappa,\lambda)_{\nu}\mathsf{s}(\kappa,\mu)\mathsf{c}_{\nu}\mathsf{c}_{\rho}(x\cdot\mathsf{d}_{\nu\sigma}\cdot\mathsf{d}_{\rho\tau})\\
&= y\cdot_{\nu}\mathsf{s}(\lambda,\mu)_{\nu}\mathsf{s}(\kappa,\lambda)\mathsf{c}_{\nu}\mathsf{c}_{\rho}(x\cdot\mathsf{d}_{\nu\sigma}\cdot\mathsf{d}_{\rho\tau}) \text{ by 1.5.18}\\
&= \mathsf{s}[\lambda,\mu]\mathsf{s}[\kappa,\lambda](x\cdot y) \text{ by symmetry.}
\end{aligned}$$

Finally, suppose that $\{\kappa,\lambda,\mu\}\cap\{\rho,\sigma\} = 0$ and $\rho \neq \sigma$. Then, using 3.2.23,

$$\begin{aligned}
\mathsf{s}[\kappa,\lambda]\mathsf{s}[\kappa,\mu](x\cdot f_{\rho\sigma}) &= \mathsf{s}[\kappa,\lambda](f_{\rho\sigma}\cdot_{\rho}\mathsf{s}(\kappa,\mu)\mathsf{c}_{\rho}\mathsf{c}_{\sigma}(x\cdot f_{\rho\sigma}))\\
&= f_{\rho\sigma}\cdot_{\rho}\mathsf{s}(\kappa,\lambda)\mathsf{s}^{\rho}_{\sigma}{}_{\rho}\mathsf{s}(\kappa,\mu)\mathsf{c}_{\rho}\mathsf{c}_{\sigma}(x\cdot f_{\rho\sigma})\\
&= f_{\rho\sigma}\cdot_{\rho}\mathsf{s}(\lambda,\mu)_{\rho}\mathsf{s}(\kappa,\lambda)\mathsf{c}_{\rho}\mathsf{c}_{\sigma}(x\cdot f_{\rho\sigma})\\
&= \mathsf{s}[\lambda,\mu]\mathsf{s}[\kappa,\lambda](x\cdot y) \text{ by symmetry.}
\end{aligned}$$

Finally we are ready for the auxiliary result of Comer and Henkin:

THEOREM 3.2.51. *Suppose $\alpha<\omega$, and $\mathfrak{A}$ is a CA_{α} of positive characteristic. Then there is a function* s *assigning to every $\tau\in{}^{\alpha}\alpha$ an endomorphism s_{τ} of $\mathfrak{Rl}\mathfrak{A}$ such that the following conditions hold for any $\sigma,\tau\in{}^{\alpha}\alpha$ and $\kappa,\lambda\in\alpha$:*

(i) $\mathsf{s}_{\sigma\circ\tau} = \mathsf{s}_{\sigma}\circ\mathsf{s}_{\tau}$;

(ii) $\mathsf{s}_{\sigma} = A1\,Id$ *if* $\sigma = \alpha 1\,Id$;

(iii) $\mathsf{s}_{[\kappa/\lambda]} = \mathsf{s}^{\kappa}_{\lambda}$;

(iv) $\mathsf{s}_{\tau}\mathsf{d}_{\kappa\lambda} = \mathsf{d}_{\tau\kappa,\tau\lambda}$;

(v) *if* $(\alpha\sim\{\kappa\})1\sigma\subseteq\tau$ *then* $\mathsf{s}_{\sigma}\mathsf{c}_{\kappa}x = \mathsf{s}_{\tau}\mathsf{c}_{\kappa}x$;

(vi) *if* $\kappa\notin Rg\sigma$, *then* $\mathsf{c}_{\kappa}\mathsf{s}_{\sigma}x = \mathsf{s}_{\sigma}x$;

(vii) *if* $\sigma^{-1}{}^{*}\{\kappa\} = \{\lambda\}$, *then* $\mathsf{c}_{\kappa}\mathsf{s}_{\sigma}x = \mathsf{s}_{\sigma}\mathsf{c}_{\lambda}x$.

PROOF. By 3.2.18–3.2.50 and Jónsson's theorem in 3.2.17 we obtain a function s assigning to every $\tau\in{}^{\alpha}\alpha$ an endomorphism s_{τ} of $\mathfrak{Rl}\mathfrak{A}$ such that (i)–(iii) hold and

$s_{[\kappa,\lambda]} = s[\kappa,\lambda]$ for any distinct $\kappa,\lambda<\alpha$. Now (iv) follows from 3.2.25(v)–(vii). For (v), we may assume that $\alpha \geqq 2$. Choose $\lambda \in \alpha \sim \{\kappa\}$. Then $\sigma \circ [\kappa/\lambda] = \tau \circ [\kappa/\lambda]$, and hence

$$s_\sigma c_\kappa x = s_\sigma s_\lambda^\kappa c_\kappa x = s_{\sigma \circ [\kappa/\lambda]} c_\kappa x = s_\tau c_\kappa x.$$

For (vi), choose $\lambda \in \alpha \sim \{\kappa\}$. Then $[\kappa/\lambda] \circ \sigma = \sigma$, so

$$c_\kappa s_\sigma x = c_\kappa s_{[\kappa/\lambda]\circ\sigma} x = c_\kappa s_\lambda^\kappa s_\sigma x = s_\lambda^\kappa s_\sigma x = s_\sigma x.$$

Finally, for (vii) note that $([\kappa,\lambda]\circ\sigma)^{-1*}\{\lambda\} = \{\lambda\}$. Hence $[\kappa,\lambda]\circ\sigma$ can be written as a composition of replacements and transpositions none of which involve λ. Hence by 3.2.45 we have $c_\lambda s_{[\kappa,\lambda]\circ\sigma} x = s_{[\kappa,\lambda]\circ\sigma} c_\lambda x$, hence $c_\lambda s[\kappa,\lambda] s_\sigma x = s[\kappa,\lambda] s_\sigma c_\lambda x$. Hence by 3.2.34 and 3.2.49, $c_\kappa s_\sigma x = s_\sigma c_\lambda x$, as desired.

The following result is closely related to 3.2.51, but will not be needed for the positive characteristic representation theorem.

THEOREM 3.2.52. *Let* $\mathfrak{C}$ *be a* $\mathsf{CA}_{\alpha+2}$ *and* $\mathfrak{A} = \mathfrak{Rr}_\alpha \mathfrak{C}$. *Then there is a function* s *assigning to every finite transformation* σ *of* α *an endomorphism* s_σ *of* $\mathfrak{BA}$ *such that the conditions of* 3.2.51 *hold for all finite transfomations* σ,τ *of* α *and all* $\kappa,\lambda<\alpha$.

PROOF. This time for distinct $\kappa,\lambda<\alpha$ we let $s[\kappa,\lambda]x = {}_\alpha s(\kappa,\lambda)x$, for all $x \in A$. The conditions in 3.2.17 and 3.2.51 follow easily using results in section 1.5.

Both 3.2.51 and 3.2.52 express relationships between cylindric and polyadic algebras. In this connection, see also section 5.4. Andréka, Comer and Németi have proved the following theorem relevant to these considerations: one can define a substitution function as above in every ${}_{<\alpha+2}\mathsf{Gs}_\alpha$, for $\alpha<\omega$; this result does not extend even to ${}_{\alpha+2}\mathsf{Cs}_\alpha$'s.

Now we turn to the promised representation theorem of Henkin.

THEOREM 3.2.53 *Let* $\alpha<\omega$. *If* $\mathfrak{A}$ *is a subdirect product of* CA_α*'s of positive characteristic, then* $\mathfrak{A}$ *is representable.*

PROOF. By 2.4.52 we may assume that $\mathfrak{A}$ is simple. By 3.2.51 we have available a substitution operation s satisfying (i)–(vii) there. Let κ be the characteristic of $\mathfrak{A}$. Thus $\bar{d}(\kappa\times\kappa) \neq 0$, and so there is an ultrafilter F on $\mathfrak{BA}$ such that $\bar{d}(\kappa\times\kappa) \in F$. (Recall that $\bar{d}(\kappa\times\kappa) = \prod_{\xi,\nu<\kappa,\xi\neq\nu} -d_{\xi\nu}$.) Now for any $x \in A$ let

$$fx = \{\tau \in {}^\alpha\kappa : s_\tau x \in F\}.$$

From this definition it is clear that f preserves $+$ and $-$. Since $\mathfrak{A}$ is simple, it suffices to show that F preserves $d_{\xi\nu}$ and c_ξ for all $\xi,\nu<\alpha$. We have $\tau \in fd_{\xi\nu}$ iff $s_\tau d_{\xi\nu} \in F$ iff $d_{\tau\xi,\tau\nu} \in F$ iff $\tau\xi = \tau\nu$, since $-d_{\mu\nu} \in F$ for all distinct $\mu,\nu<\kappa$. Thus f preserves $d_{\xi\nu}$.

To show that f preserves c_ξ, we need several steps.

(1) If $\kappa \leqq \xi < \alpha$ and $c_\xi x \in F$, then $s_\nu^\xi x \in F$ for some $\nu<\kappa$.

For, $c_\xi x = c_\xi(x \cdot \Sigma_{\mu,\nu<\kappa,\mu\neq\nu} d_{\mu\nu} + x \cdot \Sigma_{\mu<\kappa} d_{\mu\xi}) = \Sigma_{\mu,\nu<\kappa,\mu\neq\nu} d_{\mu\nu} \cdot c_\xi x + \Sigma_{\mu<\kappa} s^\xi_\mu x$. Since $d_{\mu\nu} \notin F$ for all distinct $\mu,\nu<\kappa$, the conclusion of (1) follows. Let $V = {}^\alpha\kappa$.

(2) If $\xi<\kappa$ and $\tau \in fc_\xi x$, then $\tau \in \mathbf{C}^{[V]}_\xi fx$.

For, under the hypothesis of (2) we have $s_\tau c_\xi x \in F$. Let $\sigma = \tau^\xi_\kappa$. Then $\sigma^{-1*}\{\kappa\} = \{\xi\}$, $(\alpha \sim \{\xi\}) \upharpoonright \tau \subseteq \sigma$, so by 3.2.51(v) and (vi) we get $s_\tau c_\xi x = s_\sigma c_\xi x = c_\kappa s_\sigma x$. Hence by (1) choose $\eta<\kappa$ with $s^\kappa_\eta s_\sigma x \in F$. Thus $[\kappa/\eta] \circ \sigma \in fx$ and $(\alpha \sim \{\xi\}) \upharpoonright \tau \subseteq [\kappa/\eta] \circ \sigma$, so $\tau \in \mathbf{C}_\xi fx$.

(3) If $\kappa \leqq \xi < \alpha$ and $\tau \in fc_\xi x$, then $\tau \in \mathbf{C}^{[V]}_\xi fx$.

The proof is similar to that of (2), using $\sigma = \tau^\xi_\xi$.

(4) If $\xi<\alpha$ and $\tau \in \mathbf{C}^{[V]}_\xi fx$, then $\tau \in fc_\xi x$.

For, say $\tau^\xi_\eta \in fx$, with $\eta<\kappa$. Let $\sigma = \tau^\xi_\eta$. Then $s_\sigma x \in F$, so $s_\sigma c_\xi x \in F$. But $s_\sigma c_\xi x = s_\tau c_\xi x$ by 3.2.51(v), so $s_\tau c_\xi x \in F$, hence $\tau \in fc_\xi x$. This completes the proof of 3.2.53.

Next we consider representability for CA_0's and CA_1's, both cases being rather trivial. Note from 3.1.17 that IGs_α does not have its intended meaning as the class of representable CA_α's, for $\alpha \leqq 1$. Instead, the class $\mathbf{SP}\mathsf{Cs}_\alpha$ is more appropriate.

THEOREM 3.2.54. $\mathsf{CA}_0 = \mathbf{SP}\mathsf{Cs}_0$.

PROOF. The inclusion $\subseteq$ is all that needs to be shown. By 2.4.52 it suffices to take the case of a simple CA_0 $\mathfrak{A}$. Thus $|A| = 2$, so $\mathfrak{A}$ is clearly isomorphic to a Cs_0 (see 3.1.17).

THEOREM 3.2.55. $\mathsf{CA}_1 = \mathbf{SP}\mathsf{Cs}_1$.

PROOF. Again only $\subseteq$ needs to be shown, and it suffices to take a simple CA_1 $\mathfrak{A}$. Thus by 2.3.14 the cylindrification c_0 is given by: $c_0 0 = 0$, $c_0 a = 1$ if $a \neq 0$. By the Boolean representation theorem, say f is an isomorphism from $\mathfrak{B}\mathfrak{A}$ onto a field of subsets of some set U. Define $ga = \{\langle u \rangle : u \in fa\}$ for all $a \in A$. Clearly g is an isomorphism from $\mathfrak{A}$ onto a Cs_1.

COROLLARY 3.2.56. *For* $\alpha \leqq 1$ *we have* $\mathsf{CA}_\alpha = \mathbf{SP}\mathsf{Cs}_\alpha = \mathbf{S}\mathsf{Nr}_\alpha \mathsf{CA}_{\alpha+\omega}$.

PROOF. By 3.1.128, 3.2.54, and 3.2.55.

Representation of CA_2's

Our last two positive representation results in this section concern CA_2's. The first result is that any CA_2 is isomorphic to a Crs_2. This theorem is a corollary of the proof of the second result; it has been given a shorter independent proof by Andréka and Németi. They used it to give a somewhat simpler proof of the second result too. See also section 5.5. To formulate the other result, recall from 2.6.42 that for each $\alpha \geqq 2$ there is a CA_α in which the equation

$$\mathsf{c}_1(x \cdot y \cdot \mathsf{c}_0(x \cdot -y)) \cdot -\mathsf{c}_0(\mathsf{c}_1 x \cdot -\mathsf{d}_{01}) = 0$$

fails; on the other hand, by 2.6.41 this equation holds for every representable CA_α. Of course the same two statements hold if we replace 0 and 1 by arbitrary distinct $\kappa, \lambda < \alpha$. The second result is that if the above equation and its symmetric form with 0 and 1 interchanged hold in a CA_2 $\mathfrak{A}$, then $\mathfrak{A}$ is representable. These two results are due to Henkin and Tarski, but the basic idea of the proof goes back to Everett, Ulam [46]. We shall derive both results from a rather technical lemma, 3.2.59. For it we need some preparation.

LEMMA 3.2.57. *For any non-empty set U there is a partition $\mathcal{P}$ of ${}^2U$ such that $|\mathcal{P}| = |U|$, $\mathsf{D}_{01}^{[V]} \in \mathcal{P}$, where $V = {}^2U$, and $\mathsf{C}_\kappa^{[V]} X = {}^2U$ for each $X \in \mathcal{P}$ and each $\kappa < 2$.*

PROOF. Let $\circ$ be a group operation on U. For each $u \in U$ let $X_u = \{\langle v, u \circ v\rangle : v \in U\}$. If $\langle v, w\rangle \in X_u \cap X_{u'}$, then $w = u \circ v = u' \circ v$, hence $u = u'$. Given any $\langle v, w\rangle \in {}^2U$ we have $\langle v, w\rangle \in X_u$, where $u = w \circ v^{-1}$. Clearly $X_u \neq 0$ for all $u \in U$. Thus $\{X_u : u \in U\}$ is a partition of ${}^2U$, and $|\{X_u : u \in U\}| = |U|$. If e is the identity of the group $\langle U, \circ\rangle$, then $X_e = \mathsf{D}_{01}^{[V]}$. Finally, let $u \in U$; we show that $\mathsf{C}_0 X_u = \mathsf{C}_1 X_u = V$. Let $\langle v, w\rangle \in V$ be arbitrary. Then $\langle u^{-1} \circ w, w\rangle \in X_u$ and $\langle v, u \circ v\rangle \in X_u$, so $\langle v, w\rangle \in \mathsf{C}_0 X_u \cap \mathsf{C}_1 X_u$, as desired.

LEMMA 3.2.58. *Suppose $|U| = |U'| \geqq \kappa > 0$, κ a cardinal, and $V = U \times U'$. Then there is a partition $\mathcal{P}$ of V such that $|\mathcal{P}| = \kappa$, $\mathsf{D}_{01}^{[V]} \in \mathcal{P}$ if $U = U'$ and $\kappa > 1$, and $\mathsf{C}_0^{[V]} X = \mathsf{C}_1^{[V]} X = V$ for all $X \in \mathcal{P}$.*

PROOF. Let $\mathcal{P}$ be as in 3.2.57. Choose $\mathcal{P}' \subseteq \mathcal{P}$ with $|\mathcal{P}'| = \kappa$ and $\mathsf{D}_{01}^{[W]} \in \mathcal{P}'$, where $W = {}^2U$, and fix $X \in \mathcal{P}'$ with $X \neq \mathsf{D}_{01}^{[W]}$ if $\kappa > 1$. Let $Y = X \cup \bigcup_{Z \in \mathcal{P} \sim \mathcal{P}'} Z$, and let $\mathcal{P}'' = (\mathcal{P}' \sim \{X\}) \cup \{Y\}$. Let f be a one-one function from U' onto U, with $f = U \restriction Id$ if $U = U'$. For each $X \in \mathcal{P}''$ let $X^* = \{\langle u, v\rangle : u \in U,\ v \in U'$, and $\langle u, fv\rangle \in X\}$. Finally, let $\mathcal{P} = \{X^* : X \in \mathcal{P}''\}$. Clearly $\mathcal{P}$ is as desired.

LEMMA 3.2.59. *Let $\mathfrak{A}$ be a simple complete atomic CA_2. An atom $a \in At\mathfrak{A}$ is said to be* **defective** *if the following condition holds:*

(*) $\mathsf{c}_0 a \cdot \mathsf{c}_1 a \leqq \mathsf{d}_{01}$, *and there exist $\kappa, \lambda < 2$ and $x, y, z \in A$ such that $\{\kappa, \lambda\} = 2$, $x \leqq \mathsf{c}_\kappa a$, and $\mathsf{c}_\lambda(x \cdot y) \cdot \mathsf{c}_\lambda(x \cdot z) \cdot -\mathsf{c}_\lambda(x \cdot y \cdot z) \neq 0$.*

Let I be the set of all defective atoms of $\mathfrak{A}$. Then $\mathfrak{A}$ is isomorphic to a Crs_2 whose unit

element has the form ${}^2U \sim \bigcup_{i\in I}({}^2X_i \sim \mathsf{D}_{01}^{[T]})$, *where* U *is some set, with* $X_i \subseteq U$ *for all* $i \in I$, $X_i \cap X_j = 0$ *for* $i \neq j$, *and* $T = {}^2U$. *Furthermore,* U *is finite if* $\mathfrak{A}$ *is.*

PROOF. Let $Dat = \{a \in At\mathfrak{A} : a \leqq \mathsf{d}_{01}\}$. An element $a \in Dat$ is *small* if $\mathsf{c}_0 a \cdot \mathsf{c}_1 a \leqq \mathsf{d}_{01}$; otherwise it is *big*. Note that if a is small then $a = \mathsf{c}_0 a \cdot \mathsf{c}_1 a$, since $\mathsf{c}_0 a \cdot \mathsf{c}_1 a = \mathsf{c}_0(a \cdot \mathsf{d}_{01}) \cdot \mathsf{c}_1(a \cdot \mathsf{d}_{01}) \cdot \mathsf{d}_{01} = a$. For $a, b \in Dat$ we set $A_{ab} = \{z \in At\mathfrak{A} : z \leqq \mathsf{c}_1 a \cdot \mathsf{c}_0 b\}$. In figure 3.2.60 we illustrate these notions.

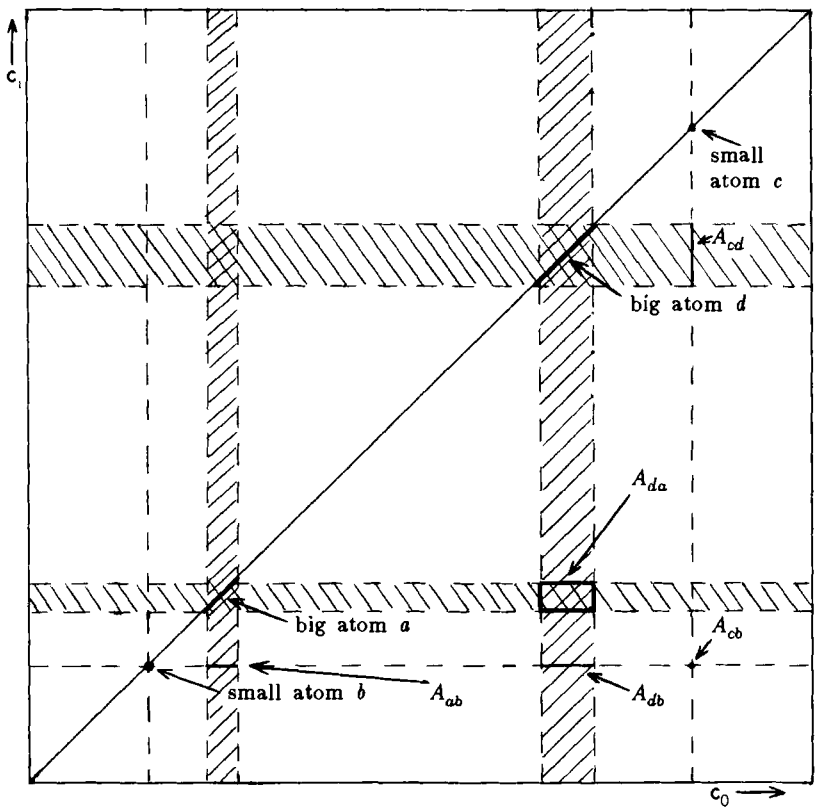

FIGURE 3.2.60

Now we establish some properties of A_{ab}:

(1) $A_{ab} \neq 0$ for all $a,b \in Dat$.

For, $\mathsf{c}_0\mathsf{c}_1(\mathsf{c}_1 a \cdot \mathsf{c}_0 b) = \mathsf{c}_0(\mathsf{c}_1 a \cdot \mathsf{c}_1\mathsf{c}_0 b) = \mathsf{c}_0\mathsf{c}_1 a \cdot \mathsf{c}_1\mathsf{c}_0 b = 1$ by simplicity.

(2) For all $a,b \in Dat$ and $z \in At\mathfrak{A}$, $z \in A_{ab}$ iff $\mathsf{c}_1 z = \mathsf{c}_1 a$ and $\mathsf{c}_0 z = \mathsf{c}_0 b$.

For, $\Leftarrow$ is clear. $\Rightarrow$: $z \leqq \mathsf{c}_1 a$, so $\mathsf{c}_1 z \leqq \mathsf{c}_1 a$. But $\mathsf{c}_1 a$ is a $\{1\}$-atom by 1.10.3, so $\mathsf{c}_1 z = \mathsf{c}_1 a$. Similarly, $\mathsf{c}_0 z = \mathsf{c}_0 b$.

(3) For all $a,b,c,d \in Dat$, if $A_{ab} \cap A_{cd} \neq 0$, then $a = c$ and $b = d$.

For, say $z \in A_{ab} \cap A_{cd}$. Then, using (2),

$$\begin{aligned} a &= \mathsf{d}_{01} \cdot \mathsf{c}_1(a \cdot \mathsf{d}_{01}) \text{ (since } a \leqq \mathsf{d}_{01}) \\ &= \mathsf{d}_{01} \cdot \mathsf{c}_1 a = \mathsf{d}_{01} \cdot \mathsf{c}_1 z = \mathsf{d}_{01} \cdot \mathsf{c}_1 c = c. \end{aligned}$$

Similarly, $b = d$.

(4) $At\mathfrak{A} = \bigcup_{a,b \in Dat} A_{ab}$.

For, let $z \in At\mathfrak{A}$. By 1.10.4, $a = \mathsf{c}_1 z \cdot \mathsf{d}_{01}$ and $b = \mathsf{c}_0 z \cdot \mathsf{d}_{01}$ are atoms; thus $a,b \in Dat$. Clearly $\mathsf{c}_1 a = \mathsf{c}_1 z$ and $\mathsf{c}_0 b = \mathsf{c}_0 z$, so $z \in A_{ab}$ by (2).

Now for each $a \in Dat$ we let

$$X_a = \begin{cases} \{\langle a,x\rangle : x \in At\mathfrak{A}\} & \text{if } a \text{ is big, or small and defective,} \\ \{\langle a,0\rangle\} & \text{if } a \text{ is small and not defective.} \end{cases}$$

Set $U = \bigcup_{a \in Dat} X_a$, $T = {}^2U$. With I as in the statement of the lemma, let $V = {}^2U \sim \bigcup_{a \in I}({}^2X_a \sim \mathsf{D}_{01}^{[T]})$. Thus

(5) $V = \bigcup_{a,b \in Dat, a \neq b} X_a \times X_b \cup \bigcup_{a \in Dat \sim I} X_a \times X_a \cup \bigcup_{a \in I}((X_a \times X_a) \cap \mathsf{D}_{01}^{[T]})$;
(6) $(X_a \times X_b) \cap (X_c \times X_d) \neq 0$ implies $a = c$ and $b = d$.

Now we shall define φ mapping $At\mathfrak{A}$ into SbV, by defining for all $a,b \in Dat$ its restriction $A_{ab}1\varphi$. We shall do this so that the following conditions hold:

(7) if $x,y \in A_{ab}$ and $x \neq y$, then $\varphi x \cap \varphi y = 0 \neq \varphi x$;
(8) if $a,b \in Dat$ and $a \neq b$, then $\bigcup_{z \in A_{ab}} \varphi z = X_a \times X_b$;
(9) if $a \in Dat \sim I$, then $\bigcup_{z \in A_{aa}} \varphi z = X_a \times X_a$;
(10) if $a \in I$, then $\bigcup_{z \in A_{aa}} \varphi z = X_a \times X_a \cap \mathsf{D}_{01}^{[T]}$;
(11) $\mathsf{C}_0^{[V]} \varphi z = V \cap (U \times X_b)$ and $\mathsf{C}_1^{[V]} \varphi z = V \cap (X_a \times U)$ for all $z \in A_{ab}$.

We consider several cases. We call a *large* if it is big or defective.

Case 1. a and b are large, and $a \neq b$. Thus $|X_a| = |X_b| = |At\mathfrak{A}| \geqq |A_{ab}| > 0$. By 3.2.58, let $A_{ab} 1 \varphi$ be a one–one function from A_{ab} onto a partition of $X_a \times X_b$ such that $\mathsf{C}_0^{[W]}\varphi z = \mathsf{C}_1^{[W]}\varphi z = X_a \times X_b$ for all $z \in A_{ab}$, where $W = X_a \times X_b$. Clearly (7)–(11) hold.

Case 2. a is big, $a = b$. Thus $\mathsf{c}_1 a \cdot \mathsf{c}_0 a \nleqq \mathsf{d}_{01}$, so $|A_{aa}| > 1$. Hence by 3.2.58 we can carry through the construction in Case 1 with $\varphi a = \mathsf{D}_{01}^{[T]} \mathsf{n} (X_a \times X_a)$.

Case 3. a defective, $a = b$. Then $A_{aa} = \{a\}$, and we set $\varphi a = (X_a \times X_a) \mathsf{n} \mathsf{D}_{01}^{[T]}$. Clearly (7)–(11) hold.

Case 4. a small non–defective, or b small non–defective. Say a small non–defective. We shall show that $\mathsf{c}_1 a \cdot \mathsf{c}_0 b$ is an atom. Let $x = \mathsf{c}_1 a \cdot \mathsf{c}_0 b$ and suppose that x is not an atom. Say $x = y + z$ where $0 \neq y, z$ and $y \cdot z = 0$. Since a is small and non–defective we have $\mathsf{c}_0 y \cdot \mathsf{c}_0 z \leqq \mathsf{c}_0 (y \cdot z) = 0$. But $y \leqq \mathsf{c}_0 b$ so $\mathsf{c}_0 y = \mathsf{c}_0 b$, and similarly $\mathsf{c}_0 z = \mathsf{c}_0 b$, so $\mathsf{c}_0 b = 0$, contradiction.

Thus $A_{ab} = \{\mathsf{c}_1 a \cdot \mathsf{c}_0 b\}$. We set $\varphi(\mathsf{c}_1 a \cdot \mathsf{c}_0 b) = X_a \times X_b$. Clearly (7)–(11) hold.

This finishes the definition of φ. By (1)–(11), φ is a one–one function from $At\mathfrak{A}$ onto a partition of V. Hence if we let

$$fx = \bigcup \{\varphi a : a \leqq x, a \in At\mathfrak{A}\}$$

for all $x \in A$, we obtain an isomorphism of $\mathfrak{B}\mathfrak{l}\mathfrak{A}$ into the BA $\mathfrak{B}\mathfrak{l}\mathfrak{G}\mathfrak{b}V$, because $\mathfrak{A}$ is complete and atomic. It remains to check that f preserves cylindrifications and diagonal elements.

By symmetry we prove only that f preserves c_0. First suppose $\langle u,v \rangle \in f\mathsf{c}_0 x$. Say $\langle u,v \rangle \in \varphi a$ with $a \leqq \mathsf{c}_0 x$, $a \in At\mathfrak{A}$. Then there is a $b \in At\mathfrak{A}$ with $b \leqq x$ and $a \leqq \mathsf{c}_0 b$. By (4), say $a \leqq \mathsf{c}_1 c \cdot \mathsf{c}_0 e$ with $c, e \in Dat$, and $b \leqq \mathsf{c}_1 s \cdot \mathsf{c}_0 t$ with $s, t \in Dat$. Then $\mathsf{c}_0 e = \mathsf{c}_0 a = \mathsf{c}_0 b = \mathsf{c}_0 t$, so $e = t$. Since $\langle u,v \rangle \in \varphi a$, by (8)–(10) we have $v \in X_e$, hence $\langle u,v \rangle \in V \mathsf{n} (U \times X_e)$ and so by (11),

$$\langle u,v \rangle \in \mathsf{C}_0^{[V]} \varphi b \subseteq \mathsf{C}_0^{[V]} fx,$$

as desired. Conversely, suppose $\langle u,v \rangle \in \mathsf{C}_0^{[V]} fx$. Say $\langle w,v \rangle \in fx$, $\langle w,v \rangle \in \varphi a$ with $a \leqq x$, $a \in At\mathfrak{A}$, $a \in A_{bc}$ with $b, c \in Dat$. Say $\langle u,v \rangle \in \varphi e$, $e \in A_{st}$. Then $v \in X_c$, $v \in X_t$, so $c = t$. Since $\mathsf{c}_0 t = \mathsf{c}_0 a \leqq \mathsf{c}_0 x$, we have $e \leqq \mathsf{c}_0 x$, hence $\langle u,v \rangle \in f\mathsf{c}_0 x$, as desired.

To show that f preserves d_{01}, first suppose that $\langle u,v \rangle \in f\mathsf{d}_{01}$. Say $\langle u,v \rangle \in \varphi a$ with $a \leqq \mathsf{d}_{01}$, $a \in At\mathfrak{A}$. Thus $a \in Dat$ and $a \in A_{aa}$. If a is big, then by Case 2, $u = v$; if a is small and defective then $u = v$ by Case 3; and if a is small and non–defective then $\varphi a = X_a \times X_a$ by Case 4, and $|X_a| = 1$, so $u = v$. Thus $\langle u,v \rangle \in \mathsf{D}_{01}^{[V]}$. Conversely, suppose that $\langle u,v \rangle \in \mathsf{D}_{01}^{[V]}$. Thus $u = v$. Say $\langle u,u \rangle \in \varphi a$, where $a \in At\mathfrak{A}$. Say $a \in A_{bc}$.
By construction it is clear that $b = c = a$, and hence $a \in Dat$. This completes the proof of 3.2.59.

THEOREM 3.2.61. $\mathsf{CA}_2 \subseteq \mathsf{ICrs}_2$.

PROOF. By the semisimplicity of CA_2's (2.4.53) and simply proved facts about ICrs_2, it suffices to show that an arbitrary simple CA_2 $\mathfrak{A}$ is isomorphic to a Crs_2. By

2.7.15 and 2.7.17, $\mathfrak{A} \subseteq \mathfrak{B}$ for some simple complete atomic CA_2 $\mathfrak{B}$. The desired result now follows from 3.2.59.

REMARKS 3.2.62. As was mentioned earlier, Theorem 3.2.61 has been given a direct proof by Andréka and Németi. The theorem gives a general geometric representation theorem for CA_2's; see the end of this section. We know from 2.6.41 and 2.6.42 that not every CA_2 is representable. Thus there is a simple CA_2 $\mathfrak{A}$ which is not isomorphic to a Cs_2. By 3.2.61, $\mathfrak{A}$ is isomorphic to a Crs_2 with unit element V. The set V does not have the form ${}^2U$, but the diagonal element $\mathsf{D}_{01}^{[V]}$ consists of various elements of the form $\langle u,u\rangle$. It is also possible to represent $\mathfrak{A}$ isomorphically as a field of subsets of some set ${}^2U$ with the natural cylindrifications $\mathsf{C}_\kappa^{[T]}$, $T={}^2U$, $\kappa<2$, but with an equivalence relation on U in place of $\mathsf{D}_{01}^{[T]}$. In fact, in the proof of 3.2.59, Case 3, one simply takes $\varphi a = X_a \times X_a$, and otherwise the proof of 3.2.59 remains the same.

To establish our other representation theorem concerning CA_2's we need two elementary lemmas.

LEMMA 3.2.63 *Suppose* $\alpha \geqq 2$, $\kappa,\lambda<\alpha$, $\kappa \neq \lambda$, and the equation

(i) $\mathsf{c}_\lambda(x\cdot y\cdot \mathsf{c}_\kappa(x\cdot -y))\cdot -\mathsf{c}_\kappa(\mathsf{c}_\lambda x\cdot -\mathsf{d}_{\kappa\lambda})=0$

holds in a CA_α $\mathfrak{A}$. *Then the following condition holds for all* $x,y,z\in A$:

(ii) *If* $\mathsf{c}_\lambda x\cdot \mathsf{s}_\lambda^\kappa \mathsf{c}_\lambda x \leqq \mathsf{d}_{\kappa\lambda}$, *then* $\mathsf{c}_\kappa(x\cdot y)\cdot \mathsf{c}_\kappa(x\cdot z)\leqq \mathsf{c}_\kappa(x\cdot y\cdot z)$.

PROOF. Assume the hypothesis of the lemma and of (ii). Then $\mathsf{c}_\lambda x\cdot -\mathsf{d}_{\kappa\lambda}\leqq \mathsf{s}_\lambda^\kappa -\mathsf{c}_\lambda x$, and so for any $w\in A$ we have

$$\begin{gathered}
\mathsf{c}_\lambda(x\cdot w\cdot \mathsf{c}_\kappa(x\cdot -w))\leqq \mathsf{c}_\kappa(\mathsf{c}_\lambda x\cdot -\mathsf{d}_{\kappa\lambda})\leqq \mathsf{s}_\lambda^\kappa -\mathsf{c}_\lambda x;\\
\mathsf{c}_\lambda(x\cdot w\cdot \mathsf{c}_\kappa(x\cdot -w))\cdot \mathsf{c}_\kappa(\mathsf{d}_{\kappa\lambda}\cdot \mathsf{c}_\lambda x)=0;\\
\mathsf{c}_\lambda(\mathsf{c}_\kappa(x\cdot w)\cdot \mathsf{c}_\kappa(x\cdot -w))\cdot \mathsf{d}_{\kappa\lambda}\cdot \mathsf{c}_\lambda x=0;\\
\mathsf{c}_\kappa(x\cdot w)\cdot \mathsf{c}_\kappa(x\cdot -w)\cdot \mathsf{c}_\lambda x=0;\\
\mathsf{c}_\kappa(\mathsf{c}_\kappa(x\cdot w)\cdot \mathsf{c}_\kappa(x\cdot -w)\cdot \mathsf{c}_\lambda x)=0;\\
\mathsf{c}_\kappa(x\cdot w)\cdot \mathsf{c}_\kappa(x\cdot -w)\cdot \mathsf{c}_\kappa \mathsf{c}_\lambda x=0;\\
\mathsf{c}_\kappa(x\cdot w)\cdot \mathsf{c}_\kappa(x\cdot -w)=0.
\end{gathered}$$

Hence if we let $fw=\mathsf{c}_\kappa(x\cdot w)$ for all $w\in A$, we see that f is a homomorphism from $\mathfrak{B}\mathfrak{l}\mathfrak{A}$ into $\mathfrak{B}\mathfrak{l}\mathfrak{R}\mathfrak{l}_y\mathfrak{A}$, with $y=\mathsf{c}_\kappa x$. Hence the conclusion of (ii) follows.

LEMMA 3.2.64. *Suppose* $\alpha\geqq 2$, $\kappa,\lambda<\alpha$, *and* $\kappa\neq\lambda$. *Let* $\mathfrak{A}$ *be a* CA_α. *Then* 3.2.63 (ii) *holds for all* $x,y,z\in A$ *iff the following inequality holds for all* $x,y,z\in A$:

(*) $\mathsf{c}_\kappa(x\cdot y)\cdot \mathsf{c}_\kappa(x\cdot z)\leqq \mathsf{c}_\kappa(x\cdot y\cdot z)+\mathsf{c}_\kappa\mathsf{c}_\lambda(\mathsf{c}_\lambda x\cdot \mathsf{s}_\lambda^\kappa \mathsf{c}_\lambda x\cdot -\mathsf{d}_{\kappa\lambda})$.

PROOF. Clearly (*) implies 3.2.63(ii). Now assume that 3.2.63(ii) holds for all $x,y,z\in A$, and let $x,y,z\in A$ be arbitrary. Set $u=\mathsf{c}_\lambda x\cdot \mathsf{s}_\lambda^\kappa \mathsf{c}_\lambda x\cdot -\mathsf{d}_{\kappa\lambda}$, $x_1=x\cdot \mathsf{c}_\kappa\mathsf{c}_\lambda u$, $x_2=x\cdot -\mathsf{c}_\kappa\mathsf{c}_\lambda u$. Then

$$\begin{aligned} \mathsf{c}_\lambda x_2 \cdot \mathsf{s}_\lambda^\kappa \mathsf{c}_\lambda x_2 \cdot -\mathsf{d}_{\kappa\lambda} &= \mathsf{c}_\lambda x \cdot \mathsf{s}_\lambda^\kappa \mathsf{c}_\lambda x \cdot -\mathsf{d}_{\kappa\lambda} \cdot -\mathsf{c}_\kappa \mathsf{c}_\lambda u \\ &= u \cdot -\mathsf{c}_\kappa \mathsf{c}_\lambda u = 0; \end{aligned}$$

hence by 3.2.63(ii) we have

(1) $\mathsf{c}_\kappa(x_2 \cdot y) \cdot \mathsf{c}_\kappa(x_2 \cdot z) \leqq \mathsf{c}_\kappa(x_2 \cdot y \cdot z)$.

Since $x_1 \leqq \mathsf{c}_\kappa \mathsf{c}_\lambda u$, we have $\mathsf{c}_\kappa x_1 \leqq \mathsf{c}_\kappa \mathsf{c}_\lambda u$ and hence

(2) $\mathsf{c}_\kappa(x_1 \cdot y) \cdot \mathsf{c}_\kappa(x_1 \cdot z) \leqq \mathsf{c}_\kappa \mathsf{c}_\lambda u$.

Now note that $\mathsf{c}_\kappa x_1 \cdot \mathsf{c}_\kappa x_2 = 0 = \mathsf{c}_\lambda x_1 \cdot \mathsf{c}_\lambda x_2$. Hence

$$\begin{aligned} \mathsf{c}_\kappa(x \cdot y) \cdot \mathsf{c}_\kappa(x \cdot z) &= (\mathsf{c}_\kappa(x_1 \cdot y) + \mathsf{c}_\kappa(x_2 \cdot y)) \cdot (\mathsf{c}_\kappa(x_1 \cdot z) + \mathsf{c}_\kappa(x_2 \cdot z)) \\ &= \mathsf{c}_\kappa(x_1 \cdot y) \cdot \mathsf{c}_\kappa(x_1 \cdot z) + \mathsf{c}_\kappa(x_1 \cdot y) \cdot \mathsf{c}_\kappa(x_2 \cdot z) \\ &\qquad + \mathsf{c}_\kappa(x_2 \cdot y) \cdot \mathsf{c}_\kappa(x_1 \cdot z) + \mathsf{c}_\kappa(x_2 \cdot y) \cdot \mathsf{c}_\kappa(x_2 \cdot z) \\ &= \mathsf{c}_\kappa(x_1 \cdot y) \cdot \mathsf{c}_\kappa(x_1 \cdot z) + \mathsf{c}_\kappa(x_2 \cdot y) \cdot \mathsf{c}_\kappa(x_2 \cdot z) \\ &\leqq \mathsf{c}_\kappa \mathsf{c}_\lambda u + \mathsf{c}_\kappa(x_2 \cdot y \cdot z) \\ &\leqq \mathsf{c}_\kappa \mathsf{c}_\lambda u + \mathsf{c}_\kappa(x \cdot y \cdot z), \end{aligned}$$

as desired.

We are now ready for the second representation theorem for CA_2's:

THEOREM 3.2.65. *The following are equivalent, for any* $\mathfrak{A} \in \mathsf{CA}_2$:

(i) $\mathfrak{A} \in \mathsf{SNr}_2\mathsf{CA}_3$;

(ii) *for any* $x, y \in A$, *the following two equations hold*:

$$\mathsf{c}_1(x \cdot y \cdot \mathsf{c}_0(x \cdot -y)) \cdot -\mathsf{c}_0(\mathsf{c}_1 x \cdot -\mathsf{d}_{01}) = 0;$$
$$\mathsf{c}_0(x \cdot y \cdot \mathsf{c}_1(x \cdot -y)) \cdot -\mathsf{c}_1(\mathsf{c}_0 x \cdot -\mathsf{d}_{01}) = 0.$$

(iii) $\mathfrak{A}$ *is representable.*

PROOF. (i)$\Rightarrow$(ii) by 2.6.41 and its proof, and (iii)$\Rightarrow$(i) by 3.1.122. To show that (ii)$\Rightarrow$(iii) it suffices to take the case $\mathfrak{A}$ simple, by 2.4.52 and 3.1.108. Let $\mathfrak{B} = \mathfrak{Em}\mathfrak{A}$. By 2.7.16, 2.7.17, 3.2.63, and 3.2.64, $\mathfrak{B}$ is a simple complete and atomic CA_2 in which the inequalities 3.2.64(*) hold for $\kappa \neq \lambda$, $\kappa, \lambda \in 2$; hence also the implications 3.2.63(ii) hold. It suffices to show that $\mathfrak{B}$ is isomorphic to a Cs_2. By 3.2.59 it suffices to show that for an arbitrary atom a of $\mathfrak{B}$, a is not defective. So assume that $\mathsf{c}_0 a \cdot \mathsf{c}_1 a \leqq \mathsf{d}_{01}$, $\{\kappa, \lambda\} = 2$, $x, y, z \in B$, and $0 \neq x \leqq \mathsf{c}_\kappa a$. Then $\mathsf{c}_\kappa x \cdot \mathsf{s}_\kappa^\lambda \mathsf{c}_\kappa x = \mathsf{c}_\kappa a \cdot \mathsf{c}_\lambda(\mathsf{d}_{\kappa\lambda} \cdot \mathsf{c}_\kappa a) = \mathsf{c}_\kappa a \cdot \mathsf{c}_\lambda a \leqq \mathsf{d}_{01}$. Hence by 3.2.63(ii), $\mathsf{c}_\lambda(x \cdot y) \cdot \mathsf{c}_\lambda(x \cdot z) \leqq \mathsf{c}_\lambda(x \cdot y \cdot z)$, as desired. This finishes the proof.

Note that the proof of 3.2.65 also gives the following result: if $\mathfrak{A}$ is a simple complete atomic CA_2 in which the implications 3.2.63(ii) hold, then $\mathfrak{A} \in \mathsf{ICs}_2$.

Finally, we mention the following corollary, proved using the last statement in Lemma 3.2.59. (It is also true with Gs_2 replaced by Cs_2.)

COROLLARY 3.2.66. *Any finite* Gs_2 *is isomorphic to a* Gs_2 *with a finite base.*

Non–representable algebras

REMARK 3.2.67. In the next part of this section we shall describe various methods for constructing non–representable CA_α's. In Part I we gave one such method: The method of splitting elements, pp. 386–394, pp. 407–408. Here we shall describe six more methods: permutation models (3.2.68), dilation (3.2.69), twisting (3.2.71), and three methods proceeding from combinatorial structures – quasigroups (3.2.72), projective geometries (3.2.75), and graphs (3.2.76). The first three methods are due to Henkin, and the last three to Monk.

CONSTRUCTION 3.2.68. (*Permutation models*). First we describe the general framework and then we make a specific construction. Let U be a non–empty set and $\mathfrak{A}$ the full Cs_α with base U, where α is arbitrary. Recall from 3.1.36 and 3.1.37 that to every permutation f of U there corresponds a base–automorphism $\tilde{f}$ of $\mathfrak{A}$. Throughout this construction, if G is a set of permutations of U we shall denote by $\tilde{G}$ the set of all $\tilde{f}$ for $f \in G$. If $\mathfrak{B}$ is any CA_α and H is a set of automorphisms of $\mathfrak{B}$, then we let $Fx_H\mathfrak{B} = \{b \in B : fb = b \text{ for all } f \in H\}$. Clearly $Fx_H\mathfrak{B}$ is a subuniverse of $\mathfrak{B}$. We denote by $\mathfrak{Sg}_H\mathfrak{B}$ the subalgebra of $\mathfrak{B}$ with universe $Fx_H\mathfrak{B}$. The permutation model method for constructing a CA_α consists in choosing suitable $U, \mathfrak{A}, G$ as above, forming $\mathfrak{C} = \mathfrak{Sg}_{\tilde{G}}\mathfrak{A}$, selecting an appropriate $c \in C$, and finally forming $\mathfrak{Rl}_c\mathfrak{C}$.

We shall now give a particular construction of this kind. We assume that $3 \leq \alpha$. Let W be a set such that $\alpha \cap W = 0$ and $|\alpha| = |W|$; say that $'$ is a one–one function from α onto W. Thus $\kappa' \in W$ for all $\kappa < \alpha$. For each $\kappa < \alpha$ let $a\kappa = \kappa$ and $a\kappa' = \kappa$. Given a permutation τ of α we define a permutation τ' of W by setting $\tau'\kappa' = (\tau\kappa)'$ for all $\kappa < \alpha$.

Let $U = \alpha \cup W$, and let $\mathfrak{A}$ be the full Cs_α with base U. Set $G = \{\tau \cup \tau' : \tau$ is a permutation of α, and $|\{\kappa < \alpha : \tau\kappa \neq \kappa\}| < \omega\}$. Let $\mathfrak{C} = \mathfrak{Sg}_{\tilde{G}}\mathfrak{A}$. For each $s \in {}^\alpha U$ let $s^- = \{g \circ s : g \in G\}$. Clearly $s^- \in C$ for every $s \in {}^\alpha U$. Let

$$V = (\alpha \upharpoonright Id)^- \cup \bigcup\{s^- : a \circ s \text{ is not one–one}\}.$$

Thus $V \in C$. Finally, let $\mathfrak{D} = \mathfrak{Rl}_V\mathfrak{C}$. We shall show that $\mathfrak{D}$ is a non–representable CA_α.

To show that $\mathfrak{D}$ is a CA_α, it suffices by 2.2.3 to check (C_4) and (C_5). To prove (C_4) it suffices to show the following:

(1) If $\kappa, \lambda < \alpha$ and $s \in V$, then $\mathsf{c}_\kappa^{\mathfrak{D}}\mathsf{c}_\lambda^{\mathfrak{D}} s^- = \{t \in V : \text{there exist } u, v \in U \text{ such that } t_{uv}^{\kappa\lambda} \in s^-\}$.

To prove (1), we may assume that $\kappa \neq \lambda$. The inclusion $\subseteq$ is obvious. For $\supseteq$, suppose $t \in V$, $u, v \in U$, and $t_{uv}^{\kappa\lambda} \in s^-$. If $t_u^\kappa \in V$ the desired conclusion is clear, so assume that $t_u^\kappa \notin V$. This implies that $t\mu \neq au$ and $t\mu \neq at\lambda$ for any $\mu \in \alpha \sim \{\kappa, \lambda\}$. Let τ be the transposition $[au/at\lambda, at\lambda/au]$ of α and let $g = \tau \cup \tau'$. Thus $g \in G$. Let $r = g \circ t_{uv}^{\kappa\lambda}$; so $r \in s^-$. Now if $\mu \in \alpha \sim \{\kappa, \lambda\}$ then, by the remark above, $r\mu = gt\mu = t\mu$. Furthermore, $at_{\tau\kappa}^\kappa \kappa = ar\kappa = agu = at\lambda = at_{\tau\kappa}^\kappa \lambda$. Therefore $t_{\tau\kappa}^\kappa \in V$ and $t_{\tau\kappa, \tau\lambda}^{\kappa,\lambda} = r \in s^-$, so $t \in \mathsf{c}_\kappa^{\mathfrak{D}}\mathsf{c}_\lambda^{\mathfrak{D}} s^-$.

Condition (C_5) is obvious, so $\mathfrak{D}$ is a CA_α. To show that $\mathfrak{D}$ is not representable, we consider the following inequality, first discussed in Thompson [52], and mentioned in

1.5.22:

(2) $\mathsf{c}_\kappa x \cdot \mathsf{c}_\lambda y \cdot \mathsf{c}_\mu z \leqq \mathsf{c}_\kappa \mathsf{c}_\lambda \mathsf{c}_\mu(\mathsf{c}_\mu(\mathsf{c}_\lambda x \cdot \mathsf{c}_\kappa y) \cdot \mathsf{c}_\lambda(\mathsf{c}_\mu x \cdot \mathsf{c}_\kappa z) \cdot \mathsf{c}_\kappa(\mathsf{c}_\mu y \cdot \mathsf{c}_\lambda z))$

Here κ,λ,μ are arbitrary distinct ordinals $<\alpha$. It is easy to check that (2) holds in every representable CA_α. We show that it fails in $\mathfrak{D}$. For simplicity take $\kappa=0$, $\lambda=1$, $\mu=2$. We let

$$\begin{aligned} x &= \langle 1', 1, 2, 3, \ldots\rangle^{\smile}, \\ y &= \langle 0, 2', 2, 3, \ldots\rangle^{\smile}, \\ z &= \langle 0, 1, 0', 3, \ldots\rangle^{\smile}, \\ f &= \alpha \upharpoonright Id. \end{aligned}$$

Clearly $f \in \mathsf{c}_0 x \cdot \mathsf{c}_1 y \cdot \mathsf{c}_2 z$ in $\mathfrak{D}$. Suppose f is in the right side of (2). We then obtain elements $u,v,w,a,b,c,m,n,p,q,r,s \in U$ and $g_0,\ldots,g_5 \in G$ with the following properties:

$$f^{012}_{uvw} \in \mathsf{c}_2(\mathsf{c}_1 x \cdot \mathsf{c}_0 y) \cdot \mathsf{c}_1(\mathsf{c}_2 x \cdot \mathsf{c}_0 z) \cdot \mathsf{c}_0(\mathsf{c}_2 y \cdot \mathsf{c}_1 z),$$

$$\begin{array}{lll} f^{012}_{uvc} \in \mathsf{c}_1 x \cdot \mathsf{c}_0 y, & f^{012}_{ubw} \in \mathsf{c}_2 x \cdot \mathsf{c}_0 z, & f^{012}_{avw} \in \mathsf{c}_2 y \cdot \mathsf{c}_1 z, \\ f^{012}_{umc} = g_0 \circ \langle 1', 1, 2, 3, \ldots\rangle, & f^{012}_{nvc} = g_1 \circ \langle 0, 2', 2, 3, \ldots\rangle, & f^{012}_{ubp} = g_2 \circ \langle 1', 1, 2, 3, \ldots\rangle, \\ f^{012}_{qbw} = g_3 \circ \langle 0, 1, 0', 3, \ldots\rangle, & f^{012}_{avr} = g_4 \circ \langle 0, 2', 2, 3, \ldots\rangle, & f^{012}_{asw} = g_5 \circ \langle 0, 1, 0', 3, \ldots\rangle. \end{array}$$

Therefore $g_0,\ldots,g_5$ permute $\{0,1,2\}$, and $u = g_0 1'$, $v = g_1 2'$, $w = g_3 0'$, so $u,v,w \in W$. Since $f^{012}_{uvw} \in V$, it follows that $|\{u,v,w\}| \leqq 2$. By symmetry, say $u = v$. Then $g_0 1' = u = v = g_1 2' = (g_1 2)' = c' = (g_0 2)' = g_0 2'$, contradiction.

CONSTRUCTION 3.2.69. (*Dilation*) The method in 3.2.68, aside from the use of permutations, was relativization. Since we relativized an atomic Cs_α, we can say that we *deleted* atoms. Here we want to do the opposite, *add* atoms.

First we explain the general procedure. We start with some Ca_α $\mathfrak{B} = \langle B, T_\kappa, E_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha}$; recall from 2.7.38 that a Ca_α is a relational structure which is the atom structure of some complete and atomic CA_α. Suppose $a \in {}^\alpha B$, and the following two conditions hold (recall that T_κ is an equivalence relation on B for every $\kappa<\alpha$):

(1) $(a_\kappa/T_\lambda) \cap (a_\lambda/T_\kappa) \neq 0$ for all $\kappa,\lambda<\alpha$;
(2) $a_\mu \notin E_{\kappa\lambda}$ if κ,λ,μ are distinct ordinals $<\alpha$.

Then we choose some element $n \notin B$ and form a relational structure $\mathfrak{B}' = \langle B', T'_\kappa, E'_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha}$ as follows. We set $B' = B \cup \{n\}$. For any $\kappa<\alpha$, T'_κ is an equivalence relation on B', $T'_\kappa \cap (B \times B) = T_\kappa$, and for any $b \in B$, $bT'_\kappa n$ iff $bT_\kappa a_\kappa$. Finally, $E'_{\kappa\lambda} = E_{\kappa\lambda}$ for κ and λ distinct ordinals $<\alpha$, while $E'_{\kappa\kappa} = B'$ for all $\kappa<\alpha$. We claim that $\mathfrak{B}'$ is a Ca_α. To prove this it suffices to check the conditions of 2.7.40. Of these conditions, (i), (iii) and (v) are obvious. To prove (ii) it suffices to show that $T'_\kappa|T'_\lambda \subseteq T'_\lambda|T'_\kappa$ for distinct $\kappa,\lambda<\alpha$. Assume that $b(T'_\kappa|T'_\lambda)c$, say $bT'_\kappa eT'_\lambda c$. We may assume that $n \in \{b,e,c\}$. By symmetry it suffices to consider the following two cases.
Case 1. $b,e \in B$, $c = n$. Thus $bT_\kappa eT_\lambda a_\lambda$, and by (1) there is a $g \in B$ with $a_\kappa T_\lambda g T_\kappa a_\lambda$, so

by (ii) for $\mathfrak{B}$, $bT_\lambda hT_\kappa a_\kappa$ for some $h \in B$. Hence $bT'_\lambda hT'_\kappa n$, as desired.
Case 2. $b,c \in B$, $e = n$. Thus $bT_\kappa a_\kappa$ and $cT_\lambda a_\lambda$. By (1), $a_\kappa T_\lambda gT_\kappa a_\lambda$ for some $g \in B$. Hence by (ii) for $\mathfrak{B}$, $bT_\lambda hT_\kappa c$ for some $h \in B$, as desired.

Now we check 2.7.40(iv). Assume $\kappa,\lambda,\mu<\alpha$ and $\mu \neq \kappa,\lambda$. Suppose $bT'_\mu c \in E'_{\kappa\mu} \cap E'_{\mu\lambda}$. If $\kappa = \lambda$, obviously $b \in E'_{\kappa\lambda}$. Assume $\kappa \neq \lambda$. If $b \in B$, obviously $b \in E'_{\kappa\lambda}$. Suppose $b = n$. Thus $a_\mu T_\mu c \in E_{\kappa\mu} \cap E_{\mu\lambda}$, so $a_\mu \in E_{\kappa\lambda}$, contradicting (2). We have now shown that $T'^*_\mu(E'_{\kappa\mu} \cap E'_{\mu\lambda}) \subseteq E'_{\kappa\lambda}$.

That $E'_{\kappa\lambda} \subseteq T'^*_\mu(E'_{\kappa\mu} \cap E'_{\mu\lambda})$ is clear if $\kappa \neq \lambda$. To check this for $\kappa = \lambda$ it suffices to show that $n \in T'^*_\mu E'_{\kappa\mu}$. Since $a_\mu \in E_{\kappa\kappa} = T^* E_{\kappa\mu}$, choose $b \in E_{\kappa\mu}$ with $a_\mu T_\mu b$. Thus $nT'_\mu b$, so $n \in T'^*_\mu E'_{\kappa\mu}$. This finishes the general description.

Now we shall construct a non–representable CA_α using the method of dilation. Assume that $3 \leq \alpha < \omega$. Let W, G and $\mathfrak{C}$ be as in 3.2.68. Let $\mathfrak{B} = \mathfrak{At}\mathfrak{C}$, the atom structure of $\mathfrak{C}$. So $\mathfrak{B} \in \mathsf{Ca}_\alpha$; we write $\mathfrak{B} = \langle B, T_\kappa, E_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha}$ as above. For each $\kappa < \alpha - 1$ let s_κ be the member of ${}^\alpha\alpha$ such that, for all $\mu < \alpha$,

$$s_\kappa \mu = \begin{cases} \mu & \text{if } \mu \leq \kappa, \\ \mu - 1 & \text{if } \kappa < \mu < \alpha, \end{cases}$$

and let $s_{\alpha-1}$ be such that for all $\mu < \alpha$,

$$s_{\alpha-1}\mu = \begin{cases} \mu & \text{if } \mu < \alpha - 1, \\ 0 & \text{if } \mu = \alpha - 1. \end{cases}$$

Let $a_\kappa = s_\kappa^-$ for each $\kappa < \alpha$. We now check the conditions (1), (2). For (1), it suffices to take distinct $\kappa,\lambda < \alpha$. Say $\kappa < \lambda$. We treat only the case $\kappa + 1 < \lambda < \alpha - 1$ and leave the other possibilities to the reader. Let t be the member of ${}^\alpha\alpha$ such that for all $\mu < \alpha$,

$$t\mu = \begin{cases} \mu & \text{if } \mu \leq \kappa, \\ \mu - 1 & \text{if } \kappa < \mu < \lambda, \\ \lambda & \text{if } \mu = \lambda, \\ \mu - 1 & \text{if } \lambda < \mu < \alpha. \end{cases}$$

Clearly $t^- T_\lambda a_\kappa$. Let τ be the cyclic permutation $(\kappa, \lambda-1, \lambda-2, \ldots, \kappa+1)$ of α and set $g = \tau \cup \tau'$. Then $g \circ s_\lambda \in s_\lambda^-$ and for any $\mu < \alpha$,

$$(g \circ s_\lambda)\mu = \begin{cases} \mu & \text{if } \mu < \kappa, \\ \lambda - 1 & \text{if } \mu = \kappa, \\ \mu - 1 & \text{if } \kappa < \mu < \lambda, \\ \lambda & \text{if } \mu = \lambda, \\ \mu - 1 & \text{if } \lambda < \mu < \alpha. \end{cases}$$

Thus $t^- T_\kappa a_\lambda$, as desired. So we take (1) as established. Condition (2) is obvious.

Thus by the general procedure we obtain a Ca_α $\mathfrak{B}' = \langle B', T'_\kappa, E'_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha}$ by adjoining a new element n. Let $\mathfrak{D} = \mathfrak{C}\mathfrak{m}\mathfrak{B}'$. Thus $\mathfrak{D}$ is a CA_α by 2.7.39. We show that $\mathfrak{D}$ is non–representable by considering a new equation, which is an algebraic version of the associativity of relative product of binary relations. To formulate it, let $\mathfrak{E}$ be an

arbitrary CA_β, $3 \leqq \beta$. We define a binary operation ; on E by setting, for any $x, y \in E$,

$$x;y = \mathsf{c}_2(\mathsf{s}_2^1\mathsf{c}_2 x \cdot \mathsf{s}_2^0\mathsf{c}_2 y).$$

Note that if $\mathfrak{F}$ is a Cs_β with base X and $x, y \in F$, then

$$x;y = \{z \in {}^\beta X : \text{there exists } u \in X \text{ with } z_u^1 \in \mathsf{c}_2 x \text{ and } z_u^0 \in \mathsf{c}_2 y\},$$

which shows the relationship of ; with relative product; see also section 5.3. This also shows that the equation

(3) $x;(y;z) = (x;y);z$

holds in every representable CA_β. Now we shall show that it does not hold in $\mathfrak{D}$.

To this end, let $x = \langle 0,0',1,2,3,\ldots\rangle^-$, $y = \langle 0',1,1,2,3,\ldots\rangle^-$, $z = \langle 1,0,1,2,3,\ldots\rangle^-$. Note that $\mathsf{s}_2^1\mathsf{c}_2 y = \mathsf{c}_1(\mathsf{d}_{12} \cdot \mathsf{c}_2 y) = \mathsf{c}_1 y$ and similarly $\mathsf{s}_2^0\mathsf{c}_2 z = \mathsf{c}_0 z$. Hence

(4) $x;(y;z) = \mathsf{c}_2(\mathsf{s}_2^1\mathsf{c}_2 x \cdot \mathsf{s}_2^0\mathsf{c}_2(\mathsf{c}_1 y \cdot \mathsf{c}_0 z))$,
(5) $(x;y);z = \mathsf{c}_2(\mathsf{s}_2^1\mathsf{c}_2(\mathsf{s}_2^1\mathsf{c}_2 x \cdot \mathsf{s}_2^0\mathsf{c}_2 y) \cdot \mathsf{c}_0 z)$.

(All of the operations above are in $\mathfrak{D}$. For simplicity we treat n as well as each element s^- for $s \in {}^\alpha U$ as an atom of $\mathfrak{D}$.) Now we claim

(6) If $t \in {}^\alpha U \mathbf{n}(x;(y;z))$, then $t_0 = t_1$.

For, $\mathsf{d}_{12} \cdot \mathsf{c}_2 x = \langle 0,0',0',2,3,\ldots\rangle^-$, hence $a_1 \nleqq \mathsf{s}_2^1\mathsf{c}_2 x$ and so $n \nleqq \mathsf{s}_2^1\mathsf{c}_2 x$. It follows that there is a $u \in U$ such that $t_u^2 \in \mathsf{s}_2^1\mathsf{c}_2 x \cdot \mathsf{s}_2^0\mathsf{c}_2(\mathsf{c}_1 y \cdot \mathsf{c}_0 z)$. Now $\mathsf{c}_1 y \cdot \mathsf{c}_0 z = \langle 0',0,1,2,3,\ldots\rangle^-$, so $\mathsf{d}_{02} \cdot \mathsf{c}_2(\mathsf{c}_1 y \cdot \mathsf{c}_0 z) = \langle 0',0,0',2,3,\ldots\rangle^-$. Therefore $\mathsf{s}_2^1\mathsf{c}_2 x \cdot \mathsf{s}_2^0\mathsf{c}_2(\mathsf{c}_1 y \cdot \mathsf{c}_0 z) = \langle 0,0,0',2,3,\ldots\rangle^-$. Hence $t_0 = t_1$, as desired.

(7) $\langle 0,1,0,2,3,\ldots\rangle^- \leqq \mathsf{c}_2 n$.

For, $\langle 0,1,0,2,3,\ldots\rangle^- \leqq \mathsf{c}_2 a_2$, and (7) follows. By (6) and (7) we have $n \nleqq x;(y;z)$. Now we show that $n \leqq (x;y);z$, so that (3) fails. Clearly $n \leqq \mathsf{c}_0 z$. Now $\mathsf{d}_{12} \cdot \mathsf{c}_2 x = \langle 0,0',0',2,3,\ldots\rangle^-$ and $\mathsf{d}_{02} \cdot \mathsf{c}_2 y = \langle 0',1,0',2,3,\ldots\rangle^-$. Hence $\mathsf{s}_2^1\mathsf{c}_2 x \cdot \mathsf{s}_2^0\mathsf{c}_2 y = \langle 0,1,0',2,3,\ldots\rangle^-$, and hence $\mathsf{d}_{12} \cdot \mathsf{c}_2(\mathsf{s}_2^1\mathsf{c}_2 x \cdot \mathsf{s}_2^0\mathsf{c}_2 y) = \langle 0,1,1,2,3,\ldots\rangle^-$. Thus $n \leqq \mathsf{s}_2^1\mathsf{c}_2(\mathsf{s}_2^1\mathsf{c}_2 x \cdot \mathsf{s}_2^0\mathsf{c}_2 y)$. So $n \leqq (x;y);z$, as desired.

REMARK 3.2.70. By combining many algebras using ultraproducts we can obtain infinite dimensional CA_α's in which the equation 3.2.69(3) fails. This is a general method, enabling one always to restrict oneself to the case $\alpha < \omega$ when considering such equations. The method is essentially described in the proof of 2.6.4, but we sketch it here. Suppose $\alpha \geqq \omega$. Let $I = \{\Gamma : 3 \subseteq \Gamma \subseteq \alpha \text{ and } |\Gamma| < \omega\}$. For each $\Gamma \in I$ let $\beta_\Gamma = |\Gamma|$ and let ρ_Γ be a one–one function from β_Γ onto Γ such that $3 \restriction Id \subseteq \rho_\Gamma$; moreover, let $M_\Gamma = \{\Delta \in I : \Gamma \subseteq \Delta\}$. Furthermore, let $\mathfrak{A}_\Gamma$ be a $\mathsf{CA}_{\beta_\Gamma}$ in which 3.2.69(3) fails. Let $\mathfrak{B}_\Gamma$ be

an algebra similar to CA_α's such that $\mathfrak{A}_\Gamma = \mathfrak{Rb}^{\rho^\Gamma}\mathfrak{B}_\Gamma$, for each $\Gamma \in I$ (extending 2.6.1 in the natural way). Let U be an ultrafilter on I such that $M_\Gamma \in U$ for every $\Gamma \in I$. Then $\mathsf{P}_{\Gamma\in I}\mathfrak{B}_\Gamma/\bar{U}$ is easily seen to be the desired algebra.

CONSTRUCTION 3.2.71. (*Twisting*) This method, roughly speaking, consists of starting from a complete atomic CA_α $\mathfrak{A}$, selecting atoms $a,b \in A$ and an ordinal $\kappa<\alpha$, and redefining c_κ on a and b by interchanging the action of c_κ on a and b, *in part* ("twisting").

Again we describe first the general framework; this description is due to Andréka and Németi. Suppose given a Ca_α $\mathfrak{B} = \langle B, T_\lambda, E_{\lambda\mu}\rangle_{\lambda,\mu<\alpha}$, $\kappa<\alpha$, $x,y \in B$ with not $(xT_\kappa y)$, and two partitions $x/T_\kappa = X_0 \cup X_1$, $y/T_\kappa = Y_0 \cup Y_1$, where $X_0 \cap X_1 = 0 = Y_0 \cap Y_1$ and the following conditions hold; for brevity we write $M = (x/T_\kappa) \cup (y/T_\kappa)$:

(1) $(M \upharpoonright T_\lambda) \sim Id \subseteq (X_0 \times Y_0) \cup (Y_0 \times X_0) \cup (X_1 \times Y_1) \cup (Y_1 \times X_1)$ for all $\lambda \in \alpha \sim \{\kappa\}$.
(2) If $\lambda \in \alpha \sim \{\kappa\}$ and $a \in M$, then there is a $b \in M \sim \{a\}$ such that $aT_\lambda b$.
(3) If $i \in 2$ and $\lambda,\mu \in \alpha$, then: $X_i \cap E_{\kappa\lambda} \cap E_{\kappa\mu} \neq 0$ iff $Y_i \cap E_{\kappa\lambda} \cap E_{\kappa\mu} \neq 0$.

Then we form a new relational structure $\mathfrak{B}' = \langle B, T'_\lambda, E_{\lambda\mu}\rangle_{\lambda,\mu<\alpha}$ as follows: $T'_\lambda = T_\lambda$ if $\lambda \neq \kappa$, while T'_κ is the equivalence relation on B with equivalence classes k/T_κ for $(k/T_\kappa) \cap M = 0$, along with $X_0 \cup Y_1$ and $X_1 \cup Y_0$. We claim that $\mathfrak{B}'$ is a Ca_α, and again we use 2.7.40. Conditions (i) and (iii) are obvious. To prove (ii) it suffices to show that $T'_\kappa | T_\lambda = T_\lambda | T'_\kappa$ for an arbitrary $\lambda \in \alpha \sim \{\kappa\}$. We first claim

(4) $T'_\kappa \subseteq T_\kappa | T_\lambda$.

For, let $aT'_\kappa b$. The non-trivial cases are illustrated by assuming that $a \in X_0$ and $b \in Y_1$. By (2) there is a $c \in M \sim \{b\}$ with $cT_\lambda b$, and then by (1) we have $c \in X_1$. Thus $aT_\kappa cT_\lambda b$, as desired.

(5) $T_\kappa \subseteq (T_\lambda | T'_\kappa) \cap (T'_\kappa | T_\lambda)$.

For, let $aT_\kappa b$. We may assume that $a,b \in M$, say $a,b \in x/T_\kappa$. If $a,b \in X_0$ or $a,b \in X_1$, then $\langle a,b\rangle \in T'_\kappa$ and so $\langle a,b\rangle$ is in the right side of (5). By symmetry it is enough to treat still $a \in X_0$, $b \in X_1$. By (2), say $aT_\lambda c \neq a$. Thus by (1), $c \in Y_0$ and hence $cT'_\kappa b$ and so $a(T_\lambda | T'_\kappa)b$. Similarly $a(T'_\kappa | T_\lambda)b$. So, (5) holds.

Using (4) and (5), we can now establish $T'_\kappa | T_\lambda = T_\lambda | T'_\kappa$:

$$T'_\kappa | T_\lambda \subseteq T_\kappa | T_\lambda | T_\lambda = T_\lambda | T_\kappa \subseteq T_\lambda | T_\lambda | T'_\kappa = T_\lambda | T'_\kappa;$$

similarly, $T_\lambda | T'_\kappa \subseteq T'_\kappa | T_\lambda$.

To prove 2.7.40(iv) it suffices to show that

(6) $T'^*_\kappa(E_{\lambda\kappa} \cap E_{\kappa\mu}) = T^*_\kappa(E_{\lambda\kappa} \cap E_{\kappa\mu})$ if $\kappa \neq \lambda,\mu$.

To show (6), suppose first that $a \in T'^{*}_{\kappa}(E_{\lambda\kappa} \cap E_{\kappa\mu})$; say $aT'_{\kappa}b \in E_{\lambda\kappa} \cap E_{\kappa\mu}$. The non-trivial cases are illustrated by $a \in X_0$, $b \in Y_1$. Thus $Y_1 \cap E_{\lambda\kappa} \cap E_{\kappa\mu} \neq 0$, so by (3), $X_1 \cap E_{\lambda\kappa} \cap E_{\kappa\mu} \neq 0$. Obviously, then, $a \in T^{*}_{\kappa}(E_{\lambda\kappa} \cap E_{\kappa\mu})$. The converse is similar.

To prove 2.7.40(v), it suffices to show that $T'_{\kappa} \cap (E_{\kappa\lambda} \times E_{\kappa\lambda}) \subseteq Id$ for $\kappa \neq \lambda$. Suppose, on the contrary, that $aT'_{\kappa}b$, $a \neq b$, and $a,b \in E_{\kappa\lambda}$. Since $\mathfrak{B}$ satisfies 2.7.40(v), the only way this could possibly happen is when, say, $a \in X_0$ and $b \in Y_1$. But then by (3) we get $X_1 \cap E_{\kappa\lambda} \neq 0$, contradicting (v) for $\mathfrak{B}$.

We have thus shown that $\mathfrak{B}'$ is a Ca_α.

Now we give a specific construction. Assume that $3 \leqq \alpha < \omega$. Let $\mathfrak{B}$ be the full Cs_α with base $2\alpha-2$, and let G be the set of all permutations of $2\alpha-2$ of the form $[0/1,1/0] \circ [2\kappa/(2\kappa+1),(2\kappa+1)/2\kappa]$ for $0<\kappa<\alpha-1$. Set $\mathfrak{A} = \mathfrak{Sr}_{\tilde{G}}\mathfrak{B}$ (see 3.2.68). Let $\mathfrak{C} = \langle At\mathfrak{A}, T_{\kappa}, E_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha}$ be the atom structure of $\mathfrak{A}$. We apply the above general procedure to $\mathfrak{C}$, the ordinal $1<\alpha$, the elements $x = \langle 3,0,0,4,6,8,\ldots\rangle^{-}$ and $y = \langle 3,1,1,4,6,8,\ldots\rangle^{-}$, and the following partitions of x/T_1 and y/T_1 (note that $\mathrm{not}(xT_1y)$):

$$
\begin{aligned}
X_0 &= \{\langle 3,i,0,4,6,8,\ldots\rangle^{-} : i \in 2\},\\
X_1 &= \{\langle 3,\lambda,0,4,6,8,\ldots\rangle^{-} : 2 \leqq \lambda < 2\alpha-2\},\\
Y_0 &= \{\langle 3,i,1,4,6,8,\ldots\rangle^{-} : i \in 2\},\\
Y_1 &= \{\langle 3,\lambda,1,4,6,8,\ldots\rangle^{-} : 2 \leqq \lambda < 2\alpha-2\}.
\end{aligned}
$$

It is tedious but routine to check that (1)–(3) hold. We let $\mathfrak{D}$ be the Ca_α obtained by our general process.

Thus $\mathfrak{Em}\mathfrak{D}$ is a CA_α. We now show that it is not representable. To do this we first consider the equation

(7) ${}_2\mathsf{s}(0,1)\mathsf{c}_2 z = {}_2\mathsf{s}(1,0)\mathsf{c}_2 z.$

As is easily checked, it holds identically in every representable CA_α. We show that it fails in $\mathfrak{Em}\mathfrak{D}$. (See 1.5.14 and the comments following it.) Take $z = \langle 0,3,0,4,6,8,\ldots\rangle^{-}$. Then

$$
\begin{aligned}
{}_2\mathsf{s}(0,1)\mathsf{c}_2 z &= \mathsf{s}^2_0\mathsf{s}^0_1\mathsf{s}^1_2\mathsf{c}_2 z = \mathsf{s}^2_0\mathsf{s}^0_1\mathsf{c}_1\langle 0,3,3,4,6,8,\ldots\rangle^{-}\\
&= \mathsf{s}^2_0\mathsf{c}_0\langle 0,0,3,4,6,8,\ldots\rangle^{-}\\
&= \mathsf{c}_2\langle 3,0,3,4,6,8,\ldots\rangle^{-};
\end{aligned}
$$

$$
\begin{aligned}
{}_2\mathsf{s}(1,0)\mathsf{c}_2 z &= \mathsf{s}^2_1\mathsf{s}^1_0\mathsf{s}^0_2\mathsf{c}_2 z = \mathsf{s}^2_1\mathsf{s}^1_0\mathsf{c}_0\langle 0,3,0,4,6,8,\ldots\rangle^{-}\\
&= \mathsf{s}^2_1\mathsf{c}_1\langle 3,3,0,4,6,8,\ldots\rangle^{-}\\
&= \mathsf{c}_2\langle 3,1,1,4,6,8,\ldots\rangle^{-}.
\end{aligned}
$$

Since clearly $\langle 3,1,1,4,6,8,\ldots\rangle \notin \mathsf{c}_2\langle 3,0,3,4,6,8,\ldots\rangle^{-}$, we see that (7) does fail in $\mathfrak{Em}\mathfrak{D}$. $\mathfrak{Em}\mathfrak{D}$ can also be used to show the failure of two further similar equations (8) and (9) which follow.

(8) ${}_2\mathsf{s}(0,1)\,{}_2\mathsf{s}(0,1)\,\mathsf{c}_2 z = \mathsf{c}_2 z.$

Again it is easy to check that (8) holds identically in every representable CA_α. It fails in $\mathfrak{Em}\mathfrak{D}$ with the same element z as above. It is in fact routine to check that

$$ {}_2\mathsf{s}(0,1)\,{}_2\mathsf{s}(0,1)\,\mathsf{c}_2 z = \mathsf{c}_2\langle 1,3,1,4,6,8,\ldots\rangle^{\frown}, $$

while clearly $\langle 0,3,0,4,6,8,\ldots\rangle \notin \mathsf{c}_2\langle 1,3,1,4,6,8,\ldots\rangle^{\frown}$. Finally, assume that $4 \leqq \alpha$, and consider the equation

(9) ${}_2\mathsf{s}(0,1)\,{}_2\mathsf{s}(0,3)\,\mathsf{c}_2 z = {}_2\mathsf{s}(1,3)\,{}_2\mathsf{s}(0,1)\,\mathsf{c}_2 z.$

Again it is easy to check that this equation holds in every representable CA_α. Let $z = \langle 4,3,0,0,6,8,\ldots\rangle^{\frown}$. Then one can check that

$$ {}_2\mathsf{s}(0,1)\,{}_2\mathsf{s}(0,3)\,\mathsf{c}_2 z = \mathsf{c}_2\langle 3,0,3,4,6,8,\ldots\rangle^{\frown}, $$
$$ {}_2\mathsf{s}(1,3)\,{}_2\mathsf{s}(0,1)\,\mathsf{c}_2 z = \mathsf{c}_2\langle 3,1,1,4,6,8,\ldots\rangle^{\frown}, $$

so (9) does fail in $\mathfrak{Em}\mathfrak{D}$.

In all of the above the indices $0,1,2,3<\alpha$ played a special role. Of course one can modify the construction of $\mathfrak{D}$ in an obvious way to take care similarly of other indices.

The above three constructions use ideas that have been well–developed in our previous work: starting with a set algebra in each case, deletions, additions or modifications are made. The remaining constructions which we shall discuss use new ideas which relate cylindric algebras to more familiar algebraic or combinatorial structures.

CONSTRUCTION 3.2.72. (*Quasigroups*) We first describe this construction in general, and then give two specific constructions: CA_α's using quasigroups (following Monk [74']) and CA_4's using Boolean groups.

Suppose that α and β are ordinals with $\alpha \geqq 3$, U is a non–empty set, and $W \subseteq {}^\beta U$, $f \in W$. Also assume that $\Gamma_\kappa \subseteq \beta$ for each $\kappa<\alpha$. We now define a relational structure $\mathfrak{A} = \langle W, T_\kappa, E_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha}$. For any $\kappa<\alpha$ we let $x T_\kappa y$ iff $x,y \in W$ and $\Gamma_\kappa \upharpoonright x \subseteq y$. Further, let $E_{\kappa\kappa} = W$ for any $\kappa<\alpha$, while for $\kappa \neq \lambda$ and $\kappa,\lambda<\alpha$ we let

$$ E_{\kappa\lambda} = \{y \in W : (\textstyle\bigcap_{\mu\neq\kappa,\lambda}\Gamma_\mu)\upharpoonright f \subseteq y\}. $$

Now we make the following additional assumptions:

(1) $T_\kappa \mid T_\lambda = T_\lambda \mid T_\kappa$ for all $\kappa,\lambda<\alpha$.
(2) For all distinct $\kappa,\lambda<\alpha$ and all $u,v \in W$, there is a $w \in W$ such that $\Gamma_\kappa \upharpoonright u \subseteq w$ and $(\bigcap_{\mu\neq\kappa,\lambda}\Gamma_\mu)\upharpoonright v \subseteq w$.
(3) For all distinct $\kappa,\lambda,\mu<\alpha$ and all $u,v \in W$, if $(\bigcap_{\nu\neq\kappa,\lambda}\Gamma_\nu)\upharpoonright u \subseteq v$ and $(\bigcap_{\nu\neq\kappa,\mu}\Gamma_\nu)\upharpoonright u \subseteq v$ then $(\bigcap_{\nu\neq\lambda,\mu}\Gamma_\nu)\upharpoonright u \subseteq v$.
(4) For all distinct $\kappa,\lambda<\alpha$ and all $u,v \in W$, if $(\Gamma_\kappa \cup \bigcap_{\mu\neq\kappa,\lambda}\Gamma_\mu)\upharpoonright u \subseteq v$, then $u = v$.

Under these assumptions we claim that $\mathfrak{A}$ is a Ca_α. To prove this we shall again apply 2.7.40. Conditions (i), (ii), (iii) are obvious. As to condition (iv), we first show that $W = T_\kappa^* E_{\kappa\lambda}$ if κ and λ are distinct ordinals $<\alpha$. Let $z \in W$ be arbitrary. By condition (2) choose $w \in W$ such that $\Gamma_\kappa 1 z \subseteq w$ and $(\bigcap_{\mu\neq\kappa,\lambda}\Gamma_\mu)1f \subseteq w$. Thus $zT_\kappa w \in E_{\kappa\lambda}$, as desired. Next, suppose that $\kappa,\lambda,\mu<\alpha$ are distinct. To show that $E_{\kappa\lambda} \subseteq T_\mu^*(E_{\kappa\mu} \cap E_{\mu\lambda})$, let $z \in E_{\kappa\lambda}$ be arbitrary. By (2) choose $w \in W$ so that $\Gamma_\mu 1 z \subseteq w$ and $(\bigcap_{\nu\neq\kappa,\mu}\Gamma_\nu)1f \subseteq w$. Since $\bigcap_{\nu\neq\kappa,\lambda}\Gamma_\nu \subseteq \Gamma_\mu$, we also have $(\bigcap_{\nu\neq\kappa,\lambda}\Gamma_\nu)1f = (\bigcap_{\nu\neq\kappa,\lambda}\Gamma_\nu)1z \subseteq w$. Hence by (3), $(\bigcap_{\nu\neq\mu,\lambda}\Gamma_\nu)1f \subseteq w$. Thus $zT_\mu w \in E_{\kappa\mu} \cap E_{\mu\lambda}$, as desired. To show that $T_\mu^*(E_{\kappa\mu} \cap E_{\mu\lambda}) \subseteq E_{\kappa\lambda}$, suppose $z \in T_\mu^*(E_{\kappa\mu} \cap E_{\mu\lambda})$. Say $zT_\mu w \in E_{\kappa\mu} \cap E_{\mu\lambda}$. Thus $\Gamma_\mu 1 z \subseteq w$, $(\bigcap_{\nu\neq\kappa,\mu}\Gamma_\nu)1f \subseteq w$, and $(\bigcap_{\nu\neq\mu,\lambda}\Gamma_\nu)1f \subseteq w$. Hence by (3), $(\bigcap_{\nu\neq\kappa,\lambda}\Gamma_\nu)1f \subseteq w$. Since $\bigcap_{\nu\neq\kappa,\lambda}\Gamma_\nu \subseteq \Gamma_\mu$, it follows that $(\bigcap_{\nu\neq\kappa,\lambda}\Gamma_\nu)1f \subseteq z$, i.e. $z \in E_{\kappa\lambda}$, as desired. It remains only to check (v). So, suppose $\kappa,\lambda<\alpha$, $\kappa \neq \lambda$, $x,y \in E_{\kappa\lambda}$, and $xT_\kappa y$. It is clear that (4) then yields $x = y$, as desired.

Now we apply this general construction to a quasigroup as follows. Let $\mathfrak{G} = \langle G,\cdot\rangle$ be an arbitrary quasigroup. Recall that this means that $\cdot$ is a binary operation on G such that for any $a,b \in G$, the equations $a\cdot x = b$ and $y\cdot a = b$ have unique solutions in G. We consider $\cdot$ as a subset of ${}^3G$. Fix some element f of $\cdot$. We now apply the above construction with 3, 3, G, $\cdot$, f, $\{\kappa\}$ in place of α, β, U, W, f, Γ_κ (for each $\kappa<3$). To see that we get a CA_3 we need to check conditions (1)–(4). In fact, it is trivial to check them; note in particular that $T_\kappa | T_\lambda = {}^2W$ for all distinct $\kappa,\lambda<3$. We formulate the main property of the CA_3's so constructed in the following theorem. (Recall that a loop is a quasigroup with identity, i.e., with an element e such that $e\cdot a = a\cdot e = a$ for all $a \in G$.) For the purpose of this theorem we denote the CA_3 $\mathfrak{Cm}\mathfrak{A}$, with $\mathfrak{A}$ as above, by $\mathfrak{Cm}\mathfrak{A}_{\mathfrak{G}f}$.

THEOREM 3.2.73. *Let $\mathfrak{G} = \langle G,\cdot\rangle$ be a loop with identity element e, and let $f = \langle e,e,e\rangle$. Then $\mathfrak{Cm}\mathfrak{A}_{\mathfrak{G}f}$ is representable iff $\mathfrak{G}$ is a group.*

PROOF. For the direction $\Rightarrow$, we assume that $\mathfrak{G}$ is not a group and show that $\mathfrak{Cm}\mathfrak{A}_{\mathfrak{G}f}$ is not representable. For this purpose we shall again use the equation 3.2.69(3), which we know holds in all representable CA_α's. Since $\mathfrak{G}$ is not a group, there are elements $a,b,c \in G$ such that $a\cdot(b\cdot c) \neq (a\cdot b)\cdot c$. Let $x = \{\langle e,c,c\rangle\}$, $y = \{\langle e,b,b\rangle\}$, $z = \{\langle a,e,a\rangle\}$. Then it is easily checked that 3.2.69(3) fails.

For the direction $\Leftarrow$, we shall assign to each $g \in \cdot$ a subset Fg of ${}^3G$:

$$Fg = \{x \in {}^3G : g_0 = x_1x_2^{-1} \text{ and } g_1 = x_2x_0^{-1}\}.$$

Note that if $x \in Fg$, then $g_2 = g_0g_1 = x_1x_0^{-1}$; thus g is uniquely determined by any element of Fg. Hence $Fg \cap Fg' = 0$ for distinct members g,g' of G. Furthermore, $Fg \neq 0$ for any $g \in \cdot$; for example, $\langle e,g_2,g_1\rangle \in Fg$. Also, $\bigcup_{g\in\cdot} Fg = {}^3G$, for if $x \in {}^3G$ then $x \in F\langle x_1x_2^{-1}, x_2x_0^{-1}, x_1x_0^{-1}\rangle$. Therefore F maps $\cdot$ onto a partition of ${}^3G$, and F is one–one. Hence there is an isomorphism H of $\mathfrak{Bl}\mathfrak{Cm}\mathfrak{A}_{\mathfrak{G}f}$ into $\mathfrak{Bl}\mathfrak{Cb}\,{}^3G$ such that

$$HX = \bigcup_{g\in X} Fg$$

for each $X \in Bl\,\mathfrak{Cm}\mathfrak{A}_{\mathfrak{G}f}$. Now we show that H preserves c_0; the argument is similar for c_1 and c_2. Suppose first $x \in H\mathsf{c}_0X$ with $X \in Cm\mathfrak{A}_{\mathfrak{G}f}$. Say $x \in Fg$ with $g \in \mathsf{c}_0X$, and gT_0h for some $h \in X$. Thus $g_0 = x_1x_2^{-1}$, $g_1 = x_2x_0^{-1}$, and $g_0 = h_0$. Let $a = h_1^{-1}x_2$. Now $x_1x_2^{-1} = g_0 = h_0$, and $x_2(x_a^0)_0^{-1} = x_2a^{-1} = x_2x_2^{-1}h_1 = h_1$; hence $x_a^0 \in Fh$. Therefore $x_a^0 \in HX$ and so $x \in \mathsf{C}_0HX$. Conversely, suppose that $x \in \mathsf{C}_0HX$; say $x_a^0 \in HX$ and $x_a^0 \in Fh$ with $h \in X$. Then $h_0 = x_1x_2^{-1}$ and $h_1 = x_2a^{-1}$. Let $g = \langle h_0, h_1ax_0^{-1}, h_0h_1ax_0^{-1}\rangle$. Then $g \in \mathsf{c}_0X$, $g_0 = h_0 = x_1x_2^{-1}$, and $g_1 = h_1ax_0^{-1} = x_2a^{-1}ax_0^{-1} = x_2x_0^{-1}$. So $x \in Fg$. Thus $x \in H\mathsf{c}_0X$, as desired.

Finally, we check that H preserves d_{01}; other diagonal elements are treated similarly. We have $x \in H\mathsf{d}_{01}$ iff $x \in Fg$ for some $g \in \mathsf{d}_{01}$. Now $g \in \mathsf{d}_{01}$ iff $g_2 = e$, so $x \in H\mathsf{d}_{01}$ iff $x_1x_2^{-1} \cdot x_2x_0^{-1} = e$, which is true iff $x_0 = x_1$, i.e., iff $x \in \mathsf{D}_{01}$.

This finishes the proof.

CONSTRUCTION 3.2.74. We now specialize the general construction of 3.2.72 in a more complicated way in order to get certain CA_4's. Let $\mathfrak{G} = \langle G, +\rangle$ be an arbitrary Boolean group, i.e., an (abelian) group with every element of order 2. In the construction 3.2.72 we take $\alpha = 4$, $\beta = 6$, $U = G$. The sets $\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3$ are given as follows:

$$\Gamma_0 = \{0,1,2\},\ \Gamma_1 = \{0,3,4\},$$
$$\Gamma_2 = \{1,3,5\},\ \Gamma_3 = \{2,4,5\}.$$

We let

$$W = \{x \in {}^6G : x_0 + x_1 = x_2,\ x_0 + x_3 = x_4,\ x_1 + x_3 = x_5, \text{ and } x_2 + x_4 = x_5\}.$$

Pick any element f of W. This specifies everything needed to obtain the relational structure $\mathfrak{A}$ described in 3.2.72. It is easy to check that $\mathfrak{A}$ is a Ca_4 by checking 3.2.72(1)–(4). Actually $\mathfrak{Cm}\mathfrak{A}$ is representable, and this gives another way to prove that $\mathfrak{A}$ is a Ca_4. In fact, for any $X \in Cm\mathfrak{A}$ let

$$HX = \{u \in {}^4G : \text{there is an } x \in X \text{ such that } u_0 + u_1 = x_5 + f_5,\ u_0 + u_2 = x_4 + f_4,$$
$$u_0 + u_3 = x_3 + f_3,\ u_1 + u_2 = x_2 + f_2,\ u_1 + u_3 = x_1 + f_1, \text{ and } u_2 + u_3 = x_0 + f_0\}.$$

Then it is easy to check that H is an isomorphism from $\mathfrak{Cm}\mathfrak{A}$ onto a Cs_4 with base G.

In our construction of non–representable CA_α's so far, the non–representability was shown by finding a specific equation holding in all representable algebras but failing in the constructed one. Of course, for any non–representable CA_α there is some such equation, since the representable CA_α's form an algebraically closed class. In our remaining constructions we shall prove non–representability in a different way; for these constructions it is an interesting open problem to find such equations.

CONSTRUCTION 3.2.75 (*Projective geometries*) This construction has been carried out in detail in Monk [74'], generalizing Monk [65]. Since we will not use this construction later in this work, we content ourselves with describing the construction

and stating without proof its main properties.

A *projective geometry* is a pair $\mathfrak{G} = \langle P, L \rangle$ consisting of a non-empty set P (of "points") and a non-empty set L of subsets of P (called "lines"), subject to certain conditions; see, e.g., Seidenberg [62**]. From $\mathfrak{G}$ we construct a CA_3 $\mathfrak{R}_{\mathfrak{G}}$ as follows. $\mathfrak{B}(\mathfrak{R}_{\mathfrak{G}}$ is the Boolean algebra of all subsets of $W_{\mathfrak{G}}$, where $W_{\mathfrak{G}}$ consists of all pairs $\langle R, f \rangle$ satisfying the following conditions:

(1) R is an equivalence relation on 3;
(2) f maps $\{\langle i,j \rangle : i,j < 3,\ i \not R j\}$ into P;
(3) For all $i,j < 3$, if $i \not R j$ then $fij = fji$;
(4) For all $i,j,k < 3$, if $i \not R j R k$, then $fij = fik$;
(5) If $3 \upharpoonright Id = R$, then either $f01 = f02 = f12$, or else $f01$, $f02$, $f12$ are distinct collinear points.

The operations on $\mathfrak{R}_{\mathfrak{G}}$ are defined as follows. Suppose $i,j < 3$, $3 = \{i,k,l\}$, and $X \subseteq W_{\mathfrak{G}}$. We define

$$\mathsf{c}_i X = \{\langle R,f \rangle \in W_{\mathfrak{G}} : \text{there is an } \langle S,g \rangle \in X \text{ such that } R \cap {}^2(3 \sim \{i\}) = S \cap {}^2(3 \sim \{i\}) \text{ and if } k \not R l \text{ then } fkl = gkl\},$$
$$\mathsf{d}_{ij} = \{\langle R,f \rangle \in W_{\mathfrak{G}} : iRj\}.$$

The following is then routine to check (see Monk [74']):

(1) $\mathfrak{R}_{\mathfrak{G}}$ is a CA_3.

Every projective geometry has a *dimension*. If $\mathfrak{G}$ is n-dimensional and is a subspace of an $(n+1)$-dimensional space $\mathfrak{G}'$, we say that $\mathfrak{G}$ is a *hyperplane* in $\mathfrak{G}'$. The following two facts are established in Monk [74']:

(2) If $\mathfrak{G}$ is a hyperplane in some space $\mathfrak{G}'$, then $\mathfrak{R}_{\mathfrak{G}}$ is a representable CA_3.
(3) If $\mathfrak{G}$ is finite and $\mathfrak{R}_{\mathfrak{G}}$ is representable, then $\mathfrak{G}$ is a hyperplane in some space $\mathfrak{G}'$.

Condition (3) enables one to specify various non-representable CA_3's. Thus if $\mathfrak{G}$ is one-dimensional and is not a line in a projective plane, then $\mathfrak{R}_{\mathfrak{G}}$ is non-representable. There are infinitely many integers n such that there is no projective plane with a line of size n, so this gives many non-representable CA_3's. Similarly, a projective plane $\mathfrak{G}$ is a hyperplane in some three-dimensional space iff $\mathfrak{G}$ satisfies Desargue's condition. So a finite projective plane $\mathfrak{G}$ satisfies that condition iff $\mathfrak{R}_{\mathfrak{G}}$ is representable. Recall that it is an open problem whether there is a projective plane with exactly 11 points on every line. Thus if we apply our construction to $\mathfrak{G} = \langle 11, \{11\} \rangle$ we obtain a CA_3 $\mathfrak{R}_{\mathfrak{G}}$ whose representability is equivalent to this problem. Note that the construction of $\mathfrak{R}_{\mathfrak{G}}$ does not involve notions of projective geometry in this case.

CONSTRUCTION 3.2.76 (*Graphs*). We now give a construction, following Monk [69], which enables us to establish in section 4.1 one of the main model-theoretic results about cylindric algebras: for $3 \leq \alpha < \omega$, the class of representable CA_α's is not finitely axiomatizable. For variants on this construction, see also Demaree [70'],

Johnson [69], Monk [70], and Monk [74']. The construction is similar to that in 3.2.75.

Assume that $3 \leqq \alpha,\beta<\omega$, with $\alpha-1 \leqq \beta$. We shall define a cylindric atom structure $\mathfrak{G}'_{\alpha\beta} = \langle G'_{\alpha\beta}, T_\kappa, E_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha}$ of dimension α and the associated CA_α $\mathfrak{G}_{\alpha\beta}$. Then we shall prove some theorems about their representability, culminating in Theorem 3.2.85, giving non-representable CA_α's with a neat embedding property. First of all, $G'_{\alpha\beta}$ consists of all pairs $\langle R,f\rangle$ satisfying the following conditions:

(1) R is an equivalence relation on α;
(2) f maps $\{\langle\kappa,\lambda\rangle:\kappa,\lambda<\alpha,\ \kappa \not R \lambda\}$ into β;
(3) For all $\kappa,\lambda<\alpha$, if $\kappa \not R \lambda$ then $f\kappa\lambda = f\lambda\kappa$;
(4) For all $\kappa,\lambda,\mu<\alpha$, if $\kappa \not R \lambda R \mu$ then $f\kappa\lambda = f\kappa\mu$;
(5) For all $\kappa,\lambda,\mu<\alpha$, if $\kappa \not R \lambda \not R \mu \not R \kappa$ then $|\{f\kappa\lambda, f\kappa\mu, f\lambda\mu\}| \neq 1$.

Note that in graph-theoretic terminology, a member $\langle R,f\rangle$ of $G'_{\alpha\beta}$ amounts to an edge-coloring of α/R with β colors and with no monochromatic triangles. Thus, e.g., for certain α,β there is no f such that $\langle \alpha \upharpoonright Id, f\rangle \in G'_{\alpha\beta}$, by Ramsey's theorem. If there *is* no such f, then our algebra $\mathfrak{G}_{\alpha\beta}$ turns out to be of positive characteristic and hence is representable by 3.2.53.

Now for $\kappa<\alpha$ and $\langle R,f\rangle,\langle S,g\rangle \in G'_{\alpha\beta}$ we define

$$\langle R,f\rangle T_\kappa \langle S,g\rangle \text{ iff } R\mathsf{n}^2(\alpha\sim\{\kappa\}) = S\mathsf{n}^2(\alpha\sim\{\kappa\}) \text{ and for all } \lambda,\mu \in \alpha\sim\{\kappa\},$$
$$\text{if } \lambda \not R \mu \text{ then } f\lambda\mu = g\lambda\mu.$$

For any $\kappa,\lambda<\alpha$ we set

$$E_{\kappa\lambda} = \{\langle R,f\rangle \in G'_{\alpha\beta} : \kappa R \lambda\}.$$

This finishes the definition of $\mathfrak{G}'_{\alpha\beta}$. We set $\mathfrak{G}_{\alpha\beta} = \mathfrak{Cm}\mathfrak{G}'_{\alpha\beta}$. Now we show that $\mathfrak{G}'_{\alpha\beta}$ is a Ca_α (and hence $\mathfrak{G}_{\alpha\beta}$ is a CA_α), by checking the conditions (i)–(v) of 2.7.40. Of these conditions, (i), (iii) and (v) are obvious.

To prove (ii) we shall prove the following more general statement, which will also be useful later.

(6) Suppose $\langle R,f\rangle \in G'_{\alpha\beta}$. For any $\Omega \subseteq \alpha$ let $M\Omega = \{\langle S,g\rangle \in G'_{\alpha\beta} : S\mathsf{n}^2(\alpha\sim\Omega) = R\mathsf{n}^2(\alpha\sim\Omega)$, and for all $\lambda,\mu \in \alpha\sim\Omega$, if $\lambda \not S \mu$ then $f\lambda\mu = g\lambda\mu\}$. Now suppose $\Gamma \subseteq \alpha$ and $\kappa \in \alpha\sim\Gamma$. Then $T^*_\kappa(M\Gamma) = M(\Gamma \cup \{\kappa\})$.

Now (ii) follows from (6), since if $\kappa,\lambda<\alpha$ and $\kappa \neq \lambda$ then $T^*_\kappa T^*_\lambda\{\langle R,f\rangle\} = T^*_\kappa M\{\lambda\} = M\{\kappa,\lambda\} = T^*_\lambda M\{\kappa\} = T^*_\lambda T^*_\kappa\{\langle R,f\rangle\}$, and so, since $\langle R,f\rangle$ is arbitrary, $T_\kappa|T_\lambda = T_\lambda|T_\kappa$.

The inclusion $\subseteq$ in (6) is clear. Now suppose $\langle S,g\rangle \in M(\Gamma \cup \{\kappa\})$. We want to find $\langle T,k\rangle \in G'_{\alpha\beta}$ such that $\langle S,g\rangle T_\kappa \langle T,k\rangle \in M\Gamma$. We set

$$T = [S\mathsf{n}^2(\alpha\sim\{\kappa\})] \cup \{\langle\kappa,\kappa\rangle\} \cup \{\langle\kappa,\lambda\rangle : \text{for some } \mu \in \alpha\sim(\Gamma \cup \{\kappa\}) \text{ we have } \kappa R \mu S \lambda\}$$
$$\cup \{\langle\lambda,\kappa\rangle : \text{for some } \mu \in \alpha\sim(\Gamma \cup \{\kappa\}) \text{ we have } \kappa R \mu S \lambda\}.$$

It is easily checked that T is an equivalence relation on α. Now suppose $\nu,\rho<\alpha$ and $\nu T \rho$. If $\nu,\rho \neq \kappa$ we let $k\nu\rho = g\nu\rho$. Suppose $\nu = \kappa$. If $\kappa T \sigma \neq \kappa$ for some σ we let

$k\nu\rho = k\rho\nu = g\sigma\rho$ (clearly this does not depend on the choice of σ). If there is no such σ, but $\rho S\xi \in \alpha\sim(\Gamma \cup \{\kappa\})$ for some ξ, we let $k\nu\rho = k\rho\nu = f\nu\xi$ (clearly this does not depend on the choice of ξ). Still if there is no such σ, let $\Omega = \{\eta : \nu \not\Gamma \eta$, and there is no ξ with $\eta S\xi \in \alpha\sim(\Gamma \cup \{\kappa\})\}$ and $\Theta = \{\eta : \nu \not\Gamma \eta\}\sim\Omega$. Choose $\langle k\nu\eta : \eta \in \Omega\rangle$ so that for all $\eta,\eta' \in \Omega$, $k\nu\eta = k\nu\eta'$ iff $\eta S\eta'$, and so that if $\eta \in \Omega$ and $\eta' \in \Theta$ then $k\nu\eta \neq k\nu\eta'$. This is possible since $\alpha - 1 \leqq \beta$. Finally, for any $\eta \in \Omega$ let $k\eta\nu = k\nu\eta$. This finishes the definition of k. It is routine to check the desired properties of $\langle T,k\rangle$. Thus (6) holds and 2.7.40(ii) is established.

Next we check (iv). The inclusion $\supseteq$ is clear. Now suppose $\langle R,f\rangle \in E_{\kappa\lambda}$. Let

$$S = [R \cap {}^2(\alpha\sim\{\mu\})] \cup \{\langle\mu,\nu\rangle : \kappa R\nu\} \cup \{\langle\nu,\mu\rangle : \kappa R\nu\} \cup \{\langle\mu,\mu\rangle\}.$$

It is easily verified that S is an equivalence relation on α. Now suppose $\nu \not S \rho$ with $\nu,\rho < \alpha$. If $\nu,\rho \neq \mu$, let $h\nu\rho = f\nu\rho$. If $\nu = \mu$ let $h\nu\rho = h\rho\nu = f\kappa\rho$. It is easily checked that $\langle S,h\rangle \in G'_{\alpha\beta}$ and $\langle R,f\rangle T_\mu \langle S,h\rangle \in E_{\kappa\mu} \cap E_{\mu\lambda}$, as desired.

This verifies all the conditions of 2.7.40, and hence $\mathfrak{G}'_{\alpha\beta}$ is a Ca_α and $\mathfrak{G}_{\alpha\beta}$ is a CA_α.

Condition (6) implies that $\mathfrak{G}_{\alpha\beta}$ is simple. In fact, given any $\langle R,f\rangle \in G'_{\alpha\beta}$, by induction (6) yields that

(7) $\mathsf{c}_{(\Gamma)}\{\langle R,f\rangle\} = M\Gamma$ for every $\Gamma \subseteq \alpha$.

Hence $\mathsf{c}_{(\alpha)}\{\langle R,f\rangle\} = M\alpha$; but clearly $M\alpha = G'_{\alpha\beta}$; simplicity of $\mathfrak{G}_{\alpha\beta}$ follows by 2.3.14.

THEOREM 3.2.77. *Suppose* $3 \leqq \alpha \leqq \beta < \omega$, *and* $\beta - 1 \leqq \gamma < \omega$. *Then* $\mathfrak{G}_{\alpha\gamma}$ *is neatly embeddable in* $\mathfrak{G}_{\beta\gamma}$.

PROOF. For each $X \in G_{\alpha\gamma}$ let

$$FX = \{\langle R,f\rangle \in G_{\beta\gamma} : \text{there is an } \langle S,g\rangle \in X \text{ such that } S = R \cap {}^2\alpha \text{ and for all } \kappa,\lambda < \alpha, \text{ if } \kappa \not R \lambda \text{ then } f\kappa\lambda = g\kappa\lambda\}.$$

Clearly F is an isomorphism from $\mathfrak{B}\mathfrak{A}\mathfrak{G}_{\alpha\gamma}$ into $\mathfrak{B}\mathfrak{A}\mathfrak{G}_{\beta\gamma}$, F preserves $\mathsf{d}_{\kappa\lambda}$ for all $\kappa,\lambda < \alpha$, and $\mathsf{c}_\kappa FX = FX$ for all $X \in G_{\alpha\gamma}$ and all $\kappa \in \beta\sim\alpha$. It remains to show that F preserves c_κ for $\kappa < \alpha$. So, suppose $X \in G_{\alpha\gamma}$.

First suppose $\langle T,h\rangle \in \mathsf{c}_\kappa FX$; say $\langle T,h\rangle T_\kappa \langle S,g\rangle$ and $\langle S,g\rangle \in F\{\langle R,f\rangle\}$ with $\langle R,f\rangle \in X$. Let $U = T \cap {}^2\alpha$ and for all $\mu,\nu < \alpha$ with $\mu \not T \nu$ let $k\mu\nu = h\mu\nu$. Then $\langle T,h\rangle \in F\{\langle U,k\rangle\}$, and $\langle U,k\rangle T_\kappa \langle R,f\rangle$ (as is easily checked), so $\langle T,h\rangle \in F\mathsf{c}_\kappa X$.

Second, suppose $\langle T,h\rangle \in F\mathsf{c}_\kappa X$. Say $\langle T,h\rangle \in F\{\langle S,g\rangle\}$ and $\langle S,g\rangle T_\kappa \langle R,f\rangle \in X$. Let

$$U = [T \cap {}^2(\beta\sim\{\kappa\})] \cup \{\langle\kappa,\kappa\rangle\} \cup \{\langle\kappa,\lambda\rangle : \text{for some } \mu \in \alpha\sim\{\kappa\},\ \kappa R\mu T\lambda\} \cup \{\langle\lambda,\kappa\rangle : \text{for some } \mu \in \alpha\sim\{\kappa\},\ \kappa R\mu T\lambda\}.$$

It is easily checked that U is an equivalence relation on β. Now suppose $\nu \not U \rho$, with $\nu,\rho < \beta$. If $\nu,\rho \neq \kappa$, let $k\nu\rho = h\nu\rho$. Suppose $\nu = \kappa$. If $\kappa R\mu \neq \kappa$ for some μ, let $k\nu\rho = h\mu\rho$. Suppose there is no such μ. If $\rho U\xi$ for some $\xi < \alpha$, let $k\nu\rho = f\nu\xi$; this does not depend on the choice of ξ. Let $\Omega = \{\eta \in \beta\sim\alpha : \eta \not U \xi$ for all $\xi < \alpha\}$. Choose $\langle k\nu\eta : \eta \in \Omega\rangle$ so that $k\nu\eta \neq k\nu\eta'$ for all $\eta \in \Omega$ and $\eta' < \alpha$ and so that if $\eta,\eta' \in \Omega$, then $k\nu\eta = k\nu\eta'$ iff $\eta U\eta'$. Then

it is routine to check that $\langle U,k\rangle \in G'_{\beta\gamma}$ and $\langle T,h\rangle T_\kappa \langle U,k\rangle \in F\{\langle R,f\rangle\}$. This finishes the proof of 3.2.77.

Now we shall modify the algebras $\mathfrak{G}_{\alpha\beta}$ in order to obtain stronger results below.

CONSTRUCTION 3.2.78. Suppose $3 \leqq \alpha,\beta < \omega$ and $\beta \geqq \alpha - 1$. We set

$$K\alpha\beta = \{\langle R,f\rangle \in G'_{\alpha\beta} : R \mathsf{n}^2(\alpha\sim 3) = Id\,\mathsf{n}^2(\alpha\sim 3), \text{ and for all } \kappa,\lambda<\alpha, \text{ if } 3 \leqq \kappa < \lambda < \alpha \text{ then } f\kappa\lambda = \kappa\}.$$

Then we set $\mathfrak{P}_{\alpha\beta} = \mathfrak{Rl}_a \mathfrak{Rb}_3 \mathfrak{G}_{\alpha\beta}$, where $a = K\alpha\beta$. We claim that $\mathfrak{P}_{\alpha\beta}$ is a CA_3. In fact, clearly $\mathsf{c}_\kappa K\alpha\beta = K\alpha\beta$ for all $\kappa<3$, so $\mathfrak{P}_{\alpha\beta}$ is a homomorphic image of $\mathfrak{Rb}_3\mathfrak{G}_{\alpha\beta}$ and hence is a CA_3. Moreover, $\mathfrak{P}_{\alpha\beta}$ is simple. For, if $\langle R,f\rangle \in K\alpha\beta$ then

$$\begin{aligned} \mathsf{c}_{(3)}^{\mathfrak{P}_{\alpha\beta}}\{\langle R,f\rangle\} &= \mathsf{c}_{(3)}^{\mathfrak{G}_{\alpha\beta}}\{\langle R,f\rangle\} \mathsf{n}\, K\alpha\beta \\ &= M3 \mathsf{n}\, K\alpha\beta \text{ by } 3.2.76(7) \\ &= K\alpha\beta, \end{aligned}$$

and simplicity follows by 2.3.14.

COROLLARY 3.2.79. *If $3 \leqq \alpha < \omega$ and $\kappa < \omega$, then $\mathfrak{Rb}_3\mathfrak{G}_{\alpha,\alpha+\kappa}$ can be neatly embedded in a $\mathsf{CA}_{3+\kappa+1}$.*

PROOF. By 3.2.77 there is a neat embedding F of $\mathfrak{G}_{\alpha,\alpha+\kappa}$ into $\mathfrak{G}_{\alpha+\kappa+1,\alpha+\kappa}$. Let ρ be the function mapping $3+\kappa+1$ into $\alpha+\kappa+1$ defined by the following conditions: $\rho\lambda = \lambda$ for all $\lambda<3$; $\rho\lambda = \alpha+\lambda-3$ for all $\lambda \in (3+\kappa+1)\sim 3$. Let $\mathfrak{B} = \mathfrak{Rb}^\rho \mathfrak{G}_{\alpha+\kappa+1,\alpha+\kappa}$. Thus $\mathfrak{B}$ is a $\mathsf{CA}_{3+\kappa+1}$. It is easily verified that F is a neat embedding of $\mathfrak{Rb}_3\mathfrak{G}_{\alpha,\alpha+\kappa}$ into $\mathfrak{B}$, as desired.

LEMMA. 3.2.80. *Suppose $3 \leqq \alpha,\beta < \omega$ and $\beta \geqq \alpha - 1$. Suppose F is an isomorphism of $\mathfrak{P}_{\alpha\beta}$ onto a Cs_3 with base U. For each $\gamma<\beta$ let $N_\gamma = \{\langle R,f\rangle \in K\alpha\beta : 0\not R1\, R2$ and $f01 = \gamma\}$. Then*

(i) *For all $u,v \in U$ with $u \neq v$ all $\gamma<\beta$, and all $\langle R,f\rangle \in N_\gamma$, if $\langle u,v,v\rangle \in F\{\langle R,f\rangle\}$ then $\langle v,u,u\rangle \in F\{\langle S,g\rangle\}$ for some $\langle S,g\rangle \in N_\gamma$.*

(ii) *For all $u,v \in U$ with $u \neq v$ there is a unique $\gamma<\beta$ such that $\langle u,v,v\rangle \in F\{\langle R,f\rangle\}$ for some $\langle R,f\rangle \in N_\gamma$.*

Furthermore, for any $u,v \in U$ with $u \neq v$ let $k\{u,v\}$ be the γ given by (ii) *and* (i). *Then k is an edge coloring of the complete graph on U with β colors, without a monochromatic triangle, i.e., there do not exist distinct $u,v,w \in U$ such that $k\{u,v\} = k\{u,w\} = k\{v,w\}$.*

PROOF. (i): Clearly $\langle v,u,u\rangle \in F(\mathsf{d}_{12}\cdot\mathsf{c}_1(\mathsf{d}_{01}\cdot\mathsf{c}_0(\mathsf{d}_{02}\cdot\mathsf{c}_2\{\langle R,f\rangle\})))$; say $\langle v,u,u\rangle \in F\{\langle S,g\rangle\}$ with $\langle S,g\rangle \in \mathsf{d}_{12}\cdot\mathsf{c}_1(\mathsf{d}_{01}\cdot\mathsf{c}_0(\mathsf{d}_{02}\cdot\mathsf{c}_2\{\langle R,f\rangle\}))$. Clearly $\langle S,g\rangle \in N_\gamma$, so (i) holds. (ii): Clearly in $\mathfrak{P}_{\alpha\beta}$ we have $-\mathsf{d}_{01}\cdot\mathsf{d}_{12} = \bigcup_{\gamma<\beta} N_\gamma$, so (ii) holds.

Now for the final conclusion, suppose $u,v,w \in U$ are distinct and $k\{u,v\} = k\{u,w\} = k\{v,w\} = \gamma$. Say $\langle u,v,v\rangle \in F\{\langle R,f\rangle\}$, $\langle u,w,w\rangle \in F\{\langle S,g\rangle\}$, $\langle v,w,w\rangle \in F\{\langle T,h\rangle\}$, with $\langle R,f\rangle, \langle S,g\rangle, \langle T,h\rangle \in N_\gamma$. Then $\bar{\mathsf{d}}(3\times 3)\cdot\mathsf{c}_2\{\langle R,f\rangle\}\cdot\mathsf{c}_1\{\langle S,g\rangle\}\cdot\mathsf{c}_0(\mathsf{d}_{01}\cdot\mathsf{c}_1\{\langle T,h\rangle\}) = 0$ by the "triangle" part in the definition of $G'_{\alpha\beta}$, but

$$\langle u,v,w\rangle \in \bar{\mathsf{d}}(3\times 3)\cap \mathsf{C}_2F\{\langle R,f\rangle\}\cap \mathsf{C}_1F\{\langle S,g\rangle\}\cap \mathsf{C}_0(\mathsf{D}_{01}\cap \mathsf{C}_1F\{\langle T,h\rangle\}),$$

a contradiction.

REMARK 3.2.81. We shall now make use of the following sharpened form of part of Ramsey's theorem, due to Greenwood, Gleason [55**]:

(1) If G is a finite set, and the complete graph on G is edge colored with β colors, $3\leqq\beta<\omega$, with no monochromatic triangles, then $|G|\leqq 3\cdot\beta!$

This result gives the following corollary to 3.2.80.

COROLLARY 3.2.82. *Suppose* $3\leqq\alpha,\beta<\omega$ *and* $\beta\geqq\alpha-1$. *Suppose* $\Phi_{\alpha\beta}$ *is isomorphic to a* Cs_3 *with base* U. *Then* $|U|\leqq 3\cdot\beta!$

Now we shall obtain a lower bound for $|U|$:

LEMMA 3.2.83. *Suppose* $3\leqq\alpha,\beta<\omega$ *and* $\beta\geqq\alpha-1$. *Suppose* $\Phi_{\alpha\beta}$ *is isomorphic to a* Cs_3 *with base* U. *Then* $|U|\geqq(\beta-2)^{\alpha-3}$.

PROOF. Let F be an isomorphism from $\Phi_{\alpha\beta}$ onto a Cs_3 with base U. We shall use the notation of 3.2.80. First we shall obtain a lower bound for $|k^{-1}*\{\gamma\}|$ for an arbitrary $\gamma<\beta$. Let R be the equivalence relation on α associated with the partition whose only non-singleton class is $\{1,2\}$. Then we set

$$M=\{\langle R,f\rangle\in K\alpha\beta: f01=\gamma \text{ and for all } \kappa\in\alpha\sim 3,\ \{f0\kappa,f1\kappa\}\cap\{\kappa,\gamma\}=0\}.$$

Note that if g is the function from $\{\langle\kappa,\lambda\rangle:\kappa,\lambda\in\alpha\sim 3,\ \kappa\neq\lambda\}$ into β such that $g\kappa\lambda=g\lambda\kappa=\kappa$ whenever $3\leqq\kappa<\lambda<\alpha$, then for any $\mu,\nu\in\mathsf{P}_{\kappa\in\alpha\sim 3}(\beta\sim\{\kappa,\gamma\})$ there is a unique $f\supseteq g$ such that $\langle R,f\rangle\in M$ and $f0\kappa=\mu\kappa$ while $f1\kappa=\nu\kappa$ for all $\kappa\in\alpha\sim 3$. Thus

(1) $|M|\geqq(\beta-2)^{2\alpha-6}$.

Now for any $\langle R,f\rangle\in M$ let $G\langle R,f\rangle=\{\{u,v\}: u,v\in U,\ u\neq v,$ and $\langle u,v,v\rangle\in F\{\langle R,f\rangle\}$ or $\langle v,u,u\rangle\in F\{\langle R,f\rangle\}\}$. Since $F\{\langle R,f\rangle\}\neq 0$, clearly $G\langle R,f\rangle\neq 0$. Next, we claim that if $G\langle R,f\rangle\cap G\langle R,g\rangle\neq 0$ then $G\langle R,f\rangle=G\langle R,g\rangle$. To show this we may assume that $f\neq g$. Since $F\{\langle R,f\rangle\}\cap F\{\langle R,g\rangle\}=0$, this means that there are distinct $u,v\in U$ such that $\langle u,v,v\rangle\in F\{\langle R,f\rangle\}$ and $\langle v,u,u\rangle\in F\{\langle R,g\rangle\}$. To show that $G\langle R,f\rangle=G\langle R,g\rangle$ it suffices to show that for all $s,t\in U$, $\langle s,t,t\rangle\in F\{\langle R,f\rangle\}$ iff $\langle t,s,s\rangle\in F\{\langle R,g\rangle\}$. Now it is easy to check the following:

$$\mathsf{d}_{12}\cdot\mathsf{c}_2(\mathsf{d}_{02}\cdot\mathsf{c}_0(\mathsf{d}_{01}\cdot\mathsf{c}_1\{\langle R,g\rangle\}))\text{ is an atom;}$$
$$\langle u,v,v\rangle\in F\{\langle R,f\rangle\}\cap F[\mathsf{d}_{12}\cdot\mathsf{c}_2(\mathsf{d}_{02}\cdot\mathsf{c}_0(\mathsf{d}_{01}\cdot\mathsf{c}_1\{\langle R,g\rangle\}))].$$

Hence $\{\langle R,f\rangle\}=\mathsf{d}_{12}\cdot\mathsf{c}_2(\mathsf{d}_{02}\cdot\mathsf{c}_0(\mathsf{d}_{01}\cdot\mathsf{c}_1\{\langle R,g\rangle\}))$, from which the above easily follows.

Clearly also for any $\langle R,f\rangle\in M$ there are at most two $\langle S,g\rangle\in M$ such that $G\langle R,f\rangle=G\langle S,g\rangle$. All of these facts show that $|k^{-1}*\{\gamma\}|\geqq|M|/2$. Hence, using (1),

$$|\{X\subseteq U: |X|=2\}|=\Sigma_{\gamma<\beta}|k^{-1}*\{\gamma\}|\geqq(\beta/2)\cdot(\beta-2)^{2\alpha-6}\geqq(\beta-2)^{2\alpha-5}.$$

Hence $|U| \geqq (\beta-2)^{\alpha-3}$, as desired.

Note that 3.2.82 and 3.2.83 imply that certain of the algebras $\mathfrak{H}_{\alpha\beta}$ are non-representable; by 3.1.108 and 3.1.126, the same applies to $\mathfrak{Rd}_3\mathfrak{G}_{\alpha\beta}$ and to $\mathfrak{G}_{\alpha\beta}$ itself. For example, a simple computation shows that $(11-2)^{12-3} > 3\cdot 11!$, so $\mathfrak{H}_{12,11}$ is non-representable. Similarly, $\mathfrak{H}_{13,13}$ and $\mathfrak{H}_{20,22}$ are non-representable. Thus using 3.2.79 and 3.2.77, $\mathfrak{Rd}_3\mathfrak{G}_{13,13}$ is a non-representable CA_3 neatly embeddable in a CA_4, and $\mathfrak{G}_{20,22}$ is a non-representable CA_{20} neatly embeddable in a CA_{23}. Now we generalize these considerations.

COROLLARY 3.2.84. *For every $\kappa \in \omega$ there is an $\alpha \in \omega \sim 3$ such that $\mathfrak{Rd}_3\mathfrak{G}_{\alpha,\alpha+\kappa}$ is a non-representable CA_3.*

PROOF. By 3.1.108 it suffices to find an α such that $\mathfrak{H}_{\alpha,\alpha+\kappa}$ is non-representable. Let $\alpha = 10\cdot(\kappa+4)!$ Suppose $\mathfrak{H}_{\alpha,\alpha+\kappa}$ is representable, say it is isomorphic to a Cs_3 with base U. Note the following:

$$\begin{aligned} &3\cdot(\kappa+4)! < \alpha+\kappa-2, \\ &(\kappa+5)\cdot(\kappa+\alpha) < (\alpha+\kappa-2)^2, \\ &(\kappa+6)\cdot(\kappa+\alpha-1) < (\alpha+\kappa-2)^2. \end{aligned}$$

Hence by 3.2.82 and 3.2.83,

$$|U| \leqq 3\cdot(\alpha+\kappa)! = 3\cdot(\kappa+4)!\cdot(\kappa+5)\cdot(\kappa+6)\cdot\ldots\cdot(\kappa+\alpha) < (\alpha+\kappa-2)^{\alpha-3} \leqq |U|,$$

contradiction.

This corollary gives one of the main results about non-representable CA_α's:

THEOREM 3.2.85. *If $3 \leqq \alpha < \omega$ and $\kappa < \omega$, then there is a non-representable CA_α which can be neatly embedded in a $\mathsf{CA}_{\alpha+\kappa}$.*

PROOF. By 3.2.84, choose $\beta \in \omega \sim 3$ so that $\mathfrak{Rd}_3\mathfrak{G}_{\beta,\beta+\alpha+\kappa-4}$ is a non-representable CA_3. By 3.2.79, say $\mathfrak{Rd}_3\mathfrak{G}_{\beta,\beta+\alpha+\kappa-4} \subseteq \mathfrak{Nr}_3\mathfrak{B}$, $\mathfrak{B}$ a $\mathsf{CA}_{\alpha+\kappa}$. Let $\mathfrak{A} = \mathfrak{Nr}_\alpha\mathfrak{B}$. Clearly $\mathfrak{A}$ is as desired.

Our final task concerning these constructions is to extend 3.2.85 to the case $\alpha \geqq \omega$; first we need a lemma.

LEMMA 3.2.86. *Suppose $3 \leqq \alpha \leqq \beta < \omega$ and $\kappa < \omega$. Then $\mathfrak{G}_{\alpha,\alpha+\kappa} = \mathfrak{Rl}_x\mathfrak{Rd}_\alpha\mathfrak{G}_{\beta,\beta+\kappa}$ for some $x \in G_{\beta,\beta+\kappa}$ such that for all distinct $\xi,\eta < \alpha$, $\mathsf{c}_\xi x \cdot \mathsf{c}_\eta x = x \leqq \mathsf{c}_\xi(\mathsf{d}_{\xi\eta}\cdot x)$.*

PROOF. Let

$$\begin{aligned} x = \{\langle R,f\rangle \in G'_{\beta,\beta+\kappa} : {} & R = (R\mathsf{n}^2\alpha)\mathsf{u}(Id\mathsf{n}^2(\beta\sim\alpha)), \text{ for all } \mu,\nu<\alpha \text{ with } \mu \not\mathrel{R} \nu \\ & \text{we have } f\mu\nu \in \alpha+\kappa, \text{ and for all } \mu \in \beta\sim\alpha \text{ and } \nu<\mu \text{ we have } f\mu\nu = \mu+\kappa\}. \end{aligned}$$

First we check that $\mathsf{c}_\xi x \mathbf{n} \mathsf{c}_\eta x \subseteq x$, where $\xi,\eta<\alpha$, $\xi \neq \eta$. Suppose $\langle R,f\rangle \in \mathsf{c}_\xi x \mathbf{n} \mathsf{c}_\eta x$, say $\langle R,f\rangle T_\xi \langle S,g\rangle \in x$ and $\langle R,f\rangle T_\eta \langle T,h\rangle \in x$. Clearly $R=(R\mathbf{n}^2\alpha)\mathbf{u}(Id\mathbf{n}^2(\beta\sim\alpha))$. Suppose $\mu\in\beta\sim\alpha$ and $\nu<\mu$. If $\nu\neq\xi$, then $f\mu\nu=g\mu\nu=\mu+\kappa$. If $\nu=\xi$, then $f\mu\nu=h\mu\nu=\mu+\kappa$. To finish showing that $\langle R,f\rangle\in x$ it clearly suffices to assume that $\xi \not\mathbf{R} \eta$ and show that $f\xi\eta\in\alpha+\kappa$. Suppose this is not true; then $f\xi\eta=\mu+\kappa$ for some $\mu\in\beta\sim\alpha$. Then $f\xi\eta=f\xi\mu=f\eta\mu$, contradicting the fact that $\langle R,f\rangle\in G'_{\beta,\beta+\kappa}$.

Thus $\mathsf{c}_\xi x\cdot\mathsf{c}_\eta x=x$. It is clear that $x\leqq\mathsf{c}_\xi(\mathsf{d}_{\xi\eta}\cdot x)$.

Now we define a function F mapping $G'_{\alpha,\alpha+\kappa}$ into x. For any $\langle S,g\rangle\in G'_{\alpha,\alpha+\kappa}$, let $F\langle S,g\rangle=\langle R,f\rangle$, where $R=S\mathbf{u}(Id\mathbf{n}^2(\beta\sim\alpha))$, $g\subseteq f$, and for all $\mu\in\beta\sim\alpha$ and all $\nu<\mu$ we have $f\mu\nu=f\nu\mu=\mu+\kappa$. Clearly $\langle R,f\rangle\in x$. In fact, it is clear that F is a one–one mapping of $G'_{\alpha,\alpha+\kappa}$ onto x. Let $HX=F^*X$ for any $X\in G_{\alpha,\alpha+\kappa}$. It is easily checked that H is the desired isomorphism of $\mathfrak{G}_{\alpha,\alpha+\kappa}$ onto $\mathfrak{Rl}_x\mathfrak{Rb}_\alpha\mathfrak{G}_{\beta,\beta+\kappa}$.

THEOREM 3.2.87. *If $3\leqq\alpha$ and $\kappa<\omega$, then there is a non–representable* CA_α *which can be neatly embedded in a* $\mathsf{CA}_{\alpha+\kappa}$.

PROOF. Since 3.2.85 disposes of the case $\alpha<\omega$, we assume $\alpha\geqq\omega$. By 3.2.84 choose $\beta\in\omega\sim3$ so that $\mathfrak{Rb}_3\mathfrak{G}_{\beta,\beta+\kappa}$ is non–representable. Thus by 3.1.118 $\mathfrak{G}_{\beta,\beta+\kappa}$ is also non–representable. Now let

$$I=\{\Gamma:\beta\subseteq\Gamma\subseteq\alpha \text{ and } |\Gamma|<\omega\}.$$

For each $\Gamma\in I$ let $|\Gamma|=\gamma_\Gamma$ and let ρ_Γ be a one–one function from γ_Γ onto Γ such that $\beta\restriction Id\subseteq\rho_\Gamma$. Let $\mathfrak{A}_\Gamma$ be an algebra similar to CA_α's such that $\mathfrak{Rb}^{\rho^\Gamma}\mathfrak{A}_\Gamma=\mathfrak{G}_{\gamma_\Gamma,\gamma_\Gamma+\kappa}$, extending Definition 2.6.1 in the natural way. For each $\Gamma\in I$ let $M_\Gamma=\{\Delta\in I:\Gamma\subseteq\Delta\}$, and let F be an ultrafilter on I such that $M_\Gamma\in F$ for every $\Gamma\in I$. Set $\mathfrak{B}=\mathsf{P}_{\Gamma\in I}\mathfrak{A}_\Gamma/\bar{F}$. We claim that $\mathfrak{B}$ is the CA_α desired in the theorem.

It is straightforward to check that $\mathfrak{B}$ is a CA_α. To show that $\mathfrak{B}$ is neatly embeddable in a $\mathsf{CA}_{\alpha+\kappa}$, first by 3.2.77 for each $\Gamma\in I$ choose a $\mathsf{CA}_{\gamma_\Gamma+\kappa}$ $\mathfrak{D}_\Gamma$ such that $\mathfrak{G}_{\gamma_\Gamma,\gamma_\Gamma+\kappa}\subseteq\mathfrak{Rr}_{\gamma_\Gamma}\mathfrak{D}_\Gamma$. Let σ_Γ be a one–one function with domain $\gamma_\Gamma+\kappa$ such that $\rho_\Gamma\subseteq\sigma_\Gamma$ and $\sigma_\Gamma(\gamma_\Gamma+\xi)=\alpha+\xi$ for every $\xi<\kappa$. Let $\mathfrak{A}'_\Gamma$ be an algebra similar to $\mathsf{CA}_{\alpha+\kappa}$'s such that $\mathfrak{Rb}^{\sigma_\Gamma}\mathfrak{A}'_\Gamma=\mathfrak{D}_\Gamma$. Then it is easily seen that $\mathsf{P}_{\Gamma\in I}\mathfrak{A}'_\Gamma/\bar{F}$ is a $\mathsf{CA}_{\alpha+\kappa}$ in which $\mathfrak{B}$ is neatly embedded.

It remains only to show that $\mathfrak{B}$ is non–representable. By 3.1.118 it suffices to show that $\mathfrak{Rb}_\beta\mathfrak{B}$ is non–representable. Now for each $\Gamma\in I$ choose $a_\Gamma\in A_\Gamma$ such that $\mathfrak{G}_{\beta,\beta+\kappa}\cong\mathfrak{Rl}_{a_\Gamma}\mathfrak{Rb}_\beta\mathfrak{G}_{\gamma_\Gamma,\gamma_\Gamma+\kappa}$, using 3.2.86, with $\mathsf{c}_\xi a_\Gamma\cdot\mathsf{c}_\eta a_\Gamma=a_\Gamma\leqq\mathsf{c}_\xi(\mathsf{d}_{\xi\eta}\cdot a_\Gamma)$ for all distinct $\xi,\eta<\beta$, and let F_Γ be the indicated isomorphism. Let $x=a/\bar{F}$ and let $Hb=\langle F_\Gamma b:\Gamma\in I\rangle/\bar{F}$ for every $b\in G_{\beta,\beta+\kappa}$. It is easy to see that H is an isomorphism from $\mathfrak{G}_{\beta,\beta+\kappa}$ into $\mathfrak{Rl}_x\mathfrak{Rb}_\beta\mathfrak{B}$. Since $\mathsf{c}_\xi x\cdot\mathsf{c}_\eta x=x\leqq\mathsf{c}_\xi(\mathsf{d}_{\xi\eta}\cdot x)$ for all distinct $\xi,\eta<\beta$, by 2.2.10 and 2.6.38 we infer that $\mathfrak{Rb}_\beta\mathfrak{B}$ is non–representable. This completes the proof of 3.2.87.

Note the following interesting consequence of 3.2.10 and 3.2.87: if $3\leqq\alpha$ and $\kappa<\omega$, then for some $\lambda<\omega$ there is a CA_α $\mathfrak{A}$ such that $\mathfrak{A}$ can be neatly embedded in some $\mathsf{CA}_{\alpha+\kappa}$ but not in a $\mathsf{CA}_{\alpha+\kappa+\lambda}$.

The last three constructions of non–representable algebras given above illustrate an important connection between combinatorial structures and algebraic logic. This connection was first developed, for relation algebras, by Jónsson and Lyndon. For recent work in this area see, e.g., Comer [76'] and [83b'].

Other representations

We close this section with some comments on other possible ways of representing cylindric algebras. See also the concluding pages of Part I, where representation by complex algebras is discussed, and section 4.3, where representation in terms of first–order logic is considered.

REMARKS 3.2.88. The class ICrs_α of cylindric–relativized set algebras has entered into our discussion so far in an auxiliary role, since its members are not in general cylindric algebras. Still, we may ask about representability of CA_α's as cylindric–relativized set algebras. Recall from Theorem 3.2.61 that every CA_2 *is* so representable. This result does not extend to any $\alpha>2$. For example, the algebras in Construction 3.2.71 are not isomorphic to cylindric–relativized set algebras, since the equations (7)–(9) there, which fail to hold in those algebras, are easily seen to hold in all cylindric–relativized set algebras which are cylindric algebras. Algebras in Construction 3.2.72 (from quasigroups) can also be used for this purpose, for $\alpha=3$. The class $\mathsf{CA}_\alpha\mathsf{n}\mathsf{Crs}_\alpha$ may be considered to be a class of concrete cylindric algebras, and we can ask about properties of $\mathsf{CA}_\alpha\mathsf{n}\mathsf{ICrs}_\alpha$ – a natural class of abstract cylindric algebras. In this connection we mention a result of D. Resek [75'], which adds considerable interest to this class.

Suppose that α is an ordinal, $n\in\omega\sim 2$, $\kappa\in{}^n\alpha$ is one–one, and $\lambda\in\alpha\sim Rg\kappa$. The *$\kappa,\lambda$–merry–go–round identity* is the equation

(1) $\mathsf{s}^\lambda_{\kappa 0}\mathsf{s}^{\kappa 0}_{\kappa 1}\cdots\mathsf{s}^{\kappa(n-2)}_{\kappa(n-1)}\mathsf{s}^{\kappa(n-1)}_\lambda\mathsf{c}_\lambda x=\mathsf{s}^\lambda_{\kappa(n-1)}\mathsf{s}^{\kappa(n-1)}_{\kappa 0}\mathsf{s}^{\kappa 0}_{\kappa 1}\cdots\mathsf{s}^{\kappa(n-3)}_{\kappa(n-2)}\mathsf{s}^{\kappa(n-2)}_\lambda\mathsf{c}_\lambda x.$

Then the result from Resek [75'] states that $\mathsf{ICrs}_\alpha\mathsf{n}\mathsf{CA}_\alpha$ is the variety characterized by the cylindric algebra axioms along with all possible merry–go–round identities. Unfortunately, the proof is too long to be included here.

More recently, Richard Thompson has shown that the merry–go–round identities can be replaced in Resek's theorem by the following two instances:

(2) $\mathsf{s}^\lambda_\kappa\mathsf{s}^\kappa_\mu\mathsf{s}^\mu_\lambda\mathsf{c}_\lambda x=\mathsf{s}^\lambda_\mu\mathsf{s}^\mu_\kappa\mathsf{s}^\kappa_\lambda\mathsf{c}_\lambda$ for all distinct $\kappa,\lambda,\mu<\alpha$,

(3) $\mathsf{s}^\lambda_\kappa\mathsf{s}^\kappa_\mu\mathsf{s}^\mu_\nu\mathsf{s}^\nu_\lambda\mathsf{c}_\lambda x=\mathsf{s}^\lambda_\nu\mathsf{s}^\nu_\kappa\mathsf{s}^\kappa_\mu\mathsf{s}^\mu_\lambda\mathsf{c}_\lambda x$ for distinct $\kappa,\lambda,\mu,\nu<\alpha$.

Note that thus $\mathsf{ICrs}_\alpha\mathsf{n}\mathsf{CA}_\alpha$ is finite schema axiomatizable in the sense of section 4.1.

REMARKS 3.2.89. Another kind of representation for cylindric algebras which has appeared in the literature involves sheaf–theoretic notions first applied in ordinary algebra in Pierce [67**]. The notion was applied to cylindric algebras by Comer [72']. A *sheaf of* CA_α's is a quadruple $\langle X,S,\pi,\mathfrak{A}\rangle$ with the following properties:

(1) X and S are topological spaces and π is a local homeomorphism from S onto X.

(2) $\mathfrak{A} = \langle \mathfrak{A}_x : x \in X \rangle$ is a system of CA_α's, and $A_x = \pi^{-1*}\{x\}$ for each $x \in X$.
(3) The operations on the algebras of $\mathfrak{A}$ are continuous.

A *section* of $\langle X, S, \pi, \mathfrak{A} \rangle$ is a continuous function σ from X to S such that $\pi \circ \sigma = X \restriction \mathrm{Id}$. These sections form a natural CA_α.

The central result of Comer [72'] is that corresponding to every CA_α $\mathfrak{B}$ there is an especially simple sheaf $\langle X, S, \pi, \mathfrak{A} \rangle$ such that $\mathfrak{B}$ is isomorphic to the CA_α of its sections.

PROBLEMS

We include here not only natural questions which were already raised in this chapter, but also some (but not all) problems mentioned in Henkin, Monk, Tarski Andréka, Németi [81']; again the two parts of this book will be denoted by [HMT81] and [AN81]. We mention in passing the status of the problems in that book.

PROBLEM 3.1. *For which infinite cardinals* κ,λ *is it true that* ${}_\kappa\mathsf{Cs}_\alpha \subseteq \mathsf{I}_\lambda\mathsf{Cs}_\alpha$?

With regard to this problem see 3.1.45 and 3.1.46; it coincides with Problem 17 of [AN81]. The related Problem 1 of [HMT81] has been solved affirmatively by Richard Thompson (unpublished).

PROBLEM 3.2. *Let* $\alpha \geqq \omega$. *If* $\mathfrak{A},\mathfrak{B} \in {}_\infty\mathsf{Cs}_\alpha^{reg}$ *and* $\mathfrak{A} \cong \mathfrak{B}$, *is there a* $\mathfrak{C} \in \mathsf{Cs}_\alpha^{reg}$ *which is ext–base–isomorphic to both* $\mathfrak{A}$ *and* $\mathfrak{B}$?

This problem is related to the logical fact that elementarily equivalent structures always have a common elementary extension. See [AN81] 3.10; the problem is formulated as number 6 in that article. Also related to it are Problems 3 and 4 there (still open), and Problem 5, solved affirmatively in Sain [82'], [84'].

PROBLEM 3.3. *If* $4<\alpha<\omega$, *is there a* Cs_α *with base* $\alpha+2$ *not generated by a single element*?

For an extensive discussion of this problem, see 3.1.68 and the references cited there.

PROBLEM 3.4. *For* $2 \leqq \alpha,\beta<\omega$ *let* $q(\alpha,\beta)$ *be the smallest* $\gamma<\omega$ *such that every* Cs_α *with base* β *can be generated by* γ *elements. Give an arithmetic characterization of this function* q.

Again, cf. 3.1.68. For related problems (still open), see [HMT81] Problem 2 and [AN81] Problem 8.

PROBLEM 3.5. *For* $\alpha \geqq \omega$, *is* $\mathsf{HCs}_\alpha = \mathsf{HCs}_\alpha^{reg}$?

Cf. 3.1.74 for this problem. Of course the equality in Problem 3.5 is equivalent to the inclusion $\mathsf{Cs}_\alpha \subseteq \mathsf{HCs}_\alpha^{reg}$. Note from 3.1.139 that only Cs_α's with finite bases need to be considered. The problem is number 4 in [HMT81]. Some related problems restricting to infinite bases are solved positively, namely Problems 5,7,8 of [HMT81];

see 3.1.139 again. The related problem 6 in [HMT81] was solved consistently negatively by Andréka and Németi; see Sain [82'] and 3.1.82.

PROBLEM 3.6. *If* $\kappa>0$ *and* $\alpha\geq\omega$, *is* ${}_\kappa\mathsf{Gs}_\alpha\not\subseteq\mathsf{HP}_\kappa\mathsf{Cs}_\alpha$?

This is the first part of Problem 12 of [AN81]. Németi (unpublished) has shown that the answer is consistently yes, namely it is true if there are no uncountable measurable cardinals.

PROBLEM 3.7. *If* $1<\kappa<\omega\leq\alpha$, $\mathfrak{A}\in{}_\kappa\mathsf{Gs}_\alpha\cap\mathsf{Dc}_\alpha$, *and* $\mathfrak{A}$ *has only finitely many subbases, is* $\mathfrak{A}\in\mathsf{ICs}_\alpha$?

This is Problem 9 of [AN81]; see also 4.14–4.18 there.

PROBLEM 3.8. *For* $\alpha\geq\omega$, *is every weakly subdirectly indecomposable* Cs_α *isomorphic to a* Cs_α^{reg}?

Cf. 3.1.88; this is Problem 9 of [HMT81]. The related Problem 10 was solved in Andréka, Németi [84'], while the related Problem 13 of [AN81] is open.

PROBLEM 3.9. *Let* $|\alpha|>\omega$. *Is the class of weakly subdirectly indecomposable* IGs_α*'s the same as the class of directed unions of members of* IWs_α?

The conclusion of this problem is valid for $|\alpha|=\omega$, as has been shown by Andréka and Németi (unpublished).

PROBLEM 3.10 *For which values of* α *and* β *do we have* $\mathsf{Rd}_\alpha(\mathsf{Cs}_\beta^{reg}\cap\mathsf{Lf}_\beta)\subseteq\mathsf{ICs}_\alpha$, $\mathsf{Rd}_\alpha(\mathsf{Cs}_\beta\cap\mathsf{Dc}_\beta)\subseteq\mathsf{ICs}_\alpha$, *or* $\mathsf{Rd}_\alpha(\mathsf{Cs}_\beta^{reg}\cap\mathsf{Dc}_\beta)\subseteq\mathsf{ICs}_\alpha$?

Cf. [AN81] 8.10(4). This problem coincides with Problems 19 and 23 there.

PROBLEM 3.11. *Suppose* $\omega\leq\alpha<\beta$, $\mathfrak{A}\in\mathsf{Gws}_\beta$, $\mathfrak{B}\subseteq\mathfrak{Nr}_\alpha\mathfrak{A}$, *and every* $b\in B$ *is regular (in* $\mathfrak{A}$*). Is* $\mathfrak{Cg}^{\mathfrak{A}}B$ *regular?*

This is Problem 29 of [AN81].

PROBLEM 3.12. *For* $\alpha\geq\omega$, *give an elementary characterization of regularity of an element of a* Gws_α.

Cf. here 3.1.23–3.1.25. In [AN81] 1.6.2, a simple characterization of regularity of *every* element of a Gws_α is given. (In its simplest version, that result says that a Gws_α $\mathfrak{A}$ is regular if *every* element $a\notin Zd\mathfrak{A}$ satisfies the same simple regularity condition given in 3.1.23 – 3.1.24.)

PROBLEM 3.13. *Let* $\omega\leq\alpha<\beta$. *Is* $\mathsf{ICs}_\alpha^{reg}=\mathsf{SNr}_\alpha\mathsf{Cs}_\beta^{reg}$?

For this problem see [AN81] 8.18, 8.18.1. The related problems 18,24,26,27,28 there are open; 3.13 is a part of Problem 27.

PROBLEM 3.14. *For various non–representable algebras discussed in* 3.2.76 – 3.2.87, *find explicit equations which hold in all representable algebras but fail in those algebras. For various such algebras* $\mathfrak{A}$, *find explicit* $\kappa \in \omega$ *such that* $\mathfrak{A}$ *cannot be neatly embedded in a* CA of κ *more dimensions.*

See the comment following 3.2.74.

We briefly indicate the status of the other problems from [HMT81] and [AN81] which are relevant to this chapter but were not mentioned above. Problems 3 and 10 of [HMT81] were solved negatively in Andréka, Németi [84'], while Problem 11 is open. Problems 1,11,14,15,16,20,21,22, and 25 of [AN81] are still open. (The first part of 11 is solved by 3.1.139, however.) Problem 2 was solved positively by Andréka and Németi (unpublished), Problem 7 negatively by Németi (unpublished). Problem 10 was solved negatively by Andréka, Comer, Németi [83'] for $\alpha<\omega$; it is still open for $\alpha \geqq \omega$.

CHAPTER 4

NOTIONS OF LOGIC RELATED TO CYLINDRIC ALGEBRAS

4. NOTIONS OF LOGIC RELATED TO CYLINDRIC ALGEBRAS

In this chapter we apply logical notions to cylindric algebras and also indicate the connections between logic and cylindric algebras. The applications of logical notions to CA's are of the usual sort when discussing any algebraic structures: we consider their model theory, and then various decision problems, in sections 4.1 and 4.2 respectively. The connections between logic and cylindric algebras discussed in section 4.3 present a new aspect of the theory of these algebras as opposed to more common algebraic structures. Cylindric algebras were designed primarily to algebraically reflect logical properties, and in that section we explore this motivating connection in some depth.

4.1 MODEL THEORY OF CYLINDRIC ALGEBRAS

In this section we study cylindric algebras from a model–theoretic point of view. Of greatest interest are various equational classes of cylindric algebras, but we also consider universal and elementary classes. The class IGs_α of representable CA_α's deserves special attention. We have already seen, in 3.1.108, that for $\alpha \geqq 2$ $\mathsf{HSPGs}_\alpha = \mathsf{IGs}_\alpha$ and hence IGs_α is an equational class. Also, by 3.2.65 IGs_2 is also characterized by only finitely many equations, although $\mathsf{IGs}_2 \neq \mathsf{CA}_2$. Two of the main results of this section are that for $3 \leqq \alpha < \omega$, IGs_α cannot be characterized by finitely many equations, while for $\omega \leqq \alpha$, IGs_α cannot be characterized by a finite schema of equations of the kind defining the class CA_α. We do give a rather complicated system of equations for IGs_α, and also a simpler but non–standard schema for deriving all the equations holding in the class IGs_α.

Beyond these main results we consider for many of the classes K_α of CA_α's which we have introduced the questions concerning the nature of the smallest equational, universal, and elementary classes containing K_α. Some such questions were considered in Part I; see, e.g., 2.6.53. Here we proceed systematically, and recall in due course those model–theoretic results mentioned earlier, as well as giving many additional results.

We begin by introducing some abbreviations which will be useful below.

DEFINITION 4.1.1. *If* K *is any class of similar structures, then* EqK *is the smallest equational class containing* K, UnK *the smallest universal class, and* ElK *the smallest elementary class.*

REMARK 4.1.2. We have $\mathsf{EqK} = \mathsf{HSPK}$, $\mathsf{UnK} = \mathsf{SUpK}$, and $\mathsf{ElK} = \mathsf{UfUpK}$; see section 0.3.

Non–finite axiomatizability

Now we give the first main non–finitizability result, due to Monk [69].

THEOREM 4.1.3. *For* $3 \leqq \alpha < \omega$, IGs_α *is not finitely axiomatizable.*

PROOF. By 3.2.85, for each $\kappa < \omega$ choose $\mathfrak{A}_\kappa \in \mathsf{CA}_\alpha \sim \mathsf{IGs}_\alpha$ and $\mathfrak{B}_\kappa \in \mathsf{CA}_{\alpha+\kappa}$ such that $\mathfrak{A}_\kappa \subseteq \mathfrak{Nr}_\alpha \mathfrak{B}_\kappa$. We can also find an algebra $\mathfrak{C}_\kappa$ similar to CA_ω's such that $\mathfrak{B}_\kappa = \mathfrak{Rd}_{\alpha+\kappa} \mathfrak{C}_\kappa$ (in the sense of 2.6.1, extended in a natural way to structures merely similar to CA_α's), for each $\kappa < \omega$. Let F be a non–principal ultrafilter over ω. Then $\mathsf{P}_{\kappa<\omega} \mathfrak{C}_\kappa / \bar{F}$ is a CA_ω, as is easily checked, and $\mathsf{P}_{\kappa<\omega} \mathfrak{A}_\kappa$ can be neatly embedded in $\mathsf{P}_{\kappa<\omega} \mathfrak{C}_\kappa / \bar{F}$. Thus by 3.2.10, $\mathsf{P}_{\kappa<\omega} \mathfrak{A}_\kappa \in \mathsf{IGs}_\alpha$. This shows that $\mathsf{CA}_\alpha \sim \mathsf{IGs}_\alpha$ is not closed under ultraproducts, and the theorem follows.

Now for $\alpha \geqq \omega$ it is clear that CA_α, IGs_α and most of our other equational classes of CA_α's are not finitely axiomatizable. The reason behind this is simple: there are infinitely many operations, all subject to certain non–trivial equational conditions, while a finite set of equations can only make restrictions on finitely many operations. Nevertheless, our axioms for CA_α's in 1.1.1 are finite, in a two–sorted sense: one sort for the ordinals $< \alpha$, the other sort for the usual first–order situation. We now want to make this more precise, in order to formulate our second non–finitizability result.

DEFINITION 4.1.4. (i) *Suppose* α *is any ordinal. We denote by* $\mathcal{L}_\alpha$ *a discourse language for* CA_α *(see p.43 of Part I). The non–logical symbols of* $\mathcal{L}_\alpha$ *will be denoted by the bold–face counterparts of the symbols used for the operations in a* CA_α; thus, e.g., $\mathbf{+}, \mathbf{\cdot}, \mathbf{c}_\kappa$ *correspond to* $+, \cdot, \mathsf{c}_\kappa$. $T\mu^\alpha$ *is the set of terms of* $\mathcal{L}_\alpha$ *and* $E\nu^\alpha$ *the set of equations of* $\mathcal{L}_\alpha$.

(ii). *Suppose* α *and* β *are ordinals and* $\gamma \in {}^\alpha\beta$. *We define* γ^+ *mapping* $T\mu^\alpha$ *into* $T\mu^\beta$ *as follows.* $\gamma^+ v_\kappa = v_\kappa$ *for each* $\kappa < \omega$. *For any terms* σ, τ *of* $\mathcal{L}_\alpha$ *and any* $\kappa, \lambda < \alpha$ *we set*

$$\gamma^+(\sigma \mathbf{+} \tau) = \gamma^+\sigma \mathbf{+} \gamma^+\tau;$$
$$\gamma^+(\sigma \mathbf{\cdot} \tau) = \gamma^+\sigma \mathbf{\cdot} \gamma^+\tau;$$
$$\gamma^+(-\sigma) = -\gamma^+\sigma;$$
$$\gamma^+\mathbf{0} = \mathbf{0};$$
$$\gamma^+\mathbf{1} = \mathbf{1};$$
$$\gamma^+\mathbf{c}_\kappa\sigma = \mathbf{c}_{\gamma\kappa}\gamma^+\sigma;$$
$$\gamma^+\mathbf{d}_{\kappa\lambda} = \mathbf{d}_{\gamma\kappa,\gamma\lambda}.$$

(iii) *Suppose* K *is a class of algebras similar to* CA_α's. *We say that* K *is* **axiomatizable by a finite schema of equations** *provided that there is a finite set* $E \subseteq E\nu^\omega$ *such* that $\mathsf{K} = \mathsf{Md}\{\gamma^+\sigma = \gamma^+\tau : \sigma = \tau \in E,\ \gamma \in {}^\omega\alpha,\ \gamma$ *one–one*$\}$; *if such a set* E *exists but is not necessarily finite, we say that* K is **axiomatizable by a countable schema of equations.**

The definition 4.1.4(ii),(iii) for finite schemas is taken from Monk [69]. It has

been generalized in Andréka, Németi [80a'].

THEOREM 4.1.5. *For $\alpha \geqq \omega$, CA_α is axiomatizable by a finite schema of equations.*

PROOF. For the desired finite set $E \subseteq Ev^\omega$ we can take the 8 equations for Boolean algebras given in section 1.1 together with the instances of $(C_1)-(C_7)$ in 1.1.1 in which $\kappa = 0$, $\lambda = 1$, $\mu = 2$, and the instance $\mathbf{d}_{11} = \mathbf{c}_0(\mathbf{d}_{10} \cdot \mathbf{d}_{01})$ of (C_6).

THEOREM 4.1.6. *For $\alpha \geqq \omega$, IGs_α is axiomatizable by a countable schema of equations.*

PROOF. Let $E = \Theta\rho\mathsf{Gs}_\omega \cap Ev^\omega$, the set of all equations holding in Gs_ω. We claim that E satisfies the condition in 4.1.4(iii) with $\mathsf{K} = \mathsf{IGs}_\omega$. To show this, first suppose $\mathfrak{A} \in \mathsf{Gs}_\alpha$. Let $\sigma = \tau \in E$ and $\gamma \in {}^\omega\alpha$, γ one–one. By 3.1.118, $\mathfrak{Rb}^\gamma\mathfrak{A} = \mathfrak{B}$ is representable. Hence $\sigma = \tau$ holds in $\mathfrak{B}$. But $(\gamma^+\rho)^{\mathfrak{A}} = \rho^{\mathfrak{B}}$ for any term ρ of $\mathcal{L}_\alpha$, so $\gamma^+\sigma = \gamma^+\tau$ holds in $\mathfrak{A}$. (Here we write $\rho^{\mathfrak{B}}$ instead of $\tilde{\rho}^{(\mathfrak{B})}$, etc.). Thus $\mathsf{IGs}_\alpha \subseteq \mathsf{Md}\{\gamma^+\sigma = \gamma^+\tau : \sigma = \tau \in E,\ \gamma \in {}^\omega\alpha,\ \gamma$ one–one$\}$. The opposite inclusion is established similarly, by showing that for any member $\mathfrak{A}$ of the right side and any one–one $\gamma \in {}^\omega\alpha$, $\mathfrak{Rb}^\gamma\mathfrak{A} \in \mathsf{IGs}_\omega$.

Now we give our second non–finitizability result:

THEOREM 4.1.7. *For $\alpha \geqq \omega$, IGs_α is not axiomatizable by a finite schema of equations.*

PROOF. Suppose on the contrary that $E \subseteq Ev^\omega$ is finite and $\mathsf{IGs}_\alpha = \mathsf{Md}\{\gamma^+\sigma = \gamma^+\tau : \sigma = \tau \in E,\ \gamma \in {}^\omega\alpha,\ \gamma$ one–one$\}$. By 3.2.87, for each $\kappa < \omega$ let $\mathfrak{A}_\kappa \in \mathsf{CA}_\alpha \sim \mathsf{IGs}_\alpha$ and $\mathfrak{B}_\kappa \in \mathsf{CA}_{\alpha+\kappa}$ be such that $\mathfrak{A}_\kappa \subseteq \mathfrak{Rr}_\alpha\mathfrak{B}_\kappa$. Also, for each $\kappa < \omega$ let $\mathfrak{C}_\kappa$ be any algebra similar to $\mathsf{CA}_{\alpha+\omega}$'s such that $\mathfrak{B}_\kappa = \mathfrak{Rb}_{\alpha+\kappa}\mathfrak{C}_\kappa$. Choose $\lambda < \omega$ and terms $\sigma_0, \ldots \sigma_{\lambda-1}, \tau_0, \ldots, \tau_{\lambda-1}$ in $\mathcal{L}_\omega$ such that $E = \{\sigma_0 = \tau_0, \ldots, \sigma_{\lambda-1} = \tau_{\lambda-1}\}$. Then for each $\kappa < \omega$, since $\mathfrak{A}_\kappa \notin \mathsf{IGs}_\alpha$, there is a one–one $\gamma_\kappa \in {}^\omega\alpha$ such that the formula

$$\gamma_\kappa^+\sigma_0 = \gamma_\kappa^+\tau_0 \wedge \ldots \wedge \gamma_\kappa^+\sigma_{\lambda-1} = \gamma_\kappa^+\tau_{\lambda-1}$$

does not hold in $\mathfrak{A}_\kappa$. For each $\kappa < \omega$ choose $\gamma'\kappa \in {}^\alpha\alpha$ such that $\gamma'\kappa$ is one–one and $\gamma_\kappa\xi = (\gamma'\kappa)\xi$ for all ξ such that c_ξ, $\mathsf{d}_{\xi\eta}$, or $\mathsf{d}_{\eta\xi}$ occurs in the formula (for some η), and let $\delta\kappa = \gamma'\kappa \cup ([(\alpha+\omega) \sim \alpha] \upharpoonright Id)$. Now we let $\mathfrak{A}'_\kappa = \mathfrak{Rb}^{\gamma'\kappa}\mathfrak{A}_\kappa$ and $\mathfrak{C}'_\kappa = \mathfrak{Rb}^{\delta\kappa}\mathfrak{C}_\kappa$. Note that

(1) $\sigma_0 = \tau_0 \wedge \ldots \wedge \sigma_{\lambda-1} = \tau_{\lambda-1}$

does not hold in $\mathfrak{A}'_\kappa$ for any $\kappa < \omega$. Let F be any non–principal ultrafilter on ω. Then the formula (1) fails to hold in $\mathsf{P}_{\kappa<\omega}\mathfrak{A}'_\kappa/\bar{F}$; since it is a conjunction of members of E, it follows that $\mathsf{P}_{\kappa<\omega}\mathfrak{A}'_\kappa/\bar{F}$ is non–representable. But $\mathsf{P}_{\kappa<\omega}\mathfrak{A}'_\kappa/\bar{F}$ can be neatly embedded in the $\mathsf{CA}_{\alpha+\omega}$ $\mathsf{P}_{\kappa<\omega}\mathfrak{C}'_\kappa/\bar{F}$, contradicting 3.2.10.

Theorem 4.1.7 is due to Monk [69]. It is one of the most important model–

theoretic results about CA_α's. If it had turned out, contrary to 4.1.7, that IGs_α is axiomatizable by a finite schema of equations, such a schema would probably have been taken as the definition of the class CA_α, relegating our present class CA_α to a minor role.

Nevertheless, IGs_α *is* an equational class, and it is natural to try to describe in a simple fashion a set of equations characterizing it. We give three descriptions of such sets; as will be seen, each description is somewhat unsatisfactory. The first description is found in Monk [69], and generalizes a method used by Ralph McKenzie to characterize the class of involuted semigroups $\langle A,;,^{\smile}\rangle$ isomorphic to algebras $\langle A,|,^{-1}\rangle$ of binary relations.

Equations characterizing IGs_α

DEFINITION 4.1.8. *Suppose* $3\leqq\alpha$ *and* Γ *is a finite subset of* α; *say* $|\Gamma|=\kappa$. *We define a sequence* λ^Γ *of natural numbers by recursion:* $\lambda^\Gamma_0=1$, *and for* $\mu<\omega$, $\lambda^\Gamma_{\mu+1}=\lambda^\Gamma_\mu+(\mu+1)^2\cdot\kappa\cdot(\lambda^\Gamma_\mu)^\kappa$. *Let* $\Gamma=\{\nu_0,\ldots,\nu_{\kappa-1}\}$ *with* $\nu_0<\ldots<\nu_{\kappa-1}$, *and for brevity set*

$$\mathbf{c}_{(\Gamma)}=\mathbf{c}_{\nu_0}\ldots\mathbf{c}_{\nu(\kappa-1)},$$
$$\sigma\oplus\tau=(\sigma\cdot-\tau)+(\tau\cdot-\sigma) \text{ for any terms } \sigma,\tau \text{ in } \mathcal{L}_\alpha,$$
$$V^\Gamma_\mu={}^\alpha(\lambda^\Gamma_\mu) \text{ for any } \mu<\omega.$$

Now for any $\mu<\omega$ *and any* $\psi\in{}^{\mu+1}Sb^\alpha(\lambda^\Gamma_\mu)$ *we let* σ^Γ_ψ *be a formal sum of the following terms, where* $V=V^\Gamma_\mu$ *and* $\lambda=\lambda^\Gamma_\mu$:

$\mathbf{0}$;

$-\mathbf{c}_{(\Gamma)}[\mathbf{v}_\nu\oplus(-\mathbf{v}_\rho)]$ *if* $\nu<\rho\leqq\mu$ *and* $\psi_\nu\neq{}^\alpha\lambda\sim\psi_\rho$;

$-\mathbf{c}_{(\Gamma)}[(\mathbf{v}_\nu+\mathbf{v}_\rho)\oplus\mathbf{v}_\tau]$ *if* $\nu,\rho,\tau\leqq\mu$ *and* $\psi_\tau\neq\psi_\nu\mathbf{u}\psi_\rho$;

$-\mathbf{c}_{(\Gamma)}[\mathbf{c}_\xi\mathbf{v}_\nu\oplus\mathbf{v}_\rho]$ *if* $\xi\in\Gamma$, $\nu\leqq\mu$, $\rho\leqq\mu$, *and there is a* $\tau\leqq\mu$ *such that* $\nu,\rho<\tau$ *and* $\psi_\rho\mathbf{n}^\alpha(\lambda^\Gamma_{\tau-1})\neq\mathbf{C}^{[V]}_\xi(\psi_\nu\mathbf{n}^\alpha(\lambda^\Gamma_\tau))\mathbf{n}^\alpha(\lambda^\Gamma_{\tau-1})$;

$-\mathbf{c}_{(\Gamma)}(\mathbf{v}_\nu\oplus\mathbf{d}_{\xi\eta})$ *if* $\xi\in\Gamma$, $\eta\in\Gamma$, $\nu\leqq\mu$, *and* $\mathbf{D}^{[V]}_{\xi\eta}\mathbf{n}^\alpha(\lambda^\Gamma_\nu)\not\leqq\psi_\nu$;

$-\mathbf{c}_{(\Gamma)}(\mathbf{v}_\nu\oplus\mathbf{d}_{01})$ *if* $0\in\Gamma$, $1\in\Gamma$, $\nu\leqq\mu$, *and there are* $\rho\leqq\mu$, $x\in\psi_\rho$, $y\in\psi_\nu$, *and* $\xi\in\Gamma$ *such that* $y_0=x_\xi$ *and* $x^\xi_{y_1}\notin\psi_\rho$.

Next, for each $\mu<\omega$ *let* φ^Γ_μ *be the following equation:*

$$\prod\{\sigma^\Gamma_\psi:\psi\in{}^{\mu+1}Sb({}^\alpha(\lambda^\Gamma_\mu))\}=\mathbf{0}.$$

Finally, let Δ_α *be the set of equations in* $\mathcal{L}_\alpha$ *implicit in* 1.1.1 *such that* $\mathsf{CA}_\alpha=\mathsf{Md}\Delta_\alpha$.

We note that for $3\leqq\alpha<\omega$, the set $\{\varphi^\alpha_\mu:\mu\in\omega\}\mathbf{u}\Delta_\alpha$ is an explicit, effectively given set; under a natural Godel numbering, it corresponds to a primitive recursive set of natural numbers. The same applies to the set $\{\varphi^\Gamma_\mu:\mu\in\omega,\ \Gamma\subseteq\omega,\ |\Gamma|<\omega\}\mathbf{u}\Delta_\omega$.

THEOREM 4.1.9. *If* $3\leqq\alpha<\omega$, *then* $\mathsf{IGs}_\alpha=\mathsf{Md}(\{\varphi^\alpha_\mu:\mu\in\omega\}\mathbf{u}\Delta_\alpha)$.

PROOF. We omit the superscript α on λ,ψ, etc. in what follows. To prove the

inclusion $\subseteq$ it suffices to take an arbitrary $\mu \in \omega$, an arbitrary Cs_α $\mathfrak{A}$, say with non-empty base U, and show that φ_μ holds in $\mathfrak{A}$. For all $\xi,\nu<\alpha$ let $\mathsf{C}_\xi = \mathsf{C}_\xi^{[V]}$ with $V = V_\mu^\alpha$, $\mathsf{D}_{\xi\eta} = \mathsf{D}_{\xi\eta}^{[V]}$, $\mathsf{C}_\xi' = \mathsf{C}_\xi^{[V']}$ with $V' = {}^\alpha U$, and $\mathsf{D}_{\xi\eta}' = \mathsf{D}_{\xi\eta}^{[V']}$. Let $a_0,\ldots,a_\mu \in A$; we wish to show that under this assignment of values to the free variables $v_0,\ldots,v_\mu$ of φ_μ, φ_μ is true in $\mathfrak{A}$. To do this we first construct functions $y_\nu \in {}^{\lambda_\nu}U$ for each $\nu \leqq \mu$ by induction. Recall that $\lambda_0 = 1$; we let $y_0 \in {}^1U$ be arbitrary. Now assume that $\nu<\mu$ and y_ν has been defined so that

(1) if $\rho,\gamma<\tau\leqq\nu$, $\xi<\alpha$, and $\mathsf{C}_\xi' a_\rho = a_\gamma$, then $\mathsf{C}_\xi'(a_\rho \mathsf{n}^\alpha(y_\nu^*\lambda_\tau))\mathsf{n}^\alpha(y_\nu^*\lambda_{\tau-1}) = a_\gamma \mathsf{n}^\alpha(y_\gamma^*\lambda_{\tau-1})$

Then we can pick a non-empty subset S of U with at most $(\nu+1)^2\cdot\alpha\cdot(\lambda_\nu)^\alpha$ elements such that if $\rho,\gamma\leqq\nu$, $\xi<\alpha$, $\mathsf{C}_\xi' a_\rho = a_\gamma$, and $x \in a_\gamma \mathsf{n}^\alpha(y_\nu^*\lambda_\nu) \sim \mathsf{C}_\xi'(a_\rho \mathsf{n}^\alpha(y_\nu^*\lambda_\nu))$, then there is a $u \in S$ such that $x_u^\xi \in a_\rho$. Let $y_{\nu+1}$ be y_ν together with a map from $\lambda_{\nu+1}\sim\lambda_\nu$ onto S. Clearly then (1) holds with ν replaced by $\nu+1$. This finishes the construction of the functions y_ν, $\nu\leqq\mu$.

Now let $x = y_\mu$, and set

(2) $\psi = \langle\{t \in {}^\alpha\lambda_\mu : x\circ t \in a_\nu\} : \nu \leqq \mu\rangle$.

Thus $\psi \in {}^{\mu+1}Sb({}^\alpha\lambda_\mu)$. Now we can complete the proof of the inclusion $\subseteq$ by showing that σ_ψ is zero under our assignment. To do this we look at the five kinds of terms making up σ_ψ that are not formally $\mathbf{0}$.

Case 1. Suppose $\nu<\rho\leqq\mu$ and $\psi_\nu \neq {}^\alpha\lambda_\mu\sim\psi_\rho$. First suppose that $\psi_\nu \mathsf{n} \psi_\rho \neq 0$, say $t \in \psi_\nu \mathsf{n} \psi_\rho$. Then by (2), $x\circ t \in a_\nu \mathsf{n} a_\rho$, hence $a_\nu \neq -a_\rho$, hence $-\mathsf{c}_{(\alpha)}[a_\nu \oplus (-a_\rho)] = 0$ as desired. Second, if ${}^\alpha\lambda_\mu \sim (\psi_\nu \mathsf{u} \psi_\rho) \neq 0$, (2) yields $a_\nu \mathsf{u} a_\rho \neq {}^\alpha U$, and again the desired conclusion follows.

Case 2. $\nu,\rho,\tau\leqq\mu$ and $\psi_\tau \neq \psi_\nu \mathsf{u} \psi_\rho$. This is treated like Case 1.

Case 3. $\xi<\alpha$, $\nu,\rho<\tau\leqq\mu$, and $\psi_\rho \mathsf{n}^\alpha\lambda_{\tau-1} \neq \mathsf{C}_\xi(\psi_\nu \mathsf{n}^\alpha\lambda_\tau)\mathsf{n}^\alpha\lambda_{\tau-1}$. Then

$$a_\rho \mathsf{n}^\alpha x^*\lambda_{\tau-1} \neq \mathsf{C}_\xi'(a_\nu \mathsf{n}^\alpha x^*\lambda_\tau)\,\mathsf{n}^\alpha x^*\lambda_{\tau-1},$$

so (1) yields $\mathsf{C}_\xi' a_\nu \neq a_\rho$, and hence $-\mathsf{C}_{(\alpha)}'(\mathsf{C}_\xi' a_\nu \oplus a_\rho) = 0$, as desired.

Case 4. $\xi,\eta<\alpha$, $\nu\leqq\mu$, and $\mathsf{D}_{\xi\eta}\mathsf{n}^\alpha\lambda_\nu \notin \psi_\nu$. Let $t \in \mathsf{D}_{\xi\eta}\mathsf{n}^\alpha\lambda_\gamma \sim \psi_\nu$. Then $t\xi = t\eta$ and $x\circ t \notin a_\nu$, so $x\circ t \in \mathsf{D}_{\xi\eta}' \sim a_\nu$ and the desired conclusion follows.

Case 5. $\nu,\rho\leqq\mu$, $u \in \psi_\rho$, $v \in \psi_\nu$, $\xi<\alpha$, $v_0 = u_\xi$, and $u_{\nu 1}^\xi \notin \psi_\rho$. Since $u \in \psi_\rho$ and $u_{\nu 1}^\xi \notin \psi_\rho$, we have $xu\xi \neq xv1$, so $xv_0 \neq xv_1$ and so $a_\nu \neq D_{01}'$ since $v \in \psi_\nu$; the desired conclusion follows.

This finishes the proof of the inclusion $\subseteq$ in the theorem. For $\supseteq$, it suffices by 0.1.27(ii), 0.3.71, and 2.4.53 to show that if $\mathfrak{A}$ is a countable simple CA_α which is a model of $\{\varphi_\mu^\alpha : \mu \in \omega\}$, then $\mathfrak{A}$ is representable. Let $\langle a_0, a_1, \ldots\rangle$ be an enumeration of the elements of A. Then

(3) for every $\mu \in \omega$ there exists a $\psi \in {}^{\mu+1}Sb({}^{\alpha}\lambda_{\mu})$ such that the following conditions hold (with $\mathsf{C}_{\xi} = \mathsf{C}_{\xi}^{[V]}$, $\mathsf{D}_{\xi\eta} = \mathsf{D}_{\xi\eta}^{[V]}$ for $V = V_{\mu}^{\alpha}$):

(a) if $\nu < \rho \leqq \mu$ and $a_{\nu} = -a_{\rho}$, then $\psi_{\nu} = {}^{\alpha}\lambda_{\mu} \sim \psi_{\rho}$;

(b) if $\nu, \rho, \tau \leqq \mu$ and $a_{\tau} = a_{\nu} + a_{\rho}$, then $\psi_{\tau} = \psi_{\nu} \mathbf{u} \psi_{\rho}$;

(c) if $\xi < \alpha$, $\nu, \rho < \tau \leqq \mu$, and $\mathsf{c}_{\xi} a_{\nu} = a_{\rho}$, then $\psi_{\rho} \mathbf{n}^{\alpha} \lambda_{\tau-1} = \mathsf{C}_{\xi}(\psi_{\nu} \mathbf{n}^{\alpha} \lambda_{\tau}) \mathbf{n}^{\alpha} \lambda_{\tau-1}$;

(d) if $\xi, \eta < \alpha$, $\nu \leqq \mu$, and $a_{\nu} = \mathsf{d}_{\xi\eta}$, then $\mathsf{D}_{\xi\eta} \mathbf{n}^{\alpha} \lambda_{\nu} \subseteq \psi_{\nu}$;

(e) if $\rho, \nu \leqq \mu$, $x \in \psi_{\rho}$, $y \in \psi_{\nu}$, $\xi < \alpha$, $y_0 = x_{\xi}$, and $a_{\nu} = \mathsf{d}_{01}$, then $x_{y1}^{\xi} \in \psi_{\rho}$.

This follows since $\mathfrak{A}$ satisfies φ_{μ} under our assignment. If ψ satisfies (3) and $\mu' \leqq \mu$, then $\langle \psi_{\nu} \mathbf{n}^{\alpha} \lambda_{\mu'} : \nu \leqq \mu' \rangle$ satisfies (3). Hence by Konig's lemma there is a $\psi \in {}^{\omega}Sb({}^{\alpha}\omega)$ such that the following conditions hold (with $\mathsf{C}_{\xi} = \mathsf{C}_{\xi}^{[V]}$, $\mathsf{D}_{\xi\eta} = \mathsf{D}_{\xi\eta}^{[V]}$ for $V = {}^{\alpha}\omega$):

(4) if $\nu < \rho < \omega$ and $a_{\nu} = -a_{\rho}$, then $\psi_{\nu} = {}^{\alpha}\omega \sim \psi_{\rho}$;

(5) if $\nu, \rho, \tau < \omega$ and $a_{\tau} = a_{\nu} + a_{\rho}$, then $\psi_{\tau} = \psi_{\nu} \mathbf{u} \psi_{\rho}$;

(6) if $\xi < \alpha$, $\nu, \rho < \omega$, and $\mathsf{c}_{\xi} a_{\nu} = a_{\rho}$, then $\psi_{\rho} = \mathsf{C}_{\xi} \psi_{\nu}$;

(7) if $\xi, \eta < \alpha$, $\nu < \omega$, and $a_{\nu} = \mathsf{d}_{\xi\eta}$, then $\mathsf{D}_{\xi\eta} \subseteq \psi_{\nu}$;

(8) if $\rho, \nu < \omega$, $a_{\nu} = \mathsf{d}_{01}$, $x \in \psi_{\rho}$, $y \in \psi_{\nu}$, $\xi < \alpha$, and $y_0 = x_{\xi}$, then $x_{y1}^{\xi} \in \psi_{\rho}$.

Now we define a binary relation E on ω by letting $\mu E \nu$ iff $\mu, \nu \in \omega$ and there exist $\rho < \omega$ and $x \in \psi_{\rho}$ such that $a_{\rho} = \mathsf{d}_{01}$ and $x_0 = \mu$, $x_1 = \nu$. Now E is an equivalence relation on ω. For, given $\mu < \omega$, choose $\rho < \omega$ with $a_{\rho} = \mathsf{d}_{01}$ and let $x \in \mathsf{D}_{01}$ with $x_0 = x_1 = \mu$. By (7) we have $x \in \psi_{\rho}$, so $\mu E \mu$. Thus E is reflexive on ω. Next, suppose $\mu E \nu$; let ρ and x be as above. Fix $y \in \mathsf{D}_{01}$ with $y_0 = \mu = y_1$. Thus $y \in \psi_{\rho}$ by (7). Then by (8), $y_{x1}^{0} \in \psi_{\rho}$, i.e. $\nu E \mu$. So E is symmetric. Finally, suppose that $\mu E \nu E \rho$. Say $\xi, \eta < \omega$, $x \in \psi_{\xi}$, $y \in \psi_{\eta}$, $a_{\xi} = a_{\eta} = \mathsf{d}_{01}$, and $x_0 = \mu$, $x_1 = \nu$, $y_0 = \nu$, $y_1 = \rho$. Thus $y_0 = x_1$, so $x_{y1}^{1} \in \psi_{\xi}$, i.e., $\mu E \rho$. So E is transitive. Thus E is an equivalence relation on ω.

Let k be a function mapping ω/E into ω such that $k(\xi/E) \in \xi/E$ for every $\xi \in \omega$. Set $U = \omega/E$. Now for any $b \in A$ we set

$$fb = \{u \in {}^{\alpha}U : \text{for some } \mu < \omega \text{ we have } a_{\mu} = b \text{ and } k \circ u \in \psi_{\mu}\}.$$

We claim that f is the desired isomorphism from $\mathfrak{A}$ onto a Cs_{α} with base U. To show that $f(b+c) = fb \mathbf{u} fc$, first suppose $u \in fb \mathbf{u} fc$; say $u \in fb$. Say $\mu < \omega$, $a_{\mu} = b$, and $k \circ u \in \psi_{\mu}$. Choose $\nu, \rho < \omega$ so that $a_{\nu} = c$ and $a_{\rho} = b + c$. Then by (5) $\psi_{\rho} = \psi_{\mu} \mathbf{u} \psi_{\nu}$, so $k \circ u \in \psi_{\rho}$. Hence $u \in f(b+c)$. Therefore $fb \mathbf{u} fc \subseteq f(b+c)$. The other inclusion is established similarly. And it is similarly checked that f preserves $-$ and c_{ξ} for each $\xi < \alpha$. Finally, suppose $\xi, \eta < \alpha$. Then $\mathsf{D}_{\xi\eta} \subseteq f\mathsf{d}_{\xi\eta}$ by (7). Suppose $u \in f\mathsf{d}_{\xi\eta}$. Say $\mu < \alpha$, $a_{\mu} = \mathsf{d}_{\xi\eta}$ and $k \circ u \in \psi_{\mu}$. We may assume that $\xi \neq \eta$. If $\{\xi, \eta\} = \{0,1\}$, then $(ku_0) E (ku_1)$, hence $u_0 = u_1$ and $u \in \mathsf{D}_{\xi\eta}$, as desired. Suppose $|\{\xi, \eta\} \mathbf{n} \{0,1\}| = 1$, say $\xi = 0$ while $\eta \neq 0,1$. Now $\mathsf{d}_{01} = \mathsf{c}_{\eta}(\mathsf{d}_{1\eta} \cdot \mathsf{c}_1 \mathsf{d}_{0\eta})$. We have $u_{u\eta}^{1} \in \mathsf{D}_{1\eta} \subseteq f\mathsf{d}_{1\eta}$ by (7), and $u_{u\eta}^{1} \in \mathsf{C}_1 f\mathsf{d}_{0\eta}$. Since f preserves $\cdot$ and c_{η}, it follows that $u_{u\eta}^{1} \in f\mathsf{d}_{01}$. Hence $k \circ u_{u\eta}^{1} \in \psi_{\rho}$ for some $\rho < \omega$ with $a_{\rho} = \mathsf{d}_{01}$, and so $(ku_0) E (ku_{\eta})$. Thus $u_0 = u_{\eta}$, so $u \in \mathsf{D}_{0\eta}$, as desired. If $\{\xi, \eta\} \mathbf{n} \{0,1\} = 0$ we proceed similarly, using the identity $\mathsf{d}_{01} = \mathsf{c}_{\xi} \mathsf{c}_{\eta}(\mathsf{c}_0 \mathsf{c}_1 \mathsf{d}_{\xi\eta} \cdot \mathsf{c}_1 \mathsf{d}_{0\xi} \cdot \mathsf{c}_0 \mathsf{d}_{1\eta})$. This checks that f preserves $\mathsf{d}_{\xi\eta}$, and hence is a homomorphism. Since $\mathfrak{A}$ is simple, f is an isomorphism. The proof of 4.1.9 is complete.

THEOREM 4.1.10 *If* $\omega \leqq \alpha$, *then* $\mathsf{IGs}_\alpha = \mathsf{Md}(\{\varphi^\Gamma_\mu : \mu \in \omega,\ \Gamma \subseteq \alpha,\ |\Gamma| < \omega\} \cup \Delta_\alpha)$.

PROOF. By 4.1.9, 3.1.126, and 2.6.47.

The second explicit characterization of the equations holding in IGs_α depends on a characterization of the elementary theory of Lf_α's. $\alpha \geqq \omega$; these results are due to Andréka, Németi [81'], solving a problem stated on p. 418 of Part I.

DEFINITION 4.1.11. *Let* α *be any ordinal.* (i) *For any formula* φ *of* $\mathcal{L}_\alpha$ *we let* $Occ\varphi = \{\kappa : \kappa < \alpha$, *and* c_κ, $\mathsf{d}_{\kappa\lambda}$, *or* $\mathsf{d}_{\lambda\kappa}$ *occurs in* φ *for some* $\lambda < \alpha\}$.

(ii) *Let* Ax_α *consist of the axioms for* CA_α's *along with all sentences of* $\mathcal{L}_\alpha$ *of the following form:*

$$\forall v_1 \ldots \forall v_\kappa (\exists v_0 \varphi(v_0, \mathsf{c}_\xi v_1, \ldots, \mathsf{c}_\xi v_\kappa) \rightarrow \exists v_0 \varphi(\mathsf{c}_\xi v_0, \mathsf{c}_\xi v_1, \ldots, \mathsf{c}_\xi v_\kappa)),$$

where φ *is any formula of* $\mathcal{L}_\alpha$ *with free variables among* $v_0, \ldots, v_\kappa$, *and* $\xi \in \alpha \sim Occ\varphi$.

Our objective is to show that for $\alpha \geqq \omega$, $\mathsf{EILf}_\alpha = \mathsf{EIDc}_\alpha = \mathsf{Md}\,Ax_\alpha$, and that an equation ε holds in IGs_α iff $Ax_\alpha \models \varepsilon$.

LEMMA 4.1.12. *If* $\alpha \geqq \omega$, *then* $\mathsf{Dc}_\alpha \models Ax_\alpha$.

PROOF. Let $\mathfrak{A}$ be a Dc_α, and let φ, κ, ξ be as in 4.1.11(ii). Assume that $a_0, a_1, \ldots, a_\kappa \in A$ satisfy $\varphi(v_0, \mathsf{c}_\xi v_1, \ldots, \mathsf{c}_\xi v_\kappa)$ when assigned to $v_0, \ldots, v_\kappa$ respectively. Since $\mathfrak{A} \in \mathsf{Dc}_\alpha$, choose $\eta \in \alpha \sim (Occ\varphi \cup \{\xi\} \cup \Delta a_0 \cup \Delta \mathsf{c}_\xi a_1 \cup \ldots \cup \Delta \mathsf{c}_\xi a_\kappa)$. Let τ be the transposition $[\xi/\eta, \eta/\xi]$ of α, and consider the substitution operator s_τ given by 1.11.9. Now s_τ is an automorphism of the reduct of $\mathfrak{A}$ to denotations of symbols occurring in φ. Hence clearly $\varphi(v_0, \ldots, v_\kappa)$ holds in $\mathfrak{A}$ under the assignment of $\mathsf{s}_\tau a_0, \mathsf{s}_\tau \mathsf{c}_\xi a_1, \ldots, \mathsf{s}_\tau \mathsf{c}_\xi a_\kappa$ to $v_0, \ldots, v_\kappa$ respectively. Now if $1 \leqq \lambda \leqq \kappa$ then $\mathsf{s}_\tau \mathsf{c}_\xi a_\lambda = \mathsf{s}_\tau \mathsf{c}_\eta \mathsf{c}_\xi a_\lambda = \mathsf{c}_\eta \mathsf{c}_\xi a_\lambda = \mathsf{c}_\xi a_\lambda$ by 1.11.12 (iv),(i). Also, $\mathsf{s}_\tau a_0 = \mathsf{s}_\tau \mathsf{c}_\eta a_0 = \mathsf{c}_\xi \mathsf{s}_\tau a_0$ by 1.11.12(vi). Hence $\varphi(\mathsf{c}_\xi v_0, \ldots, \mathsf{c}_\xi v_\kappa)$ holds in $\mathfrak{A}$ under the assignment of $\mathsf{s}_\tau a_0, a_1, \ldots, a_\kappa$ to $v_0, \ldots, v_\kappa$ respectively, as desired.

LEMMA 4.1.13. *Let* $\alpha \geqq \omega$. *Suppose that* φ *is a formula of* $\mathcal{L}_\alpha$ *with free variables among* $v_0, \ldots, v_\kappa$, $\Gamma \subseteq \alpha \sim Occ\varphi$, *and* $|\Gamma| < \omega$. *Then*

$$Ax_\alpha \models \forall v_1 \ldots \forall v_\kappa (\exists v_0 \varphi(v_0, \mathsf{c}_{(\Gamma)} v_1, \ldots, \mathsf{c}_{(\Gamma)} v_\kappa) \rightarrow \exists v_0 \varphi(\mathsf{c}_{(\Gamma)} v_0, \ldots, \mathsf{c}_{(\Gamma)} v_\kappa)).$$

PROOF. We proceed by induction on $|\Gamma|$. The case $\Gamma = 0$ is trivial. Assume the lemma true for Γ, and let $\Delta = \Gamma \cup \{\xi\}$ with $\xi \notin \Gamma$. Suppose $\exists v_0 \varphi(v_0, \mathsf{c}_{(\Delta)} v_1, \ldots, \mathsf{c}_{(\Delta)} v_\kappa)$ holds in some model $\mathfrak{A}$ of Ax_α under an assignment of $a_1, \ldots, a_\kappa$ to $v_1, \ldots, v_\kappa$. Then $\exists v_0 \varphi(v_0, \mathsf{c}_{(\Gamma)} v_1, \ldots, \mathsf{c}_{(\Gamma)} v_\kappa)$ holds in $\mathfrak{A}$ under the assignment of $\mathsf{c}_\xi a_1, \ldots, \mathsf{c}_\xi a_\kappa$ to $v_1, \ldots, v_\kappa$; so by the inductive hypothesis, $\exists v_0 \varphi(\mathsf{c}_{(\Gamma)} v_0, \ldots, \mathsf{c}_{(\Gamma)} v_\kappa)$ holds in $\mathfrak{A}$ under that assignment. Thus $\exists v_0 \varphi(\mathsf{c}_{(\Gamma)} v_0, \mathsf{c}_\xi v_1, \ldots, \mathsf{c}_\xi v_\kappa)$ holds in $\mathfrak{A}$ under the assignment of $\mathsf{c}_{(\Delta)} a_1, \ldots, \mathsf{c}_{(\Delta)} a_\kappa$ to $v_1, \ldots, v_\kappa$. Hence $\mathfrak{A} \models Ax_\alpha$ yields that $\exists v_0 \varphi(\mathsf{c}_\xi \mathsf{c}_{(\Gamma)} v_0, \mathsf{c}_\xi v_1, \ldots, \mathsf{c}_\xi v_\kappa)$ holds under this assignment. Therefore $\exists v_0 \varphi(\mathsf{c}_{(\Delta)} v_0, \ldots, \mathsf{c}_{(\Delta)} v_\kappa)$ holds in $\mathfrak{A}$ under the assignment of $a_1, \ldots, a_\kappa$ to $v_1, \ldots, v_\kappa$, as desired.

THEOREM 4.1.14. *Let* $\alpha \geqq \omega$. *Then* $\mathsf{ElLf}_\alpha = \mathsf{ElDc}_\alpha = \mathsf{Md}Ax_\alpha$, *and every* Dc_α *is elementarily equivalent to some* Lf_α.

PROOF. It suffices by Lemma 4.1.12 to show that every model $\mathfrak{A}$ of Ax_α is elementarily equivalent to some Lf_α. Let $\mathfrak{B}$ be a $|\alpha|^+$–saturated elementary extension of $\mathfrak{A}$ (see Chang, Keisler [73**]). Let $\mathfrak{C}$ be the subalgebra of $\mathfrak{B}$ with universe $\{b \in B : |\Delta b| < \omega\}$ (clearly this set *is* a subuniverse of $\mathfrak{B}$). It suffices to show that $\mathfrak{C}$ is an elementary subalgebra of $\mathfrak{B}$. To this end, let φ be a formula of $\mathcal{L}_\alpha$ with free variables among $v_0,\ldots,v_\kappa$, and suppose $\exists v_0 \varphi(v_0,\ldots,v_\kappa)$ holds in $\mathfrak{B}$ under an assignment of elements $a_1,\ldots,a_\kappa$ of C to $v_1,\ldots,v_\kappa$. We want to find $c \in C$ such that $\varphi(v_0,\ldots,v_\kappa)$ holds in $\mathfrak{B}$ under the assignment of $c,a_1,\ldots,a_\kappa$ to $v_0,\ldots,v_\kappa$. Let $\Gamma = 0cc\varphi \mathbf{u} \Delta a_1 \mathbf{u} \ldots \mathbf{u} \Delta a_\kappa$. Thus Γ is finite. Let

$$\Delta = \{\varphi(v_0,\mathbf{k}_{a1},\ldots,\mathbf{k}_{a\kappa})\} \mathbf{u} \{c_\xi v_0 = v_0 : \xi \in \alpha \sim \Gamma\},$$

where $\mathbf{k}_{a1},\ldots,\mathbf{k}_{a\kappa}$ are constants denoting $a_1,\ldots,a_\kappa$ respectively. Any element $c \in B$ which satisfies Δ will clearly be in C, and will be as desired. Thus it remains only to show that Δ is finitely satisfiable in $(\mathfrak{B},a_1,\ldots,a_\kappa)$. Thus let Ω be a finite subset of Δ. Let $\Lambda = \{\xi \in \alpha \sim \Gamma : \mathbf{c}_\xi v_0 = v_0 \in \Omega\}$. For any λ, $1 \leqq \lambda \leqq \kappa$, we have $\mathsf{c}_{(\Lambda)} a_\lambda = a_\lambda$, since $\Delta a_\lambda \subseteq \Gamma$. Thus $\exists v_0 \varphi(v_0,\mathbf{c}_{(\Lambda)} v_1,\ldots,\mathbf{c}_{(\Lambda)} v_\kappa)$ holds in $\mathfrak{B}$ under the assignment of $a_1,\ldots,a_\kappa$ to $v_1,\ldots,v_\kappa$. Hence by 4.1.13 the formula $\exists v_0 \varphi(\mathbf{c}_{(\Lambda)} v_0,\ldots,\mathbf{c}_{(\Lambda)} v_\kappa)$ holds under the same assignment. So choose $a_0 \in B$ such that $\varphi(\mathbf{c}_{(\Lambda)} v_0,\ldots,\mathbf{c}_{(\Lambda)} v_\kappa)$ holds in $\mathfrak{B}$ under the assignment of $a_0,\ldots,a_\kappa$ to $v_0,\ldots,v_\kappa$. Then $\mathsf{c}_{(\Lambda)} a_0$ satisfies all members of Ω in $(\mathfrak{B},a_1,\ldots,a_\kappa)$, as desired.

COROLLARY 4.1.15. *If* $\alpha \geqq \omega$ *and* ε *is an equation in* $\mathcal{L}_\alpha$, *then* $\mathsf{IGs}_\alpha \models \varepsilon$ *iff* $Ax_\alpha \models \varepsilon$.

PROOF. $\Rightarrow$: Since $\mathsf{ElLf}_\alpha \subseteq \mathsf{IGs}_\alpha$ by 3.2.8, this follows from 4.1.14. $\Leftarrow$. By 2.6.52, 3.2.10 we have $\mathsf{IGs}_\alpha = \mathsf{HSPLf}_\alpha$, so again the result follows from 4.1.14.

COROLLARY 4.1.16. *If* $\alpha < \omega$ *and* ε *is an equation in* $\mathcal{L}_\alpha$, *then* $\mathsf{IGs}_\alpha \models \varepsilon$ *iff* $Ax_\omega \models \varepsilon$.

PROOF. By 3.1.126, $\mathsf{IGs}_\alpha \models \varepsilon$ iff $\mathsf{IGs}_\omega \models \varepsilon$, so the result follows from 4.1.15.

REMARK 4.1.17. Ax_α is given by a rather simple collection of schemas for first order sentences, and so any of the usual proof systems for valid first order sentences yields a simple method for generating all equations holding in all representable CA_α's.

Now we turn to our third method for describing the equations holding in IGs_α. It is the simplest of the three, but involves a non–standard equational calculus described in the following definition.

DEFINITION 4.1.18. *Let* α *be any ordinal. A term* σ *of* $\mathcal{L}_{\alpha+\omega}$ *is* **special** *if the only variables occurring in it are of the form* $v_{2\kappa}$, $\kappa < \omega$. *For any special term* σ *we define* $Dim\sigma$ *by induction on the length of* σ: $Dim v_{2\kappa} = \alpha$, $Dim\mathbf{0} = Dim\mathbf{1} = 1$, $Dim\mathbf{d}_{\kappa\lambda} = \{\kappa,\lambda\}$ *for any* $\kappa,\lambda < \alpha+\omega$, $Dim(\sigma+\tau) = Dim\sigma \mathbf{u} Dim\tau = Dim(\sigma\cdot\tau)$, $Dim(-\sigma) = Dim\sigma$, $Dim(\mathsf{c}_\kappa\sigma) =$

$Dim\sigma \sim \{\kappa\}$ for any $\kappa<\alpha+\omega$.

A **special equational sequence** *is a finite sequence $\varepsilon_0,\ldots,\varepsilon_\kappa$ of equations in $\mathcal{L}_{\alpha+\omega}$ such that for all $\lambda\leqq\kappa$ one of the following holds:*

(1) *there exist $\xi,\eta<\omega$ such that ε_λ is $\mathsf{c}_{\alpha+\xi}v_{2\eta}=v_{2\eta}$,*

(2) *ε_λ is an axiom for $\mathsf{CA}_{\alpha+\omega}$'s, as naturally formalized from section 1.1, using the variables v_1,v_3,v_5;*

(3) *there is a term σ of $\mathcal{L}_{\alpha+\omega}$ such that ε_λ is $\sigma=\sigma$;*

(4) *there exist $\mu<\lambda$ and terms σ,τ such that ε_μ is $\sigma=\tau$ and ε_λ is $\tau=\sigma$;*

(5) *there exist $\mu,\nu<\lambda$ and terms σ,τ,ρ such that ε_μ is $\sigma=\tau$, ε_ν is $\tau=\rho$, and ε_λ is $\sigma=\rho$;*

(6) *there exist $\mu<\lambda$, $\xi<\omega$, and a term σ such that ε_λ is obtained from ε_μ by replacing all occurrences of $v_{2\xi+1}$ in ε_μ by σ;*

(7) *there exist $\mu<\lambda$, $\xi<\omega$, and a term σ with $Dim\sigma\subseteq\alpha$ such that ε_λ is obtained from ε_μ by replacing all occurrences of $v_{2\xi}$ in ε_μ by σ;*

(8) *there exist $\mu<\lambda$ and terms σ,ν,τ,ρ such that ε_μ is $\tau=\rho$ and ε_λ is $\sigma=\nu$, where ν is obtained from σ by replacing an occurrence of τ by ρ.*

We write $\vdash_{eq}\varepsilon$ if there is a special equational sequence $\varepsilon_0,\ldots,\varepsilon_\kappa$ such that $\varepsilon_\kappa=\varepsilon$.

LEMMA 4.1.19. *Let α be any ordinal, and let σ and τ be terms of $\mathcal{L}_{\alpha+\omega}$. Then $\vdash_{eq}\sigma=\tau$ iff for every $\mathsf{CA}_{\alpha+\omega}$ $\mathfrak{A}$ and every $f\in{}^\omega A$ such that $\Delta f(2\kappa)\subseteq\alpha$ for all $\kappa<\omega$ we have $\sigma^{\mathfrak{A}}f=\tau^{\mathfrak{A}}f$.*

PROOF. $\Rightarrow$. It suffices to show that for every special equational sequence $\varepsilon_0,\ldots,\varepsilon_\kappa$, every $\lambda\leqq\kappa$, and every f as above, $\sigma^{\mathfrak{A}}f=\tau^{\mathfrak{A}}f$, where ε_λ is $\sigma=\tau$. This is easily done by induction on λ, and we leave it to the reader. $\Leftarrow$. Suppose that $\nvdash_{eq}\sigma=\tau$. We show that the statement on the right also fails. Let $\mathfrak{B}$ be the absolutely free algebra similar to $\mathsf{CA}_{\alpha+\omega}$'s built from the set of terms of $\mathcal{L}_{\alpha+\omega}$. Thus, e.g., $\mathsf{c}_\xi\sigma=\mathsf{c}_\xi\sigma$ for any $\xi<\alpha$ and any term σ. Set $R=\{(\rho,\tau):\vdash_{eq}\rho=\tau\}$. It is easily verified that $R\in Co\mathfrak{B}$ and $\mathfrak{B}/R$ is a $\mathsf{CA}_{\alpha+\omega}$. Let $\mathfrak{A}=\mathfrak{B}/R$. For any $\xi<\omega$ let $f\xi=v_\xi/R$. Thus $f\in{}^\omega A$, and it is clear that $\Delta f(2\xi)\subseteq\alpha$ for all $\xi<\omega$. Also it is easy to check that $\rho^{\mathfrak{A}}f=\rho/R$ for any term ρ. Since $\sigma/R\neq\tau/R$, we have $\sigma^{\mathfrak{A}}f\neq\tau^{\mathfrak{A}}f$, as desired.

THEOREM 4.1.20. *Let α be any ordinal, and let σ and τ be special terms of $\mathcal{L}_{\alpha+\omega}$ such that $Occ(\sigma=\tau)\subseteq\alpha$. Then $\vdash_{eq}\sigma=\tau$ iff $\mathsf{IGs}_\alpha\models\sigma=\tau$.*

PROOF. Assume that $\vdash_{eq}\sigma=\tau$, and let $\mathfrak{A}\in\mathsf{IGs}_\alpha$. By 3.1.127, say $\mathfrak{A}\subseteq\mathfrak{Nr}_\alpha\mathfrak{B}$, where $\mathfrak{B}$ is a $\mathsf{CA}_{\alpha+\omega}$. Let $f\in{}^\omega A$ be arbitrary. By Lemma 4.1.19 we have $\sigma^{\mathfrak{B}}f=\tau^{\mathfrak{B}}f$, and so $\sigma^{\mathfrak{A}}f=\tau^{\mathfrak{A}}f$, as desired.

Assume that $\mathsf{IGs}_\alpha\models\sigma=\tau$. In order to apply 4.1.19, assume that $\mathfrak{A}$ is a $\mathsf{CA}_{\alpha+\omega}$ and $f\in{}^\omega A$ with $\Delta f(2\kappa)\subseteq\alpha$ for all $\kappa<\omega$. Let $\mathfrak{B}=\mathfrak{Nr}_\alpha\mathfrak{A}$, and let $g\in{}^\omega B$ be arbitrary such that $f(2\kappa)=g(2\kappa)$ for all $\kappa<\omega$. By 3.2.10 we have $\mathfrak{B}\in\mathsf{IGs}_\alpha$. Hence $\sigma^{\mathfrak{B}}g=\tau^{\mathfrak{B}}g$. But no variable $v_{2\kappa+1}$, $\kappa<\omega$, occurs in $\sigma=\tau$. Hence $\sigma^{\mathfrak{A}}f=\sigma^{\mathfrak{A}}g=\sigma^{\mathfrak{B}}g=\tau^{\mathfrak{B}}g=\tau^{\mathfrak{A}}f$, as desired.

Another result involving the notion of a finite schema of equations, due to Németi [81'], Theorem 13, goes to the very core of the motivation for our set of axioms for

CA_α's. We can summarize that motivation as follows: (1) The axioms should be simple and natural. (2) For each $\alpha \geqq \omega$, CA_α should be axiomatizable by a finite schema of equations, the same finite set for each α. (3) For each $\alpha \geqq \omega$, $\mathsf{IGs}_\alpha^{reg} \cap \mathsf{Lf}_\alpha = \mathsf{CA}_\alpha \cap \mathsf{Lf}_\alpha$, and $\mathsf{IGs}_\alpha = \mathsf{HSPDc}_\alpha = \mathsf{SNr}_\alpha \mathsf{CA}_{\alpha+\omega}$. (4) The equational axioms should apply to $\alpha < \omega$ also. Now (2), (3), (4) have a purely mathematical character, while (1) is vague. The result to be presented now shows that if only (2) and (3) are considered, then there are many other possible axioms. To formulate the theorem we introduce some notation which will not be used later. We shall be considering classes K similar to CA_α's but not contained in CA_α. We let Alg_α be the class of all algebras similar to CA_α's. If $\langle \mathsf{K}_\alpha : \alpha$ an infinite ordinal$\rangle$ is a system of classes with $\mathsf{K}_\alpha \subseteq \mathsf{Alg}_\alpha$ for each α, then it is said to be *uniformly axiomatizable by a finite schema of equations* provided that there is a finite set $E \subseteq Ev^\omega$ such that for each $\alpha \geqq \omega$, $\mathsf{K}_\alpha = \mathsf{Md}\{\gamma^+\sigma = \gamma^+\tau : \sigma = \tau \in E,\ \gamma \in {}^\omega\alpha,\ \gamma$ one–one$\}$. (This definition, which involves a class–indexed system of classes, can easily be rephrased in terms of a system of equations so as to be set–theoretically precise.) Let $\mathfrak{A} \in \mathsf{Alg}_\alpha$, $a \in A$. Then Δa is defined as in 1.6.1. We call $\mathfrak{A}$ *locally finite–dimensional*, in symbols $\mathfrak{A} \in \mathsf{Af}_\alpha$, if $|\Delta a| < \omega$ for all $a \in A$. $\mathfrak{A}$ is *dimension–complemented*, in symbols $\mathfrak{A} \in \mathsf{Ac}_\alpha$, if $\alpha \sim \Delta x$ is infinite for every $x \in A$. If $\omega \leqq \alpha \leqq \beta$ and $\mathfrak{A} \in \mathsf{Alg}_\beta$, the neat α–reduct $\mathfrak{Nr}_\alpha \mathfrak{A}$ is defined as in 2.6.28. Note, however, that it is not generally true that $Nr_\alpha \mathfrak{A}$ is closed under the relevant operations. We let $\mathsf{Nr}_\alpha \mathsf{K} = \{\mathfrak{Nr}_\alpha \mathfrak{A} : \mathfrak{A} \in \mathsf{K},\ \mathfrak{Nr}_\alpha \mathfrak{A} \in \mathsf{Alg}_\alpha\}$ for any $\mathsf{K} \subseteq \mathsf{Alg}_\beta$.

THEOREM 4.1.21. *Let* $\mathbf{Z}$ be the set of integers. Then there is a system $\langle \langle \mathsf{K}_\alpha^n : \alpha$ an infinite ordinal$\rangle : n \in \mathbf{Z}\rangle$ *with the following properties:*

(i) *For each* $n \in \mathbf{Z}$, $\langle \mathsf{K}_\alpha^n : \alpha$ *an infinite ordinal*$\rangle$ *is uniformly axiomatizable by a finite schema of equations.*

(ii) *For every* $\alpha \geqq \omega$, $\mathsf{K}_\alpha^0 = \mathsf{CA}_\alpha$.

(iii) *If* $\alpha \geqq \omega$, $m, n \in \mathbf{Z}$, *and* $m < n$, *then* $\mathsf{K}_\alpha^m \subset \mathsf{K}_\alpha^n$.

(iv) *If* $\alpha \geqq \omega$ *and* $m \in \mathbf{Z}$, *then* $\mathsf{IGs}_\alpha^{reg} \cap \mathsf{Lf}_\alpha = \mathsf{K}_\alpha^m \cap \mathsf{Af}_\alpha$ *and* $\mathsf{IGs}_\alpha = \mathsf{HSP}(\mathsf{K}_\alpha^m \cap \mathsf{Ac}_\alpha) = \mathsf{SNr}_\alpha \mathsf{K}_{\alpha+\omega}^m$.

(v) *If* $\omega \leqq \alpha \leqq \alpha + \omega \leqq \beta$, $m \in \mathbf{Z}$, *and* $\mathfrak{A} \in \mathsf{K}_\beta^m$, *then* $\mathfrak{Nr}_\alpha \mathfrak{A} \in \mathsf{Alg}_\alpha$.

PROOF. We first set $\mathsf{K}_\alpha^0 = \mathsf{CA}_\alpha$ for each $\alpha \geqq \omega$; then (iv) and (v) for $m = 0$ are clear. Now suppose that $\langle \mathsf{K}_\alpha^m : \alpha \geqq \omega\rangle$ has been defined, where $m \leqq 0$, so that the relevant conditions (i)–(v) hold; we define $\langle \mathsf{K}_\alpha^{m-1} : \alpha \geqq \omega\rangle$. Say that $E \subseteq Ev^\omega$ is a finite set of equations given by (i) for m. Now $\mathsf{IGs}_\omega \subseteq \mathsf{K}_\omega^m$ by (iv), and by 4.1.7 IGs_ω is not axiomatizable by a finite schema of equations. Hence there is an equation $e \in Ev^\omega$ such that $\mathsf{Gs}_\omega \models e$ but $E \not\models e$. Let $\langle \mathsf{K}_\alpha^{m-1} : \alpha \geqq \omega\rangle$ be determined by the finite schema $E \cup \{e\}$. The conditions (i)–(v) remain clear.

Now we must define $\langle \mathsf{K}_\alpha^m : \alpha \geqq \omega\rangle$ for $m > 0$. Let F consist of the six equations described in (B_0)–(B_2) of section 1.1, p. 161, together with equations (C_1)–(C_7) of 1.1.1 where $\kappa = 0$, $\lambda = 1$, $\mu = 2$, and the instance $\mathbf{d}_{11} = \mathbf{c}_0(\mathbf{d}_{10} \cdot \mathbf{d}_{01})$ of (C_5). For each $m \in \omega \sim 1$ let G_m consist of the two equations

$$\mathsf{c}_1\ldots\mathsf{c}_m x + -\mathsf{c}_1\ldots\mathsf{c}_m x = \mathbf{1}\,,$$
$$\mathsf{c}_1\ldots\mathsf{c}_m x \cdot -\mathsf{c}_1\ldots\mathsf{c}_m x = \mathbf{0}\,.$$

For each $m>0$ let $\langle \mathsf{K}_\alpha^m : \alpha\geqq\omega\rangle$ be determined by the finite schema $F \cup G_m$. To check (iii), suppose that $\alpha\geqq\omega$, $m,n\in\omega$, and $m<n$. Clearly $\mathsf{K}_\alpha^m\subseteq\mathsf{K}_\alpha^n$. Now let $\mathfrak{A}=\mathfrak{Cb}({}^\alpha\alpha)$, $f=\alpha\restriction Id$, $y=\mathsf{c}_0\ldots\mathsf{c}_{m-1}\{f\}$. Let $\mathfrak{B}$ be the member of Alg_α which is just like $\mathfrak{A}$ with one exception: the operation $-$ is changed, only on the element y, by setting $-y=\{f\}$. To check that $\mathfrak{B}\in\mathsf{K}_\alpha^n\sim\mathsf{K}_\alpha^m$, note that (C_7) holds in $\mathfrak{B}$ since $\{f\}\leqq-\mathsf{d}_{\kappa\lambda}$ for any distinct $\kappa,\lambda<\alpha$; $\mathfrak{B}\notin\mathsf{K}_\alpha^m$ since

$$\mathsf{c}_0\ldots\mathsf{c}_{m-1}\{f\}\cdot-\mathsf{c}_0\ldots\mathsf{c}_{m-1}\{f\}=\{f\};$$

and $\mathfrak{B}\in\mathsf{K}_\alpha^n$ since $-^{\mathfrak{B}}a=-^{\mathfrak{A}}a$ for any a different from y. Thus (iii) holds. Next we claim

(1) If $\alpha\geqq\omega$ and $m\in\omega$ then $\mathsf{K}_\alpha^m\cap\mathsf{Ac}_\alpha\subseteq\mathsf{CA}_\alpha$.

For, assume that $\mathfrak{A}\in\mathsf{K}_\alpha^m\cap\mathsf{Ac}_\alpha$ and $x\in A$; we show that $x+-x=1$ and $x\cdot-x=0$ (hence $\mathfrak{A}\in\mathsf{CA}_\alpha$). For, $\alpha\sim\Delta x$ is infinite, so $x=\mathsf{c}_{\kappa 1}\ldots\mathsf{c}_{\kappa m}x$ for some distinct $\kappa 0,\ldots,\kappa m<\alpha$, so $x+-x=1$ and $x\cdot-x=0$ by the assumption $\mathfrak{A}\in\mathsf{K}_\alpha^m$. Thus (1) holds.

From (1) it is clear that $\mathsf{IGs}_\alpha^{reg}\cap\mathsf{Lf}_\alpha=\mathsf{K}_\alpha^m\cap\mathsf{Af}_\alpha$ and $\mathsf{IGs}_\alpha=\mathsf{HSP}(\mathsf{K}_\alpha^m\cap\mathsf{Ac}_\alpha)$ for any $\alpha\geqq\omega$ and any $m\in\omega$.

Next note that if $\mathfrak{A}\in\mathsf{K}_{\alpha+\omega}^m$ then if $\Gamma\subseteq\omega$ and $|\omega\sim\Gamma|\geqq m$ we have $\mathfrak{Nr}_\alpha\mathfrak{A}\in\mathsf{CA}_\alpha$, $\mathfrak{Nr}_{\alpha+\Gamma}\mathfrak{A}\in\mathsf{CA}_{\alpha+\Gamma}$, and $\mathfrak{Nr}_\alpha\mathfrak{A}\subseteq\mathfrak{Nr}_\alpha\mathfrak{Nr}_{\alpha+\Gamma}\mathfrak{A}$, where we understand $\mathfrak{Nr}_{\alpha+\Gamma}\mathfrak{A}$ and $\mathsf{CA}_{\alpha+\Gamma}$ in the natural senses extending the "official" notions. Hence $\mathfrak{A}\in\mathsf{SNr}_\alpha\mathsf{CA}_{\alpha+\omega}$. Therefore $\mathsf{IGs}_\alpha=\mathsf{SNr}_\alpha\mathsf{K}_{\alpha+\omega}^m$, and also (v) holds. This completes the proof of 4.1.21.

Varieties of CA_α's

Now we turn to a detailed study of equational, universal, and elementary classes of CA_α's. It is natural to begin with the consideration of the lattice of equational classes, or varieties of CA_α's. Recall from 2.3.11 that every CA_α has a distributive congruence lattice. Hence several properties of varieties of CA_α's follow on general grounds. See, e.g., Jónsson [67**] and Baker [77**]. For $\alpha<\omega$, every subdirectly indecomposable CA_α is simple; this simplifies the study of the varieties. Now, starting with the case $\alpha=0$, we recall that $\mathsf{CA}_0=\mathsf{BA}$, and there are exactly two varieties of BA's, namely the class BA itself, and the class of all one-element BA's. Next we consider the case $\alpha=1$. The following simple description of the lattice of varieties of CA_1's is due to Monk [70']

THEOREM 4.1.22. *The lattice of varieties of* CA_1*'s forms a chain of order type* $\omega+1$.

PROOF. Recall from 2.3.14 that $\mathfrak{A} \in \mathsf{CA}_1$ is simple iff $|A|>1$ and the cylindrification c is given by the following simple rule:

$$\mathsf{c}_0 x = \begin{cases} 1 & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

for all $x \in A$. Now for each $\kappa \in \omega \sim 1$ let $\mathfrak{A}_\kappa$ be a simple CA_1 with exactly 2^κ elements, and set $\mathsf{K}_\kappa = \mathsf{HSP}\{\mathfrak{A}_\kappa\}$. Furthermore, let K_0 be the class of all one-element CA_1's. Clearly then $\mathsf{K}_0 \subsetneq \mathsf{K}_1 \subsetneq \mathsf{K}_2 \subsetneq \ldots \subsetneq \mathsf{CA}_1$. By Jónsson [67**] 3.4, all of these classes are distinct. We finish the proof by taking an arbitrary variety L of CA_1's and showing that it is equal to one of these. Assume that $\mathsf{L} \neq \mathsf{K}_0$. Then L has simple members. If all simple members of L are finite, then by an ultraproduct argument, L has a simple member of largest cardinality 2^κ. Clearly then $\mathsf{L} = \mathsf{K}_\kappa$. Hence suppose that some simple member of L is infinite. Taking subalgebras of this member of L, we see that all finite simple CA_1's are in L. But every CA is the union of its finitely generated subalgebras, and every finitely generated CA_1 is finite, by 2.1.10, so $\mathsf{L} = \mathsf{CA}_1$.

REMARKS 4.1.23. Each of the classes $\mathsf{K}_0, \mathsf{K}_1, \ldots$ mentioned in the proof of 4.1.22 can be characterized by a single equation in addition to $(C_0)-(C_7)$. For K_0 and K_1 we can take $v_0 = v$ and $\mathsf{c}_0 v_0 = v_0$ respectively. For K_κ with $\kappa \in \omega \sim 2$ we can take

$$\Pi_{\lambda<\mu<\nu} \mathsf{c}_0(v_\lambda \oplus v_\mu) = \mathbf{0},$$

where $\nu = 2^\kappa + 1$ and again $\sigma \oplus \tau$ abbreviates $(-\sigma \cdot \tau) + (-\tau \cdot \sigma)$.

Lucas [76'] has given a characterization of the lattice of universal classes of CA_1's. In particular, he showed that every universal class of CA_1's is finitely axiomatizable.

Turning to the case $\alpha \geq 2$, we shall give only one result about varieties of CA_α's, due to J. Johnson (unpublished): there are at least 2^ω varieties of CA_α's. For α countable, there are thus exactly 2^ω of them. Further investigations of the lattice of varieties of CA_α's have not been made. First we assume that $\alpha \geq \omega$.

THEOREM 4.1.24. *If $\alpha \geq \omega$, then there are at least 2^ω varieties of CA_α's.*

PROOF. For each $\kappa \in \omega \sim 2$ let σ_κ be the following term in the language $\mathcal{L}_\alpha$:

$$\mathbf{c}_{\{\kappa+1\}} \bar{\mathbf{d}}((\kappa+1) \times (\kappa+1)) + -\mathbf{c}_{\{\kappa\}} \bar{\mathbf{d}}(\kappa \times \kappa).$$

Clearly then for any CA_α $\mathfrak{A}$, the equation $\sigma_\kappa = \mathbf{1}$ holds in $\mathfrak{A}$ iff $\mathfrak{A}$ has no quotient of characteristic κ. For each $\Gamma \subseteq \omega \sim 2$, let $\mathsf{K}_\Gamma = \mathsf{CA}_\alpha \cap \mathsf{Md}\{\sigma_\kappa = \mathbf{1} : \kappa \in \Gamma\}$. Clearly $\mathsf{K}_\Gamma \neq \mathsf{K}_\Delta$ if $\Gamma \neq \Delta$, so the proof is complete.

To extend this theorem to the case $\alpha \in \omega \sim 2$ we need two lemmas. The first one is proved just as 2.6.32(ii) was.

LEMMA 4.1.25. *If $\alpha \leqq \beta$ and K is a variety of CA_β's, then $\mathsf{SNr}_\alpha\mathsf{K}$ is a variety of CA_α's.*

To proceed, we need the following lemma, which is a special case of Lemma 3 in Comer [69a].

LEMMA 4.1.26. *Suppose $2 \leqq \alpha < \omega$ and $|U_0| < |U_1| \cap \omega$. Let $\mathfrak{A}_i$ be the full Cs_α with base U_i for each $i<2$. Then $\mathfrak{A}_0$ cannot be isomorphically embedded in $\mathfrak{A}_1$.*

PROOF. Suppose, on the contrary, that $f \in Ism(\mathfrak{A}_0, \mathfrak{A}_1)$. For each $u \in U_0$ and $v \in U_1$ let $a_u = \{\langle u:\kappa<\alpha\rangle\}$ and $b_v = \{\langle v:\kappa<\alpha\rangle\}$. Then a_u is an atom of $\mathfrak{A}_0$ and $a_u \leqq \mathsf{d}_\alpha$; b_v is an atom of $\mathfrak{A}_1$ and $b_v \leqq \mathsf{d}_\alpha$. Furthermore, $\Sigma_{u\in U_0} a_u = \mathsf{d}_\alpha^{\mathfrak{A}_0}$ and $\Sigma_{v\in U_1} b_v = \mathsf{d}_\alpha^{\mathfrak{A}_1}$. Therefore $\Sigma_{u\in U_0} fa_u = \mathsf{d}_\alpha^{\mathfrak{A}_1}$, and so there exist $u \in U_0$ and distinct $v, w \in U_1$ such that $fa_u \cdot b_v \neq 0 \neq fa_u \cdot b_w$. Now we set

$$x = \mathsf{c}_{(\alpha\sim\{0\})}(fa_u \cdot b_v) \cdot \Pi_{0<\kappa<\alpha} \mathsf{c}_{(\alpha\sim\{\kappa\})}(fa_u \cdot b_w).$$

Now

$$x \leqq \mathsf{c}_{(\alpha\sim\{0\})} fa_u \cdot \Pi_{0<\kappa<\alpha} \mathsf{c}_{(\alpha\sim\{\kappa\})} fa_u = f(\mathsf{c}_{(\alpha\sim\{0\})} a_u \cdot \Pi_{0<\kappa<\alpha} \mathsf{c}_{(\alpha\sim\{\kappa\})} a_u) = fa_u$$

and

$$x \leqq \mathsf{c}_{(\alpha\sim\{0\})} b_v \cdot \mathsf{c}_{(\alpha\sim\{1\})} b_w \leqq -\mathsf{d}_{01}.$$

But $a_u \leqq \mathsf{d}_{01}$, so it follows that $x=0$. But

$$\mathsf{c}_{(\alpha)} x = \mathsf{c}_{(\alpha)}(fa_u \cdot b_v) \cdot \Pi_{0<\kappa<\alpha} \mathsf{c}_{(\alpha)}(fa_u \cdot b_w) = 1$$

since $\mathfrak{A}_1$ is simple. This is a contradiction.

Now we can extend 4.1.24 to $\alpha \in \omega\sim 2$.

THEOREM 4.1.27. *If $\alpha \in \omega\sim 2$, then there are exactly 2^ω varieties of CA_α's.*

PROOF. We assume the notation of the proof of 4.1.24, with α replaced by ω there. By 4.1.25, $\mathsf{SNr}_\alpha\mathsf{K}_\Gamma$ is a variety of CA_α's for every $\Gamma \subseteq \omega\sim 2$. To show that these are distinct, it suffices to show the following:

(1) For every $\kappa \in \omega\sim 2$ let $\mathfrak{A}_\kappa$ be the full Cs_α with base κ. Then $\mathfrak{A}_\kappa \in \mathsf{SNr}_\alpha\mathsf{K}_\Gamma$ iff $\kappa \notin \Gamma$.

To prove (1), if $\kappa \notin \Gamma$, then the full Cs_ω $\mathfrak{B}$ with base κ is clearly a member of K_Γ. By 3.1.117, $\mathfrak{A}_\kappa$ is a homomorphic image of $\mathfrak{Nr}_\alpha\mathfrak{B}$, so $\mathfrak{A}_\kappa \in \mathsf{SNr}_\alpha\mathsf{K}_\Gamma$ by 4.1.25. Conversely suppose $\mathfrak{A}_\kappa \in \mathsf{SNr}_\alpha\mathsf{K}_\Gamma$; but suppose that $\kappa \in \Gamma$. Say $\mathfrak{A}_\kappa \subseteq \mathfrak{Nr}_\alpha\mathfrak{B}$ with $\mathfrak{B} \in \mathsf{K}_\Gamma$. Thus $\mathfrak{B}$ has no quotient of characteristic κ. Let I be a maximal ideal in $\mathfrak{B}$. Then $\mathfrak{B}/I$ is simple,

so by 3.2.11 $\mathfrak{B}/I$ is isomorphic to a Cs_ω $\mathfrak{C}$. Let U be the base of $\mathfrak{C}$. By the above, $|U| \neq \kappa$. Let $f \in Ho(\mathfrak{B}, \mathfrak{C})$. Then $(Nr_\alpha \mathfrak{B}) \restriction f \subset Ho(\mathfrak{Nr}_\alpha \mathfrak{B}, \mathfrak{Nr}_\alpha \mathfrak{C})$. By 3.1.117, there is a homomorphism g from $\mathfrak{Nr}_\alpha \mathfrak{C}$ onto a Cs_α $\mathfrak{D}$ with base U. Since $\mathfrak{A}_\kappa$ is simple, $A_\gamma \restriction (g \circ f)$ is an isomorphism from $\mathfrak{A}_\kappa$ into $\mathfrak{D}$. Clearly $\alpha \geq \kappa$ (otherwise $\mathfrak{A}_\kappa$, hence $\mathfrak{D}$, would have characteristic κ, hence $|U| = \kappa$). Since $|A_\kappa| \geq |D|$, we have $\kappa < |U|$. This contradicts 4.1.26.

REMARK 4.1.28. Theorems 4.1.24 and 4.1.27 clearly also extend to varieties of IGs_γ's.

Miscellaneous results

Now we shall consider successively most of the classes K of CA_α's which we have introduced, indicating what EqK, UnK, and ElK look like in each case. We begin with $\mathsf{K} = \mathsf{Lf}_\alpha$, and restrict to $\alpha \geq \omega$, since $\mathsf{Lf}_\alpha = \mathsf{CA}_\alpha$ for $\alpha < \omega$, by 1.11.3(ii).

THEOREM 4.1.29. *If* $\alpha \leq \omega$, *then* $\mathsf{EqLf}_\alpha = \mathsf{IGs}_\alpha = \mathsf{UnLf}_\alpha$.

PROOF. By 2.6.52 and 3.2.10 we have $\mathsf{EqLf}_\alpha = \mathsf{SNr}_\alpha \mathsf{CA}_{\alpha+\omega} = \mathsf{IGs}_\alpha = \mathsf{SUpLf}_\alpha$.

REMARK 4.1.30. By 2.6.52 we also have $\mathsf{EqK} = \mathsf{IGs}_\alpha = \mathsf{UnLf}_\alpha$ for any class K such that $\mathsf{Lf}_\alpha \subseteq \mathsf{K} \subseteq \mathsf{SNr}_\alpha \mathsf{CA}_{\alpha+\omega}$, in particular for the important class Dc_α (still assuming $\alpha \geq \omega$). We have already observed above that $\mathsf{ElLf}_\alpha = \mathsf{ElDc}_\alpha$ for $\alpha \geq \omega$, and ElLf_α has been described in a simple fashion; see 4.1.14.

THEOREM 4.1.31. *For* $\alpha \geq \omega$ *we have* $\mathsf{UnLf}_\alpha \neq \mathsf{ElLf}_\alpha$.

PROOF. See 2.6.53.

Now we turn to the class Mn_α of minimal CA_α's. Note that Mn_0 consists of the one- and two-element BA's, while $\mathsf{SPCs}_0 = \mathsf{CA}_0 = \mathsf{BA}$. Hence $\mathsf{EqMn}_0 = \mathsf{CA}_0$. For $\alpha > 0$ we have the following unpublished result of Andréka and Németi:

THEOREM 4.1.32. *If* $\alpha \geq 1$ *then* $\mathsf{EqMn}_\alpha \subset \mathsf{SPCs}_\alpha$.

PROOF. We have $\mathsf{EqMn}_\alpha \subseteq \mathsf{SPCs}_\alpha$ by 3.2.12. First take the case $2 \leq \alpha < \omega$. Here we can use the equation $\mathbf{c}_{\{\alpha \sim 1\}} x = \mathbf{c}_{\{\alpha\}} x$, recalling 2.1.22. The case $\alpha = 1$ is clear. It remains to consider the case $\alpha \geq \omega$. The following proof actually works for any $\alpha \geq 3$. (For this proof, cf. 3.1.38.) Let δ be the term $\mathbf{c}^\partial_{(3)}(-\mathbf{c}_2 x + -\mathbf{s}^0_1 \mathbf{s}^1_2 \mathbf{c}_2 x + \mathbf{s}^1_2 x)$ and ρ the term $\mathbf{c}^\partial_{(2)}(-\mathbf{d}_{01} + -\mathbf{c}_2 x)$. Then let τ be the term $\delta \cdot \rho \cdot \mathbf{c}^\partial_0 \mathbf{c}_1 x$. We claim that $\tau = \mathbf{0}$ holds in Mn_α but not in IGs_α.

To show that $\tau = \mathbf{0}$ holds in Mn_α, by 3.2.12 it suffices to take any $\mathfrak{A} \in \mathsf{Mn}_\alpha \cap \mathsf{Cs}_\alpha$ and show that it holds in $\mathfrak{A}$. Suppose not, and choose $a \in A$ such that $\tau(a) \neq 0$. Say $\mathfrak{A}$ has base U. Choose $u \in \tau(a)$. Now we define

$$R = \{(r,s) \in {}^2U : u^{01}_{rs} \in \mathsf{c}_2 a\}.$$

Now R is transitive. For assume that $rRsRt$. Thus $u^{01}_{rs} \in \mathsf{c}_2 a$ and $u^{01}_{st} \in \mathsf{c}_2 a$, so $u^{012}_{rst} \in \mathsf{c}_2 a \cdot \mathsf{s}^0_1 \mathsf{s}^1_2 \mathsf{c}_2 a$. Since $u \in \delta(a)$, it follows that $u^{012}_{rst} \in \mathsf{s}^1_2 a$. Hence $u^{01}_{rt} \in \mathsf{c}_2 a$. i.e., rRt, as desired. Also R is antireflexive. For, $u \in \rho(a)$, and so for any $r \in U$ we have $u^{01}_{rr} \notin \mathsf{c}_2 a$ and so $r \not R r$. Next, $DoR = U$, for if we are given $r \in U$ then $u^0_r \in \mathsf{c}_1 a$ since $u \in \mathsf{c}^\partial_0 \mathsf{c}_1 a$, and so there is an $s \in U$ with $u^{01}_{rs} \in a$, hence rRs.

By these three properties of R there is a one–one $v \in {}^\omega U$ such that $v_\kappa R v_{\kappa+1}$ for all $\kappa < \omega$. Now by 2.1.17 and 3.1.70 there is a finite $\Gamma \subseteq \alpha$ such that $\mathsf{c}_2 a \in Sg^{\mathfrak{B}\restriction\mathfrak{A}} B$, where $B = \{\mathsf{d}_{\kappa\lambda} : \kappa, \lambda \in \Gamma\}$. Let $W = \{u\kappa : \kappa \in \Gamma \cup 3\}$. Thus W is a finite subset of U. Hence we may choose $\kappa < \lambda < \omega$ such that $v_\kappa, v_\lambda \notin W$. Set $r = v_\kappa$, $s = v_\lambda$. Thus rRs and so $u^{01}_{rs} \in \mathsf{c}_2 a$. Now for any $\mu, \nu \in \Gamma$ we have $u^{01}_{rs} \in \mathsf{d}_{\mu\nu}$ iff $u^{01}_{sr} \in \mathsf{d}_{\mu\nu}$, by the choice of κ and λ. Hence for any $b \in Sg^{\mathfrak{B}\restriction\mathfrak{A}} B$ we have $u^{01}_{rs} \in b$ iff $u^{01}_{rs} \in b$. Therefore $u^{01}_{sr} \in \mathsf{c}_2 a$. But then sRr, hence rRr. contradiction.

It remains only to show that $\tau = \mathbf{0}$ does not hold in IGs_α. Let $\mathfrak{A}$ be the full Cs_α with base ω, and set $a = \{u \in {}^\alpha\omega : u0 < u1\}$. It is easily checked that $\tau(a) = 1$.

THEOREM 4.1.33. *Mn_α is not an equational class.*

PROOF. If $\mathfrak{A} \in \mathsf{Mn}_\alpha$ and $|A| > 1$, then $\mathfrak{A} \times \mathfrak{A} \notin \mathsf{Mn}_\alpha$. The theorem follows.

THEOREM 4.1.34. If $\alpha < \omega$, *then Mn_α is a universal class.*

PROOF. By 2.4.69, there is an upper bound $\prod_{\kappa \subset \alpha} 2^{\pi(\kappa,\alpha)}$ for $|A|$, for any $\mathfrak{A} \in \mathsf{Mn}_\alpha$, where $\pi(\kappa,\alpha) = \pi_\kappa \alpha$ is defined in 2.4.68. Hence the theorem follows by a general model–theoretic argument.

This theorem does not extend to the case $\alpha \geqq \omega$, since we have already observed in 2.4.59 that Mn_α is not closed under ultraproducts for $\alpha \geqq \omega$:

THEOREM 4.1.35. *For $\alpha \geqq \omega$, Mn_α is not an elementary class.*

THEOREM 4.1.36. $\mathsf{UnMn}_\alpha \subset \mathsf{EqMn}_\alpha$.

PROOF. By 4.1.33 and 4.1.34 we may assume that $\alpha \geqq \omega$. Now the universal sentence

$$\forall x (x \cdot \mathsf{c}^\partial_0 \mathsf{d}_{01} = \mathbf{0} \vee x \cdot \mathsf{c}^\partial_0 \mathsf{d}_{01} = \mathsf{c}^\partial_0 \mathsf{d}_{01})$$

holds in every Mn_α, by 2.1.20, but it fails in $\mathfrak{A} \times \mathfrak{A} \in \mathsf{EqMn}_\alpha$ for any two–element

discrete $\mathfrak{A} \in \mathsf{Mn}_\alpha$.

The next theorem is due to Andréka, Németi [81'] 8.21.2.

THEOREM 4.1.37. $\mathsf{ElMn}_\alpha \subset \mathsf{UnMn}_\alpha$ *for* $\alpha \geqq \omega$.

PROOF. Recall from 2.1.20 that the sentence

$$\forall x(\mathcal{A}(x) \rightarrow x = \mathsf{c}_0^\partial \mathsf{d}_{0;})$$

holds in every Mn_α, where $\mathcal{A}(x)$ expresses that x is an atom. Hence we can prove 4.1.37 by finding a subalgebra $\mathfrak{A}$ of an ultrapower of some Mn_α such that $\mathfrak{A}$ has an atom different from $\mathsf{c}_0^\partial \mathsf{d}_{01}$. Let $\mathfrak{B}$ be an $\mathsf{Mn}_\alpha \cap \mathsf{Cs}_\alpha$ with base 2. Let $I = \{\Gamma \subseteq \alpha : |\Gamma| < \omega\}$, and let F be an ultrafilter over I such that $\{\Omega \in I : \Gamma \subseteq \Omega\} \in F$ for every $\Gamma \in I$. Let $\mathfrak{C} = {}^I\mathfrak{B}/\bar{F}$. For each $\Gamma \in I$ let $x_\Gamma = \mathsf{d}_\Gamma$, and set $y = x/\bar{F}$. Let $\mathfrak{A} = \mathfrak{Sg}^{\mathfrak{C}}\{y\}$. We claim that y is an atom of $\mathfrak{A}$ different from $\mathsf{c}_0^\partial \mathsf{d}_{0;}$. Now $\mathsf{c}_0^\partial \mathsf{d}_{01} = 0$ in $\mathfrak{B}$, and hence also in $\mathfrak{A}$. Clearly $y \neq 0$, so $y \neq \mathsf{c}_0^\partial \mathsf{d}_{01}$. To show that y is an atom in $\mathfrak{A}$ we shall make use of the fact that $\mathfrak{A}$ is representable (by 3.1.97). Let f be an isomorphism of $\mathfrak{A}$ into a CA_α $\mathsf{P}_{j \in J} \mathfrak{D}_j$, each $\mathfrak{D}_j$ a Ws_α, say with base U_j. Note that $\mathsf{c}_{\{3\}}\bar{\mathsf{d}}(3 \times 3) + -\mathsf{c}_{\{2\}}\bar{\mathsf{d}}(2 \times 2) = 0$ in $\mathfrak{B}$, hence in $\mathfrak{A}$, hence in each $\mathfrak{D}_j$. So $|U_j| = 2$. We may assume that $(pj_j \circ f)^* \mathfrak{A} = \mathfrak{D}_j$ for each $j \in J$. Let $K = \{j \in J : (fy)_j \neq 0\}$. Note that $y \leqq \mathsf{d}_{\kappa\lambda}$ for all $\kappa, \lambda < \alpha$. Hence for any $j \in K$, $(fy)_j$ has the form $\{(u : \kappa < \alpha)\}$ for some $u \in U_j$. Since $\mathfrak{D}_j$ is generated by $(fy)_j$, it follows that for any $j, k \in K$ there is an isomorphism g_{jk} of $\mathfrak{D}_j$ onto $\mathfrak{D}_k$ such that $g_{jk}((fy)_j) = (fy)_k$. Now suppose $0 \neq z \leqq y$, with $z \in A$. Choose $j \in K$ such that $(fz)_j \neq 0$. There is a term τ in $\mathcal{L}_\alpha$ with one variable $\mathbf{x}$ such that $z = \tau^{\mathfrak{A}}(y)$. Then $(fz)_k = \tau((fy)_k)$ for every $k \in K$. Now let $k \in K$ be arbitrary. We have $(fz)_j = (fy)_j = \tau((fy)_j)$ so, applying g_{jk}, $(fy)_k = \tau((fy)_k)$. Hence $(fz)_k = (fy)_k$. Therefore $z = y$, as desired.

REMARK 4.1.38. Andréka and Németi [81'] have also shown that $\mathsf{SPMn}_\alpha = \mathsf{HSPMn}_\alpha$ for $0 < \alpha < \omega$, and $\mathsf{SPMn}_\alpha \neq \mathsf{SPUpMn}_\alpha$ for $\alpha \geqq \omega$. It is not known if $\mathsf{HSPMn}_\alpha = \mathsf{SPUpMn}_\alpha$ or $\mathsf{SUpMn}_\alpha = \mathsf{HSUpMn}_\alpha$ for $\alpha \geqq \omega$.

Now we turn to the class Ss_α of semisimple CA_α's, which is only of interest for $\alpha \geqq \omega$, since $\mathsf{Ss}_\alpha = \mathsf{CA}_\alpha$ for $\alpha < \omega$ by 2.4.53. By 2.6.50 and 2.6.53 we have:

THEOREM 4.1.39. *If* $\alpha \geqq \omega$, *then* $\mathsf{EqSs}_\alpha = \mathsf{IGs}_\alpha = \mathsf{UnSs}_\alpha$.

By 2.4.59 we have:

THEOREM 4.1.40. *If* $\alpha \geqq \omega$, *then* $\mathsf{Ss}_\alpha \subset \mathsf{ElSs}_\alpha$.

The argument used to prove 4.1.37 also yields the following, making use of 2.4.54(ii):

THEOREM 4.1.41. *If* $\alpha \geqq \omega$, *then* $\mathsf{ElSs}_\alpha \subset \mathsf{UnSs}_\alpha$.

The next classes we consider are $\mathsf{SRd}_\alpha\mathsf{CA}_\beta$, where $\alpha<\beta$. By 2.6.8 and 2.6.9 we have $\mathsf{CA}_\alpha=\mathsf{SRd}_\alpha\mathsf{CA}_\beta$ if $\alpha\leqq 1$ or $\alpha\leqq\omega$, for any $\beta>\alpha$. By 2.6.14(i) we have $\mathsf{EqSRd}_\alpha\mathsf{CA}_\beta\subset\mathsf{CA}_\alpha$ if $2\leqq\alpha<\omega$ and $\beta>\alpha$. We restrict our further discussion to the case $2\leqq\alpha<\omega$. Since clearly $\mathsf{UpRd}_\alpha\mathsf{CA}_\beta=\mathsf{Rd}_\alpha\mathsf{CA}_\beta$ whenever $\alpha\leqq\beta$, we have $\mathsf{UnSRd}_\alpha\mathsf{CA}_\beta=\mathsf{SRd}_\alpha\mathsf{CA}_\beta$. On the other hand, by 2.6.61 we have $\mathsf{EqSRd}_\alpha\mathsf{CA}_\beta\neq\mathsf{SRd}_\alpha\mathsf{CA}_\beta$ whenever $2\leqq\alpha<\omega$ and $\alpha+2\leqq\beta$ (it is not known whether this extends to the case $\beta=\alpha+1$). This implies that $\mathsf{SRd}_\alpha\mathsf{CA}_\beta\neq\mathsf{IGs}_\alpha$ whenever $3\leqq\alpha<\omega$ and $\alpha<\beta$, by 2.6.16(i).

It is natural to also consider the class $\mathsf{Rd}_\alpha\mathsf{CA}_\beta$ itself. Clearly $\mathsf{Rd}_0\mathsf{CA}_\beta=\mathsf{CA}_0=\mathsf{BA}$ for every ordinal β. For $\alpha>0$ we have the following two results of Andréka and Németi [81'], 8.5.

THEOREM 4.1.42. *If* $0<\alpha<\omega$ *and* $\alpha<\beta$, *then* $\mathsf{ElRd}_\alpha\mathsf{CA}_\beta\subset\mathsf{CA}_\alpha$.

PROOF. First we suppose $\alpha>1$. We consider the following sentence φ of $\mathcal{L}_\alpha$:

$$\exists x(\mathbf{c}_0x\neq x)\rightarrow\exists x(\mathbf{c}_0x\neq x\wedge\mathbf{c}_1x=x\wedge\ldots\wedge\mathbf{c}_{\alpha-1}x=x)$$

We claim that φ holds in $\mathsf{Rd}_\alpha\mathsf{CA}_\beta$ but not in CA_α. Suppose $\mathfrak{B}\in\mathsf{CA}_\beta$ and $\mathfrak{A}=\mathfrak{Rb}_\alpha\mathfrak{B}$. Assuming $\mathfrak{A}$ is non-discrete, so is $\mathfrak{B}$, and $\mathsf{d}_{0\alpha}\in A$ satisfies the conclusion of φ. Thus φ holds in $\mathsf{Rd}_\alpha\mathsf{CA}_\beta$. If we take $\mathfrak{A}$ to be a $\mathsf{Mn}_\alpha\mathbf{n}\mathsf{Cs}_\alpha$ with base 2, then φ fails in $\mathfrak{A}$ by 2.1.22.

For $\alpha=1$ we can take the sentence $\exists x(\mathbf{c}_0x\neq x)\rightarrow\exists x(\mathbf{c}_0x\neq\mathbf{1})$.

THEOREM 4.1.43. *If* $\omega\leqq\alpha\leqq\beta$, *then* $\mathsf{ElRd}_\alpha\mathsf{CA}_\beta=\mathsf{CA}_\alpha$.

PROOF. It suffices to show that any CA_α $\mathfrak{A}$ is elementarily equivalent to a member of $\mathsf{Rd}_\alpha\mathsf{CA}_\beta$. Let $I=\{\Gamma:\Gamma\subseteq\beta,\ |\Gamma|<\omega\}$. Let F be an ultrafilter on I such that $\{\Omega:\Gamma\subseteq\Omega\in I\}\in F$ for all $\Gamma\in I$. For each $\Gamma\in I$, let $\rho\Gamma$ be a one-one function from Γ into α such that $(\Gamma\mathbf{n}\alpha)1Id\subseteq\rho\Gamma$. Furthermore, let $\mathfrak{B}_\Gamma$ be an algebra similar to CA_β's such that for all $\kappa,\lambda\in\Gamma$,

$$\mathsf{c}_\kappa^{\mathfrak{B}_\Gamma}=\mathsf{c}_{(\rho\Gamma)\kappa}^{\mathfrak{A}}\text{ and }\mathsf{d}_{\kappa\lambda}^{\mathfrak{B}_\Gamma}=\mathsf{d}_{(\rho\Gamma)\kappa,(\rho\Gamma)\lambda}^{\mathfrak{A}}.$$

It is easily verified that $\mathsf{P}_{\Gamma\in I}\mathfrak{B}_\Gamma/\bar{F}$ is a CA_β, and $\langle\langle a:\Gamma\in I\rangle/\bar{F}:a\in A\rangle$ is an isomorphism from $\mathfrak{A}$ onto an elementary subalgebra of $\mathfrak{Rb}_\alpha\mathsf{P}_{\Gamma\in I}\mathfrak{B}_\Gamma/\bar{F}$.

Next we discuss the classes $\mathsf{SNr}_\alpha\mathsf{CA}_\beta$. For $\beta\geqq\alpha+\omega$ we have $\mathsf{SNr}_\alpha\mathsf{CA}_\beta=\mathsf{IGs}_\alpha$ by 3.2.10, for $\alpha\leqq 1$ we have $\mathsf{SNr}_\alpha\mathsf{CA}_\beta=\mathsf{CA}_\alpha$ for any $\beta\geqq\alpha$ by 2.6.39, while $\mathsf{CA}_2\subset\mathsf{SNr}_2\mathsf{CA}_\beta=\mathsf{IGs}_2$ for any $\beta>2$ by 3.2.65. Each class $\mathsf{SNr}_\alpha\mathsf{CA}_\beta$ is a variety by 2.6.32(ii). If $\alpha\leqq\beta<\gamma$, then $\mathsf{SNr}_\alpha\mathsf{CA}_\gamma\subseteq\mathsf{SNr}_\alpha\mathsf{CA}_\beta$. We know that $\mathsf{SNr}_\alpha\mathsf{CA}_{\alpha+\omega}=\bigcap_{\kappa<\omega}\mathsf{SNr}_\alpha\mathsf{CA}_{\alpha+\kappa}$, and the ω-sequence $\mathsf{CA}_\alpha\supseteq\mathsf{SNr}_\alpha\mathsf{CA}_{\alpha+1}\supseteq\mathsf{SNr}_\alpha\mathsf{CA}_{\alpha+2}\supseteq\ldots$ is *not* eventually constant for $\alpha\geqq 3$; see 2.6.34 and 3.2.10. It is still open whether all these classes are distinct (Problem 2.12 of Part I). (The abstract Maddux [77'] asserts a solution, but this claim has been withdrawn by the author; the statement in Henkin, Monk, Tarski [81'], p.127, was based on this abstract, and so is also erroneous.)

Concerning the classes $\mathsf{Nr}_\alpha\mathsf{CA}_\beta$ themselves, we should first mention that $\mathsf{HNr}_\alpha\mathsf{CA}_\beta = \mathsf{Nr}_\alpha\mathsf{CA}_\beta$ whenever $\alpha \leqq \beta$, $\mathsf{SNr}_\alpha\mathsf{CA}_\beta = \mathsf{Nr}_\alpha\mathsf{CA}_\beta$ whenever $\alpha \leqq 1$ and $\alpha \geqq \beta$, while $\mathsf{SNr}_\alpha\mathsf{CA}_\beta \neq \mathsf{Nr}_\alpha\mathsf{CA}_\beta$ whenever $1 < \alpha < \beta$. These are results of Németi [83c'] which solve Problem 2.11 of Part I. Andréka and Németi [81'] 8.6 showed that $\mathsf{Nr}_2\mathsf{CA}_\beta \subset \mathsf{ElNr}_2\mathsf{CA}_\beta$ whenever $2 < \beta$; their proof is too long to include here. Finally, $\mathsf{ElNr}_\alpha\mathsf{CA}_\beta \subset \mathsf{SNr}_\alpha\mathsf{CA}_\beta$ for all $2 \leqq \alpha < \beta$ by Andréka, Németi [81'] 8.8(i); again we omit the proof because of it length. It is open whether $\mathsf{Nr}_\alpha\mathsf{CA}_\beta$ is an elementary class for $3 \leqq \alpha < \beta$.

The remaining classes we consider are various classes of set algebras considered in section 3.1. We begin with the class Cs_α for $\alpha < \omega$. For $2 \leqq \alpha < \omega$ we have $\mathsf{EqCs}_\alpha = \mathsf{IGs}_\alpha$ by 3.1.108, and $\mathsf{ICs}_\alpha = \mathsf{UnCs}_\alpha \subset \mathsf{EqCs}_\alpha$ by 3.1.70 and 3.1.107.

Still taking $\alpha < \omega$, it is natural to look at the classes ${}_\kappa\mathsf{Cs}_\alpha$ for various κ. If $\kappa < \omega$ then every ${}_\kappa\mathsf{Cs}_\alpha$ has power at most 2^λ, where $\lambda = \kappa^\alpha$, so we get:

THEOREM 4.1.44. *If* $0 < \alpha, \kappa < \omega$, *then* $\mathsf{I}_\kappa\mathsf{Cs}_\alpha$ *is a universal class.*

Clearly we have $\mathsf{I}_\kappa\mathsf{Cs}_\alpha \subset \mathsf{Eq}_\kappa\mathsf{Cs}_\alpha$ for $0 < \kappa, \alpha < \omega$. Furthermore:

THEOREM 4.1.45. *If* $2 \leqq \alpha < \omega$ *and* $0 < \kappa < \omega$, *then* $\mathsf{Eq}_\kappa\mathsf{Cs}_\alpha = \mathsf{I}_\kappa\mathsf{Gs}_\alpha$.

PROOF. By 3.1.77 and its proof we have $\mathsf{SP}_\kappa\mathsf{Cs}_\alpha = \mathsf{I}_\kappa\mathsf{Gs}_\alpha$. Now by the first part of the proof of 3.1.103 we have $\mathsf{H}_\kappa\mathsf{Gs}_\alpha \subseteq \mathsf{SUp}_\kappa\mathsf{Gs}_\alpha$, and by the proof of 3.1.92 we have $\mathsf{SUp}_\kappa\mathsf{Gs}_\alpha \subseteq \mathsf{I}_\kappa\mathsf{Gs}_\alpha$, so the theorem follows.

Theorem 4.1.45 does not extend to $\alpha = 1$; see 3.1.17.

Now we consider $\alpha < \omega$, but $\kappa \geqq \omega$. Clearly $\mathsf{I}_\infty\mathsf{Cs}_\alpha$ is not a variety. The following result is due to Andréka and Németi.

THEOREM 4.1.46. *Suppose* $0 < \alpha < \omega$ *and* $\kappa \geqq \omega$. *Then* $\mathsf{El}_\kappa\mathsf{Cs}_\alpha = \mathsf{Un}_\kappa\mathsf{Cs}_\alpha = \mathsf{I}_\infty\mathsf{Cs}_\alpha$.

PROOF. Clearly $\mathsf{El}_\kappa\mathsf{Cs}_\alpha \subseteq \mathsf{Un}_\kappa\mathsf{Cs}_\alpha \subseteq \mathsf{I}_\infty\mathsf{Cs}_\alpha$. Hence it suffices to take an arbitrary $\mathfrak{A} \in {}_\infty\mathsf{Cs}_\alpha$ and show that $\mathfrak{A}$ is elementarily equivalent to some $\mathfrak{B} \in {}_\kappa\mathsf{Cs}_\alpha$. By 3.1.112, $\mathfrak{A}$ is isomorphic to some Cs_α $\mathfrak{C}$ with base of power $\geqq \kappa$. Since $\alpha < \omega$, the language $\mathcal{L}_\alpha$ is countable. Hence $\mathfrak{C}$ has a countable elementary substructure $\mathfrak{D}$. Note that $\mathfrak{D}$ is a Cs_α with the same base as $\mathfrak{C}$. Hence by 3.1.45(i)(b), $\mathfrak{D}$ is isomorphic to a ${}_\kappa\mathsf{Cs}_\alpha$, as desired.

THEOREM 4.1.47. *If* $1 < \alpha < \omega$ *and* $\kappa \geqq \omega$, *then* $\mathsf{I}_\infty\mathsf{Cs}_\alpha \subset \mathsf{I}_\infty\mathsf{Gs}_\alpha = \mathsf{HSP}_\kappa\mathsf{Cs}_\alpha$.

PROOF. Clearly $\mathsf{I}_\infty\mathsf{Cs}_\alpha \subset \mathsf{I}_\infty\mathsf{Gs}_\alpha$. Now, using the same results cited in the proof of 4.1.45, we have

$$\begin{aligned}\mathsf{HSP}_\kappa\mathsf{Cs}_\alpha &= \mathsf{H}_\kappa\mathsf{Gs}_\alpha \subseteq \mathsf{SUp}_\infty\mathsf{Gs}_\alpha \\ &\subseteq \mathsf{I}_\infty\mathsf{Gs}_\alpha = \mathsf{SP}_\infty\mathsf{Cs}_\alpha \\ &= \mathsf{SPSUp}_\kappa\mathsf{Cs}_\alpha \quad \text{by } 4.1.46 \\ &\subseteq \mathsf{HSP}_\kappa\mathsf{Cs}_\alpha.\end{aligned}$$

This finishes the proof.

Now we take $\alpha \geqq \omega$. Here we discuss the following classes: Cs_α, Cs_α^{reg}, Ws_α, $\mathsf{Cs}_\alpha^{reg} \cap \mathsf{Lf}_\alpha$, ${}_\kappa\mathsf{Cs}_\alpha$ and ${}_\kappa\mathsf{Cs}_\alpha^{reg} \cap \mathsf{Lf}_\alpha$ for various κ.

By 3.1.108 we have $\mathsf{Eq}\,\mathsf{Cs}_\alpha = \mathsf{I}\mathsf{Gs}_\alpha$ for all $\alpha \geqq 2$. The following characterization of $\mathsf{Un}\,\mathsf{Cs}_\alpha$ is due to Andréka and Németi [81'] 7.1:

THEOREM 4.1.48. *For $\alpha \geqq \omega$ we have:*

$$\begin{aligned}\mathsf{Un}\,\mathsf{Cs}_\alpha &= \mathsf{I}\{\mathfrak{A} \in \mathsf{Gs}_\alpha : |A| \leqq 2 \text{ or } \mathfrak{A} \text{ has characteristic } \kappa \neq 1\} \\ &= \mathsf{I}\mathsf{Gs}_\alpha \cap \mathsf{Md}(\{\forall x(x = \mathbf{0} \vee x = \mathbf{1} \vee \mathsf{c}_{(2)}\bar{\mathsf{d}}(2 \times 2) = \mathbf{1} \\ &\qquad \cup \{\mathsf{c}_{(\kappa)}\bar{\mathsf{d}}(\kappa \times \kappa) = \mathbf{0} \vee \mathsf{c}_{(\kappa)}\bar{\mathsf{d}}(\kappa \times \kappa) = \mathbf{1} : \kappa \in \omega \sim 2\}).\end{aligned}$$

PROOF. The second equality is clear, as is the part $\subseteq$ of the first inclusion. Now suppose $\mathfrak{A} \in \mathsf{Gs}_\alpha$ with characteristic $\kappa \neq 1$. Thus $\mathfrak{A} \in {}_\kappa\mathsf{Gs}_\alpha$ if $\kappa < \omega$, and $\mathfrak{A} \in {}_\infty\mathsf{Gs}_\alpha$ if $\kappa = 0$. By 3.1.136 it follows that $\mathfrak{A} \in \mathsf{SUpCs}_\alpha$, as desired.

From 4.1.48 it follows, of course, that $\mathsf{Un}\,\mathsf{Cs}_\alpha \subset \mathsf{Eq}\,\mathsf{Cs}_\alpha$ for $\alpha \geqq \omega$. We do not know whether $\mathsf{Un}\,\mathsf{Cs}_\alpha = \mathsf{El}\,\mathsf{Cs}_\alpha$. Note that $\mathsf{I}\mathsf{Cs}_\alpha$ is not an elementary class, since it is not closed under ultraproducts (see 3.1.100).

Next we consider the class Cs_α^{reg}. By 3.1.107 we have $\mathsf{Eq}\,\mathsf{Cs}_\alpha^{reg} = \mathsf{I}\mathsf{Gs}_\alpha$ for any $\alpha \geqq \omega$. By the argument for 4.1.48 we have $\mathsf{Un}\,\mathsf{Cs}_\alpha^{reg} = \mathsf{Un}\,\mathsf{Cs}_\alpha$ for $\alpha \geqq \omega$. Andréka and Németi have observed that $\mathsf{El}\,\mathsf{Cs}_\alpha^{reg} \subset \mathsf{Un}\,\mathsf{Cs}_\alpha^{reg}$. To prove this we need the following two general theorems about cylindric algebras.

THEOREM 4.1.49. *Let $\alpha \geqq \omega$, and suppose that $\mathfrak{A}$ is a CA_α of characteristic $\kappa > 0$. Suppose $x \in A$ is an atom. Then for any finite $\Gamma \subseteq \alpha$, $\mathsf{c}_{(\Gamma)}x$ is a finite sum of atoms.*

PROOF. Clearly it suffices to take the case $|\Gamma| = 1$; say $\Gamma = \{\gamma\}$. By 3.2.11 we may assume that $\mathfrak{A}$ is a Gs_α. We claim that $\mathsf{c}_\gamma x$ is a sum of at most κ atoms. Suppose this is not the case. Then there exist non-zero pairwise disjoint elements $y_0, \ldots, y_\kappa \leqq \mathsf{c}_\gamma x$. Since $\alpha \geqq \omega$ and $0 < \kappa < \omega$, there is a $\Gamma \subseteq \alpha \sim \{\gamma\}$ with $|\Gamma| = \kappa + 2$ such that $x \leqq \mathsf{d}_\Gamma$; say $\Gamma = \{\beta_0, \ldots, \beta_{\kappa+1}\}$; let $\Omega = \{\beta_0, \ldots, \beta_\kappa\}$ and $\delta = \beta_{\kappa+1}$. Now we set

$$z = \Pi_{\xi \leqq \kappa} \mathsf{c}_\gamma(\mathsf{c}_{(\Omega)} y_\xi \cdot \mathsf{d}_{\gamma, \beta\xi}).$$

Note that

$$\begin{aligned}\mathsf{c}_{(\Omega)} z &= \Pi_{\xi \leqq \kappa} \mathsf{c}_{\beta\xi} \mathsf{c}_\gamma(\mathsf{c}_{(\Omega)} y_\xi \cdot \mathsf{d}_{\gamma, \beta\xi}) \\ &= \Pi_{\xi \leqq \kappa} \mathsf{c}_\gamma \mathsf{c}_{(\Omega)} y_\xi \geqq x\end{aligned}$$

since $\mathsf{c}_\gamma y_\xi = \mathsf{c}_\gamma x$ for each $\xi \leqq \kappa$ (by 1.10.3, since $\mathsf{c}_\gamma x$ is a $\{\gamma\}$-atom). Thus $z \neq 0$. Let $u \in z$ (recall that $\mathfrak{A}$ is a Gs_α). Take any $\xi \leqq \kappa$. Then by the definition of z there is a $w_\xi \in y_\xi$ such that $(\alpha \sim (\{\gamma\} \cup \Omega)) \upharpoonright u \subseteq w_\xi$ and $w_\xi \gamma = u\beta\xi$. Since $y_\xi \leqq \mathsf{c}_\gamma x \leqq \mathsf{d}_\Gamma$, we have $w_\xi \eta = w_\xi \delta = u\delta$ for every $\eta \in \Omega$. Thus for distinct $\xi, \xi' \leqq \kappa$ the functions w_ξ and $w_{\xi'}$ differ

only possibly at γ. Since $y_\xi \cdot y_{\xi'} = 0$, it follows that $w_\xi\gamma \neq w_{\xi'}\gamma$. Thus $u\beta\xi \neq u\beta\xi'$ for $\xi \neq \xi'$, so $u \in \bigcap_{\xi<\xi'\leq\kappa} -\mathsf{d}_{\xi\xi'}$, contradicting $\mathfrak{A}$ being of characteristic κ.

COROLLARY 4.1.50. *Let $\alpha \geqq \omega$, and suppose that $\mathfrak{A}$ is a CA_α of characteristic $\kappa > 0$. If $\Sigma At\mathfrak{A}$ exists, then $\Delta\Sigma At\mathfrak{A} = 0$.*

PROOF. For any $\xi < \alpha$ we have

$$\mathsf{c}_\xi \Sigma At\mathfrak{A} = \Sigma_{x \in At\mathfrak{A}} \mathsf{c}_\xi x \geqq \Sigma At\mathfrak{A}$$

by 4.1.49.

THEOREM 4.1.51. *If $\alpha \geqq \omega$, then $\mathsf{EICs}_\alpha^{reg} \subset \mathsf{UnCs}_\alpha^{reg}$.*

PROOF. Let $p = \langle 0 : \xi < \alpha \rangle$, and let $\mathfrak{A}$ be the full Ws_α with unit element ${}^\alpha 2^{\langle p \rangle}$. Set $\mathfrak{B} = \mathfrak{Sg}^{\mathfrak{A}}\{\{p\}\}$. By 3.1.102, $\mathfrak{B} \in \mathsf{ICs}_\alpha^{reg}$. Let F be any non-principal ultrafilter on ω, and set $\mathfrak{C} = {}^\omega\mathfrak{B}/\bar{F}$. Let $y \in {}^\omega B$ be such that $y_i = \mathsf{d}_{i,i+1}$ for every $i < \omega$, and set $z = y/F$. Let $x = \langle \{p\} : i < \omega \rangle / \bar{F}$. Finally, let $\mathfrak{D} = \mathfrak{Sg}^{\mathfrak{C}}\{x, y\}$. We claim that $\mathfrak{D} \in \mathsf{UnCs}_\alpha^{reg} \sim \mathsf{EICs}_\alpha^{reg}$. Since $\mathfrak{D} \in \mathsf{SUpCs}_\alpha^{reg}$, clearly $\mathfrak{D} \in \mathsf{UnCs}_\alpha^{reg}$. To show that $\mathfrak{D} \notin \mathsf{EICs}_\alpha^{reg}$, we consider a natural sentence φ in $\mathcal{L}_\alpha$ that expresses the statement "for all v, if the characteristic is 2 and v is the sum of all atoms, then $v = 0$ or $v = 1$". By 4.1.50 and 3.1.87, φ holds in the class Cs_α^{reg}. Hence it suffices to show that φ fails in $\mathfrak{D}$. Note that $0 \neq y \neq 1$, $\Delta y = 0$, and $\mathfrak{D}$ has characteristic 2. So, it suffices to show that y is the sum of all atoms of $\mathfrak{D}$. Now $x \leqq y$, and $\mathfrak{Rl}_{-y}\mathfrak{D} \cong \mathfrak{D}/Ig\{y\}$. Under the natural homomorphism from $\mathfrak{D}$ onto $\mathfrak{D}/Ig\{y\}$, both x and y go to 0, so $\mathfrak{Rl}_{-y}\mathfrak{D} \in \mathsf{Mn}_\alpha$. By 2.1.20, $\mathfrak{Rl}_{-y}\mathfrak{D}$ is atomless. Thus it suffices to show that $\mathfrak{Rl}_y\mathfrak{D}$ is atomic. Now

(1) $\{x, y\} \subseteq \{z \in D : \text{for some } u \in Sg^{\mathfrak{C}}\{x\} \text{ we have } z \cdot y = u \cdot y\} \in Su\mathfrak{D}$.

This is easily established, using the fact that $\Delta y = 0$. From (1) it easily follows that $\mathfrak{Rl}_y\mathfrak{D} = \{u \cdot y : u \in Sg^{\mathfrak{C}}\{x\}\}$. Now $\mathfrak{Sg}^{\mathfrak{C}}\{x\}$ is isomorphic to $\mathfrak{B}$, and $\mathfrak{B}$ is subdirectly indecomposable. It follows easily that $\mathfrak{Rl}_y\mathfrak{D}$ is also subdirectly indecomposable. Hence from 2.4.46 we infer that, in $\mathfrak{D}$, $y = \Sigma\{\mathsf{c}_{(\Gamma)}x : \Gamma \subseteq \alpha,\ |\Gamma| < \omega\}$. By 4.1.48 it follows that $y = \Sigma At\mathfrak{D}$, as desired.

Note again that for $\alpha \geqq \omega$, $\mathsf{Cs}_\alpha^{reg} \subset \mathsf{EICs}_\alpha^{reg}$ by 3.1.100. This finishes the discussion of the class Cs_α^{reg}.

Turning to Ws_α we note that by 3.1.108 we have $\mathsf{EqWs}_\alpha = \mathsf{IGs}_\alpha$ for $\alpha \geqq \omega$. By the proof of 4.1.51 we have $\mathsf{EIWs}_\alpha \subset \mathsf{UnWs}_\alpha$ for $\alpha \geqq \omega$. Andréka and Németi [81'] 7.3.7 have noticed that $\mathsf{EIWs}_\alpha \subset \mathsf{EICs}_\alpha^{reg}$ for $\alpha \geqq \omega$:

THEOREM 4.1.52. $\mathsf{EIWs}_\alpha \subset \mathsf{EICs}_\alpha^{reg}$ *for* $\alpha \geqq \omega$.

PROOF. $\subseteq$ is given by 3.1.102. Now let φ be a natural sentence of $\mathcal{L}_\alpha$ which expresses the statement: "if the characteristic is 2, then the algebra is either atomic or atomless". We show that φ holds in Ws_α but not in Cs_α^{reg}. First let $\mathfrak{A}$ be any Ws_α

of characteristic 2; we may assume that the unit element of $\mathfrak{A}$ has the form ${}^{\alpha}2^{\{p\}}$. Suppose x is an atom of $\mathfrak{A}$, and y is an arbitrary non-zero element of $\mathfrak{A}$; we show that there is an atom $\leq y$. This will show that φ holds in $\mathfrak{A}$. Now

(1) x is a singleton.

In fact, (1) follows from the following observation:

(2) If $f \in x$, $g \in x$, $\xi < \alpha$, and $f\xi = g\xi$, then $f = g$.

To prove (2), suppose $\eta < \alpha$. If $x \leq \mathsf{d}_{\xi\eta}$ then $f\eta = f\xi = g\xi = g\eta$. If $x \leq -\mathsf{d}_{\xi\eta}$, then $f\eta = 1 - f\xi = 1 - g\xi = g\eta$. So $f = g$.

Say by (1) $x = \{f\}$. Let $h \in y$. Then there is a finite $\Gamma \subseteq \alpha$ such that $(\alpha \sim \Gamma) \upharpoonright f \neq h$. It follows that $\mathsf{c}_{(\Gamma)} x \cap y \neq 0$. Clearly $\mathsf{c}_{(\Gamma)} x$ is finite, so there is indeed some atom $\leq \mathsf{c}_{(\Gamma)} x \cap y \leq y$, as desired.

Now we construct a ${}_2\mathsf{Cs}_\alpha^{reg}$ in which φ fails. Choose $\Gamma \subseteq \alpha$ such that Γ and $\alpha \sim \Gamma$ are infinite. Let $x = \{\langle 0 : \xi < \alpha \rangle\}$ and $y = \{q \in {}^{\alpha}2 : q\xi = 1 \text{ for all } \xi \in \Gamma\}$. Let $\mathfrak{A}$ be the full Cs_α with base 2 and $\mathfrak{B} = \mathfrak{Sg}^{\mathfrak{A}}\{x, y\}$. Note that x and y are small (see 3.1.56). Hence by 3.1.63, $\mathfrak{B}$ is regular. Obviously x is an atom of $\mathfrak{B}$. Now we show that there is no atom $\leq y$; this will then establish that φ fails in $\mathfrak{B}$. Suppose that z is an atom of $\mathfrak{B}$ and $z \leq y$. Choose $p \in z$, and let $V = {}^{\alpha}2^{\{p\}}$. Then $rl_V \in Ho(\mathfrak{B}, \mathfrak{C})$ for some CA_α $\mathfrak{C}$. Since $p\xi = 1$ for all $\xi \in \Gamma$ and Γ is infinite, we have $x \cap V = 0$. Hence $\mathfrak{C}$ is generated by $\{y \cap V\}$. Since $\Delta(y \cap V) = \Gamma$ and $\alpha \sim \Gamma$ is infinite, we have $\mathfrak{C} \in \mathsf{Dc}_\alpha$. But $0 \neq V \cap z$ is an atom of $\mathfrak{C}$. This contradicts 1.11.8(ii).

Now we turn to the important class $\mathsf{Cs}_\alpha^{reg} \cap \mathsf{Lf}_\alpha$. By 3.1.123 we have $\mathsf{Eq}(\mathsf{Cs}_\alpha^{reg} \cap \mathsf{Lf}_\alpha) = \mathsf{I}\mathsf{Gs}_\alpha$ for $\alpha \geq \omega$ and by the proof of 4.1.48 we have $\mathsf{Un}(\mathsf{Cs}_\alpha^{reg} \cap \mathsf{Lf}_\alpha) = \mathsf{Un}\,\mathsf{Cs}_\alpha$. By 4.1.51 and 3.1.136, $\mathsf{El}(\mathsf{Cs}_\alpha^{reg} \cap \mathsf{Lf}_\alpha) \subset \mathsf{Un}(\mathsf{Cs}_\alpha^{reg} \cap \mathsf{Lf}_\alpha)$; and by 3.1.100 it is clear that $\mathsf{Cs}_\alpha^{reg} \cap \mathsf{Lf}_\alpha \subset \mathsf{El}(\mathsf{Cs}_\alpha^{reg} \cap \mathsf{Lf}_\alpha)$. We also note that $\mathsf{El}(\mathsf{Cs}_\alpha^{reg} \cap \mathsf{Lf}_\alpha) \subset \mathsf{El}\mathsf{Lf}_\alpha$; for example, $\mathsf{c}_{(2)}\overline{\mathsf{d}}(2 \times 2) = 0$ or $\mathsf{c}_{(2)}\overline{\mathsf{d}}(2 \times 2) = 1$ holds in any $\mathsf{Cs}_\alpha^{reg} \cap \mathsf{Lf}_\alpha$ but not in every Lf_α. Also note from 3.1.70(i) that any $\mathsf{Cs}_\alpha^{reg} \cap \mathsf{Lf}_\alpha$ with more than one element is simple; hence using 3.1.134, $\mathsf{I}(\mathsf{Cs}_\alpha^{reg} \cap \mathsf{Lf}_\alpha) = \mathsf{I}(\mathsf{Ws}_\alpha \cap \mathsf{Lf}_\alpha) \cup \{\mathfrak{A} \in \mathsf{CA}_\alpha : |A| = 1\}$. All of these facts are for $\alpha \geq \omega$.

Next we consider ${}_\kappa\mathsf{Cs}_\alpha$ for $1 < \kappa < \omega$ but $\alpha \geq \omega$. By 3.1.136 we then have $\mathsf{Eq}\,{}_\kappa\mathsf{Cs}_\alpha = \mathsf{Un}\,{}_\kappa\mathsf{Cs}_\alpha = \mathsf{I}_\kappa\mathsf{Gs}_\alpha$. We do not know whether $\mathsf{Un}\,{}_\kappa\mathsf{Cs}_\alpha = \mathsf{El}\,{}_\kappa\mathsf{Cs}_\alpha$. Of course ${}_\kappa\mathsf{Cs}_\alpha \subset \mathsf{El}\,{}_\kappa\mathsf{Cs}_\alpha$.

Taking $\alpha \geq \omega$ and $\kappa \geq \omega$, by 3.1.136 we have $\mathsf{Eq}\,{}_\kappa\mathsf{Cs}_\alpha = \mathsf{Un}\,{}_\kappa\mathsf{Cs}_\alpha = \mathsf{I}_\infty\mathsf{Cs}_\alpha$. Again we do not know whether $\mathsf{Un}\,{}_\kappa\mathsf{Cs}_\alpha = \mathsf{El}\,{}_\kappa\mathsf{Cs}_\alpha$, but clearly ${}_\kappa\mathsf{Cs}_\alpha \subset \mathsf{El}\,{}_\kappa\mathsf{Cs}_\alpha$.

For $\alpha \geq \omega$ and $\kappa < \omega$, by 3.1.136 we have $\mathsf{Eq}({}_\kappa\mathsf{Cs}_\alpha^{reg} \cap \mathsf{Lf}_\alpha) = \mathsf{Un}({}_\kappa\mathsf{Cs}_\alpha^{reg} \cap \mathsf{Lf}_\alpha) = \mathsf{I}_\kappa\mathsf{Gs}_\alpha$; for $\kappa \geq \omega$, both classes are equal to $\mathsf{I}_\infty\mathsf{Cs}_\alpha$. It is open whether $\mathsf{Un}({}_\kappa\mathsf{Cs}_\alpha \cap \mathsf{Lf}_\alpha) = \mathsf{El}({}_\kappa\mathsf{Cs}_\alpha \cap \mathsf{Lf}_\alpha)$, but clearly ${}_\kappa\mathsf{Cs}_\alpha \cap \mathsf{Lf}_\alpha \subset \mathsf{El}({}_\kappa\mathsf{Cs}_\alpha \cap \mathsf{Lf}_\alpha)$.

To conclude this section we discuss an algebraic form of relativization which was alluded to in 2.2.23.

DEFINITION 4.1.53. *For terms σ in $\mathcal{L}_\alpha$ not involving the variable v_0 we define a new term σ^{rl}:*

$$v_\iota^{rl} = v_\iota \cdot v_0 \qquad (\sigma + \tau)^{rl} = \sigma^{rl} + \tau^{rl}$$
$$(\sigma \cdot \tau)^{rl} = \sigma^{rl} \cdot \tau^{rl} \qquad (-\sigma)^{rl} = -\sigma^{rl} \cdot v_0$$
$$(\mathbf{c}_\kappa \sigma)^{rl} = \mathbf{c}_\kappa \sigma^{rl} \cdot v_0 \qquad \mathbf{d}_{\kappa\lambda}^{rl} = \mathbf{d}_{\kappa\lambda} \cdot v_0$$

THEOREM 4.1.54. *Suppose $\alpha < \omega$ and $\mathsf{CA}_\alpha \models \sigma = \tau$, where σ and τ are terms not involving v_0. Let $\mathfrak{A}$ be a CA_α, and assume that $x \in A$ and $\Delta x \subseteq 1$. Let $b = \prod_{\kappa<\alpha} \mathbf{s}_\kappa^0 x$. Suppose $a \in {}^\omega Rl_b \mathfrak{A}$ and $a_0 = b$. Then $(\sigma^{rl})^{\mathfrak{A}} a = (\tau^{rl})^{\mathfrak{A}} a$.*

PROOF. Let $\mathfrak{B} = \mathfrak{Rl}_b \mathfrak{A}$. Note that $\mathfrak{B} \in \mathsf{CA}_\alpha$ by 2.2.10. It is enough to show that $\rho^{\mathfrak{B}} a = (\rho^{rl})^{\mathfrak{A}} a$ for every term ρ not involving v_0. This is straightforward, and we leave it to the reader.

Using this theorem one can somewhat simplify the proofs of 2.2.20–2.2.22.

4.2. DECISION PROBLEMS

We consider here some natural decision problems for CA_α's. We restrict ourselves to the case $\alpha \leqq \omega$, where there are natural notions of effectiveness. Mainly we are concerned with decidability or undecidability of the equational or the full first-order theories of CA_α or IGs_α. The main results are: (1) the equational theories of CA_α and of IGs_α are decidable; (2) if $3 \geqq \alpha \leqq \omega$ and $\mathsf{IGs}_\alpha \subseteq \mathsf{K} \subseteq \mathsf{CA}_\alpha$ then the equational theory of K is undecidable; (3) the elementary theory of CA_α is undecidable.

We begin, however, with a minor result:

THEOREM 4.2.1. *If $\alpha < \omega$, then the first-order theory of Mn_α is decidable.*

PROOF. By 2.1.17, any Mn_α has at most 2^λ elements, where $\lambda = 2^\kappa$ and $\kappa = \alpha^2 + \alpha$. Hence the conclusion is obvious.

We do not know whether 4.2.1 extends to $\alpha = \omega$. We present one small result along these lines; it depends of the following lemma.

LEMMA 4.2.2. *Let σ and τ be terms of $\mathcal{L}_\omega$. Then $\mathsf{Mn}_\omega \models \sigma = \tau$ iff for all $\alpha < \omega$, if $Occ(\sigma = \tau) \subseteq \alpha$ then $\mathsf{Mn}_\alpha \models \sigma = \tau$. (Recall from 4.1.11 the definition of $Occ(\sigma = \tau)$.)*

PROOF. $\Rightarrow$. Assume that $\mathsf{Mn}_\omega \models \sigma = \tau$, $\alpha < \omega$, $Occ(\sigma = \tau) \subseteq \alpha$, and $\mathfrak{A} \in \mathsf{Mn}_\alpha$. By 2.6.57 choose $\mathfrak{B} \in \mathsf{Mn}_\omega$ such that $\mathfrak{A} \subseteq \mathfrak{Nr}_\alpha \mathfrak{B}$. Thus $\mathfrak{B} \models \sigma = \tau$, so $\mathfrak{A} \models \sigma = \tau$, as desired.

$\Leftarrow$. Assume that $\mathsf{Mn}_\omega \not\models \sigma = \tau$. Then by 2.1.16 and 2.4.52 there exist $\mathfrak{A} \in \mathsf{Mn}_\omega$ and $x \in {}^\omega A$ such that $\mathfrak{A}$ is simple and $\sigma^{\mathfrak{A}} x \neq \tau^{\mathfrak{A}} x$. Let the variables occurrring in $\sigma = \tau$ be among $v_0, \ldots, v_{\kappa-1}$, where $\kappa \in \omega \sim 1$. We may assume that $x_\lambda = x_0$ for all $\lambda \in \omega \sim \kappa$. By 2.1.17 there is an $\alpha \in \omega$ such that $Occ(\sigma = \tau) \subseteq \alpha$ and $x_0, \ldots, x_{\kappa-1} \in Sg^{\mathfrak{B}\mathfrak{r}\mathfrak{A}}\{\mathsf{d}_{\xi\eta} : \xi, \eta < \alpha\}$. Let $\mathfrak{B}$ be the minimal subalgebra of $\mathfrak{Rb}_\alpha \mathfrak{A}$. Then $\sigma^{\mathfrak{B}} x \neq \tau^{\mathfrak{B}} x$, as desired.

As an immediate corollary of 4.2.2 and the proof of 4.2.1 we have:

COROLLARY 4.2.3. *$\{\sigma = \tau : \sigma$ and τ are terms of $\mathcal{L}_\omega$ and $\mathsf{Mn}_\omega \not\models \sigma = \tau\}$ is recursively enumerable.*

For $\alpha \leqq \omega$ we do not know whether the equational theory of monadic-generated CA_α's is decidable. This is asserted as Theorem 22 for $\alpha < \omega$ in Monk [64a], but we have not been able to reconstruct the proof. The following two lemmas and corollary give some information about this question. The lemmas are proved like 4.2.2.

LEMMA 4.2.4. *Suppose* $\alpha \leqq \omega$. *Let* Mg_α *be the class of all monadic–generated* CA_α*'s, and for each* $\kappa < \omega$ *let* $\mathsf{Mg}_\alpha^\kappa$ *be the class of all* CA_α*'s generated by some set* X *with* $|X| \leqq \kappa$ *and* $\Delta x \subseteq \{0\}$ *for all* $x \in X$. *Then for any* $\alpha < \omega$ *and any terms* σ, τ *of* $\mathcal{L}_\alpha$ *we have* $\mathsf{Mg}_\alpha \models \sigma = \tau$ *iff for all* $\kappa < \omega$ $\mathsf{Mg}_\alpha^\kappa \models \sigma = \tau$.

LEMMA 4.2.5. *Assume the notation of* 4.2.4. *For any terms* σ, τ *of* $\mathcal{L}_\omega$ *we have* $\mathsf{Mg}_\omega \models \sigma = \tau$ *iff for all* $\kappa < \omega$ *and all* $\alpha < \omega$, *if* $Occ(\sigma = \tau) \subseteq \alpha$ *then* $\mathsf{Mg}_\alpha^\kappa \models \sigma = \tau$.

COROLLARY 4.2.6. *Assume the notation of* 4.2.4. *For any* $\alpha \leqq \omega$ *the set* $\{\sigma = \tau : \mathsf{Mg}_\alpha \not\models \sigma = \tau\}$ *is recursively enumerable.*

Equations in CA_2's

Now we turn to CA_2's. The first main result, due to Henkin, is an almost immediate consequence of 2.5.4.

THEOREM 4.2.7. *The equational theory of* CA_2 *is decidable.*

PROOF. For any equation $\sigma = \tau$ in $\mathcal{L}_2$ let $\kappa(\sigma, \tau)$ be the number of symbols in $\sigma = \tau$, and $\lambda(\sigma, \tau) = 9\kappa(\sigma, \tau) + 1$. Then the theorem clearly follows from the following statement:

(*) If $\not\models \sigma = \tau$, then $\sigma = \tau$ fails in some CA_2 with at most 2^μ elements, with $\mu = 2^{\lambda(\sigma, \tau)}$.

For, to prove (*), say $\mathfrak{A}$ is a CA_2, $x \in {}^\omega A$, and $\sigma^{\mathfrak{A}} x \neq \tau^{\mathfrak{A}} x$. Let

$$X = \{x_i : v_i \text{ occurs in } \sigma = \tau\} \cup \{\rho^{\mathfrak{A}} x : \mathsf{c}_\xi \rho \text{ is a subterm of } \sigma \text{ or } \tau \text{ for some } \xi = 0, 1\}.$$

Clearly $|X| \leqq \kappa(\sigma, \tau)$. Let $\mathfrak{B}$ satisfy the conditions of 2.5.4. Thus $|B| \leqq 2^\mu$. Let $y \in {}^\omega B$ such that $y_i = x_i$ if v_i occurs in $\sigma = \tau$. Clearly $\sigma^{\mathfrak{B}} y = \sigma^{\mathfrak{A}} x \neq \tau^{\mathfrak{A}} x = \tau^{\mathfrak{B}} y$, as desired.

To prove that the equational theory of IGs_2 is decidable, a result of Dana Scott [62'], we need a result analogous to 2.5.4. It turns out to be much harder to prove than 2.5.4.

LEMMA 4.2.8. *Let* $\mathfrak{A}$ *be a* Cs_2 *and* X *a finite subset of* A. *Then there is an* IGs_2 $\mathfrak{B}$ *satisfying the following conditions:*

(i) $\mathfrak{B}$ *is finite, and in fact* $|B| \leqq \kappa_5$, *where* $\kappa_0 = 2^{9 \cdot |X| + 1}$, $k_1 = 2^{\kappa_0}$, $\kappa_2 = \kappa_1 + 2 \cdot \kappa_1^2$, $\kappa_3 = 2^{9\kappa 2 + 1}$, $\kappa_4 = 2^{\kappa 3}$, *and* $\kappa_5 = 2^{\kappa 3} \cdot 2^{\kappa 4}$.

(ii) *There is a one–one function* f *from* $X \cup \{\mathsf{d}_{01}\}$ *into* $\mathfrak{B}$ *such that* $f\mathsf{d}_{01} = \mathsf{d}_{01}$, $f(x + y) = fx + fy$ *if* $x, y, x + y \in X \cup \{\mathsf{d}_{01}\}$, $f(-x) = -fx$ *if* $x, -x \in X \cup \{\mathsf{d}_{01}\}$, *and* $f(\mathsf{c}_\xi x) = \mathsf{c}_\xi fx$ *if* $x, \mathsf{c}_\xi x \in X \cup \{\mathsf{d}_{01}\}$, *for any* $\xi \in 2$.

PROOF. For any CA_2 $\mathfrak{C}$ we define

(1) $Thin_0\mathfrak{C} = \{x \in C: c_1 x \cdot s_1^0 c_1 x \leq d_{01}\}$;

$\mathrm{Thin}_1\mathfrak{C}$ is defined similarly, interchanging 0 and 1. Note:

(2) If $\mathfrak{C}$ is a Cs_2 with base U, then $\mathrm{Thin}_0\mathfrak{C} = \{x \in C$: for some $u \in U$ we have $x \subseteq \{u\} \times U\}$.

An analogous statement is true for $\mathrm{Thin}_1\mathfrak{C}$.

Now we begin the construction. We apply 2.5.4 to $\mathfrak{A}$ and X to obtain a CA_2 $\mathfrak{A}_1$ with the following properties (and other properties in that proof):

(3) $|A_1| \leq \kappa_1$;

(4) $X \subseteq A_1$ and $\mathfrak{B}\mathfrak{l}\mathfrak{A}_1 \subseteq \mathfrak{B}\mathfrak{l}\mathfrak{A}$;

(5) $d_{01}^{\mathfrak{A}} = d_{01}^{\mathfrak{A}_1}$ and if $x \in X$, then $c_\lambda^{\mathfrak{A}} x = c_\lambda^{\mathfrak{A}_1} x$ for $\lambda = 0,1$.

The cylindrifications of $\mathfrak{A}$ and $\mathfrak{A}_1$ will be denoted by c_λ and c_λ^1 respectively, for $\lambda = 0,1$. Now we let

$$X' = A_1 \cup \{c_0(x \cdot c_1 z): z \in A_1 \cap Thin_0\mathfrak{A},\ x \in A_1\} \cup \{c_1(x \cdot c_0 z): z \in A_1 \cap Thin_1 z,\ x \in A_1\}.$$

Thus $|X'| \leq \kappa_2$. Now we apply 2.5.4 to $\mathfrak{A}$ and X' to obtain a CA_2 $\mathfrak{A}_2$ with the following properties: (and other properties in that proof):

(6) $|A_2| \leq \kappa_4$;

(7) $X' \subseteq A_2$ and $\mathfrak{B}\mathfrak{l}\mathfrak{A}_2 \subseteq \mathfrak{B}\mathfrak{l}\mathfrak{A}$;

(8) $d_{01}^{\mathfrak{A}} = d_{01}^{\mathfrak{A}_2}$ and, if $x \in X'$, then $c_\lambda^{\mathfrak{A}} x = c_\lambda^{\mathfrak{A}_2} x$ for $\lambda = 0,1$.

Note from the proof of 2.5.4 that $\mathfrak{A}_2$ is simple. The cylindrifications of $\mathfrak{A}_2$ will be denoted by c_λ^2 for $\lambda = 0,1$. Now

(9) $z \in Thin_0\mathfrak{A}_1$ iff $z \in A_1$, $c_1 z \in A_1$, and $z \in Thin_0\mathfrak{A}$.

For $\Rightarrow$, obviously $z \in A_1$ and $z \in Thin_0\mathfrak{A}$. Since $c_1^1 z \in c_1^* A$ we have $c_1^1 z \in Thin_0\mathfrak{A}$. Also $c_1 z \in Thin_0\mathfrak{A}$, and $c_1 z \leq c_1^1 z$. Hence $c_1 z = c_1^1 z \in A_1$ by (2) (both $c_1 z$ and $c_1^1 z$ have the form $\{u\} \times U$, U the base of $\mathfrak{A}$). For $\Leftarrow$, we have $c_1 z = c_1^1 z$ and, by a statement at the end of the proof of 2.5.4, $c_1^1(d_{01} \cdot c_1 z) = c_1(d_{01} \cdot c_1 z)$. Hence, clearly, $z \in Thin_0\mathfrak{A}_1$. A statement analogous to (9) holds for $Thin_1\mathfrak{A}_1$.

Next we claim:

(10) If $z \in A_1 \cap Thin_0\mathfrak{A}$ and $x \in A_2$, then $c_0(x \cdot c_1 z) \in A_2$.

To prove (10), assume that $z \in A_1 \cap Thin_0\mathfrak{A}$, $z \neq 0$. For every $x \in A$ let $Tx = c_0(x \cdot c_1 z)$. Since $c_0 c_1 z = 1$, it follows from 3.2.3 that T is an endomorphism of $\mathfrak{B}\mathfrak{l}\mathfrak{A}$. (This can also easily be checked directly.) Also, the following are easy to check:

(11) If $u \in A$ then $Tc_0 u = c_0 u$.
(12) If $u \in A$ then $Tc_1 u = 1$ or $Tc_1 u = 0$.

Now to prove (10), recall from the proof of 2.5.4 that $A_2 = Sg^{\mathfrak{BA}}(Y \cup (s_1^0)^* Y \cup (s_0^1)^* Y)$, where $Y = X' \cup c_0^* X' \cup c_1^* X'$ (since $d_{01} \in X'$). Thus, since T is an endomorphism of $\mathfrak{BA}$ and $\mathfrak{BA}_2 \subseteq \mathfrak{BA}$ it suffices to show that $Tx \in A_2$ for any $x \in Y \cup (s_1^0)^* Y \cup (s_0^1)^* Y$. By (11) and (12) it suffices to prove this for any $x \in X'$: recalling the definition of X', we may assume that $x \in A_2$. Then $Tx = c_0(x \cdot c_1 z) \in X' \subseteq A_2$, as desired. So (10) holds. A corresponding statement holds for $Thin_1 \mathfrak{A}$. Next note:

(13) If $z \leq d_{01}$, then $z \in Thin_0 \mathfrak{A}_2$ iff $z \in Thin_1 \mathfrak{A}_2$.

For, $c_1^2 z \cdot c_0^2(d_{01} \cdot c_1^2 z)) = c_1^2 z \cdot c_0^2 z = c_0^2 z \cdot c_1^2(d_{01} \cdot c_0^2 z)$, so (13) is clear.

Now we shall construct a Ca_2 $\mathfrak{C}$; the CA_2 we are after will be $\mathfrak{Cm}\mathfrak{C}$ (cf. 2.7.38). Let $D = \{z : z \in Thin_0 \mathfrak{A}_2 \cap At\mathfrak{A}_2,\ z \leq d_{01}$, and $z \notin A_1\}$. Let g be a one–one function mapping D onto a set D' disjoint from $At\mathfrak{A}_2$ (the set of atoms of $\mathfrak{A}_2$). Let $C = At\mathfrak{A}_2 \cup D'$. We now define equivalence relations T_0, T_1 on C by specifying their equivalence classes. Assume that $\lambda < 2$. For each $z \in D$ we let the following be a T_λ–class:

$$\{x \in At\mathfrak{A}_2 : c_\lambda^2 x = c_\lambda^2 z\} \cup \{gz\}.$$

If $y \in At\mathfrak{A}_2$ and $c_\lambda y \neq c_\lambda z$ for all $z \in D$, we take the following as an equivalence class of T_λ:

$$\{x \in At\mathfrak{A}_2 : c_\lambda^2 x = c_\lambda^2 y\}.$$

Finally, we set $E_{00} = E_{11} = C$, and $E_{01} = E_{10} = \{x \in At\mathfrak{A}_2 : x \leq d_{01}\}$. Let $\mathfrak{C} = (C, T_\lambda, E_{\lambda\mu})_{\lambda,\mu<2}$. Note that $|C| \leq \kappa_3 + 2^{\kappa_3}$, and hence $|Cm\mathfrak{C}| \leq \kappa_5$. Thus with $\mathfrak{B} = \mathfrak{Cm}\mathfrak{C}$, condition (i) in our lemma has been checked.

Now we show that $\mathfrak{C}$ is a Ca_2. To do this we check the conditions (i)–(v) of 2.7.40. Conditions (i) and (iii) are clear. For (v), assume that $\{\lambda,\mu\} = \{0,1\}$ and $\langle x,y\rangle \in T_\lambda \cap (E_{\lambda\mu} \times E_{\lambda\mu})$. Thus $x,y \leq d_{01}$, $x,y \in At\mathfrak{A}_2$, and $c_\lambda^2 x = c_\lambda^2 y$. So $x = d_{01} \cdot c_\lambda^2 x = d_{01} \cdot c_\lambda^2 y = y$. So (v) holds. To check (iv), by symmetry it suffices to show that $C = T_0^* E_{01}$. The inclusion $\supseteq$ is obvious. Suppose $a \in C$. If $a \in At\mathfrak{A}_2$, let $y = c_0^2 a \cdot d_{01}$. Thus by 1.10.3, $y \in At\mathfrak{A}_2$, $y \in E_{01}$, and $c_0^2 y = c_0^2 a$ hence $\langle y,a\rangle \in T_0$ and so $a \in T_0^* E_{01}$. Suppose $a \notin At\mathfrak{A}_2$; say $a = gz$ with $z \in D$. Then $aT_0 z \in E_{01}$ hence $a \in T_0^* E_{01}$. So, (iv) holds. To prove (ii) we establish more generally:

(14) $T_0 | T_1 = T_1 | T_0 = C \times C$.

To prove (14) it suffices to show that $C \times C \subseteq T_0 | T_1$. Let $a,b \in C$. If $a,b \in At\mathfrak{A}_2$, then $c_0^2 c_1^2 b = 1$ and so $a(T_0|T_1)b$. Suppose $a \in At\mathfrak{A}_2$ and $b \notin At\mathfrak{A}_2$; say $b = gz$ with $z \in D$. Then $c_0^2 c_1^2 z = 1$, so $a(T_0|T_1)z$ and hence $a(T_0|T_1)b$. The case $a \notin At\mathfrak{A}_2$, $b \in At\mathfrak{A}_2$ is similar.

Finally, suppose $a,b \notin At\mathfrak{A}_2$, say $a = gz$, $b = gz'$, with $z,z' \in D$. Then $z(T_0|T_1)z'$, so $a(T_0|T_1)b$. Thus (14) holds. Hence $\mathfrak{C}$ is a Ca_2 and hence $\mathfrak{B}$ is a CA_2, which by (14) is simple. We denote the operations of $\mathfrak{B}$ with primes: c_0', $s_0^{1\prime}$, d_{01}', etc.

Now to prove that $\mathfrak{B} \in \mathbf{I}Cs_2$ we shall apply the remark after 3.2.65. To this end, assume that $x,y,z \in B$ and $c_1'(x \cdot s_1^{0\prime} c_1' x) \leq d_{01}'$; we shall prove that $c_0'(x \cdot y) \cdot c_0'(x \cdot z) \leq c_0'(x \cdot y \cdot z)$. By symmetry this establishes $\mathfrak{B} \in \mathbf{I}Cs_2$. Let $a \in c_0'(x \cdot y) \cdot c_0'(x \cdot z)$; say $aT_0b \in x \mathbf{n} y$ and $aT_0c \in x \mathbf{n} z$. It suffices to show that $b = c$.

(15) $b,c \in At\mathfrak{A}_2$ and for all $u \in D$, $c_1^2 b \neq c_1^2 u \neq c_1^2 c$.

For, otherwise say $u \in D$ and bT_1gu. Then $gu \in c_1' x \cdot s_1^{0\prime} c_1' x$ but $gu \notin d_{01}'$, contradiction.

(16) $c_1^2 b, c_1^2 c \in Thin_0 \mathfrak{A}_2$.

In fact, if $u \in At\mathfrak{A}_2$ and $u \leq c_1^2(c_1^2 b) \cdot c_0^2(d_{01} \cdot c_1^2 c_1^2 b)$, then $u \in c_1' x \cdot s_1^{0\prime} c_1' x$, so $u \in d_{01}'$; thus $c_1^2 b \in Thin_0 \mathfrak{A}_2$ and similarly $c_1^2 c \in Thin_0 \mathfrak{A}_2$.

Now let $u = d_{01} \cdot c_1^2 b$ and $v = d_{01} \cdot c_1^2 c$.

(17) $u,v \in A_1$

For, if $u \notin A_1$, then $u \in D$ (since $c_1^2 u = c_1^2 b \in Thin_0 \mathfrak{A}_2$ by (16)), and $c_1^2 u = c_1^2 b$, contradicting (15). Similarly for v.

Now bT_0c, hence $c_0^2 b = c_0^2 c$. Furthermore, $c_0 b = c_0(b \cdot c_1^2 b) = c_0(b \cdot c_1^2 u) = c_0(b \cdot c_1 u) \in A_2$ by (10); similarly $c_0 c \in A_2$. Thus $c_0 b = c_0^2 b = c_0^2 c = c_0 c$. But $b + c \in Thin_0 \mathfrak{A}$, as is easily checked, so $b = c$, as desired. Thus $\mathfrak{B} \in \mathbf{I}Cs_2$.

It remains to check (ii) of the lemma. First we show:

(18) If $z \in D$, then $c_1^1 z \notin Thin_0 \mathfrak{A}_1$ (recall that c_1^1 is defined for all members of A, and $z \in A_2 \subseteq A$).

For, if $c_1^1 z \in Thin_0 \mathfrak{A}_1$, then $c_1 z \leq c_1^1 z \in Thin_0 \mathfrak{A} \mathbf{n} c_1^* A$, hence $c_1 z = c_1^1 z$ by (2); so $z = d_{01} \cdot c_1(d_{01} \cdot z) = d_{01} \cdot c_1 z = d_{01} \cdot c_1^1 z \in A_1$, contradicting $z \in D$. So, (18) holds. By (18), for every $z \in D$ we have $c_1^1 z \cdot c_0^1(d_{01} \cdot c_1^1 z) \not\leq d_{01}$, so there is a function h mapping D into $At\mathfrak{A}_1$ such that $hz \leq c_1^1 z \cdot c_0^1 z \cdot -d_{01}$ for all $z \in D$. Now for any $x \in A_2$ we set

$$fx = \{a \in At\mathfrak{A}_2 : a \leq x\} \mathbf{u} g^* h^{-1*} \{a \in At\mathfrak{A}_1 : a \leq x\}.$$

We claim that $(X \mathbf{u} \{d_{01}\}) \upharpoonright f = k$ is as desired in the lemma. Clearly k preserves $+$ and $-$, $kd_{01} = E_{01}$, and k is one-one. Now assume that $x, c_0 x \in X \mathbf{u} \{d_{01}\}$; we show that $c_0' fx = fc_0 x$; the argument for c_1 is similar. Note that $c_0^1 x = c_0 x = c_0^2 x$.

First suppose that $u \in c_0' fx$. Say $uT_0v \in fx$.

Case 1. $u,v \in A_2$. Then $u \leq c_0^2 u = c_0^2 v \leq c_0^2 x = c_0 x$, so $u \in fc_0 x$.

Case 2. $u \in A_2$, $v \notin A_2$. Say $w \in D$, $gw = v$, $hw \leq x$. Thus $c_0^1 w \cdot x \neq 0$ by the definition of h

so $w \cdot c_0 x \neq 0$, since otherwise $c_0^1 w \leq -c_0 x$; hence $w \leq c_0^2 x = c_0 x$ so $c_0^2 w \leq c_0 x$. Now $c_0^2 u = c_0^2 w$, so $u \leq c_0 x$ and $u \in f c_0 x$.

Case 3. $u \notin A_2$, $v \in A_2$. Thus $v \leq x$. Say $u = gw$ with $w \in D$. Since $u T_0 v$, we have $c_0^2 w = c_0^2 v$. Thus $w \leq c_0^2 w = c_0^2 v \leq c_0^2 x = c_0^1 x$, so $hw \leq c_0^1 w \leq c_0^1 x = c_0 x$. Hence $u \in f c_0 x$.

Case 4. $u, v \notin A_2$. Since $u T_0 v$, we have $u = v$. Say $gw = v$, $w \in D$, $hw \leq x$. Then $hw \leq c_0 x$, so $u \in f c_0 x$.

Second, suppose that $u \in f c_0 x$.

Case 1. $u \in A_2$. Thus $u \leq c_0 x$. Thus $u \cdot c_0^2 x = u \cdot c_0 x \neq 0$, so $c_0^2 u \cdot x \neq 0$; let $v \in At\mathfrak{A}_2$ with $v \leq c_0^2 u \cdot x$. Then $u T_0 v \in f x$, as desired.

Case 2. $u \notin A_2$. Say $u = gw$, $w \in D$, $hw \leq c_0 x$. Since $hw \leq c_1^1 w \cdot c_0^1 w \cdot -d_{01}$, we get $c_1^1 w \cdot c_0^1 w \cdot c_0 x \neq 0$, hence $w \cdot c_0^2 x = w \cdot c_0^1 x \neq 0$. So $c_0^2 w \cdot x \neq 0$. Let $v \in At\mathfrak{A}_2$ with $v \leq c_0^2 w \cdot x$. Then $u T_0 v \in f x$, as desired.

This completes the proof of 4.2.8.

THEOREM 4.2.9. *The equational theory of* IGs_2 *is decidable.*

PROOF. As for 4.2.7, using 4.2.8 in place of 2.5.4.

Equations in CA_α's, $3 \leq \alpha < \omega$

Our second major result of this section is that if $3 \leq \alpha \leq \omega$ and $\mathsf{IGs}_\alpha \subseteq \mathsf{K} \subseteq \mathsf{CA}_\alpha$, then the equational theory of K is undecidable. For $\alpha \geq 4$, and for $\mathsf{K} = \mathsf{IGs}_3$, this result is due to Tarski. The remaining cases are due to R. Maddux [80'], and we follow his proof, which applies to the general situation. The original proof of Tarski is much longer, using the notion of pairing elements in relation algebras and involving extensive relation–algebraic computations. The decision result upon which the present proof is based, due to Post [47**] and Markov [47**], is that the word problem for semigroups is recursively unsolvable. In this respect we shall follow the notation of Davis [58**].

DEFINITION 4.2.10. *Let* $0 < \alpha < \omega$. *We define* $\mathfrak{S}_\alpha = \langle S_\alpha, \cdot \rangle$, *where* S_α *is the set of all non–zero finite sequences of elements of* α *and* $\cdot$ *is the binary operation of concatenation.*

Here we have deviated from Davis [58**] by not allowing the empty sequence. In fact, the modification appears to be necessary in order to carry out the proof of Theorem 4.5, p.98, there (it is needed below). Clearly:

THEOREM 4.2.11. *If* $0 < \alpha < \omega$, *then* $\mathfrak{S}_\alpha$ *is* K*–freely generated by* α, *where* K *is the class of all semigroups.*

DEFINITION 4.2.12. *Let* $0 < \alpha < \omega$ *and let* $T \subseteq S_\alpha \times S_\alpha$. *We define the binary relation* $\sim_T$ *on* S_α *by setting, for any* $f, g \in S_\alpha$, $f \sim_T g$ *iff there exist finite sequences* h, k *of elements of* α ($h = 0$ *or* $k = 0$ *possible*) *and a pair* $\langle l, m \rangle \in T$ *such that* $f = h^\frown l^\frown k$ *and* $g = h^\frown m^\frown k$, *or* $f = h^\frown m^\frown k$ *and* $g = h^\frown l^\frown k$. *Furthermore,* $f \approx_T g$ *iff there is a finite sequence* $n_0, \ldots, n_k$

such that $f = n_0 \sim_T n_1 \sim_T \cdots \sim_T n_k = g$.

Let $\mathfrak{F}\mathfrak{r}_\alpha$ be the algebra introduced in 0.4.19 *(the absolutely free algebra with α generators and a single binary operation). For any $X \subseteq Fr_\alpha \times Fr_\alpha$ we set*

$$\simeq_X = \{\langle x,y\rangle \in Fr_\alpha \times Fr_\alpha : fx \approx_T fy, \text{ where } f \text{ is the homomorphism from } \mathfrak{F}\mathfrak{r}_\alpha \text{ into } \mathfrak{S}_\alpha \text{ such that } f\xi = \xi \text{ for all } \xi < \alpha \text{ and } T = \{\langle fx, fy\rangle : \langle x,y\rangle \in X\}\}.$$

We need two elementary facts about the relation $\simeq$.

THEOREM 4.2.13. *If $0 < \alpha < \omega$ and $X \subseteq Fr_\alpha \times Fr_\alpha$, then $\simeq_X = \bigcap\{R : R \in Co\mathfrak{F}\mathfrak{r}_\alpha,\ \mathfrak{F}\mathfrak{r}_\alpha/R$ is a semigroup, and $X \subseteq R\}$.*

PROOF. $\subseteq$: suppose that $x \simeq_X y$, f and T are as in 4.2.12, and R is as above. Since $\mathfrak{F}\mathfrak{r}_\alpha/R$ is a semigroup, choose by 4.2.11 $g \in Hom(\mathfrak{S}_\alpha, \mathfrak{F}\mathfrak{r}_\alpha/R)$ such that $g\xi = \xi/R$ for all $\xi < \alpha$. Thus $gfu = u/R$ for all $u \in Fr_\alpha$. Now if $u,v \in Fr_\alpha$ and $fu \sim_T fv$, say $fu = h\hat{\ }l\hat{\ }k$ and $fv = h\hat{\ }m\hat{\ }k$ with lTm. Say $l = fl'$, $m = fm'$, $l'Xm'$. Thus $l'Rm'$, so $gl = gfl' = l'/R = m'/R = gfm' = gm$. Hence $gfu = gfv$. So we have shown that $fu \sim_T fv$ implies that $gfu = gfv$. Since f maps onto S_α, this implies that $gfx = gfy$, i.e., xRy, as desired.

$\supseteq$: It suffices to show that $\simeq_X \in Co\mathfrak{F}\mathfrak{r}_\alpha$, $\mathfrak{F}\mathfrak{r}_\alpha/\simeq_X$ is a semigroup, and $X \subseteq \simeq_X$. To show that $\simeq_X \in Co\mathfrak{F}\mathfrak{r}_\alpha$, suppose $x \simeq_X y$ and $z \in Fr_\alpha$; we show that $x \cdot z \simeq_X y \cdot z$; by symmetry, this is enough. Now $f(x \cdot z) = fx\hat{\ }fz$ and $f(y \cdot z) = fy\hat{\ }fz$. Since $fx \approx_T fy$, it is clear that $fx\hat{\ }fz \approx_T fy\hat{\ }fz$. Hence $x \cdot z \approx_X y \cdot z$. Next, to show that $\mathfrak{F}\mathfrak{r}_\alpha/\simeq_X$ is a semigroup, suppose that $x,y,z \in Fr_\alpha$. Then $f(x \cdot (y \cdot z)) = fx\hat{\ }fy\hat{\ }fz = f((x \cdot y) \cdot z)$, so $x \cdot (y \cdot z) \simeq_X (x \cdot y) \cdot z$, as desired. Finally, suppose that xXy. Then $(fx)T(fy)$, so $fx \approx_T fy$, hence $x \simeq_X y$, as desired.

REMARK 4.2.14. Let $0 < \alpha < \omega$, and let $\mathbf{K}$ be the class of all semigroups. Recalling 0.4.65, we see that for any $X \subseteq Fr_\alpha \times Fr_\alpha$ we have $\simeq_X = Cr_\alpha^X\mathbf{K}$; $\mathfrak{F}\mathfrak{r}_\alpha/Cr_\alpha^X\mathbf{K}$ is freely generated under the set X of defining relations by the sequence $\langle \xi/Cr_\alpha^X\mathbf{K} : \xi < \alpha\rangle$.

THEOREM 4.2.15. *If $0 < \alpha < \omega$ and $X \subseteq Fr_\alpha \times Fr_\alpha$, then for any $f,g \in Fr_\alpha$ the following conditions are equivalent:*

(i) $f \simeq_X g$;

(ii) *If F is a homomorphism from $\mathfrak{F}\mathfrak{r}_\alpha$ into some semigroup $\mathfrak{A}$, and if $Fh = Fk$ for all $\langle h,k\rangle \in X$, then $Ff = Fg$.*

The following is the form of the unsolvability of the word problem for semigroups which we shall use:

THEOREM 4.2.16. *There exist $0 < \alpha < \omega$ and a finite $X \subseteq Fr_\alpha \times Fr_\alpha$ such that $\simeq_X$ is a non-recursive relation (under some standard Gödel numbering of the objects involved).*

PROOF. By Theorem 4.5, p.98, of Davis [58**] choose $0 < \alpha < \omega$ and a finite $T \subseteq S_\alpha \times S_\alpha$ such that $\approx_T$ is non-recursive. Choose

$T \subseteq S_\alpha \cdot S_\alpha$ such that $\approx_T$ is non-recursive. Choose $X \subseteq Fr_\alpha \cdot Fr_\alpha$, X finite, so that $T = \{\langle fx, fy\rangle : xXy\}$, with f as in 4.2.12. Then for any $x, y \in Fr_\alpha$, $x \simeq_X y$ iff $fx \approx_T fy$. There is an effective procedure for going from a member u of S_α to a member x of Fr_α such that $fx = u$. Hence $\simeq_X$ is non-recursive.

Now we return to cylindric algebras. We shall again use the binary operation ; defined in any CA_α with $\alpha \geq 3$ as follows (cf. 3.2.69):

$$x;y = \mathsf{c}_2(\mathsf{s}_2^1\mathsf{c}_2 x \cdot \mathsf{s}_2^0\mathsf{c}_2 y),$$

for any x, y. Before proceeding to the undecidability result, we need to establish some properties of this operation.

LEMMA 4.2.17. *Let $\alpha \geq 3$, and let $\mathfrak{A}$ be a CA_α. Suppose $x, y, z, w \in A$. Then:*
(i) $\mathsf{c}_1(x;y) = x;\mathsf{c}_1 y$.
(ii) $x;\mathsf{c}_1 y = \mathsf{c}_1(\mathsf{c}_2 x \cdot \mathsf{c}_0\mathsf{c}_2(\mathsf{d}_{01} \cdot \mathsf{d}_{02} \cdot \mathsf{c}_1\mathsf{c}_2 y))$.
(iii) $(x;y);\mathsf{c}_1 z = x;(y;\mathsf{c}_1 z)$.
(iv) $((x;y);z);\mathsf{c}_1 w = (x;(y;z));\mathsf{c}_1 w$.

PROOF. Condition (i) is clear. For (ii), we have

$$\begin{aligned} x;\mathsf{c}_1 y &= \mathsf{c}_2(\mathsf{c}_1(\mathsf{d}_{12} \cdot \mathsf{c}_2 x) \cdot \mathsf{c}_0(\mathsf{d}_{02} \cdot \mathsf{c}_2\mathsf{c}_1 y)) \\ &= \mathsf{c}_1\mathsf{c}_2(\mathsf{d}_{12} \cdot \mathsf{c}_2 x \cdot \mathsf{c}_0(\mathsf{d}_{02} \cdot \mathsf{c}_2\mathsf{c}_1 y)) \\ &= \mathsf{c}_1\mathsf{c}_2(\mathsf{c}_2 x \cdot \mathsf{c}_0(\mathsf{d}_{01} \cdot \mathsf{d}_{02} \cdot \mathsf{c}_1\mathsf{c}_2 y)) \\ &= \mathsf{c}_1(\mathsf{c}_2 x \cdot \mathsf{c}_0\mathsf{c}_2(\mathsf{d}_{01} \cdot \mathsf{d}_{02} \cdot \mathsf{c}_1\mathsf{c}_2 y)), \end{aligned}$$

as desired. For (iii), we have

$$\begin{aligned} (x;y);\mathsf{c}_1 z &= \mathsf{c}_1(\mathsf{c}_2(x;y) \cdot \mathsf{c}_0\mathsf{c}_2(\mathsf{d}_{01} \cdot \mathsf{d}_{02} \cdot \mathsf{c}_1\mathsf{c}_2 z)) \text{ by (ii)} \\ &= \mathsf{c}_1(\mathsf{c}_2(\mathsf{s}_2^1\mathsf{c}_2 x \cdot \mathsf{s}_2^0\mathsf{c}_2 y) \cdot \mathsf{c}_0\mathsf{c}_2(\mathsf{d}_{01} \cdot \mathsf{d}_{02} \cdot \mathsf{c}_1\mathsf{c}_2 z)) \\ &= \mathsf{c}_2\mathsf{c}_1(\mathsf{s}_2^1\mathsf{c}_2 x \cdot \mathsf{s}_2^0\mathsf{c}_2 y \cdot \mathsf{c}_0\mathsf{c}_2(\mathsf{d}_{01} \cdot \mathsf{d}_{02} \cdot \mathsf{c}_1\mathsf{c}_2 z)) \\ &= \mathsf{c}_2(\mathsf{s}_2^1\mathsf{c}_2 x \cdot \mathsf{c}_1(\mathsf{s}_2^0\mathsf{c}_2 y \cdot \mathsf{c}_0\mathsf{c}_2(\mathsf{d}_{01} \cdot \mathsf{d}_{02} \cdot \mathsf{c}_1\mathsf{c}_2 z))) \\ &= \mathsf{c}_2(\mathsf{s}_2^1\mathsf{c}_2 x \cdot \mathsf{c}_0\mathsf{c}_1(\mathsf{d}_{02} \cdot \mathsf{c}_2 y \cdot \mathsf{c}_0\mathsf{c}_2(\mathsf{d}_{01} \cdot \mathsf{d}_{02} \cdot \mathsf{c}_1\mathsf{c}_2 z))) \\ &= \mathsf{c}_2(\mathsf{s}_2^1\mathsf{c}_2 x \cdot \mathsf{s}_2^0\mathsf{c}_1(\mathsf{c}_2 y \cdot \mathsf{c}_0\mathsf{c}_2(\mathsf{d}_{01} \cdot \mathsf{d}_{02} \cdot \mathsf{c}_1\mathsf{c}_2 z))) \\ &= x;(y;\mathsf{c}_1 z) \text{ by (ii)} \end{aligned}$$

Finally,

$$\begin{aligned} ((x;y);z);\mathsf{c}_1 w &= (x;y);(z;\mathsf{c}_1 w) && \text{by (iii)} \\ &= (x;y);\mathsf{c}_1(z;w) && \text{by (i)} \\ &= x;(y;\mathsf{c}_1(z;w)) && \text{by (iii)} \\ &= x;(y;(z;\mathsf{c}_1 w)) && \text{by (i)} \\ &= x;((y;z);\mathsf{c}_1 w) && \text{by (iii)} \\ &= (x;(y;z));\mathsf{c}_1 w && \text{by (iii)}. \end{aligned}$$

THEOREM 4.2.18. *Suppose* $3 \leq \alpha \leq \omega$ *and* $\mathsf{IGs}_\alpha \subseteq \mathsf{K} \subseteq \mathsf{CA}_\alpha$. *Then the equational theory of* K *is undecidable.*

PROOF. We assume at first that $\alpha < \omega$; the case $\alpha = \omega$ will follow easily from this case, as we shall show at the end.

Suppose that $0 < \beta < \omega$. With each element x of Fr_β we associate a term τx *of* $\mathcal{L}_\alpha$ involving only the variables $v_0, \ldots, v_{\beta-1}$ as follows:

$$\tau\xi = v_\xi \text{ if } \xi < \beta; \quad \tau(x \cdot y) = (\tau x); (\tau y),$$

for any $x, y \in Fr_\beta$. Here we use $\sigma; \tau$ as an abbreviation for $\mathbf{c}_2(\mathbf{c}_1(\mathbf{d}_{12} \cdot \mathbf{c}_2\sigma) \cdot \mathbf{c}_0(\mathrm{d}_{02} \cdot \mathbf{c}_2\tau))$. We also need the following abbreviations:

$$\mathbf{c}_{(\alpha)}\sigma \text{ for } \mathbf{c}_0 \ldots \mathbf{c}_{\alpha-1}\sigma;$$
$$\sigma \oplus \tau \text{ for } (-\sigma \cdot \tau) + (-\tau \cdot \sigma);$$
$$\sigma \leq \tau \text{ for } \sigma + \tau = \tau.$$

Now we shall show:

(1) If $X \subseteq Fr_\beta \times Fr_\beta$, X finite, and $x, y \in Fr_\beta$, then $x \simeq_X y$ iff the following inequality holds in K:

$$(\tau x; \mathbf{c}_1 v_\beta) \oplus (\tau y; \mathbf{c}_1 v_\beta) \leq \Sigma_{\langle u,v \rangle \in X} \mathbf{c}_{(\alpha)}(\tau u \oplus \tau v).$$

For $\Rightarrow$, assume that $x \simeq_X y$. By 2.4.53, it suffices to prove that the indicated equation holds an arbitrary simple $\mathfrak{A} \in \mathsf{HSPK}$. Let $h \in {}^\omega A$. If $\tau u^{\mathfrak{A}} h \neq \tau v^{\mathfrak{A}} h$ for some $\langle u,v \rangle \in X$, then the right side of the inequality is 1, and there is nothing to prove. Hence we may assume that $\tau u^{\mathfrak{A}} h = \tau v^{\mathfrak{A}} h$ for all $\langle u,v \rangle \in X$. Now we set

$$R = \{\langle u,v \rangle \in Fr_\beta \times Fr_\beta : \text{for all } a \in A \text{ we have } (\tau u^{\mathfrak{A}} h); \mathsf{c}_1 a = (\tau v^{\mathfrak{A}} h); \mathsf{c}_1 a\}.$$

We shall establish that $R \in Co\mathfrak{Gr}_\beta$, $\mathfrak{Gr}_\beta / R$ is a semigroup, and $X \subseteq R$; by 4.2.13, this will show that $\langle x,y \rangle \in R$ and hence under h the left side of the inequality (1) is 0, as desired. Clearly R is an equivalence relation on Fr_α. To show that it is a congruence on $\mathfrak{Gr}_\beta$, suppose that uRv and $w \in Fr_\beta$; we show that $(u \cdot w)R(v \cdot w)$; $(w \cdot u)R(w \cdot v)$ similarly. In fact, for any $a \in A$,

$$\begin{aligned}
\tau(u \cdot w)^{\mathfrak{A}} h; \mathsf{c}_1 a &= (\tau u; \tau w)^{\mathfrak{A}} h; \mathsf{c}_1 a \\
&= (\tau u^{\mathfrak{A}} h; \tau w^{\mathfrak{A}} h); \mathsf{c}_1 a \\
&= \tau u^{\mathfrak{A}} h; (\tau w^{\mathfrak{A}} h; \mathsf{c}_1 a) \text{ by 4.2.18(iii)} \\
&= \tau v^{\mathfrak{A}} h; (\tau w^{\mathfrak{A}} h; \mathsf{c}_1 a) \text{ by 4.2.18(ii)} \\
&= (\tau(v \cdot w))^{\mathfrak{A}} h; \mathsf{c}_1 a \text{ similarly.}
\end{aligned}$$

Thus $R \in Co\mathfrak{Gr}_\beta$. To show that $\mathfrak{Gr}_\beta / R$ is a semigroup, suppose $u, v, w \in Fr_\beta$; we want to show that $((u \cdot v) \cdot w)R(u \cdot (v \cdot w))$. This is a clear consequence of 4.2.18(iv). Finally, suppose that uXv. Then $\tau u^{\mathfrak{A}} h = \tau v^{\mathfrak{A}} h$ by the above assumption, and so of course uRv.

We have now established the implication $\Rightarrow$ in (1).

For $\Leftarrow$, suppose that $x \not\equiv_X y$. By 4.2.15, let F be a homomorphism of $\mathfrak{Fr}_\beta$ into a semigroup $\mathfrak{A}$ such that $Fu = Fv$ for all $\langle u,v\rangle \in X$, while $Fx \neq Fy$. Since any semigroup can be embedded in a semigroup with identity, we may assume that $\mathfrak{A}$ has an identity e. Let $\mathfrak{B}$ be the full Cs_α with base A. Note that $\mathfrak{B} \in \mathbf{HSPK}$. We shall show that the inequality in the right side of (1) fails in $\mathfrak{B}$; this then will establish $\Leftarrow$. Let $h \in {}^\omega B$ be any function satisfying the following conditions: for $\xi < \beta$, $h\xi = \{a \in {}^\alpha A : a_1 = a_0 \cdot F\xi\}$, and $h\beta = \{a \in {}^\alpha A : a_0 = Fx\}$. Now we claim

(2) for all $u \in Fr_\beta$, $\tau u^{\mathfrak{B}} h = \{a \in {}^\alpha A : a_1 = a_0 \cdot Fu\}$.

To show this it suffices to show that the set Y of $u \in Fr_\beta$ for which the indicated equation holds is a subalgebra of $\mathfrak{Fr}_\beta$ containing β. Now obviously $\beta \subseteq Y$. Suppose $u,v \in Y$. Then

$$\begin{aligned}
\tau(u \cdot v)^{\mathfrak{B}} h &= (\tau u; \tau v)^{\mathfrak{B}} h \\
&= (\tau u^{\mathfrak{B}} h); (\tau v^{\mathfrak{B}} h) \\
&= \{a \in {}^\alpha A : a_1 = a_0 \cdot Fu\}; \{a \in {}^\alpha A : a_1 = a_0 \cdot Fv\} \\
&= \{a \in {}^\alpha A : \text{there is a } b \in A \text{ such that} \\
&\qquad b = a_0 \cdot Fu \text{ and } a_1 = b \cdot Fv\} \quad \text{(cf. 3.2.69)} \\
&= \{a \in {}^\alpha A : a_1 = a_0 \cdot Fu \cdot Fv\} \\
&= \{a \in {}^\alpha A : a_1 = a_0 \cdot F(u \cdot v)\},
\end{aligned}$$

as desired. Thus (2) holds. Now we look at the inequality in (1). If $\langle u,v\rangle \in X$, then $Fu = Fv$, and so by (2) $\tau u^{\mathfrak{B}} h = \tau v^{\mathfrak{B}} h$. It follows that

$$(\Sigma_{\langle u,v\rangle \in X} \mathsf{c}_{(\alpha)}(\tau u \oplus \tau v))^{\mathfrak{B}} h = 0$$

On the other hand,

$$\begin{aligned}
(\tau x; \mathsf{c}_1 v_\beta)^{\mathfrak{B}} h &= (\tau x)^{\mathfrak{B}} h; \mathsf{c}_1 h\beta \\
&= \{a \in {}^\alpha A : a_1 = a_0 \cdot Fx\}; c_1\{a \in {}^\alpha A : a_0 = Fx\} \\
&= \{a \in {}^\alpha A : \text{for some } s \in A,\ s = a_0 \cdot Fx \text{ and } s = Fx\} \\
&= \{a \in {}^\alpha A : a_0 \cdot Fx = Fx\}.
\end{aligned}$$

Similarly, $(\tau y; \mathsf{c}_1 v_\beta)^{\mathfrak{B}} h = \{a \in {}^\alpha A : a_0 \cdot Fy = Fx\}$. Now fix some $a \in {}^\alpha A$ with $a_0 = e$. Then $a \in (\tau x; \mathsf{c}_1 v_\beta)^{\mathfrak{B}} h \sim (\tau y; \mathsf{c}_1 v_\beta)^{\mathfrak{B}} h$. Thus the equation in (1) fails to hold under h. So, now we have established (1).

From 4.2.16 and (1) it follows that the equational theory of $\mathbf{K}$ is undecidable.

Now we take the case $\alpha = \omega$. Suppose $\mathsf{IGs}_\omega \subseteq \mathbf{K} \subseteq \mathsf{CA}_\omega$. Then:

(3) If $3 \leqq \alpha < \omega$, then $\mathsf{IGs}_\alpha \subseteq \mathbf{S}\{\mathfrak{Rb}_\alpha \mathfrak{A} : \mathfrak{A} \in \mathbf{K}\} \subseteq \mathsf{CA}_\alpha$.

For, let $\mathfrak{B} \in \mathsf{IGs}_\alpha$. By 3.1.122, say $\mathfrak{B} \subseteq \mathfrak{Rb}_\alpha \mathfrak{A}$ with $\mathfrak{A} \in \mathsf{IGs}_\omega$. Then $\mathfrak{A} \in \mathbf{K}$. So, the first inclusion of (3) holds. The second one holds by 2.6.2.

The following is clear:

(4) If $3 \leq \alpha < \omega$ and σ and τ are terms of $\mathcal{L}_\alpha$, then $\mathsf{K} \models \sigma = \tau$ iff $\mathsf{S}\{\mathfrak{Rd}_\alpha \mathfrak{A} : \mathfrak{A} \in \mathsf{K}\} \models \sigma = \tau$.

From (3), (4), and the result for all α with $3 \leq \alpha < \omega$ (or even the case $\alpha = 3$) it follows that the equational theory of K is undecidable. This finishes the proof of 4.2.18.

COROLLARY 4.2.19 *Let $3 \leq \alpha \leq \beta$, $\alpha \leq \omega$. The equational theory of each of the following classes is undecidable:* Lf_α, Ss_α, Dc_α *(if $\alpha = \omega$)*, $\mathsf{Nr}_\alpha\mathsf{CA}_\beta$, $\mathsf{Rd}_\alpha\mathsf{CA}_\beta$.

Elementary theory of CA_1

Now we turn to another major result in this section, the undecidability of the elementary theory of CA_1. (Later we will see that the elementary theory CA_2 is also undecidable (in 4.2.25)). For CA_1 the result is due to M. Rubin [76']; the slightly weaker theorem that the elementary theory of a BA with a distinguished subalgebra is undecidable was obtained independently by R. McKenzie.

We begin with some preliminaries concerning Boolean algebras.

DEFINITION 4.2.20. *For any* BA $\mathfrak{A}$ *let* $El\mathfrak{A} = \{a \in A :$ *there exist* $b, c \in A$ *such that* $\mathfrak{Rl}_b\mathfrak{A}$ *is atomic,* $\mathfrak{Rl}_c\mathfrak{A}$ *is atomless, and* $a = b + c\}$.

We count the one-element BA as both atomic and atomless. Thus $At\mathfrak{A} \subseteq El\mathfrak{A}$ for any BA $\mathfrak{A}$, and $a \in El\mathfrak{A}$ if $\mathfrak{A}$ has no atom $\leq a$.

LEMMA 4.2.21. *For any* BA $\mathfrak{A}$, $El\mathfrak{A}$ *is an ideal in* $\mathfrak{A}$. *Also, if* $a \in A$, *then*

$$\mathfrak{A}/El\mathfrak{A} \cong (\mathfrak{Rl}_a\mathfrak{A}/El(\mathfrak{Rl}_a\mathfrak{A})) \times (\mathfrak{Rl}_{-a}\mathfrak{A}/El(\mathfrak{Rl}_{-a}\mathfrak{A})).$$

PROOF. The first statement is obvious. The isomorphism desired in the second statement is determined by the condition

$$f(x/El\mathfrak{A}) = \langle (x \cdot a)/El(\mathfrak{Rl}_a\mathfrak{A}), (x \cdot -a)/El(\mathfrak{Rl}_{-a}\mathfrak{A}) \rangle$$

for any $x \in A$.

LEMMA 4.2.22. *There are formulas* $\varphi_0(v_0)$, $\varphi_1(v_0)$, $\varphi_2(v_0)$, $\varphi_3(v_0, v_1)$, *and* $\varphi_4(v_0)$ *of* $\mathcal{L}_1$, *with the indicated free variables, such that for any* CA_1 $\mathfrak{A}$ *and any* $a, b \in A$ *the following conditions hold:*

(i) $\mathfrak{A} \models \varphi_0[a]$ *iff* a *is an atom of* $\mathfrak{A}$;

(ii) $\mathfrak{A} \models \varphi_1[a]$ *iff* a *is an atom of* $\mathfrak{Sb}\mathfrak{A}$;

(iii) $\mathfrak{A} \models \varphi_2[a]$ *iff* $a \in El\mathfrak{A}$;

(iv) $\mathfrak{A} \models \varphi_3[a, b]$ *iff* $a \oplus b \in El\mathfrak{A}$;

(v) $\mathfrak{A} \models \varphi_4[a]$ *iff* $a/El\mathfrak{A}$ *is an atom of* $\mathfrak{A}/El\mathfrak{A}$;

The construction of these formulas is routine.

THEOREM 4.2.23. *The elementary theory of* CA_1 *is undecidable.*

PROOF. The proof will be based on the undecidability of the elementary theory of two equivalence relations; see, e.g., Monk [76**], p.295. Let $\mathcal{L}_e$ be a first order language with just two non-logical constants, binary relation symbols $\mathbf{E}_0, \mathbf{E}_1$. Γ is the standard set of axioms for two equivalence relations; thus an $\mathcal{L}_e$-structure $\mathfrak{A} = \langle A, E_0, E_1 \rangle$ is a model of Γ iff E_0 and E_1 are equivalence relations with field A.

Now we describe some particular BA's which will be used below. Let $\mathfrak{A}$ be the BA of finite and cofinite subsets of ω, and let F_1 be the ultrafilter of $\mathfrak{A}$ consisting of the cofinite sets. Let $\mathfrak{B}$ be the free BA on ω generators $\langle x_\kappa : \kappa < \omega \rangle$, and let F_2 be the ultrafilter on $\mathfrak{B}$ generated by $\{-x_\kappa : \kappa < \omega\}$. Now it is easy to see that the set

(1) $C = \{\langle a,b \rangle : a \in A,\ b \in B, \text{ and } a \in F_1 \text{ iff } b \in F_2\}$

is a subuniverse of $\mathfrak{A} \times \mathfrak{B}$; we denote the corresponding BA by $\mathfrak{C}$. Now the following conditions clearly hold, for any $c \in C$:

(2) $c \in At\mathfrak{C}$ iff $c = \langle \{\iota\}, 0 \rangle$ for some $\iota \in \omega$;
(3) $\mathfrak{R}\mathfrak{l}_c\mathfrak{C}$ is atomless iff $c = \langle 0, b \rangle$ for some $b \in B \sim F_2$;
(4) $\mathfrak{R}\mathfrak{l}_c\mathfrak{C}$ is atomic iff $c = \langle a, 0 \rangle$ for some finite subset a of ω;
(5) $E\mathfrak{l}\mathfrak{C} = \{\langle a,b \rangle : a \in A \sim F_1,\ b \in B \sim F_2\}$, and $E\mathfrak{l}\mathfrak{C}$ is a maximal ideal in $\mathfrak{C}$.

Next we want to describe a translation from $\mathcal{L}_e$ into $\mathcal{L}_1$. To this end we need the formulas $\varphi_1, \ldots, \varphi_5$ of Lemma 4.2.22. In terms of them it is easy to construct formulas $\varphi_6(v_0, v_1)$, $\varphi_7(v_0, v_1)$, with the indicated free variables, such that for any CA_1 $\mathfrak{D}$ and any $d, e \in D$, the following conditions hold:

(6) $\mathfrak{D} \models \varphi_6[d,e]$ iff $d/E\mathfrak{l}\mathfrak{B}\mathfrak{l}\mathfrak{D}$ and $e/E\mathfrak{l}\mathfrak{B}\mathfrak{l}\mathfrak{D}$ are atoms and there exist $d', e' \in D$ such that $d/E\mathfrak{l}\mathfrak{B}\mathfrak{l}\mathfrak{D} = d'/E\mathfrak{l}\mathfrak{B}\mathfrak{l}\mathfrak{D}$, $e/E\mathfrak{l}\mathfrak{B}\mathfrak{l}\mathfrak{D} = e'/E\mathfrak{l}\mathfrak{B}\mathfrak{l}\mathfrak{D}$, and for every atom a of $\mathfrak{B}\mathfrak{d}\mathfrak{D}$, there is exactly one atom $\leq a \cdot d'$ iff there is exactly one atom $\leq a \cdot e'$;

(7) similarly for φ_7, with "exactly one" replaced at both its occurences by "exactly three".

Now we associate with each formula ψ of $\mathcal{L}_e$ a formula ψ^t of $\mathcal{L}_1$ by the following stipulations:

$$(v_i = v_j)^t = \varphi_3(v_i, v_j); \qquad \mathbf{T}^t = \mathbf{T};$$
$$(\mathbf{E}_0 v_i v_j)^t = \varphi_6(v_i, v_j); \qquad (\neg\varphi)^t = \neg\varphi^t;$$
$$(\mathbf{E}_1 v_i v_j)^t = \varphi_7(v_i, v_j); \qquad (\varphi \vee \psi)^t = \varphi^t \vee \psi^t;$$
$$\mathbf{F}^t = \mathbf{F}; \qquad (\varphi \wedge \psi)^t = \varphi^t \wedge \psi^t;$$
$$(\forall v_i \varphi)^t = \forall v_i (\varphi_4(v_i) \rightarrow \varphi^t).$$

Now let Δ consist of the following sentences of $\mathcal{L}_1$:

axioms for CA_1, expressed naturally;

$$\exists v_0 \varphi_4(v_0);$$
$$\forall v_0 \forall v_1 \forall v_2 \forall v_3 [(v_0 = v_1)^t \wedge (v_2 = v_3)^t \wedge \varphi_6(v_0, v_2) \rightarrow \varphi_6(v_1, v_3)];$$
$$\forall v_0 [\varphi_4(v_0) \rightarrow \varphi_6(v_0, v_0)];$$
$$\forall v_0 \forall v_1 [\varphi_6(v_0, v_1) \rightarrow \varphi_6(v_1, v_0)];$$
$$\forall v_0 \forall v_1 \forall v_6 [\varphi_6(v_0, v_1) \wedge \varphi_6(v_1, v_2) \rightarrow \varphi_6(v_0, v_2)];$$

similar sentences with φ_6 replaced by φ_7.

It suffices to show that Δ is an undecidable theory in $\mathcal{L}_1$. To do this, it suffices to prove the following statement.

(8) For any sentence φ of $\mathcal{L}_e$, $\Gamma \models \varphi$ iff $\Delta \models \varphi^t$.

To prove $\Rightarrow$ in (8), the easy direction, we need the following construction. Let $\mathfrak{D}$ be any model of Δ. We form an $\mathcal{L}_e$-structure $\mathfrak{F} = \langle F, E_0, E_1 \rangle$ as follows:

$$F = At(\mathfrak{B}(\mathfrak{D}/E(\mathfrak{B}(\mathfrak{D}),$$
$$E_0 = \{\langle d/E(\mathfrak{B}(\mathfrak{D}, e/E(\mathfrak{B}(\mathfrak{D}\rangle : \mathfrak{D} \models \varphi_6[d,e]\},$$
$$E_1 = \{\langle d/E(\mathfrak{B}(\mathfrak{D}, e/E(\mathfrak{B}(\mathfrak{D}\rangle : \mathfrak{D} \models \varphi_7[d,e]\}.$$

It is easily checked that $\mathfrak{F}$ is a model of Γ. Now let $l = (E(\mathfrak{B}(\mathfrak{D})^*$; recall from 0.2.21 that $l \in Ho(\mathfrak{B}(\mathfrak{D}, \mathfrak{B}(\mathfrak{D}/E(\mathfrak{B}(\mathfrak{D})$. Then one can easily prove by induction on ψ that the following statement holds:

(9) For any formula ψ of $\mathcal{L}_e$ and any $a \in {}^{\omega}\{x \in D : \mathfrak{D} \models \varphi_4[x]\}$ we have $\mathfrak{F} \models \psi[l \circ a]$ iff $\mathfrak{D} \models \psi^t[a]$.

From this the direction $\Rightarrow$ in (8) is clear.

For the direction $\Leftarrow$, we shall associate with each model $\mathfrak{D}$ of Γ in which D is a non-zero ordinal $\leq \omega$ a model $\mathfrak{C}_{\mathfrak{D}}$ of Δ. Say $\mathfrak{D} = \langle \alpha, E_0, E_1 \rangle$, where E_0 and E_1 are equivalence relations on α, $\alpha \leq \omega$. Now we shall use the constructions of BA's mentioned at the beginning of this proof. For each $\kappa < \omega$ let $y_\kappa = x_\kappa \cdot \Pi_{\lambda < \kappa} - x_\lambda$. Then the y_κ's are non-zero, pairwise disjoint, and:

(10) $\Sigma^{\mathfrak{B}}_{\kappa < \omega} y_\kappa = 1$.

This condition is easily checked. Furthermore:

(11) If $0 \neq z \in B \sim F_2$, then $1 \leq |\{\kappa < \omega : z \cdot y_\kappa \neq 0\}| < \omega$.

For, the first inequality is clear from (10). For the second inequality, we can write

$$z = \Sigma_{M \in \mathcal{P}} (\Pi_{\kappa \in M} x_\kappa \cdot \Pi_{x \in N \sim M} - x_\kappa)$$

for some finite $N \subseteq \omega$ and some $\mathcal{P} \subseteq SbN$. Since $z \notin F_2$, we have $\Pi_{\kappa \in M} x_\kappa \cdot \Pi_{x \in N \sim M} - x_\kappa \notin F_2$ for all $M \in \mathcal{P}$, hence $M \neq 0$ for all $M \in \mathcal{P}$. Choose $\lambda \in \omega$ with $\lambda > \kappa$ for all $\kappa \in \bigcup_{M \in \mathcal{P}} M$. Then $\{\kappa < \omega : z \cdot y_\kappa \neq 0\} \subseteq \{0, \ldots, \lambda - 1\}$, so the second inequality follows.

Note that $y_\kappa \notin F_2$ for all $\kappa<\omega$. Hence $\langle 0, y_\kappa\rangle \in C$ for all $\kappa<\omega$. Let $\langle b_{\varepsilon\sigma\lambda}:\varepsilon\in 2,\ \sigma\in\alpha/E_\varepsilon,\ \lambda\in\omega\rangle$ be an enumeration of $\{\langle 0, y_\kappa\rangle:\kappa\in\omega\}$ without repetitions. Then by (3) and (11) we have:

(12) If $c\in C$ and $\mathfrak{R}\mathfrak{l}_c\mathfrak{C}$ is atomless, then $1\leqq|\{\langle\varepsilon,\sigma,\lambda\rangle:\varepsilon\in 2,\ \sigma\in\alpha/E_\varepsilon,\ \lambda\in\omega,\ c\cdot b_{\varepsilon\sigma\lambda}\neq 0\}|<\omega$.

Next, for each $\kappa<\alpha$ let $\langle M^\kappa_{\varepsilon\sigma\lambda}:\varepsilon\in 2,\ \sigma\in\alpha/E_\varepsilon,\ \lambda\in\omega\rangle$ be a partition of $At\mathfrak{C}$ such that, for any $\varepsilon\in 2,\ \sigma\in\alpha/E_\varepsilon,\ \lambda<\omega$,

$$|M^\kappa_{\varepsilon\sigma\lambda}| = \begin{cases} 1 \text{ if } \varepsilon=0 \text{ and } \kappa\in\sigma, \\ 2 \text{ if } \varepsilon=0 \text{ and } \kappa\notin\sigma, \\ 3 \text{ if } \varepsilon=1 \text{ and } \kappa\in\sigma, \\ 4 \text{ if } \varepsilon=1 \text{ and } \kappa\notin\sigma. \end{cases}$$

Finally, for any $\varepsilon\in 2,\ \sigma\in\alpha/E_\varepsilon,\ \lambda<\omega$ we let $c_{\varepsilon\sigma\lambda}$ be that member of ${}^\alpha C$ such that, for any $\kappa<\alpha$,

$$c_{\varepsilon\sigma\lambda}{}^\kappa = b_{\varepsilon\sigma\lambda}+\Sigma M^\kappa_{\varepsilon\sigma\lambda}.$$

Thus, using (10),

(13) $\langle c_{\varepsilon\sigma\lambda}:\varepsilon\in 2,\ \sigma\in\alpha/E_\varepsilon,\ \lambda\in\omega\rangle$ is a system of pairwise disjoint elements of ${}^\alpha\mathfrak{C}$ with sum 1.

Now let $\mathfrak{F}={}^\alpha\mathfrak{C}$ and $\mathfrak{G}=\mathfrak{Sg}^{\mathfrak{F}}\{c_{\varepsilon\delta\lambda}:\varepsilon\in 2,\ \delta\in\alpha/E_\varepsilon,\ \lambda\in\omega\}$. For each $\kappa<\alpha$ let t_κ be that member of F such that $t_\kappa\kappa=1$ and $t_\kappa\mu=0$ if $\mu\in\alpha\sim\{\kappa\}$. Note that $\mathfrak{G}$ is isomorphic to $\mathfrak{A}$. Also note the following concerning $\mathfrak{G}$:

(14) If $d,e\in G$, $\kappa\in\alpha$, and $d_\kappa\leqq e_\kappa$, then $d\leqq e$.

For, if $c_{\varepsilon\sigma\lambda}\leqq d$, then $c_{\varepsilon\sigma\lambda}{}^\kappa\leqq e_\kappa$, hence $c_{\varepsilon\sigma\lambda}\leqq e$ since $c_{\varepsilon\sigma\lambda}$ is an atom. Since $\{c_{\varepsilon\sigma\lambda}:\varepsilon\in 2,\ \sigma\in\alpha/E_\varepsilon,\ \lambda\in\omega\}$ is the set of atoms of $\mathfrak{G}$, (14) follows.

Next, let

$$I=\{\langle\Omega,d,e\rangle:\Omega \text{ is a finite subset of } \alpha,\ d\in{}^\Omega C, \text{ and } e\in G\}.$$

For each $\langle\Omega,d,e\rangle\in I$ we let $s_{\Omega de}$ be the member of ${}^\alpha C$ such that, for any $\lambda<\alpha$,

$$s_{\Omega de}{}^\lambda = \begin{cases} d\lambda \text{ if } \lambda\in\Omega, \\ e\lambda \text{ if } \lambda\notin\Omega. \end{cases}$$

Let $H=\{s_i:i\in I\}$. Note that if $\alpha<\omega$ then $H={}^\alpha C$. Now we claim

(15) $H\in Su\mathfrak{F}$.

To show this, let $\langle\Omega,d,e\rangle,\langle\Omega',d',e'\rangle\in I$. Then $-s_{\Omega de}=s_{\Omega uv}$ with $u\lambda=-d\lambda$ for all $\lambda\in\Omega$, and $v=-e$. And $s_{\Omega de}+s_{\Omega'd'e'}=s_{\Lambda uv}$, where $\Lambda=\Omega\cup\Omega'$, $v=e+e'$, and $u\in{}^{\Lambda}C$ is defined as follows: for any $\lambda\in\Lambda$,

$$u\lambda=\begin{cases} d\lambda+d'\lambda & \text{if } \lambda\in\Omega\cap\Omega',\\ d\lambda+e'\lambda & \text{if } \lambda\in\Omega\sim\Omega',\\ e\lambda+d'\lambda & \text{if } \lambda\in\Omega'\sim\Omega.\end{cases}$$

So (15) holds. We denote by $\mathfrak{H}$ the BA with universe H.

(16) $G\subseteq H$.

For, if $e\in G$ then $e=s_{\Omega 0e}\in H$. Next,

(17) For every $x\in H$ there is a smallest $y\in G$ such that $x\leqq y$.

To prove (17), let $\langle\Omega,d,e\rangle\in I$. We may assume, of course, that $\Omega\neq 0$. For each $\kappa\in\Omega$ let w_κ be the member of ${}^{\alpha}C$ such that, for any $\lambda<\alpha$,

$$w_\kappa\lambda=\begin{cases} 0 & \text{if } \lambda\in\Omega\sim\{\kappa\},\\ d\lambda & \text{if } \lambda=\kappa.\\ e\lambda & \text{if } \lambda\notin\Omega.\end{cases}$$

Clearly $w_\kappa\in H$. We prove (17) first for an arbitrary w_κ. Say $d_\kappa=\langle u,v\rangle$.
Case 1. u is a finite subset of ω and $v\in B\sim F_2$. Let $N=\{\langle\varepsilon,\sigma,\lambda\rangle:\varepsilon\in 2,\ \sigma\in\alpha/E_\varepsilon,\ \lambda\in\omega,$ and $\langle u,0\rangle\cdot\Sigma M^\kappa_{\varepsilon\sigma\lambda}\neq 0$ or $b_{\varepsilon\sigma\lambda}\cdot\langle 0,v\rangle\neq 0\}$. Using (12), it is clear that N is finite. Therefore $e+\Sigma_{\langle\varepsilon,\sigma,\lambda\rangle\in N}c_{\varepsilon\sigma\lambda}$ (or merely $\Sigma_{\langle\varepsilon,\sigma,\lambda\rangle\in N}c_{\varepsilon\sigma\lambda}$ if $\Omega=\alpha$) is the smallest element of G which is $\geqq w_\kappa$ (using (14)).
Case 2. u is a cofinite subset of ω and $v\in F_2$. Let $N=\{\langle\varepsilon,\sigma,\lambda\rangle:\varepsilon\in 2,\ \sigma\in\alpha/E_\varepsilon,\ \lambda\in\omega,$ $\Sigma M^\kappa_{\varepsilon\sigma\lambda}\leqq\langle -u,1\rangle$, and $b_{\varepsilon\sigma\lambda}\leqq\langle 1,-v\rangle\}$. Again, N is finite. This time $e+-\Sigma_{\langle\varepsilon,\sigma,\lambda\rangle\in N}c_{\varepsilon\sigma\lambda}$ (or $-\Sigma_{\langle\varepsilon,\sigma,\lambda\rangle\in N}c_{\varepsilon\sigma\lambda}$ if $\Omega=\alpha$) is the smallest element of G which is $\geqq w_\kappa$.

Thus (17) holds for any w_κ. Now $s_{\Omega de}=\Sigma_{\kappa\in\Omega}w_\kappa$. For each $\kappa\in\Omega$ let $y_\kappa\in G$ be smallest such that $w_\kappa\leqq y_\kappa$. Then $\Sigma_{\kappa\in\Omega}y_\kappa$ is the smallest element of G which is $\geqq s_{\Omega de}$. Hence (17) holds.

Now for each $x\in H$ *let* c_0x be the smallest $y\in G$ such that $x\leqq y$, and let $d_{00}=1^{\mathfrak{H}}$. We let $\mathfrak{C}_{\mathfrak{D}}$ be $\mathfrak{H}$ with the operation c_0 and the element d_{00} adjoined. It is easily checked that $\mathfrak{C}_{\mathfrak{D}}$ is a CA_1. Note that $\mathfrak{Bl}\mathfrak{C}_{\mathfrak{D}}=\mathfrak{H}$, the universe of $\mathfrak{C}_{\mathfrak{D}}$ is H, and $\mathfrak{Sb}\mathfrak{C}_{\mathfrak{D}}=\mathfrak{G}$. Now the following conditions are easily checked:

(18) If $\langle\Omega,d,e\rangle\in I$ and $x=s_{\Omega de}$, then the following conditions are equivalent: (a) $\mathfrak{Rl}_x\mathfrak{H}$ is atomic (resp. atomless), (b) for all $\kappa\in\Omega$, the algebra $\mathfrak{Rl}_{d_\kappa}\mathfrak{C}$ is atomic (resp. atomless), and $e=0$ if $\Omega\neq\alpha$.
(19) $x\in El\mathfrak{H}$ iff there exist finite $\Omega\subseteq\alpha$ and $d\in{}^{\Omega}C$ such that $x=s_{\Omega d0}$ and $d_\kappa\in El\mathfrak{C}$ for all $\kappa\in\Omega$.

Note that $t_\kappa\in H$ for all $\kappa\in\alpha$. Now by Lemma 4.2.22, (5), and (19) we have:

(20) For any $x \in H$ we have $\mathfrak{E}_{\mathfrak{D}} \models \varphi_4[x]$ iff there is a $\kappa \in \alpha$ such that $x \oplus t_\kappa \in El\Phi$.

Note that the κ mentioned in (20) is unique. From (20) we obtain:

(21) $\mathfrak{E}_{\mathfrak{D}} \models \exists v_0 \varphi_4(v_0)$.

Now the most important properties of $\mathfrak{E}_{\mathfrak{D}}$ are given in the following statement and the similar statement (23) below.

(22) Suppose $x, y \in H$, $\mathfrak{E}_{\mathfrak{D}} \models \varphi_4[x]$, $\mathfrak{E}_{\mathfrak{D}} \models \varphi_4[y]$, $x \oplus t_\kappa \in El\Phi$, and $y \oplus t_\lambda \in El\Phi$, where $\kappa, \lambda < \alpha$. Then $\mathfrak{E}_{\mathfrak{D}} \models \varphi_6[x, y]$ iff $\kappa E_0 \lambda$.

First we show the direction $\Leftarrow$; so suppose that $\kappa E_0 \lambda$. To check (6) we take $x, y, t_\kappa, t_\lambda$ for d, e, d', d' respectively. Suppose that z is an atom of $\mathfrak{Bb}\mathfrak{E}_{\mathfrak{D}} = \mathfrak{G}$. Thus $z = c_{\varepsilon\sigma\mu}$ for some $\varepsilon \in 2$, $\sigma \in \alpha/E_t$, $\mu \in \omega$. Thus for any $\nu < \alpha$ we have

$$(t_\kappa \cdot c_{\varepsilon\sigma\mu})\nu = \begin{cases} 0 \text{ if } \nu \neq \kappa, \\ b_{\varepsilon\sigma\mu} + \Sigma M^\nu_{\varepsilon\sigma\mu} \text{ if } \nu = \kappa. \end{cases}$$

Hence there is exactly one atom $\leq t_\kappa \cdot c_{\varepsilon\sigma\mu}$ iff $|M^\kappa_{\varepsilon\sigma\mu}| = 1$ iff $\varepsilon = 0$ and $\kappa \in \sigma$. Similarly, there is exactly one atom $\leq t_\lambda \cdot c_{\varepsilon\sigma\mu}$ iff $\varepsilon = 0$ and $\lambda \in \sigma$. Since $\kappa E_0 \lambda$, it follows that there is exactly one atom $\leq t_\kappa \cdot c_{\varepsilon\sigma\mu}$ iff there is exactly one atom $\leq t_\lambda \cdot c_{\varepsilon\sigma\mu}$. Thus $\mathfrak{E}_{\mathfrak{D}} \models \varphi_6[x, y]$ by (6).

Now we take the direction $\Rightarrow$ in (22): assume that $\mathfrak{E}_{\mathfrak{D}} \models \varphi_6[x, y]$. Then we get elements d', e' as indicated in (6), with x, y in place of d, e. Thus $d' \oplus t_\kappa \in El\Phi$. By (19) there exist finite $\Omega \subseteq \alpha$ and $d \in {}^\Omega C$ such that $d' \oplus t_\kappa = s_{\Omega d 0}$ and $d_\mu \in El\mathfrak{E}$ for all $\mu \in \Omega$. We may assume that $\kappa \in \Omega$. For each $\mu \in \alpha$ let $d'_\mu = \langle u_\mu, v_\mu \rangle$. Now $(d' \oplus t_\kappa)_\mu = \langle u_\mu, v_\mu \rangle$ if $\mu \neq \kappa$, and $(d' \oplus t_\kappa)_\kappa = \langle -u_\mu, -v_\mu \rangle$. Hence u_μ is finite for $\mu \neq \kappa$, and cofinite for $\mu = \kappa$, by Lemma 4.2.22 and (5). Now let $\sigma = \kappa/E_0$. For each $\mu \in \Omega \sim \{\kappa\}$ there is an $\iota_\mu \in \omega$ such that $M^\mu_{0\sigma\tau} \cdot d'_\mu = 0$ for all $\tau \in \omega \sim \iota_\mu$, and there is an $\iota_\kappa \in \omega$ such that $M^\kappa_{0\sigma\tau} \leq d'_\kappa$ for all $\tau \in \omega \sim \{\iota_\kappa\}$. Now we can use the same reasoning for e', obtaining a finite $\Lambda \in \alpha$ and $e \in {}^\Lambda C$ such that $e' \oplus t_\lambda = s_{\Lambda e 0}$ and $e_\mu \in El\mathfrak{E}$ for all $\mu \in \Lambda$, and also for each $\mu \in \Lambda \sim \{\lambda\}$ obtaining a $\xi_\mu \in \omega$ such that $M^\mu_{0\sigma\tau} \cdot e'_\mu = 0$ for all $\tau \in \omega \sim \xi_\mu$, and a $\xi_\lambda \in \omega$ such that $M^\lambda_{0\sigma\tau} \leq e'_\lambda$ for all $\tau \in \omega \sim \{\xi_\lambda\}$. Now fix $\tau \in \omega$ with $\tau \geq \iota_\mu$ for all $\mu \in \Omega$ and $\tau \geq \xi_\mu$ for all $\mu \in \Lambda$. Recall that $c_{0\sigma\tau}$ is an atom of $\mathfrak{Bb}\mathfrak{E}_{\mathfrak{D}} = \mathfrak{G}$. By the choice of τ, for any $\mu \in \alpha \sim \{\kappa\}$ we have $(c_{0\sigma\tau} \cdot d')_\mu = \langle 0, w \rangle$ for some w, while $(c_{0\sigma\tau} \cdot d')_\kappa = \langle \Sigma M^\kappa_{0\sigma\tau}, w \rangle$ for some w. Note that $|M^\kappa_{0\sigma\tau}| = 1$. Hence there is exactly one atom $\leq c_{0\sigma\tau} \cdot d'$. Hence by (6) there is exactly one atom $\leq c_{0\sigma\tau} \cdot e'$. Reasoning just as above, this means that $|M^\lambda_{0\sigma\tau}| = 1$. So $\lambda \in \sigma$ and $\kappa E_0 \lambda$, as desired. So (22) holds. Similarly:

(23) Suppose $x, y \in H$, $\mathfrak{E}_{\mathfrak{D}} \models \varphi_1[y]$, $x \oplus t_\kappa \in El\Phi$, and $y \oplus t_\lambda \in El\Phi$, where $\kappa, \lambda < \alpha$. Then $\mathfrak{E}_{\mathfrak{D}} \models \varphi_7[x, y]$ iff $\kappa E_1 \lambda$.

Now we can check that $\mathfrak{E}_{\mathfrak{D}}$ is a model of Δ. Of all the sentences in Δ, only the one

expressing transitivity of $\{\langle x,y\rangle : \mathfrak{C}_{\mathfrak{D}} \models \varphi_5[x,y]\}$, and the similar one for φ_7, give any difficulties. So, suppose that $\mathfrak{C}_{\mathfrak{D}} \models \varphi_6[x,y]$ and $\mathfrak{C}_{\mathfrak{D}} \models \varphi_6[y,z]$. Thus by Lemma 4.2.22(v) we have $\mathfrak{C}_{\mathfrak{D}} \models \varphi_4[x]$, $\mathfrak{C}_{\mathfrak{D}} \models \varphi_4[y]$, and $\mathfrak{C}_{\mathfrak{D}} \models \varphi_4[z]$. By (20), there exist $\kappa,\lambda,\mu < \alpha$ such that $x \oplus t_\kappa$, $y \oplus t_\lambda$, and $z \oplus t_\mu$ are all members of $El\mathfrak{H}$. Thus by (22) we have $\kappa E_0 \lambda E_0 \mu$, hence $\kappa E_0 \mu$. Then (22) again yields $\mathfrak{C}_{\mathfrak{D}} \models \varphi_6[x,y]$, as desired. The corresponding property for φ_7 is established similarly. Hence $\mathfrak{C}_{\mathfrak{D}}$ is a model of Δ.

Now with the essential use of (22) and (23) we can show:

(24) Suppose $\alpha \in {}^{\omega}\alpha$, $x \in {}^{\omega}H$, and $x_\kappa \oplus t_{\iota\kappa} \in El\mathfrak{H}$ for all $\kappa < \omega$. Then for any formula ψ of $\mathcal{L}_e$ we have $\mathfrak{D} \models \psi[\iota]$ iff $\mathfrak{C}_{\mathfrak{D}} \models \psi^t[x]$.

This finishes the construction started after (9) above. Now we prove $\Leftarrow$ in (8). Let $\mathfrak{D}$ be any model of Γ. By the downward Lowenheim, Skolem theorem we may assume that the universe of $\mathfrak{D}$ is some ordinal $\alpha \leq \omega$. By the above construction, $\mathfrak{C}_{\mathfrak{D}}$ is a model of Δ, so $\mathfrak{C}_{\mathfrak{D}} \models \varphi^t$, hence by (24) $\mathfrak{D} \models \varphi$, as desired.

This finishes the proof of 4.2.23.

REMARK 4.2.24. Comer [75a'] has shown that the elementary theory of any of the varieties of CA_i's mentioned in 4.1.21, except CA_i itself, is decidable. And in Comer [69b] p.176 it is shown that the elementary theory of finite CA_i's is decidable.

Additional results

The following is a special case of the main theorem of Comer [69b].

THEOREM 4.2.25. *The elementary theory of* CA_2 *is undecidable.*

PROOF. Recall that equivalence relations E_0 and E_1 on a set A are called *disjoint* provided that $|(x/E_0) \cap (y/E_1)| \leq 1$ for all $x,y \in A$. Let Γ be the theory of two disjoint equivalence relations (on the universe of the model), formulated in the language $\mathcal{L}_e$ introduced in the proof of 4.2.23. We shall make use of the fact that Γ is *finitely inseparable*, i.e., there is no recursive set containing $\{\varphi : \varphi$ a sentence of $\mathcal{L}_e$, $\models \varphi\}$ and disjoint from $\{\varphi : \varphi$ a sentence of $\mathcal{L}_e$, $\mathfrak{A} \models \neg\varphi$ in some finite model $\mathfrak{A}$ of $\Gamma\}$ (see, e.g., Comer [69b]). Now adjoin a new individual constant $\mathbf{k}$ to $\mathcal{L}_2$, forming a language $\mathcal{L}_2'$. Thus $\mathcal{L}_2'$–structures are systems $(\mathfrak{A},a)$ with $\mathfrak{A}$ similar to CA_2's, and $a \in A$. Let $\theta(v_0)$ be the formula in $\mathcal{L}_2'$ expressing that v_0 is an atom $\leq \mathbf{k}$. With each formula φ of $\mathcal{L}_e$ we associate a formula φ' of $\mathcal{L}_2'$: $(\mathbf{E}_k v_i v_j)' = \varphi(v_i) \wedge \varphi(v_j) \wedge \mathbf{c}_k v_i = \mathbf{c}_k v_j$ for $k = 0,1$; $(v_i = v_j)' = v_i = v_j$; $'$ commutes with sentential connectives; and $(\forall v_i \varphi)' = \forall v_i(\theta(v_i) \rightarrow \varphi')$. Let φ^* be $\exists v_0 \theta(v_0) \rightarrow \varphi'$. Then clearly $\models \varphi \Rightarrow \models \varphi^*$. If $\mathfrak{A}$ is any atomic CA_2 and $0 \neq a \in A$, we define $\mathfrak{B}_{\mathfrak{A}a} = \langle B, E_0, E_1\rangle$ by setting $B = \{b \in A : (\mathfrak{A},a) \models \theta[b]\}$ and $E_k = \{\langle a,b\rangle \in {}^2B : \mathsf{c}_k a = \mathsf{c}_k b\}$ for $k = 0,1$. Now we show that if $\mathfrak{C}$ is any finite model of Γ then there is a finite CA_2 $\mathfrak{A}$ and $0 \neq a \in A$ with $\mathfrak{B}_{\mathfrak{A}a} = \mathfrak{C}$. Standard arguments then show that $\Theta\rho\mathsf{CA}_2$ is undecidable. Let $\mathfrak{C} = \langle C, E_0, E_1\rangle$; say the E_k classes are

$K_0^k, \ldots, K_{mk}^k$, $k=0,1$. Let $n = max(m0, m1)$ and $\mathfrak{A} = \mathfrak{Sb}({}^2n)$. For each $c \in C$ let $fc = \{\langle i,j\rangle\}$, where $\langle i,j\rangle$ is the (unique) pair such that $c \in K_i^1 \cap K_j^0$. Let $a = f^*C$. It is easily verified that $f \in Is(\mathfrak{C}, \mathfrak{B}_{\mathfrak{A}a})$, as desired.

REMARK 4.2.26. We conclude this section with a brief discussion of the recursive axiomatizability of certain elementary theories discussed in section 4.1. By the result of Resek [75'] mentioned in 3.2.88, $\Theta\rho(\mathsf{Crs}_\alpha \cap \mathsf{CA}_\alpha)$ is recursively axiomatizable by the equations mentioned there, since $\mathsf{ICrs}_\alpha \cap \mathsf{CA}_\alpha$ is the variety characterized by those equations. Similarly, $\Theta\rho\mathsf{Gs}_\alpha$ and $\Theta\rho_\infty\mathsf{Cs}_\alpha$ are recursively axiomatizable (see the discussion of equations characterizing Gs_α in section 4.1). By Theorem 4.1.14, $\Theta\rho\mathsf{Lf}_\alpha = \Theta\rho\mathsf{Dc}_\alpha$ is recursively axiomatizable for $\alpha \leqq \omega$. There is one more major result along these lines, namely that $\Theta\rho\mathsf{Cs}_\alpha$ is recursively axiomatizable. This result is due to Andréka, Németi [81'], 8.13.4 and 8.13.6. We now present their interesting proof of this fact. Their proof is similar to some other known proofs in this domain; see, e.g., Tarski [55*], comment after Theorem 2.4. and Schein [70'] (which is discussed in section 5.6).

DEFINITION 4.2.27. (i) *A* **many–sorted** Cs *is a structure*

$$\mathfrak{M} = \langle A, U, V, I, Proj, E, C, D, +, -, 1\rangle$$

such that A, U, V, and I are non–empty sets, Proj is a function mapping $V \times I$ *into* U. $E \subseteq V \times A$, C *is a function mapping* $I \times A$ *into* A, D *is a function mapping* $I \times I$ *into* A, $+$ *is a binary, and* $-$ *a unary, operation on* A, $1 \in A$, *and the following conditions hold for all* $a, b \in A$, $u \in U$, $x, y \in V$, *and* $i, j \in I$:

(M_1) *If* $Proj(x,i) = Proj(y,i)$ *for all* $i \in I$, *then* $x = y$.

(M_2) *If for all* $z \in V$ (zEa *iff* zEb), *then* $a = b$.

(M_3) $xEC(i,a)$ *iff there is a* z *such that* zEa *and for all* $j \in I \sim \{i\}$ *we have* $Proj(x,j) = Proj(z,j)$.

(M_4) $xED(i,j)$ *iff* $Proj(x,i) = Proj(x,j)$.

(M_5) $xE(a+b)$ *iff* xEa *or* xEb.

(M_6) $xE(-a)$ *iff it is not the case that* xEa.

(M_7) $xE1$.

(M_8) *There is a* $z \in V$ *such that* $Proj(z,i) = u$ *and for all* $l \in \alpha \sim \{i\}$ *we have* $Proj(z,l) = Proj(x,l)$.

(ii) *Let* $\mathfrak{A}$ *be a* Cs_α *with base* $U \neq 0$. *We associate with* $\mathfrak{A}$ *the many–sorted* Cs

$$\mathfrak{M}\mathfrak{f}_{\mathfrak{A}} = \langle A, U, V, \alpha, Proj, E, C, D, +, -, 1\rangle$$

defined as follows: $V = {}^\alpha U$, $Proj(x,\kappa) = x\kappa$ *for any* $x \in V$, $\kappa < \alpha$, $E = \{\langle x,a\rangle : x \in V,\ a \in A,\ x \in a\}$, $C(\kappa,a) = \mathbf{C}_\kappa a$ *for any* $\kappa < \alpha$ *and* $a \in A$, $D(\kappa,\lambda) = \mathbf{D}_{\kappa\lambda}$ *for any* $\kappa, \lambda < \alpha$, $+ = +^{\mathfrak{A}}$, $- = -^{\mathfrak{A}}$, *and* $1 = 1^{\mathfrak{A}}$; *clearly* $\mathfrak{M}\mathfrak{f}_{\mathfrak{A}}$ *is a many–sorted* Cs.

(iii) *Let* $\mathfrak{M}$ *be a many–sorted* Cs *as in* (i), *and suppose that* ξ *is a one–one function mapping* α *onto* I. *We associate with* $\mathfrak{M}$ *and* ξ *an algebra* $\mathfrak{Ca}_{\mathfrak{M}\xi}$ *similar to* CA_α*'s as follows*:

$$\mathfrak{Ca}_{\mathfrak{M}\xi} = \langle A, +, \cdot, -, 0, 1, \mathsf{c}_\kappa, \mathsf{d}_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha},$$

where A, $+$, $-$, *and* 1 *come from* $\mathfrak{M}$, $a \cdot b = -(-a + -b)$ *for any* $a, b \in A$, $0 = -1$, $\mathsf{c}_\kappa a = C^{\mathfrak{M}}(\xi\kappa, a)$ *for any* $\kappa < \alpha$ *and* $a \in A$, *and* $\mathsf{d}_{\kappa\lambda} = D^{\mathfrak{M}}(\xi\kappa, \xi\lambda)$ *for any* $\kappa, \lambda < \alpha$.

LEMMA 4.2.28. *Let $\alpha \geqq 2$. If $\mathfrak{M}$ is a many-sorted* Cs *and ξ is a one-one function mapping α onto I, then $\mathfrak{Ca}_{\mathfrak{M}_\xi}$ is isomorphic to a* Cs_α.

PROOF. We assume notation as in 4.2.27(i),(iii). Let $W = \{\langle Proj(x,\xi\nu) : \nu < \alpha\rangle : x \in V\}$. Thus $W \subseteq {}^\alpha U$. Our object is to show that $\mathfrak{Ca}_{\mathfrak{M}}$ is isomorphic to a Gws_α^{cm} with unit element W: then 3.1.104 will give the desired result. Let $\mathfrak{P}$ be the Cs_α $\mathfrak{Sb}({}^\alpha U)$.

For any $x \in V$ let $fx = \langle Proj(x,\xi\nu) : \nu < \alpha\rangle$. For any $a \in A$ let $Fa = \{fx : xEa\}$. Note that f is one-one, by (M_1). From (M_5), (M_6) and (M_7) it now follows easily that F preserves $+$, $\cdot$, $-$, 0, and 1 of $\mathfrak{Ca}_{\mathfrak{M}}$, mapping into the Crs_α of all subsets of W. By (M_1), F is one-one. Next, (M_3) and (M_4) can easily be used to show that F preserves c_κ and $\mathsf{d}_{\kappa\lambda}$ too, for all $\kappa,\lambda < \alpha$.

Next we claim

(1) $(\mathsf{s}_\kappa^\kappa)^{\mathfrak{P}} W = W$.

For, first let $u \in W$; say $u = fx$ with $x \in V$. By (M_8), choose $z \in V$ such that $Proj(z,\xi\kappa) = Proj(x,\xi\lambda)$ and $Proj(z,\xi\mu) = Proj(x,\xi\mu)$ for all $\mu \in \alpha \sim \{\kappa\}$. Then $fz \in W \cap \mathsf{D}_{\kappa\lambda}^{[W]}$ and $\alpha \sim \{\kappa\} \upharpoonright fx \subseteq fz$, so $u = fx \in (\mathsf{s}_\kappa^\kappa)^{\mathfrak{P}} W$. Thus $\supseteq$ in (1) holds: $\subseteq$ is proved similarly.

By (1) and 3.1.31, W is the unit element of a Gws_α. Next we show that $F^*\mathfrak{Ca}_{\mathfrak{M}_\xi}$ is compressed. To this end, write

$$W = \bigcup_{\iota \in J} {}^\alpha Y_\iota^{\langle p\iota\rangle}$$

with ${}^\alpha Y_\iota^{\langle p\iota\rangle} \cap {}^\alpha Y_j^{\langle pj\rangle} = 0$ for distinct $i,j \in J$; take $i,j \in J$ and $u \in Y_\iota$ — we show that $u \in Y_j$. This will show that all Y_κ's are equal, as desired. Fix $w \in {}^\alpha Y_j^{\langle pj\rangle}$; say $w = fx$. By (M_8), choose $z \in V$ such that $Proj(z,\xi 0) = u$ and $Proj(z,\xi\lambda) = Proj(x,\xi\lambda)$ for all $\lambda \in \alpha \sim \{0\}$. Thus $(\alpha \sim \{0\}) \upharpoonright fx \subseteq fz$. Say $fz \in {}^\alpha Y_k^{\langle pk\rangle}$. Now $(fz)^c_{\langle fz\rangle} \in {}^\alpha Y_j^{\langle pj\rangle} \cap {}^\alpha Y_k^{\langle pk\rangle}$, so $j = k$. Since $(fz)0 = u$, it follows that $u \in Y_j$.

Thus $F^*\mathfrak{Ca}_{\mathfrak{M}_\xi} \in \mathsf{Gws}_\alpha^{cm}$. By 3.1.104, $F^*\mathfrak{Ca}_{\mathfrak{M}}$ is isomorphic to a Cs_α, as desired.

The following lemma is obvious.

LEMMA 4.2.29. *Let $\alpha \geqq 2$. If $\mathfrak{A} \in \mathsf{Cs}_\alpha$ and $\xi = \alpha \upharpoonright Id$, then $\mathfrak{Ca}\mathfrak{M}\Gamma_{\mathfrak{A}_\xi} = \mathfrak{A}$.*

DEFINITION 4.2.30. *Let $\alpha \geqq 2$.* (i) *$\mathfrak{M}_\alpha$ is a many-sorted language suitable for many-sorted* Cs*'s, having also individual constants $\mathbf{k}_\kappa$ for each $\kappa < \alpha$. Its symbols are* **A**, **U**, **V**, **I**, **Proj**, **E**, **C**, **D**, $+$, $-$, **1**, $\mathbf{k}_\kappa$ $(\kappa < \alpha)$. *If $\mathfrak{M}$ is a many-sorted* Cs *as in* 4.2.27(i), *and ξ is a one-one function from α onto I, then $(\mathfrak{M},\xi\kappa)_{\kappa<\alpha}$ is the expansion of $\mathfrak{M}$ to a possible model of $\mathfrak{M}_\alpha$.*

(ii) *We define a function $trans_\alpha = trans$ which assigns to each term or formula σ of $\mathcal{L}_\alpha$ a term or formula, respectively, of $\mathfrak{M}_\alpha$, as follows, where $\kappa,\lambda < \alpha$, $\mu < \omega$, σ and τ are terms of $\mathcal{L}_\alpha$, and φ and ψ are formulas of $\mathcal{L}_\alpha$:*

$trans\, v_\mu = v_\mu$	$trans(\sigma = \tau) = trans\,\sigma = trans\,\tau$
$trans\,\mathbf{0} = -\mathbf{1}$	$trans(\varphi \vee \psi) = trans\,\varphi \vee trans\,\tau$
$trans\,\mathbf{1} = \mathbf{1}$	$trans(\varphi \wedge \psi) = trans\,\varphi \wedge trans\,\psi$

$$trans(\sigma+\tau) = trans\sigma + trans\tau \qquad trans(\neg\varphi) = \neg trans\varphi$$
$$trans(\sigma\cdot\tau) = -(-trans\sigma + -trans\tau) \qquad trans\mathbf{F} = \mathbf{F}$$
$$trans(-\tau) = -trans\sigma \qquad trans\mathbf{T} = \mathbf{T}$$
$$trans(\mathbf{c}_\kappa\sigma) = \mathbf{C}(\mathbf{k}_\kappa, trans\sigma) \qquad trans\exists v_\mu\varphi = \exists v_\mu trans\varphi$$
$$trans\mathbf{d}_{\kappa\lambda} = \mathbf{D}(\mathbf{k}_\kappa, \mathbf{k}_\lambda)$$

The following lemma is easy to check.

LEMMA 4.2.31. *Let $\mathfrak{M}$ be a many sorted* Cs, *as in* 4.2.27(i), *let ξ be a one–one function mapping α onto I, and set $\mathfrak{R} = (\mathfrak{M}, \xi\kappa)_{\kappa<\alpha}$, and $\mathfrak{A} = \mathfrak{Ca}_{\mathfrak{M}_\xi}$. Suppose that $a \in {}^\omega A$. Then:*

(i) *For any term σ of $\mathcal{L}_\alpha$ we have $\sigma^{\mathfrak{A}} a = (trans\sigma)^{\mathfrak{R}} a$.*

(ii) *For any formula φ of $\mathcal{L}_\alpha$ we have $\mathfrak{A} \models \varphi[a]$ iff $\mathfrak{R} \models trans\varphi[a]$.*

THEOREM 4.2.32. *$\Theta\rho\mathsf{Cs}_\alpha$ is recursively axiomatizable.*

PROOF. (We assume implicitly that $\alpha \geq \omega$.) We may assume that $\alpha = \omega$. Let Γ be the natural formulation of the axioms (M_1) – (M_8) in the language $\mathcal{M}_\alpha$, together with all the sentences $\mathbf{Ik}_\kappa$ for $\kappa<\alpha$, and $\neg\mathbf{k}_\kappa = \mathbf{k}_\lambda$ for $\kappa<\lambda<\alpha$. We claim that for any sentence φ of $\mathcal{L}_\alpha$,

(1) $\mathsf{Cs}_\alpha \models \varphi$ iff $\Gamma \models trans\varphi$.

To prove (1), first suppose that $\mathsf{Cs}_\alpha \models \varphi$. To prove that $\Gamma \models trans\varphi$, it suffices to show that every countable model $(\mathfrak{M}, m_\kappa)_{\kappa<\alpha}$ of Γ is a model of $trans\varphi$. Here $\mathfrak{M}$ is a many–sorted Cs, while m_κ is the denotation of $\mathbf{k}_\kappa$ for each $\kappa<\alpha$. Assume the notation of 4.2.27. Then $|\alpha| = \omega = |I|$, so we can choose a one–one function ξ from α onto I such that $\xi\kappa = m_\kappa$ for all $\kappa \in Occ\varphi$. (Recall the definition of $Occ\varphi$ from 4.1.11.) Now by 4.2.28, $\mathfrak{Ca}_{\mathfrak{M}_\xi} \in \mathsf{ICs}_\alpha$ and hence our supposition above implies that $\mathfrak{Ca}_{\mathfrak{M}_\xi} \models \varphi$. Then by Lemma 4.2.31, $(\mathfrak{M}, \xi\kappa)_{\kappa<\alpha} \models trans\varphi$. Hence obviously $(\mathfrak{M}, m_\kappa)_{\kappa<\alpha} \models trans\varphi$ as well.

Now suppose that $\Gamma \models trans\varphi$, and let $\mathfrak{A} \in \mathsf{Cs}_\alpha$. Then $(\mathfrak{M}\Gamma_{\mathfrak{A}}, \kappa)_{\kappa<\alpha}$ is clearly a model of Γ, so $(\mathfrak{M}\Gamma_{\mathfrak{A}}, \kappa)_{\kappa<\alpha} \models trans\varphi$. Then by Lemmas 4.2.29 and 4.2.31 if follows that $\mathfrak{A} \models \varphi$, as desired. Thus (1) holds.

From (1) it follows that $\Theta\rho\mathsf{Cs}_\alpha$ is recursively enumerable, and hence recursively axiomatizable.

4.3 CONNECTIONS BETWEEN LOGIC AND CYLINDRIC ALGEBRAS

Cylindric algebras arise by abstracting from two more concrete mathematical situations: the set–theoretical operations on spaces of various dimensions, and ordinary first–order logic with its provability and model–theoretic notions. Having treated the set–theoretical source of CA_α's in sections 3.1 and 3.2, we turn to the source in first–order logic now.

We shall begin the section with a careful description of the basic connections between logic and CA_α's (4.3.1–4.3.31), culminating in the logical representation theorem mentioned in the preface to Part I, p.18: for $\alpha \geqq \omega$, every Lf_α is isomorphic to a formula algebra associated with some theory (4.3.28). Then we shall describe various important facts concerning these connections: homomorphisms between Lf_α's correspond to interpretations between theories (4.3.38), isomorphisms to definitional equivalence (4.3.43), equations holding in all Lf_α's correspond to validity of sentences (4.3.55–4.3.64). In a concluding remark, 4.3.68, we summarize the section and state without proof some further connections.

Many of the theorems we give are very easy to prove and so their proofs are omitted.

The reader may wish to consult various papers in which this connections material is treated differently or further elaborated: Andréka, Gergely, Németi [77'], [81'], Andréka, Németi [75'], Andréka, Sain [81'], Gergely [80'], Griffin [73'], Henkin [67], Henkin, Tarski [61], Hoehnke [66], Monk [65a], [71'], [76'], Németi [81'], [83'], Sanerib [69'].

As we have indicated several times in informal remarks, ordinary first–order logic corresponds only to a part of the theory of cylindric algebras – to Lf_α's for $\alpha \geqq \omega$ and the set algebras restricted to Lf_α's. This is, indeed, the real source of the theory of CA_α's and the most important connections are found here. We want to also indicate connections between certain more general logics and arbitrary CA_α's. Two major novelties occur: for α finite we deal with logics supplied with only finitely many variables, while for α infinite we allow infinitely long atomic formulas (but only the usual two–place conjunctions and disjunctions, and quantification over single variables only). For ordinary first–order logic we have the completeness theorem available, and so we can use either the syntactic notion $\vdash$ or the semantic one $\models$ in defining various notions. For the more general logics we consider we do not in general have a completeness theorem, and thus for our purposes we must discuss both $\vdash$ and $\models$ for them. (*Some* completeness theorems exist; see Andréka, Gergely, Németi [77'] 4.4 and 6.3.)

Basic connections

Now we shall describe the basic notions — languages, formulas, proofs, models, satisfaction — for a certain general kind of logic, and at the same time go into their algebraic counterparts. We have described the basic notions of ordinary logic in the Preliminaries of Part I, but besides generalizing these we wish to modify our treatment slightly in order to have simpler formulations of results in this section. First of all, *our languages will not have operation symbols.* Exceptions are the first-order discourse languages *for* cylindric algebras (see section 4.1), which will play a small role here, and some languages illustrating some notions below. A language with operation symbols is equivalent in a sense with another language in which these operation symbols are replaced by relation symbols with rank increased by one. Operation symbols can be treated algebraically too, but we shall not enter into this.

The *logical constants* are as in Part I, p.40; they are common to all of the languages which we consider. They form a set $C=\{\mathbf{A},\mathbf{K},\mathbf{N},\mathbf{Q},\mathbf{E},\mathbf{T},\mathbf{F}\}$ of seven distinct objects. Next, we fix upon a sequence $\langle v_\xi : \xi \in Ord\rangle$ of distinct objects; we assume that $Rgv \cap C = 0$ and $V \sim Rgv$ is a proper class, V being the universe of all sets, and Ord the class of all ordinals. The v_ξ's are the *variables*, and in a particular language the variables used will be a proper initial segment of the above list. Now we define a *language* to be a triple $\Lambda = \langle \alpha, \mathbf{R}, \rho\rangle$ such that α is ordinal, $\mathbf{R}$ and ρ are functions with the same domain β (a cardinal), $\mathbf{R}$ is a one-one function with $Rg\mathbf{R} \cap (C \cup Rgv) = 0$, and $Rg\rho \subseteq \alpha+1$. Here α specifies the length of the sequence $\langle v_\xi : \xi<\alpha\rangle$ of variables, $\mathbf{R}$ is the sequence of relation symbols of Λ, and $\mathbf{R}_\xi$ has rank $\rho\xi$ for each $\xi<\beta$. Two special types of languages will be considered: an *ordinary* language, in which $\alpha \geqq \omega$ and $Rg\rho \subseteq \omega$, and a *full* language, in which $Rg\rho \subseteq \{\alpha\}$.

For any language Λ we define $\vee$, $\wedge$, $\neg$, $\exists$, $\rightarrow$, $\leftrightarrow$, $\forall$ as on pp. 40–42 of Part I. We write $\exists v_\kappa$ in place of the earlier notation, however; similarly for $\forall$. *Atomic formulas, formulas, free and bound occurrences,* and *sentences* are defined as usual (see p.42 of Part I), making allowance for the possibility of relation symbols of infinite rank. Thus an atomic formula has either the form $v_\kappa = v_\lambda$ for $\kappa,\lambda<\alpha$ (an *equation*), or the form

$$\mathbf{R}_\xi(v_{\kappa 0},\ldots,v_{\kappa\eta},\ldots)_{\eta<\rho\xi},$$

where $\kappa \in {}^{\rho\xi}\alpha$ (a *relational atomic formula*). A formula is called *restricted* if all of its relational atomic subformulas have the form $\mathbf{R}_\xi(v_0,\ldots,v_\eta,\ldots)_{\eta<\rho\xi}$. It turns out that for most purposes in ordinary logic one can work exclusively with restricted formulas. This is illustrated by the validity of the following sentence, where $\mathbf{R}$ is a ternary relation symbol:

$$\forall v_1 \forall v_2 [\mathbf{R}(v_1,v_2,v_1) \leftrightarrow \exists v_0 \exists v_3 \exists v_4 (v_0 = v_1 \wedge v_3 = v_2 \wedge v_4 = v_1 \wedge \exists v_1 \exists v_2 [v_3 = v_1 \wedge v_4 = v_2 \wedge \mathbf{R}(v_0,v_1,v_2)]\,)].$$

Note that if φ is a restricted *sentence* in a full language $\Lambda = \langle \alpha, \mathbf{R}, \rho\rangle$ with $\alpha \geqq \omega$, then all atomic subformulas of φ are equations.

For any language Λ, $\Phi\mu^\Lambda$ denotes the collection of all formulas of Λ, and $\Phi\mu^\Lambda_\tau$ the

collection of all restricted formulas of Λ. Now we consider two algebras:

$$\mathfrak{Fm}^{\Lambda} = \langle \Phi\mu^{\Lambda}, \mathbf{v}, \mathbf{\wedge}, \neg, \mathbf{F}, \mathbf{T}, \exists v_\kappa, v_\kappa = v_\lambda \rangle_{\kappa,\lambda<\alpha};$$
$$\mathfrak{Fm}^{\Lambda}_{r} = \langle \Phi\mu^{\Lambda}_{r}, \mathbf{v}, \mathbf{\wedge}, \neg, \mathbf{F}, \mathbf{T}, \exists v_\kappa, v_\kappa = v_\lambda \rangle_{\kappa,\lambda<\alpha}.$$

THEOREM 4.3.1. *Let $\Lambda = \langle \alpha, \mathbf{R}, \rho \rangle$ be a language, with $Do\mathbf{R} = \beta$, and let γ be the number of relational atomic formulas of Λ. Then:*

(i) $\mathfrak{Fm}^{\Lambda}_{r} \subseteq \mathfrak{Fm}^{\Lambda}$;

(ii) $\mathfrak{Fm}^{\Lambda} \cong {}_{\alpha}\mathfrak{Fr}_{\gamma}$;

(iii) $\mathfrak{Fm}^{\Lambda}_{r} \cong {}_{\alpha}\mathfrak{Fr}_{\beta}$, *where there is an isomorphism from ${}_{\alpha}\mathfrak{Fr}_{\beta}$ onto $\mathfrak{Fm}^{\Lambda}_{r}$ extending the function $\langle \mathbf{R}_\xi(v_0,\ldots,v_\eta,\ldots)_{\eta<\rho\xi} : \xi<\beta \rangle$.*

In this theorem, ${}_{\alpha}\mathfrak{Fr}_{\beta}$ is the absolutely free algebra similar to CA_α's, with β generators; the presubscript α is usually omitted.

REMARKS 4.3.2. Because of the infinitely long atomic formulas when $\alpha \geqq \omega$, the formulas of a full language Λ are in general infinitely long. But in a restricted formula φ, in a full language, an atomic subformula $\mathbf{R}_\xi(v_0,\ldots,v_\eta,\ldots)_{\eta<\alpha}$ *could* be replaced just by $\mathbf{R}_\xi$ for all practical purposes, since no information would be lost. Then all restricted formulas would have finite length. This idea has been carried out by Jaśkowski [48']. In the generality considered here, the languages we have introduced have not been studied very much; but see Henkin, Tarski [61] and Andréka, Gergely, Németi [77'].

According to 4.3.1, absolutely free algebras are the algebraic counterpart of languages. In the case of $\mathfrak{Fm}^{\Lambda}_{r}$ for a full language Λ, all features of Λ which are of interest can be decoded from its isomorphic image ${}_{\alpha}\mathfrak{Fr}_{\beta}$. Some aspects of Λ are lost if Λ is not full, namely the rank of relation symbols, and more is lost if we consider $\mathfrak{Fm}^{\Lambda}$: substitution of variables in atomic formulas has no reflection in ${}_{\alpha}\mathfrak{Fr}_{\gamma}$. Further algebraic apparatus will enable us to reflect these aspects of Λ too; see below.

In 4.3.1 it is possible to have $\beta = 0$ (and hence $\gamma = 0$). The language then has no non-logical constants, and is uniquely determined by α. The free algebras considered in 4.3.1 then have the empty set of generators. In section 2.5 of Part I we did not treat this case. We can take for ${}_{\alpha}\mathfrak{Fr}_0$ the subalgebra of ${}_{\alpha}\mathfrak{Fr}_1$ generated by the empty set. Note that $Hom({}_{\alpha}\mathfrak{Fr}_0, \mathfrak{A}) \neq 0$ for every algebra $\mathfrak{A}$ similar to CA_α's.

We turn to semantic notions. Let $\Lambda = \langle \alpha, \mathbf{R}, \rho \rangle$ be a language, with $Do\mathbf{R} = \beta$. A *possible model* for Λ is a pair $\mathfrak{M} = \langle M, R \rangle$ where M is a non-empty set and R is a function with domain β assigning to each $\xi<\beta$ a subset R_ξ of ${}^{\rho\xi}M$; we sometimes denote R_ξ by $\mathbf{R}^{\mathfrak{M}}_\xi$. With each formula φ of Λ we correlate the set $\varphi^{\mathfrak{M}\alpha} = \varphi^{\mathfrak{M}}$ of all $x \in {}^{\alpha}M$ which satisfy φ in $\mathfrak{M}$; see Part I, pp.42–44. In particular, if φ is $\mathbf{R}_\xi(v_{\kappa 0},\ldots,v_{\kappa\eta},\ldots)_{\eta<\rho\xi}$ then

$$\varphi^{\mathfrak{M}} = \{x \in {}^{\alpha}M : x \circ \kappa \in \mathbf{R}^{\mathfrak{M}}_\xi\}.$$

In the case of a restricted atomic formula $\mathbf{R}_\xi(v_0,\ldots,v_\eta,\ldots)_{\eta<\alpha}$ in a full language, this takes the simpler form $\varphi^{\mathfrak{R}} = \mathbf{R}_\xi^{\mathfrak{R}}$.

THEOREM 4.3.3. *If Λ is a language and $\mathfrak{R}$ is a possible model of Λ, then $\{\varphi^{\mathfrak{R}}:\varphi$ is a (restricted) formula of $\Lambda\}$ is a cylindric field of sets of dimension α.*

The proof of 4.3.3 is quite easy.

DEFINITION 4.3.4. *If Λ is a language and $\mathfrak{R}$ is a possible model of Λ, then we denote by $\mathfrak{Cf}_\alpha^{\mathfrak{R}}$ (resp. ${}^r\mathfrak{Cf}_\alpha^{\mathfrak{R}}$) the cylindric set algebra with universe $\{\varphi^{\mathfrak{R}}:\varphi$ a (restricted) formula of $\Lambda\}$.*

The following theorem is an algebraic version of the definition of satisfaction.

THEOREM 4.3.5. *Let $\Lambda = \langle\alpha,\mathbf{R},\rho\rangle$ be an ordinary language and $\mathfrak{R}$ a possible model of Λ. Then $\mathfrak{Cf}_\alpha^{\mathfrak{R}} \in \mathsf{Cs}_\alpha^{reg} \cap \mathsf{Lf}_\alpha$. Furthermore, there is a unique homomorphism h of $\mathfrak{Fm}^\Lambda$ onto $\mathfrak{Cf}_\alpha^{\mathfrak{R}}$ such that for any $\xi < Do\mathbf{R}$ and any $\kappa \in {}^{\rho\xi}\alpha$, $h\mathbf{R}_\xi(v_{\kappa 0},\ldots,v_{\kappa\eta},\ldots)_{\eta<\rho\xi} = \{a \in {}^\alpha M: a\circ\kappa \in \mathbf{R}_\xi^{\mathfrak{R}}\}$, and $h\varphi = \varphi^{\mathfrak{R}}$, for every $\varphi \in Fm^\Lambda$.*

Because of substitutions in atomic formulas, it is difficult to give 4.3.5 a more algebraic form using the free algebra ${}_\alpha\mathfrak{Fr}_\gamma$ of 4.3.1. This is a case in which restricted formulas are useful.

THEOREM 4.3.6. *Let Λ be an ordinary language. Then there is an $f \in Ho(\mathfrak{Fm}^\Lambda, \mathfrak{Fm}_r^\Lambda)$ such that $\varphi^{\mathfrak{R}} = (f\varphi)^{\mathfrak{R}}$ for every formula φ and every possible model $\mathfrak{R}$ of Λ.*

PROOF. We define f as follows. Let φ be a formula of Λ. Then $f\varphi$ is obtained from φ by replacing every non-restricted relational atomic subformula

$$\mathbf{R}_\xi(v_{\kappa 0},\ldots,v_{\kappa\eta},\ldots)_{\eta<\rho\xi}$$

of φ by the following formula, where $v_{\lambda 0},\ldots,v_{\lambda\eta},\ldots$, $\eta<\rho\xi$, are the first $\rho\xi$ variables not in the set $\{v_0,\ldots,v_{\rho\xi-1}\} \cup \{v_{\kappa 0},\ldots,v_{\rho\xi-1}\}$:

$$\begin{gathered}\exists v_{\lambda 0}\ldots\exists v_{\lambda(\rho\xi-1)}[v_{\kappa 0} = v_{\lambda 0} \wedge \ldots \wedge v_{\kappa(\rho\xi-1)} = v_{\lambda(\rho\xi-1)} \wedge \\ \exists v_0\ldots\exists v_{\rho\xi-1}(v_0 = v_{\lambda 0} \wedge \ldots \wedge v_{\rho\xi-1} = v_{\lambda(\rho\xi-1)} \wedge \mathbf{R}_\xi(v_0,\ldots,v_{\rho\xi-1}))].\end{gathered}$$

Then the desired conclusions are easily checked.

COROLLARY 4.3.7. *Let $\Lambda = \langle\alpha,\mathbf{R},\rho\rangle$ be an ordinary language and $\mathfrak{R}$ a possible model of Λ. Then $\mathfrak{Cf}_\alpha^{\mathfrak{R}} = {}^r\mathfrak{Cf}_\alpha^{\mathfrak{R}}$.*

REMARK 4.3.8. By Corollary 4.3.7 we can work exclusively with restricted formulas in discussing the important set algebras $\mathfrak{Cf}_\alpha^{\mathfrak{R}}$, for Λ ordinary. We shall mainly be interested in the algebras ${}^r\mathfrak{Cf}_\alpha^{\mathfrak{R}}$ even for Λ arbitary.

Suppose that $\Lambda = \langle\alpha,\mathbf{R},\rho\rangle$ is any language, with $Do\mathbf{R} = \beta$, and $\mathfrak{R}$ is a possible model of Λ. Then by Theorem 4.3.1 there is a unique homomorphism $h \in Ho(\mathfrak{Fm}_r^\Lambda, {}^r\mathfrak{Cf}_\alpha^{\mathfrak{R}})$ such that $h(\mathbf{R}_\xi(v_0,\ldots,v_\eta,\ldots)_{\eta<\rho\xi}) = \{a \in {}^\alpha M: \rho\xi \upharpoonright a \in \mathbf{R}_\xi^{\mathfrak{R}}\}$ for all $\xi<\beta$. This fact will be used

several times in what follows. By 4.3.1 it can be given a purely algebraic formulation: there is a unique $h \in Ho({}_{\alpha}\mathfrak{Tr}_{\beta}, {}^{r}\mathfrak{Cf}_{\alpha}^{\mathfrak{M}})$ such that $h\xi = \{a \in {}^{\alpha}M : \rho\xi \restriction a \in \mathbf{R}_{\xi}^{\mathfrak{M}}\}$ for all $\xi < \beta$. This can be used to give a purely algebraic definition of the set algebra ${}^{r}\mathfrak{Cf}_{\alpha}^{\mathfrak{M}}$ associated with $\mathfrak{M}$. Note that if Λ is a full language then the last property of h takes the even simpler form $h\xi = \mathbf{R}_{\xi}^{\mathfrak{M}}$ for each $\xi < \beta$.

The passage from a model $\mathfrak{M}$ to the Cs_{α} $\mathfrak{Cf}_{\alpha}^{\mathfrak{M}}$ has an inverse, in a sense. The qualification comes about because this associated Cs_{α} is not provided with any simple structure which enables one to recapture the actual distinguished relations of $\mathfrak{M}$. This leads us to the notion of a Cs_{α} with generators. The general idea of an algebra or CA_{α} with generators has been used successfully in Pigozzi [71'] for another purpose.

DEFINITION 4.3.9. *A* Cs_{α} *with generators is a pair* $\langle \mathfrak{A}, x \rangle$ *such that* $\mathfrak{A}$ *is a* Cs_{α} *with non-empty base* M *and* x *is a function from some cardinal* β *onto a set of generators of* $\mathfrak{A}$.

Assume additionally that $\Lambda = \langle \alpha, \mathbf{R}, \rho \rangle$ *is a language. We call* $\langle \mathfrak{A}, x \rangle$ suitable for Λ *if the following three conditions hold:* (a) $Do\rho = \beta$, (b) $\Delta x_{\xi} \subseteq \rho\xi$ *for each* $\xi < \beta$, (c) *for all* $\xi < \beta$ *and all* f, g, *if* $f \in {}^{\alpha}M$, $g \in x_{\xi}$, *and* $\rho\xi \restriction f \subseteq g$, *then* $f \in x_{\xi}$.

Now if $\langle \mathfrak{A}, x \rangle$ *is suitable for* Λ, *we define a possible model* $\mathfrak{Mb}_{\mathfrak{A}x}^{\Lambda} = \langle M, R \rangle$ *of* Λ *as follows: for each* $\xi < \beta$, $R_{\xi} = \{\rho\xi \restriction a : a \in x_{\xi}\}$.

Note that (b) follows from (c) in this definition; we retain (b) since it is simpler. Condition (c) expresses a kind of regularity.

THEOREM 4.3.10. *Let* $\Lambda = \langle \alpha, \mathbf{R}, \rho \rangle$ *be a language, with* $Do\mathbf{R} = \beta$.

(i) *Suppose that* $\mathfrak{M}$ *is a possible model of* Λ. *Let* $\mathfrak{A} = {}^{r}\mathfrak{Cf}_{\alpha}^{\mathfrak{M}}$ *and* $x = \langle \{a \in {}^{\alpha}M : \rho\xi \restriction a \in \mathbf{R}_{\xi}^{\mathfrak{M}}\} : \xi < \beta \rangle$. *Then* $\langle \mathfrak{A}, x \rangle$ *is a* Cs_{α} *with generators which is suitable for* Λ, *and* $\mathfrak{Mb}_{\mathfrak{A}x}^{\Lambda} = \mathfrak{M}$.

(ii) *Let* $\langle \mathfrak{A}, x \rangle$ *be a* Cs_{α} *with generators which is suitable for* Λ. *Set* $\mathfrak{M} = \mathfrak{Mb}_{\mathfrak{A}x}^{\Lambda}$. *Then* $\mathfrak{A} = {}^{r}\mathfrak{Cf}_{\alpha}^{\mathfrak{M}}$ *and* $x = \langle \{a \in {}^{\alpha}M : \rho\xi \restriction a \in \mathbf{R}_{\xi}^{\mathfrak{M}}\} : \xi < \beta \rangle$.

REMARK 4.3.11. The above discussion provides an algebraic translation of the logical concept $\mathfrak{M}$ *is a possible model of* Λ. In complete form, the algebraic concept is a pair $\langle \mathfrak{A}, x \rangle$ which is a $\mathsf{Cs}_{\omega}^{reg} \cap \mathsf{Lf}_{\omega}$ with generators. In less complete form, it is just an arbitrary Cs_{α}. In considering algebraic translations of logical concepts and theorems, a situation of this kind usually arises: the complete algebraic notion involves Lf_{ω}'s with generators, and regular set algebras. Then immediate generalizations are natural to consider: to *generator-free* notions or theorems, arbitrary CA_{α}'s, and non-regular set algebras. We shall return to this idea several times below.

We now recall the definition of the semantical consequence relation $\models$. Unlike in Part I, pp.44–45, we shall distinguish carefully between this notion and the corresponding proof-theoretical notion $\vdash$, since they are not identical for some of the situations we consider. Let $\Lambda = \langle \alpha, \mathbf{R}, \rho \rangle$ be a language. If $\mathfrak{M}$ is a possible model of Λ and $x \in {}^{\alpha}M$, then we write $\mathfrak{M} \models \varphi[x]$ for $x \in \varphi^{\mathfrak{M}}$. Furthermore, $\mathfrak{M} \models \varphi$ if $\mathfrak{M} \models \varphi[x]$ for all $x \in {}^{\alpha}M$; then $\mathfrak{M}$ is called a *model* of φ. If K is a class of possible models of Λ, then $\mathsf{K} \models \varphi$ if $\mathfrak{M} \models \varphi$ for all $\mathfrak{M} \in \mathsf{K}$. For any set Σ of formulas, $\mathsf{Md}_{\Lambda}\Sigma = \mathsf{Md}\Sigma$ is the class of all

models of Σ. If $\Sigma \cup \{\varphi\}$ is a set of formulas of Λ, then we write $\Sigma \models \varphi$ if $\mathsf{Md}\Sigma \models \varphi$. Finally, $\models \varphi$ means $\mathsf{K} \models \varphi$ where K is the class of all possible models of Λ. Note that there is apparently an ambiguity in the notation $\models$, in that the language Λ is not referred to. An easy argument shows, however, that in all of the above cases the meaning of $\models$ is the same for all languages. It is more conventional to restrict $\Sigma \models \varphi$ to the case in which $\Sigma \cup \{\varphi\}$ is a set of sentences, but we need the general case of formulas for our extended languages. Note that if Λ is an ordinary language, then $\Sigma \models \varphi$ iff $\{[\psi] : \psi \in \Sigma\} \models [\varphi]$, where for any formula χ, $[\chi]$ is the closure of χ (see Part I, p.42).

We use these notions to define an important kind of cylindric algebras.

DEFINITION 4.3.12. *Let Λ be a language and Σ a set of formulas of Λ. We set*

$$_s\equiv^{\Lambda}_{\Sigma} = \{\langle\varphi,\psi\rangle : \varphi \text{ and } \psi \text{ are formulas of } \Lambda \text{ and } \Sigma \models \varphi \leftrightarrow \psi\},$$

$$_s\equiv^{\Lambda}_{r\Sigma} = \{\langle\varphi,\psi\rangle : \varphi \text{ and } \psi \text{ are restricted formulas of } \Lambda \text{ and } \Sigma \models \varphi \leftrightarrow \psi\}.$$

THEOREM 4.3.13. *Let Λ be a language and Σ a set of formulas of Λ. Then $_s\equiv^{\Lambda}_{\Sigma} \in Co\mathfrak{Fm}^{\Lambda}$ and $_s\equiv^{\Lambda}_{r\Sigma} \in Co\mathfrak{Fm}^{\Lambda}_{r}$.*

DEFINITION 4.3.14. *Let Λ be a language and Σ a set of formulas of Λ. We set*

$$_s\mathfrak{Fm}^{\Lambda}_{\Sigma} = \mathfrak{Fm}^{\Lambda}/_s\equiv^{\Lambda}_{\Sigma}, \quad _s\mathfrak{Fm}^{\Lambda}_{r\Sigma} = \mathfrak{Fm}^{\Lambda}_{r}/_s\equiv^{\Lambda}_{r\Sigma}.$$

We give some basic properties of these algebras.

THEOREM 4.3.15. *Let Λ be an ordinary language and Σ a set of formulas of Λ. Then*

(i) $_s\mathfrak{Fm}^{\Lambda}_{\Sigma} \cong {}_s\mathfrak{Fm}^{\Lambda}_{r\Sigma}$.

(ii) *There is a set Γ of restricted formulas such that $_s\mathfrak{Fm}^{\Lambda}_{\Sigma} \cong {}_s\mathfrak{Fm}^{\Lambda}_{r\Gamma}$.*

PROOF: obvious from Theorem 4.3.6.

Because of this theorem we now work just with the algebras $_s\mathfrak{Fm}^{\Lambda}_{r\Sigma}$.

LEMMA 4.3.16. *If $\Lambda = \langle\alpha,\mathbb{R},\rho\rangle$ is a language and Σ is a set of formulas of Λ, then $_s\mathfrak{Fm}^{\Lambda}_{r\Sigma} \in \mathsf{SPCs}_{\alpha}$.*

PROOF. Suppose that $0 \neq a \in {}_sFm^{\Lambda}_{r\Sigma}$; it suffices to find a homomorphism g from $_s\mathfrak{Fm}^{\Lambda}_{r\Sigma}$ into a Cs_{α} such that $ga \neq 0$. Write $a = \varphi/\equiv$ for some restricted formula φ of Λ. Since $a \neq 0$, there is a model $\mathfrak{M}$ of Σ and an $x \in {}^{\alpha}M$ such that $\mathfrak{M} \models \varphi[x]$. Now if $\psi \equiv \chi$ then $\psi^{\mathfrak{M}} = \chi^{\mathfrak{M}}$. Hence there is a function g from $_sFm^{\Lambda}_{r\Sigma}$ into $Cs^{\mathfrak{M}}_{\alpha}$ such that $g(\psi/\equiv) = \psi^{\mathfrak{M}}$ for every restricted formula ψ of Λ. Clearly g is as desired.

We shall see below, in Theorem 4.3.28, that a converse of 4.3.16 holds: every Gs_{α} is isomorphic to $_s\mathfrak{Fm}^{\Lambda}_{r\Sigma}$ for some Λ and Σ.

THEOREM 4.3.17. *Let $\Lambda = \langle\alpha,\mathbb{R},\rho\rangle$ be a language, with $\mathsf{Do}\mathbb{R} = \beta$. Then $_s\mathfrak{Fm}^{\Lambda}_{r0} \cong \mathfrak{Fr}^{\rho}_{\beta}\mathsf{Gs}_{\alpha}$.* (Recall that $\mathfrak{Fr}^{\rho}_{\beta}\mathsf{Gs}_{\alpha}$ is a *dimension-restricted* free algebra; see Part I, p.349.)

PROOF. Assume $\alpha \geq 2$. By 3.1.107 we have $\mathsf{IGs}_\alpha = \mathsf{SPCs}_\alpha^{reg}$. Hence by 2.5.34, 4.3.16, and section 0.4 it suffices to take $\mathfrak{X} \in \mathsf{Cs}_\alpha^{reg}$ and $x \in {}^\beta A$ such that $\Delta x_\xi \subseteq \rho\xi$ for all $\xi < \beta$ and find $h \in Hom({}_s\mathfrak{Fm}_{\tau 0}^\Lambda, \mathfrak{X})$ such that, for any $\xi < \beta$, $h(\mathbf{R}_\xi(v_0, \ldots, v_\eta, \ldots)_{\eta < \rho\xi}/=) = x_\xi$. Let $\mathfrak{B} = \mathfrak{Cg}^{\mathfrak{X}} Rgx$. Then $\langle \mathfrak{B}, x \rangle$ is a Cs_α with generators which is suitable for Λ. Let $\mathfrak{R} = \mathfrak{Rb}_{\mathfrak{B}x}^\Lambda$. By Theorem 4.3.10(ii) we have $\mathfrak{B} = {}^\tau\mathfrak{Cl}_\alpha^{\mathfrak{R}}$ and $x = \langle \{a \in {}^\alpha M : \rho\xi \upharpoonright a \in \mathbf{R}_\xi^{\mathfrak{R}}\} : \xi < \beta \rangle$. Then by Theorem 4.3.1 choose $k \in Ho(\mathfrak{Fm}_\tau^\Lambda, \mathfrak{B})$ such that $k\mathbf{R}_\xi(v_0, \ldots, v_\eta, \ldots)_{\eta < \rho\xi} = x_\xi$ for all $\xi < \beta$. Then $k\varphi = \varphi^{\mathfrak{R}}$ for every restricted formula φ. Note that if $\varphi = \psi$, then $\models \varphi \leftrightarrow \psi$ and hence $k\varphi = k\psi$. Hence the existence of the desired function h is clear.

COROLLARY 4.3.18. *Let* $\Lambda = \langle \alpha, \mathbf{R}, \rho \rangle$, *with* $Do\rho = \beta$. (i) *If* Λ *is an ordinary language, then* ${}_s\mathfrak{Fm}_{\tau 0}^\Lambda \in \mathsf{Lf}_\alpha$.

(ii) *If* Λ *is a full language, then* ${}_s\mathfrak{Fm}_{\tau 0}^\Lambda \cong \mathfrak{Fr}_\beta \mathsf{Gs}_\alpha$.

A converse of Corollary 4.3.18 also holds; see 4.3.28.

Returning to the ideas of Remark 4.3.11, we may say that the complete algebraic form of a theory in a language is a pair $\langle \mathfrak{X}, x \rangle$ such that $\mathfrak{X}$ is an Lf_ω and x is an indexed system of generators of $\mathfrak{X}$; the natural generalization is simply an arbitrary CA_α. The complete algebraic form of a language itself is a dimension-restricted free algebra $\mathfrak{Fr}_\beta^\rho \mathsf{Gs}_\omega$, with $\rho\xi < \omega$ for all $\xi < \beta$, and generalization yields the free algebras $\mathfrak{Fr}_\beta \mathsf{CA}_\alpha$.

Now we turn to the syntactic method of defining cylindric algebras. For ordinary logic one defines $\Gamma \vdash \varphi$ in a proof-theoretic way and then shows, as a form of the completeness theorem, that $\Gamma \vdash \varphi$ iff $\Gamma \models \varphi$. Thus (assuming the completeness theorem), this method does not yield any new cylindric algebras. We shall discuss the notion $\vdash$ only in some methodological remarks in which the completeness theorem is not assumed.

For an arbitrary language we introduce a proof-theoretical notion $\vdash_\tau$ suitable for restricted formulas. Let $\Lambda = \langle \alpha, \mathbf{R}, \rho \rangle$ with $Do\mathbf{R} = \beta$. The set $\Lambda\gamma^\Lambda$ *of logical axioms* for Λ consists of all *restricted* formulas of the following kinds, where φ and ψ are arbitrary restricted formulas and κ, λ, μ arbitrary ordinals $< \alpha$:

(1) φ, a propositional tautology,

(2) $\forall v_\kappa(\varphi \to \psi) \to (\forall v_\kappa \varphi \to \forall v_\kappa \psi)$,

(3) $\forall v_\kappa \varphi \to \varphi$,

(4) $\varphi \to \forall v_\kappa \varphi$, if v_κ does not occur free in φ,

(5) $v_\kappa = v_\kappa$,

(6) $\exists v_\kappa(v_\kappa = v_\lambda)$,

(7) $v_\kappa = v_\lambda \to (v_\kappa = v_\mu \to v_\lambda = v_\mu)$,

(8) $v_\kappa = v_\lambda \to [\varphi \to \forall v_\kappa(v_\kappa = v_\lambda \to \varphi)]$ if $\kappa \neq \lambda$,

(9) $\exists v_\kappa \varphi \leftrightarrow \neg \forall v_\kappa \neg \varphi$.

Now let $\Gamma \cup \{\varphi\}$ be a set of restricted formulas of Λ. We write $\Gamma \vdash_{\tau\Lambda} \varphi$, or simply $\Gamma \vdash_\tau \varphi$, if φ belongs to every set Ω of restricted formulas such that $\Lambda\gamma^\Lambda \cup \Gamma \subseteq \Omega$, $\chi \in \Omega$ whenever $\psi \to \chi \in \Omega$ and $\psi \in \Omega$, and $\forall v_\kappa \psi \in \Omega$ whenever $\psi \in \Omega$ and $\kappa < \alpha$. As usual, the last two conditions are non-symbolically expressed by saying that Ω is closed under detachment and generalization. We write $\vdash_\tau \varphi$ for $0 \vdash_\tau \varphi$.

DEFINITION 4.3.19. *Let* Λ *be a language,* Γ *a set of restricted formulas of* Λ. *We set*

$$ {}_p\equiv^{\Lambda}_{r\Gamma} = \{\langle\varphi,\psi\rangle : \varphi \text{ and } \psi \text{ are restricted formulas of } \Lambda \text{ and } \Gamma\vdash_r \varphi\leftrightarrow\psi\}. $$

THEOREM 4.3.20. *Let* Λ *and* Γ *be as in* 4.3.19. *Then* ${}_p\equiv^{\Lambda}_{r\Gamma} \in Co\mathfrak{Sm}^{\Lambda}_r$.

DEFINITION 4.3.21. *Let* Λ *and* Γ *be as in* 4.3.19. *We set* ${}_p\mathfrak{Sm}^{\Lambda}_{r\Gamma} = \mathfrak{Sm}^{\Lambda}_r / {}_p\equiv^{\Lambda}_{r\Gamma}$.

LEMMA 4.3.22. *Let* Λ *and* Γ *be as in* 4.3.18, *with* $\Lambda = \langle\alpha,\mathbf{R},\rho\rangle$. *Then* ${}_p\mathfrak{Sm}^{\Lambda}_{r\Gamma} \in \mathsf{CA}_\alpha$.

PROOF. We write $\vdash$ in place of $\vdash_r$. We must check the postulates (C_0)–(C_7) for cylindric algebras given on p.162 of Part I. (C_0) is clear. For (C_1), recall that $0 = \mathbf{F}/\equiv$. We have $\Gamma\vdash\neg\mathbf{F}$ by (1) above, so $\Gamma\vdash\forall v_\kappa\neg\mathbf{F}$, hence easily $\Gamma\vdash\exists v_\kappa\mathbf{F}\leftrightarrow\mathbf{F}$, which proves (C_1). Now we will frequently use in this proof the fact that $\Gamma\vdash\varphi\rightarrow\psi$ implies that $\varphi/\equiv\ \leqq \psi/\equiv$. Applying this to (3) easily gives (C_2). To prove (C_3) we need several auxiliary statements. We use x,y for arbitrary elements of ${}_pFm^{\Lambda}_{\Gamma}$, φ,ψ for arbitrary restricted formulas of Λ, and κ,λ,μ for arbitrary ordinals $<\alpha$.

(a) If $x\leqq y$, then $c_\kappa x \leqq c_\kappa y$.

This is clear, since $\Gamma\vdash\varphi\rightarrow\psi$ implies $\Gamma\vdash\forall v_\kappa(\varphi\rightarrow\psi)$ and hence $\Gamma\vdash\forall v_\kappa\varphi\rightarrow\forall v_\kappa\psi$.

(b) $c_\kappa c_\kappa x = c_\kappa x$.

For, $c_\kappa x \leqq c_\kappa c_\kappa x$ by (C_2), which has already been established. We have $\Gamma\vdash\neg\exists v_\kappa\varphi\rightarrow\forall v_\kappa\neg\exists v_\kappa\varphi$ by (4), and this easily gives $c_\kappa c_\kappa x \leqq c_\kappa x$.

Using (a) and (b) we can get half of (C_3):

(c) $c_\kappa(x\cdot c_\kappa y) \leqq c_\kappa x\cdot c_\kappa y$.

For, $x\cdot c_\kappa y \leqq x$, so $c_\kappa(x\cdot c_\kappa y)\leqq c_\kappa x$ by (a). And $x\cdot c_\kappa y\leqq c_\kappa y$, so $c_\kappa(x\cdot c_\kappa y)\leqq c_\kappa c_\kappa y = c_\kappa y$ by (a) and (b). Thus (c) holds.

For the other direction, note that (2) directly yields:

(d) $-c_\kappa(x\cdot y)\leqq c_\kappa -x + -c_\kappa y$.

Using (C_2) we establish the following similarly to (b):

(e) $c_\kappa(-c_\kappa x) = -c_\kappa x$.

It follows that

$$ -c_\kappa(x\cdot c_\kappa y) = -c_\kappa(c_\kappa y\cdot x) \leqq c_\kappa(-c_\kappa y) + -c_\kappa x = -c_\kappa y + -c_\kappa x, $$

which gives the other half of (C_3).

Note the following consequence of (4):

(f) $c_\lambda c_\kappa c_\lambda x = c_\kappa c_\lambda x$.

Hence we have $x \leq c_\kappa c_\lambda x$ by (C_2), $c_\kappa x \leq c_\kappa c_\kappa c_\lambda x = c_\kappa c_\lambda x$ by (a), (b), so $c_\lambda c_\kappa x \leq c_\kappa c_\lambda x$ by (f). Symmetry yields (C_4). (C_5) follows from (5).

Next note from (5) and (7) that $\Gamma \vdash v_\kappa \equiv v_\lambda \Rightarrow v_\lambda \equiv v_\kappa$. Hence

(g) $d_{\kappa\lambda} = d_{\lambda\kappa}$.

Now using (g), (C_6) is a clear consequence of (6) if $\lambda = \mu$.

Next note a further consequence of (4):

(h) $c_\kappa d_{\lambda\mu} = d_{\lambda\mu}$ if κ,λ,μ are distinct.

Hence half of (C_6) for κ,λ,μ distinct easily follows: $d_{\kappa\lambda} \leq -d_{\kappa\mu} + d_{\lambda\mu}$ by (7), hence $d_{\lambda\kappa} \cdot d_{\kappa\mu} \leq d_{\lambda\mu}$ using (g), so $c_\kappa(d_{\lambda\kappa} \cdot d_{\kappa\mu}) \leq c_\kappa d_{\lambda\mu} = d_{\lambda\mu}$ by (a), (h). Now note from (8):

(i) $d_{\kappa\lambda} \cdot c_\kappa(d_{\kappa\lambda} \cdot x) \leq x$ if $\kappa \neq \lambda$.

Now assume that κ,λ,μ are distinct. Then

$$d_{\lambda\mu} \cdot c_\lambda(d_{\lambda\mu} \cdot c_\kappa(d_{\lambda\kappa} \cdot d_{\kappa\mu})) \leq c_\kappa(d_{\lambda\kappa} \cdot d_{\kappa\mu})$$

by (i), and

$$\begin{aligned} c_\lambda(d_{\lambda\mu} \cdot c_\kappa(d_{\lambda\kappa} \cdot d_{\kappa\mu})) &= c_\lambda c_\kappa(d_{\lambda\kappa} \cdot d_{\kappa\mu}) \text{ by the first half of } (C_6) \\ &= c_\kappa c_\lambda(d_{\lambda\kappa} \cdot c_\lambda d_{\kappa\mu}) \text{ by } (C_4), \text{ (h)} \\ &= c_\kappa(c_\lambda d_{\lambda\kappa} \cdot c_\lambda d_{\kappa\mu}) = c_\kappa c_\lambda d_{\kappa\mu} = c_\lambda 1 = 1, \end{aligned}$$

and (C_6) follows. Finally, (C_7) follows from what we have shown: by (i) we have $d_{\kappa\lambda} \cdot x \cdot c_\kappa(d_{\kappa\lambda} \cdot -x) = 0$, and $(C_1),(C_3)$ yield (C_7).

Again, a converse of 4.3.22 holds; see 4.3.28. Lemma 4.3.22 yields a connection between our syntactical and semantical definitions of cylindric algebras, and two completeness theorems:

THEOREM 4.3.23. *Let* $\Lambda = \langle \alpha, \mathbf{R}, \rho \rangle$ *be a language and* $\Gamma \cup \{\varphi\}$ *a set of restricted formulas of* Λ, *with* $\mathrm{Do}\mathbf{R} = \beta$.

(i) *If* Λ *is ordinary, then* ${}_p\mathfrak{Fm}^\Lambda_{r\Gamma} = {}_s\mathfrak{Fm}^\Lambda_{r\Gamma} \in \mathsf{Lf}_\alpha$.

(ii) *Suppose that* α *is infinite, and* $\alpha \sim \rho\xi$ *is infinite for every* $\xi < \beta$. *Then* ${}_p\mathfrak{Fm}^\Lambda_{r\Gamma} = {}_s\mathfrak{Fm}^\Lambda_{r\Gamma} \in \mathsf{Dc}_\alpha$.

(iii) *Under either hypothesis* (i) *or* (ii), *if* $\Gamma \nvdash_r \psi$ *for a certain formula* ψ, *then* Γ *has a model* $\mathfrak{R}$ *such that* $\mathfrak{R} \nvDash \psi[x]$ *for some* $x \in {}^\alpha M$; *and* $\Gamma \vdash_r \varphi$ *iff* $\Gamma \vDash \varphi$, *for every formula* φ.

PROOF. We indicate the proof for Λ ordinary; the hypothesis of (ii) is treated similarly. Clearly ${}_p\mathfrak{Fm}^\Lambda_{r\Gamma} \in \mathsf{Lf}_\alpha$. Next we show that if $\Gamma \nvdash_r \psi$, then Γ has a model with the additional property indicated. Since $\mathsf{Lf}_\alpha \subseteq \mathsf{SPCs}^{reg}_\alpha$ by 3.2.8, there is a homomorphism h of ${}_p\mathfrak{Fm}^\Lambda_{r\Gamma}$ onto a $\mathsf{Cs}^{reg}_\alpha \cap \mathsf{Lf}_\alpha$ $\mathfrak{A}$ with non-empty base M, such that

$h(\psi/\equiv) \neq 1$. Let $x_\xi = h(\mathbf{R}_\xi(v_0,\ldots,v_\eta,\ldots)_{\eta<\rho\xi}/\equiv)$ for each $\xi<\beta$. Thus $\langle\mathfrak{A},x\rangle$ is a Cs_α with generators which is suitable for Λ. Let $\mathfrak{R} = \mathfrak{Rb}^\Lambda_{\mathfrak{A}x}$. Then by 4.3.10(ii) we have $\mathfrak{A} = {}^r\mathfrak{Cf}^{\mathfrak{R}}_\alpha$, and clearly $h(\varphi/\equiv) = \varphi^{\mathfrak{R}}$ for each $\varphi \in \Phi\mu^\Lambda_\tau$. Since $\varphi/\equiv = 1$ for each $\varphi \in \Gamma$ it follows that $\mathfrak{R}$ is a model of Γ; and the desired property for ψ is clear too.

In the usual way one now obtains: $\Gamma \vdash_\tau \varphi$ iff $\Gamma \models \varphi$, for every formula φ. Hence clearly ${}_p\mathfrak{Sm}^\Lambda_{\tau\Gamma} = {}_s\mathfrak{Sm}^\Lambda_{\tau\Gamma}$. This finishes the proof.

REMARK 4.3.24. In the proof of 4.3.23 we have used the representation theory to establish two unusual completeness theorems; the first one is due to Monk [65a]. One can also use our representation theorems to establish completeness theorems for more conventional proof systems. The proofs follow the same lines. If one deals with all formulas instead of just restricted ones, then one must use 4.3.6 and also the fact that $\varphi \leftrightarrow f\varphi$ is provable, with f as in the proof 4.3.6.

Conversely, completeness theorems imply representation theorems; see 4.3.29.

THEOREM 4.3.25. *If $\Lambda = \langle\alpha,\mathbf{R},\rho\rangle$ is any language and $Do\mathbf{R} = \beta$, then ${}_p\mathfrak{Sm}^\Lambda_{\tau 0} \cong \mathfrak{Tr}^\rho_\beta \mathsf{CA}_\alpha$.*

PROOF. By Lemma 4.3.22 we have ${}_p\mathfrak{Sm}^\Lambda_{\tau 0} \in \mathsf{CA}_\alpha$, so by 2.5.37 it suffices to show that if $\mathfrak{A} \in \mathsf{CA}_\alpha$, $a \in {}^\beta A$, and $\Delta a_\xi \subseteq \rho\xi$ for all $\xi<\beta$, then there is a $k \in Hom({}_p\mathfrak{Sm}^\Lambda_{\tau 0},\mathfrak{A})$ such that $k(\mathbf{R}_\xi(v_0,\ldots,v_\eta,\ldots)_{\eta<\rho\xi}/\equiv) = a_\xi$ for each $\xi<\beta$. By Theorem 4.3.1(iii) there is an $h \in Hom(\mathfrak{Sm}^\Lambda_\tau,\mathfrak{A})$ such that $h\mathbf{R}_\xi(v_0,\ldots,v_\eta,\ldots)_{\eta<\rho\xi} = a_\xi$ for each $\xi<\beta$. Thus by Definition 4.3.19 it suffices to show that $\varphi \underset{p}{=}^\Lambda_{\tau 0} \psi$ implies that $h\varphi = h\psi$; equivalently, we want to show that $\vdash_\tau \varphi$ implies $h\varphi = 1$.

Let $\Delta = \{\varphi : \varphi \in \Phi\mu^\Lambda_\tau$ and $h\varphi = 1\}$. It suffices to show that $\Lambda\gamma^\Lambda \subseteq \Delta$ and Δ is closed under detachment and generalization. To show that $\Lambda\gamma^\Lambda \subseteq \Delta$, we check only (1), (2), (4), and (8), the others being very easy. Suppose that φ is a propositional tautology but $h\varphi \neq 1$. Then there is a homomorphism l from $\mathfrak{B}\mathfrak{A}$ onto a two-element BA $\mathfrak{B}$ such that $lh\varphi = 0$. But then $l \circ h$ assigns to every $\psi \in \Phi\mu^\Lambda_\tau$ a value 0 or 1, and $lh(\psi \vee \chi)$, $lh(\psi \wedge \chi)$, $lh(\neg\psi)$, $lh\mathbf{F}$, $lh\mathbf{T}$ are evaluated by the familiar truth tables. This contradicts φ being a propositional tautology. Next, given an instance of (2) we apply h to get

$$\begin{aligned}
&\mathsf{c}_\kappa(h\varphi \cdot -h\psi) + \mathsf{c}_\kappa(-h\varphi) + -\mathsf{c}_\kappa(-h\psi)\\
&\geq \mathsf{c}_\kappa(h\varphi \cdot -h\psi) + \mathsf{c}_\kappa(-h\varphi \cdot -h\psi) + -\mathsf{c}_\kappa(-h\psi)\\
&= \mathsf{c}_\kappa(-h\psi) + -\mathsf{c}_\kappa(-h\psi) = 1.
\end{aligned}$$

Concerning (4), one can easily show by induction on φ that $h\varphi = \mathsf{c}_\kappa h\varphi$ if v_κ does not occur free in φ; hence $h\psi = 1$ for any instance ψ of (4). Applying h to an instance of (8), we get

$$\begin{aligned}
-\mathsf{d}_{\kappa\lambda} + -h\varphi + -\mathsf{c}_\kappa(\mathsf{d}_{\kappa\lambda} \cdot -h\varphi) &= -(\mathsf{d}_{\kappa\lambda} \cdot h\varphi \cdot \mathsf{c}_\kappa(\mathsf{d}_{\kappa\lambda} \cdot -h\varphi))\\
&= -(\mathsf{d}_{\kappa\lambda} \cdot h\varphi \cdot \mathsf{d}_{\kappa\lambda} \cdot -h\varphi) = 1.
\end{aligned}$$

Thus $\Lambda\gamma^\Lambda \subseteq \Delta$. If $\psi \to \chi \in \Delta$ and $\psi \in \Delta$, then $-h\psi + h\chi = 1$ and $h\psi = 1$, so $h\chi = 1$ and $\chi \in \Delta$. If $\psi \in \Delta$ and $\kappa<\alpha$, then $h\psi = 1$, so $1 = \mathsf{c}^\partial_\kappa h\psi = h\forall v_\kappa\psi = 1$ and hence $\forall v_\kappa\psi \in \Delta$. This finishes the proof.

COROLLARY 4.3.26. *If* $\Lambda = \langle \alpha, \mathbf{R}, \rho \rangle$ *is a full language, then* ${}_p\mathfrak{Fm}^{\Lambda}_{r0} \cong \mathfrak{Fr}_\beta \mathsf{CA}_\alpha$, *where* $Do\mathbf{R} = \beta$.

From these results it is rather easy to get our logical representation theorem. We need first a simple auxiliary result.

THEOREM 4.3.27. *Let* Λ *be a language and* $\Sigma \subseteq \Phi\mu^{\Lambda}_r$; *suppose further that* $t \in \{s,p\}$. *Let* F *be a filter on* ${}_t\mathfrak{Fm}^{\Lambda}_{r\Sigma}$, *and set* $T = \{\varphi \in \Phi\mu^{\Lambda}_r : \varphi/{}_t\!\equiv^{\Lambda}_{r\Sigma} \in F\}$. *Then* ${}_t\mathfrak{Fm}^{\Lambda}_{r\Sigma}/F \cong {}_t\mathfrak{Fm}^{\Lambda}_{rT}$.

PROOF. Denote equivalence classes under F by x/F. Then the desired isomorphism f is determined by the condition that $f((\varphi/\equiv_\Sigma)/F) = \varphi/\equiv_T$; since $\varphi/\equiv_\Sigma \in F$ iff $\varphi \in T$, f exists and is an isomorphism.

Now we can prove the desired logical representation theorem:

THEOREM 4.3.28. (i) $\mathsf{CA}_\alpha = \mathsf{I}\{{}_p\mathfrak{Fm}^{\Lambda}_{r\Sigma} : \Lambda$ *is a (full) language,* $\Sigma \subseteq \Phi\mu^{\Lambda}_r\}$.
(ii) $\mathsf{IGs}_\alpha = \mathsf{I}\{{}_s\mathfrak{Fm}^{\Lambda}_{r\Sigma} : \Lambda$ *is a (full) language,* $\Sigma \subseteq \Phi\mu^{\Lambda}_r\}$, *for* $\alpha \geqq 2$.
(iii) *For* $\alpha \geqq \omega$, $\mathsf{Lf}_\alpha = \mathsf{I}\{{}_p\mathfrak{Fm}^{\Lambda}_{r\Sigma} : \Lambda$ *is an ordinary language,* $\Sigma \subseteq \Phi\mu^{\Lambda}_r\}$.

PROOF. We prove only (i), the statements (ii), (iii) being proved similarly. The inclusion $\supseteq$ follows from Lemma 4.3.22. For $\subseteq$, let $\mathfrak{A}$ be any CA_α. Let $\Lambda = \langle \alpha, \mathbf{R}, \rho \rangle$ be a full language with $Do\mathbf{R} = |A|$. Then by Corollary 4.3.26 there is a filter F on ${}_p\mathfrak{Fm}^{\Lambda}_{r0}$ such that ${}_p\mathfrak{Fm}^{\Lambda}_{r0}/F \cong \mathfrak{A}$. Then by Theorem 4.3.27 there is a set $\Sigma \subseteq \Phi\mu^{\Lambda}_r$ with ${}_r\mathfrak{Fm}^{\Lambda}_{r\Sigma} \cong \mathfrak{A}$, as desired.

REMARK 4.3.29. In 4.3.23 – 4.3.24 we observed that one can easily prove completeness theorems from representation theorems. The converse is also true. Thus, note that all parts of Theorem 4.3.28 were established by elementary arguments. The completeness theorem for $\vdash_r \Lambda$, Λ ordinary, yields that ${}_p\mathfrak{Fm}^{\Lambda}_{r\Sigma} = {}_s\mathfrak{Fm}^{\Lambda}_{r\Sigma}$, and hence by 4.3.28 $\mathsf{Lf}_\alpha \subseteq \mathsf{IGs}_\alpha$ if $\alpha \geqq \omega$. If a more conventional proof notion is used, this line of reasoning becomes somewhat more involved but still elementary; the main steps are analogs of 4.3.22 (${}_p\mathfrak{Fm}^{\Lambda}_{r\Gamma} \in \mathsf{CA}_\alpha$) and 4.3.25 (${}_p\mathfrak{Fm}^{\Lambda}_{r0} \cong \mathfrak{Fr}^{\rho}_\beta \mathsf{CA}_\alpha$).

REMARK 4.3.30. Theorem 4.3.27 can be extended to establish a one–one correspondence between filters and theories. We modify the notion of a theory from Part I, p.45, as follows. Let Λ be a language. An *s–theory* (resp. *p–theory*) *in* Λ is a set Σ of restricted formulas of Λ such that for any restricted formula φ, if $\Sigma \models \varphi$ (resp. $\Sigma \vdash_r \varphi$), then $\varphi \in \Sigma$. Note that every s–theory is also a p–theory. To establish the correspondence for proof notions we will need the following form of the well–known *deduction theorem*: if $\Gamma \cup \{\varphi\} \vdash_r \psi$, then there is an $i \in \omega$ and a $\kappa \in {}^i\alpha$ such that $\Gamma \vdash_r \forall v_{\kappa 0} \ldots \forall v_{\kappa(i-1)} \varphi \rightarrow \psi$. This is proved in the usual way.

THEOREM 4.3.31. *Let* Λ *be a language,* $\Sigma \subseteq \Phi\mu^{\Lambda}_r$, *and* $t \in \{s,p\}$. *Let* $\mathcal{F}$ *be the set of all filters in* ${}_t\mathfrak{Fm}^{\Lambda}_{r\Sigma}$ *and* $\mathcal{T}$ *the set of all those* t*–theories in* Λ *which contain* Σ. *For each*

$F \in \mathcal{F}$ let $fF = \{\varphi \in \Phi\mu_r^\Lambda : \varphi/_t\equiv_\Sigma^\Lambda \in F\}$. Then f is a one–one function mapping $\mathcal{F}$ onto $\mathcal{T}$. Furthermore, for each $F \in \mathcal{F}$ we have ${}_t\mathfrak{Fm}_{r\Sigma}^\Lambda/F \cong {}_t\mathfrak{Fm}_{r(fF)}^\Lambda$.

PROOF. Everything is routine (using 4.3.27) except checking that fF is a t–theory when F is a filter. Here we treat the cases $t = s$, $t = p$ separately.

First suppose that $t = p$; assume that $fF \vdash_r \varphi$. By the deduction theorem and the fact that F is closed under each operation c_κ^∂ we obtain $i \in \omega \sim \{0\}$ and $\varphi \in {}^i fF$ such that $\vdash_r \psi_0 \wedge \ldots \wedge \psi_{i-1} \rightarrow \varphi$. It follows that

$$(\psi_0/\equiv)\cdot\ldots\cdot(\psi_{i-1}/\equiv) \leq \varphi/\equiv,$$

so $\varphi/\equiv \in F$ and hence $\varphi \in fF$, as desired.

Now take $t = s$; assume that $fF \models \varphi$. Suppose that $\varphi \notin fF$. Thus $\varphi/\equiv \notin F$, and so $(\varphi/\equiv)/F \neq 1$. Now by 4.3.16 and its proof there is a homomorphism h of ${}_\alpha\mathfrak{Fm}_{r\Sigma}^\Lambda/F$ into a Cs_α^{reg} $\mathfrak{X}$ such that $h((\varphi/\equiv)/F) \neq 1$. Let $x_\xi = h((\mathbf{R}_\xi(v_0,\ldots,v_\eta,\ldots)_{\eta<\rho\xi}/\equiv)/F)$ for each $\xi<\beta$ (where $\Lambda = \langle\alpha,\mathbf{R},\rho\rangle$ with $Do\mathbf{R} = \beta$). Thus $\langle\mathfrak{X},x\rangle$ is a Cs_α generators suitable for Λ (see 4.3.9). Let $\mathfrak{M} = \mathfrak{Rb}_{\mathfrak{X}x}^\Lambda$. Clearly $h((\psi/\equiv)/F) = \psi^{\mathfrak{M}}$ for every restricted formula ψ. Hence $\mathfrak{M}$ is a model of fF, but not of φ, contadicting $fF \models \varphi$.

This finishes our description of most of the basic connections between logic and CA_α's. One further important basic connection – between formulas and terms – will be treated later.

An application

At this point we want to indicate how a logical theorem can be used to prove an algebraic result, namely the following theorem about free algebras.

THEOREM 4.3.32. *If $3 \leq \alpha < \omega$ and $4 \leq \beta < \omega$, then $\mathfrak{Fr}_\beta\mathsf{Gs}_\alpha$ is not atomic.*

PROOF. We shall use a result of Ehrenfeucht, Fuhrken [71**]. They work in an ordinary language Ω with ω variables $\langle v_\kappa : \kappa<\omega\rangle$ and with four non–logical constants: unary function symbols $\mathbf{f}_0,\mathbf{f}_1,\mathbf{g}$ and a binary relation symbol $\mathbf{R}$. They construct a sentence φ in this language with the following properties:

(1) The variables occurring in φ are v_0,v_1,v_2.

(2) φ is consistent.

(3) If ψ is a formula of Ω which has only the variables $v_0,\ldots,v_{\alpha-1}$ in it, is consistent, and $\models \psi \rightarrow \varphi$, then there is a formula χ with at most the variables $v_0,\ldots,v_{\alpha-1}$ in it, such that $\psi \wedge \chi$ and $\psi \wedge \neg\chi$ are consistent.

Our theorem follows in a routine way from this result. Namely, let $\Lambda = \langle\alpha,\mathbf{R},\rho\rangle$ be a full language, with $Do\mathbf{R} = \beta$. By Corollary 4.3.18 we have $\mathfrak{Fr}_\beta\mathsf{Gs}_\alpha \simeq {}_p\mathfrak{Fm}_{r0}^\Lambda$. Hence it

suffices to show that ${}_{\mathfrak{s}}\mathfrak{Fm}^{\Lambda}_{\tau 0}$ is not atomic. To do this we carry out in this special case the general procedure of replacing a language with function symbols by one without, converting to restricted formulas too. Since only finitely many variables are available, some care must be taken.

A possible model $\mathfrak{R}$ of Λ is called *normal* if the following conditions hold:

(4) $\mathbf{C}_\kappa^{[V]}\mathbf{R}_i^{\mathfrak{R}} = \mathbf{R}_i^{\mathfrak{R}}$ for all $\kappa \in \alpha \sim 2$ and all $i<4$, where $V = {}^\alpha M$.

(5) $\mathbf{R}_i^{\mathfrak{R}} = {}^\alpha M$ for all $i \in \beta \sim 4$.

Now for any $i<4$ and any $\mu,\nu<\alpha$ one can associate a restricted formula $\mathbf{R}_i^! v_\mu v_\nu$ of Λ having only v_μ and v_ν free such that for any normal $\mathfrak{R}$ and any $a \in {}^\alpha M$, $\mathfrak{R} \models \mathbf{R}_i^! v_\mu v_\nu[a]$ iff $\langle a_\mu, a_\nu, a_\nu, \ldots \rangle \in \mathbf{R}_i^{\mathfrak{R}}$. For example, take $\mu = 1$ and $\nu = 0$ (one of the hardest possibilities). Then we let $\mathbf{R}_i^! v_1 v_0$ be the following formula:

$$\exists v_2(v_2 = v_0 \wedge \exists v_0(v_0 = v_1 \wedge \exists v_1(v_1 = v_2 \wedge \exists v_2 \ldots \exists v_{\alpha-1} \mathbf{R}_i v_0 \ldots v_{\alpha-1})))$$

Next, we associate with each formula φ of Ω with variables among $v_0, \ldots, v_{\alpha-1}$ a formula φ' of Λ, by induction, where t and r are terms of Ω with variables v_κ, v_λ respectively:

$(\mathbf{R} v_\mu v_\nu)' = \mathbf{R}_3^! v_\mu v_\nu$; $\quad (v_\mu = v_\nu)' = v_\mu = v_\nu$;
$(\mathbf{f}_0 v_\mu = v_\nu)' = \mathbf{R}_0^! v_\mu v_\nu$; $\quad (\mathbf{f}_1 v_\mu = v_\nu)' = \mathbf{R}_1^! v_\mu v_\nu$;
$(\mathbf{g} v_\mu = v_\nu)' = \mathbf{R}_2^! v_\mu v_\nu$;
$(\mathbf{f}_0 t = v_\nu)' = \exists v_\rho((t = v_\rho)' \wedge \mathbf{R}_0^! v_\rho v_\nu)$, with $\rho = \min(\alpha \sim \{v_\kappa, v_\nu\})$;
$(\mathbf{f}_1 t = v_\nu)' = \exists v_\rho((t = v_\rho)' \wedge \mathbf{R}_1^! v_\rho v_\nu)$, ρ as above;
$(\mathbf{g} t = v_\nu)' = \exists v_\rho((t = v_\rho)' \wedge \mathbf{R}_2^! v_\rho v_\nu)$, ρ as above;
$(t = r)' = \exists v_\rho((t = v_\rho)' \wedge (r = v_\rho)')$, with $\rho = \min(\alpha \sim \{v_\kappa, v_\lambda\})$, if r is not a variable;
$(\mathbf{R} t v_\mu)' = \exists v_\rho((t = v_\rho)' \wedge \mathbf{R}_3^! v_\rho v_\mu)$ with $\rho = \min(\alpha \sim \{v_\kappa, v_\mu\})$, if t is not a variable;
$(\mathbf{R} t r)' = \exists v_\rho((r = v_\rho)' \wedge (\mathbf{R} t v_\rho)')$ with $\rho = \min(\alpha \sim \{v_\kappa, v_\lambda\})$, if r is not a variable;
$(\neg \varphi)' = \neg \varphi'$; $(\varphi \wedge \psi)' = \varphi' \wedge \psi'$; $(\varphi \vee \psi)' = \varphi' \vee \psi'$; $\mathbf{F}' = \mathbf{F}$; $\mathbf{T}' = \mathbf{T}$;
$(\exists v_\rho \varphi)' = \exists v_\rho \varphi'$.

Next, let $\mathfrak{R}$ be a possible model of Ω. We associate with it a possible model $\mathfrak{R}^\Lambda$ of Λ: (with $\mathfrak{R} = \mathfrak{R}^\Lambda$)

$M^\Lambda = M$; $\quad \mathbf{R}_2^{\mathfrak{R}} = \{a \in {}^\alpha M : \mathbf{g}^{\mathfrak{R}} a_0 = a_1\}$;
$\mathbf{R}_0^{\mathfrak{R}} = \{a \in {}^\alpha M : \mathbf{f}_0^{\mathfrak{R}} a_0 = a_1\}$; $\quad \mathbf{R}_3^{\mathfrak{R}} = \{a \in {}^\alpha M : a_0 \mathbf{R}^{\mathfrak{R}} a_1\}$;
$\mathbf{R}_1^{\mathfrak{R}} = \{a \in {}^\alpha M : \mathbf{f}_1^{\mathfrak{R}} a_0 = a_1\}$; $\quad \mathbf{R}_i^{\mathfrak{R}} = {}^\alpha M$ if $i \in \beta \sim 4$.

Clearly $\mathfrak{R}^\Lambda$ is normal. Now the following is easy to check:

(6) Let $\mathfrak{R}$ be a possible model of Ω, $a \in {}^\alpha M$, φ a formula of Ω with variables among $v_0, \ldots, v_{\alpha-1}$. Then $\mathfrak{R} \models \varphi[a]$ iff $\mathfrak{R}^\Lambda \models \varphi'[a]$.

With each restricted formula ψ of Λ we now associate a formula ψ^* of Ω:

$$
\begin{aligned}
&(\mathbf{R}_0 v_0 \ldots v_{\alpha-1})^* = \mathbf{f}_0 v_0 = v_1 ;\\
&(\mathbf{R}_1 v_0 \ldots v_{\alpha-1})^* = \mathbf{f}_1 v_0 = v_1 ;\\
&(\mathbf{R}_2 v_0 \ldots v_{\alpha-1})^* = \mathbf{g} v_0 = v_1 ;\\
&(\mathbf{R}_3 v_0 \ldots v_{\alpha-1})^* = \mathbf{R} v_0 v_1 ;\\
&(\mathbf{R}_i v_0 \ldots v_{\alpha-1})^* = \mathbf{T} \text{ if } i \in \beta \sim 4 ;\\
&(v_\kappa = v_\lambda)^* = v_\kappa = v_\lambda ;\\
&(\neg \varphi)^* = \neg \varphi^*;\ \mathbf{F}^* = \mathbf{F};\ \mathbf{T}^* = \mathbf{T};\\
&(\varphi \wedge \psi)^* = \varphi^* \wedge \psi^* ;\ (\varphi \vee \psi)^* = \varphi^* \vee \psi^*;\\
&(\exists v_\kappa \varphi)^* = \exists v_\kappa \varphi^* ;
\end{aligned}
$$

Next, let γ be the conjunction of the following formulas of Λ:

$$
\begin{gathered}
\mathbf{R}_i v_0 \ldots v_{\alpha-1} \leftrightarrow \exists v_2 \ldots \exists v_{\alpha-1} \mathbf{R}_i v_0 \ldots v_{\alpha-1},\ i<4,\\
\forall v_0 \exists v_1 [\mathbf{R}_i' v_0 v_1 \wedge \forall v_2 (\mathbf{R}_i' v_0 v_2 \rightarrow v_1 = v_2)],\ i<3,\\
\mathbf{R}_i v_0 \ldots v_{\alpha-1},\ 4 \leqq i < \beta.
\end{gathered}
$$

One can easily show:

(7) If ψ is a restricted formula of Λ, then $\gamma \models \psi \leftrightarrow \psi^{*\prime}$

(8) If $\mathfrak{M}$ is a possible model of Ω, then $\mathfrak{M}^\Lambda$ is a model of γ.

With each model $\mathfrak{M}$ of γ we associate a possible model $\mathfrak{M}^\Omega$ of Ω as follows, with $\mathfrak{N} = \mathfrak{M}^\Omega$ and $a \in M$:

$$
\begin{aligned}
&M^\Omega = M;\\
&\mathbf{f}_0^{\mathfrak{N}} a = \text{the } b \text{ such that } \mathfrak{M} \models \mathbf{R}_0' v_0 v_1 [a,b] ;\\
&\mathbf{f}_1^{\mathfrak{N}} a = \text{the } b \text{ such that } \mathfrak{M} \models \mathbf{R}_1' v_0 v_1 [a,b] ;\\
&\mathbf{g}^{\mathfrak{N}} a = \text{the } b \text{ such that } \mathfrak{M} \models \mathbf{R}_2' v_0 v_1 [a,b] ;\\
&\mathbf{R}^{\mathfrak{N}} = \{\langle a,b\rangle : \langle a,b,b,\ldots\rangle \in \mathbf{R}_3^{\mathfrak{M}}\} ;
\end{aligned}
$$

Note that:

(9) For any model $\mathfrak{M}$ of γ we have $\mathfrak{M}^{\Omega\Lambda} = \mathfrak{M}$.

After these lengthy preliminaries we are ready to establish the theorem. We should emphasize that the above preliminaries could be formulated in a more general setting, making the present proof much shorter.

Recall from the beginning of this proof that φ is the Ehrenfeucht, Fuhrken sentence in Ω. Let ψ be the formula

$$\varphi' \wedge \forall v_0 \ldots \forall v_{\alpha-1} \gamma$$

of Λ. Let $\equiv$ be ${}_s\equiv^\Lambda_{\tau 0}$, used in defining ${}_s\mathfrak{Fm}^\Lambda_{\tau 0}$. We show that $\psi/\equiv$ is a non-zero atomless element of ${}_s\mathfrak{Fm}^\Lambda_{\tau 0}$, completing the proof.

By (2), let $\mathfrak{M}$ be a model of φ. Then by (6) and (8), $\mathfrak{M}^\Lambda$ is a model of ψ. Hence $\psi/\equiv \neq 0$.

Now suppose that $0 \neq \chi/\equiv \leqq \psi/\equiv$. We shall find a restricted formula θ such that

$(\chi \wedge \theta)/\equiv \neq 0 \neq (\chi \wedge \neg\theta)/\equiv$, thus showing that $\psi/\equiv$ is atomless. Since $\chi/\equiv \neq 0$, there is a possible model $\mathfrak{M}$ of Λ and an $a \in {}^{\alpha}M$ such that $\mathfrak{M} \models \chi[a]$. Now $\models \chi \rightarrow \psi$, so $\mathfrak{M}$ is a model of γ. Hence by (7) we have $\mathfrak{M} \models \chi^{*\prime}[a]$, and so by (6) and (9),

(10) $\mathfrak{M}^{\Omega} \models \chi^{*}[a]$.

Next we claim

(11) $\models \chi^{*} \rightarrow \varphi$.

For, let $\mathfrak{N}$ be any possible model of Ω and suppose that $b \in {}^{\alpha}N$ and $\mathfrak{N} \models \chi^{*}[b]$. By (6) we have $\mathfrak{N}^{\Lambda} \models \chi^{*\prime}[b]$, so by (7) and (8), $\mathfrak{N}^{\Lambda} \models \chi[b]$. Since $\models \chi \rightarrow \psi$, we get $\mathfrak{N}^{\Lambda} \models \varphi'$, and so $\mathfrak{N} \models \varphi$ by (6), proving (11).

By (3), (10), and (11), there is a formula θ of Ω with at most the variables $v_0, \ldots, v_{\alpha-1}$ in it such that both $\chi^{*} \wedge \theta$ and $\chi^{*} \wedge \neg\theta$ are consistent. Let $\mathfrak{N}$ be a possible model of Ω and let $b \in {}^{\alpha}N$ be such that $\mathfrak{N} \models (\chi^{*} \wedge \theta)[b]$. By (6), (7), (8) we have $\mathfrak{N}^{\Lambda} \models (\chi \wedge \theta')[b]$. Thus $(\chi \wedge \theta')/\equiv \neq 0$. Similarly $(\chi \wedge \neg\theta')/\equiv \neq 0$, completing the proof.

REMARK 4.3.33. We do not have a purely algebraic proof for 4.3.32, and we do not know whether it extends to $1 \leq \beta \leq 3$, or whether an analogous result holds for $\mathfrak{Sr}_{\beta}\mathsf{CA}_{\alpha}$. (We have not been able to reconstruct the proof of the result, due to Tarski, mentioned in 2.5.8.) Note from 2.5.7(i) and 4.3.32 that for $3 \leq \alpha < \omega$ we have $\mathsf{IGs}_{\alpha} \neq \mathsf{HSP}\{\mathfrak{A} \in \mathsf{IGs}_{\alpha} : |A| < \omega\}$.

Interpretations

We now turn to one of the most important connections between logic and algebra, namely the duality between interpretations and homomorphisms, and between definitional equivalence and isomorphisms. The former is due to Németi (see Andréka, Gergely, Németi, Sain [80]), the latter to Hoehnke [66].

DEFINITION 4.3.34. *Let $\Lambda = \langle \alpha, \mathbf{R}, \rho \rangle$ and $\Omega = \langle \alpha, \mathbf{S}, \sigma \rangle$ be two languages. An* **interpretation** *of Λ into Ω is a function f mapping $Do\mathbf{R}$ into Φu_{τ}^{Ω}. We denote by f^{+} the homomorphism from $\mathfrak{Fm}_{\tau}^{\Lambda}$ into $\mathfrak{Fm}_{\tau}^{\Omega}$ such that, for each $\xi \in Do\mathbf{R}$,*

$$f^{+}\mathbf{R}_{\xi}(v_0, \ldots, v_{\eta}, \ldots)_{\eta < \rho\xi} = f\xi.$$

Now assume additionally that $t \in \{s, p\}$, and Γ, Σ are t-theories in Λ, Ω respectively. An interpretation f of Λ into Ω is a called an **interpretation of Γ into Σ** *provided that $f^{+}\varphi \in \Sigma$ for all $\varphi \in \Gamma$.*

This definition takes a slightly more conventional form in the case of ordinary languages, as follows.

THEOREM 4.3.35. *Let $\Lambda = \langle \alpha, \mathbf{R}, \rho \rangle$ and $\Omega = \langle \alpha, \mathbf{S}, \sigma \rangle$ be languages, with Ω ordinary, and let $t \in \{s, p\}$ with Γ, Σ t-theories in Λ, Ω respectively. Let f be an interpretation of Γ into Σ. Then there is an interpretation g of Γ into Σ with the following two properties:*

(i) *For any $\xi \in Do\mathbf{R}$ the free variables of $g\xi$ are among $v_0, \ldots, v_{\eta}, \ldots, \eta < \rho\xi$.*

(ii) *$f\xi \leftrightarrow g\xi \in \Sigma$ for any $\xi \in Do\mathbf{R}$.*

PROOF. Note that if $\xi \in Do\mathbf{R}$ and $\rho\xi \leqq \eta < \alpha$ then

$$\mathbf{R}_\xi(v_0,\ldots,v_\iota,\ldots)_{\iota<\rho\xi} \leftrightarrow \exists v_\eta \mathbf{R}_\xi(v_0,\ldots,v_\iota,\ldots)_{\iota<\rho\xi} \in \Gamma.$$

Hence $f\xi \leftrightarrow \exists v_\eta f\xi \in \Sigma$. Hence if we let

$$g\xi = \exists v_{\eta 1}\ldots\exists v_{\eta m} f\xi$$

for each $\xi \in Do\mathbf{R}$, where $\eta 1,\ldots,\eta m$ lists all ν such that $\rho\xi \leqq \nu < \alpha$ and v_ν occurs free in $f\xi$, the desired conclusions follow.

EXAMPLE 4.3.36. There are many instances of interpretations in logic. For example, the theory of groups can be interpreted in the theory of Boolean algebras, as follows. Let $\Lambda = \langle \omega, \mathbf{R}, \rho \rangle$ where $Do\mathbf{R} = Do\rho = 1$ and $\rho 0 = 3$; $\mathbf{R}_0$ is a ternary relation symbol used to express properties of the group operation. Let Γ consist of all restricted formulas which follow from the group axioms formulated in this language. For the theory of Boolean algebras, let $\Omega = \langle \omega, \mathbf{S}, \sigma \rangle$ where $Do\mathbf{S} = Do\sigma = 5$ and $\sigma 0 = 3$, $\sigma 1 = 3$, $\sigma 2 = 2$, $\sigma 3 = 1$, $\sigma 4 = 1$; $\mathbf{S}_0,\ldots,\mathbf{S}_4$ are used to express properties of $+,\cdot,-,0,1$, respectively. Let Σ consist of all restricted formulas which follow from BA axioms formulated in this language. Then the following function f interprets the group operation as symmetric difference in a BA (if we dealt with languages with operation symbols and with unrestricted formulas, $v_0 \cdot -v_1 + v_1 \cdot -v_0$ would be assigned to $v_0 \cdot v_1$): $Dof = 1$, and $f0$ is the formula

$$\exists v_2 \exists v_3 \exists v_4 \exists v_5 [\mathbf{S}_2(v_1,v_2) \wedge \mathbf{S}_2(v_0,v_3) \wedge \mathbf{S}_1(v_0,v_2,v_4) \wedge \mathbf{S}_1(v_3,v_1,v_5) \wedge \mathbf{S}_0(v_4,v_5,v_2)],$$

except that $\mathbf{S}_2(v_1,v_2),\ldots$ should be replaced by restricted formulas by the prescription given in the proof of Theorem 4.3.6. Then f is an interpretation of Γ into Σ in the sense of Definition 4.3.34.

That interpretations give rise to homomorphisms follows almost immediately from the definitions involved:

THEOREM 4.3.37. *Let $\Lambda = \langle \alpha, \mathbf{R}, \rho \rangle$ and $\Omega = \langle \alpha, \mathbf{S}, \sigma \rangle$ be languages. Suppose that $t \in \{s,p\}$, Γ, Σ are t-theories in Λ, Ω respectively, and f is an interpretation of Γ into Σ. Then there is a $g \in Ho({}_t\mathfrak{Fm}^\Lambda_{r\Sigma}, {}_t\mathfrak{Fm}^\Omega_{r\Sigma}))$ such that $g(\varphi/_t\equiv^\Lambda_{r\Gamma}) = (f^+\varphi)/_t\equiv^\Lambda_{r\Sigma}$ for every $\varphi \in \Phi\mu^\Lambda_r$.*

It is perhaps surprising that a kind of converse of Theorem 4.3.37 holds; every homomorphism between CA_α's is representable as an interpretation between theories:

THEOREM 4.3.38. *Let $\mathfrak{A}, \mathfrak{B} \in \mathsf{CA}_\alpha$ and $f \in Hom(\mathfrak{A}, \mathfrak{B})$. Then there exist languages Λ, Ω, p-theories Γ, Σ in Λ, Ω respectively, an interpretation g of Γ into Σ, and $a \in Is(\mathfrak{A}, {}_p\mathfrak{Fm}^\Lambda_{r\Gamma})$, $b \in Is(\mathfrak{B}, {}_p\mathfrak{Fm}^\Omega_{r\Sigma})$ such that $f = b^{-1} \circ \{\langle \varphi/_p\equiv^\Lambda_{r\Gamma}, g^+\varphi/_p\equiv^\Omega_{r\Sigma}\rangle : \varphi \in \Phi\mu^\Lambda_r\} \circ a$.*

PROOF. By 4.3.28(i) we choose Λ, Ω, Γ, Σ, a, b so that Γ and Σ are p-theories in Λ, Ω respectively and $a \in Is(\mathfrak{A}, \mathfrak{Fm}^\Lambda_{r\Gamma})$, $b \in Is(\mathfrak{B}, \mathfrak{Fm}^\Omega_{r\Sigma})$. Say $\Lambda = \langle \alpha, \mathbf{R}, \rho \rangle$. Now for each $\xi \in Do\mathbf{R}$ choose $g\xi \in bfa^{-1}(\mathbf{R}_\xi(v_0,\ldots,v_\eta,\ldots)_{\eta<\rho\xi}/\equiv^\Lambda_{r\Gamma})$. Clearly the desired conditions hold.

REMARKS 4.3.39. By using other parts of 4.3.28, one can obtain theorems analogous to 4.3.38 (as well as 4.3.41 and 4.3.44 below) for homomorphisms between IGs_α's and interpretations between s–theories, and for homomorphisms between Lf_α's and interpretations between p–theories in ordinary languages for $\alpha \geqq \omega$. This last result has been strengthened by Gergely [80'] using category notions. Namely, let α be infinite, and let Lf_α be the category of Lf_α's and homomorphisms between them. On the other hand, let Th be the category whose objects are all pairs $\langle \Lambda, \Gamma \rangle$, $\Lambda = \langle \alpha, \mathbf{R}, \rho \rangle$ an ordinary language and Γ a p–theory in Λ, and whose morphisms from $\langle \Lambda, \Gamma \rangle$ to $\langle \Omega, \Sigma \rangle$ are interpretations of Γ into Σ. The result of Gergely [80'] is that the categories Lf_α and Th are isomorphic.

It is clear that one–one homomorphisms, i.e., isomorphic embeddings, correspond to *faithful* interpretations, that is, interpretations f as in 4.3.34 such that $\varphi \in \Gamma$ iff $f^+\varphi \in \Sigma$, for all $\varphi \in \Phi\mu_r^\Lambda$.

We now extend the last part of Remarks 4.3.39 to indicate the logical counterpart of the algebraic notion of subalgebra.

DEFINITION 4.3.40. *Let $\Lambda = \langle \alpha, \mathbf{R}, \rho \rangle$ and $\Omega = \langle \alpha, \mathbf{S}, \sigma \rangle$ be languages. We call Ω an* **expansion** *of Λ if $Do\mathbf{R} \subseteq Do\mathbf{S}$ and $\mathbf{R}_\xi = \mathbf{S}_\xi$, $\rho\xi = \sigma\xi$, for all $\xi \in Do\mathbf{R}$. Now let Γ and Σ be t–theories in Λ, Ω respectively, where $t \in \{s,p\}$. The pair $\langle \Omega, \Sigma \rangle$ is a* **conservative expansion** *of $\langle \Lambda, \Gamma \rangle$ provided that Ω is an expansion of Λ and $\varphi \in \Gamma$ iff $\varphi \in \Sigma$, for all $\varphi \in \Phi u_r^\Lambda$.*

THEOREM 4.3.41. *Let $\mathfrak{A}, \mathfrak{B} \in \mathsf{CA}_\alpha$. Then the following conditions are equivalent:*

(i) $\mathfrak{A} \subseteq \mathfrak{B}$.

(ii) *$A \subseteq B$ and there exist Λ, Ω, Γ, Σ, g such that Γ and Σ are p–theories in Λ, Ω respectively, $\langle \Omega, \Sigma \rangle$ is a* conservative expansion of *$\langle \Lambda, \Gamma \rangle$, $g \in Is(\mathfrak{B}, {}_p\mathfrak{Fm}_{r\Sigma}^\Omega)$, $A \upharpoonright g \in Ism(\mathfrak{A}, {}_p\mathfrak{Fm}_{r\Sigma}^\Omega)$, and $\{\langle \varphi/\equiv_{r\Gamma}^\Lambda, \varphi/\equiv_{r\Sigma}^\Omega \rangle : \varphi \in \Phi\mu_r^\Lambda\} \in Is({}_p\mathfrak{Fm}_{r\Gamma}^\Lambda, g^*\mathfrak{A})$.*

PROOF. Clearly (ii)$\Rightarrow$(i). Now assume that $\mathfrak{A} \subseteq \mathfrak{B}$. Then we can choose full languages $\Lambda = \langle \alpha, \mathbf{R}, \rho \rangle$ and $\Omega = \langle \alpha, \mathbf{S}, \sigma \rangle$ such that Ω is an expansion of Λ and there is a function f mapping $Do\mathbf{S}$ onto B such that $f^*Do\mathbf{R} = A$. By Corollary 4.3.26 there is then a homomorphism $h \in Ho({}_p\mathfrak{Fm}_{r0}^\Omega, \mathfrak{B})$ such that $h(\mathbf{R}_\xi(v_0, \ldots, v_\eta, \ldots)_{\eta<\alpha}) = f\xi$ for all $\xi \in Do\mathbf{S}$. Let $\Sigma = \{\varphi \in \Phi\mu_r^\Omega : h\varphi = 1\}$ and $\Gamma = \Sigma \cap \Phi\mu_r^\Omega$. The desired conditions are now easily checked, with $g^{-1}(\varphi/\equiv_\Sigma) = h\varphi$ for all $\varphi \in \Phi\mu_r^\Omega$.

Now we turn to the notion of definitional equivalence; see also Part I, pp.56–57.

DEFINITION 4.3.42. *Let $\Lambda = \langle \alpha, \mathbf{R}, \rho \rangle$ and $\Omega = \langle \alpha, \mathbf{S}, \sigma \rangle$ be languages, $t \in \{s,p\}$, and Γ and Σ t–theories in Λ, Ω respectively. We say that Γ and Σ are* **definitionally equivalent** *provided that there are interpretations f of Γ into Σ and g of Σ into Γ such that for all $\xi \in Do\mathbf{R}$ and $\nu \in Do\mathbf{S}$ we have*

$$[\mathbf{R}_\xi(v_0, \ldots, v_\eta, \ldots)_{\eta<\rho\xi} \leftrightarrow g^+f\xi] \in \Gamma,$$
$$[\mathbf{S}_\nu(v_0, \ldots, v_\eta, \ldots)_{\eta<\sigma\nu} \leftrightarrow f^+g\nu] \in \Sigma.$$

THEOREM 4.3.43. *Let Λ, Ω, t, Γ, and Σ be as in the first part of Definition 4.3.34, and let h be arbitrary. Then the following conditions are equivalent:*

(i) $h \in Is({}_t\mathfrak{Fm}_{r\Sigma}^\Lambda, {}_t\mathfrak{Fm}_{r\Sigma}^\Omega)$.

(ii) Γ *and* Σ *are definitionally equivalent using interpretations* f *and* g *as in* 4.3.42, *such that* $h(\varphi/_t\equiv^{\Lambda}_{\tau\Gamma}) = (f^+\varphi)/_t\equiv^{\Omega}_{\tau\Sigma}$ *for all* $\varphi \in \Phi\mu^{\Lambda}_{\tau}$.

(iii) *There is a faithful interpretation* f *of* Γ *into* Σ *such that for every* $\psi \in \Phi\mu^{\Omega}_{\tau}$ *there is a* $\varphi \in \Phi\mu^{\Lambda}_{\tau}$ *with* $(f^+\varphi \leftrightarrow \psi) \in \Sigma$, *and such that* $h(\varphi/_t\equiv^{\Lambda}_{\tau\Gamma}) = (f^+\varphi)/_t\equiv^{\Omega}_{\tau\Sigma}$ *for all* $\varphi \in \Phi\mu^{\Lambda}_{\tau}$.

COROLLARY 4.3.44. *Let* $\mathfrak{A},\mathfrak{B} \in \mathsf{CA}_\alpha$. *Then* $\mathfrak{A} \simeq \mathfrak{B}$ *iff there exist languages* $\Lambda = \langle \alpha, \mathbf{R}, \rho\rangle$ *and* $\Omega = \langle \alpha, \mathbf{S}, \sigma\rangle$ *and* p*—theories* Γ *and* Σ *in* Λ *and* Ω *respectively such that* $\mathfrak{A} \simeq_p \mathfrak{Fm}^{\Lambda}_{\tau\Gamma}$, $\mathfrak{B} \simeq_p \mathfrak{Fm}^{\Omega}_{\tau\Sigma}$, *and* Γ *and* Σ *are definitionally equivalent.*

REMARK 4.3.45. It is of some interest to apply the notions of interpretation and definitional equivalence to models rather then theories; this is what we did in Part I, p.56. See also Németi [81']. Let $\Lambda = \langle \alpha, \mathbf{R}, \rho\rangle$ and $\Omega = \langle \alpha, \mathbf{S}, \sigma\rangle$ be languages, and let $\mathfrak{M}$ and $\mathfrak{N}$ be possible models for Λ and Ω repectively, with the same universe $M = N$. We say that $\mathfrak{M}$ can be *interpreted in* $\mathfrak{N}$ if every relation of $\mathfrak{M}$ is definable in terms of those of $\mathfrak{N}$. This means just that $\mathfrak{Cf}^{\mathfrak{M}}_\alpha \subseteq \mathfrak{Cf}^{\mathfrak{N}}_\alpha$. Similarly, $\mathfrak{M}$ and $\mathfrak{N}$ are definitionally equivalent iff $\mathfrak{Cf}^{\mathfrak{M}}_\alpha = \mathfrak{Cf}^{\mathfrak{N}}_\alpha$. Now assume that Γ and Σ are s—theories in Λ and Ω respectively, and f is an interpretation of Γ into Σ. Then with each model $\mathfrak{M}$ of Σ one can associate a possible model $\mathfrak{N}$ of Λ as follows. We set $N = M$, the universe of $\mathfrak{M}$. For each $\xi \in DoR$ we set

$$\mathbf{R}^{\mathfrak{N}}_\xi = \{x \in {}^{\rho\xi}N : \text{for some } y \in {}^{\alpha}N \text{ with } x \subseteq y \text{ we have } \mathfrak{M} \models (f\xi)[y]\}.$$

In case Λ and Ω are ordinary languages, we can analyze this situation further, as follows. Let f have the additional property mentioned in 4.3.35. Then for any $\varphi \in \Phi\mu^{\Lambda}_{\tau}$ and any $x \in {}^{\alpha}M$ we have $\mathfrak{N} \models \varphi[x]$ iff $\mathfrak{M} \models (f^+\varphi)[x]$. Therefore, $\mathfrak{N}$ is a model of Γ. If we assume additionally that Γ and Σ are definitionally equivalent via the function f, in the sense of Theorem 4.3.43, then it is easy to check that this correlation of a model of Γ with a model of Σ is one—one and onto.

Logic and Gs_α's

We now turn to a discussion of connections between Gs_α's and logic; see also Németi [81'].

THEOREM 4.3.46. *Let* Λ *be a language and* K *a set of possible models of* Λ *with pairwise disjoint universes. Then*

$$\{\bigcup_{\mathfrak{M} \in \mathsf{K}} \varphi^{\mathfrak{M}} : \varphi \in \Phi\mu^{\Lambda}_{\tau}\}$$

is a generalized cylindric field of sets of dimension α.

DEFINITION 4.3.47. *With* Λ *and* K *as in* 4.3.46, *we denote by* ${}^{\tau}\mathfrak{Cf}^{\mathsf{K}}_\alpha$ *the* Gs_α *with universe*

$$\{\bigcup_{\mathfrak{M} \in \mathsf{K}} \varphi^{\mathfrak{M}} : \varphi \in \Phi\mu^{\Lambda}_{\tau}\}.$$

THEOREM 4.3.48. *If $\Lambda=\langle\alpha,\mathbf{R},\rho\rangle$ is an ordinary language and K is a set of possible models of Λ with pairwise disjoint universes, then ${}^{r}\mathfrak{Gf}_\alpha^{\mathsf{K}}\in\mathsf{Gs}_\alpha^{reg}\cap\mathsf{Lf}_\alpha$.*

PROOF. Clearly ${}^{r}\mathfrak{Gf}_\alpha^{\mathsf{K}}\in\mathsf{Lf}_\alpha$. To prove that it is regular, suppose that $x\in{}^{r}Gs_\alpha^{\mathsf{K}}$, $f\in x$, $g\in\bigcup_{\mathfrak{M}\in\mathsf{K}}{}^{\alpha}M$, and $(\Delta x\cup 1)\,1f\subseteq g$. Say $x=\bigcup_{\mathfrak{M}\in\mathsf{K}}\varphi^{\mathfrak{M}}$. Choose $\mathfrak{M}\in\mathsf{K}$ so that $f\in\varphi^{\mathfrak{M}}$. Thus $f\in{}^{\alpha}M$ so, since $f0=g0$, also $g\in{}^{\alpha}M$. Now let $\Gamma=\{\kappa<\alpha:v_\kappa$ occurs free in φ but $\kappa\notin\Delta x\}$. Let $h\in{}^{\alpha}M$ be such that $(\Delta a\cup 1\cup\Gamma)\,1h=(\Delta a\cup 1\cup\Gamma)\,1f$ while $(\alpha\sim(\Delta a\cup 1\cup\Gamma))\,1h\subseteq g$. Then $h\kappa=f\kappa$ for every κ for which v_κ occurs free in φ, so $h\in\varphi^{\mathfrak{M}}$. Hence $g\in\mathsf{c}_{(\Gamma)}x=x$, as desired.

It is convenient to extend the notion of elementary equivalence to our languages as follows. Let $\Lambda=\langle\alpha,\mathbf{R},\rho\rangle$ be a language, and $\mathfrak{M}$ and $\mathfrak{N}$ possible models of Λ. We call $\mathfrak{M}$ and $\mathfrak{N}$ *elementarily equivalent* provided that for every restricted formula φ of Λ the following conditions are equivalent: (a) there is an $a\in{}^{\alpha}M$ such that $\mathfrak{M}\models\varphi[a]$, (b) there is an $a\in{}^{\alpha}N$ such that $\mathfrak{N}\models\varphi[a]$.

THEOREM 4.3.49. *Assume that $\alpha\geq 2$. Let $\Lambda=\langle\alpha,\mathbf{R},\rho\rangle$ be a language and Σ an s-theory in Λ. Suppose that K is a set of models of Σ with pairwise disjoint universes such that for every model $\mathfrak{M}$ of Σ there is an elementarily equivalent $\mathfrak{N}\in\mathsf{K}$. Then ${}_{s}\mathfrak{Fm}_{r\Sigma}^{\Lambda}\simeq{}^{r}\mathfrak{Gf}_\alpha^{\mathsf{K}}$.*

REMARKS 4.3.50. Let $\Lambda=\langle\alpha,\mathbf{R},\rho\rangle$ be a language and Σ an s-theory in Λ. Then a set K with the properties described in Theorem 4.3.49 always exists. In fact, for each $\Gamma\subseteq\Phi\mu_r^{\Lambda}$ let $\mathfrak{M}_\Gamma$ be a model of Σ such that $\Gamma=\{\varphi\in\Phi\mu_r^{\Lambda}:$ there is an $a\in{}^{\alpha}M_\Gamma$ such that $\mathfrak{M}_\Gamma\models\varphi[a]\}$, if there is such a model; otherwise Γ is not in the domain of this function $\mathfrak{M}$. We may assume that $M_\Gamma\cap M_\Delta=0$ for all distinct $\Gamma,\Delta\in Do\mathfrak{M}$. Then $\mathsf{K}=Rg\mathfrak{M}$ is as desired.

Thus Theorem 4.3.49 gives an algebraic expression for the notion of a theory; recall from 4.3.31 that filters are another such expression.

REMARK 4.3.51. (This remark, and Theorem 4.3.53, are essentially due to Andréka and Németi.) Interpretations of one theory into another can be given an algebraic expression in terms of base homomorphisms between Gs's, as follows. To conveniently formulate the result we shall introduce some special terminology for this remark only. If Λ and Ω are ordinary languages, an interpretation f of Λ into Ω is *ordinary* if it satisfies the additional condition 4.3.35(i). For any set Γ of formulas, ${}_{A}\mathsf{Md}\Gamma$ is the *set* of all models $\mathfrak{M}$ of Γ with $M\subseteq A$. If K is a set of possible models of Λ, with possibly non-disjoint universes, we modify the notation ${}^{r}\mathfrak{Gf}_\alpha^{\mathsf{K}}$ as follows: it is the Gs_α with universe $\{\bigcup_{\mathfrak{M}\in\mathsf{K}}\bar{\varphi}^{\mathfrak{M}}:\varphi\in\Phi\mu_r^{\Lambda}\}$, where for any $\mathfrak{M}\in\mathsf{K}$ and $\varphi\in\Phi\mu_r^{\Lambda}$ we set

$$\bar{\varphi}^{\mathfrak{M}}=\{a\in{}^{\alpha}(M\times\{\mathfrak{M}\}):\mathfrak{M}\models\varphi[\langle\bigcap\bigcap a_\kappa:\kappa<\alpha\rangle]\}.$$

(Note that $\bigcap\bigcap\langle x,y\rangle=x$ for any ordered pair $\langle x,y\rangle$. The above procedure is just "disjointing" the universes of all $\mathfrak{M}\in\mathsf{K}$.) Next, we want to formalize a procedure in Remark 4.3.45. Let $\Lambda=\langle\alpha,\mathbf{R},\rho\rangle$ and $\Omega=\langle\alpha,\mathbf{S},\sigma\rangle$ be ordinary languages, and f an ordinary interpretation of Λ into Ω. With each possible model $\mathfrak{M}$ of Ω we associate a

possible model $\mathfrak{Rb}_f\mathfrak{M}$ of Λ as follows. For brevity let $\mathfrak{N}=\mathfrak{Rb}_f\mathfrak{M}$. Set $N=M$. For each $\xi\in Do\mathbf{R}$ let

$$\mathbf{R}_\xi^{\mathfrak{N}}=\{x\in{}^{\rho\xi}N:\text{ for some } y\in{}^{\alpha}N \text{ with } x\subseteq y \text{ we have } \mathfrak{M}\models(f\xi)[y]\}.$$

Now we can state our new algebraic version of interpretations: let $\Lambda=\langle\alpha,\mathbf{R},\rho\rangle$ and $\Omega=\langle\alpha,\mathbf{R},\sigma\rangle$ be ordinary languages, and let f be an ordinary interpretation of Λ into Ω. Let Γ and Σ be s-theories in Λ and Ω respectively. Then the following conditions are equivalent:

(1) f is an interpretation of Γ into Σ;
(2) For any $\kappa\geqq|\Gamma|+|\Sigma|$, let $\mathsf{K}={}_\kappa\mathsf{Md}\Gamma$ and $\mathsf{L}={}_\kappa\mathsf{Md}\Sigma$. Define $g\langle u,\mathfrak{M}\rangle=\langle u,\mathfrak{Rb}_f\mathfrak{M}\rangle$ for any $\mathfrak{M}\in\mathsf{L}$ and $u\in M$. Then $\hat{g}$ is a base-homomorphism of ${}^{\tau}\mathfrak{Gf}_\alpha^{\mathsf{K}}$ into ${}^{\tau}\mathfrak{Gf}_\alpha^{\mathsf{L}}$.

It is routine to prove this.

From the above it is clear that Gs_α's form an algebraic version of classes of models. We now want to make this precise, and give a theorem indicating when a Gs_α corresponds to an elementary class of models. The following definition gives a kind of inverse to the process of Definition 4.3.47.

DEFINITION 4.3.52. *Let* $\mathfrak{A}\in\mathsf{Gs}_\alpha^{reg}\cap\mathsf{Lf}_\alpha$. *A language* $\Lambda=\langle\alpha,\mathbf{R},\rho\rangle$ **fits** $\mathfrak{A}$ *provided that there is a one-one function* f (*a* **fitting function**) *from* $Do\mathbf{R}$ *into* A *such that* $Rgf=\{x\in A:\Delta x\in\omega\}$ *and for every* $\xi\in Do\mathbf{R}$ *we have* $\Delta f\xi=\rho\xi$. *Then we set*

$$\mathsf{Mc}\mathfrak{A}=\{\langle M,\{\rho\xi\restriction u:u\in f\xi\}\rangle_{\xi\in Do\rho}:M \text{ a subbase of } \mathfrak{A}\}.$$

Thus $\mathsf{Mc}\mathfrak{A}$ is a set of possible models of Λ.

THEOREM 4.3.53. *Let* $\alpha\geqq\omega$, $\mathfrak{A}\in\mathsf{Gs}_\alpha^{reg}\cap\mathsf{Lf}_\alpha$, *and* $\kappa\geqq|U|$ *for each subbase* U *of* $\mathfrak{A}$, κ *a cardinal. Let* Λ *fit* $\mathfrak{A}$ *with fitting function* f. *Then* Λ *is an ordinary language, and the following conditions are equivalent:*

(i) *There is a set* Γ *of sentences of* Λ *such that* $\mathsf{IMc}\mathfrak{A}=\{\mathfrak{M}:\mathfrak{M}$ *is a model of* Γ *and* $|M|\leqq\kappa\}$.

(ii) *If* $h\in Ho(\mathfrak{A},\mathfrak{B})$, $\mathfrak{B}\in{}_\lambda\mathsf{Cs}_\alpha^{reg}$, *with* $\lambda\leqq\kappa$, *then* h *is a base-homomorphism* $\hat{g}$, *where* Rgg *is a subbase of* $\mathfrak{A}$.

PROOF. For each subbase M of $\mathfrak{A}$ we let $\mathfrak{M}$ be the possible model for Λ indicated in Definition 4.3.52.

Now suppose that (i) holds, and let $h\in Ho(\mathfrak{A},\mathfrak{B})$, where $\mathfrak{B}\in{}_\lambda\mathsf{Cs}_\alpha^{reg}$ and $\lambda\leqq\kappa$. By 4.3.1(iii), let k be the homomorphism from $\mathfrak{Fm}_\tau^\Lambda$ into $\mathfrak{A}$ such that $k\mathbf{R}_\xi v_0\ldots v_{\rho\xi-1}=f\xi$ for every $\xi\in Do\mathbf{R}$ (with $\Lambda=\langle\alpha,\mathbf{R},\rho\rangle$). Then it is easy to check that

(1) $k\varphi\cap{}^{\alpha}M=\varphi^{\mathfrak{M}}$ for each subbase M of $\mathfrak{A}$ and each $\varphi\in\Phi\mu_\tau^\Lambda$, and k maps onto $\mathfrak{A}$.

By (i) we thus have $k\varphi=1$ for every $\varphi\in\Gamma$.

Next, note that Rgf generates $\mathfrak{A}$. Hence $Rg(h\circ f)$ generates $\mathfrak{B}$. It is then easy to check that $\langle\mathfrak{B},h\circ f\rangle$ is a Cs_α with generators suitable for Λ. Let $\mathfrak{N}=\mathfrak{Mb}_{\mathfrak{B},h\circ f}^\Lambda$. It is easy to check that $hk\varphi=\varphi^{\mathfrak{N}}$ for every $\varphi\in\Phi\mu_\tau^\Lambda$. Since $k\varphi=1$ for every $\varphi\in\Gamma$, it follows that $\mathfrak{N}$ is a model of Γ. Hence by (i) there is an isomorphism g of $\mathfrak{N}$ onto some

$\mathfrak{R} \in Mc\mathfrak{A}$. To show that $h = \hat{g}$, let $a \in A$ and $t \in {}^{\alpha}N$. Say $a = k\varphi$ with $\varphi \in \Phi\mu_{\tau}^{\Lambda}$. Then $t \in \hat{g}a$ iff $g \circ t \in a$ iff $g \circ t \in \varphi^{\mathfrak{R}}$ (by (1)) iff $t \in \varphi^{\mathfrak{R}}$ iff $t \in hk\varphi$ iff $t \in ha$, as desired.

Now assume (ii). Let $k \in Ho(\mathfrak{F}\mathfrak{m}_{\tau}^{\Lambda}, \mathfrak{A})$ be such that $k\mathbf{R}_{\xi}v_0 \ldots v_{\rho\xi-1} = f\xi$ for every $\xi \in Do\mathbf{R}$. Let Γ be the set of sentences φ such that $k\varphi = 1$ To establish (i), first suppose that M is a subbase of $\mathfrak{A}$. Let $W = {}^{\alpha}M$. Then $(rl_{W}^{\mathfrak{A}} \circ k)\varphi = \varphi^{\mathfrak{R}}$ for every $\varphi \in \Phi\mu_{\tau}^{\Lambda}$, and hence $\mathfrak{R}$ is a model of Γ. This proves $\subseteq$ in (i). Conversely, suppose that $\mathfrak{R}$ is a model of Γ and $|N| \leq \kappa$. Let $\mathfrak{B} = {}^{\tau}\mathfrak{C}\mathfrak{f}_{\alpha}^{\mathfrak{R}}$. Now $k\varphi = 1$ implies that $\varphi^{\mathfrak{R}} = 1$, so there is an $h \in Ho(\mathfrak{A}, \mathfrak{B})$ such that $hk\varphi = \varphi^{\mathfrak{R}}$ for all $\varphi \in \Phi\mu_{\tau}^{\Lambda}$. By (ii), h is a base homomorphism $\hat{g}$, where g is a one–one function from N onto some subbase M of $\mathfrak{A}$ (see 3.1.52). We claim that $g \in Is(\mathfrak{R}, \mathfrak{R})$, hence $\mathfrak{R}$ is in the left side of (i). For, let $\xi \in Do\mathbf{R}$ and $a \in {}^{\rho\xi}N$. Choose $b \in {}^{\alpha}N$ with $a \subseteq b$. Let φ be the formula $\mathbf{R}_{\xi}v_0 \ldots v_{\rho\xi-1}$. Then $a \in \mathbf{R}_{\xi}^{\mathfrak{R}}$ iff $b \in \varphi^{\mathfrak{R}}$ iff $b \in hk\varphi$ iff $g \circ b \in k\varphi$ iff $g \circ b \in \varphi^{\mathfrak{R}}$ iff $g \circ a \in \mathbf{R}_{\xi}^{\mathfrak{R}}$, as desired.

In connection with Theorem 4.3.53, see Theorem 3.1.111.

REMARK 4.3.54. We shall not discuss the logical counterpart of Ws_{α}'s and Gws_{α}'s. For some indications along these lines, see Tarski, Vaught [57*].

Terms and formulas

We turn to another basic connection between logic and CA_{α}'s, namely a kind of duality between terms in a discourse language for the class CA_{α} and formulas in the kind of languages we have been considering. The basic situation is very simple, as we shall see. It becomes more complicated and interesting when we establish a connection with ordinary languages.

DEFINITION 4.3.55. *Let α be any ordinal. Recall that $\mathcal{L}_{\alpha}$ is the standard discourse language for the class CA_{α}; it has ω variables $v_0, v_1, \ldots$ and operation symbols $+, \ldots, \mathbf{d}_{\kappa\lambda}$ $(\kappa, \lambda < \alpha)$. Let $\Lambda_{\alpha} = \langle \alpha, \mathbf{R}, \rho \rangle$ be a full language with $Do\mathbf{R} = \omega$.*

With each term σ of $\mathcal{L}_{\alpha}$ we associate a formula $\xi_{\alpha}\sigma$, or simply $\xi\sigma$, of Λ_{α}, as follows (for arbitrary $i \in \omega$ and $\kappa, \lambda < \alpha$):

$$\xi v_i = \mathbf{R}_i(v_0 \ldots v_{\eta} \ldots)_{\eta < \alpha} \text{ for each } i \in \omega;$$
$$\xi \mathbf{d}_{\kappa\lambda} = v_{\kappa} = v_{\lambda};$$
$$\xi(\sigma + \tau) = \xi\sigma \vee \xi\tau;$$
$$\xi(\sigma \cdot \tau) = \xi\sigma \wedge \xi\tau;$$
$$\xi(-\sigma) = \neg\xi\sigma;$$
$$\xi\mathbf{0} = \mathbf{F};$$
$$\xi\mathbf{1} = \mathbf{T};$$
$$\xi\mathbf{c}_{\kappa}\sigma = \exists v_{\kappa}\xi\sigma;$$

Clearly ξ is a one–one function onto the set of all restricted formulas of Λ_{α}, and we let $\tau\mu$ be its inverse.

THEOREM 4.3.56. *Let $\mathfrak{R} = \langle M, R \rangle$ be a possible model of Λ_{α} and set $\mathfrak{A} = {}^{\tau}\mathfrak{C}\mathfrak{f}_{\alpha}^{\mathfrak{R}}$. Then for any term σ of $\mathcal{L}_{\alpha}$ we have $\sigma^{\mathfrak{A}}R = (\xi\sigma)^{\mathfrak{R}}$.*

Theorem 4.3.56 leads to the following important connection result.

THEOREM 4.3.57. *For any terms σ,τ of $\mathcal{L}_\alpha$ the following conditions are equivalent:*
(i) $\models \xi\sigma \leftrightarrow \xi\tau$,
(ii) $\mathsf{Gs}_\alpha \models \sigma = \tau$.

PROOF. (i)$\Rightarrow$(ii). Let $\mathfrak{A} \in \mathsf{Cs}_\alpha$ and $a \in {}^\omega A$; we want to show that $\sigma^{\mathfrak{A}} a = \tau^{\mathfrak{A}} a$. Let $\mathfrak{B} = \mathfrak{Cg} Rga$. Then $(\mathfrak{B}, a)$ is a Cs_α with generators suitable for Λ_α. Let $\mathfrak{M} = \mathfrak{Mb}^{\Lambda\alpha}_{\mathfrak{B}a}$. Then by (i) $(\xi\sigma)^{\mathfrak{M}} = (\xi\tau)^{\mathfrak{M}}$, so by Theorem 4.3.56, $\sigma^{\mathfrak{B}} a = \tau^{\mathfrak{B}} a$, giving the desired result.

(ii)$\Rightarrow$(i). Let $\mathfrak{M}$ be a possible model of Λ_α; we show that $(\xi\sigma)^{\mathfrak{M}} = (\xi\tau)^{\mathfrak{M}}$. Let $\mathfrak{A} = {}^{\tau}\mathfrak{C}\mathfrak{f}^{\mathfrak{M}}_\alpha$. Then $\sigma^{\mathfrak{A}} R = \tau^{\mathfrak{A}} R$, where $\mathfrak{M} = \langle M, R\rangle$. Now use 4.3.56 again.

REMARK 4.3.58. Theorem 4.3.57 can be given the following equivalent form: for any restricted formula φ of Λ_α the following conditions are equivalent: (i) $\models \varphi$, (ii) $\mathsf{Gs}_\alpha \models \tau\mu\varphi = 1$. A similar remark applies to 4.3.59.

THEOREM 4.3.59. *For any terms σ,τ of $\mathcal{L}_\alpha$ the following conditions are equivalent:*
(i) $\vdash_\tau \xi\sigma \leftrightarrow \xi\tau$.
(ii) $\mathsf{CA}_\alpha \models \sigma = \tau$.

PROOF. It suffices to show that for any restricted formula φ of Λ_α we have: $\vdash_\tau \varphi$ iff $\mathsf{CA}_\alpha \models \tau\mu\varphi = 1$. Let $\equiv$ be ${}_p\equiv^{\Lambda\alpha}_{\tau 0}$ (see Definition 4.3.19). Assume that $\vdash_\tau \varphi$. Then $\varphi/\equiv = 1$. Let $\mathfrak{A} \in \mathsf{CA}_\alpha$ and $a \in {}^\omega A$. By Theorem 4.3.25, let $h \in Hom({}_p\mathfrak{Fm}^{\Lambda\alpha}_{\tau 0}, \mathfrak{A})$ such that $h(\mathbf{R}_i(v_0,\ldots,v_\kappa,\ldots)_{\kappa<\alpha}/\equiv) = a_i$ for all $i \in \omega$. Then $h(\psi/\equiv) = (\tau\mu\psi)^{\mathfrak{A}} a$ for any $\psi \in \Phi^{\Lambda\alpha}_\tau$. Therefore $\mathfrak{A} \models \tau\mu\varphi = 1\,[a]$. This shows that (i)$\Rightarrow$(ii), and the converse is similar.

The above theorems, 4.3.57 and 4.3.59, give satisfactory logical versions for validity of equations in the classes Gs_α or CA_α. Note that for α finite we may express 4.3.57 as follows: $\mathsf{Gs}_\alpha \models \sigma = \tau$ iff $\vdash \xi\sigma \leftrightarrow \xi\tau$, where $\vdash$ is the ordinary provability notion (with *infinitely* many variables available). For $\alpha = \omega$, let σ and τ be terms in $\mathcal{L}_\alpha$. Choose $\beta<\omega$ so that they are terms in $\mathcal{L}_\beta$. Then $\mathsf{Gs}_\omega \models \sigma = \tau$ iff $\mathsf{Gs}_\beta \models \sigma = \tau$ iff $\vdash \xi_\beta\sigma \leftrightarrow \xi_\beta\tau$. For $\alpha>\omega$ a similar but more complicated procefure, involving reduct functions, can be used.

These results are, however, not completely satifactory when considered as algebraic forms of logical results, primarily because the formulas involved have their atomic relational parts all of the same rank. For this reason we now consider a different kind of translation, starting from a countable ordinary language which expands every such language.

DEFINITION 4.3.60. $\Lambda_u = \langle \omega, \mathbf{R}, \rho\rangle = \Lambda u$ *is an ordinary language with* $Do\mathbf{R} = \omega$, *having, for each* $i \in \omega$, *infinitely many relation symbols of rank* i. *We define* $\xi'\sigma$, *for* σ *a term of* $\mathcal{L}_\omega$, *analogously to Definition* 4.3.55; *for arbitrary* $i \in \omega$ *and* $\kappa,\lambda<\omega$,

$$\xi' v_i = \mathbf{R}_i v_0 \ldots v_{\rho i-1};$$
$$\xi' \mathbf{d}_{\kappa\lambda} = v_\kappa = v_\lambda;$$
$$\xi'(\sigma + \tau) = \xi'\sigma \vee \xi'\tau;$$

$$\xi'(\sigma \cdot \tau) = \xi'\sigma \wedge \xi'\tau;$$
$$\xi'(-\sigma) = \neg\xi'\sigma;$$
$$\xi'\mathbf{0} = \mathbf{F};$$
$$\xi'\mathbf{1} = \mathbf{T};$$
$$\xi'\mathsf{c}_\kappa\sigma = \exists u_\kappa \xi'\sigma;$$

Again, ξ' is a one–one function onto the set of all restricted formulas of Λ_u, and we let $\tau\mu'$ be its inverse.

The results we present about this notion are due to Andréka and Németi.

THEOREM 4.3.61. *Let $\varphi,\psi \in \Phi\mu_r^{\Lambda u}$. Let $H = \{i \in \omega : \boldsymbol{v}_i$ occurs in $\varphi \vee \psi\}$ and $G = \{i \in \omega : \mathbf{R}_i$ occurs in $\varphi \vee \psi\}$. Then the following conditions are equivalent:*

(i) $\models \varphi \leftrightarrow \psi$.

(ii) $\mathsf{Gs}_\omega \models \bigwedge\{\mathsf{c}_i \boldsymbol{v}_j = \boldsymbol{v}_j : j \in G,\ i \in H \sim \rho j\} \rightarrow \tau\mu'\varphi = \tau\mu'\psi$.

PROOF. For brevity, let $\theta = \bigwedge\{\mathsf{c}_i \boldsymbol{v}_j = \boldsymbol{v}_j : i \in G,\ i \in H \sim \rho j\}$. First we take the easy direction (ii)$\Rightarrow$(i). Assume that (i) fails. Let $\mathfrak{M}$ be a possible model of Λ_u such that $\varphi^{\mathfrak{M}} \neq \psi^{\mathfrak{M}}$. For each $i \in \omega$ let $a_i = \{u \in {}^\omega M : \rho i \upharpoonright u \in \mathbf{R}_i^{\mathfrak{M}}\}$, where $\Lambda_u = \langle \omega, \mathbf{R}, \rho \rangle$. Let $\mathfrak{A} = \mathfrak{Cf}_\omega^{\mathfrak{M}}$. Then, as is easily checked,

(1) For any $\chi \in \Phi\mu_r^{\Lambda u}$ we have $(\tau\mu'\chi)^{\mathfrak{A}} a = \chi^{\mathfrak{M}}$.

From (1) it is clear that

$$\mathfrak{A} \not\models (\theta \rightarrow \tau\mu'\varphi = \tau\mu'\psi)[a],$$

showing that (ii) fails.

Now assume (i). Then we claim

(2) If $\mathfrak{A} \in \mathsf{Gs}_\omega \cap \mathsf{Lf}_\omega$, $a \in {}^\omega A$, and $\mathsf{c}_i a_j = a_j$ whenever $j \in G$ and $i \in \omega \sim \rho j$, then $\mathfrak{A} \models (\tau\mu'\varphi = \tau\mu'\psi)[a]$.

For, by 3.1.107 we may assume that $\mathfrak{A} \in \mathsf{Cs}_\omega^{reg} \cap \mathsf{Lf}_\omega$. Assume the hypothesis of (2). We may assume that $a_j = 0$ if $j \in \omega \sim G$. Let $\mathfrak{B} = \mathfrak{Cg}Rga$. Then $\langle \mathfrak{B}, a \rangle$ is a Cs_α with generators suitable for Λ_u. Let $\mathfrak{M} = \mathfrak{Nb}_{\mathfrak{B}a}^{\Lambda u}$. Again one can verify (1). Since $\varphi^{\mathfrak{M}} = \psi^{\mathfrak{M}}$ by (i), it follows that $\mathfrak{A} \models (\tau\mu'\varphi = \tau\mu'\psi)[a]$, as desired in (2).

Now, to prove (ii), take any $\mathfrak{A} \in \mathsf{Gs}_\omega$. By 3.1.123 it suffices to take the case $\mathfrak{A} \in \mathsf{Gs}_\omega \cap \mathsf{Lf}_\omega$. Suppose $a \in {}^\omega A$ and $\mathfrak{A} \models \theta[a]$. Let $L = \bigcup_{j \in G} \Delta a_j \sim H$. Thus L is finite. Let ρ be a one–one function from ω onto $\omega \sim L$ such that $H \upharpoonright \rho \subseteq Id$, and set $\mathfrak{B} = \mathfrak{Rb}^\rho \mathfrak{A}$. Then $\mathfrak{B} \in \mathsf{IGs}_\omega \cap \mathsf{Lf}_\omega$ by 3.1.118. Now for any $j \in G$ and $\kappa < \omega$ we have

$$\begin{aligned}
&\mathsf{c}_\kappa^{\mathfrak{B}} a_j \neq a_j \text{ iff } \mathsf{c}_{\rho\kappa}^{\mathfrak{A}} a_j \neq a_j \text{ iff } \rho\kappa \in \Delta^{\mathfrak{A}} a_j \\
&\text{iff } \rho\kappa \in \Delta^{\mathfrak{A}} a_j \text{ and } \rho\kappa \in H \quad \text{(definition of } \rho) \\
&\text{iff } \kappa \in \Delta^{\mathfrak{A}} a_j \cap H \quad (H \upharpoonright \rho \subseteq Id) \\
&\text{iff } \kappa \in \Delta^{\mathfrak{A}} a_j \text{ and } \kappa < \rho j \quad \text{(since } \mathfrak{A} \models \theta[a]).
\end{aligned}$$

Thus $\Delta^{\mathfrak{B}} a_j \subseteq \rho j$ for all $j \in G$. Hence by (2) we have $(\tau\mu'\varphi)^{\mathfrak{B}}[a] = (\tau\mu'\psi)^{\mathfrak{B}}[a]$. Now $H \upharpoonright \rho \subseteq Id$, and if $\mathbf{c}_\kappa$ or $\mathbf{d}_{\kappa\lambda}$ occurs in $\tau\mu'\varphi = \tau\mu'\psi$ then $\kappa,\lambda \in H$. Hence $\mathfrak{A} \models (\tau\mu'\varphi = \tau\mu'\psi)[a]$, finishing the proof.

THEOREM 4.3.62. *Let φ and ψ be restricted formulas of Λ_u. Suppose that if $\mathbf{v}_i$ and $\mathbf{R}_j$ occur in $\varphi \vee \psi$, then $i < \rho j$ (where $\Lambda_u = \langle \omega, \mathbf{R}, \rho \rangle$). Then the following conditions are equivalent:*

(i) $\mathsf{Gs}_\omega \models \tau\mu'\varphi = \tau\mu'\psi$.

(ii) $\models \varphi \leftrightarrow \psi$.

PROOF. Using the notation of 4.3.61, $H \subseteq \rho j$ for each $j \in G$, so 4.3.62 follows from 4.3.61.

DEFINITION 4.3.63. *Let f be a permutation of ω. We denote by f^+ the permutation of $\Phi\mu_r^{\Lambda u}$ induced by f as follows: for any $\varphi \in \Phi\mu_r^{\Lambda u}$, $f^+\varphi$ is the formula obtained from φ by replacing each atomic subformula $\mathbf{R}_i \mathbf{v}_0 \ldots \mathbf{v}_{\rho i - 1}$ of φ $(i \in \omega)$ by $\mathbf{R}_{fi} \mathbf{v}_0 \ldots \mathbf{v}_{\rho f i - 1}$.*

THEOREM 4.3.64. *For any $\varphi, \psi \in \Phi\mu_r^{\Lambda u}$ the following conditions are equivalent:*

(i) $\mathsf{Gs}_\omega \models \tau\mu'\varphi = \tau\mu'\psi$.

(ii) *For every permutation f of ω we have $\models f^+\varphi \leftrightarrow f^+\psi$.*

(iii) *There is a permutation f of ω such that $\models f^+\varphi \leftrightarrow f^+\psi$, and if $\mathbf{v}_i$ and $\mathbf{R}_j$ occur in $\varphi \vee \psi$ then $i < \rho f j$ (with $\Lambda_u = \langle \omega, \mathbf{R}, \rho \rangle$).*

PROOF. (i)$\Rightarrow$(ii). Assume (i), let f be a permutation of ω, and let $\mathfrak{M}$ be a possible model of Λ_u; we want to show that $(f^+\varphi)^{\mathfrak{M}} = (f^+\psi)^{\mathfrak{M}}$. Say $\mathfrak{M} = \langle M, R \rangle$. Let $\mathfrak{A} = \mathfrak{Cf}_\omega^{\mathfrak{M}}$. Now for each $i \in \omega$ let

$$a_i = \{u \in {}^\omega M : \rho f i \upharpoonright u \in \mathbf{R}_{fi}^{\mathfrak{M}}\}.$$

Then the following is easily checked: $(\tau\mu'\chi)^{\mathfrak{A}} a = (f^+\chi)^{\mathfrak{M}}$ for every $\chi \in \Phi\mu_r^{\Lambda u}$. Hence $(f^+\varphi)^{\mathfrak{M}} = (f^+\psi)^{\mathfrak{M}}$, as desired.

(ii)$\Rightarrow$(iii) is obvious. Now assume that f satisfies the conditions of (iii). By Theorem 4.3.62 we then have $\mathsf{Gs}_\omega \models \tau\mu' f^+\varphi = \tau\mu' f^+\psi$. Now if $\mathfrak{A} \in \mathsf{Gs}_\omega$, $a \in {}^\omega A$, and $\chi \in \Phi\mu_r^{\Lambda u}$, then $(\tau\mu'\chi)^{\mathfrak{A}} a = (\tau\mu' f^+\chi)^{\mathfrak{A}}(a \circ f^{-1})$. It follows that (i) holds.

REMARK 4.3.65. Theorem 4.3.64 leads to a natural new logical notion. We call a restricted formula φ of Λ_u *type–free valid* if $\models f^+\varphi$ for every permutation f of ω. By 4.3.64 we know that this is equivalent to $\mathsf{Gs}_\omega \models \tau\mu'\varphi = 1$. Furthermore, 4.3.64(iii) shows that there is a natural proof theory connected with this notion: given φ, let f be a permutation of ω such that if $\mathbf{v}_i$ and R_j occur in φ, then $i < \rho f j$. Then φ is type–free valid iff $f^+\varphi$ is provable in the usual sense. It would be interesting, however, to have a proof notion involving exclusively type–free valid fomulas. For more on the notion type–free valid, see Andréka, Gergely, Németi [77'].

REMARK 4.3.66. We consider the logical meaning of the product of CA_α's. For brevity we consider only Lf_ω's. By Theorem 4.3.28 it suffices to deal with two

algebras $\mathfrak{A} = {}_p\mathfrak{Fm}^{\Lambda}_{r\Gamma}$ and $\mathfrak{B} = {}_p\mathfrak{Fm}^{\Omega}_{r\Sigma}$, where $\Lambda = \langle \omega, \mathbf{R}, \rho \rangle$ and $\Omega = \langle \omega, \mathbf{S}, \sigma \rangle$ are ordinary languages; furthermore, we clearly may assume that $Rg\mathbf{R} \cap Rg\mathbf{S} = 0$. Let $\Xi = \langle \omega, \mathbf{T}, \mu \rangle$ be an arbitrary ordinary language whose non-logical symbols are those of Λ *and* Ω plus a new one-place relation symbol $\mathbf{P}$. By Theorem 4.3.17, let $h \in Hom({}_s\mathfrak{Fm}^{\Xi}_{r0}, \mathfrak{A} \times \mathfrak{B})$ be such that:

(1) $h(\mathbf{R}_\xi v_0 \ldots v_{\rho\xi-1}/\equiv) = \langle \mathbf{R}_\xi v_0 \ldots v_{\rho\xi-1}/\equiv, 0 \rangle$ for $\xi \in Do\mathbf{R}$,
(2) $h(\mathbf{S}_\eta v_0 \ldots v_{\sigma\eta-1}/\equiv) = \langle 0, \mathbf{S}_\eta v_0 \ldots v_{\sigma\eta-1}/\equiv \rangle$ for $\eta \in Do\mathbf{S}$,
(3) $h(\mathbf{P}v_0/\equiv) = \langle 1, 0 \rangle$.

Clearly, in fact, $h \in Ho({}_s\mathfrak{Fm}^{\Xi}_{r0}, \mathfrak{A} \times \mathfrak{B})$. Let $\Delta = \{\varphi \in \Phi\mu^{\Xi}_r : h(\varphi/\equiv) = \langle 1, 1 \rangle\}$. By Theorem 4.3.27 we infer that $\mathfrak{A} \times \mathfrak{B} \cong {}_s\mathfrak{Fm}^{\Xi}_{r\Delta}$. We can characterize Δ in terms of Γ and Σ as follows, thus giving a logical meaning to $\mathfrak{A} \times \mathfrak{B}$. This characterization is due to Andréka and Németi. Let Δ' be the set of all formulas of Ξ of the following kinds:

$$\mathbf{P}v_0 \Rightarrow \forall v_0 \mathbf{P}v_0,$$
$$\mathbf{P}v_0 \Rightarrow \neg \mathbf{S}_\eta v_0 \ldots v_{\sigma\eta-1} \text{ for } \eta \in Do\mathbf{S},$$
$$\neg \mathbf{P}v_0 \Rightarrow \neg \mathbf{R}_\xi v_0 \ldots v_{\rho\xi-1} \text{ for } \xi \in Do\mathbf{R},$$
$$\mathbf{P}v_0 \Rightarrow \varphi \text{ if } \varphi \in \Gamma,$$
$$\neg \mathbf{P}v_0 \Rightarrow \psi \text{ if } \psi \in \Sigma.$$

We claim

(4) $\Delta = \{\varphi : \varphi \in \Phi\mu^{\Xi}_r \text{ and } \Delta' \models \varphi\}$.

In fact, clearly $h\varphi = 1$ for each $\varphi \in \Delta'$, and hence $\supseteq$ holds. For $\subseteq$, we first associate with each restricted formula φ of Ξ a restricted formula φ' of Λ by replacing each subformula $\mathbf{P}v_0$ by $\mathbf{T}$, and each subformula $\mathbf{S}_\eta v_0 \ldots v_{\sigma\eta-1}$ by $\mathbf{F}$. Similarly, we obtain $\varphi^* \in \Phi\mu^{\Omega}_r$ by replacing $\mathbf{P}v_0$ by $\mathbf{F}$ and each $\mathbf{R}_\xi v_0 \ldots v_{\rho\xi-1}$ by $\mathbf{F}$. Then

(5) $h(\varphi/\equiv) = \langle \varphi'/\equiv, \varphi^*/\equiv \rangle$ for any $\varphi \in \Phi\mu^{\Xi}_r$;
(6) $\Delta' \models \varphi \Leftrightarrow (\varphi' \wedge \mathbf{P}v_0) \vee (\varphi^* \wedge \neg \mathbf{P}v_0)$ for any $\varphi \in \Phi\mu^{\Xi}_r$.

Both of these statements are easily verified by induction on φ. Now to prove $\subseteq$ in (4), let $\varphi \in \Delta$. Thus $h(\varphi/\equiv) = 1$, so by (5) we have $\Gamma \models \varphi'$ and $\Sigma \models \varphi^*$. Hence by the definition of Δ' we have $\Delta' \models \mathbf{P}v_0 \Rightarrow \varphi'$ and $\Delta' \models \neg \mathbf{P}v_0 \Rightarrow \varphi^*$. Hence by (6), $\Delta' \models \varphi$, as desired.

We give one result connecting ultraproducts of models with ultraproducts of the associated Cs_α's.

THEOREM 4.3.67. *Let $\Lambda = \langle \alpha, \mathbf{R}, \rho \rangle$ be an ordinary language, $\langle \mathfrak{R}_i : i \in I \rangle$ a system of possible models of Λ, and F an ultrafilter on I. Let $\mathfrak{R} = \mathrm{P}_{i \in I} \mathfrak{R}_i / F$. Then there is an $f \in Ism(\mathfrak{Cf}^{\mathfrak{R}}_\alpha, \mathrm{P}_{i \in I} \mathfrak{Cf}^{\mathfrak{R}_i}_\alpha / F)$ such that $f\varphi^{\mathfrak{R}} = \langle \varphi^{\mathfrak{R}_i} : i \in I \rangle / F$ for any $\varphi \in \Phi\mu^{\Lambda}_r$, and $Rep(c) \circ f = Cs^{\mathfrak{R}}_\alpha \restriction Id$ for any $\langle F, \langle M_i : i \in I \rangle, \alpha \rangle$-choice function c.* (Cf. 3.1.89.)

PROOF. Clearly there is an isomorphism f satisfying the first condition. Now let c be any $(F,\langle M_i : i \in I\rangle,\alpha)$–choice funstion, let $\varphi \in \Phi\mu_r^{\Lambda}$, and let $q \in {}^{\alpha}N$. Let $r = \langle c(\kappa,q_\kappa) : \kappa < \alpha\rangle$. Thus $pj_i \circ r = (c^+q)_i$ and $r\kappa/\bar{F} = q\kappa$ for all $i \in I$ and $\kappa < \alpha$. Then $q \in Rep(c)f\varphi$ iff $\{i \in I : (c^+q)_i \in \varphi^{\mathfrak{R}}\} \in F$ iff $\{i \in I : pj_i \circ r \in \varphi^{\mathfrak{R}i}\} \in F$ iff $\bar{F}^* \circ r \in \varphi^{\mathfrak{R}}$ iff $q \in \varphi^{\mathfrak{R}}$, as desired.

Concluding remarks

REMARKS 4.3.68. In these concluding remarks of the section we want to briefly summarize the section and indicate without proof some other connections which can be established. For simplicity we restrict ourselves to ordinary languages. Recall Remark 4.3.11; we indicate below several generator–free ideas.

(1) The algebraic version of an ordinary language is a dimension–restricted free algebra $\mathfrak{Fr}_\kappa^\Omega \mathsf{CA}_\alpha$ with $\alpha \geqq \omega$ and $Rg\Omega \subseteq \omega$.

(2) A theory in a language has several algebraic expressions: a filter in a free algebra, a homomorphism from a free algebra to some Lf_α, merely an Lf_α (a generator–free version), or a Gs_α – also a generator–free version.

(3) The most immediate algebraic form of the compactness theorem is as follows. Suppose $\alpha \geqq \omega$, κ is a cardinal, and $\Omega \in {}^{\kappa}\omega$. Let F be a filter in $\mathfrak{Fr}_\kappa^\Omega \mathsf{CA}_\alpha$, and suppose that for each $a \in F$ there is a homomorphism h from $\mathfrak{Fr}_\kappa^\Omega \mathsf{CA}_\alpha$ into a Cs_α such that $ha \neq 0$. Then there is a homomorphism k from $\mathfrak{Fr}_\kappa^\Omega \mathsf{CA}_\alpha$ into a Cs_α such that $ka \neq 0$ for all $a \in F$. This algebraic theorem is an immediate consequence of the representability of Lf_α's, just like the compactness theorem is immediate from the completeness theorem. In fact, the assumptions imply that F is a proper filter, and hence $\mathfrak{Fr}_\kappa^\Omega \mathsf{CA}_\alpha/F$ is an Lf_α with at least two elements. Therefore there is a homomorphism of $\mathfrak{Fr}_\kappa^\Omega \mathsf{CA}_\alpha/F$ onto some Cs_α with a non–empty base; this homomorphism naturally induces the desired homomorphism k. Of course this algebraic form of the compactness theorem can be proved more directly using ultraproducts; the easy proof based on 3.1.92 will be left to the reader.

(4) An interpretation between theories corresponds to a homomorphism between Lf_α's; definitional equivalence to isomorphism. Alternatively, interpretations correspond to base–homomorphisms of Gs_α's.

(5) Possible models of languages correspond to cylindric set algebras with generators; Cs_α's themselves are the generator–free version.

(6) We have already used, in section 3.2, an algebraic version of an individual constant, and discussed other versions. Operation symbols can be treated similarly.

(7) The algebraic version of the basic logical notion of elementary equivalence is the isomorphism of $\langle \mathfrak{A},x\rangle$ and $\langle \mathfrak{B},y\rangle$, where $\langle \mathfrak{A},x\rangle$ and $\langle \mathfrak{B},y\rangle$ are Cs_α's with generators (which can be be considered as algebraic structures in their own right). The generator–free notion is just that $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic. Note that $\mathfrak{Cr}_\alpha^{\mathfrak{B}} \cong \mathfrak{Cr}_\alpha^{\mathfrak{A}}$ iff $\mathfrak{B}$ is elementarily equivalent to some model which is definitionally equivalent to $\mathfrak{A}$ Note also that 3.1.38(1) is a generator–free version of the logical theorem that if $\mathfrak{B}$

and $\mathfrak{N}$ are elementarily equivalent and $|M|<\omega$, then $\mathfrak{M}\cong\mathfrak{N}$.
(8) Recall that a set Γ of sentences in Λ is *complete* if for every sentence φ of Λ, either $\Gamma\models\varphi$ or $\Gamma\models\neg\varphi$. This means exactly that ${}_s\mathfrak{Fm}_\Gamma^\Lambda$ is *simple*, or else has only one element. An equivalent way of saying the Γ is complete is that any two models of Γ are elementarily equivalent. If we take the algebraic version of this simple equivalence we are led to the following theorem (a generalization of the obvious translation):

(*) For any Ss_α $\mathfrak{C}$ the following conditions are equivalent:
(i) $|C|=1$, or $\mathfrak{C}$ is simple;
(ii) for all $f,g\in Ho\mathfrak{C}$, if $f^*\mathfrak{C}$ and $g^*\mathfrak{C}$ are simple, then there is an $h\in Is(f^*\mathfrak{C},g^*\mathfrak{C})$ with $g=h\circ f$.

Andréka and Németi have shown that "Ss_α" cannot be omitted in (*), but that (*) continues to hold if "Ss_α" is omitted and "simple" is replaced by "subdirectly indecomposable". (*) is a generalization of the generator–version of the above equivalent definition of simplicity. *The generator–free version of this result does not hold.* It would say that the following conditions are equivalent:

(iii) $|C|=1$ or $\mathfrak{C}$ is simple.
(iv) If $\mathfrak{A},\mathfrak{B}\in\mathsf{Lf}_\alpha\cap\mathsf{H}\mathfrak{C}$ are simple, then $\mathfrak{A}\cong\mathfrak{B}$.

It is easy to give a counterexample to this.
(9) *κ–categoricity.* The algebraic version of this notion is formulated as follows. A CA_α $\mathfrak{C}$ is *κ–categorical* if for any $\mathfrak{A},\mathfrak{B}\in\mathsf{Cs}_\alpha^{reg}\cap\mathsf{Lf}_\alpha$ and any $f\in Ho(\mathfrak{C},\mathfrak{A})$, $g\in Ho(\mathfrak{C},\mathfrak{B})$, if both $\mathfrak{A}$ and $\mathfrak{B}$ have bases of size κ, then there is a base–isomorphism h of $\mathfrak{A}$ onto $\mathfrak{B}$ such that $h\circ f=g$. Then the Łoś–Vaught test for completeness takes the following form:

(*) Let $\alpha\geqq\omega$, and let $\mathfrak{C}$ be an Lf_α of characteristic 0 (see 2.4.61). Suppose that κ is a cardinal $\geqq|C|$, and $\mathfrak{C}$ is κ–categorical. Then $\mathfrak{C}$ is simple.

On the other hand, the well–known characterization theorem for ω–categorical theories looks like this algebraically:

(**) Let $\mathfrak{A}\in\mathsf{Lf}_\omega$ be simple, countable, and have characteristic 0. Then the following conditions are equivalent:
(i) $\mathfrak{A}$ is ω–categorical.
(ii) For every $n\in\omega$, $\mathfrak{Nr}_n\mathfrak{A}$ is finite.

(10) *Elementary extensions.* Let Λ be an ordinary language and $\mathfrak{M},\mathfrak{N}$ possible models of Λ. To say that $\mathfrak{N}$ is an elementary extension of $\mathfrak{M}$ is to say that $\mathfrak{Cf}_\alpha^{\mathfrak{N}}$ is ext–isomorphic to $\mathfrak{Cf}_\alpha^{\mathfrak{M}}$ with the natural generators preserved (see Definition 3.1.41). Thus many of the results in section 3.1 have a logical meaning for our general languages. Going the other way, the many logical results concerning elementary extensions give rise to algebraic theorems. To illustrate the ideas, we consider one specific elementary result.

Given a possible model $\mathfrak{M}$ for a language Λ, expand Λ to Λ' by adjoining an individual constant c_m for each $m\in M$. Then $(\mathfrak{M},m)_{m\in M}$ denotes the expansion of $\mathfrak{M}$

to a possible model of Λ' in which each constant c_m has denotation m. Let Θ be the set of all sentences of Λ' which hold in $(\mathfrak{M},m)_{m\in M}$. Then for any possible model $\mathfrak{N}$ of Λ the following conditions are equivalent:

(i) $\mathfrak{N}$ is isomorphic to an elementary extension of $\mathfrak{M}$.
(ii) $\mathfrak{N}$ can be expanded to a model of Θ.

The natural algebraic theorem corresponding to this result is as follows. Let $\alpha\geqq 1$, and let $\mathfrak{A}$ be a Crs_α with base U and unit element V, and suppose that $u\in U$. We define

$$k_u=\{f\in V: f_0=u\}.$$

For any $Y\subseteq U$ we let $\mathfrak{A}_Y$ be the subalgebra of $\mathfrak{Cb}V$ generated by $A\cup\{k_u: u\in Y\}$. The result (due to Andréka and Németi) is then as follows:

(*) Suppose that $\alpha\geqq 1$, and $\mathfrak{A},\mathfrak{B}\in \mathsf{Cs}_\alpha^{reg}\cap\mathsf{Lf}_\alpha$ with bases U,W restectively. Then the following conditions are equivalent:
(i) $\mathfrak{B}$ is base isomorphic to some Cs_α which is ext–isomorphic to $\mathfrak{A}$.
(ii) There is an $X\subseteq W$ and an $f\in Is(\mathfrak{B}_X,\mathfrak{A}_U)$ such that $f^*B=A$.

This result has been considerably generalized by Andréka and Németi.
(11) *Omitting types.* We mention one algebraic result which is a generator–free version of an omitting types theorem. Let $\mathfrak{A}$ be a countable Lf_ω, $X\subseteq Nr_1\mathfrak{A}$, and $\Pi^{\mathfrak{A}}X=0$. Then there is a $\mathfrak{B}\in\mathsf{Cs}_\omega^{reg}\cap\mathsf{Lf}_\omega$ with countable base and an $f\in Ho(\mathfrak{A},\mathfrak{B})$ such that $\bigcap f^*X=0$.

In general, omitting types theorems correapond to preservations of meets, or joins, by homomorphisms.
(12) *Saturated models.* The following theorem is due to Németi [83b']. He has also proved related theorems concerning universal models.

(*) Let $\mathfrak{M}$ be a possible model of an ordinary language $\Lambda=\langle\alpha,\mathbf{R},\rho\rangle$. Then $\mathfrak{M}$ is $|\alpha|^+$–saturated iff $\mathfrak{Cf}_\alpha^{\mathfrak{M}}$ has the following property: if $F\subseteq\mathfrak{Cf}_\alpha^{\mathfrak{M}}$ is directed by $\supseteq$, then $c_\kappa\bigcap F=\bigcap c_\kappa^*F$ for every $\kappa<\alpha$.

(13) *Beth's theorem.* Németi [83'] has shown that this theorem corresponds to the statement that epis are surjective for Lf_ω's; this equivalence generalizes to more general Λ and associated classes.
(14) *Prime models.* The algebraic counterpart of this notion is as follows. Let $\mathfrak{A}\in\mathsf{Cs}_\alpha^{reg}\cap\mathsf{Lf}_\alpha$. We call $\mathfrak{A}$ CA–prime if for all $\mathfrak{B}\in\mathsf{Cs}_\alpha^{reg}\cap\mathsf{Lf}_\alpha$, $\mathfrak{A}\cong\mathfrak{B}$ implies that $\mathfrak{A}$ is base–isomorphic to some $\mathfrak{C}$ which is sub–isomorphic to $\mathfrak{B}$. One basic theorem about prime models has the following algebraic form:

(*) Let $\mathfrak{A}$ be a countable $\mathsf{Cs}_\omega^{reg}\cap\mathsf{Lf}_\omega$. Then the following conditions are equivalent:
(i) $\mathfrak{A}$ is CA–prime.
(ii) The base U of $\mathfrak{A}$ is countable, and for every $n<\omega$, $\mathfrak{Nr}_n\mathfrak{A}$ is atomic and

$${}^\omega U=\bigcup\{x: x \text{ is an atom in } \mathfrak{Nr}_n\mathfrak{A}\}.$$

PROBLEMS

PROBLEM 4.1. *For* $\alpha \geqq 3$, *give a simple equational basis for* IGs_α.

Cf. the part of section 4.1 dealing with equations characterizing IGs_α.

PROBLEM 4.2. *For* $\alpha \geqq \omega_1$, *are there* $2^{|\alpha|}$ *varieties of* ICs_α*'s? Of* IGs_α*'s?*

See Theorem 4.1.24.
In connection with 4.1.32–4.1.38 the following problem arises.

PROBLEM 4.3. *Is* $\mathsf{HSPMn}_\alpha = \mathsf{SPUpMn}_\alpha$ *or* $\mathsf{SUpMn}_\alpha = \mathsf{HSUpMn}_\alpha$ *for* $\alpha \geqq \omega$?

PROBLEM 4.4. *For* $3 \leqq \alpha < \beta$, *is* $\mathsf{Nr}_\alpha\mathsf{CA}_\beta$ *an elementary class?*

For this problem, see the comments preceeding Theorem 4.1.44.

PROBLEM 4.5. *For* $\alpha \geqq \omega$ *is it true that* $\mathsf{UnCs}_\alpha = \mathsf{ElCs}_\alpha$?

Cf. here Theorem 4.1.48.
For the next four problems see Andréka, Németi [81'] 7.4–7.9.

PROBLEM 4.6. *If* $1 < \kappa < \omega \leqq \alpha$ *and* $\mathfrak{A} \in {}_\kappa\mathsf{Cs}_\alpha^{reg}$, *is* $\mathfrak{A} \times \mathfrak{A} \in \mathsf{El}_\kappa\mathsf{Cs}_\alpha^{reg}$? *Similarly with* "$\mathsf{Ws}_\alpha$" *in place of* "${}_\kappa\mathsf{Cs}_\alpha^{reg}$".

PROBLEM 4.7. *If* $1 < \kappa < \omega \leqq \alpha$, *is* $\mathsf{El}({}_\kappa\mathsf{Gs}_\alpha \cap \mathsf{Lf}_\alpha) = \mathsf{El}({}_\kappa\mathsf{Cs}_\alpha^{reg} \cap \mathsf{Lf}_\alpha)$?

This is Problem 14 of Andréka, Németi [81'].

PROBLEM 4.8. *If* $\omega \leqq \alpha$, *is* $\mathsf{El}_\infty\mathsf{Ws}_\alpha = \mathsf{Un}_\infty\mathsf{Ws}_\alpha$ *or* $\mathsf{El}_\infty\mathsf{Cs}_\alpha^{reg} = \mathsf{El}_\infty\mathsf{Ws}_\alpha \cup \mathsf{I}_0\mathsf{Cs}_\alpha$?

This is Problem 16 of Andréka, Németi [81'].

PROBLEM 4.9. *If* $\omega \leqq \alpha$, *is* $\mathsf{El}_\infty\mathsf{Cs}_\alpha^{reg} = \mathsf{I}_\infty\mathsf{Cs}_\alpha$?

Some consistency results are known concerning the next problem; see Andréka, Németi [81'] Problem 20.

PROBLEM 4.10. *If* $\omega \leqq \alpha < \beta$, *is* $\mathsf{El}(\mathsf{Ws}_\alpha \cap \mathsf{Lf}_\alpha) = \mathsf{ElRd}_\alpha(\mathsf{Ws}_\beta \cap \mathsf{Lf}_\beta)$ *or* $\mathsf{ElWs}_\alpha = \mathsf{ElRd}_\alpha \mathsf{Ws}_\beta$?

Related problems (still open) are numbers 20,21,25 Of Andréka, Németi [81'].

PROBLEM 4.11. *Is the first–order theory of* Mn_ω *decidable?*

For Problem 4.11, see 4.2.1–4.2.3.

PROBLEM 4.12. *For* $\alpha \leqq \omega$, *is the equational theory of monadic–generated* CA_α*'s decidable?*

See p. 131 and also Monk [64a]

PROBLEM 4.13. *For* $\alpha = \omega$, *give a simple recursive axiomatization of* $\Theta\rho\mathsf{Cs}_\alpha$.

See Theorem 4.2.31.

PROBLEM 4.14. *Give a purely algebraic proof that if* $3 \leqq \alpha < \omega$ *and* $4 \leqq \beta < \omega$ *then* $\mathfrak{S}\mathfrak{r}_\beta \mathsf{Gs}_\alpha$ *is not atomic. Does this result extend to* $1 \leqq \beta \leqq 3$, *and is it also true with* "Gs_α" *replaced with* "CA_α"?

PROBLEM 4.15. *For* $3 \leqq \alpha < \omega$, *let* K *be the class of all finite* CA_α*'s. Is* $\mathsf{CA}_\alpha = \mathsf{EqK}$?

Cf. the introduction to this volume, and also 4.3.32–4.3.33, for information on Problems 4.14 and 4.15.

PROBLEM 4.16. *Give a proof calculus for type–free valid formulas which involves* **only** *type–free valid formulas.*

See 4.3.64–4.3.65.

CHAPTER 5

OTHER ALGEBRAIC VERSIONS OF LOGIC

5. OTHER ALGEBRAIC VERSIONS OF LOGIC

This chapter is devoted to the descriptions of other algebraic versions of logic which have appeared in the literature. The first five sections deal with versions closely related to cylindric algebras: diagonal–free cylindric algebras, which differ from CA_α's only in not having diagonal elements – they have been mentioned at various places in Part I; projective algebras, which are a two–dimensional precursor to CA_α's; relation algebras, an independently interesting and well–developed kind of algebras intermediate between CA_2's and CA_3's; polyadic algebras, developed mainly by Halmos and very roughly equivalent to CA_α's; and relativized CA_α's, which are not in general CA_α's but extend the class CA_α – again they have been mentioned from time to time already. The last section of the chapter surveys additional algebraic versions of logic, in particular the general algebraic logic developed mainly by Andréka and Németi. In all of this chapter our interest is mainly in the connection of the new notions with CA_α's, and, of course, we do not go into the theory of these new algebras to the depth that we have for CA_α's.

5.1 DIAGONAL–FREE CYLINDRIC ALGEBRAS

Recall, from 1.1.2, that a *diagonal–free cylindric algebra of dimension* α, for brevity a Df_α, is an algebraic structure $\mathfrak{B}=\langle B,+,\cdot,-,0,1,\mathsf{c}_\kappa\rangle_{\kappa<\alpha}$ satisfying (C_0)–(C_4) of 1.1.1, i.e., all of the postulates for cylindric algebras not involving the diagonal elements. For any CA_α $\mathfrak{A}$, $\mathfrak{Df}\mathfrak{A}$ is the diagonal–free part of $\mathfrak{A}$, i.e., the reduct of $\mathfrak{A}$ obtained by deleting the diagonal elements. In this section we sketch the theory of Df_α's, which parallels the theory of CA_α's, and we indicate relationships between these classes.

The elementary arithmetic of cylindrifications, expressed in Chapter 1, extends with practically no changes to Df_α's. Specifically, 1.2.1–1.2.17 hold for Df_α's, with the same proofs. The results in section 1.3 concerning discrete CA_α's do *not* extend to Df_α's, as we shall now indicate.

DEFINITION 5.1.1. *A* Df_α $\mathfrak{A}$ *is* **discrete** *if* $\mathsf{c}_\kappa x = x$ *for all* $\kappa<\alpha$ *and all* $x\in A$.

Now 1.3.11 and 1.3.16 still hold when formulated for Df_α's. This is *not* true of 1.3.13 and 1.3.15, even though these results do not involve diagonal elements. This follows from the following obvious lemma.

LEMMA 5.1.2. *Suppose* $\alpha \leqq \beta$ *and* $\mathfrak{A}$ *is a* Df_α. *Let* $\mathfrak{B} = \langle A, +, \cdot, -, 0, 1, \mathsf{c}_\kappa^{\mathfrak{B}} \rangle_{\kappa<\beta}$, *where* $\mathsf{c}_\kappa^{\mathfrak{B}} = \mathsf{c}_\kappa^{\mathfrak{A}}$ *if* $\kappa < \alpha$, *and* $\mathsf{c}_\kappa^{\mathfrak{B}} = A \upharpoonright Id$ *if* $\kappa \in \beta \sim \alpha$. *Then* $\mathfrak{B}$ *is a* Df_β.

Note that in 5.1.2 we could take $\mathsf{c}_\kappa^{\mathfrak{B}}$ for $\kappa \in \beta \sim \alpha$ to be such that $\mathsf{c}_\kappa^{\mathfrak{B}} x = 1$ if $x \neq 0$, $\mathsf{c}_\kappa^{\mathfrak{B}} 0 = 0$.

DEFINITION 5.1.3. *If* K *is a class of* CA_α*'s, then* $\mathsf{Df}\mathsf{K} = \{\mathfrak{Df}\mathfrak{A} : \mathfrak{A} \in \mathsf{K}\}$.

COROLLARY 5.1.4. (i) *If* $\alpha \leqq 1$, *then* $\mathsf{Df}_\alpha = \mathsf{Df}\mathsf{CA}_\alpha$.
(ii) *If* $\alpha \geqq 2$, *then* $\mathsf{Df}\mathsf{CA}_\alpha \subset \mathsf{Df}_\alpha$.

PROOF. By 5.1.2 and 1.3.13.

REMARK 5.1.5. In connection with 5.1.4 it is natural to ask whether for every $\mathfrak{A} \in \mathsf{Df}\mathsf{CA}_\alpha$ there is a unique $\mathfrak{B} \in \mathsf{CA}_\alpha$ such that $\mathfrak{A} = \mathfrak{Df}\mathfrak{B}$. For $\alpha \leqq 1$ this is obviously the case. For $\alpha \geqq 2$ we can easily produce counterexamples, as follows. Let $U = \alpha \times 2$, let $\mathfrak{B}$ be the full Cs_α with base U, and let $\mathfrak{A} = \mathfrak{Df}\mathfrak{B}$. We produce a different CA_α $\mathfrak{C}$ such that $\mathfrak{A} = \mathfrak{Df}\mathfrak{C}$ as follows. For each $\kappa < \alpha$ let t_κ be the permutation of U such that for any $\lambda < \alpha$ and $\varepsilon \in 2$ we have

$$t_\kappa \langle \lambda, \varepsilon \rangle = \begin{cases} \langle \lambda, \varepsilon \rangle & \text{if } \kappa \neq \lambda, \\ \langle \lambda, 1-\varepsilon \rangle & \text{if } \kappa = \lambda, \end{cases}$$

For all $\kappa, \lambda < \alpha$ let $\mathsf{d}'_{\kappa\lambda} = \{x \in {}^\alpha U : t_\kappa x_\kappa = t_\lambda x_\lambda\}$, and let $\mathfrak{C} = \langle A, +, \cdot, -, 0, 1, \mathsf{c}_\kappa^{\mathfrak{A}}, \mathsf{d}'_{\kappa\lambda} \rangle_{\kappa,\lambda<\alpha}$. Now $\mathfrak{C} \neq \mathfrak{B}$, since $x \in \mathsf{d}'_{01} \sim \mathsf{d}_{01}^{\mathfrak{B}}$, where $x_0 = \langle 0,1 \rangle$ and $x_\lambda = \langle 0,0 \rangle$ for all $\lambda \in \alpha \sim 1$. Clearly also $\mathfrak{A} = \mathfrak{Df}\mathfrak{C}$. Finally, $\mathfrak{C}$ is a CA_α; to prove this, we must check (C_5)–(C_7) of 1.1.1. Obviously (C_5) holds. To check (C_6), suppose that $\kappa, \lambda, \mu < \alpha$ and $\kappa \neq \lambda, \mu$; we want to show that $\mathsf{d}'_{\lambda\mu} = \mathsf{c}_\kappa(\mathsf{d}'_{\lambda\kappa} \cdot \mathsf{d}'_{\kappa\mu})$. First suppose that $x \in \mathsf{d}'_{\lambda\mu}$. Let $y_\xi = x_\xi$ for all $\xi \in \alpha \sim \{\kappa\}$ and $y_\kappa = t_\kappa t_\lambda x_\lambda$. Then $t_\kappa y_\kappa = t_\lambda x_\lambda = t_\lambda y_\lambda$ and $t_\kappa y_\kappa = t_\lambda x_\lambda = t_\mu x_\mu = t_\mu y_\mu$, so $y \in \mathsf{d}'_{\lambda\kappa} \cap \mathsf{d}'_{\kappa\mu}$, hence $x \in \mathsf{c}_\kappa(\mathsf{d}'_{\lambda\kappa} \cdot \mathsf{d}'_{\kappa\mu})$. Conversely, suppose $x \in \mathsf{c}_\kappa(\mathsf{d}'_{\lambda\kappa} \cdot \mathsf{d}'_{\kappa\mu})$. Say $x_u^\kappa \in \mathsf{d}'_{\lambda\kappa} \cdot \mathsf{d}'_{\kappa\mu}$, where $u \in U$. Then $t_\lambda x_\lambda = t_\lambda(x_u^\kappa)_\lambda = t_\kappa(x_u^\kappa)_\kappa = t_\mu(x_u^\kappa)_\mu = t_\mu x_\mu$, so $x \in \mathsf{d}'_{\lambda\mu}$, as desired. Finally we check (C_7). Suppose $\kappa \neq \lambda$, $X \in A$, but $x \in \mathsf{c}_\kappa(\mathsf{d}'_{\kappa\lambda} \cdot X) \cap \mathsf{c}_\kappa(\mathsf{d}'_{\kappa\lambda} \cdot -X)$. say $x_u^\kappa \in \mathsf{d}'_{\kappa\lambda} \cdot X$ and $x_v^\kappa \in \mathsf{d}'_{\kappa\lambda} \cdot -X$, where $u, v \in U$. Then $t_\kappa u = t_\kappa(x_u^\kappa)_\kappa = t_\lambda(x_u^\kappa)_\lambda = t_\lambda x_\lambda$ and $t_\kappa v = t_\lambda x_\lambda$ similarly, so $t_\kappa u = t_\kappa v$. Hence $u = v$; but $x_u^\kappa \in X$ and $x_v^\kappa \notin X$, contradiction.

We do not know how to characterize those $\mathfrak{A} \in \mathsf{Df}\mathsf{CA}_\alpha$ for which there is a unique $\mathfrak{B} \in \mathsf{CA}_\alpha$ such that $\mathfrak{A} = \mathfrak{Df}\mathfrak{B}$.

The duality results in section 1.4 hold for Df_α's also. Concerning 1.5.23 and 1.5.24 we have the following result.

THEOREM 5.1.6. *For each* $\alpha \geqq 4$ *there is an equation not involving diagonal elements which holds in every* CA_α *but not in every* Df_α. *An example of such an equation is:*
$\mathsf{c}_0\mathsf{c}_3 x \cdot \mathsf{c}_1\mathsf{c}_3 y \cdot \mathsf{c}_2\mathsf{c}_3 z \cdot \mathsf{c}_0\mathsf{c}_1\mathsf{c}_2[\mathsf{c}_2(\mathsf{c}_1\mathsf{c}_3 x \cdot \mathsf{c}_0\mathsf{c}_3 y) \cdot \mathsf{c}_1(\mathsf{c}_2\mathsf{c}_3 x \cdot \mathsf{c}_0\mathsf{c}_3 z) \cdot \mathsf{c}_0(\mathsf{c}_2\mathsf{c}_3 y \cdot \mathsf{c}_1\mathsf{c}_3 z)] = \mathsf{c}_0\mathsf{c}_3 x \cdot \mathsf{c}_1\mathsf{c}_3 y \cdot \mathsf{c}_2\mathsf{c}_3 z$.

In particular, there is a Df_α *in which this equation fails while* $x \neq \mathsf{c}_3 x$, $y \neq \mathsf{c}_3 y$, *and* $z \neq \mathsf{c}_3 z$.

PROOF. By 3.2.68, let $\mathfrak{A}$ be a CA_3 having elements x, y, z such that

$$\mathsf{c}_0 x \cdot \mathsf{c}_1 y \cdot \mathsf{c}_2 z \nleq \mathsf{c}_0 \mathsf{c}_1 \mathsf{c}_2 [\mathsf{c}_2(\mathsf{c}_1 x \cdot \mathsf{c}_0 y) \cdot \mathsf{c}_1(\mathsf{c}_2 x \cdot \mathsf{c}_0 z) \cdot \mathsf{c}_0(\mathsf{c}_2 y \cdot \mathsf{c}_1 z)].$$

Now if we apply 5.1.2, we get the desired Df_α. Note that the equation given in 5.1.6 holds in every CA_α by 1.5.23.

For the final sentence of the theorem, we can apply the remark following 5.1.2, as soon as we check that x, y, and z are all different from 0 and 1. Clearly they are all different from 0. By symmetry it remains only to show, say, that $x \neq 1$. Suppose, to the contrary, that $x = 1$. Then

$$\begin{aligned}
\mathsf{c}_1 y \cdot \mathsf{c}_2 z &\nleq \mathsf{c}_0 \mathsf{c}_1 \mathsf{c}_2 [\mathsf{c}_2 \mathsf{c}_0 y \cdot \mathsf{c}_1 \mathsf{c}_0 y \cdot \mathsf{c}_0(\mathsf{c}_2 y \cdot \mathsf{c}_1 z)] \\
&= \mathsf{c}_1 \mathsf{c}_2 [\mathsf{c}_2 \mathsf{c}_0 y \cdot \mathsf{c}_0(\mathsf{c}_1 \mathsf{c}_0 y \cdot \mathsf{c}_2 y \cdot \mathsf{c}_1 z)] \\
&= \mathsf{c}_1 \mathsf{c}_2 [\mathsf{c}_2 \mathsf{c}_0 y \cdot \mathsf{c}_0(\mathsf{c}_2 y \cdot \mathsf{c}_1 z)] \\
&= \mathsf{c}_0 \mathsf{c}_1 \mathsf{c}_2 (\mathsf{c}_2 \mathsf{c}_0 y \cdot \mathsf{c}_2 y \cdot \mathsf{c}_1 z) \\
&= \mathsf{c}_0 \mathsf{c}_1 \mathsf{c}_2 (\mathsf{c}_2 y \cdot \mathsf{c}_1 z) \\
&= \mathsf{c}_0 (\mathsf{c}_1 \mathsf{c}_2 y \cdot \mathsf{c}_1 \mathsf{c}_2 z) \geqq \mathsf{c}_1 y \cdot \mathsf{c}_2 z,
\end{aligned}$$

a contradiction.

We do not know whether the first part of 5.1.6 extends to $\alpha = 3$. It would be interesting to find other equations with the property mentioned in 5.1.6.

The notion of dimension set naturally extends to Df_α's, and Theorems 1.6.2, 1.6.3, 1.6.5 – 1.6.8, and 1.6.17 hold for Df_α's also; this does not apply to 1.6.11, of course (apply 5.1.2). Definition 1.6.18 can be given for Df's in the same form, and Theorem 1.6.19 extends also.

The results of section 1.7 all carry over to Df_α's. The same is true of the results in section 1.10 which do not involve diagonal elements, except for 1.10.5. Locally finite–dimensional and dimension–complemented Df_α's can be defined as in section 1.11, but only the trivial properties expressed in 1.11.3(i), (ii), (iv) carry over to Df_α's. In fact, these classes are not as important for Df_α's as they are for CA_α's.

Algebraic theory

Now we turn to the proper algebraic theory of Df_α's.

THEOREM 5.1.7. (i) $\mathsf{SDf}_\alpha = \mathsf{Df}_\alpha$.

(ii) *If* $\alpha \geqq 2$, *then there is a* Df_α $\mathfrak{A}$ *with one generator such that* $|A| = |\alpha| \mathbf{u} \omega$.

(iii) *If* $\alpha \geqq 2$ *and* $\omega \leqq \kappa \leqq |\alpha| \mathbf{u} \omega$, κ *a cardinal, then there is a* CA_α $\mathfrak{A}$ *with one generator such that* $|A| = \kappa$.

PROOF. (i) is obvious, and (ii) follows from the proof of 2.1.11. For (iii), use the proof of 2.1.11 and 5.1.2.

THEOREM 5.1.8. *Every* Df_α $\mathfrak{A}$ *has exactly one minimal subalgebra, namely a one– or two–element subalgebra depending on whether* $|A|=1$ *or* $|A|>1$.
All one–element Df_α*'s are isomorphic, and all two–element* Df_α*'s are isomorphic.*

Now we consider the class SDfCA_α.

THEOREM 5.1.9. (i) *If* $\alpha\leqq 1$, *then* $\mathsf{SDfCA}_\alpha=\mathsf{DfCA}_\alpha=\mathsf{CA}_\alpha=\mathsf{Df}_\alpha$.
(ii) *If* $\alpha\geqq 2$, *then* $\mathsf{DfCA}_\alpha\subset\mathsf{SDfCA}_\alpha$.
(iii) *If* $\alpha\geqq 2$, *then* $\mathsf{SDfCA}_\alpha\subseteq\mathsf{Df}_\alpha$; $\mathsf{SDfCA}_\alpha\neq\mathsf{Df}_\alpha$ *if* $\alpha\geqq 4$.

PROOF. (i) is obvious, as is $\subseteq$ in (ii) and (iii). To show that $\mathsf{DfCA}_\alpha\neq\mathsf{SDfCA}_\alpha$ for $\alpha\geqq 2$, let $\mathfrak{A}$ be the full Cs_α with base 2, let $a=\{u\in{}^\alpha 2: u0=0\}$, and let $\mathfrak{B}=\mathfrak{Cg}^{\mathfrak{Df}\mathfrak{A}}\{a\}$. Then $B=\{0,a,-a,1\}$, $\mathsf{c}_0 a=1$, and $B\restriction\mathsf{c}_\kappa\subseteq Id$ for $\kappa\in\alpha\sim 1$. Thus by 1.3.13, $\mathfrak{B}\in\mathsf{SDfCA}_\alpha\sim\mathsf{DfCA}_\alpha$. The proper inclusion $\subset$ in (iii) for $\alpha\geqq 4$ follows from 5.1.6.

We do not know whether the last part of 5.1.9(iii) extends to $\alpha=3$. We show in 5.1.47 that $\mathsf{SDfCA}_2=\mathsf{Df}_2$.

Now we consider relativized Df_α's.

DEFINITION 5.1.10. *Let* $\mathfrak{A}$ *be a* Df_α *and* $b\in A$; *say* $\mathfrak{A}=\langle A,+,\cdot,-,0,1,\mathsf{c}_\kappa\rangle_{\kappa<\alpha}$. *We let* $Rl_b\mathfrak{A}$, $+'$, $\cdot'$, $-'$, $0'$, $1'$, c'_κ *be as in* 2.2.1, *and set* $\mathfrak{Rl}_b\mathfrak{A}=\langle Rl_b\mathfrak{A},+',\cdot',-',0',1',\mathsf{c}'_\kappa\rangle_{\kappa<\alpha}$; $\mathfrak{Rl}_b\mathfrak{A}$ *is a* **diagonal–free–relativized algebra of dimension** α. Dr_α *is the class of all diagonal–free relativized algebras of dimension* α.

The following results are analogous to 2.2.2–2.2.4 and have similar proofs.

THEOREM 5.1.11. $\mathsf{Df}_\alpha\subseteq\mathsf{Dr}_\alpha$.

THEOREM 5.1.12. *If* $\mathfrak{A}\in\mathsf{Df}_\alpha$ *and* $b\in A$, *then* $\mathfrak{Rl}_b\mathfrak{A}$ *satisfies all of the axioms* (C_0)–(C_4) *for* Df*'s except possibly* (C_4). $\mathfrak{Rl}_b\mathfrak{A}$ *is a* Df_α *iff the following condition holds (in* $\mathfrak{A}$*) for all* $\kappa,\lambda<\alpha$ *and all* $x\in Rl_b\mathfrak{A}$: $\mathsf{c}_\kappa(\mathsf{c}_\lambda x\cdot b)\leqq\mathsf{c}_\lambda(\mathsf{c}_\kappa x\cdot b)$.

THEOREM 5.1.13. (i) *If* $\alpha\leqq 1$, *then* $\mathsf{Df}_\alpha=\mathsf{Dr}_\alpha$.
(ii) If $\alpha\geqq 2$, *then* $\mathsf{Df}_\alpha\subset\mathsf{Dr}_\alpha$.

Analogously to 2.2.8 we have:

THEOREM 5.1.14. (i) $\mathsf{PDr}_\alpha=\mathsf{Dr}_\alpha$.

(ii) $\mathsf{UpDr}_\alpha = \mathsf{Dr}_\alpha$.
(iii) $\mathsf{UpSDr}_\alpha = \mathsf{SDr}_\alpha$.

We do not know whether Dr_α is an equational class for $\alpha \geqq 2$; in particular, we do not know whether $\mathsf{SDr}_\alpha = \mathsf{Dr}_\alpha$ or $\mathsf{HDr}_\alpha = \mathsf{Dr}_\alpha$. However, SDr_α *is* an equational class, characterized by $(C_0)-(C_3)$, as we shall see in 5.1.32 below.

The proof of 2.2.10 gives

THEOREM 5.1.15. *Let* $\mathfrak{A} \in \mathsf{Df}_\alpha$ *and* $b \in A$. *If* $\mathsf{c}_\kappa b \cdot \mathsf{c}_\lambda b = b$ *for any two distinct* $\kappa, \lambda < \alpha$, *then* $\mathfrak{Rl}_b \mathfrak{A} \in \mathsf{Df}_\alpha$.

COROLLARY 5.1.16. *Let* $\mathfrak{A} \in \mathsf{Df}_\alpha$ *and* $b \in A$. *If* $b \in Zd\mathfrak{A}$, *then* $\mathfrak{Rl}_b \mathfrak{A} \in \mathsf{Df}_\alpha$ *and* $\mathfrak{Zb}\mathfrak{Rl}_b \mathfrak{A} = \mathfrak{Rl}_b \mathfrak{Zb}\mathfrak{A}$.

THEOREM 5.1.17. *If* $\mathfrak{A} \in \mathsf{Df}_\alpha$ *and* $b \in Zd\mathfrak{A}$, *then* $\mathfrak{Rl}_b \mathfrak{A}$ *is discrete iff for all* $x \leqq b$, $x \in Zd\mathfrak{A}$.

Theorem 2.2.15 also naturally extends to Df's.

The properties of monadic-generated CA_α's derived in the last part of section 2.2 carry over in a simplified form for Df_α's as follows. (These proofs do not use the notion of relativization.)

DEFINITION 5.1.18. *A* Df_α $\mathfrak{A}$ *is* **monadic-generated** *if there is a set* X *such that* $\mathfrak{A} = \mathfrak{Cg}X$ *and* $|\Delta x| \leqq 1$ *for every* $x \in X$.

THEOREM 5.1.19. *Let* $\mathfrak{A}$ *be a monadic-generated* Df_α, *say* $\mathfrak{A} = \mathfrak{Cg}X$ *with* $|\Delta x| \leqq 1$ *for all* $x \in X$. *Let*

$$C = X \cup \{a_\kappa(Y,Z) : Y \cup Z \subseteq X,\ |Y \cup Z| < \omega,\ Y \cap Z = 0,\ \kappa < \alpha, \text{ and } \Delta y = \{\kappa\} \text{ for all } y \in Y \cup Z\},$$

where

$$a_\kappa(Y,Z) = \mathsf{c}_\kappa(\Pi_{y \in Y}\, y \cdot \Pi_{z \in Z} - z).$$

Then $A = Sg^{\mathfrak{B}\mathfrak{Rl}\mathfrak{A}} C$.

COROLLARY 5.1.20. *If* $\mathfrak{A} \in \mathsf{Df}_\alpha$ *and* $\mathfrak{A} = \mathfrak{Cg}X$ *with* $|\Delta x| \leqq 1$ *for every* $x \in X$, *then* $\mathfrak{A}$ *is finite.*

The notions and results of section 2.3, concerning homomorphisms, isomorphisms, and ideals, carry over to Df's with the same proofs, except for 2.3.22, 2.3.30, 2.3.31, 2.3.32, and 2.3.33. Concerning these theorems we have the following results.

THEOREM 5.1.21. *For each* $\alpha \geqq 1$ *there is a locally finite-dimensional simple* Df_α *generated by two elements but not by one element.*

PROOF. Let $\mathfrak{A}$ be the full Cs_1 with base 3, $\mathfrak{B} = \mathfrak{Df}\mathfrak{A}$, and $\mathfrak{C}$ the Df_α obtained from $\mathfrak{B}$ by the procedure of 5.1.2. It is easily checked that $\mathfrak{C}$ is as desired.

Note that Theorem 5.1.21 shows that 2.3.22 does not hold for Df_α's. A similar easy argument shows that 2.3.30 does not extend to Df_α's. 2.3.31 essentially involves

diagonal elements. We already remarked on p.295 of Part I that 2.3.32 applies also to Df's, as noticed by Don Pigozzi:

THEOREM 5.1.22. *For every* Df_α $\mathfrak{A}$, *if* $\mathfrak{A}=\mathfrak{Cg}X$ *and* $|X|<\omega$, *then* $|At\mathfrak{A}\cap Zd\mathfrak{A}|\leq 2^{|X|}$.

PROOF. For each $a\in At\mathfrak{A}\cap Zd\mathfrak{A}$ let $fa=\{x\in X: x\cdot a=0\}$. It suffices to show that f is one–one. Suppose that $fa=fb$, where $a,b\in At\mathfrak{A}\cap Zd\mathfrak{A}$. Let $B=\{z\in A: z\cdot a=0$ iff $z\cdot b=0\}$. Thus $X\subseteq B$. It is easily checked that $B\in Su\mathfrak{A}$. Hence $B=A$, and so of course $a=b$, as desired.

Corollary 2.3.33(i) does not extend to Df_α's, if $\alpha\geq 2$, as one can see using the proof of 2.1.11(ii) and 5.1.2, and using 5.1.19 for the case $\alpha=2$.

Note that if $\mathfrak{A}$ is a CA_α, then a subset I of A is an ideal in $\mathfrak{A}$ iff it is an ideal in $\mathfrak{Df}\mathfrak{A}$. Furthermore, $\mathfrak{A}$ is simple iff $\mathfrak{Df}\mathfrak{A}$ is.

The results in section 2.4 (direct products and related notions) which do not involve diagonal elements extend with the same proofs to Df's with these exceptions: 2.4.5(ii), 2.4.17, and 2.4.54(i). Counterexamples are easily provided for 2.4.5(ii) and 2.4.54(i) in their versions for Df's. On the other hand, 2.4.17 *does* extend to Df's, with a different proof:

THEOREM 5.1.23.(i) *Every finite directly indecomposable* Df_α *is simple.*

(ii) *Every finite* Df_α *is isomorphic to a direct product of simple* Df_α's.

PROOF. Let $\mathfrak{A}$ be a finite directly indecomposable Df_α. In order to apply the Df–version of 2.3.14, suppose that $0\neq x\in A$. Let y be a maximal element of $\{\mathsf{c}_{(\Gamma)}x:\Gamma\subseteq\alpha,\ |\Gamma|<\omega\}$. Then $\Delta y=0$, so $y=1$. Thus $\mathsf{c}_{(\Gamma)}x=1$ for some finite $\Gamma\subseteq\alpha$, as desired. This proves (i). (ii) follows by (i) and 0.3.31.

Now we turn to free Df's. The analogs of 2.5.1 and 2.5.3 hold for Df's. Lemma 2.5.4 holds in the following somewhat stronger form, with essentially the same proof:

LEMMA 5.1.24. *For every simple* $\mathfrak{A}\in\mathsf{Df}_2$ *and every finite* $X\subseteq A$ *there is a* $\mathfrak{B}\in\mathsf{Df}_2$ *satisfying the conditions:*

(i) $\mathfrak{B}$ *is finite, and in fact,* $|B|\leq 2^a$ *with* $a=2^{3\cdot|X|}$;

(ii) $X\subseteq B$ *and* $\mathfrak{Bl}\mathfrak{B}\subseteq\mathfrak{Bl}\mathfrak{A}$;

(iii) *if* $x\in X$, *then* $\mathsf{c}_\kappa^{\mathfrak{B}}x=\mathsf{c}_\kappa^{\mathfrak{A}}x\in B$ *for* $\kappa=0,1$.

Of the other results in section 2.5 not involving diagonal elements, all but the following carry over to Df's, with the same proofs: 2.5.11, 2.5.15, 2.5.16, 2.5.24, 2.5.43, 2.5.51, 2.5.52, 2.5.53, 2.5.61, 2.5.66, 2.5.67, 2.5.68. We now consider these results in succession.

Concerning 2.5.11, the situation differs for $\alpha<\omega$ and $\alpha\geq\omega$:

THEOREM 5.1.25. *If* $\alpha,\beta<\omega$, *then* $\mathfrak{Sr}_\beta\mathsf{Df}_\alpha$ *has exactly* 2^β *zero–dimensional atoms.*

PROOF. Let $\mathfrak{A}=\mathfrak{Sr}_\beta\mathsf{Df}_\alpha$ and for each $\xi<\beta$ let $g_\xi=\xi/Cr_\beta\mathsf{Df}_\alpha$. The case $\alpha\leq 1$ is treated in 2.5.11, so assume that $\alpha\geq 2$. By 5.1.22 it suffices to exhibit 2^β zero–dimensional atoms. For each $\Gamma\subseteq\beta$ let

$$x_\Gamma = \mathsf{c}^{\partial}_{(\alpha)}(\Pi_{\xi\in\Gamma} g_\xi \cdot \Pi_{\xi\in\beta\sim\Gamma} - g_\xi).$$

It is easy to verify that each x_Γ is non–zero, they are zero–dimensional, and they are all distinct. Now fix $\Gamma \subseteq \beta$. Let

$$B = \{y \in A : y \cdot x_\Gamma = 0 \text{ or } -y \cdot x_\Gamma = 0\}.$$

It is easily seen that $g_\xi \in B$ for each $\xi < \beta$, and $B \in Su\mathfrak{A}$. Hence $B = A$, and this shows that x_Γ is an atom of $\mathfrak{A}$, as desired.

THEOREM 5.1.26. *If $\alpha \geqq \omega$ and β is an arbitrary non–zero cardinal, then $\mathfrak{Sr}_\beta \mathsf{Df}_\alpha$ is directly indecomposable and hence has only the two zero–dimensional elements* 0 *and* 1.

PROOF. It suffices to show that if $0 < a < 1$ in $\mathfrak{Sr}_\beta \mathsf{Df}_\alpha$ then $\Delta a \neq 0$. There is a finite set $\Gamma \subseteq \alpha$ and a term τ in the first–order language for Df_α's involving c_ξ only for $\xi \in \Gamma$ such that $a = \tau^{\mathfrak{A}} x$ for some $x \in {}^\omega Rgg$, where $\mathfrak{A} = \mathfrak{Sr}_\beta \mathsf{Df}_\alpha$ and $g_\xi = \xi / Cr_\beta \mathsf{Df}_\alpha$ for all $\xi < \beta$. Let $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, \mathsf{c}_\kappa \rangle_{\kappa<\alpha}$, and set $\mathfrak{B} = \langle A, +, \cdot, -, 0, 1, \mathsf{c}'_\kappa \rangle_{\kappa<\alpha}$, where $\mathsf{c}'_\kappa = \mathsf{c}_\kappa$ for all $\kappa \in \Gamma$, while for any $\kappa \in \alpha \sim \Gamma$ and any $y \in A$ we set

$$\mathsf{c}'_\kappa y = \begin{cases} 1 \text{ if } y \neq 0, \\ 0 \text{ if } y = 0. \end{cases}$$

Then clearly $\mathfrak{B} \in \mathsf{Df}_\alpha$. Let $f \in Hom(\mathfrak{A}, \mathfrak{B})$ such that $fg_\xi = g_\xi$ for all $\xi < \beta$. Then $fa = a$. Choose $\xi \in \alpha \sim \Gamma$. Then $\mathsf{c}'_\xi a = 1$, so also $\mathsf{c}_\xi a \neq a$, as desired.

Theorem 2.5.15 does not carry over to Df's in its most general form, but we have:

THEOREM 5.1.27. *In the algebra $\mathfrak{Sr}_\beta \mathsf{Df}_\alpha$ we have $\Delta(\xi / Cr_\beta \mathsf{Df}_\alpha) = \alpha$ for every $\xi < \beta$.*

Similarly for 2.5.16:

THEOREM 5.1.28. *$\mathfrak{Sr}_\beta \mathsf{Df}_\alpha$ is not dimension–complemented.*

Theorem 2.5.24 extends to Df's in a more general result, with a trivial proof not involving free algebras:

THEOREM 5.1.29. *There is a simple Df_α which is not dimension–complemented.*

Corollary 2.5.43 is trivially true for Df's, since the minimal subalgebra of $\mathfrak{Sr}^\Delta_\beta \mathsf{Df}_\alpha$ has exactly two elements. Theorem 2.5.51 and its two corollaries 2.5.52, 2.5.53 do not hold for Df's. For example, if $\beta = 1$, $\alpha \geqq 2$, $\Delta, \Delta' \in {}^\beta Sb\alpha$ with $\Delta 0 = \{0\}$, $\Delta 1 = \{1\}$, then all elements of $\mathfrak{Sr}^\Delta_\beta \mathsf{Df}_\alpha$ have dimension set $\subseteq \{0\}$ and all elements of $\mathfrak{Sr}^{\Delta'}_\beta \mathsf{Df}_\alpha$ have dimension set $\subseteq \{1\}$, with $\Delta(0 / Cr^\Delta_\beta \mathsf{Df}_\alpha) = \{0\}$, so $\mathfrak{Sr}^\Delta_\beta \mathsf{Df}_\alpha \not\cong \mathfrak{Sr}^{\Delta'}_\beta \mathsf{Df}_\alpha$. Finally, 2.5.61, 2.5.66, 2.5.67, 2.5.68 take the following form for Df's:

THEOREM 5.1.30. *Assume that* $\alpha \geqq 2$, $\Delta \in {}^{\beta}Sb\alpha$, $\beta < \omega$, *and* $|\Delta\xi| \leqq 1$ *for all* $\xi < \beta$. *Then* $\mathfrak{Sr}_{\beta}^{\Delta}\mathsf{Df}_{\alpha}$ *is finite.*

More specifically, for each $\kappa < \alpha$ *let* $\Gamma_{\kappa} = \{\xi < \beta : \Delta\xi = \{\kappa\}\}$, *and let* $\Theta = \{\xi < \beta : \Delta\xi = 0\}$. *Let* $\Omega = \{\kappa < \alpha : \Gamma_{\kappa} \neq 0\}$. *Set* $\gamma = |\Theta| + \Sigma\{|\Gamma_{\kappa}| + 2^{|\Gamma\kappa|} - 1 : \kappa \in \Omega\}$. *Then* $|Fr_{\beta}^{\Delta}\mathsf{Df}_{\alpha}| = 2^{\delta}$, *where* $\delta = |At\mathfrak{Sr}_{\beta}^{\Delta}\mathsf{Df}_{\alpha}| = 2^{\gamma}$.

PROOF. Let $\delta\kappa = |\Gamma_{\kappa}|$ for each $\kappa \in \Omega$. Let $\mathfrak{A}_{\kappa} = \mathfrak{Sr}_{\delta\kappa}\mathsf{Df}_1$ for each $\kappa \in \Omega$, and let $g(\xi, \kappa) = \xi / Cr_{\delta\kappa}\mathsf{Df}_1$ for each $\xi < \delta\kappa$. Let $\mathfrak{B}$ be the free BA on free generators $h\xi$ for $\xi \in \Theta$. Let f_{κ} be a one–one function from $\delta\kappa$ onto Γ_{κ} for each $\kappa \in \Omega$. Note by 2.5.62 that:

(1) $|B| = 2^{a}$ with $a = 2^{|\Theta|}$;
(2) $|A_{\kappa}| = 2^{b}$, with $b = 2^{c}$ and $c = \delta\kappa + 2^{\delta\kappa} - 1$, for each $\kappa \in \Omega$.

Let $\mathfrak{C}$ be the BA–free product of $\mathfrak{B}$ and all the algebras $\mathfrak{B}(\mathfrak{A}_{\kappa}$. By (1) and (2) we see that $|C| = 2^{\delta}$. We shall define cylindrifications on C so that the resulting algebra is a Df_{α} isomorphic to $\mathfrak{Sr}_{\beta}^{\Delta}\mathsf{Df}_{\alpha}$, thereby completing the proof. If $\lambda \in \alpha \sim \Omega$, we let $\mathsf{c}_{\lambda} = C1Id$. Suppose now that $\lambda \in \Omega$. Let $d \in C$. Then we can write d in the form

$$d = \Sigma_{\iota<\mu} b_{\iota} \cdot \Pi_{\kappa\in\Omega} a_{\kappa\iota},$$

where $\mu < \omega$ and for each $\iota < \mu$ we have $b_{\iota} \in B$ and $a_{\kappa\iota} \in A_{\kappa}$ for each $\kappa \in \Omega$. We would like to set

(3) $\mathsf{c}_{\lambda} d = \Sigma_{\iota<\mu} b_{\iota} \cdot \mathsf{c}_0^{\mathfrak{A}\lambda} a_{\lambda\iota} \cdot \Pi_{\kappa\in\Omega\sim\{\lambda\}} a_{\kappa\iota}$.

To see that this is possible, suppose also that

$$d = \Sigma_{\eta<\nu} b'_{\eta} \cdot \Pi_{\kappa\in\Omega} a'_{\kappa\eta}$$

with $\nu < \omega$ and $b'_{\eta} \in B$, $a'_{\kappa\eta} \in A_{\kappa}$ for each $\eta < \nu$, $\kappa \in \Omega$. We need to show that the right side of (3) is equal to

(4) $\Sigma_{\eta<\nu} b'_{\eta} \cdot \mathsf{c}_0^{\mathfrak{A}\lambda} a'_{\lambda\eta} \cdot \Pi_{\kappa\in\Omega\sim\{\lambda\}} a'_{\kappa\eta}$.

By symmetry, it suffices to take an arbitrary $\iota < \mu$ and show that

(5) $b_{\iota} \cdot \mathsf{c}_0^{\mathfrak{A}\lambda} a_{\lambda\iota} \cdot \Pi_{\kappa\in\Omega\sim\{\lambda\}} a_{\kappa\iota}$

is $\leqq$ the expression (4). Now from the two expressions for d we see that

$$b_{\iota} \cdot \Pi_{\kappa\in\Omega} a_{\kappa\iota} \cdot \Pi_{\eta<\nu}(-b'_{\eta} + \Sigma_{\kappa\in\Omega} - a'_{\kappa\eta}) = 0.$$

Using the free product property it follows that for every $\Xi \subseteq \nu$ and every $h \in {}^{\Xi}\Omega$, either $b_{\iota} \leqq \Sigma_{\eta\in\nu\sim\Xi} b'_{\eta}$ or there is a $\kappa \in \Omega$ such that $a_{\kappa\iota} \leqq \Sigma\{a'_{\kappa\eta} : h\eta = \kappa\}$. Hence for every $\Xi \subseteq \nu$ and every $h \in {}^{\Xi}\Omega$, either $b_{\iota} \leqq \Sigma_{\eta\in\nu\sim\Xi} b'_{\eta}$, or $\mathsf{c}_0^{\mathfrak{A}\lambda} a_{\lambda\iota} \leqq \Sigma\{\mathsf{c}_0^{\mathfrak{A}\lambda} a'_{\lambda\eta} : h\eta = \lambda\}$, or there is a

$\kappa\in\Omega\sim\{\lambda\}$ such that $a_{\kappa\iota}\leqq\Sigma\{a'_{\kappa\eta}: h\eta=\kappa\}$. Hence, as desired, $(5)\leqq(4)$.

Now let $\mathfrak{D}$ be the structure similar to Df_α's such that $\mathfrak{B}\mathfrak{l}\mathfrak{D}=\mathfrak{C}$ and $\mathsf{c}_\lambda^{\mathfrak{D}}=\mathsf{c}_\lambda$ for all $\lambda<\alpha$. It is easily checked that $\mathfrak{D}$ *is* a Df_α. Next we define $k\in{}^\beta\mathfrak{D}$; k is to be our sequence of free generators under Δ (cf. 2.5.34). Let $\xi<\beta$. If $\Delta\xi=0$, set $k\xi=h\xi$. If $\Delta\xi=\{\kappa\}$, where $\kappa<\alpha$, let $k\xi=g(f_\kappa^{-1}\xi,\kappa)$. Note that $\mathfrak{D}$ is generated by Rgk, and $\Delta k\xi\subseteq\Delta\xi$ for every $\xi<\beta$. Now let $\mathfrak{E}$ be a Df_α, $y\in{}^\beta E$, and suppose that $\Delta y\xi\subseteq\Delta\xi$ for each $\xi<\beta$. Let $l\in Hom(\mathfrak{B},\mathfrak{B}\mathfrak{l}\mathfrak{E})$ be such that $lh\xi=y\xi$ for each $\xi\in\Theta$. Note that $Rgl\subseteq Zd\mathfrak{E}$. Now say $\Delta\xi=\{\kappa\}$. Set $\rho=\{\langle 0,\kappa\rangle\}$. Let $s_\kappa\in Hom(\mathfrak{A}_\kappa,\mathfrak{R}\mathfrak{b}^\rho\mathfrak{E})$ be such that $s_\kappa g(\xi,\kappa)=yf_\kappa\xi$ for each $\xi<\delta\kappa$. Note that $\Delta s_\kappa x\subseteq\{\kappa\}$ for all $x\in A\kappa$. By the free product property there is a $t\in Hom(\mathfrak{C},\mathfrak{B}\mathfrak{l}\mathfrak{E})$ such that $th\xi=y\xi$ for each $\xi\in\Delta$, and $tg(\xi,\kappa)=yf_\kappa\xi$ whenever $\kappa\in\Omega$ and $\xi<\delta\kappa$. Then if $\xi\in\Theta$ we have $tk\xi=th\xi=y\xi$. If $\xi\in\Gamma_\kappa$ with $\kappa\in\Omega$ then $tk\xi=tg(f_\kappa^{-1}\xi,\kappa)=yf_\kappa f_\kappa^{-1}\xi=y\xi$. So, $t\circ k=y$. It remains to show that t preserves cylindrifications. Suppose d is as above, before (3), and $\lambda\in\Omega$. Then

$$\begin{aligned}tc_\lambda d&=t(\Sigma_{\iota<\mu}b_\iota\cdot c_0^{\mathfrak{A}\lambda}a_{\lambda\iota}\cdot\Pi_{\kappa\in\Omega\sim\{\lambda\}}a_{\kappa\iota})\\&=\Sigma_{\iota<\mu}lb_\iota\cdot s_\lambda c_0^{\mathfrak{A}\lambda}a_{\lambda\iota}\cdot\Pi_{\kappa\in\Omega\sim\{\lambda\}}s_\kappa a_{\kappa\iota}\\&=\Sigma_{\iota<\mu}lb_\iota\cdot c_\lambda^{\mathfrak{E}}s_\lambda a_{\lambda\iota}\cdot\Pi_{\kappa\in\Omega\sim\{\lambda\}}s_\kappa a_{\kappa\iota}\\&=c_\lambda td.\end{aligned}$$

Next, suppose $\lambda\in\alpha\sim\Omega$. Then $\mathsf{c}_\lambda d=d$. Now by the choice of t and the noted properties of l and the s_κ, $\mathsf{c}_\lambda td=td$. Thus $tc_\lambda d=\mathsf{c}_\lambda td$ again, as desired.

By the Df–analog of 2.5.37 we have $\mathfrak{D}\cong\mathfrak{Tr}_\beta^\Delta\mathsf{Df}_\alpha$, as desired.

In view of 5.1.2, the notions of reducts and neat embeddings are trival for Df's:

THEOREM 5.1.31. *If* $\alpha\leqq\beta$, *then* $\mathsf{Df}_\alpha=\mathsf{Rd}_\alpha\mathsf{Df}_\beta=\mathsf{Nr}_\alpha\mathsf{Df}_\beta$.

The notions and results of section 2.7 transfer to Df's with no difficulty. Thus Definition 2.7.1 is modified by omitting reference to diagonal elements; Bof and Bof_α are used for the corresponding notions. Similarly, we use df (or df_α) for the class of all relational structures (of dimension α) which are atom structures of complete and atomic Df's (Df_α's). Other notions applied to Df's have the natural diagonal–free interpretation. For example, if $\mathfrak{A}$ is a Df_α then $\mathfrak{C}\mathfrak{m}\mathfrak{A}$ is defined as in 2.7.4, but omitting reference to diagonal elements. Note that in the Df–version of Theorem 2.7.40, only conditions (i), (ii) occur.

Now we can use these notions to establish a result about relativized Df's referred to above.

THEOREM 5.1.32. *Let* $\mathfrak{A}$ *be similar to* Df_α*'s. Then* $\mathfrak{A}\in\mathsf{SDr}_\alpha$ *iff* $\mathfrak{A}$ *satisfies* (C_0)–(C_3) *of* 1.1.1.

PROOF. The direction $\Rightarrow$ is clear from the definitions. Now suppose that $\mathfrak{A}$ satisfies (C_0)–(C_3). By the Df–version of 2.7.13, we may assume that $\mathfrak{A}$ is complete and atomic. Now we call a relational structure $\mathfrak{B}=\langle B,T_\kappa\rangle_{\kappa<\alpha}$ *regular* if T_κ is an

equivalence relation on B for every $\kappa<\alpha$.

(1) If $\mathfrak{B}=\langle B,T_\kappa\rangle_{\kappa<\alpha}$ is a regular relational structure, $\kappa,\lambda<\alpha$, and $a(T_\kappa|T_\lambda)b$, then there is a regular relational structure $\mathfrak{C}\supseteq\mathfrak{B}$, $\mathfrak{C}=\langle C,T'_\kappa\rangle_{\kappa<\alpha}$, such that $a(T'_\lambda|T'_\kappa)b$.

For, we may assume that $\kappa\neq\lambda$. Let x be any element $\notin B$ and set $C=B\cup\{x\}$. Let T'_κ be the equivalence relation on C whose equivalence classes are $(b/T_\kappa)\cup\{x\}$ and c/T_κ for $c\not T_\kappa b$; T'_λ the equivalence relation on C with classes $a/T_\lambda\cup\{x\}$ and d/T_λ for $d\not T_\lambda a$; and for $\mu\neq\kappa,\lambda$, T'_μ is the equivalence relation on C with classes $\{x\}$ and all T_μ–classes. Clearly (1) holds.

Now $\mathfrak{A}\mathfrak{t}\mathfrak{A}$ is clearly a regular relational structure, and by (1) and an easy transfinite construction we easily obtain a df_α $\mathfrak{B}$ such that $\mathfrak{A}\mathfrak{t}\mathfrak{A}\subseteq\mathfrak{B}$, using the Df–version of 2.7.40. Let $a=\Sigma At\mathfrak{A}$. Now clearly $\mathfrak{Rf}_a\mathfrak{Cm}\mathfrak{B}\simeq\mathfrak{Cm}\mathfrak{A}\mathfrak{t}\mathfrak{A}\simeq\mathfrak{A}$ by the Df–version of 2.7.34, so $\mathfrak{A}\in\mathsf{SDr}_\alpha$, as desired.

Representable Df_α's

Now we turn to the notions of diagonal–free set algebras, as in section 3.1. Recall Definition 3.1.1.

DEFINITION 5.1.33. (i) *A is an* **α–dimensional df–cylindric–relativized field of sets** *iff there is a set V of α–termed sequences such that A is a non–empty family of subsets of V closed under all the operations $\cup,\cap,{}_V\sim$ and $\mathsf{C}_\kappa^{|V|}$ (for each $\kappa<\alpha$).*

(ii) *$\mathfrak{A}$ is a* **df–cylindric–relativized set algebra of dimension** *α iff there is a set V of α–termed sequences such that $\mathfrak{A}=\langle A,\cup,\cap,{}_V\sim,0,V,\mathsf{C}_\kappa^{|V|}\rangle_{\kappa<\alpha}$, where A is as in* (i).

For the rest of the definition, let A, $\mathfrak{A}$, and V be as in (i) *and* (ii).

(iii) *$\mathfrak{A}$ is an* **α–dimensional df–cylindric set algebra** *if there is a system $\langle U_\kappa:\kappa<\alpha\rangle$ of sets such that $V=\mathsf{P}_{\kappa<\alpha}U_\kappa$; Csdf_α is the class of all such. $\langle U_\kappa:\kappa<\alpha\rangle$ is called the* **base system** *of $\mathfrak{A}$. In case there is a set W such that $U_\kappa=W$ for all $\kappa<\alpha$, we call $\mathfrak{A}$* **uniform** *with base W; Csudf_α is the class of all uniform Csdf_α's.*

(iv) *$\mathfrak{A}$ is an* **α–dimensional generalized df–cylindric set algebra** *if V has the form*

$$\bigcup_{i\in I}\mathsf{P}_{\kappa<\alpha}U_{i\kappa},$$

where for distinct $i,j\in I$ and any $\kappa<\alpha$ we have $U_{i\kappa}\cap U_{j\kappa}=0$. If for every $i\in I$ there is a W_i such that $U_{i\kappa}=W_i$ for all $\kappa<\alpha$, we call $\mathfrak{A}$ **uniform**. *W is then called the* **base** *of $\mathfrak{A}$. Gsdf_α and Gsudf_α are the corresponding classes.*

(v) *A Df_α is* **representable** *if it is isomorphic to a Gsdf_α.*

(vi) *If $\langle U_\kappa:\kappa<\alpha\rangle$ is a system of sets and $p\in\mathsf{P}_{\kappa<\alpha}U_\kappa$, then we set*

$$\mathsf{P}^{(p)}_{\kappa<\alpha}U_\kappa=\{q\in\mathsf{P}_{\kappa<\alpha}U_\kappa:\{\xi<\alpha:q\xi\neq p\xi\}\ \textit{is finite}\}.$$

(vii) *$\mathfrak{A}$ is an* **α–dimensional weak df–cylindric set algebra**, *$\mathfrak{A}\in\mathsf{Wsdf}_\alpha$, if V has the form given in* (vi).

(viii) *$\mathfrak{A}$ is an* **α–dimensional generalized weak df–cylindric set algebra**, *$\mathfrak{A}\in\mathsf{Gwsdf}_\alpha$, if V has the form*

$$\bigcup_{i\in I}\mathsf{P}^{(pi)}_{\kappa<\alpha}U_{i\kappa},$$

with $\mathsf{P}^{(pi)}_{\kappa<\alpha}U_{i\kappa}\cap\mathsf{P}^{(pj)}_{\kappa<\alpha}U_{j\kappa}=0$ *for* $i\neq j$. *We call* $\mathfrak{A}$ **normal** *if for all* $i,j\in I$, $U_{i\kappa}=U_{j\kappa}$ *for all* $\kappa<\alpha$, *or* $U_{i\kappa}\cap U_{j\kappa}=0$ *for all* $\kappa<\alpha$; **widely-distributed** *if* $U_{i\kappa}\cap U_{j\kappa}=0$ *whenever* $i,j\in I$, $i\neq j$, *and* $\kappa<\alpha$; **compressed** *if* $U_{i\kappa}=U_{j\kappa}$ *for all* $i,j\in I$ *and all* $\kappa<\alpha$. $\mathsf{Gwsdf}^{nm}_\alpha$, $\mathsf{Gwsdf}^{wd}_\alpha$, $\mathsf{Gwsdf}^{cm}_\alpha$ *are the corresponding classes.*

We state some facts about relationships between these notions. The first one depends on the following algebraic version of the upward Lowenheim, Skolem theorem; it is due to James S. Johnson [69] (see also Halmos [57], Lemma 6.4).

THEOREM 5.1.34. *Let* $\mathfrak{A}$ *be a* Csdf_α *with base system* $U=\langle U_\lambda:\lambda<\alpha\rangle$, *with* $U_\lambda\neq 0$ *for all* $\lambda<\alpha$. *Suppose* $V=\langle V_\lambda:\lambda<\alpha\rangle$ *and* $|U_\lambda|\leq|V_\lambda|$ *for every* $\lambda<\alpha$. *Then* $\mathfrak{A}$ *is isomorphic to a* Csdf_α *with base system* V.

PROOF. For each $\lambda<\alpha$ let f_λ be a function mapping V_λ onto U_λ. Now for each $X\in A$ let

$$FX=\{x\in\mathsf{P}V:\langle f_\lambda x_\lambda:\lambda\in\alpha\rangle\in X\}.$$

It is routine to check that F is the desired isomorphism.

COROLLARY 5.1.35. $\mathsf{ICsdf}_\alpha=\mathsf{ICsudf}_\alpha$.

THEOREM 5.1.36. (i) $\mathsf{Csdf}_\alpha\subseteq\mathsf{Gsdf}_\alpha$.
(ii) $\mathsf{Gsdf}_\alpha\subseteq\mathsf{Gwsdf}^{nm}_\alpha$.
(iii) $\mathsf{Gwsdf}^{nm}_\alpha\subseteq\mathsf{Df}_\alpha$.

PROOF. (i) and (ii) are obvious. For (iii) it suffices to check (C_4). There is no problem for $\alpha\leq 1$. Let $\mathfrak{A}$ have unit element V as in 5.1.33(viii). Suppose $\kappa,\lambda<\alpha$, $\kappa\neq\lambda$. If $\alpha=2$, then for any $X\in A$,

(1) $\mathsf{C}^{[V]}_\kappa\mathsf{C}^{[V]}_\lambda X=\bigcup\{\mathsf{P}_{\mu<\alpha}U_{i\mu}:\mathsf{P}_{\mu<\alpha}U_{i\mu}\cap X\neq 0\}$.

For, suppose $u\in\mathsf{C}^{[V]}_\kappa\mathsf{C}^{[V]}_\lambda X$. Say $u^k_s\in\mathsf{C}^{[V]}_\lambda X$, $u^{\kappa\lambda}_{st}\in X$. Say $u^{\kappa\lambda}_{st}\in\mathsf{P}_{\mu<\alpha}U_{i\mu}$, $u^\kappa_s\in\mathsf{P}_{\mu<\alpha}U_{j\mu}$, $u\in\mathsf{P}_{\mu<\alpha}U_{k\mu}$. Then $s\in U_{i\kappa}\cap U_{j\kappa}$, so $i=j$; and $u\lambda\in U_{j\lambda}\cap U_{k\lambda}$, so $j=k$. Thus $u\in\mathsf{P}_{\mu<\alpha}U_{i\mu}$ with $\mathsf{P}_{\mu<\alpha}U_{i\mu}\cap X\neq 0$, i.e., u is in the right side of (1). Now suppose $u\in\mathsf{P}_{\mu<\alpha}U_{i\mu}$ with $\mathsf{P}_{\mu<\alpha}U_{i\mu}\cap X\neq 0$. Say $v\in\mathsf{P}_{\mu<\alpha}U_{i\mu}\cap X$. Then $v^\lambda_{u\lambda}\in V$ hence $v^\lambda_{u\lambda}\in\mathsf{C}^{[V]}_\lambda X$, and $u=(v^\lambda_{u\lambda})^\kappa_{u\kappa}\in\mathsf{C}^{[V]}_\kappa\mathsf{C}^{[V]}_\lambda X$, as desired.

From (1) it is clear that (C_4) holds in $\mathfrak{A}$.

Now assume that $\alpha\geq 3$. Then

(2) $\mathsf{C}^{[V]}_\kappa\mathsf{C}^{[V]}_\lambda X=\{u\in V:\text{there is a } v\in X \text{ such that } (\alpha\sim\{\kappa,\lambda\})\mathsf{1}v\subseteq u\}$.

In fact, $\subseteq$ is obvious. Now let $u\in V$, $v\in X$, $(\alpha\sim\{\kappa,\lambda\})\mathsf{1}v\subseteq u$. Say $u\in\mathsf{P}^{(pi)}_{\mu<\alpha}U_{i\mu}$, $v\in\mathsf{P}^{(pj)}_{\mu<\alpha}U_{j\mu}$. Choose $\mu\in\alpha\sim\{\kappa,\lambda\}$. Then $u\mu=v\mu\in U_{i\mu}\cap U_{j\mu}$, so $i=j$. Hence clearly

$u \in \mathsf{C}_\kappa^{[V]}\mathsf{C}_\lambda^{[V]}X$, and (2), hence ($C_4$), holds.

REMARK 5.1.37. Not every Gwsdf_α is a Df_α. For, take any $\alpha \geqq \omega$. Let $I = \{i,j\}$ with $i \neq j$; $U_{i0} = \{0\}$, $U_{i1} = \{0,1\}$, and $U_{i\kappa} = \omega$ if $\kappa > 1$; $U_{j0} = \{1\} = U_{j1}$ and $U_{jk} = \omega$ if $\kappa > 1$. Let $(pi)\kappa = 0$ for all $\kappa \in \alpha$. Let $(pj)0 = (pj)1 = 1$, $(pj)\kappa = 0$ for all $\kappa \in \alpha \sim 2$. Set

$$V = \mathsf{P}_{\kappa<\alpha}^{(pi)} U_{i\kappa} \mathbf{u} \mathsf{P}_{\kappa<\alpha}^{(pj)} U_{j\kappa}.$$

Let $X = \{pi\}$. Then $\mathsf{C}_1^{[V]}\mathsf{C}_0^{[V]}X = \{q \in V : q\kappa = 0$ for all $\kappa \neq 1\}$, while $\mathsf{C}_0^{[V]}\mathsf{C}_1^{[V]}X = \{q \in V : q\kappa = 0$ for all $\kappa \neq 0,1\}$. Thus (C_4) fails.

The following facts are proved like the corresponding ones for cylindric set algebras (see 3.1.76–3.1.80).

THEOREM 5.1.38. (i) *If* $\alpha \geqq 2$, *then* $\mathsf{IGsdf}_\alpha = \mathsf{SPCsdf}_\alpha$.
(ii) *If* $\alpha < \omega$, *then* $\mathsf{Gsdf}_\alpha = \mathsf{Gwsdf}_\alpha^{nm}$.
(iii) *If* $\alpha \geqq \omega$, *then* $\mathsf{Gwsdf}_\alpha^{nm} = \mathsf{SPWsdf}_\alpha$.

We need below the following algebraic version of the downward Lowenheim, Skolem theorem (cf. Halmos [57], proof of Theorem (6.7)):

THEOREM 5.1.39. *Suppose* $\alpha < \omega$. *Let* $\mathfrak{A}$ *be a* Csudf_α *with base* V. *For any* $u,v \in V$ *we define* uSv *iff for all* $x,y \in {}^\alpha V$, *all* $\kappa < \alpha$, *and all* $X \in A$, *if* $(\alpha \sim \{\kappa\}) \upharpoonright x \subseteq y$ *and* $x_\kappa = u$, $y_\kappa = v$, *then* $x \in X$ *iff* $y \in X$.

Then S *is an equivalence relation on* V, *and for any equivalence relation* R *on* V *such that* $R \subseteq S$, $\mathfrak{A}$ *is isomorphic to a* Csudf_α *with base* V/R.

PROOF. Obviously S is an equivalence relation on V. Now let R be as indicated. Recall that R^* is the natural map from V onto V/R (see p.28 of Part I). For each $X \in Sb^\alpha(V/R)$ let

$$FX = \{x \in {}^\alpha V : R^* \circ x \in X\}.$$

Let $\mathfrak{B}$ be the Csudf_α of all subsets of ${}^\alpha(V/R)$. By the proof of 5.1.34, F is an isomorphism from $\mathfrak{B}$ onto a Csudf_α with base V. Therefore it suffices to show that $A \subseteq RgF$. *Let* $Y \in A$. Set $X = \{R^* \circ x : x \in Y\}$. Thus $X \in Sb^\alpha(V/R)$ and $Y \subseteq FX$. Now suppose that $x \in FX$. Then $R^* \circ x \in X$, so $R^* \circ x = R^* \circ y$ for some $y \in Y$. For each $\lambda \leqq \alpha$ let $z_\lambda = \langle x_\kappa : \kappa < \lambda \rangle \mathbf{u} \langle y_\kappa : \lambda \leqq \kappa < \alpha \rangle$. Since $y = z_0 \in Y$, it is clear by induction that $z_\lambda \in Y$ for all $\lambda \leqq \alpha$, since $R \subseteq S$. Thus $x = z_\alpha \in Y$. Hence $FX = Y$, as desired.

THEOREM 5.1.40. *For* $\alpha < \omega$, *any* Csdf_α $\mathfrak{A}$ *with* $|A| > 1$ *is simple.*

LEMMA 5.1.41 $\mathsf{UpGsdf}_\alpha = \mathsf{IGsdf}_\alpha$.

PROOF. Let $\mathfrak{A}_i \in \mathsf{Gsdf}_\alpha$ for all $i \in I$ and let F be an ultrafilter on I. Take any $a \in \mathsf{P}\mathfrak{A}$, and suppose that $a/\bar{F} \neq 0$. We want to find a Csdf_α $\mathfrak{B}$ and a homomorphism f

from $\mathsf{P}_{i\in I}\mathfrak{A}_i/\bar{F}$ into $\mathfrak{B}$ such that $f(a/\bar{F})\neq 0$. Clearly we may assume that each $\mathfrak{A}_i$ is a Csudf_α with non–empty base, say with base U_i. Let $Z=\{i\in I: a_i\neq 0\}$, and for each $i\in Z$ choose $s_i\in a_i$; let $s_i\in {}^\alpha U_i$ be arbitrary for $i\in I\sim Z$. Set $w=\langle\langle s_i\kappa: i\in I\rangle:\kappa<\alpha\rangle$, and let c' be an (F,U,α)–choice function such that $c'(\kappa,w\kappa/\bar{F})=w\kappa$ for all $\kappa<\alpha$ (see 3.1.89). By the proof of 3.1.90, $Rep(c')$ is a homomorphism from $\mathsf{P}_{i\in I}\mathfrak{A}_i/\bar{F}$ into a Csudf_α $\mathfrak{B}$ such that $Rep(c')(a/\bar{F})\neq 0$, as desired.

LEMMA 5.1.42. $\mathsf{HGwsdf}_\alpha^{nm}\subseteq\mathsf{IGsdf}_\alpha$.

PROOF. We follow closely the proof of 3.1.103. In fact, the case $\alpha<\omega$ is exactly as in that proof. Now assume that $\alpha\geqq\omega$, and that $\mathfrak{A}$ is a $\mathsf{Gwsdf}_\alpha^{nm}$, with unit element V as in 5.1.33(viii), but with I repaced by J. Let L, a, I, F, r, j be as in the proof of 3.1.103. Thus in particular $r_i\in\mathsf{P}_{\kappa<\alpha}^{(pji)}U_{ji,\kappa}$ for all $i\in J$. Let W be the base of V (see 3.1.1(ii)), and let $X={}^I W/\bar{F}$. For each $\kappa<\alpha$ let

$$Q_\kappa=(k/\bar{F}: k\in\mathsf{P}_{i\in I}U_{ji,\kappa}\},$$

where each element $k/\bar{F}$ is considered as a member of X. Furthermore, let w be as in the proof of 3.1.103. Note that $w_\kappa\in\mathsf{P}_{i\in I}U_{ji,\kappa}$. Fix κ. Now for each $y\in X$ choose $k_y\in{}^I W$ so that $y=k_y/\bar{F}$; if $y\in Q_\kappa$ we may assume that $k_y\in\mathsf{P}_{i\in I}U_{ji,\kappa}$, and if $y=w\kappa/\bar{F}$ we may assume that $k_y=w\kappa$. Define $c(\kappa,y)\in{}^I W$ as follows. Let $\langle z,\Gamma\rangle\in I$. We set

$$c(\kappa,y)_{z\Gamma}=\begin{cases} r(z,\Gamma)_\kappa & \text{if } \kappa\notin\Gamma,\\ k_y(z,\Gamma) & \text{if } \kappa\in\Gamma.\end{cases}$$

Then c is an $(F,\langle W: i\in I\rangle,\alpha)$–choice function satisfying (2), (3) in the proof of 3.1.103 as well as:

(1′) If $\kappa<\alpha$ and $y\in Q_\kappa$, then $c(\kappa,y)\in\mathsf{P}_{i\in I}U_{ji,\kappa}$.

Let $f=Rep(F,\langle W: i\in I\rangle,\alpha,\langle A: i\in I\rangle,c)$ and let $\delta a=\langle a: i\in I\rangle/\bar{F}$ for all $a\in A$. Then as in the proof of 3.1.103, the remaining details follow; note that (4) should be changed to:

(4′) For any $q\in{}^\alpha X$ and any $i\in I$, $(c^+q)_i\in V$ iff $(c^+q)_i\in\mathsf{P}_{\kappa<\alpha}U_{ji,\kappa}$.

In the last part of the proof one shows that $f\delta V=\mathsf{P}_{\kappa<\alpha}Q_\kappa$.

THEOREM 5.1.43. $\mathsf{HSPGsdf}_\alpha=\mathsf{IGsdf}_\alpha=\mathsf{IGwsdf}_\alpha^{nm}=\mathsf{IGwsdf}_\alpha^{wd}$ *for* $\alpha\geqq 2$.

Now we turn to the representation theory for Df's. We do not know of any result comparable in strength and importance to 3.2.5 and its corollaries. The *df*–version of 3.2.14 *does* hold, however, but its proof must be slightly modified:

THEOREM 5.1.44. *Let* $\mathfrak{A}$ *be any atomic* Df_α, $\alpha\geqq 2$. *Then the following conditions are equivalent:*

(i) *Every atom of* $\mathfrak{A}$ *is rectangular.*

(ii) *There is an isomorphism of* $\mathfrak{A}$ *onto a* $\mathsf{Gwsdf}_\alpha^{nm}$ $\mathfrak{B}$ *that carries each atom of* $\mathfrak{A}$ *to a singleton element of* $\mathfrak{B}$.

PROOF. As in the CA–case, the implication (ii)⇒(i) is trivial. Now assume (i). We define a binary relation E on $At\mathfrak{A}\times\alpha$ as follows. For any $a,b\in At\mathfrak{A}$ and $\xi,\eta\in\alpha$, $\langle a,\xi\rangle E\langle b,\eta\rangle$ iff there is some finite $\Gamma\subseteq\alpha$ with $\xi,\eta\in\Gamma$ such that $\mathsf{c}_{(\Gamma\sim\{\xi\})}a=\mathsf{c}_{(\Gamma\sim\{\eta\})}b$. One checks as in the proof of 3.2.14 that E is an equivalence relation of $At\mathfrak{A}\times\alpha$. Then h, V, $\mathfrak{C}$, and j are defined as before, and it is checked as before that j is an isomorphism of $\mathfrak{A}$ onto a df–cylindric–relativized set algebra with unit element V. Moreover, (2) and (3) in that proof remain valid. The remainder of the proof is modified as follows.

First assume that $\alpha<\omega$. For each $\kappa<\alpha$ and each α–atom k, let $U_{k\kappa}=\{(ha)\kappa : a$ is an atom $\leqq k\}$. Then as in the proof of 3.2.14 one shows:

(4′) If $\kappa<\alpha$ and k and l are distinct α–atoms, then $U_{k\kappa}\cap U_{l\kappa}=0$.
(5′) $V=\bigcup\{\mathsf{P}_{\kappa<\alpha}U_{k\kappa} : k$ an α–atom$\}$.
(6′) Suppose that k is an α–atom, a is an atom $\leqq k$, $\kappa<\alpha$, $x\in\mathsf{P}_{\lambda<\alpha}U_{k\lambda}$, and $x\lambda=(ha)\lambda$ for all $\lambda\in\alpha\sim\kappa$. Then $x\in V$.

For the case $\alpha\geqq\omega$, the proof in 3.2.14 goes through with no changes; the subbases are "uniform".

THEOREM 5.1.45. *For* $\alpha\geqq 2$, *a* Df_α *is representable iff it can be embedded in an atomic* Df_α *in which all atoms are rectangular.*

Another positive representation theorem is that $\mathsf{Df}_2=\mathsf{IGsdf}_2$. The proof is similar to, but simpler than, the proof of 3.2.65:

LEMMA 5.1.46. *If* $\mathfrak{A}$ *is a simple complete atomic* Df_2 *then* $\mathfrak{A}$ *is representable.*

PROOF. Let U be an infinite set such that $|U|\geqq|At\mathfrak{A}|$. Fix $a\in At\mathfrak{A}$. Now let $M_0\subseteq At\mathfrak{A}\cap Rl_u\mathfrak{A}$ and $M_1\subseteq At\mathfrak{A}\cap Rl_v\mathfrak{A}$, $u=\mathsf{c}_0a$, $v=\mathsf{c}_1a$, satisfy the following conditions:

(1) For all $b\in At\mathfrak{A}\cap Rl_u\mathfrak{A}$ there is a $c\in M_0$ such that $\mathsf{c}_1b=\mathsf{c}_1c$.
(2) For all $b\in At\mathfrak{A}\cap Rl_v\mathfrak{A}$ there is a $c\in M_1$ such that $\mathsf{c}_0b=\mathsf{c}_0c$.
(3) If $b,c\in M_0$ and $b\neq c$, then $\mathsf{c}_1b\neq\mathsf{c}_1c$.
(4) If $b,c\in M_1$ and $b\neq c$, then $\mathsf{c}_0b\neq\mathsf{c}_0c$.

For each $\varepsilon\in 2$ let f_ε be a one–one mapping of M_0 onto a partition of U having each element of size $|U|$. Now we claim:

(5) For every atom b there are unique $c \in M_0$ and $d \in M_1$ such that $b \leqq \mathsf{c}_1 c \cdot \mathsf{c}_0 d$.

For, since $\mathfrak{A}$ is simple we have

$$1 = \mathsf{c}_1\mathsf{c}_0 a = \Sigma\{\mathsf{c}_1 x : x \in At\mathfrak{A} \cap Rl_u\mathfrak{A}\} = \Sigma\{\mathsf{c}_1 x : x \in M_0\}.$$

This gives a unique $c \in M_0$ such that $b \leqq \mathsf{c}_1 c$. The existence and uniqueness of d is proved similarly. We denote c, d by $(b)_0, (b)_1$ respectively.

Now we define a function h mapping $At\mathfrak{A}$ into $Sb({}^2U)$ by defining $(At\mathfrak{A} \cap Rl_w\mathfrak{A}) \upharpoonright h$, with $w = \mathsf{c}_0 b \cdot \mathsf{c}_1 c$, for an arbitrary $b \in M_0$ and $c \in M_1$. In fact, let this restriction be a one-one function onto a partition $\mathcal{P}$ of $V = f_0 b \times f_1 c$ such that for any $X \in \mathcal{P}$, $\mathsf{C}_0^{|V|} X = \mathsf{C}_1^{|V|} X = V$; this is possible by 3.2.58. Now for any $x \in A$ let $jx = \bigcup\{hb : b \in At\mathfrak{A}, b \leqq x\}$. It is clear that j is a Boolean isomorphism from $\mathfrak{A}$ onto a Boolean set algebra with unit element ${}^2U$. By symmetry it suffices to show that j preserves c_0. First suppose that $\langle u,v\rangle \in j\mathsf{c}_0 x$. Say $\langle u,v\rangle \in hb$, where $b \in At\mathfrak{A}$ and $b \leqq \mathsf{c}_0 x$; say $b \leqq \mathsf{c}_0 e$, where $e \in At\mathfrak{A}$ and $e \leqq x$. Thus $\mathsf{c}_0 b = \mathsf{c}_0 e$, and so $(b)_1 = (e)_1$. We have $\langle u,v\rangle \in \mathsf{C}_0 hb = U \times f_1(b)_1 = U \times f_1(e)_1 = \mathsf{C}_0 he$, so there is a $w \in U$ such that $\langle w,v\rangle \in he$. Thus $\langle u,v\rangle \in \mathsf{C}_0 jx$, as desired. Second suppose that $\langle u,v\rangle \in \mathsf{C}_0 jx$. Say $\langle w,v\rangle \in he$ with $e \in At\mathfrak{A}$, $e \leqq x$. Choose $f \in M_0$ so that $u \in f_0 b$. Note that $\langle u,v\rangle \in \mathsf{C}_0 he = U \times f_1(e)_1$. Therefore $\langle u,v\rangle \in f_0 b \times f_1(e)_1$. Hence there is an atom $c \leqq \mathsf{c}_1 b \cdot \mathsf{c}_0(e)_1$ such that $\langle u,v\rangle \in hc$. Now $c \leqq \mathsf{c}_0(e)_1 = \mathsf{c}_0 e \leqq \mathsf{c}_0 x$, so $\langle u,v\rangle \in j\mathsf{c}_0 x$, as desired.

Applying the df-versions of results in section 2.7, we obtain:

THEOREM 5.1.47. $\mathsf{Df}_2 = \mathsf{IGsdf}_2$; *hence* $\mathsf{Df}_2 = \mathsf{SDfCA}_2$.

CA_α's and Df_α's

Next we discuss non-representable Df_α's. The construction given in 3.2.68 gives a non-representable Df_α for each $\alpha \geqq 3$; in fact, the Df_α constructed is also a DfCA_α. In order to extend the non-finite axiomatizability results for CA_α's to Df_α's, we now must discuss the relationships between CA_α's and Df_α's further. These results, 5.1.48 to 5.1.63, are due to James S. Johnson [69].

LEMMA 5.1.48. *Assume $\alpha < \omega$. Suppose $\mathfrak{A}$ is a CA_α and $\mathfrak{Df}\mathfrak{A}$ is a Csdf_α. Then there is an isomorphism f of $\mathfrak{Df}\mathfrak{A}$ onto a Csudf_α with base U such that $\mathsf{D}_{\kappa\lambda}^{|U|} \subseteq f\mathsf{d}_{\kappa\lambda}$ for all $\kappa, \lambda < \alpha$.*

PROOF. By 5.1.34 we may assume that $\mathfrak{Df}\mathfrak{A}$ has a pairwise disjoint base system $\langle V_\kappa : \kappa < \alpha\rangle$. Let $U = \bigcup_{\kappa<\alpha} V_\kappa$. Now by 1.8.7 we have $\mathsf{c}_{(\alpha \sim \{\kappa\})}\mathsf{d}_\alpha = 1$ for any $\kappa < \alpha$, so that for any $u \in V_\kappa$ there is an $x \in \mathsf{d}_\alpha$ such that $x_\kappa = u$. Hence for each $\kappa < \alpha$ there is a function g_κ mapping V_κ into d_α such that $(g_\kappa u)_\kappa = u$ for all $u \in V_\kappa$. Now define for each $\kappa < \alpha$ a function t_κ mapping U onto V_κ by setting $t_\kappa v = (g_\lambda v)_\kappa$, where λ is such that $v \in V_\lambda$. Finally, for any $x \in A$ let

$$fx = \{u \in {}^{\alpha}U : \langle t_\kappa u_\kappa : \kappa < \alpha \rangle \in x\}.$$

By the proof of 5.1.34, f is an isomorphism from $\mathfrak{Df}\mathfrak{A}$ onto a Csudf_α with base U. Now suppose that $\kappa,\lambda<\alpha$ and $x \in \mathsf{D}_{\kappa\lambda}^{|U|}$. Let $y_\mu = x_\kappa$ for all $\mu<\alpha$. Say $x_\kappa \in V_\nu$. Then $g_\nu x_\kappa \in \mathsf{d}_\alpha$, so for each $\mu<\alpha$ we have

$$t_\mu y_\mu = t_\mu x_\kappa = (g_\nu x_\kappa)_\mu,$$

hence $\langle t_\mu y_\mu : \mu<\alpha\rangle = g_\nu x_\kappa \in \mathsf{d}_\alpha$. Thus $y \in f\mathsf{d}_\alpha$ and so $x \in \mathsf{c}_{(\alpha\sim\{\kappa,\lambda\})} f\mathsf{d}_\alpha = f\mathsf{c}_{(\alpha\sim\{\kappa,\lambda\})}\mathsf{d}_\alpha = f\mathsf{d}_{\kappa\lambda}$, as desired.

LEMMA 5.1.49. *Suppose* $3 \leq \alpha < \omega$, $\mathfrak{A}$ *is a* CA_α, $\mathfrak{Df}\mathfrak{A}$ *is a* Csudf_α *with base* U, *and* $\mathsf{D}_{\kappa\lambda}^{|U|} \subseteq \mathsf{d}_{\kappa\lambda}$ *for all* $\kappa,\lambda<\alpha$. *Let* $R = \{\langle u,v\rangle \in {}^2U$: *there is an* $x \in \mathsf{d}_{01}$ *such that* $x_0 = u$ *and* $x_1 = v\}$. *Then* R *is an equivalence relation on* U, *and for any distinct* $\kappa,\lambda<\alpha$ *we have* $R = \{\langle u,v\rangle \in {}^2U$: *there is an* $x \in \mathsf{d}_{\kappa\lambda}$ *such that* $x_\kappa = u$ *and* $x_\lambda = v\}$.

PROOF. For any distinct $\kappa,\lambda<\alpha$ let $S_{\kappa\lambda} = \{\langle u,v\rangle \in {}^2U$: there is an $x \in \mathsf{d}_{\kappa\lambda}$ such that $x_\kappa = u$ and $x_\lambda = v\}$. To prove the second part of the conclusion it suffices to show that $S_{\kappa\lambda} \subseteq S_{\kappa\mu}$ whenever $|\{\kappa,\lambda,\mu\}| = 3$. Let $\langle u,v\rangle \in S_{\kappa\lambda}$; say $x \in \mathsf{d}_{\kappa\lambda}$ with $x\kappa = u$, $x\lambda = v$. Since $\mathsf{C}_\mu \mathsf{d}_{\kappa\lambda} = \mathsf{d}_{\kappa\lambda}$, there is a $y \in {}^\alpha U$ such that $y\kappa = u$, $y\lambda = v$, $y\mu = v$, $y \in \mathsf{d}_{\kappa\lambda}$. Then $y \in \mathsf{D}_{\lambda\mu} \subseteq \mathsf{d}_{\lambda\mu}$, so $y \in \mathsf{d}_{\kappa\mu}$. Hence $\langle u,v\rangle \in S_{\kappa\mu}$, as desired.

Now R is reflexive on U since $\mathsf{D}_{01} \subseteq \mathsf{d}_{01}$, and it is symmetric since $S_{01} = S_{10}$. Now suppose that $uRvRw$. Then $\langle u,v\rangle \in S_{01}$ and $\langle v,w\rangle \in S_{12}$. Say $x \in \mathsf{d}_{01}$, $x0 = u$, $x1 = v$, $y \in \mathsf{d}_{12}$, $y1 = v$, $y2 = w$. Let $z0 = u$, $z1 = v$, $z2 = w$, $z \in {}^\alpha U$. Then $z \in \mathsf{C}_{(\alpha\sim 2)}\{x\} \subseteq \mathsf{C}_{(\alpha\sim 2)}\mathsf{d}_{01} = \mathsf{d}_{01}$ and $z \in \mathsf{d}_{12}$ similarly. Hence $z \in \mathsf{d}_{02}$, so $\langle u,w\rangle \in S_{02} = R$, as desired.

LEMMA 5.1.50. *Assume the hypotheses of* 5.1.49, *and let* $E = \{X \in A$: *for all* $x,y \in {}^\alpha U$, *if* $x_\lambda R y_\lambda$ *for all* $\lambda<\alpha$, *then* $x \in X$ *iff* $y \in X\}$.

Then $\{X \in A : \Delta X \neq \alpha\} \subseteq E$, *and* $E \in S\mathfrak{u}\mathfrak{A}$.

PROOF. Let $X \in A$ and $\Delta X \neq \alpha$. Say $\kappa \in \alpha\sim\Delta X$. To show that $X \in E$ it suffices to prove the following statement:

(1) If $\lambda<\alpha$, $x,y \in {}^\alpha U$, $x_\lambda R y_\lambda$, $(\alpha\sim\{\lambda\}) \upharpoonright x \subseteq y$, and $x \in X$, then $y \in X$.

Assume the hypothesis of (1). If $\kappa = \lambda$ then obviously $y \in X$, so assume that $\kappa \neq \lambda$. Since $x_\lambda R y_\lambda$, choose by 5.1.49 $w \in \mathsf{d}_{\kappa\lambda}$ such that $w_\kappa = x_\lambda$ and $w_\lambda = y_\lambda$. Then $(y_{x\lambda}^\kappa)\kappa = x_\lambda = w_\kappa$ and $(y_{x\lambda}^\kappa)\lambda = y_\lambda = w_\lambda$, so $y_{x\lambda}^\kappa \in \mathsf{C}_{(\alpha\sim\{\kappa,\lambda\})}\{w\} \subseteq \mathsf{C}_{(\alpha\sim\{\kappa,\lambda\})}\mathsf{d}_{\kappa\lambda} = \mathsf{d}_{\kappa\lambda}$. Also, $(\alpha\sim\{\lambda\}) \upharpoonright y_{x\lambda}^\kappa \subseteq x_{x\lambda}^\kappa$ and $x_{x\lambda}^\kappa \in \mathsf{C}_\kappa X = X$, $x_{x\lambda}^\kappa \in \mathsf{D}_{\kappa\lambda} \subseteq \mathsf{d}_{\kappa\lambda}$, so $y_{x\lambda}^\kappa \in \mathsf{C}_\lambda(X \cap \mathsf{d}_{\kappa\lambda}) \cap \mathsf{d}_{\kappa\lambda} = X \cap \mathsf{d}_{\kappa\lambda}$. Hence $y \in \mathsf{C}_\kappa X = X$, as desired.

Now E is obviously closed under $+$ and $-$, and $\mathsf{d}_{\kappa\lambda} \in E$ for all $\kappa,\lambda<\alpha$ since $\Delta\mathsf{d}_{\kappa\lambda} \subseteq \{\kappa,\lambda\}$ and $\alpha \geq 3$ (using what was established above). It remains to show that E is closed under C_κ for an arbitrary $\kappa<\alpha$. By symmetry it suffices to assume that $X \in E$, $x \in \mathsf{C}_\kappa X$, $y \in {}^\alpha U$, and $x_\lambda E y_\lambda$ for all $\lambda<\alpha$, and prove that $y \in \mathsf{C}_\kappa X$. Say $x_u^\kappa \in X$. Now $(x_u^\kappa)_\lambda R (y_u^\kappa)_\lambda$ for each $\lambda<\alpha$. Hence $y_u^\kappa \in X$ since $X \in E$. Thus $y \in \mathsf{C}_\kappa X$, as desired.

Now the above lemmas enable us to establish two important connections between representablility of $\mathfrak{A}$ and of $\mathfrak{Df}\mathfrak{A}$ for a CA_α $\mathfrak{A}$:

THEOREM 5.1.51. *Suppose* $\mathfrak{A}$ *is a* CA_α *with* $3 \leqq \alpha < \omega$, $\mathfrak{A}$ *is generated by* $\{x \in A : \Delta x \neq \alpha\}$, *and* $\mathfrak{Df}\mathfrak{A} \in \mathsf{IGsdf}_\alpha$. *Then* $\mathfrak{A}$ *is representable.*

PROOF. By 5.1.48 we may assume that $\mathfrak{Df}\mathfrak{A}$ is a Csudf_α with base U such that $\mathsf{D}_{\kappa\lambda}^{[U]} \subseteq \mathsf{d}_{\kappa\lambda}$ for all $\kappa,\lambda < \alpha$. Now let R and E be as in 5.1.49 and 5.1.50, and let S be as in 5.1.39. By 5.1.39 and its proof, $\mathfrak{Df}\mathfrak{A}$ is isomorphic to a Csudf_α $\mathfrak{B}$ with base U/R, where the isomorphism from $\mathfrak{B}$ onto $\mathfrak{A}$ is the restriction to $\mathfrak{B}$ of the following isomorphism F of $\mathfrak{Gb}({}^\alpha(U/R))$ into $\mathfrak{Gb}({}^\alpha U)$ (both considered as Df_α's): $F = \langle \{x \in {}^\alpha U : R^* \circ x \in X\} : X \subseteq {}^\alpha(U/R) \rangle$. It suffices to show that $F\mathsf{D}_{\kappa\lambda}^{[U/R]} = \mathsf{d}_{\kappa\lambda}$ for arbitrary $\kappa,\lambda < \alpha$. Suppose that $x \in F\mathsf{D}_{\kappa\lambda}$. Then $R^* \circ x \in \mathsf{D}_{\kappa\lambda}$, so $x_\kappa R x_\lambda$. From 5.1.49 it follows easily that $x \in \mathsf{d}_{\kappa\lambda}$. If, conversely, $x \in \mathsf{d}_{\kappa\lambda}$, then $x_\kappa R x_\lambda$ by 5.1.49, and so $x \in F\mathsf{D}_{\kappa\lambda}$ easily.

THEOREM 5.1.52. *If* $3 \leqq \alpha < \omega$, $\mathfrak{B}$ *is a* $\mathsf{CA}_{\alpha+1}$, *and* $\mathfrak{Df}\mathfrak{B} \in \mathsf{IGsdf}_{\alpha+1}$, *then* $\mathfrak{Rr}_\alpha\mathfrak{B} \in \mathsf{IGs}_\alpha$.

PROOF. Let $\mathfrak{C} = \mathfrak{Gg}^{\mathfrak{B}} N r_\alpha \mathfrak{B}$. Then $\mathfrak{Df}\mathfrak{C} \subseteq \mathfrak{Df}\mathfrak{B}$, so $\mathfrak{Df}\mathfrak{C} \in \mathsf{IGsdf}_{\alpha+1}$. By 5.1.51, $\mathfrak{C} \in \mathsf{IGs}_{\alpha+1}$, so by an easy argument, $\mathfrak{Rr}_\alpha\mathfrak{C} = \mathfrak{Rr}_\alpha\mathfrak{B} \in \mathsf{IGs}_\alpha$.

REMARK 5.1.53. In connection with the above results it should be noted that for every $\alpha \geqq 2$ there is a non–representable CA_α $\mathfrak{A}$ such that $\mathfrak{Df}\mathfrak{A}$ is representable. For $\alpha = 2$ we can take the algebra $\mathfrak{A}$ of 2.6.13; the equation

$$\mathsf{c}_0(x_0 \cdot x_2) \cdot \mathsf{c}_0(x_0 \cdot -x_2) \cdot [\textstyle\prod_{\lambda<2} -\mathsf{c}_0\mathsf{c}_1(\mathsf{c}_1 x_\lambda \cdot \mathsf{s}_1^0 \mathsf{c}_1 x_\lambda \cdot -\mathsf{d}_{01})] \cdot -\mathsf{c}_0 \textstyle\prod_{\lambda<2} -\mathsf{c}_1 x_\lambda = \mathsf{0}$$

fails in $\mathfrak{A}$ but holds in every IGs_2, so $\mathfrak{A}$ is non–representable. On the other hand, $\mathfrak{Df}\mathfrak{A}$ is representable by 5.1.47. Now assume that $3 \leqq \alpha$. We shall define a non–representable CA_α $\mathfrak{A}$ with $\mathfrak{Df}\mathfrak{A}$ representable. Set $\beta = (\alpha \cup \omega)^+$, let a and b be distinct objects not in β, and let $U = \beta \cup \{a,b\}$. Now $\mathfrak{Df}\mathfrak{A}$ will be a Csudf_α with base U. To define $\mathfrak{A}$ we first describe certain elements of $Sb({}^\alpha U)$. For each $\kappa < \alpha$ let

$$\begin{aligned}
Q_\kappa &= \{x \in {}^{\alpha\sim\{k\}}\beta : x \text{ is one–one}\};\\
P_{0\kappa} &= \{x \in Q_\kappa : \textstyle\bigcup_{\lambda\in\alpha\sim\{\kappa\}} x_\lambda \text{ is a limit ordinal}\};\\
P_{1\kappa} &= Q_\kappa \sim P_{0\kappa};\\
M_\kappa &= \{x \in {}^\alpha U : x_\kappa = a \text{ and } (\alpha\sim\{\kappa\}) \upharpoonright x \in P_{0\kappa}\} \cup\\
&\qquad \{x \in {}^\alpha U : x_\kappa = b \text{ and } (\alpha\sim\{\kappa\}) \upharpoonright x \in P_{1\kappa}\};
\end{aligned}$$

N_κ is like M_κ with a and b interchanged. Finally, suppose $\Gamma \subseteq \alpha$ and R is an equivalence relation on $\alpha\sim\Gamma$; we set

$$T_{\Gamma R} = \{x \in {}^\alpha U : x^*\Gamma \subseteq \{a,b\},\ x^*(\alpha\sim\Gamma) \subseteq \beta, \text{ and } [(\alpha\sim\Gamma) \upharpoonright x] \mid [(\alpha\sim\Gamma) \upharpoonright x]^{-1} = R\}.$$

Now we let

$$\mathcal{P} = \{M_\kappa : \kappa < \alpha\} \cup \{N_\kappa : \kappa < \alpha\}$$
$$\cup (\{T_{\Gamma R} : \Gamma \subseteq \alpha,\ R \text{ an equivalence relation on } \alpha \sim \Gamma\} \sim \{T_{\Gamma R} : |\Gamma| = 1,\ R \subseteq Id\}).$$

It is easily checked that $\mathcal{P}$ is a partition of ${}^\alpha U$. Let A be the set of all unions of members of $\mathcal{P}$. We claim that A is closed under all cylindrifications $\mathsf{C}_\kappa^{|V|}$ for $\kappa < \alpha$, where $V = {}^\alpha U$. This follows from the following easily checked facts, where $\kappa < \alpha$ is arbitrary:

(1) $M_\kappa \cup N_\kappa = T_{\{\kappa\},R}$, where $R = (\alpha \sim \{\kappa\}) \upharpoonright Id$.

(2) Let R be an equivalence relation on α with $R \cap {}^2(\alpha \sim \{\kappa\}) \subseteq Id$. Then
$$\mathsf{C}_\kappa M_\kappa = \mathsf{C}_\kappa N_\kappa = \mathsf{C}_\kappa T_{0R} = M_\kappa \cup N_\kappa \cup \bigcup\{T_{0S} : S \text{ is an equivalence relation on } \alpha \text{ and } S \cap {}^2(\alpha \sim \{\kappa\}) \subseteq Id\}.$$

(3) If $\lambda \in \alpha \sim \{\kappa\}$, $R = (\alpha \sim \{\kappa, \lambda\}) \upharpoonright Id$, and S is an equivalence relation on $\alpha \sim \{\lambda\}$ with $S \cap {}^2(\alpha \sim \{\kappa, \lambda\}) \subseteq Id$, then
$$\mathsf{C}_\kappa M_\lambda = \mathsf{C}_\kappa N_\lambda = \mathsf{C}_\kappa T_{\{\kappa,\lambda\},R} = \mathsf{C}_\kappa T_{\{\lambda\}S} = T_{\{\kappa,\lambda\},R} \cup \bigcup\{T_{\{\lambda\}W} : W \text{ is an equivalence relation on } \alpha \sim \{\lambda\} \text{ and } W \cap {}^2(\alpha \sim \{\kappa, \lambda\}) \subseteq Id\}.$$

(4) Let R be an equivalence relation on α with $R \cap {}^2(\alpha \sim \{\kappa\}) \not\subseteq Id$, and set $S = R \cap {}^2(\alpha \sim \{\kappa\})$. Then
$$\mathsf{C}_\kappa T_{0R} = \mathsf{C}_\kappa T_{\{\kappa\}S} = T_{\{\kappa\}S} \cup \bigcup\{T_{0W} : W \text{ is an equivalence relation on } \alpha, \text{ and } W \cap {}^2(\alpha \sim \{\kappa\}) = S\}.$$

(5) If $\lambda \in \alpha \sim \{\kappa\}$, S is an equivalence relation on $\alpha \sim \{\lambda\}$, and $R = S \cap {}^2(\alpha \sim \{\kappa, \lambda\}) \not\subseteq Id$, then
$$\mathsf{C}_\kappa T_{\{\lambda\}S} = \mathsf{C}_\kappa T_{\{\kappa,\lambda\}R} = T_{\{\kappa,\lambda\}R} \cup \bigcup\{T_{\{\lambda\}W} : W \text{ is an equivalence relation on } \alpha \sim \{\lambda\} \text{ and } W \cap {}^2(\alpha \sim \{\kappa, \lambda\}) = R\}.$$

(6) If $\Gamma \subseteq \alpha$, R is an equivalence relation on $\alpha \sim \Gamma$, and $|\Gamma \sim \{\kappa\}| \geqq 2$, then
$$\mathsf{C}_\kappa T_{\Gamma R} = \bigcup\{T_{\Delta S} : \Delta \subseteq \alpha,\ \Delta \sim \{\kappa\} = \Gamma \sim \{\kappa\},\ S \text{ is an equivalence relation on } \alpha \sim \Delta, \text{ and } S \cap {}^2[\alpha \sim (\Delta \cup \{\kappa\})] = R \cap {}^2[\alpha \sim (\Delta \cup \{\kappa\})]\}.$$

Now for any $\kappa, \lambda < \alpha$ we set

$$\mathsf{d}_{\kappa\lambda} = \{x \in {}^\alpha U : x_\kappa = x_\lambda \text{ or } x_\kappa, x_\lambda \in \{a, b\}\}.$$

Finally, we let $\mathfrak{A} = \langle A, \cup, \cap, \sim, 0, V, \mathsf{C}_\kappa^{|V|}, \mathsf{d}_{\kappa\lambda} \rangle_{\kappa,\lambda<\alpha}$. Thus by the above, $\mathfrak{Df}\mathfrak{A}$ is a Csudf_α with base U. We still have to check that each $d_{\kappa\lambda}$ is in $\mathfrak{A}$, and $\mathfrak{A} \in \mathsf{CA}_\alpha$. Note that $d_{\kappa\kappa} = V$ for all $\kappa < \alpha$. Suppose $\kappa \neq \lambda$. Then

(7) $\mathsf{d}_{\kappa\lambda} = \bigcup\{T_{\Gamma R} : \Gamma \subseteq \alpha,\ R \text{ is an equivalence relation on } \alpha \sim \Gamma, \text{ and } \kappa, \lambda \in \Gamma \text{ or } \kappa R \lambda\}$.

Thus $\mathsf{d}_{\kappa\lambda} \in A$. Clearly (C_5) holds, and (C_6) is routine. To check (C_7), assume that $\kappa \neq \lambda$. It suffices to show that if X and Y are distinct members of $\mathcal{P}$ with $X, Y \subseteq \mathsf{d}_{\kappa\lambda}$ then $\mathsf{C}_\kappa X \cap \mathsf{C}_\kappa Y = 0$. This is routine, using (1)–(7). So, $\mathfrak{A}$ is a CA_α.

Now suppose that $\mathfrak{A}$ is representable. Then there is a homomorphism f from $\mathfrak{A}$ onto a Cs_α such that $fM_0 \neq 0$. Choose $x \in fM_0$. Now $\mathsf{C}_1 M_0 \cap \mathsf{s}_1^0 \mathsf{C}_1 M_0 \subseteq \mathsf{d}_{01}$. Also, $x \in fM_0 \subseteq f\mathsf{C}_0 M_0 = f\mathsf{C}_0 N_0 = \mathsf{C}_0 fN_0$, so there is a w such that $x_w^0 \in fN_0$. Since $\mathsf{C}_1 M_0 = \mathsf{C}_1 N_0$, it follows that $x_w^1 \in \mathsf{C}_1 fM \cap \mathsf{s}_1^0 \mathsf{C}_1 fM_0$, and so by the above $x_w^1 \in f\mathsf{d}_{01} = \mathsf{D}_{01}$. Hence $x_0 = w$ and so $x = x_w^0 \in fN_0$. Thus $x \in fM_0 \cap fN_0$, contradiction.

Now we return to the discussion of non–finitizability for the classes Df_α. We use notation introduced in 3.2.76.

LEMMA 5.1.54. *Suppose* $3 \leqq \alpha, \beta < \omega$ *and* $\alpha - 1 \leqq \beta$. *Then* $\mathfrak{Rl}\mathfrak{G}_{\alpha\beta}$ *is generated by* $\{x \in G_{\alpha\beta} : |\Delta x| \leqq 2\}$.

PROOF. Since $\mathfrak{G}_{\alpha\beta}$ is finite, it suffices to note that for any $\langle R, f\rangle \in G'_{\alpha\beta}$ we have

$$\{\langle R, f\rangle\} = \bigcap\{\mathsf{c}_{\alpha \sim \{\kappa,\lambda\}}\{\langle R, f\rangle\} : \kappa, \lambda < \alpha,\ \kappa \neq \lambda\}.$$

This is immediate from 3.2.76(7).

Combining this lemma with 5.1.51, we can extend 3.2.84 and 3.2.85 as follows.

LEMMA 5.1.55. *For every* $\kappa \in \omega$ *there is an* $\alpha \in \omega \sim 3$ *such that* $\mathfrak{Df}\mathfrak{Rb}_3\mathfrak{G}_{\alpha,\alpha+\kappa}$ *is a non–representable* Df_3.

THEOREM 5.1.56. *If* $3 \leqq \alpha < \omega$ *and* $\kappa < \omega$, *then there is a* CA_α $\mathfrak{A}$ *neatly embeddable in a* $\mathsf{CA}_{\alpha+\kappa}$ *such that* $\mathfrak{Df}\mathfrak{A}$ *is non–representable.*

Finally, repeating the proof of 4.1.3, we obtain:

THEOREM 5.1.57. *For* $3 \leqq \alpha < \omega$, IGsdf_α *is not finitely axiomatizable; neither is the class of representable* DfCA_α's.

Now we want to generalize 5.1.56 to the case $\alpha \geqq \omega$. To this end we need two lemmas.

LEMMA 5.1.58. *Suppose* $3 \leqq \alpha < \omega$, $\mathfrak{A}$ *is a* Gsdf_α, $x \in A$, *and* $\mathsf{c}_\kappa x \cdot \mathsf{c}_{(\alpha \sim \{\kappa\})} x = x$ *for all* $\kappa < \alpha$. *Then* $\mathfrak{Rl}_x \mathfrak{A} \in \mathsf{IGsdf}_\alpha$.

PROOF. Let $\mathfrak{A}$ have unit element as in 5.1.33(iv). Let $J = \{i \in I : x \cap \mathsf{P}_{\kappa<\alpha} U_{i\kappa} \neq 0\}$. For each $i \in J$ and $\kappa < \alpha$ let $W_{i\kappa} = \{u\kappa : u \in x \cap \mathsf{P}_{\lambda<\alpha} U_{i\lambda}\}$. Now:

(1) $\mathsf{P}_{\kappa<\alpha} W_{i\kappa} \subseteq x$ for any $i \in J$.

In fact, it suffices to assume that $u \in (\mathsf{P}_{\kappa<\alpha} W_{i\kappa}) \cap x$, $\kappa < \alpha$, $t \in W_{i\kappa}$, and show that $u_t^\kappa \in x$. We have $u_t^\kappa \in V$, $u_t^\kappa \in \mathsf{c}_\kappa x$. Say $t = v\kappa$, $v \in x$. Thus $u_t^\kappa \in \mathsf{c}_{(\alpha \sim \{\kappa\})} x$. Hence $u_t^\kappa \in x$ by the hypothesis of the lemma.

By (1) it follows that x is a Gsdf_α unit element, and so the lemma holds.

By the proof of 3.2.86 we have:

LEMMA 5.1.59. *Suppose* $3 \leqq \alpha \leqq \beta < \omega$ *and* $\kappa < \omega$. *Then* $\mathfrak{G}_{\alpha,\alpha+\kappa} = \mathfrak{Rl}_x \mathfrak{Rd}_\alpha \mathfrak{G}_{\beta,\beta+\kappa}$ *for some* $x \in G_{\beta,\beta+\kappa}$ *such that* $\mathsf{c}_\kappa x \cdot \mathsf{c}_{(\alpha \sim \{\kappa\})} x = x$ *for all* $\kappa < \alpha$.

Then the proof of 3.2.87 yields:

THEOREM 5.1.60. *If* $3 \leqq \alpha$ *and* $\kappa < \omega$, *then there is a* CA_α $\mathfrak{A}$ *neatly embeddable in a* $\mathsf{CA}_{\alpha+\kappa}$ *such that* $\mathfrak{Df}\mathfrak{A}$ *is not representable.*

Now we can repeat the proofs of 4.1.5—4.1.7 to obtain the following results (the notions in 4.1.4 being modified in the natural way):

THEOREM 5.1.61. *For* $\alpha \geqq \omega$, Df_α *is axiomatizable by a finite schema of equations.*

THEOREM 5.1.62. *For* $\alpha \geqq \omega$, IGsdf_α *is axiomatizable by a countable schema of equations.*

THEOREM 5.1.63. *For* $\alpha \geqq \omega$, IGsdf_α *is not axiomatizable by a finite schema of equations.*

Now we turn to decidability questions. By 5.1.24 and the proof of 4.2.7 we have:

THEOREM 5.1.64. *The equational theory of* Df_2 *is decidable.*

Recall from 5.1.47 that $\mathsf{Df}_2 = \mathsf{IGsdf}_2$. Now we consider Df_α, $\alpha \geqq 3$. As for CA_α's, the equational theory of Df_α is undecidable for $3 \leqq \alpha \leqq \omega$. The proof is due to Maddux [80'], and follows closely 4.2.17—4.2.18. We need a substitute for the operation ; defined there, since its definition involves diagonal elements. This substitute, and the analog of 4.2.17, are given in the following lemma:

LEMMA 5.1.65. *Let* $\mathfrak{A}$ *be a* Df_α, $\alpha \geqq 3$. *Suppose* $u, v \in A$ *and* $\mathsf{c}_0 u = u$, $\mathsf{c}_1 v = v$. *For any* $x, y \in A$ *define*

$$x : y = \mathsf{c}_2(\mathsf{c}_1(u \cdot \mathsf{c}_2 x) \cdot \mathsf{c}_0(v \cdot \mathsf{c}_2 y)).$$

Now let $x, y, z, w \in A$. *Then the conditions of* 4.2.17 *hold with* ; *replaced by* :, *and* d_{12}, d_{02} *by* u, v *respectively.*

PROOF. By the proof of 4.2.17.

THEOREM 5.1.66. *Suppose* $3 \leqq \alpha \leqq \omega$ *and* $\mathsf{IGsdf}_\alpha \subseteq \mathsf{K} \subseteq \mathsf{Df}_\alpha$. *Then the equational theory of* K *is undecidable.*

PROOF. We slightly modify the proof of 4.2.18. First assume that $\alpha < \omega$. Suppose that $0 < \beta < \omega$. For any terms σ, τ of $\mathcal{L}_\alpha$ we let $\sigma : \tau$ be an abbreviation for $\mathsf{c}_2(\mathsf{c}_1(v_{\beta+1} \cdot \mathsf{c}_2 \sigma) \cdot \mathsf{c}_0(v_{\beta+2} \cdot \mathsf{c}_2 \tau))$. We also use the abbreviations $\mathsf{c}_{(\alpha)}$, $\oplus$, $\leq$ from the proof of 4.2.19. Now with each element x of Fr_β we associate a term τx involving only the variables $v_0, \ldots, v_{\beta-1}$, $v_{\beta+1}, v_{\beta+2}$, as follows:

$$\tau\xi = v_\xi \text{ if } \xi < \beta;$$
$$\tau(x \cdot y) = \tau x : \tau y,$$

for any $x, y \in Fr_\beta$. Now we claim:

(1) If $X \subseteq Fr_\beta \times Fr_\beta$, X finite, and $x, y \in Fr_\beta$, then $x \simeq_X y$ iff the following inequality holds in K:

$$(\tau x : \mathsf{c}_1 v_\beta) \oplus (\tau y : \mathsf{c}_1 v_\beta) \leq \Sigma_{\langle u,v\rangle \in x} \mathsf{c}_{(\alpha)}(\tau u \oplus \tau v) + \mathsf{c}_{(\alpha)}(v_{\beta+1} \oplus \mathsf{c}_0 v_{\beta+1}) + \mathsf{c}_{(\alpha)}(v_{\beta+2} \oplus \mathsf{c}_1 v_{\beta+2}).$$

The proof of (1), and the rest of the proof of 5.1.66, follows the lines of the proof of 4.2.18. (In the second part of the proof of (1), let additionally $h(\beta+1) = \{a \in {}^\alpha A : a_1 = a_2\}$ and $h(\beta+2) = \{a \in {}^\alpha A : a_0 = a_2\}$.) The case $\alpha = \omega$ is treated as in 4.2.18.

The connections with logic for Df's are like those for CA's; we sketch enough of this theory to establish a logical representation theorem analogous to 4.3.28. Although in the absence of the equality symbol we cannot reduce arbitrary formulas to restricted ones we shall deal here only with restricted formulas. ${}^-\Phi\mu_r^\Lambda$ denotes the set of all restricted formulas of Λ not involving equality. The corresponding absolutely free algebra is

$${}^-\mathfrak{Fm}_r^\Lambda = \langle {}^-\Phi\mu_r^\Lambda, \mathbf{v}, \mathbf{\wedge}, \mathbf{\neg}, \mathbf{F}, \mathbf{T}, \exists v_\kappa \rangle_{\kappa<\alpha}.$$

An analog of 4.3.1 holds. Given a possible model $\mathfrak{M}$ of Λ, $\mathfrak{Cf}\mathfrak{d}\mathfrak{f}_\alpha^{\mathfrak{M}}$ denotes the Csdf$_\alpha$ with universe $\{\varphi^{\mathfrak{M}} : \varphi \in {}^-\Phi\mu_r^\Lambda\}$. The analog of 4.3.5 holds. Introducing the notation of 4.3.9 for Csdf$_\alpha$'s also, we obtain 4.3.10 for Df's. We can then introduce the semantical equivalence relation analogous to ${}_s\equiv_{r\Sigma}^\Lambda$ for equality-free formulas, and an algebra corresponding to ${}_s\mathfrak{Fm}_{r\Sigma}^\Lambda$; we denote them by ${}^-_s\equiv_{r\Sigma}^\Lambda$ and ${}^-_s\mathfrak{Fm}_{r\Sigma}^\Lambda$ respectively. Theorems analogous to 4.3.16, 4.3.17, and 4.3.18 then hold. The proof theoretic notions are treated similarly; we take as axioms only the formulas in schemas (1)–(4) (see the discussion prior to 4.3.19.) This leads to notions ${}^-_p\equiv_{r\Gamma}^\Lambda$ and ${}^-_p\mathfrak{Fm}_{r\Gamma}^\Lambda$, and an analog of 4.3.22. The results leading up to the analog of 4.3.28 are then easy to check.

5.2 PROJECTIVE ALGEBRAS

The notion of a projective algebra was introduced by Everett and Ulam [46] to abstractly express Boolean algebras of subsets of a plane and their projections on the two coordinate axes (see also Remark 5.2.2 below.) Their further theory was developed by McKinsey [48], Bednarek and Ulam [78'], and Larson [83'], [83a']. The relationships to cylindric algebras were discussed in Chin, Tarski [48] and Jónsson, Tarski [51]. In this section we show that projective algebras are definitionally equivalent to certain Df_2's. We do not go into the theory of projective algebras, but to keep our development self–contained we develop some of their elementary arithmetic.

DEFINITION 5.2.1. *A* **projective algebra** (PA) *is an algebraic system* $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, \mathsf{p}_0, \mathsf{p}_1, \Box, a\rangle$ *satisfying the following conditions, for all* $x, y, z, u \in A$ *and* $\varepsilon \in 2$:

(P_1) $\langle A, +, \cdot, -, 0, 1\rangle$ *is a* BA.

(P_2) p_0 *and* p_1 *are unary operations on* A, $\Box$ *is a binary operation on* A, *and* a *is an atom of* $\mathfrak{A}$.

(P_3) $\mathsf{p}_\varepsilon(x+y) = \mathsf{p}_\varepsilon x + \mathsf{p}_\varepsilon y$.

(P_4) $\mathsf{p}_0\mathsf{p}_1 = \mathsf{p}_1\mathsf{p}_0 1 = a$.

(P_5) $\mathsf{p}_\varepsilon x = 0$ *iff* $x = 0$.

(P_6) $\mathsf{p}_\varepsilon\mathsf{p}_\varepsilon x = \mathsf{p}_\varepsilon x$.

(P_7) *If* $x \leq \mathsf{p}_\varepsilon 1$, *then* $\mathsf{p}_\varepsilon x = x$.

(P_8) *If* $x \neq 0 \neq y$ *then* $\mathsf{p}_0(x \Box y) = \mathsf{p}_0 x$ *and* $\mathsf{p}_1(x \Box y) = \mathsf{p}_1 y$.

(P_9) $x \Box 0 = 0 \Box x = 0$.

(P_{10}) *If* $\mathsf{p}_0 z = \mathsf{p}_0 x$ *and* $\mathsf{p}_1 z = \mathsf{p}_1 y$, *then* $z \leq x \Box y$.

(P_{11}) $(\mathsf{p}_0 1) \Box a = \mathsf{p}_0 1$ *and* $a \Box (\mathsf{p}_1 1) = \mathsf{p}_1 1$.

(P_{12}) $(x+y) \Box (\mathsf{p}_1 1) = [x \Box (\mathsf{p}_1 1)] + [y \Box (\mathsf{p}_1 1)]$ *and* $(\mathsf{p}_0 1) \Box (x+y) = [(\mathsf{p}_0 1) \Box x] + [(\mathsf{p}_0 1) \Box y]$.

REMARK 5.2.2. The motivation for the introduction of this notion is as follows. $\mathfrak{A}$ is a *projective set algebra* if there exist sets U_0, U_1 and $u \in U_0$, $v \in U_1$ such that, with $W = U_0 \times U_1$,

$$\mathfrak{A} = \langle A, \cup, \cap, {}_W\sim, 0, W, P_0, P_1, \Box, a\rangle,$$

where $\langle A, \cup, \cap, {}_W\sim, 0, W\rangle$ is a Boolean set algebra of subsets of W, $a = \{\langle u, v\rangle\} \in A$, *and* P_0 and P_1 are the unary and $\Box$ the binary operations on subsets of W given as follows, with $X, Y \subseteq W$:

$$P_0X = \{\langle w,v\rangle : \langle w,t\rangle \in X \text{ for some } t \in U_1\},$$
$$P_1X = \{\langle u,w\rangle : \langle t,w\rangle \in X \text{ for some } t \in U_0\},$$
$$X \Box Y = \{\langle s,t\rangle : \langle s,w\rangle \in X \text{ for some } w \in U_1, \text{ and } \langle w,t\rangle \in Y \text{ for some } w \in U_0\}.$$

It is easily checked that any projective set algebra is a projective algebra. These set algebras are the motivating source for the definition of abstract projective algebras, and the axioms in 5.2.1 are complete in the sense that any projective algebra is isomorphic to a projective set algebra; see Everett, Ulam [46] and McKinsey [48].

Now we develop some of the arithmetic of projective algebras.

THEOREM 5.2.3. *Let* $\mathfrak{A}$ *be a* PA, *as in* 5.2.1, *and let* $w,x,y,z \in A$ *and* $\varepsilon \in 2$. *Then*
(i) $(\mathsf{p}_0 1) \Box (\mathsf{p}_1 1) = 1$.
(ii) *If* $x \leqq y$ *then* $\mathsf{p}_\varepsilon x \leqq \mathsf{p}_\varepsilon y$.
(iii) $\mathsf{p}_\varepsilon a = a$.
(iv) *If* $x \leqq \mathsf{p}_\varepsilon 1$, *then* $x = \mathsf{p}_\varepsilon y$ *for some* y.
(v) *If* $w \leqq x$ *and* $y \leqq z$, *then* $w \Box y \leqq x \Box z$.
(vi) *If* $w,x \leqq \mathsf{p}_0 1$ *and* $y,z \leqq \mathsf{p}_1 1$, *then* $(w \cdot x) \Box (y \cdot z) = (w \Box y) \cdot (x \Box z)$.
(vii) $x \Box y = (\mathsf{p}_0 x) \Box y = x \Box (\mathsf{p}_1 y) = (\mathsf{p}_0 x) \Box (\mathsf{p}_1 y)$.
(viii) $x \Box y = (x \Box (\mathsf{p}_1 1)) \cdot ((\mathsf{p}_0 1) \Box y)$.
(ix) $(x \Box (\mathsf{p}_1 1)) \cdot \mathsf{p}_0 1 = \mathsf{p}_0 x$, $((\mathsf{p}_0 1) \Box x) \cdot \mathsf{p}_1 1 = \mathsf{p}_1 x$.
(x) $x \Box a = \mathsf{p}_0 x$; $a \Box x = \mathsf{p}_1 x$.
(xi) *If* $x \leqq \mathsf{p}_0 1$, *then* $-(x \Box (\mathsf{p}_1 1)) = (-x \cdot \mathsf{p}_0 1) \Box (\mathsf{p}_1 1)$.
(xii) *If* $y \leqq \mathsf{p}_1 1$, *then* $-((\mathsf{p}_0 1) \Box y) = (\mathsf{p}_0 1) \Box (-y \cdot \mathsf{p}_1 1)$.

PROOF. (i) is immediate from (P_{10}). For (ii), if $x \leqq y$ then $\mathsf{p}_\varepsilon y = \mathsf{p}_\varepsilon(x+y) = \mathsf{p}_\varepsilon x + \mathsf{p}_\varepsilon y$ and so $\mathsf{p}_\varepsilon x \leqq \mathsf{p}_\varepsilon y$, using (P_3). To prove (iii), by symmetry it suffices to assume $\varepsilon = 0$. Then

$$\begin{aligned} \mathsf{p}_0 a &= \mathsf{p}_0 \mathsf{p}_0 \mathsf{p}_1 1 && \text{by } (\mathrm{P}_4) \\ &= \mathsf{p}_0 \mathsf{p}_1 1 && \text{by } (\mathrm{P}_6) \\ &= a && \text{by } (\mathrm{P}_4). \end{aligned}$$

Now to show (iv), again assume $\varepsilon = 0$. Suppose $x \leqq \mathsf{p}_0 1$. By (P_5) we may assume that $x \neq 0$. Thus $\mathsf{p}_1 1 \neq 0$ by (P_5). Hence $\mathsf{p}_0(x \Box \mathsf{p}_1 1) = \mathsf{p}_0 x = x$ by (P_8) and (P_7). Next, for (v) assume that $w \leqq x$ and $y \leqq z$. By (P_9) we may assume that $w \neq 0 \neq y$. Then

$$\begin{aligned} \mathsf{p}_0((w \Box y) + (x \Box z)) &= \mathsf{p}_0(w \Box y) + \mathsf{p}_0(x \Box z) && \text{by } (\mathrm{P}_3) \\ &= \mathsf{p}_0 w + \mathsf{p}_0 x && \text{by } (\mathrm{P}_8) \\ &= \mathsf{p}_0(w+x) && \text{by } (\mathrm{P}_3) \\ &= \mathsf{p}_0 x. \end{aligned}$$

Similarly, $\mathsf{p}_1((w \Box y) + (x \Box z)) = \mathsf{p}_1 z$. Hence by (P_{10}) we get $w \Box y \leqq (w \Box y) + (x \Box z) \leqq x \Box z$, as desired. Next, assume the hypotheses of (vi). The inequality $\leqq$ is clear by (v). Let $t = (w \Box y) \cdot (x \Box z)$. We may assume that $w,x,y,z \neq 0$. Then $\mathsf{p}_0 t$

$\leqq p_0(w \square y) = p_0 w = w$ by (ii), (P_7), (P_8). Similarly, $p_0 t \leqq x$, $p_1 t \leqq y$, and $p_1 t \leqq z$. Thus $p_0 t \leqq x \cdot w$ and $p_1 t \leqq y \cdot z$. Hence $t \leqq (p_0 t) \square (p_1 t) \leqq (x \cdot w) \square (y \cdot z)$ by (P_{10}) and (v).

For (vii) we may assume that $x, y, p_0 x, p_1 y \neq 0$, by ($P_5$) and ($P_9$). Then $p_0((p_0 x) \square y) = p_0 p_0 x = p_0 x$ by (P_8) and (P_6), and $p_1((p_0 x) \square y) = p_1 y$ by (P_8), so $(p_0 x) \square y \leqq x \square y$ by (P_{10}). The other parts of (vii) are established similarly. As to (viii), we have

$$\begin{aligned} x \square y &= (p_0 x) \square (p_1 y) && \text{by (vii)} \\ &= (p_0 x \cdot p_0 1) \square (p_1 y \cdot p_1 1) && \text{by (ii)} \\ &= ((p_0 x) \square (p_1 1)) \cdot ((p_0 1) \square (p_1 y)) && \text{by (vi), (ii)} \\ &= (x \square (p_1 1)) \cdot ((p_0 1) \square y) && \text{by (vii).} \end{aligned}$$

To prove (ix), we may assume by (P_5) and (P_9) that $x \neq 0$. Note then that $0 \neq p_1 p_0 x \leqq p_1 p_0 1 = a$ by (P_5), (ii) and (P_4), hence $p_1 p_0 x = p_1 p_0 1$. Thus

$$\begin{aligned} p_0 x &\leqq (p_0 x) \square (p_1 p_0 x) && \text{by } (P_{10}) \\ &= (p_0 x) \square (p_1 p_0 1) \\ &\leqq (p_0 x) \square (p_1 1) = x \square (p_1 1) && \text{by (ii), (vii)} \end{aligned}$$

Also, $p_0 x \leqq p_0 1$ by (ii), so $p_0 x \leqq (x \square (p_1 1)) \cdot p_0 1$. Now let $t = (x \square (p_1 1)) \cdot p_0 1$. Then $t \leqq p_0 1$, so

$$t = p_0 t \leqq p_0(x \square (p_1 1)) = p_0 x \quad \text{by } (P_7), (P_8), (P_5).$$

The other part of (ix) follows by symmetry. Next,

$$\begin{aligned} x \square a &= (x \square (p_1 1)) \cdot ((p_0 1) \square a) && \text{by (viii)} \\ &= (x \square (p_1 1)) \cdot p_0 1 && \text{by } (P_{12}) \\ &= p_0 x && \text{by (ix),} \end{aligned}$$

so (x) holds, the proof of the other part being similar. Assume that $x \leqq p_0 1$. Then

$$\begin{aligned} (x \square (p_1 1)) + ((-x \cdot p_0 1) \square (p_1 1)) &= (x + -x \cdot p_0 1) \square (p_1) && \text{by } (P_{11}) \\ &= (p_0 1) \square (p_1 1) = 1 && \text{by (i)} \end{aligned}$$

Also,

$$(x \square (p_1 1)) \cdot ((-x \cdot p_0 1) \square (p_1 1)) = (x \cdot -x \cdot p_0 1) \square (p_1 1) = 0 \quad \text{by (vi), } (P_9).$$

Hence (xi) holds. Finally, (xii) is established similarly to (xi).

Now we are ready for the main theorem of this section. The notion of polynomial equivalence is discussed in Part I, pp.125ff.

THEOREM 5.2.4. *Let* K *be the class of all algebras* $\mathfrak{A}=\langle A,+,\cdot,-,0,1,\mathsf{c}_\kappa,a\rangle_{\kappa<2}$ *satisfying the following conditions:*

(i) $\langle A,+,\cdot,-,0,1,\mathsf{c}_\kappa\rangle_{\kappa<2}$ *is a simple* Df_2.

(ii) a *is an atom of* $\langle A,+,\cdot,-,0,1\rangle$.

(iii) *If* $x\in A$ *and* $x\leqq \mathsf{c}_0 a$ *or* $x\leqq \mathsf{c}_1 a$, *then* $x=\mathsf{c}_0 x\cdot \mathsf{c}_1 x$.

Then K *is polynomially equivalent to the class* PA *of projective algebras.*

In more detail, if $\mathfrak{A}\in\mathsf{K}$ *with notation as above, define for any* $x,y\in A$

$$\mathsf{p}_0 x=\mathsf{c}_0 x\cdot \mathsf{c}_1 a,\quad \mathsf{p}_1 x=\mathsf{c}_1 x\cdot \mathsf{c}_0 a,$$
$$x\,\square\, y=\mathsf{c}_0 x\cdot \mathsf{c}_1 y;$$

let $\mathfrak{A}^p=\langle A,+,\cdot,-,0,1,\mathsf{p}_0,\mathsf{p}_1,\square,a\rangle$. *Conversely, given any* PA $\mathfrak{A}$ *as in* 5.2.1, *define for any* $x\in A$

$$\mathsf{c}_0 x=(\mathsf{p}_0 x)\,\square\,(\mathsf{p}_1 1),\quad \mathsf{c}_1 x=(\mathsf{p}_0 1)\,\square\,(\mathsf{p}_1 x),$$

and let $\mathfrak{A}^c=\langle A,+,\cdot,-,0,1,\mathsf{c}_\kappa,a\rangle_{\kappa<2}$. *Then*

(iv) *If* $\mathfrak{A}\in\mathsf{K}$, *then* $\mathfrak{A}^p\in\mathsf{PA}$, *and* $\mathfrak{A}^{pc}=\mathfrak{A}$.

(v) *If* $\mathfrak{A}\in\mathsf{PA}$, *then* $\mathfrak{A}^c\in\mathsf{K}$, *and* $\mathfrak{A}^{cp}=\mathfrak{A}$.

PROOF. To establish (iv), let $\mathfrak{A}\in\mathsf{K}$. Clearly (P_1) and (P_2) hold for $\mathfrak{A}^p$. We prove (P_3) only in case $\varepsilon=0$; for any $x,y\in A$,

$$\mathsf{p}_0(x+y)=\mathsf{c}_0(x+y)\cdot \mathsf{c}_1 a=(\mathsf{c}_0 x+\mathsf{c}_0 y)\cdot \mathsf{c}_1 a=\mathsf{p}_0 x+\mathsf{p}_0 y.$$

As to (P_4),

$$\begin{aligned}\mathsf{p}_0\mathsf{p}_1 1&=\mathsf{p}_0(\mathsf{c}_1 1\cdot \mathsf{c}_0 a)=\mathsf{c}_0(\mathsf{c}_1 1\cdot \mathsf{c}_0 a)\cdot \mathsf{c}_1 a\\&=\mathsf{c}_0 a\cdot \mathsf{c}_1 a=a\qquad \text{by (iii)}.\end{aligned}$$

Similarly, $\mathsf{p}_1\mathsf{p}_0 1=a$. Next, $\mathsf{p}_0 x=0$ iff $\mathsf{c}_0 x\cdot \mathsf{c}_1 a=0$ iff $x\cdot \mathsf{c}_0\mathsf{c}_1 a=0$ iff $x=0$, by simplicity. Similarly, $\mathsf{p}_1 x=0$ iff $x=0$, so (P_5) holds. We prove (P_6) only for $\varepsilon=0$:

$$\mathsf{p}_0\mathsf{p}_0 x=\mathsf{p}_0(\mathsf{c}_0 x\cdot \mathsf{c}_1 a)=\mathsf{c}_0(\mathsf{c}_0 x\cdot \mathsf{c}_1 a)\cdot \mathsf{c}_1 a=\mathsf{c}_0 x\cdot \mathsf{c}_1 a=\mathsf{p}_0 x.$$

For (P_7) we again suppose $\varepsilon=0$. By (P_5) we may assume that $x\neq 0$. Suppose $x\leqq \mathsf{p}_0 1$. Thus $x\leqq \mathsf{c}_1 a$, so $\mathsf{c}_1 x=\mathsf{c}_1 a$ by 1.10.3. Hence

$$\mathsf{p}_0 x=\mathsf{c}_0 x\cdot \mathsf{c}_1 a=\mathsf{c}_0 x\cdot \mathsf{c}_1 x=x \text{ by (iii)}.$$

Next, suppose $x\neq 0\neq y$. Then

$$\begin{aligned}\mathsf{p}_0(x\,\square\, y)&=\mathsf{p}_0(\mathsf{c}_0 x\cdot \mathsf{c}_1 y)=\mathsf{c}_0(\mathsf{c}_0 x\cdot \mathsf{c}_1 y)\cdot \mathsf{c}_1 a\\&=\mathsf{c}_0 x\cdot \mathsf{c}_0\mathsf{c}_1 y\cdot \mathsf{c}_1 a=\mathsf{c}_0 x\cdot \mathsf{c}_1 a\qquad \text{by simplicity}\\&=\mathsf{p}_0 x.\end{aligned}$$

Similarly, $\mathsf{p}_1(x\,\square\, y)=\mathsf{p}_1 y$. Thus (P_8) holds. (P_9) is clear. Assume the hypotheses of (P_{10}). Then $\mathsf{c}_0 x\geqq \mathsf{c}_0 x\cdot \mathsf{c}_1 a=\mathsf{p}_0 x=\mathsf{p}_0 z=\mathsf{c}_0 z\cdot \mathsf{c}_1 a$, so $\mathsf{c}_0 x\geqq \mathsf{c}_0(\mathsf{c}_0 z\cdot \mathsf{c}_1 a)=\mathsf{c}_0 z\cdot \mathsf{c}_0\mathsf{c}_1 a=\mathsf{c}_0 z$ by simplicity. Thus $z\leqq \mathsf{c}_0 x$. By symmetry, $z\leqq \mathsf{c}_1 y$, so $z\leqq x\,\square\, y$. For (P_{11}),

$$(p_0 1)\,\square\, a = c_0(c_0 1\cdot c_1 a)\cdot c_1 a = c_1 a = p_0 1;$$

similarly, $a\,\square\,(p_1 1) = p_1 1$. Finally,

$$\begin{aligned}(x+y)\,\square\,(p_1 1) &= c_0(x+y)\cdot c_1 p_1 1 = c_0 x\cdot c_1 p_1 1\cdot c_0 y\cdot c_1 p_1 1\\ &= (x\,\square\,(p_1 1))+(y\,\square\,(p_1 1)).\end{aligned}$$

The other part of (P_{12}) is similarly proved. Thus $\mathfrak{A}^p \in \mathsf{PA}$. Denote the cylindrifications of $\mathfrak{A}^{pc}$ by c_0', c_1'. Then we have

$$\begin{aligned}c_0' x &= (p_0 x)\,\square\,(p_1 1) = (c_0 x\cdot c_1 a)\,\square\,(c_1 1\cdot c_0 a)\\ &= c_0(c_0 x\cdot c_1 a)\cdot c_1(c_1 1\cdot c_0 a)\\ &= c_0 x\cdot c_0 c_1 a\cdot c_1 c_0 a = c_0 x\end{aligned}$$

by simplicity. Similarly, $c_1' = c_1$. This proves (iv).

Now suppose $\mathfrak{A}\in\mathsf{PA}$. First we show that $\mathfrak{A}^c \in \mathsf{Df}_2$. Conditions (C_0) and (C_1) are clear. For (C_2), note that

$$x \leqq (p_0 x)\,\square\,(p_1 x) \leqq (p_0 x)\,\square\,(p_1 1) = c_0 x;$$

similarly, $x \leqq c_1 x$. To establish (C_3) we need some auxiliary statements.

(1) If $x \leqq y$, then $c_\varepsilon x \leqq c_\varepsilon y$, for $\varepsilon \in 2$.

This follows from 5.2.3(ii), 5.2.3(v).

(2) $c_\varepsilon c_\varepsilon x = c_\varepsilon x$ for $\varepsilon \in 2$.

For example, if $\varepsilon = 0$ we have, if $x \neq 0$,

$$\begin{aligned}c_0 c_0 x &= c_0((p_0 x)\,\square\,(p_1 1)) = (p_0[(p_0 x)\,\square\,(p_1 1)])\,\square\,(p_1 1)\\ &= (p_0 x)\,\square\,(p_1 1) = c_0 x.\end{aligned}$$

(3) $c_\varepsilon(-c_\varepsilon x) = -c_\varepsilon x$ for $\varepsilon \in 2$.

Again we take only $\varepsilon = 0$, and assume that $-p_0 x\cdot p_0 1 \neq 0$:

$$\begin{aligned}c_0(-c_0 x) &= c_0(-[(p_0 x)\,\square\,(p_1 1)])\\ &= c_0[(-p_0 x\cdot p_0 1)\,\square\,(p_1 1)] && \text{by } 5.2.3(\text{xi})\\ &= (p_0[(-p_0 x\cdot p_0 1)\,\square\,(p_1 1)])\,\square\,(p_1 1)\\ &= (-p_0 x\cdot p_0 1)\,\square\,(p_1 1) && \text{by } (P_8),(P_9)\\ &= -c_0 x && \text{by } 5.2.3(\text{xi}).\end{aligned}$$

Now, returning to the verification of (C_3), we have $x\cdot c_0 y \leqq c_0 x$ by (C_2), hence $c_0(x\cdot c_0 y) \leqq c_0 x$ by (1),(2). Similarly, $c_0(x\cdot c_0 y) \leqq c_0 y$, so $c_0(x\cdot c_0 y) \leqq c_0 x\cdot c_0 y$. Conversely,

$$\begin{aligned}
c_0x &= (p_0x)\,\square\,(p_1 1) = [p_0(x\cdot c_0y)]\,\square\,(p_1 1) + [p_0(x\cdot -c_0y)]\,\square\,(p_1 1) \quad \text{by } (P_3),(P_{12})\\
&= c_0(x\cdot c_0y) + c_0(x\cdot -c_0y)\\
&\leqq c_0(x\cdot c_0y) + c_0(-c_0y) \quad \text{by } (1)\\
&= c_0(x\cdot c_0y) + -c_0y.
\end{aligned}$$

Hence (C_3) follows (using symmetry for c_1).

To establish (C_4) and simplicity it suffices to show that $c_0c_1x = 1$ if $x \neq 0$. We have

$$\begin{aligned}
c_0c_1x &= c_0[(p_0 1)\,\square\,(p_1x)]\\
&= (p_0[(p_0 1)\,\square\,(p_1x)])\,\square\,(p_1 1)\\
&= (p_0 1)\,\square\,(p_1 1) = 1.
\end{aligned}$$

Thus $\mathfrak{A}^c$ is a simple Df_2. Next we check (iii). Suppose, for example, that $x \leqq c_0a$. We may assume that $x \neq 0$. Now $x \leqq (p_0a)\,\square\,(p_1 1) = a\,\square\,(p_1 1) = p_1 1$. Hence $p_0x \leqq p_0p_1 1 = a$, so $p_0x = a$. Hence

$$\begin{aligned}
c_0x\cdot c_1x &= [(p_0x)\,\square\,(p_1 1]\cdot[(p_0 1)\,\square\,(p_1x)]\\
&= [a\,\square\,(p_1 1)]\cdot[(p_0 1)\,\square\,(p_1x)]\\
&= (p_1 1)\cdot[(p_0 1)\,\square\,(p_1x)]\\
&= p_1x \quad \text{by } 5.2.3(\text{ix})\\
&= x \quad \text{by } (P_7).
\end{aligned}$$

So $\mathfrak{A}^c \in K$. Finally, let p_0', p_1', $\square'$ be the operations in $\mathfrak{A}^{cp}$. For any x we have

$$\begin{aligned}
p_0'x &= c_0x\cdot c_1a = [(p_0x)\,\square\,(p_1 1)]\cdot[(p_0 1)\,\square\,(p_1a)]\\
&= (p_0x)\,\square\,(p_1a) = x\,\square\,a = p_0x.
\end{aligned}$$

Similarly, $p_1'x = p_1x$. Finally,

$$\begin{aligned}
x\,\square'\,y &= c_0x\cdot c_1y = [(p_0x)\,\square\,(p_1 1)]\cdot[(p_0 1)\,\square\,(p_1y)]\\
&= (p_0x)\,\square\,(p_1y) = x\,\square\,y.
\end{aligned}$$

5.3 RELATION ALGEBRAS

Relation algebras were historically the first algebraic version of a portion of predicate logic to be studied. In fact, their extensive study dates back at least to Schroder [1890], although the modern abstract development dates from Tarski [41]. Some of the most important works concerning these algebras are Chin, Tarski [51], Jónsson, Tarski [52], Jónsson [59], [82'], Lyndon [50], [56], [61], Maddux [81'], [82'], Monk [64], and McKenzie [70']. Their theory is rather extensive and well-developed. Our purpose in this section is modest. We show that relation algebras correspond closely to certain CA_3's, so that their study could, in principle, be reduced to the study of a simply defined class of CA_3's; see 5.3.17.

The development here is self-contained. In particular, before working to establish this connection with CA_3's we shall develop enough of the arithmetic and algebraic theory of relation algebras to support the development of this connection.

DEFINITION 5.3.1. *A relation algebra* (RA) *is a system* $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, ;, \breve{}, 1' \rangle$ *such that* $\langle A, +, \cdot, -, 0, 1 \rangle$ *is a* BA, ; *is a binary operation on* A, $\breve{}$ *is a unary operation on* A, $1' \in A$, *and for all* $x, y, z \in A$ *the following conditions hold:*

(R_1) $(x;y);z = x;(y;z)$

(R_2) $(x+y);z = (x;z)+(y;z)$

(R_3) $x;1' = x$

(R_4) $x\breve{}\breve{} = x$

(R_5) $(x+y)\breve{} = x\breve{}+y\breve{}$

(R_6) $(x;y)\breve{} = y\breve{};x\breve{}$

(R_7) $x\breve{};[-(x;y)] \leq -y$

To reduce parentheses, let us assume that ; binds more closely than $+$ or $\cdot$. Thus $(x;z)+(y;z) = x;z+y;z$ and $y;1\cdot 1' = (y;1)\cdot 1'$.

The motivation for the definition of RA comes from the set-theoretical calculus of binary relations, as indicated in the following definition.

DEFINITION 5.3.2. *Let* U *be any set. A relation set algebra* (Rs) *with base* U *is a system* $\mathfrak{A} = \langle A, \cup, \cap, {}_V\sim, 0, {}^2U, |, {}^{-1}, U \upharpoonright Id \rangle$ *such that* A *is a collection of subsets of* $V = {}^2U$ *closed under* $\cup$, $\cap$, ${}_V\sim$, $|$ *and* ${}^{-1}$ *and with* $U \upharpoonright Id \in A$ *(recall the definitions of* $|$ *and* ${}^{-1}$ *from p.28 of Part I). If* A *consists of all subsets of* ${}^2U$ *we call* $\mathfrak{A}$ **full**. *An* RA $\mathfrak{A}$ *is* **representable** *if* $\mathfrak{A}$ *is isomorphic to a subdirect product of* Rs's; RRA *is the class of all representable relation algebras.*

It is easy to check that every Rs and every RRA is an RA.

Elementary arithmetic

Now we give the elementary arithmetic needed for the rest of this section, in 5.3.3–5.3.5. Further arithmetic is found in Schroder [1890], Chin, Tarski [51], Monk [61b], and Maddux [82']. For these results we assume that we deal with an arbitrary RA and elements in it.

LEMMA 5.3.3.(i) $x;(y+z) = x;y+x;z$.
(ii) $1'^{\smile} = 1'$.
(iii) $1';x = x$.
(iv) *If* $x \leqq y$, *then* $x^{\smile} \leqq y^{\smile}$.
(v) $(x \cdot y)^{\smile} = x^{\smile} \cdot y^{\smile}$.
(vi) $0^{\smile} = 0$.
(vii) $1^{\smile} = 1$.
(viii) $^{\smile}$ *is a Boolean automorphism.*
(ix) $[-(y;x)];x^{\smile} \leqq -y$.

PROOF. For (i) we calculate:

$$\begin{aligned} x;(y+z) &= [x;(y+z)]^{\smile\smile} \\ &= [(y+z)^{\smile};x^{\smile}]^{\smile} \\ &= [(y^{\smile}+z^{\smile});x^{\smile}]^{\smile} \\ &= [y^{\smile};x^{\smile}+z^{\smile};x^{\smile}]^{\smile} \\ &= (y^{\smile};x^{\smile})^{\smile}+(z^{\smile};x^{\smile})^{\smile} \\ &= x;y+x;z. \end{aligned}$$

Concerning (ii) we have $1' = 1'^{\smile\smile} = (1'^{\smile};1')^{\smile} = 1'^{\smile};1' = 1'^{\smile}$. For (iii):

$$1';x = (1';x)^{\smile\smile} = (x^{\smile};1')^{\smile} = x^{\smile\smile} = x.$$

For (iv): $x \leqq y$ implies $x+y = y$, hence $(x+y)^{\smile} = x^{\smile}+y^{\smile} = y^{\smile}$, so $x^{\smile} \leqq y^{\smile}$. Now $x \cdot y \leqq x$, so $(x \cdot y)^{\smile} \leqq x^{\smile}$. Similarly, $(x \cdot y)^{\smile} \leqq y^{\smile}$, so $(x \cdot y)^{\smile} \leqq x^{\smile} \cdot y^{\smile}$. Applying this to $x^{\smile}$ and $y^{\smile}$, we get $(x^{\smile} \cdot y^{\smile})^{\smile} \leqq x \cdot y$, hence $x^{\smile} \cdot y^{\smile} \leqq (x \cdot y)^{\smile}$. This proves (v). For (vi) and (vii):

$$0^{\smile} = 0+0^{\smile} = 0^{\smile\smile}+0^{\smile} = (0^{\smile}+0)^{\smile} = 0^{\smile\smile} = 0.$$
$$1 = 1+1^{\smile} = 1^{\smile\smile}+1^{\smile} = (1^{\smile}+1)^{\smile} = 1^{\smile}.$$

Now (viii) follows in an obvious fashion. Finally, by (R_7) we have $x^{\smile\smile};[-(x^{\smile};y^{\smile})] \leqq -y^{\smile}$, so $[-(y;x)];x^{\smile} \leqq -y$, as desired.

LEMMA 5.3.4. *Let* $\mathfrak{A}$ *be a relation algebra, as in* 5.3.1. *For all* $x,y \in A$ *let* $x:y = y;x$. *Then* $\langle A,+,\cdot,-,0,1,:,^{\smile},1'\rangle$ *is also an* RA.

PROOF. By 5.3.3(i), (iii), (ix).

Lemma 5.3.4 implies a duality principle which we shall not formulate explicitly but will apply when needed. For example, (R_7) and 5.3.3(ix) are dual statements.

LEMMA 5.3.5. (i) *If* $u \leqq v$, *then* $u;x \leqq v;x$.
(ii) *The following conditions are equivalent:* $x;y \cdot z = 0$, $x^{\smile};z \cdot y = 0$, $z;y^{\smile} \cdot x = 0$.
(iii) *For atoms* x,y,z, *the following conditions are equivalent:* $z \leqq x;y$, $y \leqq x^{\smile};z$, $x \leqq z;y^{\smile}$.
(iv) *If* ΣX *exists, then so does* $\Sigma_{x \in X}(x;y)$, *and* $(\Sigma X);y = \Sigma_{x \in X}(x;y)$.
(v) $0;y = 0$.
(vi) $x;y \cdot z \leqq x;(x^{\smile};z \cdot y)$.
(vii) $x \leqq x;x^{\smile};x$.
(viii) $x \cdot 1' = (x \cdot 1');(x \cdot 1')^{\smile}$
(ix) $(x \cdot 1')^{\smile} = x \cdot 1'$.
(x) $x \cdot 1' = (x \cdot 1');(x \cdot 1')$.
(xi) $-[(x \cdot 1');1] = (-x \cdot 1');1$.
(xii) *If* u *is an atom, then so is* $u;1 \cdot 1'$.
(xiii) $(x;1 \cdot 1');1 = x;1$.

PROOF. For (i), assume that $u \leqq v$. So, $u + v = v$ and $u;x \leqq v;x$ follows easily from (R_2). Next, to prove (ii) first assume that $x;y \cdot z = 0$. Thus $z \leqq -(x;y)$, so

$$x^{\smile};z \leqq x^{\smile};[-(x;y)] \leqq -y,$$

and hence $x^{\smile};z \cdot y = 0$. This argument, applied to $x^{\smile}$, z, y rather than x, y, z shows that $x^{\smile};z \cdot y = 0$ implies $x;y \cdot z = 0$. Thus the equivalence $x;y \cdot z = 0$ iff $x^{\smile};z \cdot y = 0$ holds in all RA's, so its dual "$x;y \cdot z = 0$ iff $z;y^{\smile} \cdot x = 0$" does too, and (ii) holds. (iii) is an obvious consequence of (ii).

Now assume that ΣX exists. For any $x \in X$ we have $x \leqq \Sigma X$, hence $x;y \leqq (\Sigma X);y$. Thus $(\Sigma X);y$ is an upper bound for $\{x;y : x \in X\}$. Suppose z is any other upper bound. Then $x;y \cdot -z = 0$ hence $-z;y^{\smile} \cdot x = 0$ for any $x \in X$. Thus $-z;y^{\smile} \cdot \Sigma X = 0$, so $(\Sigma X);y \cdot -z = 0$, or $(\Sigma X);y \leqq z$, establishing (iv). Taking $X = 0$, we get (v). For (vi), note that

$$x;[-(x^{\smile};z) \cdot y] \leqq x;y \cdot -z$$

using (R_7), and hence

$$\begin{aligned} x;y \cdot z &= (x;[-(x^{\smile};z) \cdot y] + x;[(x^{\smile};z) \cdot y]) \cdot z \\ &= x;[(x^{\smile};z) \cdot y] \cdot z \leqq x;(x^{\smile};z \cdot y). \end{aligned}$$

Now (vii) follows:

$$x = x;1' \cdot x \leqq x;(x^{\smile};x \cdot 1') \leqq x;x^{\smile};x.$$

For (viii) we have:

$$x \cdot 1' \leqq (x \cdot 1');(x \cdot 1')^{\smile};(x \cdot 1') \leqq (x \cdot 1');(x \cdot 1')^{\smile} = (x \cdot 1');(x^{\smile} \cdot 1') \leqq x \cdot 1',$$

from which the desired conclusion follows. Applying $^{\smile}$ to (viii), (ix) follows. (x) is a consequence of (viii) and (ix).

To prove (xi), note that $(x \cdot 1');(-x \cdot 1') \leqq x;1' = x$ and similarly $(x \cdot 1');(-x \cdot 1') \leqq -x$, so $(x \cdot 1);(-x \cdot 1') = 0$. Since $1;1 = 1$, we get $(x \cdot 1');(-x \cdot 1') \cdot 1;1 = 0$, *hence* $(x \cdot 1');1;1 \cdot -x \cdot 1' = 0$ hence $(-x \cdot 1');1 \cdot (x \cdot 1');1 = 0$. But also

$$(-x\cdot 1');1+(x\cdot 1');1=1';1=1\,,$$

so (xi) is true. For (xii), note that $u^{\smile}$ is also an atom, and $u^{\smile};1'\cdot 1=u^{\smile}\neq 0$, so $u;1\cdot 1'\neq 0$. Now suppose that $u;1\cdot 1'\cdot x\neq 0$. Hence $(x\cdot 1');1\cdot u\neq 0$, so $u\leqq(x\cdot 1');1$. By (xi), $u\cdot(-x\cdot 1');1=0$, so $-x\cdot 1'\cdot u;1=0$, proving (xii). Finally, we have $-(x;1)\cdot 1'\cdot x;1=0$, hence $-(x;1)\cdot 1'\cdot x;1;1=0$ and so $x;1\cdot[-(x;1)\cdot 1'];1=0$. Hence by (xi),

$$x;1\leqq(x;1\cdot 1');1\leqq x;1\,,$$

and (xiii) is proved.

Algebraic theory

This finishes our brief development of the arithmetic of relation algebras. We need one major result in the algebraic theory of RA's. It is the RA version of one of the main theorems in section 2.7. This result was first proved in Jónsson, Tarski [52], but we follow a suggestion in McKenzie [66].

THEOREM 5.3.6. *Any* RA *can be embedded in a complete atomic* RA.

PROOF. Let $\mathfrak{A}$ be an arbitrary RA, and let X be the set of all ultrafilters on $\mathfrak{B}\mathfrak{l}\mathfrak{A}$. (In analogy to CA's, $\mathfrak{B}\mathfrak{l}\mathfrak{A}$ is the Boolean reduct of $\mathfrak{A}$.) We denote by f the Stone isomorphism of $\mathfrak{B}\mathfrak{l}\mathfrak{A}$ into the complete atomic BA $\mathfrak{S}\mathfrak{b}X$. Recall that f is defined as follows: for any $a\in A$,

$$fa=\{F\in X: a\in F\}.$$

Now we define operations ; and $^{\smile}$ on SbX. For any $M,N\subseteq X$ let

$$M;N=\{F\in X:\text{there exist } G\in M \text{ and } H\in N \text{ such that } \{x;y: x\in G,\ y\in H\}\subseteq F\}.$$

For any $M\subseteq X$ let

$$M^{\smile}=\{F\in X:\text{there is a } G\in M \text{ such that } \{x^{\smile}: x\in G\}\subseteq F\}.$$

Finally, we let $\mathfrak{B}=\langle SbX,\cup,\cap,{}_X\sim,0,X,;,^{\smile},f1'\rangle$. The rest of the proof will consist in showing that f is an isomorphism of $\mathfrak{A}$ into $\mathfrak{B}$, and $\mathfrak{B}$ is an RA (which is obviously complete and atomic.)

For the first part of this task, we only need to check that f preserves $^{\smile}$ and ;. Let $a\in A$, and first suppose that $F\in fa^{\smile}$. Set $G=\{x\in A: x^{\smile}\in F\}$. Since $^{\smile}$ is an automorphism of $\mathfrak{B}\mathfrak{l}\mathfrak{A}$, G is an ultrafilter on $\mathfrak{B}\mathfrak{l}\mathfrak{A}$. Now $a^{\smile}\in F$, so $a\in G$ and hence $G\in fa$. Therefore $F\in(fa)^{\smile}$. The converse is even easier; so f preserves $^{\smile}$. To show that f preserves ;, suppose that $a,b\in A$. First suppose that $F\in(fa);(fb)$. Choose $G\in fa$ and $H\in fb$ so that $\{x;y: x\in G,\ y\in H\}\subseteq F$. Now $a\in G$ and $b\in H$, so $a;b\in F$ and hence $F\in f(a;b)$, as desired. Conversely, suppose that $F\in f(a;b)$, so that $a;b\in F$. Let G be a proper filter, maximal subject to the conditions $a\in G$ and $x;b\in F$ for all $x\in G$. (G exists by Zorn's lemma.) We claim that G is an ultrafilter. For, suppose that $x\notin G$; we show that $-x\in G$. Let H be the filter generated by $G\cup\{x\}$. If $\forall y\in G((x\cdot y);b\in F)$, then H is a proper filter such that $z;b\in F$ for all $z\in H$, contra–

dicting the maximal condition on G. So, choose $y \in G$ such that $(x \cdot y);b \notin F$. Now for any $z \in G$ we have $(x \cdot y \cdot z);b \leq (x \cdot y);b$, so $(x \cdot y \cdot z);b \notin F$; but $(y \cdot z);b = (x \cdot y \cdot z);b + (-x \cdot y \cdot z);b$ and $(y \cdot z);b \in F$ since $y \cdot z \in G$, so $(-x \cdot y \cdot z);b \in F$. By the maximal condition on G it follows that $-x \in G$. Thus, indeed, G is an ultrafilter.

Let H be a proper filter, maximal subject to the conditions $b \in H$ and $\forall x \in G \forall y \in H$ $(x;y \in F)$. Again we claim that H is an ultrafilter. Suppose that $y \notin H$. As above we infer that there exist $x \in G$ and $z \in H$ such that $x;(y \cdot z) \notin F$. For any $u \in G$ and $v \in H$ we have $(u \cdot x);(y \cdot z \cdot v) \notin F$ while $(u \cdot x);(z \cdot v) \in F$, so $(u \cdot x);(-y \cdot z \cdot v) \in F$. Hence $-y \in H$ by the maximal condition on H.

Since $G \in fa$ and $H \in fb$, we have $F \in (fa);(fb)$, as desired.

Next we check the axioms $(R_1)-(R_7)$.

(R_1). Suppose $M,N,P \subseteq X$, and $F \in M;(N;P)$. Say $G \in M$, $H \in N;P$, and $\{x;y : x \in G, y \in H\} \subseteq F$. Also say $K \in N$, $L \in P$, $\{x;y : x \in K, y \in L\} \subseteq H$. Now

(1) $\{x;y : x \in G, y \in K\}$ has the finite intersection property.

For, if $x \in G$ and $y \in K$, then $1 \in L$ implies that $y;1 \in H$, hence $x;y;1 \in F$, hence $x;y \neq 0$. Now if $x_1,\ldots,x_n \in G$ and $y_1,\ldots,y_n \in K$, then

$$x_1;y_1 \cdot \ldots \cdot x_n;y_n \geq (x_1 \cdot \ldots \cdot x_n);(y_1 \cdot \ldots \cdot y_n) \neq 0,$$

so (1) holds.

Now note that if $x \in G$, $y \in K$, and $z \in L$, then $y;z \in H$ and $x;y;z \in F$. Hence by (1) and Zorn's lemma there is a proper filter J maximal subject to the conditions $\forall x \in G \forall y \in K(x;y \in J)$ and $\forall x \in J \forall y \in L(x;y \in F)$. Once more we claim that J is an ultrafilter. The proof goes as before. This shows that $F \in (M;N);P$. Thus $M;(N;P) \subseteq (M;N);P$. The other inclusion is entirely analogous, so (R_1) is proved.

(R_2) is completely straightforward.

(R_3). Suppose $M \subseteq X$. We want to show that $M;f1' = M$. Let $F \in M;f1'$. Say $G \in M$, $H \in f1'$, and $\{x;y : x \in G, y \in H\} \subseteq F$. Then $1' \in H$. Hence for any $x \in G$ we have $x = x;1' \in F$. Therefore $F = G \in M$. Now let $F \in M$. Let H be a proper filter maximal subject to the conditions $1' \in H$ and $\forall x \in F \forall y \in H(x;y \in F)$. As above, H is an ultrafilter. Hence $F \in M;f1'$, and (R_3) holds.

(R_4) and (R_5) are obvious since $\breve{}$ is an automorphism of $\mathfrak{BA}$.

(R_6). Suppose $M,N \subseteq X$. It suffices, by (R_4), to show that $(M;N)\breve{} \subseteq N\breve{};M\breve{}$. Let $F \in (M;N)\breve{}$. Say $G \in M;N$ and $\{x\breve{} : x \in G\} \subseteq F$. Say $H \in M$, $K \in N$, and $\{x;y : x \in H, y \in K\} \subseteq G$. Let $H' = \{x\breve{} : x \in H\}$, $K' = \{x\breve{} : x \in K\}$. Since $\breve{}$ is an automorphism of $\mathfrak{BA}$, we have $H',K' \in X$, hence $H' \in M\breve{}$ and $K' \in N\breve{}$. If $x \in H'$ and $y \in K'$, then $x\breve{} \in H$, $y\breve{} \in K$, hence $x\breve{};y\breve{} \in G$, and $y;x = (x\breve{};y\breve{})\breve{} \in F$. Thus $F \in N\breve{};M\breve{}$, as desired.

(R_7). Suppose $M,N \subseteq X$. We want to show that $M\breve{};[X \sim (M;N)] \cap N = 0$; suppose on the contrary that F is a member of this set. Thus $F \in N$, and there exist $G \in M\breve{}$, $H \in X \sim (M;N)$ such that $\{x;y : x \in G, y \in H\} \subseteq F$. Let $K = \{x\breve{} : x \in G\}$. Again, $K \in X$ and hence $K \in M$. Since $H \notin M;N$, it follows that there is an $x \in K$ and a $y \in F$ such that $x;y \notin H$. Hence $-(x;y) \in H$. Since $x\breve{} \in G$, we have $x\breve{};[-(x;y)] \in F$. But $x\breve{};[-(x;y)] \leq -y$, so $-y \in F$, contradiction.

This completes the proof of Theorem 5.3.6.

CA_α's and RA's

Now we introduce the standard method of associating an RA with a CA_α, due to Henkin and Tarski.

DEFINITION 5.3.7. *Let $\mathfrak{A}$ be a CA_α, $\alpha \geqq 3$. We set $\mathfrak{Ra}\mathfrak{A} = \langle Nr_2\mathfrak{A}, +, \cdot, -, 0, 1, ;, \breve{}, \mathsf{d}_{01}\rangle$, where for any $x, y \in Nr_2\mathfrak{A}$ we set*

$$x;y = \mathsf{c}_2(\mathsf{s}_2^1 x \cdot \mathsf{s}_2^0 y),$$
$$x\breve{} = {}_2\mathsf{s}(0,1)x.$$

The following result is due to Henkin and Tarski.

THEOREM 5.3.8. *If $\mathfrak{A}$ is a CA_α, $\alpha \geqq 4$, then $\mathfrak{Ra}\mathfrak{A}$ is an RA.*

PROOF. Clearly $Nr_2\mathfrak{A}$ is closed under ; and $\breve{}$, and so $\mathfrak{Ra}\mathfrak{A}$ is an algebra similar to RA's. We must check (R_1)–(R_7). We use some theorems from section 1.5 without explicit citation. Let x, y, z be arbitrary elements of $Nr_2\mathfrak{A}$. First we note:

(*) $\mathsf{c}_2(\mathsf{s}_2^1 x \cdot \mathsf{s}_2^0 y) = \mathsf{c}_3(\mathsf{s}_3^1 x \cdot \mathsf{s}_3^0 y)$

For,

$$\begin{aligned}
\mathsf{c}_2(\mathsf{s}_2^1 x \cdot \mathsf{s}_2^0 y) &= \mathsf{c}_2(\mathsf{s}_2^1 \mathsf{c}_2\mathsf{c}_3 x \cdot \mathsf{s}_2^0 \mathsf{c}_2\mathsf{c}_3 y)\\
&= \mathsf{c}_2(\mathsf{s}_2^3\mathsf{s}_3^1 \mathsf{c}_2\mathsf{c}_3 x \cdot \mathsf{s}_2^3\mathsf{s}_3^0 \mathsf{c}_2\mathsf{c}_3 y)\\
&= \mathsf{c}_2\mathsf{s}_2^3(\mathsf{s}_3^1 x \cdot \mathsf{s}_3^0 y)\\
&= \mathsf{c}_3(\mathsf{s}_3^1 x \cdot \mathsf{s}_3^0 y).
\end{aligned}$$

Now we use (*) to check (R_1)–(R_7):

$$\begin{aligned}
x;(y;z) &= \mathsf{c}_2[\mathsf{s}_2^1 x \cdot \mathsf{s}_2^0 \mathsf{c}_2(\mathsf{s}_2^1 y \cdot \mathsf{s}_2^0 z)]\\
&= \mathsf{c}_2[\mathsf{s}_2^1 x \cdot \mathsf{s}_2^0 \mathsf{c}_3(\mathsf{s}_3^1 y \cdot \mathsf{s}_3^0 z)]\\
&= \mathsf{c}_2\mathsf{c}_3[\mathsf{s}_2^1 x \cdot \mathsf{s}_2^0\mathsf{s}_3^1 y \cdot \mathsf{s}_3^0 z]\\
&= \mathsf{c}_3[\mathsf{c}_2(\mathsf{s}_2^1 x \cdot \mathsf{s}_2^0\mathsf{s}_3^1 y) \cdot \mathsf{s}_3^0 z]\\
&= \mathsf{c}_3[\mathsf{s}_3^1\mathsf{c}_2(\mathsf{s}_2^1 x \cdot \mathsf{s}_2^0 y) \cdot \mathsf{s}_3^0 z]\\
&= (x;y);z.
\end{aligned}$$

$$(x+y);z = \mathsf{c}_2[\mathsf{s}_2^1(x+y) \cdot \mathsf{s}_2^0 z] = x;z + y;z.$$

$$x;1' = \mathsf{c}_2(\mathsf{s}_2^1 x \cdot \mathsf{s}_2^0 \mathsf{d}_{01}) = \mathsf{c}_2(\mathsf{s}_2^1 x \cdot \mathsf{d}_{12}) = \mathsf{c}_2(x \cdot \mathsf{d}_{12}) = x.$$

$$(x+y)\breve{} = {}_2\mathsf{s}(0,1)(x+y) = x\breve{} + y\breve{}.$$

$$x\breve{}\breve{} = {}_2\mathsf{s}(0,1)\,{}_2\mathsf{s}(0,1)x = x.$$

$$\begin{aligned}(x;y)^{\smile} &= {}_2\mathsf{s}(0,1)\mathsf{c}_2(\mathsf{s}_2^1x\cdot\mathsf{s}_2^0y)\\ &= {}_2\mathsf{s}(0,1)\mathsf{c}_3(\mathsf{s}_3^1x\cdot\mathsf{s}_3^0y)\\ &= \mathsf{c}_3({}_2\mathsf{s}(0,1)\mathsf{s}_3^1x\cdot{}_2\mathsf{s}(0,1)\mathsf{s}_3^0y)\\ &= \mathsf{c}_3(\mathsf{s}_0^2\mathsf{s}_1^0\mathsf{s}_2^1\mathsf{s}_3^1x\cdot\mathsf{s}_0^2\mathsf{s}_1^0\mathsf{s}_2^1\mathsf{s}_3^0y)\\ &= \mathsf{c}_3(\mathsf{s}_1^0\mathsf{s}_3^1x\cdot\mathsf{s}_0^1\mathsf{s}_3^0y)\\ &= \mathsf{c}_2\mathsf{s}_2^3(\mathsf{s}_1^0\mathsf{s}_3^1x\cdot\mathsf{s}_0^1\mathsf{s}_3^0y)\\ &= \mathsf{c}_2(\mathsf{s}_1^0\mathsf{s}_2^1x\cdot\mathsf{s}_0^1\mathsf{s}_2^0y)\\ &= \mathsf{c}_2(\mathsf{s}_2^0\mathsf{s}_0^2\mathsf{s}_1^0\mathsf{s}_2^1x\cdot\mathsf{s}_2^1\mathsf{s}_1^2\mathsf{s}_0^1\mathsf{s}_2^0y)\\ &= \mathsf{c}_2(\mathsf{s}_2^0x^{\smile}\cdot\mathsf{s}_2^1y^{\smile}) = y^{\smile};x^{\smile}.\end{aligned}$$

Finally, for (R_7) we note that each of the following conditions are equivalent:

$$\begin{gathered}x^{\smile};[-(x;y)]\cdot y=0.\\ \mathsf{c}_2[\mathsf{s}_2^1\mathsf{s}_1^2\mathsf{s}_0^1\mathsf{s}_2^0x\cdot\mathsf{s}_2^0-\mathsf{c}_2(\mathsf{s}_2^1x\cdot\mathsf{s}_2^0y)]\cdot y=0.\\ \mathsf{s}_0^1\mathsf{s}_2^0x\cdot\mathsf{s}_2^0-\mathsf{c}_2(\mathsf{s}_2^1x\cdot\mathsf{s}_2^0y)\cdot y=0.\\ \mathsf{s}_2^0(\mathsf{s}_3^1x\cdot-\mathsf{c}_2(\mathsf{s}_2^1x\cdot\mathsf{s}_2^0y))\cdot\mathsf{s}_3^0y=0.\\ \mathsf{d}_{02}\cdot\mathsf{s}_3^1x\cdot-\mathsf{c}_2(\mathsf{s}_2^1x\cdot\mathsf{s}_2^0y)\cdot\mathsf{s}_3^0y=0.\end{gathered}$$

Now $\mathsf{d}_{02}\cdot\mathsf{s}_3^1x\cdot\mathsf{s}_3^0y\leqq\mathsf{c}_3(\mathsf{s}_3^1x\cdot\mathsf{s}_3^0y)=\mathsf{c}_2(\mathsf{s}_2^1x\cdot\mathsf{s}_2^0y)$, so (R_7) follows.

COROLLARY 5.3.9. *If* $\mathfrak{A}\in\mathsf{SNr}_3\mathsf{CA}_\alpha$ *with* $\alpha\geqq4$, *then* $\mathfrak{Ra}\mathfrak{A}$ *is an* RA.

PROOF. Say $\mathfrak{A}\subseteq\mathfrak{B}=\mathfrak{Nr}_3\mathfrak{C}$, $\mathfrak{C}\in\mathsf{CA}_\alpha$. Then $\mathfrak{Ra}\mathfrak{A}\subseteq\mathfrak{Ra}\mathfrak{B}=\mathfrak{Ra}\mathfrak{C}\in\mathsf{RA}$ by 5.3.8.

The following elementary result will also be useful in what follows.

THEOREM 5.3.10. Let $\mathfrak{A}$ *be a* CA_α, $\alpha\geqq4$, *and let* $x,y\in\mathfrak{Nr}_2\mathfrak{A}$. Then
(i) $\mathsf{c}_0(\mathsf{s}_2^1x\cdot y)=\mathsf{s}_2^0(x^{\smile};y)$.
(ii) $\mathsf{c}_1(\mathsf{s}_2^0x\cdot y)=\mathsf{s}_2^1(y;x^{\smile})$.

PROOF. We have

$$\begin{aligned}\mathsf{s}_2^0(x^{\smile};y) &= \mathsf{s}_2^0\mathsf{c}_2(\mathsf{s}_2^1\mathsf{s}_1^2\mathsf{s}_0^1\mathsf{s}_2^0x\cdot\mathsf{s}_2^0y)\\ &= \mathsf{s}_2^0\mathsf{c}_2(\mathsf{s}_0^1\mathsf{s}_2^0x\cdot\mathsf{s}_2^0y)\\ &= \mathsf{s}_2^0\mathsf{c}_3\mathsf{s}_3^2(\mathsf{s}_0^1\mathsf{s}_2^0x\cdot\mathsf{s}_2^0y)\\ &= \mathsf{c}_3\mathsf{s}_2^0(\mathsf{s}_0^1\mathsf{s}_3^0x\cdot\mathsf{s}_3^0y)\\ &= \mathsf{c}_0(\mathsf{s}_2^1x\cdot y).\end{aligned}$$

Condition (ii) is proved similarly.

Now we need two lemmas concerning algebraic properties of the operation $\mathfrak{Ra}$.

LEMMA 5.3.11. *Let* $\mathsf{K} \subseteq \mathsf{CA}_\alpha$, $\alpha \geqq 3$. *Then* $\mathfrak{Ra}^*\mathsf{PK} = \mathsf{P}\mathfrak{Ra}^*\mathsf{K}$.

PROOF. First suppose $\mathfrak{A} \in {}^I\mathsf{K}$, so that $\mathsf{P}_{\iota \in I}\mathfrak{A}_\iota$ is a typical member of PK. Now clearly $\mathfrak{Nr}_2\mathsf{P}_{\iota \in I}\mathfrak{A}_\iota = \mathsf{P}_{\iota \in I}\mathfrak{Nr}_2\mathfrak{A}_\iota$. Furthermore, for any $x, y \in Nr_2\mathsf{P}_{\iota \in I}\mathfrak{A}_\iota$ and any $i \in I$ we have

$$\begin{aligned}(x;y)_\iota &= [\mathsf{c}_2(\mathsf{s}_2^1 x \cdot \mathsf{s}_2^0 y)]_\iota \\ &= \mathsf{c}_2(\mathsf{s}_2^1 x_\iota \cdot \mathsf{s}_2^0 y_\iota) \\ &= x_\iota ; y_\iota ,\end{aligned}$$

and similarly for $x^{\smile}$. Hence $\mathfrak{Ra}\mathsf{P}_{\iota \in I}\mathfrak{A}_i = \mathsf{P}_{i \in I}\mathfrak{Ra}\mathfrak{A}_i$, establishing $\mathfrak{Ra}^*\mathsf{PK} \subseteq \mathsf{P}\mathfrak{Ra}^*\mathsf{K}$. The other inclusion is similar.

LEMMA 5.3.12. *Let* $\mathfrak{A} \in \mathsf{SNr}_3\mathsf{CA}_4$ *and* $X \subseteq Nr_2\mathfrak{A}$. *Then* $\mathfrak{Cg}^{\mathfrak{Ra}\mathfrak{A}}X = \mathfrak{Ra}\mathfrak{Cg}^{\mathfrak{A}}X$.

PROOF. Clearly it suffices to show that $Sg^{\mathfrak{Ra}\mathfrak{A}}X = Nr_2\mathfrak{Cg}^{\mathfrak{A}}X$. Let

$$E = \{\mathsf{s}_2^0 x_0 \cdot \mathsf{s}_2^1 x_1 \cdot x_2 : x_0, x_1, x_2 \in Sg^{\mathfrak{Ra}\mathfrak{A}}X\}.$$

Note that E is closed under $\cdot$. Now we claim

(1) $Sg^{\mathfrak{A}}X = \{\Sigma Y : Y \subseteq E,\ Y \text{ finite}\}$.

Clearly $Sg^{\mathfrak{Ra}\mathfrak{A}}X \subseteq Sg^{\mathfrak{A}}X$, and hence the inclusion $\supseteq$ in (1) holds. For the other inclusion, let F be the right side of (1). It suffices to show that $X \subseteq F \in Su\mathfrak{A}$.

Since $1 \in Sg^{\mathfrak{Ra}\mathfrak{A}}X$, it is clear that $X \subseteq E \subseteq F$. To show that $\mathsf{d}_{\kappa\lambda} \in F$ for all $\kappa, \lambda < 3$, observe that $\mathsf{d}_{\kappa\kappa} = 1 \in E$ for any $\kappa < 3$, $\mathsf{d}_{01} \in Sg^{\mathfrak{Ra}\mathfrak{A}}X \in E$, $\mathsf{d}_{02} = \mathsf{s}_2^0 1 \cdot \mathsf{s}_2^1 \mathsf{d}_{01} \cdot 1 \in E$, and $\mathsf{d}_{12} = \mathsf{s}_2^0 \mathsf{d}_{01} \cdot \mathsf{s}_2^1 1 \cdot 1 \in E$.

Clearly F is closed under $+$. To show that F is closed under $-$, it clearly suffices to show that $-y \in F$ for each $y \in E$. And we have, if $x_0, x_1, x_2 \in Sg^{\mathfrak{Ra}\mathfrak{A}}X$,

$$\begin{aligned}-(\mathsf{s}_2^0 x_0 \cdot \mathsf{s}_2^1 x_1 \cdot x_2) &= -\mathsf{s}_2^0 x_0 + -\mathsf{s}_2^1 x_1 + -x_2 \\ &= \mathsf{s}_2^0(-x_0) \cdot \mathsf{s}_2^1 1 \cdot 1 + \mathsf{s}_2^0 1 \cdot \mathsf{s}_2^1(-x_1) \cdot 1 + \mathsf{s}_2^0 1 \cdot \mathsf{s}_2^1 1 \cdot -x_2 \\ &\in F,\end{aligned}$$

as desired.

It remains to show that F is closed under c_0, c_1, and c_2. Clearly it suffices to show that if $y \in E$, then $\mathsf{c}_i y \in F$, for $i = 0,1,2$. So suppose $y = \mathsf{s}_2^0 x_0 \cdot \mathsf{s}_2^1 x_1 \cdot x_2$ with $x_0, x_1, x_2 \in Sg^{\mathfrak{Ra}\mathfrak{A}}X$. Then

$$\begin{aligned}\mathsf{c}_0 y &= \mathsf{s}_2^0 x_0 \cdot \mathsf{c}_0(\mathsf{s}_2^1 x_1 \cdot x_2) \\ &= \mathsf{s}_2^0 x_0 \cdot \mathsf{s}_2^0(x_1^{\smile}; x_2) \text{ by } 5.3.10(\mathrm{i}) \\ &= \mathsf{s}_2^0(x_0 \cdot x_1^{\smile}; x_2) \cdot \mathsf{s}_2^1 1 \cdot 1 \in E.\end{aligned}$$

Similarly, $c_1 y \in E$. Finally,

$$\begin{aligned} c_2 y &= c_2(s_2^0 x_0 \cdot s_2^1 x_1)\cdot x_2 \\ &= x_1;x_0\cdot x_2 \in E. \end{aligned}$$

We have established (1). Note that in the very last part of the proof we have also shown that if $y \in E$ then $c_2 y \in Sg^{\mathfrak{Ra}\mathfrak{A}} X$, and hence

(2) If $y \in Sg^{\mathfrak{A}} X$, then $c_2 y \in Sg^{\mathfrak{Ra}\mathfrak{A}} X$.

For the lemma itself, for any $y \in A$ we have $y \in Nr_2 \mathfrak{Cg}^{\mathfrak{A}} X$ iff $y \in Sg^{\mathfrak{A}} X$ and $c_2 y = y$. Thus by (2), $y \in Nr_2 \mathfrak{Cg}^{\mathfrak{A}} X$ implies that $y \in Sg^{\mathfrak{Ra}\mathfrak{A}} X$. The converse is clear, so the lemma holds.

COROLLARY 5.3.13. *If* $\mathsf{K} \subseteq \mathsf{SNr}_3\mathsf{CA}_4$ *then* $\mathfrak{Ra}^*\mathsf{SK} = \mathsf{S}\mathfrak{Ra}^*\mathsf{K}$.

PROOF. First let $\mathfrak{A} \in \mathfrak{Ra}^*\mathsf{SK}$. Say $\mathfrak{A} = \mathfrak{Ra}\mathfrak{B}$, $\mathfrak{B} \subseteq \mathfrak{C} \in \mathsf{K}$. Clearly $\mathfrak{A} \subseteq \mathfrak{Ra}\mathfrak{C}$, so $\mathfrak{A} \in \mathsf{S}\mathfrak{Ra}^*\mathsf{K}$. Second let $\mathfrak{A} \in \mathsf{S}\mathfrak{Ra}^*\mathsf{K}$. Say $\mathfrak{A} \subseteq \mathfrak{B} = \mathfrak{Ra}\mathfrak{C}$, $\mathfrak{C} \in \mathsf{K}$. Then $\mathfrak{A} = \mathfrak{Cg}^{\mathfrak{Ra}\mathfrak{C}} A = \mathfrak{Ra}\mathfrak{Cg}^{\mathfrak{C}} A$ by Lemma 5.3.12, and $\mathfrak{Cg}^{\mathfrak{C}} A \subseteq \mathfrak{C} \in \mathsf{K}$. Thus $\mathfrak{A} \in \mathfrak{Ra}^*\mathsf{SK}$, as desired.

DEFINITION 5.3.14. $\mathsf{M} = \{\mathfrak{A} \in \mathsf{SNr}_3\mathsf{CA}_4 : \mathfrak{A} = \mathfrak{Cg}^{\mathfrak{A}}\{x \in A : \Delta x \subseteq 2\}\}$.

Finally we can formulate precisely the main result of this section: it is that $\mathfrak{Ra}$ maps M onto RA, and induces a one–one mapping of isomorphism types of members of M onto isomorphism types of RA's. Obviously $\mathfrak{A} \cong \mathfrak{B}$ implies that $\mathfrak{Ra}\mathfrak{A} \cong \mathfrak{Ra}\mathfrak{B}$. The converse, for members of M, is given in our next result.

LEMMA 5.3.15. *Let* $\mathfrak{A},\mathfrak{B} \in \mathsf{M}$ *and suppose that* $f \in Is(\mathfrak{Ra}\mathfrak{A},\mathfrak{Ra}\mathfrak{B})$. *Then there is a* $g \in Is(\mathfrak{A},\mathfrak{B})$ *such that* $f \subseteq g$.

PROOF. Let $g = Sg^{\mathfrak{A}\times\mathfrak{B}} f$. By 0.3.47 it suffices to show that g is a one–one function with domain A and range B. Since $Dog = DoSg^{\mathfrak{A}\times\mathfrak{B}} f = Sg^{\mathfrak{A}} Dof = Sg^{\mathfrak{A}} Nr_2 A$, we have $Dog = A$ from the definition of M. Similarly, $Rgg = B$.

By symmetry it remains only to show that g is a function. This depends on the following fact:

(1) If $\langle a,b\rangle \in g$ and $c_2\langle a,b\rangle = \langle a,b\rangle$, then $fa = b$.

In fact, say (by the definition of M) $\mathfrak{A} \subseteq \mathfrak{C} \in \mathsf{Nr}_3\mathsf{CA}_4$ and $\mathfrak{B} \subseteq \mathfrak{D} \in \mathsf{Nr}_3\mathsf{CA}_4$. Note that $\mathfrak{Ra}\mathfrak{A} \subseteq \mathfrak{Ra}\mathfrak{C}$, $\mathfrak{Ra}\mathfrak{B} \subseteq \mathfrak{Ra}\mathfrak{D}$, $\mathfrak{A}\times\mathfrak{B} \subseteq \mathfrak{C}\times\mathfrak{D}$, and $\mathfrak{Ra}(\mathfrak{C}\times\mathfrak{D}) = \mathfrak{Ra}\mathfrak{C}\times\mathfrak{Ra}\mathfrak{D}$. Hence

$$\begin{aligned} \langle a,b\rangle \in Nr_2\mathfrak{Cg}^{\mathfrak{A}\times\mathfrak{B}} f &= Nr_2\mathfrak{Cg}^{\mathfrak{C}\times\mathfrak{D}} f \\ &= Sg^{\mathfrak{Ra}(\mathfrak{C}\times\mathfrak{D})} f \qquad \text{by } 5.3.12 \\ &= Sg^{\mathfrak{Ra}\mathfrak{C}\times\mathfrak{Ra}\mathfrak{D}} f = f, \end{aligned}$$

as desired.

Now suppose $\langle x,y\rangle,\langle x,z\rangle \in g$. Then

$$\begin{aligned}\langle 0,\mathsf{c}_2(y\oplus z)\rangle &= \langle \mathsf{c}_2 0,\mathsf{c}_2(y\oplus z)\rangle\\ &= \mathsf{c}_2\langle 0,y\oplus z\rangle = \mathsf{c}_2(\langle x,y\rangle\oplus\langle x,z\rangle)\in g.\end{aligned}$$

Furthermore, $\mathsf{c}_2\langle 0,\mathsf{c}_2(y\oplus z)\rangle = \langle 0,\mathsf{c}_2(y\oplus z)\rangle$. Hence by (1) we conclude that $f0=\mathsf{c}_2(y\oplus z)$, so $\mathsf{c}_2(y\oplus z)=0$, hence $y=z$, as desired.

We are now ready to prove our main result for RRA. It summarizes some results of Maddux and Monk.

THEOREM 5.3.16. *The class* RRA *of representable relation algebras is related as follows to the class* IGs_3 *of representable cylindric algebras:*

$$\mathsf{RRA}=\mathfrak{Ra}^*\mathsf{IGs}_3=\mathfrak{Ra}^*\{\mathfrak{A}\in\mathsf{IGs}_3:\mathfrak{A}=\mathfrak{Cg}^{\mathfrak{A}}Nr_2\mathfrak{A}\}=\bigcap_{\alpha<\omega}\mathfrak{Ra}^*\mathsf{SNr}_3\mathsf{CA}_{4+\alpha}.$$

PROOF. First we claim:

(1) If U is a non-empty set, $\mathfrak{A}$ is the full Rs with base U, and $\mathfrak{B}$ is the full Cs_3 with base U then $\mathfrak{A}\cong\mathfrak{Ra}\mathfrak{B}$.

In fact, for each $X\subseteq{}^2U$ let $fX=\{u\in{}^3U:2\upharpoonright u\in X\}$. It is routine to check that f is the desired isomorphism. Now let K be the class of all RA's isomorphic to full Rs's, L the class of all CA_3's isomorphic to full Cs_3's. Then $\mathsf{K}=\mathfrak{Ra}^*\mathsf{L}$ by (1). Hence

$$\begin{aligned}\mathfrak{Ra}^*\mathsf{IGs}_3 &= \mathfrak{Ra}^*\mathsf{SPL}\\ &= \mathsf{SP}\mathfrak{Ra}^*\mathsf{L} && (5.3.11,\ 5.3.13)\\ &= \mathsf{SPK}\\ &= \mathsf{RRA} && (\text{definition of RRA}).\end{aligned}$$

Next, if $\mathfrak{A}\in\mathsf{IGs}_3$ let $\mathfrak{B}=\mathfrak{Cg}^{\mathfrak{A}}Nr_2\mathfrak{A}$. Then $\mathfrak{B}\in\mathsf{IGs}_3$ and $\mathfrak{B}=\mathfrak{Cg}^{\mathfrak{B}}Nr_2\mathfrak{B}$. Furthermore, $\mathfrak{Ra}\mathfrak{A}=\mathfrak{Ra}\mathfrak{B}$. It follows that $\mathfrak{Ra}^*\mathsf{IGs}_3=\mathfrak{Ra}^*\{\mathfrak{A}\in\mathsf{IGs}_3:\mathfrak{A}=\mathfrak{Cg}^{\mathfrak{A}}Nr_2\mathfrak{A}\}$.

Next, note that $\mathsf{IGs}_3\subseteq\mathsf{SNr}_3\mathsf{CA}_{4+\alpha}$ for any $\alpha<\omega$, by 3.1.127. Therefore, $\mathfrak{Ra}^*\mathsf{IGs}_3\subseteq\bigcap_{\alpha<\omega}\mathfrak{Ra}^*\mathsf{SNr}_3\mathsf{CA}_{4+\alpha}$. Conversely suppose that $\mathfrak{A}\in\bigcap_{\alpha<\omega}\mathfrak{Ra}^*\mathsf{SNr}_3\mathsf{CA}_{4+\alpha}$. For each $\alpha<\omega$ choose $\mathfrak{B}_\alpha\in\mathsf{SNr}_3\mathsf{CA}_{4+\alpha}$ such that $\mathfrak{A}=\mathfrak{Ra}\mathfrak{B}_\alpha$. We may assume that $\mathfrak{B}_\alpha=\mathfrak{Cg}Nr_2\mathfrak{B}_\alpha$ for each $\alpha<\omega$. Hence by 5.3.15 we have $\mathfrak{B}_0\cong\mathfrak{B}_\alpha$ for all $\alpha<\omega$. Thus $\mathfrak{B}_0\in\bigcap_{\alpha<\omega}\mathsf{SNr}_3\mathsf{CA}_{4+\alpha}$, so by 3.2.10 we have $\mathfrak{B}_0\in\mathsf{IGs}_3$. Hence $\mathfrak{A}=\mathfrak{Ra}\mathfrak{B}_0\in\mathfrak{Ra}^*\mathsf{IGs}_3$, as desired.

Now we give the main theorem of this section. It is due to Maddux [78a'], improving results of Monk [61b].

THEOREM 5.3.17. $\mathsf{RA}=\mathfrak{Ra}^*\mathsf{M}$.

PROOF. The inclusion $\supseteq$ is a consequence of 5.3.8. For $\subseteq$, it suffices by 5.3.6 and 5.3.12 to show that for any complete atomic RA $\mathfrak{A}$, $\mathfrak{A}\cong\mathfrak{Ra}\mathfrak{B}$ for some $\mathfrak{B}\in\mathsf{M}$. Namely, if this is true and $\mathfrak{C}\in\mathsf{RA}$ is arbitrary, let $\mathfrak{C}\subseteq\mathfrak{A}$, $\mathfrak{A}$ complete and atomic by 5.3.6. Say $\mathfrak{A}=\mathfrak{Ra}\mathfrak{B}$, $\mathfrak{B}\in\mathsf{M}$. Then by 5.3.12, $\mathfrak{C}=\mathfrak{Cg}^{\mathfrak{A}}C=\mathfrak{Cg}^{\mathfrak{Ra}\mathfrak{B}}C=\mathfrak{Ra}\mathfrak{Cg}^{\mathfrak{B}}C$, and $\mathfrak{Cg}^{\mathfrak{B}}C=\mathfrak{Cg}^{\mathfrak{B}}Nr_2\mathfrak{Cg}^{\mathfrak{B}}C$ hence $\mathfrak{Cg}^{\mathfrak{B}}C\in\mathsf{M}$.

So, let $\mathfrak{A} \in RA$ be non-trivial, complete and atomic. We shall construct a CA_4 $\mathfrak{B}$ such that $\mathfrak{A} \cong \mathfrak{Ra}\mathfrak{B}$ by constructing the associated Ca_4 $\mathfrak{C}$ (see 2.7.38). Let

$$C = \{a \in {}^{4\times 4}At\mathfrak{A} : \text{for all } \kappa,\lambda,\mu < 4 \text{ we have: } a_{\kappa\kappa} \leq 1',\ \breve{a}_{\kappa\lambda} = a_{\lambda\kappa},\ a_{\kappa\mu} \leq a_{\kappa\lambda}; a_{\lambda\mu}\}.$$

For $\kappa,\lambda < 4$ we set

$$E_{\kappa\lambda} = \{a \in C : a_{\kappa\lambda} \leq 1'\},$$
$$T_\kappa = \{\langle a,b\rangle \in {}^2C : a_{\lambda\mu} = b_{\lambda\mu} \text{ for all } \lambda,\mu \in 4\sim\{\kappa\}\}.$$

Finally, $\mathfrak{C} = \langle C, T_\kappa, E_{\kappa\lambda}\rangle_{\kappa,\lambda<4}$. First we note that $C \neq 0$; choose an atom $x \leq 1'$, and let $a_{\kappa\lambda} = x$ for all $\kappa,\lambda < 4$. The required conditions are easily checked, using some of the arithmetic laws established at the beginning of this section. Now we proceed to check the conditions of 2.7.40.

Condition (i) is clear. For condition (ii), we need the following fact, which will also be useful later:

(1) If $4 = \{\kappa,\lambda,\mu,\nu\}$, $a,b \in C$, and $a_{\mu\nu} = b_{\mu\nu}$, then $a(T_\lambda | T_\kappa)b$.

To prove this, assume its hypotheses. Then we claim:

(2) $a_{\kappa\mu}; b_{\mu\lambda} \cdot a_{\kappa\nu}; b_{\nu\lambda} \neq 0$.

For, we have $a_{\mu\nu} \leq a_{\mu\kappa}; a_{\kappa\nu}$ and $b_{\mu\nu} \leq b_{\mu\lambda}; b_{\lambda\nu}$. Since $a_{\mu\nu} = b_{\mu\nu}$, it follows that $a_{\mu\kappa}; a_{\kappa\nu} \cdot b_{\mu\lambda}; b_{\lambda\nu} \neq 0$. Hence $\breve{a}_{\mu\kappa}; b_{\mu\lambda}; b_{\lambda\nu} \cdot a_{\kappa\nu} \neq 0$ and $a_{\kappa\nu}; \breve{b}_{\lambda\nu} \cdot \breve{a}_{\mu\kappa}; b_{\mu\lambda} \neq 0$. Thus $a_{\kappa\nu}; b_{\nu\lambda} \cdot a_{\kappa\mu}; b_{\mu\lambda} \neq 0$, as desired.

We now define an element $c \in C$ such that $aT_\lambda cT_\kappa b$, thereby proving (1). Let $c_{\rho\sigma} = a_{\rho\sigma}$ if $\rho,\sigma \neq \lambda$. For $\rho \neq \kappa$ let $c_{\rho\lambda} = b_{\rho\lambda}$, $c_{\lambda\rho} = b_{\lambda\rho}$. Finally, let $c_{\kappa\lambda}$ be an atom $\leq a_{\kappa\mu}; b_{\mu\lambda} \cdot a_{\kappa\nu}; b_{\nu\lambda}$, and set $c_{\lambda\kappa} = \breve{c}_{\kappa\lambda}$. We need to check that $c \in C$. Clearly $c_{\rho\rho} \leq 1'$ and $\breve{c}_{\rho\sigma} = c_{\sigma\rho}$ for all $\rho,\sigma < 4$. If $\rho,\sigma,\eta \in 4\sim\{\lambda\}$, then $c_{\rho\sigma} = a_{\rho\sigma} \leq a_{\rho\eta}; a_{\eta\sigma} = c_{\rho\eta}; c_{\eta\sigma}$. The remaining possibilities work out as follows:

1) If $\rho,\sigma \in 4\sim\{\lambda,\kappa\}$, then $c_{\rho\sigma} = a_{\rho\sigma} = b_{\rho\sigma} \leq b_{\rho\lambda}; b_{\lambda\sigma} = c_{\rho\lambda}; c_{\lambda\sigma}$. Thus by 5.3.5(iii), also $c_{\rho\lambda} \leq c_{\rho\sigma}; c_{\sigma\lambda}$ and $c_{\lambda\sigma} \leq c_{\lambda\rho}; c_{\rho\sigma}$. Similarly, $c_{\lambda\lambda} \leq c_{\lambda\rho}; c_{\rho\lambda}$.

2) If $\rho \in 4\sim\{\lambda,\kappa\}$, then $c_{\kappa\lambda} \cdot a_{\kappa\rho}; b_{\rho\lambda} \neq 0$, hence $c_{\kappa\lambda} \leq c_{\kappa\rho}; c_{\rho\lambda}$, $c_{\kappa\rho} \leq c_{\kappa\lambda}; c_{\lambda\rho}$, $c_{\rho\lambda} \leq c_{\rho\kappa}; c_{\kappa\lambda}$, $c_{\lambda\kappa} \leq c_{\lambda\rho}; c_{\rho\kappa}$, $c_{\rho\kappa} \leq c_{\rho\lambda}; c_{\lambda\kappa}$, $c_{\lambda\rho} \leq c_{\lambda\kappa}; c_{\kappa\rho}$.

3) We have $c_{\kappa\lambda} \leq a_{\kappa\mu}; b_{\mu\lambda} \leq a_{\kappa\kappa}; a_{\kappa\mu}; b_{\mu\lambda} \leq a_{\kappa\kappa}; 1$, so

$$\begin{aligned} c_{\kappa\lambda} &\leq a_{\kappa\kappa}; 1 \cdot c_{\kappa\lambda} \leq a_{\kappa\kappa}; (\breve{a}_{\kappa\kappa}; c_{\kappa\lambda} \cdot 1) \\ &= a_{\kappa\kappa}; a_{\kappa\kappa}; c_{\kappa\lambda} \leq 1'; c_{\kappa\kappa}; c_{\kappa\lambda} = c_{\kappa\kappa}; c_{\kappa\lambda}. \end{aligned}$$

Hence $c_{\kappa\kappa} \leq c_{\kappa\lambda}; c_{\lambda\kappa}$ and $c_{\lambda\kappa} \leq c_{\lambda\kappa}; c_{\kappa\kappa}$. Similarly, $c_{\lambda\kappa} \leq c_{\lambda\lambda}; c_{\lambda\kappa}$, $c_{\lambda\lambda} \leq c_{\lambda\kappa}; c_{\kappa\lambda}$, and $c_{\kappa\lambda} \leq c_{\kappa\lambda}; c_{\lambda\lambda}$.

4) $c_{\lambda\lambda} \leq c_{\lambda\lambda}; c_{\lambda\lambda}$ since $c_{\lambda\lambda} \leq 1'$.

It is easy to verify that we have taken care of all possibilities above, so (1) holds.

Now to establish 2.7.40(ii), it suffices to take distinct $\kappa,\lambda < 4$ and show that $T_\kappa | T_\lambda \subseteq T_\lambda | T_\kappa$. Suppose $a(T_\kappa | T_\lambda)b$. Then (1) applies to give the desired result.

2.7.40(iii) obviously holds. For 2.7.40(iv), assume that $\kappa,\lambda \neq \mu$. First suppose that $a \in T^*_\mu(E_{\kappa\mu} \cap E_{\mu\lambda})$. Say $b \in E_{\kappa\mu} \cap E_{\mu\lambda}$ and $a_{\rho\sigma} = b_{\rho\sigma}$ for all $\rho,\sigma \neq \mu$. Then

$$a_{\kappa\lambda} = b_{\kappa\lambda} \leqq b_{\kappa\mu};b_{\mu\lambda} \leqq 1';1' = 1',$$

so $a \in E_{\kappa\lambda}$, as desired. Conversely, suppose that $a \in E_{\kappa\lambda}$. Define $\tau \in {}^4 4$ by:

$$\tau\rho = \begin{cases} \rho & \text{if } \rho \neq \mu, \\ \kappa & \text{if } \rho = \mu, \end{cases}$$

for any $\rho<4$. Now we define $b \in {}^{4\times 4}At\mathfrak{A}$ by setting, for any $\rho,\sigma<4$, $b_{\rho\sigma} = a_{\tau\rho,\tau\sigma}$. It is routine to check that $b \in C$. Clearly $aT_\mu b$. Now $b_{\kappa\mu} = a_{\kappa\kappa} \leqq 1'$ and $b_{\mu\lambda} = a_{\kappa\lambda} \leqq 1'$. Thus $b \in E_{\kappa\mu} \cap E_{\mu\lambda}$, so $a \in T^*_\mu(E_{\kappa\mu} \cap E_{\mu\lambda})$, as desired.

Finally, for 2.7.40(v), assume that $\kappa \neq \lambda$, $aT_\kappa b$ and $a,b \in E_{\kappa\lambda}$. For any $\mu<4$ we then have

$$a_{\kappa\mu} \leqq a_{\kappa\lambda};a_{\lambda\mu} \leqq 1';a_{\lambda\mu} = a_{\lambda\mu},$$

so $a_{\kappa\mu} = a_{\lambda\mu}$. Similarly, $a_{\mu\kappa} = a_{\mu\lambda}$, $b_{\kappa\mu} = b_{\lambda\mu}$, and $b_{\mu\kappa} = b_{\mu\lambda}$ for any $\mu<4$. Hence if $\mu \neq \kappa$ we get $a_{\kappa\mu} = a_{\lambda\mu} = b_{\lambda\mu} = b_{\kappa\mu}$ and similarly $a_{\mu\kappa} = b_{\mu\kappa}$. Finally, $a_{\kappa\kappa} = a_{\lambda\kappa} = b_{\lambda\kappa} = b_{\kappa\kappa}$ by similar arguments. Since $aT_\kappa b$ we get $a = b$, as desired.

We have now checked all conditions of 2.7.40, so $\mathfrak{C}$ is a Ca_4. Let $\mathfrak{B} = \mathfrak{C}\mathfrak{m}\mathfrak{C}$ (see 2.7.33, 2.7.40). So $\mathfrak{B}$ is a CA_4.

Now we define a function f which will turn out to be an isomorphism from $\mathfrak{A}$ onto $\mathfrak{R}a\mathfrak{B}$. For any $x \in A$ we let

$$fx = \{a \in C : a_{01} \leqq x\}.$$

Thus $f^*A \subseteq SbC$. It is also clear that $fx \in Nr_2\mathfrak{B}$ for any $x \in A$, so f maps into $Ra\mathfrak{B}$.

Clearly f preserves $+$ and $-$, so f is a Boolean homomorphism. Also,

$$f1' = \{a \in C : a_{01} \leqq 1'\} = \mathsf{d}^{\mathfrak{B}}_{01},$$

as desired. To check that f preserves ;, we use the following fact, also needed later.

(3) If $u,v,w \in At\mathfrak{A}$ and $w \leqq u;v$, then there is a $c \in C$ such that $c_{01} = w$, $c_{02} = u$, and $c_{12} = v^{\smile}$.

For, we first define $b \in {}^{3\times 3}At\mathfrak{A}$ by the following stipulations:

$$\begin{array}{ll} b_{00} = u;1\cdot 1' & b_{02} = u \\ b_{11} = 1;w\cdot 1' & b_{20} = u^{\smile} \\ b_{22} = v;1\cdot 1' & b_{12} = v^{\smile} \\ b_{01} = w & b_{21} = v. \\ b_{10} = w^{\smile} & \end{array}$$

Note that b_{00}, b_{11}, b_{22} are atoms by 5.3.5(xii). Then it is clear that $b_{\kappa\kappa} \leqq 1'$ for all $\kappa<3$, and $b^{\smile}_{\kappa\lambda} = b_{\kappa\lambda}$ for all $\kappa,\lambda<3$. We claim that also $b_{\kappa\lambda} \leqq b_{\kappa\mu};b_{\mu\lambda}$ for all $\kappa,\lambda,\mu<3$. In fact, $b_{01} = w \leqq u;v = b_{02};b_{21}$. Hence by 5.3.5(iii) we get $b_{02} \leqq b_{01};b_{12}$, $b_{21} \leqq b_{20};b_{01}$, $b_{10} \leqq b_{12};b_{20}$, $b_{20} \leqq b_{21};b_{10}$, $b_{12} \leqq b_{10};b_{02}$. Next, note that $w \leqq u;v \leqq u;1 = (u;1\cdot 1');1$ by 5.3.5(xiii). Hence $w\cdot(u;1\cdot 1');1 \neq 0$, so $0 \neq (u;1\cdot 1');w \leqq 1';w = w$. Since w is an atom, we get $(u;1\cdot 1');w = w$. Therefore $b_{01} \leqq b_{00};b_{01}$, $b_{00} \leqq b_{01};b_{10}$, $b_{10} \leqq b_{10};b_{00}$. The other cases are treated similarly. Thus the above claim holds. Now let $\tau \in {}^4 3$ be such that $3 \upharpoonright Id \subseteq \tau$ and $\tau 3 = 0$. Define $c_{\kappa\lambda} = b_{\tau\kappa,\tau\lambda}$ for all $\kappa,\lambda<4$. Then the properties of b easily yield that $c \in C$. Hence (3) holds.

Now, to see that f preserves ; , first suppose that $a \in f(x;y)$. Thus $a_{01} \leqq x;y$, so by 5.3.5(iv) and its dual there are atoms $u \leqq x$, $v \leqq y$ such that $a_{01} \leqq u;v$. By (3), choose $c \in C$ so that $c_{01} = a_{01}$, $c_{02} = u$, $c_{12} = v^{\smile}$.

By (1) we have $a(T_2|T_3)c$, so choose e with aT_2eT_3c. Let $(4\sim\{1\})\upharpoonright Id \subseteq \sigma$, $\sigma 1 = 2$, and let $g_{\kappa\lambda} = e_{\sigma\kappa,\sigma\lambda}$ for all $\kappa,\lambda<4$. Then $g \in C$, eT_1g, and $g_{01} = e_{02} = c_{02} = u \leqq x$. Thus $g \in fx$. Also $g_{12} = e_{22} \leqq 1'$, so $g \in E_{12}$. Hence $e \in \mathsf{s}^1_2 fx$. Next, let $(4\sim\{0\})\upharpoonright Id \subseteq \tau$, $\tau 0 = 2$, and let $h_{\kappa\lambda} = e_{\tau\kappa,\tau\lambda}$ for all $\kappa,\lambda<4$. Then $h \in C$, eT_0h, and $h_{01} = e_{21} = c_{21} = v \leqq y$. Thus $h \in fy$. Also, $h_{02} = e_{22} \leqq 1'$, so $h \in E_{02}$. Hence $e \in \mathsf{s}^0_2 fy$. Thus $a \in \mathsf{c}_2(\mathsf{s}^1_2 fx\cdot \mathsf{s}^0_2 fy) = fx;fy$. So we have shown that $f(x;y) \subseteq fx;fy$.

Next, suppose that $a \in fx;fy$. Since $fx;fy = \mathsf{c}_2(\mathsf{s}^1_2 fx\cdot \mathsf{s}^0_2 fy)$, choose b with $aT_2b \in \mathsf{s}^1_2 fx \cap \mathsf{s}^0_2 fy$. Say $bT_1c \in E_{12} \cap fx$, $bT_0e \in E_{02} \cap fy$. Thus $c_{12} \leqq 1'$, $c_{01} \leqq x$, $e_{02} \leqq 1'$, $e_{01} \leqq y$. Hence

$$\begin{aligned} a_{01} = b_{01} \leqq b_{02};b_{21} &= c_{02};e_{21} \\ &\leqq c_{01};c_{12};e_{20};e_{01} \\ &\leqq x;1';1';y = x;y, \end{aligned}$$

so $a \in f(x;y)$ as desired. Therefore, f preserves ; .

Next we check that f preserves $^{\smile}$. Suppose $a \in fx^{\smile}$. Thus $a_{01} \leqq x^{\smile}$. Let $\sigma = [2/0]$, $\tau = [0/1]$, $\rho = [1/2]$, all as members of ${}^4 4$ (see p.36 of Part I). Let $b_{\kappa\lambda} = a_{\sigma\kappa,\sigma\lambda}$, $c_{\kappa\lambda} = b_{\tau\kappa,\tau\lambda}$, $e_{\kappa\lambda} = c_{\rho\kappa,\rho\lambda}$ for all $\kappa,\lambda<4$. Clearly $b,c,e \in C$. We have $aT_2b \in E_{02}$, $bT_0c \in E_{01}$, $cT_1e \in E_{12}$. Furthermore, $e_{01} = c_{02} = b_{12} = a_{10} \leqq x$, so $e \in fx$. Thus $a \in \mathsf{s}^2_0 \mathsf{s}^0_1 \mathsf{s}^1_2 fx = (fx)^{\smile}$, as desired. The converse is similar.

We have now shown that f is a homomorphism from $\mathfrak{X}$ into $\mathfrak{Ra}\mathfrak{B}$. Now if $x \neq 0$, let w be an atom $\leqq x$. Then $w \leqq w;1'$, so there is an atom v such that $w \leqq w;v$. Then (3) yields $c \in C$ such that $c_{01} = w \leqq x$. So $c \in fx$. This shows that f is one—one.

Finally, assume that $M \in Ra\mathfrak{B}$; we find $x \in A$ such that $fx = M$, thus showing that f is onto and finishing the proof. Note that $\mathsf{c}_2\mathsf{c}_3 M = M$. Let

$$x = \Sigma\{a_{01} : a \in M\}.$$

Now suppose that $a \in M$. Then $a_{01} \leqq x$, so $a \in fx$. Conversely, suppose that $a \in fx$. Then there is a $b \in M$ such that $a_{01} = b_{01}$. By (1) we have $a(E_2|E_3)B$. Hence $a \in \mathsf{c}_2\mathsf{c}_3 M = M$, as desired.

5.4 POLYADIC ALGEBRAS

Polyadic algebras were introduced and extensively studied by Halmos. His work on them is collected in Halmos [62]. Further development of their theory can be found in Andréka, Németi [84a'], [84b], Daigneault [59], [63], [63a], [64], [64a], [71a'], Daigneault, Monk [63], Fenstad [64a], Galler [55], [57], Georgescu [74'], [75'], [76'], [80a'], [82'], [82b'], [83'], Jeroslow [70'], J.S. Johnson [68], [69], [70'], [73'], Jurie [72'], Kramosil [71'], LeBlanc [59], [60a], [63], [66], Lucas [67], [68], Monk [60], [64a], [70a'], [71'], Petrescu [74'], Pinter [73b'], [73d'], Potthoff [71'].

Our aim in this section is to sketch the theory of these algebras, giving proofs for some results not in the literature, and to give a fairly complete description of the relationships between polyadic and cylindric algebras. For some of our results we assume acquaintance with some theorems in the literature; this section is not self-contained. We begin with the axioms; but readers not very familiar with polyadic algebras might want to read definition 5.4.22 first; it gives the set-theoretical concrete versions of polyadic algebras which motivate the following axioms.

Basic notions

DEFINITION 5.4.1. *Let α be an arbitrary ordinal. We denote* $[\kappa/\lambda,\lambda/\kappa]$ by $[\kappa,\lambda]$.

(i) *A* **polyadic algebra of dimension** α (*a* PA_α) *is an algebraic structure*

$$\mathfrak{A}=\langle A,+,\cdot,-,0,1,\mathsf{c}_{(\Gamma)},\mathsf{s}_\tau\rangle_{\Gamma\subseteq\alpha,\tau\in T},$$

where $T={}^\alpha\alpha$, *such that* $+$ *and* $\cdot$ *are binary operations on* A, $-$, $\mathsf{c}_{(\Gamma)}$, s_τ *are unary operations on* A, $0,1\in A$, *and the following postulates are satisfied for any* $x,y\in A$, *any* $\Gamma,\Delta\subseteq\alpha$, *and any* $\sigma,\tau\in{}^\alpha\alpha$:

($\mathbf{P}_0$) $\langle A,+,\cdot,-,0,1\rangle$ *is a* BA;

($\mathbf{P}_1$) $\mathsf{c}_{(\Gamma)}0=0$;

($\mathbf{P}_2$) $x\leqslant\mathsf{c}_{(\Gamma)}x$;

($\mathbf{P}_3$) $\mathsf{c}_{(\Gamma)}(x\cdot\mathsf{c}_{(\Gamma)}y)=\mathsf{c}_{(\Gamma)}x\cdot\mathsf{c}_{(\Gamma)}y$;

($\mathbf{P}_4$) $\mathsf{c}_{(0)}x=x$;

($\mathbf{P}_5$) $\mathsf{c}_{(\Gamma)}\mathsf{c}_{(\Delta)}x=\mathsf{c}_{(\Gamma\cup\Delta)}x$;

($\mathbf{P}_6$) $\mathsf{s}_{\alpha 1 Id}x=x$;

($\mathbf{P}_7$) $\mathsf{s}_{\sigma\circ\tau}x=\mathsf{s}_\sigma\mathsf{s}_\tau x$;

($\mathbf{P}_8$) $\mathsf{s}_\sigma(x+y)=\mathsf{s}_\sigma x+\mathsf{s}_\sigma y$;

($\mathbf{P}_9$) $\mathsf{s}_\sigma(-x)=-\mathsf{s}_\sigma x$;

($\mathbf{P}_{10}$) *if* $(\alpha\sim\Gamma)1\sigma=(\alpha\sim\Gamma)1\tau$, *then* $\mathsf{s}_\sigma\mathsf{c}_{(\Gamma)}x=\mathsf{s}_\tau\mathsf{c}_{(\Gamma)}x$;

($\mathbf{P}_{11}$) *if* $(\tau^{-1}{*}\Gamma)1\tau$ *is one-one, then* $\mathsf{c}_{(\Gamma)}\mathsf{s}_\tau x=\mathsf{s}_\tau\mathsf{c}_{(\Delta)}x$, *with* $\Delta=\tau^{-1}{*}\Gamma$.

We write c_κ *in place of* $c_{(\{\kappa\})}$.

(ii). *A* **polyadic equality algebra of dimension** α *(a* PEA_α*) is an algebraic structure*

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, c_{(\Gamma)}, s_\tau, d_{\kappa\lambda}\rangle_{\Gamma\subseteq\alpha, \tau\in T, \kappa,\lambda<\alpha}$$

where $T = {}^\alpha\alpha$, *such that* $d_{\kappa\lambda}\in A$ *for all* $\kappa,\lambda<\alpha$, $\langle A, +, \cdot, -, 0, 1, c_{(\Gamma)}, s_\tau\rangle_{\Gamma\subseteq\alpha, \tau\in T}$ *is a* PA_α, *and the following conditions hold for all* $\kappa,\lambda<\alpha$, *all* $\tau\in{}^\alpha\alpha$, *and all* $x\in A$:

(E_1) $d_{\kappa\kappa} = 1$;

(E_2) $x\cdot d_{\kappa\lambda} \leq s_{[\kappa/\lambda]}x$;

(E_3) $s_\tau d_{\kappa\lambda} = d_{\tau\kappa,\tau\lambda}$.

(iii) *For* $\mathfrak{A}$ *a* PEA_α, *as above, its* **cylindric reduct** *is*

$$\mathfrak{Rb}_{ca}\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, c_\kappa, d_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha}.$$

(iv) *For* $\mathfrak{A}$ *a* PA_α *or* PEA_α, *as above, its* **diagonal-free reduct** *is*

$$\mathfrak{Rb}_{df}\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, c_\kappa\rangle_{\kappa<\alpha}.$$

(v) *If* K *is a class of* PEA_α*'s or* PA_α*'s,* $\mathsf{Rd}_{df}\mathsf{K}$ *is the class* $\{\mathfrak{Rb}_{df}\mathfrak{A} : \mathfrak{A}\in\mathsf{K}\}$. *Similarly for* $\mathsf{Rd}_{ca}\mathsf{K}$.

One of the basic results about polyadic algebras is as follows, from Halmos [57a] or Halmos [62]: If $\mathfrak{A}$ is a complete atomic PA_α, then $\mathfrak{A}$ is the PA–reduct of a PEA_α, and any PA_α can be isomorphically embedded in the PA–reduct of a PEA_α (the PA–reduct is understood in its natural way).

The following theorem is clear:

THEOREM 5.4.2. *If* $\mathfrak{A}$ *is a* PA_α, *then* $\mathfrak{Rb}_{df}\mathfrak{A}$ *is a* Df_α.

THEOREM 5.4.3. *If* $\mathfrak{A}$ *is a* PEA_α, *then* $\mathfrak{Rb}_{ca}\mathfrak{A}$ *is a* CA_α. *Furthermore,* $s_{[\kappa/\lambda]}x = c_\kappa(d_{\kappa\lambda}\cdot x)$ *for any distinct* $\kappa,\lambda<\alpha$ *and any* $x\in A$.

PROOF. First we show

(1) $c_\kappa d_{\kappa\lambda} = 1$ for any $\kappa,\lambda<\alpha$.

For, $1 = d_{\lambda\lambda} = s_{[\kappa/\lambda]}d_{\kappa\lambda} \leq s_{[\kappa/\lambda]}c_\kappa d_{\kappa\lambda} = s_{\alpha 1 Id}c_\kappa d_{\kappa\lambda} = c_\kappa d_{\kappa\lambda}$. Next,

(2) $x\cdot d_{\kappa\lambda} = s_{[\kappa/\lambda]}x\cdot d_{\kappa\lambda}$ for any $\kappa,\lambda<\alpha$ and $x\in A$.

For, $\leq$ is given in (E_2). On the other hand,

$$s_{[\kappa,\lambda]}x\cdot d_{\kappa\lambda}\cdot -x \leq s_{[\kappa/\lambda]}x\cdot s_{[\kappa/\lambda]}-x = 0,$$

so (2) follows. Now if $\kappa\neq\lambda$ then $[\kappa/\lambda]^{-1}{}^*\{\kappa\} = 0$, so by ($P_{11}$) we have $c_\kappa s_{[\kappa/\lambda]}x = s_{[\kappa/\lambda]}x$ for any $x\in A$. Hence

$$\begin{aligned} c_\kappa(x\cdot d_{\kappa\lambda}) &= s_{[\kappa/\lambda]}x\cdot c_\kappa d_{\kappa\lambda} \text{ by (2)} \\ &= s_{[\kappa/\lambda]}x, \end{aligned}$$

giving part of what is desired in the theorem. From this it is easy to check the

axioms (C_6) and (C_7) for cylindric algebras, and the theorem follows.

REMARKS 5.4.4. In connection with 5.4.2 and 5.4.3 it is natural to ask about their converses. One may even ask whether every CA_α is a subalgebra of $\mathfrak{Rb}_{ca}\mathfrak{A}$ for some PEA_α $\mathfrak{A}$; similarly for Df_α's and PA_α's. First note that for $\alpha \leqq 1$ CA_α, Df_α, PA_α, and PEA_α are virtually identical classes. Now take $\alpha = 2$. From 5.1.47, which says that every Df_2 is representable, it follows that every Df_2 can be isomorphically embedded in $\mathfrak{Rb}_{df}\mathfrak{A}$ for some PEA_2 $\mathfrak{A}$. On the other hand, we have the following result.

THEOREM 5.4.5. *If $\kappa < \alpha$ and $\mathfrak{A}$ is a PA_α such that $\mathsf{c}_\kappa^{\mathfrak{A}} = A \upharpoonright Id$, then $\mathsf{c}_\lambda^{\mathfrak{A}} = A \upharpoonright Id$ for all $\lambda < \alpha$.*

PROOF. For any $x \in A$ we have

$$\begin{aligned} \mathsf{c}_\lambda x &= \mathsf{c}_\lambda \mathsf{s}_{[\kappa,\lambda]} \mathsf{s}_{[\kappa,\lambda]} x = \mathsf{s}_{[\kappa,\lambda]} \mathsf{c}_\kappa \mathsf{s}_{[\kappa,\lambda]} x \\ &= \mathsf{s}_{[\kappa,\lambda]} \mathsf{s}_{[\kappa,\lambda]} x = x. \end{aligned}$$

REMARK 5.4.6. Since for any $\alpha \geqq 2$ we can easily construct a Df_α $\mathfrak{A}$ such that $\mathsf{c}_0^{\mathfrak{A}} = A \upharpoonright Id$ while $\mathsf{c}_1^{\mathfrak{A}} \neq A \upharpoonright Id$, it follows from 5.4.5 that $\mathfrak{A} \neq \mathfrak{Rb}_{ca}\mathfrak{B}$ for every PA_α $\mathfrak{B}$.

We shall see, in 5.4.23, that every CA_2 can be embedded in $\mathfrak{Rb}_{ca}\mathfrak{B}$ for some PEA_2 $\mathfrak{B}$. We now give, however, an example of a CA_2 $\mathfrak{A}$ such that $\mathfrak{A} \neq \mathfrak{Rb}_{ca}\mathfrak{B}$ for every PEA_2 $\mathfrak{B}$. Let $|X| = |Y| = 8$, $X = \{x_0, \ldots, x_7\}$, $X \cap Y = 0$, $Y = \{y_0, \ldots, y_7\}$, $U = X \cup Y$. We define a Cs_2 with base U by specifying its atoms a, b, c, d, e, f, g, h:

$$\begin{aligned} a &= \{\langle x_i, x_i \rangle : i < 8\}, \\ b &= \{\langle x_0, x_1 \rangle, \langle x_1, x_2 \rangle, \ldots, \langle x_6, x_7 \rangle, \langle x_7, x_0 \rangle\}, \\ c &= X \times X \sim (a \cup b), \\ d &= \{\langle y_i, x_i \rangle : i < 8\}, \\ e &= Y \times X \sim d, \\ f &= X \times Y, \\ g &= \{\langle y_i, y_i \rangle : i < 8\}, \\ h &= Y \times Y \sim g. \end{aligned}$$

Clearly we have $\mathsf{D}_{01} = a \cup g$, $\mathsf{C}_0 a = \mathsf{C}_0 b = \mathsf{C}_0 c = \mathsf{C}_0 d = \mathsf{C}_0 e = a \cup b \cup c \cup d \cup e$, $\mathsf{C}_1 a = \mathsf{C}_1 b = \mathsf{C}_1 c = \mathsf{C}_1 f = a \cup b \cup c \cup f$, $\mathsf{C}_0 f = \mathsf{C}_0 g = \mathsf{C}_0 h = f \cup g \cup h$, and $\mathsf{C}_1 e = \mathsf{C}_1 d = \mathsf{C}_1 g = \mathsf{C}_1 h = e \cup d \cup g \cup h$. Thus we *have* defined a Cs_2 $\mathfrak{A}$. It is illustrated in Figure 5.4.7. Suppose that $\mathfrak{A} = \mathfrak{Rb}_{ca}\mathfrak{B}$ for some PEA_2 $\mathfrak{B}$. Since $\mathsf{s}_{[0,1]}\mathsf{d}_{01} = \mathsf{d}_{01}$, and $\mathsf{s}_{[0,1]}$ is an automorphism of $\mathfrak{B}(\mathfrak{A}$, we have $\mathsf{s}_{[0,1]} a = a$ or $\mathsf{s}_{[0,1]} a = g$. If $\mathsf{s}_{[0,1]} a = a$, then $a \cup b \cup c \cup d \cup e = \mathsf{c}_0 a = \mathsf{c}_0 \mathsf{s}_{[0,1]} a = \mathsf{s}_{[0,1]} \mathsf{c}_1 a = \mathsf{s}_{[0,1]}(a \cup b \cup c \cup f)$, which is impossible, since $\mathsf{s}_{[0,1]}(a \cup b \cup c \cup f)$ is the sum of only four atoms of $\mathfrak{A}$. A similar contradiction is reached assuming that $\mathsf{s}_{[0,1]} a = g$.

Now we turn to the case $3 \leqq \alpha < \omega$. We do not know whether every Df_α can be embedded in some $\mathfrak{Rb}_{df}\mathfrak{A}$, $\mathfrak{A}$ a PA_α, in this case. At least, by the above, for each such α there is a Df_α $\mathfrak{A}$ such that $\mathfrak{A} \neq \mathfrak{Rb}_{df}\mathfrak{B}$ for every PA_α $\mathfrak{B}$. Now we claim that for any such α there is a CA_α not embeddable in $\mathfrak{Rb}_{ca}\mathfrak{B}$ for any PEA_α $\mathfrak{B}$. This

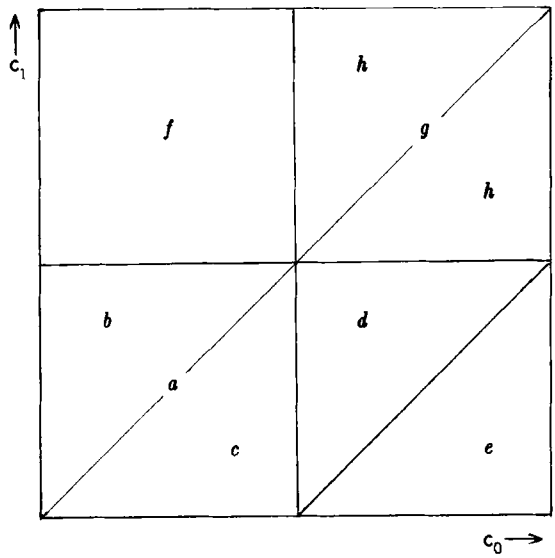

FIGURE 5.4.7

depends on the following elementary fact about PEA_α's.

THEOREM 5.4.8 *Suppose* $\alpha \geqq 3$ *and* $\mathfrak{A}$ *is a* PEA_α. *Then* $\mathsf{s}_{[0,1]}\mathsf{c}_2 a = \mathsf{s}_0^2\mathsf{s}_1^0\mathsf{s}_2^1\mathsf{c}_2 a$ *for any* $a \in A$.

PROOF. Let $\sigma \in {}^\alpha\alpha$ be such that $\sigma 0 = 1$, $\sigma 1 = 0$, $\sigma 2 = 0$, and $\sigma\kappa = \kappa$ for all $\kappa \in \alpha \sim 3$. Thus $(\alpha \sim \{2\}) \restriction [0,1] = (\alpha \sim \{2\}) \restriction \sigma$, and $\sigma = [2/0] \circ [0/1] \circ [1/2]$. Hence

$$\begin{aligned} \mathsf{s}_{[0,1]}\mathsf{c}_2 a &= \mathsf{s}_\sigma \mathsf{c}_2 a = \mathsf{s}_{[2/0]}\mathsf{s}_{[0/1]}\mathsf{s}_{[1/2]}\mathsf{c}_2 a \\ &= \mathsf{s}_0^2\mathsf{s}_1^0\mathsf{s}_2^1\mathsf{c}_2 x \end{aligned}$$

using 5.4.3, as desired.

REMARKS 5.4.9. From 5.4.8 we obtain by symmetry that $\mathsf{s}_{[0,1]}\mathsf{c}_2 a = \mathsf{s}_1^2\mathsf{s}_0^1\mathsf{s}_2^0\mathsf{c}_2 a$ for any $a \in A$ also. Now if we take a CA_α $\mathfrak{A}$, given by 3.2.71, such that $\mathsf{s}_0^2\mathsf{s}_1^0\mathsf{s}_2^1\mathsf{c}_2 a \neq \mathsf{s}_1^2\mathsf{s}_0^1\mathsf{s}_2^0\mathsf{c}_2 a$ for some $a \in A$, we see that $\mathfrak{A}$ is not embeddable in $\mathfrak{Rb}_{ca}\mathfrak{B}$ for any PEA_α $\mathfrak{B}$. Note that this is true for any $\alpha \geqq 3$, without the restriction $\alpha < \omega$.

Finally, for $\alpha \geqq \omega$ there is a Df_α $\mathfrak{A}$ not embeddable in $\mathfrak{Rb}_{df}\mathfrak{B}$ for any PA_α $\mathfrak{B}$. This follows from the following facts: (1) every PA_α, $\alpha \geqq \omega$, is representable (see below), (2) if $\mathfrak{A}$ is a representable PA_α then $\mathfrak{Rb}_{df}\mathfrak{A}$ is a representable Df_α (see below), (3) for any $\alpha \geqq 3$ there is a non-representable Df_α (see 5.1.60).

There are also relationships between CA_α's and PEA_α's in some special situations which have already been noticed in our work. Thus if $3 \leqq \alpha < \omega$, let K, resp. L, be the class of all CA_α's, resp. PEA_α's, of positive characteristic. Then by 3.2.51 we have $\mathsf{K} = \{\mathfrak{Rb}_{ca}\mathfrak{A} : \mathfrak{A} \in \mathsf{L}\}$. Actually CA_α's and PEA_α's of positive characteristic are *polynomially equivalent* in the sense of Part I, p. 125. From 3.2.52 we see that for $3 \leqq \alpha < \omega$, if $\mathfrak{A}$ is a CA_α neatly embeddable in a $\mathsf{CA}_{\alpha+2}$, then $\mathfrak{A} \subseteq \mathfrak{Rb}_{ca}\mathfrak{B}$ for some PEA_α $\mathfrak{B}$. Recall also the results of Andréka, Comer and Németi quoted after 3.2.52. Finally, locally finite–dimensional CA_α's and PEA_α's essentially coincide for $\alpha \geqq \omega$; we shall now go into this important fact in some detail.

DEFINITION 5.4.10. *A* PEA_α $\mathfrak{A}$ *is* **locally finite–dimensional**, *in symbols* $\mathfrak{A} \in \mathsf{LfPEA}_\alpha$, *if for every* $a \in A$ *there is a finite subset* Γ *of* α *such that* $\mathsf{c}_{(\alpha \sim \Gamma)}a = a$.

If $\mathfrak{A} \in \mathsf{Lf}_\alpha$, $\alpha \geqq \omega$, *we let* $\mathfrak{A}^+ = (A, +, \cdot, -, 0, 1, \mathsf{c}_{(\Gamma)}, \mathsf{s}^+_\tau, \mathsf{d}_{\kappa\lambda})_{\Gamma \subseteq \alpha, \tau \in T, \kappa, \lambda < \alpha}$, *where* $T = {}^\alpha\alpha$, s^+ *is given by* 1.11.13, *and* $\mathsf{c}_{(\Gamma)}a = \mathsf{c}_{(\Delta a \cap \Gamma)}a$ *for all* $\Gamma \subseteq \alpha$.

THEOREM 5.4.11. *Let* $\alpha \geqq \omega$. *Then*

(i) *If* $\mathfrak{A} \in \mathsf{LfPEA}_\alpha$, *then* $\mathfrak{Rb}_{ca}\mathfrak{A} \in \mathsf{Lf}_\alpha$ *and* $(\mathfrak{Rb}_{ca}\mathfrak{A})^+ = \mathfrak{A}$.

(ii) *If* $\mathfrak{A} \in \mathsf{Lf}_\alpha$, *then* $\mathfrak{A}^+ \in \mathsf{LfPEA}_\alpha$ *and* $\mathfrak{Rb}_{ca}\mathfrak{A}^+ = \mathfrak{A}$.

This theorem is immediate from 1.11.14. Looking at Definition 5.4.10, one might ask whether the definition of local finiteness can be given the simpler form: $\{\kappa : \mathsf{c}_\kappa a \neq a\}$ *is finite for all* $a \in A$, as for CA_α's. We shall see below, in 5.4.41, that this in *not* equivalent to local finiteness.

Now we shall make some remarks on the arithmetic and algebraic theory of polyadic algebras. Of course the arithmetical properties of cylindric algebras given in Chapter 1 of Part I also apply to polyadic equality algebras. There are also additional laws. We give two examples that have turned out to be useful.

THEOREM 5.4.12. *If* $\Gamma \subseteq \alpha$, $\tau \in {}^\alpha\alpha$, *and* $\mathsf{c}_{(\alpha \sim \Gamma)}a = a$, *then* $\mathsf{c}_{(\alpha \sim \Delta)}\mathsf{s}_\tau a = \mathsf{s}_\tau a$, *where* $\Delta = \tau^*\Gamma$.

PROOF. If $\Gamma = 0$, then $\Delta = 0$ and, with $\delta = \alpha 1 Id$, $\mathsf{s}_\tau a = \mathsf{s}_\tau \mathsf{c}_{(\alpha)}a = \mathsf{s}_\delta \mathsf{c}_{(\alpha)}a = \mathsf{c}_{(\alpha)}a = a$ and hence $\mathsf{c}_{(\alpha)}\mathsf{s}_\tau a = \mathsf{c}_{(\alpha)}a = a = \mathsf{s}_\tau a$, as desired. Now suppose that $\Gamma \neq 0$. Let $\sigma \in {}^\alpha\alpha$ be such that $\Gamma 1 \sigma = \Gamma 1 \tau$ and $\sigma^*(\alpha \sim \Gamma) \subseteq \tau^*\Gamma$. Thus $\sigma^{-1*}(\alpha \sim \tau^*\Gamma) = 0$. Hence

$$\begin{aligned}\mathsf{c}_{(\alpha \sim \Delta)}\mathsf{s}_\tau a &= \mathsf{c}_{(\alpha \sim \Delta)}\mathsf{s}_\tau \mathsf{c}_{(\alpha \sim \Gamma)}a \\ &= \mathsf{c}_{(\alpha \sim \Delta)}\mathsf{s}_\sigma a = \mathsf{s}_\sigma \mathsf{c}_{(0)}a = \mathsf{s}_\sigma \mathsf{c}_{(\alpha \sim \Gamma)}a = \mathsf{s}_\tau a.\end{aligned}$$

THEOREM 5.4.13. *Suppose that* $a \in A$, $\sigma, \tau \in {}^\alpha\alpha$, $\Gamma, \Delta \subseteq \alpha$, $\sigma^*\Gamma = \Omega$, $\Gamma 1 \sigma$ *is one–one*, $\tau^*(\Delta \sim \Gamma) \cap \sigma^*\Gamma = 0$, $\mathsf{c}_{(\alpha \sim \Delta)}a = a$, *and* $(\alpha \sim \Gamma) 1 \sigma = (\alpha \sim \Gamma) 1 \tau$. *Then* $\mathsf{s}_\tau \mathsf{c}_{(\Gamma)}a = \mathsf{c}_{(\Omega)}\mathsf{s}_\sigma a$.

PROOF. First we suppose that $\sigma^*\Gamma = \alpha$. Thus $\Delta \sim \Gamma = 0$ and hence $\mathsf{c}_{(\alpha \sim \Gamma)}a = a$. Now since $\Gamma 1 \sigma$ is one–one, we have $\theta \overset{\mathrm{def}}{=} (\Gamma 1 \sigma)^{-1} \in {}^\alpha\Gamma$. Hence, with $\rho = \alpha 1 Id$,

$$\begin{aligned}\mathsf{c}_{(\Omega)}\mathsf{s}_\sigma a &= \mathsf{c}_{(\alpha)}\mathsf{s}_\sigma a = \mathsf{s}_\theta \mathsf{c}_{(\alpha)}\mathsf{s}_\sigma a \\ &= \mathsf{c}_{(\Gamma)}\mathsf{s}_\theta \mathsf{s}_\sigma a = \mathsf{c}_{(\Gamma)}a = \mathsf{c}_{(\alpha)}a = \mathsf{s}_\rho \mathsf{c}_{(\alpha)}a = \mathsf{s}_\tau \mathsf{c}_{(\alpha)}a \\ &= \mathsf{s}_\tau \mathsf{c}_{(\Gamma)}a,\end{aligned}$$

as desired.

So now assume that $\sigma^*\Gamma \neq \alpha$. Let $\varphi \in {}^\alpha\alpha$ be such that $\Gamma 1\varphi = \Gamma 1\sigma$, $(\Delta \sim \Gamma)1\varphi = (\Delta \sim \Gamma)1\tau$, and $\varphi^*[\alpha \sim (\Gamma \cup \Delta)] \subseteq \alpha \sim \sigma^*\Gamma$. Then $\Delta 1\varphi = \Delta 1\sigma$, $\varphi^{-1*}\sigma^*\Gamma = \Gamma$, and $\Gamma 1\varphi$ is one–one. Hence

$$\begin{aligned} \mathsf{s}_\tau \mathsf{c}_{(\Gamma)} a &= \mathsf{s}_\tau \mathsf{c}_{(\Gamma)} \mathsf{c}_{(\alpha \sim \Delta)} a = \mathsf{s}_\tau \mathsf{c}_{(\alpha \sim (\Delta \sim \Gamma))} a \\ &= \mathsf{s}_\varphi \mathsf{c}_{(\Gamma)} a = \mathsf{c}_{(\Omega)} \mathsf{s}_\varphi a = \mathsf{c}_{(\Omega)} \mathsf{s}_\varphi \mathsf{c}_{(\alpha \sim \Delta)} a \\ &= \mathsf{c}_{(\Omega)} \mathsf{s}_\sigma a, \end{aligned}$$

as desired.

Algebraic theory

We now turn to the algebraic theory of polyadic algebras, concentrating on those aspects which differ from the situation for cylindric algebras. Relativization for polyadic algebras has not been studied much. We mention only one result, which is analogous to 2.2.3.

DEFINITION 5.4.14. *Let $\mathfrak{A}$ be a* PA_α, *with notation as in* 5.4.1, *and suppose that $a \in A$. We let $\mathfrak{Rl}_a\mathfrak{A} = \{x \in A : x \leq a\}$ and define, for $x, y \in Rl_a\mathfrak{A}$, $\Gamma \subseteq \alpha$, $\tau \in {}^\alpha\alpha$,*

$$\begin{aligned} x +' y &= x + y, \\ x \cdot' y &= x \cdot y, \\ -' x &= a \cdot - x \\ 0' &= 0, \\ 1' &= a, \\ \mathsf{c}'_{(\Gamma)} x &= \mathsf{c}_{(\Gamma)} x \cdot a, \\ \mathsf{s}'_\tau x &= \mathsf{s}_\tau x \cdot a. \end{aligned}$$

Set $\mathfrak{Rl}_a\mathfrak{A} = \langle Rl_a\mathfrak{A}, +', \cdot', -', 0', 1', \mathsf{c}'_{(\Gamma)}, \mathsf{s}'_\tau \rangle_{\Gamma \subseteq \alpha, \tau \in T}$ with $T = {}^\alpha\alpha$.

If $\mathfrak{A} \in \mathsf{PEA}_\alpha$ we extend this definition in the obvious way, with $\mathsf{d}'_{\kappa\lambda} = \mathsf{d}_{\kappa\lambda} \cdot a$ for all $\kappa, \lambda < \alpha$.

THEOREM 5.4.15. *If $\mathfrak{A} \in \mathsf{PA}_\alpha$ or $\mathfrak{A} \in \mathsf{PEA}_\alpha$ and $b \in A$, then $\mathfrak{Rl}_b\mathfrak{A}$ satisfies all of the axioms for a* PA_α, *or* PEA_α, *except perhaps* (P_5), (P_7), (P_9), (P_{10}), (P_{11}), *and* (E_3). *$\mathfrak{Rl}_b\mathfrak{A}$ is a* PA_α, *or* PEA_α, *iff the following three conditions are satisfied (in $\mathfrak{A}$) for all $x \in \mathfrak{Rl}_b\mathfrak{A}$, $\Gamma, \Delta \subseteq \alpha$, and $\tau \in {}^\alpha\alpha$:*

(i) $\mathsf{c}_{(\Gamma \cup \Delta)} x \cdot b \leq \mathsf{c}_{(\Gamma)}(\mathsf{c}_{(\Delta)} x \cdot b)$ *if $\Gamma, \Delta \subseteq \alpha$ are disjoint.*

(ii) $b \leq \mathsf{s}_\tau b$.

(iii) *If τ maps $\tau^{-1*}\Gamma$ one–one onto Γ, then $\mathsf{s}_\tau(\mathsf{c}_{(\Delta)} x \cdot b) \cdot b \leq \mathsf{c}_{(\Gamma)}(\mathsf{s}_\tau x \cdot b)$, where $\Delta = \tau^{-1*}\Gamma$.*

PROOF. As in the proof of 2.2.3, (P_0),(P_1),(P_2), and (P_3) are clear. Also, (P_4),(P_6) and (P_8) are obvious. For PEA_α's, (E_1) and (E_2) are obvious.

For the second part of the theorem, first suppose that $\mathfrak{Rl}_b\mathfrak{A}$ is a PA_α, or PEA_α. If $\Gamma, \Delta \subseteq \alpha$, then for any $x \in Rl_a\mathfrak{A}$,

$$\mathsf{c}_{(\Gamma \cup \Delta)} x \cdot b = \mathsf{c}'_{(\Gamma \cup \Delta)} x = \mathsf{c}'_{(\Gamma)} \mathsf{c}'_{(\Delta)} x = \mathsf{c}_{(\Gamma)}(\mathsf{c}_{(\Delta)} x \cdot b) \cdot b$$

$$\leqq c_{(\Gamma)}(c_{(\Delta)}x\cdot b),$$

so (i) holds. Next, let $\tau\in{}^{\alpha}\alpha$. Then $b=s'_{\tau}b=s_{\tau}b\cdot b\leqq s_{\tau}b$, so (ii) holds. Assume the hypotheses of (iii). Then

$$\begin{aligned} s_{\tau}(c_{(\Delta)}x\cdot b)\cdot b &= s'_{\tau}c'_{(\Delta)}x = c'_{(\Gamma)}s'_{\tau}x = c_{(\Gamma)}(s_{\tau}x\cdot b)\cdot b \\ &\leqq c_{(\Gamma)}(s_{\tau}x\cdot b), \end{aligned}$$

as desired.

Conversely, suppose that (i), (ii), (iii) hold. We must check (P_5), (P_7), (P_9), (P_{10}), (P_{11}), and (E_3). First we check (P_5) when $\Gamma,\Delta\subseteq\alpha$ are disjoint:

$$\begin{aligned} c'_{(\Gamma)}c'_{(\Delta)}x &= c_{(\Gamma)}(c_{(\Delta)}x\cdot b)\cdot b \\ &\leqq c_{(\Gamma)}c_{(\Delta)}x\cdot b = c_{(\Gamma\cup\Delta)}x\cdot b = c'_{(\Gamma\cup\Delta)}x \\ &= c_{(\Gamma\cup\Delta)}x\cdot b \leqq c_{(\Gamma)}(c_{(\Delta)}x\cdot b)\cdot b \\ &= c'_{(\Gamma)}c'_{(\Delta)}b, \end{aligned}$$

as desired. Next note that for any $\Omega\subseteq\alpha$ we have $c'_{(\Omega)}c'_{(\Omega)}x=c'_{(\Omega)}x$. Hence for arbitrary $\Gamma,\Delta\subseteq\alpha$ we get

$$\begin{aligned} c'_{(\Gamma)}c'_{(\Delta)}x &= c'_{(\Gamma\sim\Delta)}c'_{(\Gamma\cap\Delta)}c'_{(\Gamma\cap\Delta)}c'_{(\Delta\sim\Gamma)}x \\ &= c'_{(\Gamma\sim\Delta)}c'_{(\Gamma\cap\Delta)}c'_{(\Delta\sim\Gamma)}x \\ &= c'_{(\Gamma)}c'_{(\Delta\sim\Gamma)}x = c'_{(\Gamma\cup\Delta)}x. \end{aligned}$$

So, (P_5) holds. For (P_7), let $\sigma,\tau\in{}^{\alpha}\alpha$. Then

$$\begin{aligned} s'_{\sigma\circ\tau}x &= s_{\sigma\circ\tau}x\cdot b = s_{\sigma}s_{\tau}x\cdot b \\ &= s_{\sigma}s_{\tau}x\cdot b\cdot s_{\sigma}b = s_{\sigma}(s_{\tau}x\cdot b)\cdot b \\ &= s'_{\sigma}s'_{\tau}x. \end{aligned}$$

Next we treat (P_9):

$$\begin{aligned} s'_{\sigma}-'x &= s_{\sigma}(-x\cdot b)\cdot b = -s_{\sigma}x\cdot s_{\sigma}b\cdot b = -s_{\sigma}x\cdot b \\ &= -(s_{\sigma}x\cdot b)\cdot b = -'s'_{\sigma}x. \end{aligned}$$

We turn to (P_{10}). Assume its hypotheses. Then

$$\begin{aligned} s'_{\sigma}c'_{(\Gamma)}x &= s_{\sigma}(c_{(\Gamma)}x\cdot b)\cdot b = s_{\sigma}c_{(\Gamma)}x\cdot s_{\sigma}b\cdot b \\ &= s_{\tau}c_{(\Gamma)}x\cdot b = s'_{\tau}c'_{(\Gamma)}x. \end{aligned}$$

For (P_{11}), assume that $(\tau^{-1*}\Gamma)\upharpoonright\tau$ is one-one. Let $\Delta=\tau^{-1*}\Gamma$ and $\Omega=\tau^{*}\Delta$. First we prove $\leqq$:

$$\begin{aligned} c'_{(\Gamma)}s'_{\tau}x &= c_{(\Gamma)}(s_{\tau}x\cdot b)\cdot b \\ &\leqq c_{(\Gamma)}s_{\tau}x\cdot b = s_{\tau}c_{(\Delta)}x\cdot b\cdot s_{\tau}b \end{aligned}$$

$$= \mathsf{s}_\tau(\mathsf{c}_{(\Delta)}x \cdot b) \cdot b = \mathsf{s}'_\tau \mathsf{c}'_{(\Delta)}x.$$

Next we prove $\geqq$ assuming that $\Delta = 0$:

$$\begin{aligned} \mathsf{s}'_\tau \mathsf{c}'_{(0)}x = \mathsf{s}'_\tau x = \mathsf{s}_\tau x \cdot b &\leqq \mathsf{c}_{(\Gamma)}(\mathsf{s}_\tau x \cdot b) \cdot b \\ &= \mathsf{c}'_{(\Gamma)}\mathsf{s}'_\tau x. \end{aligned}$$

Finally we prove $\geqq$ in general:

$$\begin{aligned} \mathsf{s}'_\tau \mathsf{c}'_{(\Delta)}x &= \mathsf{c}'_{(\Gamma\sim\Omega)}\mathsf{s}'_\tau \mathsf{c}'_{(\Delta)}x \quad \text{(the case } \Delta = 0) \\ &= \mathsf{c}'_{(\Gamma\sim\Omega)}(\mathsf{s}_\tau(\mathsf{c}_{(\Delta)}x \cdot b) \cdot b) \\ &\leqq \mathsf{c}'_{(\Gamma\sim\Omega)}(\mathsf{c}_{(\Omega)}(\mathsf{s}_\tau x \cdot b) \cdot b) \quad \text{(by (iii))} \\ &= \mathsf{c}'_{(\Gamma\sim\Omega)}\mathsf{c}'_{(\Omega)}\mathsf{s}'_\tau x = \mathsf{c}'_{(\Gamma)}\mathsf{s}'_\tau x. \end{aligned}$$

Only (E_3) remains:

$$\begin{aligned} \mathsf{s}'_\tau \mathsf{d}'_{\kappa\lambda} &= \mathsf{s}_\tau(\mathsf{d}_{\kappa\lambda} \cdot b) \cdot b = \mathsf{s}_\tau \mathsf{d}_{\kappa\lambda} \cdot \mathsf{s}_\tau b \cdot b \\ &= \mathsf{d}_{\tau\kappa,\tau\lambda} \cdot b = \mathsf{d}'_{\tau\kappa,\tau\lambda}. \end{aligned}$$

An important algebraic property of PA_α's not shared by cylindric algebras is that every polyadic algebra and every PEA_α is semisimple; see Halmos [62], p.131. The notion of neat embedding is also important for polyadic algebras:

DEFINITION 5.4.16. *Let* $\beta \leqq \alpha$, *and let* $\mathfrak{A}$ *be a* PA_α, *with notation as in Definition* 5.4.1. *Let* $\mathfrak{Nr}_\beta\mathfrak{A} = \langle Nr_\beta\mathfrak{A}, +, \cdot, -, 0, 1, \mathsf{c}_{(\Gamma)}, \mathsf{s}'_\tau \rangle_{\Gamma \subseteq \beta, \tau \in T}$, *where* $T = {}^\beta\beta$, $Nr_\beta\mathfrak{A} = \{a \in A : \mathsf{c}_{(\alpha\sim\beta)}a = a\}$, *and* $\mathsf{s}'_\tau = \mathsf{s}_\sigma$ *with* $\sigma = \tau \cup \langle \kappa : \kappa \in \alpha \sim \beta \rangle$ for each $\tau \in {}^\beta\beta$. *A* PA_β $\mathfrak{B}$ *can be* **neatly embedded** *in* $\mathfrak{A}$ *provided that* $\mathfrak{B}$ *is isomorphic to a subalgebra of* $\mathfrak{Nr}_\beta\mathfrak{A}$.

These notions are extended in the obvious way to PEA_α's.

The main result about neat embeddings (in Daigneault, Monk [63], Theorem 3.3), is that if $\omega \leqq \beta$, then every PA_β can be neatly embedded in a PA_α, for any $\alpha \geqq \beta$. This result extends to PEA_α's also:

THEOREM 5.4.17. *Suppose that* $\omega \leqq \beta \leqq \alpha$. *Then every* PEA_β *can be neatly embedded in a* PEA_α.

PROOF. We may assume that $|\beta| < |\alpha|$. Let $\mathfrak{A}$ be a given PEA_β and $\mathfrak{A}'$ its PA_β–reduct obtained by deleting the diagonal elements of $\mathfrak{A}$. By Daigneault, Monk [63] Theorem 3.3, there is a PA_α $\mathfrak{B}$ such that $\mathfrak{A}' \subseteq \mathfrak{Nr}_\beta\mathfrak{B}$; furthermore, each element of B has the form $\mathsf{s}^+_\sigma a$ for some $a \in A$ and some permutation σ of α. (The operations of $\mathfrak{B}$ are denoted with a superscript $^+$.) Now we define proposed diagonal elements $\mathsf{d}^+_{\kappa\lambda}$ for $\kappa, \lambda < \alpha$: for $\kappa = \lambda$ let $\mathsf{d}^+_{\kappa\lambda} = 1$, while if $\kappa \neq \lambda$, let μ be an element of $\beta \sim \{\kappa, \lambda\}$ and set

$$\mathsf{d}^+_{\kappa\lambda} = \begin{cases} \mathsf{d}_{\kappa\lambda} \text{ if } \kappa,\lambda<\beta, \\ \mathsf{s}^+_{[\mu,\kappa]}\mathsf{d}_{\mu\lambda} \text{ if } \kappa\geqq\beta,\ \lambda<\beta, \\ \mathsf{s}^+_{[\mu,\lambda]}\mathsf{d}_{\kappa\mu} \text{ if } \kappa<\beta,\ \lambda\geqq\beta, \\ \mathsf{s}^+_{[0,\kappa]}\mathsf{s}^+_{[1,\lambda]}\mathsf{d}_{01} \text{ if } \kappa,\lambda\geqq\beta. \end{cases}$$

First we check that this definition does not depend on the particular $\mu\in\beta\sim\{\kappa,\lambda\}$ chosen. In fact, suppose also $\nu\in\beta\sim\{\kappa,\lambda,\mu\}$. Note that $\{\mu,\lambda\}\upharpoonright[\mu,\kappa]=\{\mu,\lambda\}\upharpoonright([\nu,\kappa]\circ[\mu,\nu])$. Hence for $\kappa\geqq\beta$, $\lambda<\beta$ we have

$$\begin{aligned}\mathsf{s}^+_{[\mu,\kappa]}\mathsf{d}_{\mu\lambda} &= \mathsf{s}^+_{[\nu,\kappa]}\mathsf{s}^+_{[\mu,\nu]}\mathsf{d}_{\mu\lambda} = \mathsf{s}^+_{[\nu,\kappa]}\mathsf{s}_{[\mu,\nu]}\mathsf{d}_{\mu\lambda} \\ &= \mathsf{s}^+_{[\nu,\kappa]}\mathsf{d}_{\nu\lambda},\end{aligned}$$

as desired; the other case involving μ is similar.

Now we need to check (E_1)–(E_3). (E_1) is obvious. Checking (E_3) requires several cases, depending on which of $\kappa,\lambda,\tau\kappa,\tau\lambda$ are $<\beta$. We take one typical case: $\kappa,\lambda\geqq\beta$, $\kappa\neq\lambda$, and $\tau\kappa,\tau\lambda<\beta$. Let μ,ν be distinct members of $\beta\sim\{0,1,\tau\kappa,\tau\lambda\}$. Then

$$\begin{aligned}\mathsf{d}^+_{\kappa\lambda} &= \mathsf{s}^+_{[0,\kappa]}\mathsf{s}^+_{[1,\lambda]}\mathsf{d}_{01} = \mathsf{s}^+_{[0,\kappa]}\mathsf{s}^+_{[1,\lambda]}\mathsf{s}^+_{[0,\mu]}\mathsf{s}^+_{[1,\nu]}\mathsf{d}_{\mu\nu}, \\ \mathsf{d}^+_{\tau\kappa,\tau\lambda} &= \mathsf{d}_{\tau\kappa,\tau\lambda} = \mathsf{s}^+_{[\mu,\tau\kappa]}\mathsf{s}^+_{[\nu,\tau\lambda]}\mathsf{d}_{\mu\nu}.\end{aligned}$$

Since $\{\mu,\nu\}\upharpoonright\{[\mu,\tau\kappa]\circ[\nu,\tau\lambda]\}=\{\mu,\nu\}\upharpoonright(\tau\circ[0,\kappa]\circ[1,\lambda]\circ[0,\mu]\circ[1,\nu])$, it follows easily that $\mathsf{s}^+\mathsf{d}_{\kappa\lambda}=\mathsf{d}^+_{\tau\kappa,\tau\lambda}$, as desired. To prove (E_2), we first show it for $x\in A$. If $\kappa,\lambda<\beta$, it follows from (E_2) for $\mathfrak{A}$. Suppose $\kappa\geqq\beta$, $\lambda<\beta$. Choose $\mu\in\beta\sim\{\lambda\}$, and then choose $\sigma,\tau\in{}^\alpha\alpha$ such that $(\alpha\sim\beta)\upharpoonright Id\subseteq\tau$, $\sigma\circ\tau=\alpha\upharpoonright Id$, and $\mu\notin Rg\tau$. Then

$$\begin{aligned}\mathsf{s}^+_\tau(x\cdot\mathsf{d}^+_{\kappa\lambda}) &= \mathsf{s}^+_\tau x\cdot\mathsf{s}^+_{[\mu,\kappa]}\mathsf{d}_{\mu\lambda} = \mathsf{s}^+_\tau x\cdot\mathsf{s}^+_{[\mu,\kappa]}\mathsf{s}^+_{[\lambda/\tau\lambda]}\mathsf{d}_{\mu\lambda} \\ &= \mathsf{s}^+_\tau x\cdot\mathsf{s}^+_{[\mu,\kappa]}\mathsf{d}_{\mu,\tau\lambda} = \mathsf{s}^+_{[\mu,\kappa]}(\mathsf{s}^+_\tau x\cdot\mathsf{d}_{\mu,\tau\lambda}) \\ &\leqq \mathsf{s}^+_{[\mu,\kappa]}\mathsf{s}^+_{[\mu/\tau\lambda]}\mathsf{s}^+_\tau x = \mathsf{s}^+_\tau x.\end{aligned}$$

Applying s^+_σ, we get $x\cdot\mathsf{d}^+_{\kappa\lambda}\leqq x=\mathsf{s}^+_{[\kappa/\lambda]}x$, as desired. The other cases are similar.

Now the general case of (E_2) follows: for $x\in A$ and σ a permutation of α, choose $\mu,\nu<\alpha$ so that $\sigma\mu=\kappa$ and $\sigma\nu=\lambda$; then

$$\mathsf{s}^+_\sigma x\cdot\mathsf{d}_{\kappa\lambda} = \mathsf{s}^+_\sigma(x\cdot\mathsf{d}_{\mu\nu}) \leqq \mathsf{s}^+_\sigma\mathsf{s}^+_{[\mu/\nu]}x = \mathsf{s}^+_{[\kappa/\lambda]}\mathsf{s}^+_\sigma x.$$

COROLLARY 5.4.18. *If $\beta\geqq\omega$ and $\mathfrak{A}$ is a PEA_β, then $\mathfrak{Rb}_{ca}\mathfrak{A}$ is representable.*

PROOF. By 5.4.17 and 3.2.10.

The notions and results of section 2.7 can be extended to polyadic algebras. This is a consequence of Jónsson, Tarski [51],[52] 2.15 and 2.18. Thus every PA_α (or PEA_α) can be embedded in a complete atomic PA_α (or PEA_α). The notion of a complex algebra deserves special attention:

DEFINITION 5.4.19. *Let* $\mathfrak{B}=(B,T_\Gamma,U_\tau)_{\Gamma\subseteq\alpha,\tau\in S}$ *be a relational structure, with* $S={}^\alpha\alpha$, *such that* $T_\Gamma\subseteq B\times B$ *and* $U_\tau\subseteq B\times B$ *for all* $\Gamma\subseteq\alpha$ *and* $\tau\in{}^\alpha\alpha$. *The* **complex algebra** $\mathfrak{Cm}\mathfrak{B}$ *of* $\mathfrak{B}$ *is the algebra*

$$(SbB,\cup,\cap,{}_B\sim,0,B,T_\Gamma^*,U_\tau^*)_{\Gamma\subseteq\alpha,\tau\in S}.$$

Similarly, if $\mathfrak{B}=(B,T_\Gamma,U_\tau,E_{\kappa\lambda})_{\Gamma\subseteq\alpha,\tau\in S,\kappa,\lambda<\alpha}$ *is a relational structure with* $T_\Gamma\subseteq B\times B$, $U_\tau\subseteq B\times B$, $E_{\kappa\lambda}\subseteq B$ *for all* $\Gamma\subseteq\alpha$, $\tau\in{}^\alpha\alpha$, $\kappa,\lambda<\alpha$, $\mathfrak{Cm}\mathfrak{B}$ *is*

$$(SbB,\cup,\cap,{}_B\sim,0,B,T_\Gamma^*,U_\tau^*,E_{\kappa\lambda})_{\Gamma\subseteq\alpha,\tau\in S,\kappa,\lambda<\alpha}.$$

The analog of 2.7.40 is as follows:

THEOREM 5.4.20. *For every relational structure* $\mathfrak{B}=(B,T_\Gamma,U_\tau)_{\Gamma\subseteq\alpha,\tau\in S}$ *with* $S={}^\alpha\alpha$, $T_\Gamma\subseteq B\times B$, $U_\tau\subseteq B\times B$, *we have* $\mathfrak{Cm}\mathfrak{B}\in\mathsf{PA}_\alpha$ *iff the following seven conditions hold for all* $\Gamma,\Delta\subseteq\alpha$ *and all* $\sigma,\tau\in{}^\alpha\alpha$:

(i) T_Γ *is an equivalence relation on* B,

(ii) $T_\Gamma|T_\Delta=T_{\Gamma\cup\Delta}$,

(iii) U_τ^{-1} *is a function from* B *into* B,

(iv) $U_{\alpha\upharpoonright Id}=B\upharpoonright Id$,

(v) $U_\tau|U_\sigma=U_{\sigma\circ\tau}$,

(vi) *if* $(\alpha\sim\Gamma)\upharpoonright\sigma=(\alpha\sim\Gamma)\upharpoonright\tau$ *then* $T_\Gamma|U_\sigma=T_\Gamma|U_\tau$,

(vii) *if* $(\tau^{-1}{*}\Gamma)\upharpoonright\tau$ *is one–one then* $U_\tau|T_\Gamma=T_\Delta|U_\tau$, *where* $\Delta=\tau^{-1}{*}\Gamma$.

PROOF. First assume that $\mathfrak{Cm}\mathfrak{B}\in\mathsf{PA}_\alpha$. Just as in the proof of 2.7.40, it is easily shown that (i) holds. For any $x,y\in B$ we have $x(T_\Gamma|T_\Delta)y$ iff $xT_\Gamma zT_\Delta y$ for some z iff $y\in T_\Delta^*T_\Gamma^*\{x\}$ iff $y\in T_{\Gamma\cup\Delta}^*\{x\}$ iff $xT_{\Gamma\cup\Delta}y$, so (ii) holds. Suppose that $xU_\tau y$ and $zU_\tau y$. Thus $y\in U_\tau^*\{x\}\cap U_\tau^*\{z\}$ so, since U_τ^* is an endomorphism of $\mathfrak{Rl}\mathfrak{Cm}\mathfrak{B}$, $x=z$. Thus U_τ^{-1} is a function. If $x\in B$, then $x\in B=U_\tau^*B$, so there is a $y\in B$ with $yU_\tau x$; hence $x\in DoU_\tau^{-1}$. Hence (iii) holds. (iv) is clear since $U_{\alpha\upharpoonright Id}^*X=X$ for any $X\subseteq B$. For (v), we have $x(U_\tau|U_\tau)y$ iff $xU_\tau zU_\sigma y$ for some z iff $y\in U_\sigma^*U_\tau^*\{x\}=U_{\sigma\circ\tau}^*\{x\}$ iff $xU_{\sigma\circ\tau}y$. Assume the hypothesis of (vi). Then $x(T_\Gamma|U_\sigma)y$ iff $xT_\Gamma zU_\sigma y$ iff $y\in U_\sigma^*T_\Gamma^*\{x\}$ iff $y\in U_\tau^*T_\Gamma^*\{x\}$ iff $x(T_\Gamma|U_\tau)y$. (vii) is established similarly.

The converse part of the theorem is just as routine.

THEOREM 5.4.21. *For every relational structure* $\mathfrak{B}=(B,T_\Gamma,U_\tau,E_{\kappa\lambda})_{\Gamma\subseteq\alpha,\tau\subseteq\Sigma,\kappa,\lambda<\alpha}$ *with* $S={}^\alpha\alpha$, $T_\Gamma\subseteq B\times B$, $E_{\kappa\lambda}\subseteq B$, *we have* $\mathfrak{Cm}\mathfrak{B}\in\mathsf{PEA}_\alpha$ *iff in addition to* 5.4.19(i)–(vii) *the following three conditions hold for all* $\kappa,\lambda<\alpha$ *and all* $\tau\in{}^\alpha\alpha$:

(viii) $E_{\kappa\kappa}=B$,

(ix) $E_{\kappa\lambda}\upharpoonright Id\subseteq U_{[\kappa/\lambda]}$,

(x) $U_\tau^*E_{\kappa\lambda}\subseteq E_{\tau\kappa,\tau\lambda}$.

PROOF: routine.

CA$_\alpha$'s and representable polyadic algebras

We now turn to the discussion of representable polyadic algebras.

DEFINITION 5.4.22.(i) *Let U be a set, α an ordinal. For any $\Gamma \subseteq \alpha$ and any $X \subseteq {}^\alpha U$ we set, with $V = {}^\alpha U$,*

$$\mathsf{C}^{|V|}_{(\Gamma)} X = \{u \in {}^\alpha U : \textit{there is a } v \in X \textit{ with } (\alpha \sim \Gamma) 1 u = (\alpha \sim \Gamma) 1 v\}.$$

For each $\tau \in {}^\alpha \alpha$ and each $X \subseteq {}^\alpha U$ we let

$$\mathsf{s}^{|V|}_\tau X = \{u \in {}^\alpha U : x \circ \tau \in X\}.$$

(ii) *$\mathfrak{A}$ is a* **polyadic set algebra of dimension** *α* **with base** *U, in symbols $\mathfrak{A} \in \mathsf{Ps}_\alpha$, provided that $\mathfrak{A} = (A, \mathsf{u}, \mathsf{n}, {}_V\sim, 0, {}^\alpha U, \mathsf{C}^{|V|}_{(\Gamma)}, \mathsf{s}^{|V|}_\tau)_{\Gamma \subseteq \alpha, \tau \in S}$, where $V = {}^\alpha U$, $S = {}^\alpha \alpha$, and A is a collection of subsets of ${}^\alpha U$ closed under all the indicated operations.*

(iii) *$\mathfrak{A}$ is a* **polyadic equality set algebra of dimension** *α* **with base** *U, in symbols $\mathfrak{A} \in \mathsf{Pse}_\alpha$, if $\mathfrak{A} = (A, \mathsf{u}, \mathsf{n}, {}_V\sim, 0, {}^\alpha U, \mathsf{C}^{|V|}_{(\Gamma)}, \mathsf{s}^{|V|}_\tau, \mathsf{D}^{|V|}_{\kappa\lambda})_{\Gamma \subseteq \alpha, \tau \in S, \kappa, \lambda < \alpha}$, with V and S as above, $(A, \mathsf{u}, \mathsf{n}, {}_V\sim, 0, {}^\alpha U, \mathsf{C}^{|V|}_{(\Gamma)}, \mathsf{s}^{|V|}_\tau)_{\Gamma \subseteq \alpha, \tau \in S}$ a Ps_α, and $\mathsf{D}^{|V|}_{\kappa\lambda} \in A$, for all $\kappa, \lambda < \alpha$.*

(iv) *$\mathfrak{A}$ is a* **representable polyadic (equality) algebra** *of dimension α, in symbols $\mathfrak{A} \in \mathsf{Rpp}_\alpha$ ($\mathfrak{A} \in \mathsf{Rppe}_\alpha$), if $\mathfrak{A}$ is isomorphic to a subdirect product of members of Ps_α (Pse_α).*

It is straight-forward to check that every Ps_α, Rpp_α (resp. Pse_α, Rppe_α) is a PA_α (resp. PEA_α). Note that the definition of representable polyadic algebra differs in form from the corresponding definition for cylindric algebras; but cf. 3.1.77.

Regularity of CA$_\alpha$'s has an elementary formulation in polyadic terms. Thus let $\mathfrak{A}$ be a Pse_α. Then, of course, $\mathfrak{Rb}_{ca}\mathfrak{A}$ is a Cs_α. An element $x \in A$ ($= Rd_{ca}\mathfrak{A}$) is regular iff $\mathsf{c}_{(\alpha \sim \Delta x)} x = x$. Also, if $\mathfrak{A} \in \mathsf{Pse}_\alpha$ is locally finite-dimensional (in the sense of Definition 5.4.10), then $\mathfrak{Rb}_{ca}\mathfrak{A} \in \mathsf{Cs}^{reg}_\alpha$. In fact, if $x \in A$, choose a finite $\Gamma \subseteq \alpha$ so that $\mathsf{c}_{(\alpha \sim \Gamma)} x = x$. Clearly $\Delta x \subseteq \Gamma$, so $\mathsf{c}_{(\alpha \sim \Delta x)} x = \mathsf{c}_{(\alpha \sim \Gamma)} \mathsf{c}_{(\Gamma \sim \Delta x)} x = \mathsf{c}_{(\alpha \sim \Gamma)} x = x$, so x is, indeed, regular. Hence a Pse_α $\mathfrak{A}$ is locally finite-dimensional iff $\mathfrak{Rb}_{ca}\mathfrak{A}$ is locally finite dimensional and regular.

Now we return briefly to the questions raised in 5.4.6; the following theorem was proved independently by Henkin, and by Andréka and Németi:

THEOREM 5.4.23 $\mathsf{CA}_2 = \mathsf{S}\mathsf{Rd}_{ca}\mathsf{PEA}_2$.

PROOF. The inclusion $\subseteq$ is all that needs to be proved. Clearly it suffices to show that any simple complete atomic CA_2 is embeddable in the CA-reduct of some PEA_2. Here we can apply 3.2.59. By it, it suffices to show that

$$b = {}^2U \sim \mathsf{U}_{\iota \in I}({}^2X_\iota \sim \mathsf{D}_{01})$$

has the properties described in 5.4.15, considering b as an element of the Pse_2 of all subsets of ${}^2U$ (see 3.2.59 for the notation). Condition 5.4.15(i) is immediate from 3.2.59, and 5.4.15(ii) is easy to check. The only non-trivial instance of 5.4.15(iii) is illustrated by

$$\mathsf{s}_{[0,1]}(\mathsf{c}_1 x \cdot b) \cdot b \leqq \mathsf{c}_0(\mathsf{s}_{[0,1]} x \cdot b),$$

which is also straight–forward.

In Andréka, Németi [84b'] it is shown that if $\alpha<\omega$ and $\mathfrak{A}$ is a PEA_α such that $\mathfrak{Rb}_{c\alpha}\mathfrak{A}$ is a full Cs_α, then $\mathfrak{A}$ is a Pse_α; here the hypothesis that $\mathfrak{Rb}_{c\alpha}\mathfrak{A}$ is a full Cs_α (rather than a non–full Cs_α or a full Gs_α) is necessary. We do not know if $\alpha<\omega$ is needed; as a partial extension to $\alpha \geqq \omega$ Andréka and Németi have shown that if $\mathfrak{A}$ is a simple Mn_α with $\alpha \geqq \omega$ and $\mathfrak{A} = \mathfrak{Rb}_{c\alpha}\mathfrak{B}$ for some PEA_α $\mathfrak{B}$, then $\mathfrak{B}$ is representable. Furthermore, if $2 \leqq \alpha < \omega$, then there is a Cs_α $\mathfrak{A}$ such that $\mathfrak{A} \neq \mathfrak{Rb}_{c\alpha}\mathfrak{B}$ for every PEA_α $\mathfrak{B}$ (cf. 5.4.6).

We now indicate some relationships between representable PA_α's, PEA_α's, CA_α's, and Df_α's; these results are from J.S. Johnson [69].

THEOREM 5.4.24. *Suppose that* $3 \leqq \alpha < \omega$, $\mathfrak{A}$ *is a* PEA_α *generated by* $\{x : \Delta x \neq \alpha\}$, *and the* PA_α*–reduct of* $\mathfrak{A}$ *is representable. Then* $\mathfrak{A}$ *is representable.*

PROOF. We may assume that $\mathfrak{A}$ is simple, hence that the PA_α reduct of $\mathfrak{A}$ is a Ps_α with base U. The proof now follows the lines of the proof of Theorem 5.1.51. Corresponding to Lemma 5.1.48 we claim (with $V = {}^\alpha U$):

(1) $\mathsf{D}_{\kappa\lambda}^{|V|} \subseteq \mathsf{d}_{\kappa\lambda}$ for all $\kappa, \lambda < \alpha$.

To prove this, let $a \in \mathsf{D}_{\kappa\lambda}^{|V|}$. Since $\mathsf{c}_\kappa \mathsf{d}_{\kappa\lambda} = 1$ (see the proof of 5.4.3), choose $u \in U$ such that $a_u^\kappa \in \mathsf{d}_{\kappa\lambda}$. Now by (E_2), $\mathsf{d}_{\kappa\lambda} \leqq \mathsf{s}_{[\kappa/\lambda]}\mathsf{d}_{\kappa\lambda}$, so $a_u^\kappa \circ [\kappa/\lambda] \in \mathsf{d}_{\kappa\lambda}$. But $a_u^\kappa \circ [\kappa/\lambda] = a$, so $a \in \mathsf{d}_{\kappa\lambda}$, proving (1).

By (1), we have the conclusion of Lemma 5.1.49 available. Now we want to extend Lemma 5.1.50, showing that $E \in Su\mathfrak{A}$. (Lemma 5.1.50 itself only gives that $E \in Su\mathfrak{Rb}_{c\alpha}\mathfrak{A}$.) Suppose that $X \in E$, $\tau \in {}^\alpha\alpha$, $x, y \in {}^\alpha U$, and $x_\lambda R y_\lambda$ for all $\lambda < \alpha$. Then $(x \circ \tau)\lambda = x_{\tau\lambda} R y_{\tau\lambda} = (y \circ \tau)_\lambda$ for all $\lambda < \alpha$. Hence $x \circ \tau \in X$ iff $y \circ \tau \in X$, i.e., $x \in \mathsf{s}_\tau X$ iff $y \in \mathsf{s}_\tau X$. Thus $\mathsf{s}_\tau X \in E$, as desired.

Now we are ready to define the desired isomorphism G of $\mathfrak{A}$ with a Pse_α. The base of this Pse_α is U/R, R as in Lemma 5.1.49; for any $x \in A$ we set

$$GX = \{R^* \circ x : x \in X\}.$$

We take only three non–trivial parts in the verification that G is an isomorphism: $GX \cap G({}^\alpha U \sim X) = 0$, $G\mathsf{s}_\tau X = \mathsf{s}_\tau GX$, and $G\mathsf{d}_{\kappa\lambda} = \mathsf{D}_{\kappa\lambda}$. Suppose $y \in GX \cap G({}^\alpha U \sim X)$. Say $y = R^* \circ x$, $x \in X$ and $y = R^* \circ z$, $z \in {}^\alpha U \sim X$. Thus $x_\lambda R z_\lambda$ for all $\lambda < \alpha$, so we have contadicted $E = A$ (a consequence of $E \in Su\mathfrak{A}$ and the hypothesis of the theorem). Next, suppose that $y \in G\mathsf{s}_\tau X$. Say $y = R^* \circ x$ with $x \in \mathsf{s}_\tau X$. Thus $x \circ \tau \in X$, so $y \circ \tau = R^* \circ x \circ \tau \in GX$, hence $y \in \mathsf{s}_\tau GX$. Conversely, suppose that $y \in \mathsf{s}_\tau GX$. Thus $y \circ \tau \in GX$, say $y \circ \tau = R^* \circ x$ with $x \in X$. Let $y = R^* \circ z$, $z \in {}^\alpha U$. Then $R^* \circ z \circ \tau = R^* \circ x$, so by $E = A$ we have $z \circ \tau \in X$. Thus $z \in \mathsf{s}_\tau X$ and $y \in G(\mathsf{s}_\tau X)$, as desired. Next, suppose that $y \in G\mathsf{d}_{\kappa\lambda}$. Say $y = R^* \circ x$, $x \in \mathsf{d}_{\kappa\lambda}$. By the conclusion of Lemma 5.1.49, $x_\kappa R x_\lambda$, and hence $y\kappa = y\lambda$ and $y \in \mathsf{D}_{\kappa\lambda}$. Conversely, suppose that $y \in \mathsf{D}_{\kappa\lambda}$. Say $y = R^* \circ x$. Thus $x_\kappa R x_\lambda$, so

$y\kappa = y\lambda$ and $y \in \mathsf{D}_{\kappa\lambda}$. Conversely, suppose that $y \in \mathsf{D}_{\kappa\lambda}$. Say $y = R^* \circ x$. Thus $x_\kappa R x_\lambda$, so by the conclusion of Lemma 5.1.49 there is a $z \in \mathsf{d}_{\kappa\lambda}$ such that $z_\kappa = x_\kappa$ and $z_\lambda = x_\lambda$. But $\mathsf{c}_{(\alpha \sim \{\kappa,\lambda\})}\mathsf{d}_{\kappa\lambda} = \mathsf{d}_{\kappa\lambda}$, so $x \in \mathsf{d}_{\kappa\lambda}$ too. Thus $y \in G\mathsf{d}_{\kappa\lambda}$, finishing the proof of Theorem 5.4.22.

COROLLARY 5.4.25. *Suppose that* $3 \leq \alpha < \omega$ *and* $\mathfrak{A}$ *is a* PEA$_\alpha$ *neatly embedded in a* PEA$_{\alpha+1}$ $\mathfrak{B}$ *such that the polyadic reduct of* $\mathfrak{B}$ *is representable. Then* $\mathfrak{A}$ *is representable.*

PROOF: obvious from Theorem 5.4.24.

THEOREM 5.4.26. *Suppose that* $3 \leq \alpha < \omega$, $\mathfrak{A}$ *is a* PEA$_\alpha$, $\mathfrak{Rd}_{ca}\mathfrak{A}$ *is representable, and* $\mathfrak{B}\mathfrak{l}\mathfrak{A}$ *is generated by* $\{x : \Delta x \neq \alpha\}$. *Then* $\mathfrak{A}$ *is representable.*

PROOF. It suffices to show that if $0 \neq a \in A$, then there is a homomorphism f of $\mathfrak{A}$ onto a Pse$_\alpha$ $\mathfrak{B}$ such that $fa \neq 0$. Let f be a homomorphism of $\mathfrak{Rd}_{ca}\mathfrak{A}$ onto a Cs$_\alpha$ with base U such that $fa \neq 0$. We shall show that f preserves each operation s_τ as well. Note by Theorem 5.4.3 that $f\mathsf{s}_{[\kappa/\lambda]}x = \{u \in {}^\alpha U : u \circ [\kappa/\lambda] \in x\} = \mathsf{s}_{[\kappa/\lambda]}fx)$. Hence, since α is finite, it suffices to show that f preserves $\mathsf{s}_{[\kappa,\lambda]}$ for arbitrary distinct $\kappa,\lambda < \alpha$. By hypothesis it suffices to show that $f\mathsf{s}_{[\kappa,\lambda]}x = \mathsf{s}_{[\kappa,\lambda]}fx$ for any $x \in A$ such that $\Delta x \neq \alpha$. Choose $\mu \in \alpha \sim \Delta x$. If $\kappa = \mu$, then $f\mathsf{s}_{[\kappa,\lambda]}x = f\mathsf{s}_{[\kappa,\lambda]}\mathsf{c}_\mu x = f\mathsf{s}_{[\lambda/\kappa]}x = \mathsf{s}_{[\lambda/\kappa]}fx = \mathsf{s}_{[\lambda/\kappa]}\mathsf{c}_\mu fx = \mathsf{s}_{[\kappa,\lambda]}fx$, as desired. Similarly if $\lambda = \mu$. If $\mu \neq \kappa,\lambda$, then

$$\begin{aligned} f\mathsf{s}_{[\kappa,\lambda]}x &= f\mathsf{s}_{[\mu/\lambda]}\mathsf{s}_{[\lambda/\kappa]}\mathsf{s}_{[\kappa/\mu]}\mathsf{c}_\mu x \\ &= \mathsf{s}_{[\mu/\lambda]}\mathsf{s}_{[\lambda/\kappa]}\mathsf{s}_{[\kappa/\mu]}\mathsf{c}_\mu fx = \mathsf{s}_{[\kappa,\lambda]}fx, \end{aligned}$$

as desired.

THEOREM 5.4.27. *Suppose that* $3 \leq \alpha < \omega$, $\mathfrak{A}$ *is a* PEA$_\alpha$ *neatly embedded in a* PEA$_{\alpha+1}$ $\mathfrak{B}$, *and* $\mathfrak{Rd}_{ca}\mathfrak{B}$ *is representable. Then* $\mathfrak{A}$ *is representable.*

PROOF. Let $\mathfrak{C} = \mathfrak{Sg}^{\mathfrak{B}}A$. Then also $C = Sg^{\mathfrak{B}\mathfrak{l}\mathfrak{C}}\{X \in C : |\Delta X| < \alpha+1\}$. In fact, $\supseteq$ is trivial, so it suffices to show $\subseteq$. For this, note that $A \subseteq \{X \in C : |\Delta X| < \alpha+1\}$, and $Sg^{\mathfrak{B}\mathfrak{l}\mathfrak{C}}\{X \in C : |\Delta X| < \alpha+1\}$ is closed under *all* the operations of $\mathfrak{B}$ and contains all diagonal elements. Then 5.4.26 yields that $\mathfrak{C}$, and hence $\mathfrak{A}$, are representable.

THEOREM 5.4.28. *Suppose that* $3 \leq \alpha < \omega$, $\mathfrak{A}$ *is a* PEA$_\alpha$ *neatly embedded in a* PEA$_{\alpha+2}$ $\mathfrak{B}$ *such that* $\mathfrak{Rd}_{df}\mathfrak{B}$ *is representable. Then* $\mathfrak{A}$ *is representable.*

PROOF. Let $\mathfrak{C}$ be the neat $(\alpha+1)$—reduct of $\mathfrak{B}$. By 5.1.52, $\mathfrak{Rd}_{ca}\mathfrak{C}$ is representable. Hence by 5.4.27 $\mathfrak{A}$ is representable.

THEOREM 5.4.29. *Suppose that* $3 \leq \alpha < \omega$, *and* $\mathfrak{A}$ *is a* PEA$_\alpha$, $\mathfrak{Rd}_{df}\mathfrak{A}$ *is representable, and* $\mathfrak{B}\mathfrak{l}\mathfrak{A}$ *is generated by* $\{x : \Delta x \neq \alpha\}$. *Then* $\mathfrak{A}$ *is representable.*

PROOF. By 5.1.51, $\mathfrak{Rd}_{ca}\mathfrak{A}$ is representable, so by 5.4.26 $\mathfrak{A}$ is representable.

Now we turn to some representation theorems for polyadic algebras. First we shall show that for each α with $2 \leqq \alpha < \omega$ there is a PEA_α whose CA_α–reduct is non–representable. To do this, we start with a version of Lemma 2.6.12.

LEMMA 5.4.30. *Assume that* $2 \leqq \alpha < \omega$ *and* $\mathfrak{A}$ *is a* PEA_α *with notation as in* 5.4.1, *and* a *is any element of* $\mathfrak{A}$ *such that* $a \leqq -\mathsf{d}_{\kappa\lambda}$ *whenever* $\kappa < \lambda < \alpha$ *and* $\mathsf{s}_\sigma a = a$ *for any permutation* $\sigma \in {}^\alpha\alpha$. *Let* $\mathfrak{B} = \langle B, +', \cdot', -', 0', 1', \mathsf{c}'_{(\Gamma)}, \mathsf{s}'_\tau, \mathsf{d}'_{\kappa\lambda} \rangle_{\Gamma \subseteq \alpha, \tau \in S, \kappa, \kappa, \lambda < \alpha}$, *with* $S = {}^\alpha\alpha$, *similar to* $\mathfrak{A}$, *determined by the following stipulations*:

(i) $\langle B, +', \cdot', -', 0', 1' \rangle = \mathfrak{B}\mathfrak{l}\mathfrak{A} \times \mathfrak{R}\mathfrak{l}_a \mathfrak{B}\mathfrak{l}\mathfrak{A}$;

(ii) $\mathsf{d}'_{\kappa\lambda} = \langle \mathsf{d}_{\kappa\lambda}, a \cdot \mathsf{d}_{\kappa\lambda} \rangle$ *for all* $\kappa, \lambda < \alpha$;

(iii) $\mathsf{c}'_{(\Gamma)}\langle x, y \rangle = \langle \mathsf{c}_{(\Gamma)}(x+y), a \cdot \mathsf{c}_{(\Gamma)}(x+y) \rangle$ *for all* $x, y \in A$ *with* $y \leqq a$ *and all non–empty* $\Gamma \subseteq \alpha$;

(iv) $B \upharpoonright \mathsf{c}'_{(0)} = B \upharpoonright Id$;

(v) $\mathsf{s}'_\sigma \langle x, y \rangle = \langle \mathsf{s}_\sigma x, \mathsf{s}_\sigma y \rangle$ *if* σ *is a permutation of* α, $\mathsf{s}'_\sigma \langle x, y \rangle = \langle \mathsf{s}_\sigma x, a \cdot \mathsf{s}_\sigma x \rangle$ *otherwise, for all* $x, y \in A$ *with* $y \leqq a$.

Under these assumptions, $\mathfrak{B} \in \mathsf{PEA}_\alpha$. *Furthermore,* $\mathfrak{A} \cong \mathfrak{R}\mathfrak{l}_{\langle 1,0 \rangle}\mathfrak{B}$ *and for each* $x \in A$ *such that* $x \leqq a$ *we have disjoint elements* $\langle x, 0 \rangle$ *and* $\langle 0, x \rangle$ *in* $\mathfrak{B}$ *such that* $\mathsf{c}_{(\Gamma)}\langle x, 0 \rangle = \mathsf{c}_{(\Gamma)}\langle 0, x \rangle$ *for every non–empty* $\Gamma \subseteq \alpha$.

PROOF. The axioms (P_0),(P_1),(P_2),(P_4),(P_6) and (E_1) are obvious, while (P_3) is treated like (C_3) in the proof of 2.6.12. For (P_5) we have, for non–empty $\Gamma, \Delta \subseteq \alpha$,

$$\begin{aligned} \mathsf{c}'_{(\Gamma)}\mathsf{c}'_{(\Delta)}\langle x, y \rangle &= \mathsf{c}'_{(\Gamma)}\langle \mathsf{c}_{(\Delta)}(x+y), a \cdot \mathsf{c}_{(\Delta)}(x+y) \rangle \\ &= \langle \mathsf{c}_{(\Gamma)}\mathsf{c}_{(\Delta)}(x+y), a \cdot \mathsf{c}_{(\Gamma)}\mathsf{c}_{(\Delta)}(x+y) \rangle \\ &= \mathsf{c}'_{(\Gamma \cup \Delta)}\langle x, y \rangle. \end{aligned}$$

Next we take (P_7). If σ and τ are permutations, the assertion is clear. If σ is a permutation but τ is not, then

$$\begin{aligned} \mathsf{s}'_\sigma \mathsf{s}'_\tau \langle x, y \rangle &= \mathsf{s}'_\sigma \langle \mathsf{s}_\tau x, a \cdot \mathsf{s}_\tau x \rangle \\ &= \langle \mathsf{s}_\sigma \mathsf{s}_\tau x, a \cdot \mathsf{s}_\sigma \mathsf{s}_\tau x \rangle = \mathsf{s}'_{\sigma \circ \tau} \langle x, y \rangle. \end{aligned}$$

If τ is a permutation but σ is not, then

$$\begin{aligned} \mathsf{s}'_\sigma \mathsf{s}'_\tau \langle x, y \rangle &= \mathsf{s}'_\sigma \langle \mathsf{s}_\tau x, \mathsf{s}_\tau y \rangle \\ &= \langle \mathsf{s}_\sigma \mathsf{s}_\tau x, a \cdot \mathsf{s}_\sigma \mathsf{s}_\tau x \rangle = \mathsf{s}'_{\sigma \circ \tau} \langle x, y \rangle. \end{aligned}$$

Finally, if neither is a permutation then

$$\begin{aligned} \mathsf{s}'_\sigma \mathsf{s}'_\tau \langle x, y \rangle &= \mathsf{s}'_\sigma \langle \mathsf{s}_\tau x, a \cdot \mathsf{s}_\tau x \rangle \\ &= \langle \mathsf{s}_\sigma \mathsf{s}_\tau x, a \cdot \mathsf{s}_\sigma \mathsf{s}_\tau x \rangle = \mathsf{s}'_{\sigma \circ \tau} \langle x, y \rangle. \end{aligned}$$

(P_8) and (P_9) are routine. (P_{10}) is clear if $\Gamma = 0$, while for $\Gamma \neq 0$ we have, for any $\rho \in {}^\alpha\alpha$,

$$\begin{aligned} \mathsf{s}'_\rho \mathsf{c}'_{(\Gamma)}\langle x, y \rangle &= \mathsf{s}'_\rho \langle \mathsf{c}_{(\Gamma)}(x+y), a \cdot \mathsf{c}_{(\Gamma)}(x+y) \rangle \\ &= \langle \mathsf{s}_\rho \mathsf{c}_{(\Gamma)}(x+y), a \cdot \mathsf{s}_\rho \mathsf{c}_{(\Gamma)}(x+y) \rangle; \end{aligned}$$

hence (P_{10}) follows. (P_{11}) is clear if $\Gamma=0$. Assume that $\Gamma\neq 0$. If $\Delta=0$, then τ is not a permutation, and

$$\begin{aligned} c'_{(\Gamma)}s'_\tau\langle x,y\rangle &= c'_{(\Gamma)}\langle s_\tau x, a\cdot s_\tau x\rangle \\ &= \langle c_{(\Gamma)}s_\tau x, a\cdot c_{(\Gamma)}s_\tau x\rangle \\ &= \langle s_\tau x, a\cdot s_\tau x\rangle = s'_\tau\langle x,y\rangle. \end{aligned}$$

If $\Delta\neq 0$ and τ is a permutation, then

$$\begin{aligned} c'_{(\Gamma)}s'_\tau\langle x,y\rangle &= c'_{(\Gamma)}\langle s_\tau x, s_\tau y\rangle \\ &= \langle c_{(\Gamma)}s_\tau(x+y), a\cdot c_{(\Gamma)}s_\tau(x+y)\rangle \\ &= \langle s_\tau c_{(\Delta)}(x+y), a\cdot s_\tau c_{(\Delta)}(x+y)\rangle \\ &= s'_\tau c'_{(\Delta)}\langle x,y\rangle \end{aligned}$$

For $\Delta\neq 0$ and τ not a permutation,

$$\begin{aligned} c'_{(\Gamma)}s'_\tau\langle x,y\rangle &= c'_{(\Gamma)}\langle s_\tau x, a\cdot s_\tau x\rangle \\ &= \langle c_{(\Gamma)}s_\tau x, a\cdot c_{(\Gamma)}s_\tau x\rangle, \end{aligned}$$

while

$$\begin{aligned} s'_\tau c'_{(\Delta)}\langle x,y\rangle &= s'_\tau\langle c_{(\Delta)}(x+y), a\cdot c_{(\Delta)}(x+y)\rangle \\ &= \langle s_\tau c_{(\Delta)}(x+y), a\cdot s_\tau c_{(\Delta)}(x+y)\rangle \\ &= \langle c_{(\Gamma)}(s_\tau x+s_\tau y), a\cdot c_{(\Gamma)}(s_\tau x+s_\tau y)\rangle \\ &= \langle c_{(\Gamma)}s_\tau x, a\cdot c_{(\Gamma)}s_\tau x\rangle, \end{aligned}$$

since $s_\tau y\leqq s_\tau a\leqq s_\tau(-d_{\kappa\lambda})=0$, where κ,λ are such that $\kappa\neq\lambda$ and $\tau\kappa=\tau\lambda$. (E_2) holds clearly if $\kappa=\lambda$, while for $\kappa\neq\lambda$ we have

$$\begin{aligned} \langle x,y\rangle\cdot d'_{\kappa\lambda} &= \langle x,y\rangle\cdot\langle d_{\kappa\lambda},0\rangle = \langle x\cdot d_{\kappa\lambda},0\rangle \\ &\leqq \langle s_{[\kappa/\lambda]}x, a\cdot s_{[\kappa/\lambda]}x\rangle = s_{[\kappa/\lambda]}\langle x,y\rangle. \end{aligned}$$

Finally, (E_3) clearly holds if $\kappa=\lambda$; for $\kappa\neq\lambda$ and τ a permutation we have

$$s'_\tau d'_{\kappa\lambda} = s'_\tau\langle d_{\kappa\lambda},0\rangle = \langle d_{\tau\kappa,\kappa\lambda},0\rangle = d'_{\tau\kappa,\tau\lambda}.$$

For $\kappa\neq\lambda$ and τ not a permutation we get

$$\begin{aligned} s'_\tau d'_{\kappa\lambda} &= s'_\tau\langle d_{\kappa\lambda},0\rangle = \langle s_\tau d_{\kappa\lambda}, a\cdot s_\tau d_{\kappa\lambda}\rangle \\ &= \langle d_{\tau\kappa,\tau\lambda}, a\cdot d_{\tau\lambda,\tau\lambda}\rangle = d'_{\tau\kappa,\tau\lambda}. \end{aligned}$$

The last part of the lemma is clear.

THEOREM 5.4.31. *If* $2\leqq\alpha<\omega$, *then there is a* PEA$_\alpha$ $\mathfrak{B}$ *such that* $\mathfrak{Rb}_{c\alpha}\mathfrak{B}$ *is non-representable. In fact, the following equation (which holds in all* IGs$_\alpha$*'s by* 2.6.41*) fails in* $\mathfrak{B}$:

$$\mathsf{c}_1(y\cdot\mathsf{c}_0(\mathsf{c}_1y\cdot -y))\cdot -\mathsf{c}_0(\mathsf{c}_1y\cdot -\mathsf{d}_{01})=0.$$

PROOF. Let $\mathfrak{X}$ be the Pse_α of all subsets of ${}^\alpha\alpha$, and let $a=\{u\in{}^\alpha\alpha: u \text{ is one-one}\}$. Thus $a\leqq -\mathsf{d}_{\kappa\lambda}$ for all distinct $\kappa,\lambda<\alpha$. We now apply Lemma 5.4.30, obtaining an algebra $\mathfrak{B}$ with the properties indicated there. Let $u=\alpha\upharpoonright Id$ and $y=\langle\{u\},0\rangle$. Then the following facts are easily checked:

$$\begin{aligned}
&-'y=\langle{}^\alpha\alpha\sim\{u\},a\rangle,\\
&\mathsf{c}'_1y=\langle\mathsf{c}_1\{u\},\{u\}\rangle,\\
&\mathsf{c}'_1y\cdot'-'y=\langle\mathsf{c}_1\{u\}\sim\{u\},\{u\}\rangle,\\
&\mathsf{c}'_0(\mathsf{c}'_1y\cdot'-'y)=\langle\mathsf{c}_0\mathsf{c}_1\{u\},\{u\}\rangle,\\
&y\cdot'\mathsf{c}'_0(\mathsf{c}'_1y\cdot'-'y)=\langle\{u\},0\rangle,\\
&\mathsf{c}'_1(y\cdot'\mathsf{c}'_0(\mathsf{c}'_1y\cdot'-'y))=\langle\mathsf{c}_1\{u\},\{u\}\rangle,\\
&\mathsf{d}'_{01}=\langle\mathsf{d}_{01},0\rangle,\\
&-'\mathsf{d}'_{01}=\langle-\mathsf{d}_{01},a\rangle,\\
&\mathsf{c}'_1y\cdot'-'\mathsf{d}'_{01}=\langle\mathsf{c}_1\{u\}\cdot-\mathsf{d}_{01},\{u\}\rangle\\
&\mathsf{c}'_0(\mathsf{c}'_1y\cdot'-'\mathsf{d}'_{01})=\langle\mathsf{c}_0(\mathsf{c}_1\{u\}\cdot-\mathsf{d}_{01}),a\cdot\mathsf{c}_0(\mathsf{c}_1\{u\}\cdot-\mathsf{d}_{01})\rangle,\\
&-'\mathsf{c}'_0(\mathsf{c}'_1y\cdot'-'\mathsf{d}'_{01})=\langle-\mathsf{c}_0(\mathsf{c}_1\{u\}\cdot-\mathsf{d}_{01}),a\cdot-\mathsf{c}_0(\mathsf{c}_1\{u\}\cdot-\mathsf{d}_{01})\rangle,
\end{aligned}$$

and hence $\langle\{u_0^1\},0\rangle$ is $\leqq$ the left side of the given equation, as desired.

Andréka and Németi have noticed that 5.4.31 can be improved if $3\leqq\alpha<\omega$: there is a PEA_α $\mathfrak{B}$ such that $\mathfrak{Rb}_{df}\mathfrak{B}$ is non-representable. This can be seen by using 3.2.68 and 5.4.15.

Analogously to 5.1.47 we shall now show that every PA_2 is representable. To do this we need the following lemma.

LEMMA 5.4.32. *Suppose that κ and λ are cardinals, $1\leqq 2\kappa+\lambda+1\leqq|U|\geqq\omega$. Then there is a partition $\mathcal{P}$ of ${}^2U$ with the following properties:*

(i) $|\mathcal{P}|=\kappa+\lambda+1$,

(ii) $\mathsf{D}_{01}^{|V|}\in\mathcal{P}$, *where* $V={}^2U$,

(iii) *there exist disjoint* $\mathcal{P}_0,\mathcal{P}_1\subseteq\mathcal{P}$ *with* $|\mathcal{P}_0|=\kappa$, $|\mathcal{P}_1|=\lambda$, $\mathsf{D}_{01}^{|V|}\notin\mathcal{P}_0\cup\mathcal{P}_1$, $\mathcal{P}=\mathcal{P}_0\cup\mathcal{P}_1\cup\{\mathsf{D}_{01}^{|V|}\}$ *such that* $X^{-1}=X$ *for all* $X\in\mathcal{P}_0$, $X\neq X^{-1}\in\mathcal{P}_1$ *for all* $X\in\mathcal{P}_1$,

(iv) $\mathsf{C}_\kappa^{|V|}X={}^2U$ *for all* $X\in\mathcal{P}$ *and* $\kappa<2$.

PROOF. We form the sets X_u as in the proof of 3.2.57; if e is the identity of the group, then $X_e=\mathsf{D}_{01}^{|V|}$. Note that $(X_u)^{-1}=X_v$, $v=u^{-1}$, for all $u\in U$. We can combine pairs X_u,X_u^{-1} to give κ elements Y with $Y^{-1}=Y$; at the same time we can arrange for the set $\mathcal{P}_1$, incorporating in elements of $\mathcal{P}_0$ or of $\mathcal{P}_1$ "superfluous" members of $\mathcal{P}$.

THEOREM 5.4.33. *Every* PA_2 *is representable.*

PROOF. Let $\mathfrak{C}$ be a simple PA_2. Embedding $\mathfrak{C}$ in a simple complete and atomic

PA$_2$ $\mathfrak{B}$, it suffices to show that $\mathfrak{B}$ is representable. Now $\mathfrak{B}$ is the PA reduct of a simple PEA$_2$ $\mathfrak{A}$; see the remark following 5.4.1. We shall follow the proof of 3.2.59 closely. We take the definitions of Dat, small, big, A_{ab} from there, and (1)–(4) hold. But now we let, for all $a \in Dat$,

$$X_a = \{\langle a,\alpha\rangle : \alpha < \mu\},$$

where $\mu = |At\mathfrak{A}| \mathbf{u} \omega$. Let $U = \bigcup_{a \in Dat} X_a$. Now we shall define φ mapping $At\mathfrak{A}$ into SbV, $V = {}^2U$, so that the following conditions hold:

(1) if $x, y \in A_{ab}$ and $x \neq y$ then $\varphi x \mathbf{n} \varphi y = 0 \neq \varphi x$,
(2) if $a, b \in Dat$ then $\bigcup_{x \in A_{ab}} \varphi x = X_a \times X_b$,
(3) if $a \in Dat$ then $\mathsf{D}_{01}^{|V|} \mathbf{n} (X_a \times X_a) \subseteq \varphi a$,
(4) if $x \in A_{ab}$ then $\mathsf{C}_0^{|V|} \varphi x = U \times X_b$ and $\mathsf{C}_1^{|V|} \varphi x = X_a \times U$.

Let $<$ be an ordering of Dat. For $a, b \in Dat$ we define $A_{ab} \upharpoonright \varphi$:
Case 1. $a = b$, a small. Let $\varphi a = X_a \times X_a$.
Case 2. $a = b$, a big. By Lemma 5.4.32 let $A_{ab} \upharpoonright \varphi$ be a one–one mapping of A_{aa} onto a partition of $X_a \times X_a$ such that $\varphi a = \mathsf{D}_{01}^{|V|} \mathbf{n} (X_a \times X_a)$, $\varphi \mathsf{s}_{[0,1]} u = (\varphi u)^{-1}$ for every $u \in A_{aa}$, and (4) holds.
Case 3. $a < b$. By Lemma 3.2.57 let φ be a one–one mapping of A_{ab} onto a partition of $X_a \times X_b$ such that (4) holds.
Case 4. $b < a$. For any $u \in A_{ab}$ let $\varphi u = (\varphi \mathsf{s}_{[0,1]} u)^{-1}$; note that $\mathsf{s}_{[0,1]} u \leqq \mathsf{c}_0 \mathsf{s}_{[0,1]} a \cdot \mathsf{c}_1 \mathsf{s}_{[0,1]} b = \mathsf{c}_1 b \cdot \mathsf{c}_0 a$, so $\mathsf{s}_{[0,1]} u \in A_{ba}$.

Then for any $x \in A$ let

$$fx = \bigcup \{\varphi a : a \leqq x,\ a \in Dat\}.$$

So f is an isomorphism of $\mathfrak{B}\mathfrak{A}$ into the BA $\mathfrak{S}\mathfrak{b}V$. As in the proof of 3.2.59 one shows that f preserves $\mathsf{c}_{(\Gamma)}$ for all $\Gamma \subseteq 2$.

To show that f preserves $\mathsf{s}_{[0/1]}$, suppose first that $\langle u,v\rangle \in f\mathsf{s}_{[0/1]}x$, $x \in A$. Say $\langle u,v\rangle \in \varphi a$, $a \leqq \mathsf{s}_{[0/1]} b$, $a \in At\mathfrak{A}$, $b \in At\mathfrak{A}$, $b \leqq x$. Then $b \in Dat$; otherwise $a \leqq \mathsf{s}_{[0/1]} - \mathsf{d}_{01} = 0$, contradiction. Thus $\mathsf{s}_{[0/1]} b = \mathsf{c}_0 b$. Hence $v \in X_b$ and $\langle v,v\rangle \in \mathsf{D}_{01}^{|V|} \mathbf{n} (X_b \times X_b) \subseteq \varphi b$. Thus $\langle u,v\rangle \in \mathsf{s}_{[0/1]} fx$. Conversely, suppose that $\langle u,v\rangle \in \mathsf{s}_{[0/1]} fx$. Thus $\langle v,v\rangle \in fx$; say $\langle v,v\rangle \in \varphi b$ with $b \leqq x$. It follows from (2),(3) that $b \in Dat$. Say $\langle u,v\rangle \in \varphi a$, $a \in A_{cd}$. Then $v \in X_d \mathbf{n} X_b$ so $d = b$. Thus $a \leqq \mathsf{c}_0 b$ and so $\langle u,v\rangle \in f\mathsf{c}_0 b \subseteq f\mathsf{s}_{[0/1]} x$, as desired. Similarly, f preserves $\mathsf{s}_{[1/0]}$.

Finally, suppose that $\langle u,v\rangle \in f\mathsf{s}_{[0,1]} x$. Say $\langle u,v\rangle \in \varphi a$, $a \leqq \mathsf{s}_{[0,1]} x$, $a \in At\mathfrak{A}$. If $a \in Dat$, it is clear from cases 1,2 that $\langle v,u\rangle \in \varphi a$; also $a = \mathsf{s}_{[0,1]} a \leqq x$, so $\langle u,v\rangle \in \mathsf{s}_{[0,1]} fx$. Suppose $a \notin Dat$: let $a \in A_{bc}$. If $b = c$, then Case 2 applies, and $\langle v,u\rangle \in (\varphi a)^{-1} = \varphi \mathsf{s}_{[0,1]} a$. Since $\mathsf{s}_{[0,1]} a \leqq x$, this shows that $\langle u,v\rangle \in \mathsf{s}_{[0,1]} fx$. Next suppose that $b < c$. Then $\langle v,u\rangle \in (\varphi a)^{-1} = (\varphi \mathsf{s}_{[0,1]} \mathsf{s}_{[0,1]} a)^{-1} = \varphi \mathsf{s}_{[0,1]} a$ by Case 4, since $\mathsf{s}_{[0,1]} a \in A_{cb}$. Again this shows the desired result. Similarly if $c < b$. Thus we have shown that $f\mathsf{s}_{[0,1]} x \subseteq \mathsf{s}_{[0,1]} fx$. Hence

$s_{[0,1]}fx = s_{[0,1]}fs_{[0,1]}s_{[0,1]}x \subseteq s_{[0,1]}s_{[0,1]}fs_{[0,1]}x = fs_{[0,1]}x$, finishing the proof.

The ideas of this proof can also be used to prove a representation theorem for PEA_2's analogous to 3.2.65.

Now we turn to representation questions for $3 \leq \alpha < \omega$. Here the two main positive theorems in section 3.2 extend to PEA_α's.

THEOREM 5.4.34. *Suppose* $3 \leq \alpha < \omega$ *and* $\mathfrak{A}$ *is a* PEA_α. *Then* $\mathfrak{A}$ *is representable iff* $\mathfrak{A}$ *can be embedded in a rich* PEA_α $\mathfrak{B}$ *such that in* $\mathfrak{B}$ *the equations*

$$c_\lambda(x \cdot y \cdot c_\kappa(x \cdot -y)) \cdot -c_\kappa(c_\lambda x \cdot -d_{\kappa\lambda}) = 0$$

hold, for all distinct $\kappa, \lambda < \alpha$ *and all* $x, y \in B$.

PROOF. It suffices to check that the function f defined in the proof of Theorem 3.2.5 preserves $s_{[\kappa,\lambda]}$ for all distinct $\kappa, \lambda < \alpha$. Suppose that $u \in fs_{[\kappa,\lambda]}a$; we want to show that $u \circ [\kappa,\lambda] \in fa$. We have $\prod_{\mu \in \Delta b} s^0_\mu u_\mu \leq s_{[\kappa,\lambda]}a$, with $b = s_{[\kappa,\lambda]}a$, so $\prod_{\mu \in \Delta b} s_{[\kappa,\lambda]} s^0_\mu u_\mu \leq a$. Note that $\Delta b = [\kappa,\lambda]^* \Delta a$. We want to show

$$\prod_{\mu \in \Delta a} s^0_\mu (u \circ [\kappa,\lambda])_\mu \leq a;$$

it suffices to show

(1) $\{s_{[\kappa,\lambda]} s^0_\mu u_\mu : \mu \in \Delta s_{[\kappa,\lambda]}a\} \subseteq \{s^0_\mu (u \circ [\kappa,\lambda])_\mu : \mu \in \Delta a\}$.

So, let $\mu \in \Delta s_{[\kappa,\lambda]}a$.

Case 1. $\{\kappa,\lambda\} \cap \{\mu, 0\} = 0$. Then

$$s_{[\kappa,\lambda]} s^0_\mu u_\mu = s^0_\mu s_{[\kappa,\lambda]} u_\mu = s^0_\mu u_\mu = s^0_\mu (u \circ [\kappa,\lambda])_\mu .$$

Case 2. $\kappa = 0$, $\mu \notin \{\kappa,\lambda\}$. Then

$$s_{[\kappa,\lambda]} s^0_\mu u_\mu = s^0_\mu u_\mu = s^0_\mu (u \circ [\kappa,\lambda])_\mu .$$

Case 3. $\kappa = 0$, $\mu = \lambda$. Then $\kappa \in \Delta a$ and

$$s_{[\kappa,\lambda]} s^0_\mu u_\mu = s^0_\kappa (u \circ [\kappa,\lambda])_\kappa .$$

Case 4. $\kappa = 0 = \mu$. Then $\lambda \in \Delta a$ and

$$s_{[\kappa,\lambda]} s^0_\mu u_\mu = s^0_\lambda u_0 = s^0_\lambda (u \circ [\kappa,\lambda])_\lambda .$$

Case 5. $\kappa = \mu \neq 0 \neq \lambda$. Then $\lambda \in \Delta a$ and

$$s_{[\kappa,\lambda]} s^0_\mu u_\mu = s^0_\lambda u_\kappa = s^0_\lambda (u \circ [\kappa,\lambda])_\lambda .$$

Other cases are similar to these. So we have shown that $fs_{[\kappa,\lambda]}a \subseteq s_{[\kappa,\lambda]}fa$. The other inclusion easily follows *from* this one.

We do not know whether some version of 5.4.34 extends to PA_α's.

THEOREM 5.4.35. *Let $\mathfrak{A}$ be any simple atomic* PEA_α, *with* $3 \leqq \alpha < \omega$. *Then the following conditions are equivalent:*

(i) *Every atom of $\mathfrak{A}$ is rectangular.*

(ii) *There is an isomorphism of $\mathfrak{A}$ onto a* Pse_α $\mathfrak{B}$ *that carries each atom of $\mathfrak{A}$ to a singleton element of $\mathfrak{B}$.*

PROOF. Following the proof of Theorem 3.2.14, it suffices to show that if a is an atom of $\mathfrak{A}$ and σ is a permutation of α then $\{hs_\sigma a\} \subseteq s_\sigma\{ha\}$. To do this it suffices to take any $\xi < \alpha$ and show that $(s_\sigma a, \sigma\xi) E(a, \xi)$:

$$\begin{aligned} c_{(\alpha \sim \{\sigma\xi\})} s_\sigma a \cdot d_\alpha &= s_\sigma c_{(\alpha \sim \{\xi\})} a \cdot d_\alpha \\ &= s^{\xi}_{\sigma\xi} c_{(\alpha \sim \{\xi\})} a \cdot d_\alpha \\ &= c_{(\alpha \sim \{\xi\})} a \cdot d_\alpha , \end{aligned}$$

as desired.

THEOREM 5.4.36. *If* $3 \leqq \alpha < \omega$, then a PEA_α *is representable iff it can be embedded in an atomic* PEA_α *is which the atoms are rectangular.*

To extend this theorem to PA_α's, it is convenient to have a PA_α–version of Theorem 2.7.21:

THEOREM 5.4.37. *Assume that $\alpha < \omega$. For any* PA_α $\mathfrak{A}$ *there is a complete* PA_α $\mathfrak{B}$ *such that $\mathfrak{A} \subseteq \mathfrak{B}$ and the following two conditions hold:*

(i) $\Sigma^{\mathfrak{A}} X = \Sigma^{\mathfrak{B}} X$ *for every $X \subseteq A$ for which $\Sigma^{\mathfrak{A}} X$ exists;*

(ii) $b = \Sigma_{b \geqslant x \in A} x$ *for every $b \in B$.*

PROOF. We proceed as in the proof of 2.7.21, obtaining $\mathfrak{B}\mathfrak{A}\mathfrak{B}$ and the operations $c_{(\Gamma)}$, $\Gamma \subseteq \alpha$, so that (i), (ii), and (P_1)–(P_5) hold. Then for any $\sigma \in {}^\alpha\alpha$ and any $b \in B$ we set

$$s_\sigma b = \Sigma_{b \geqslant x \in A} s_\sigma x.$$

As in the proof of 2.7.21 we easily see that $s_\sigma \Sigma X = \Sigma s^*_\sigma X$ for any $X \subseteq B$. Then the other postulates are easy to check.

THEOREM 5.4.38. *If* $3 \leqq \alpha < \omega$, *then a* PA_α *is representable iff it can be embedded in an atomic* PA_α *in which the atoms are rectangular.*

PROOF. For the non–trivial direction $\Leftarrow$ it suffices to show that any atomic PA_α $\mathfrak{A}$ in which all atoms are rectangular is representable. By 5.4.37 we may assume that $\mathfrak{A}$ is complete. Then $\mathfrak{A}$ is the PA_α–reduct of some PEA_α $\mathfrak{B}$; by 5.4.35, $\mathfrak{B}$ is representable, so $\mathfrak{A}$ is also.

The following theorem of Henkln Is proved just as the corresponding theorem

3.2.53 was.

THEOREM 5.4.39. *For $\alpha<\omega$, any PEA_α of finite characteristic is representable.*

This result can be formulated in the following way: If $\alpha<\omega$ and $\mathfrak{A}$ is a PEA_α such that $\bar{\mathsf{d}}(\alpha\times\alpha)=0$, then $\mathfrak{A}$ is representable. Andréka and Németi [84b'] generalized this as follows: If $\alpha<\omega$ and $\mathfrak{A}$ is a PEA_α such that each equation

$$\mathsf{c}_\kappa(x\cdot\bar{\mathsf{d}}(\alpha\times\alpha))\cdot\bar{\mathsf{d}}(\alpha\times\alpha)=x\cdot\bar{\mathsf{d}}(\alpha\times\alpha),\ \kappa<\alpha,$$

holds in $\mathfrak{A}$, then $\mathfrak{A}$ is representable. (In a full Pse_α, these equations are equivalent to the base being of size $\leqq\alpha$.)

Now we turn to a discussion of non-representable polyadic algebras.

REMARKS 5.4.40. Assume that $3\leqq\alpha<\omega$. J.S. Johnson [69] showed that there is a PEA_α whose PA_α-reduct is representable but whose CA_α-reduct is not. He also showed that Rpp_α and Rppe_α are not finitely axiomatizable. Andréka and Németi [84c'] showed that there is a non-representable PEA_α $\mathfrak{A}$ with $\mathfrak{Rb}_{c\alpha}\mathfrak{A}$ representable, for $4\leqq\alpha<\omega$. The case $\alpha=3$ is open.

REMARKS 5.4.41. Assume that $\alpha\geqq\omega$. Daigneault and Keisler independently showed that every PA_α is representable; see Daigneault, Monk [63] and Keisler [63]. On the other hand, Rppe_α is not even closed under ultraproducts, as was first shown by Monk; see J.S. Johnson [69]. In fact, Johnson proved a more general theorem: there is an ultraproduct $\mathfrak{A}$ of Pse_α's with infinite bases such that $\mathfrak{A}$ is non-representable. Thus by 3.1.109, $\mathfrak{A}$ is a non-representable PEA_α such that $\mathfrak{Rb}_{c\alpha}\mathfrak{A}$ is representable.

REMARK 5.4.42. Returning to the class $\mathsf{CA}_\alpha\cap\mathsf{ICrs}_\alpha$ of "generalized" representable algebras discussed at the end of section 3.2, we note that for any $\mathfrak{A}\in\mathsf{PEA}_\alpha$ we have $\mathfrak{Rb}_{c\alpha}\mathfrak{A}\in\mathsf{CA}_\alpha\cap\mathsf{ICrs}_\alpha$, by the theorem of Resek mentioned there.

We shall not discuss the connections of polyadic algebras with logic; roughly speaking, the connections are as in section 4.3, except that it is not necessary to talk only about restricted formulas. See Monk [71'] and Johnson [73'].

5.5 RELATIVIZED CYLINDRIC ALGEBRAS

We have already discussed in section 2.2 the class Cr_α of relativized cylindric algebras, and in section 3.1 the associated "concrete" class Crs_α. Both of these are classes of algebras which are not, in general, cylindric algebras. In sections 2.2 and 3.1, the part of our discussion having to do with relativized CA_α's was essentially concerned mainly with the classes $\mathsf{CA}_\alpha \cap \mathsf{Cr}_\alpha$ and $\mathsf{CA}_\alpha \cap \mathsf{Crs}_\alpha$, but now we want to focus attention on the larger classes Cr_α and Crs_α. Note that $\mathsf{Crs}_\alpha = \mathsf{SRICs}_\alpha$. The main results we present are that SCr_α and ICrs_α are equational classes, but for $\alpha \geqq 3$ they are not finitely axiomatizable.

We begin with the case $\alpha = 2$. Here we have a simple characterization of ICrs_2 due to Henkin, Resek [75']; we use a shorter proof due to Andréka and Németi, which proceeds via the following four lemmas.

LEMMA 5.5.1. *Let $\mathfrak{A}$ be similar to CA_2's and satisfy* (C_0)–(C_3) *(cf. Part I, p.162). Then there is a structure $\mathfrak{B} = \langle B, T_\kappa, E_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha}$ similar to Ca_2's (cf. 2.7.38) such that the following conditions hold:*

(i) *T_0 and T_1 are equivalence relations on B.*

(ii) *$\mathfrak{A} \cong \mathfrak{A}' \subseteq \mathfrak{Cm}\mathfrak{B}$ for some $\mathfrak{A}'$, and for all distinct $a, b \in B$ there is an $x \in A'$ such that $a \in x$ and $b \notin x$.*

PROOF. We use the terminology of section 2.7. Note that $\mathfrak{A}$ is a Bo_2. Set $\mathfrak{B} = \mathfrak{A}t\mathfrak{Cm}\mathfrak{A}$. Then by 2.7.5 and 2.7.34 we see that there is an isomorphism of $\mathfrak{A}$ onto a subalgebra $\mathfrak{A}'$ of $\mathfrak{Cm}\mathfrak{B}$ such that (ii) holds. Writing $\mathfrak{B}$ as in the formulation of the theorem, we see by 2.7.32 that $T_\kappa = \{\langle x, y\rangle \in {}^2At\mathfrak{C} : y \leqq c_\kappa x\}$, where $\mathfrak{C} = \mathfrak{Cm}\mathfrak{A}$, for each $\kappa < 2$. Clearly then T_κ is transitive and reflexive on $At\mathfrak{C}$, where $At\mathfrak{C} = B$. Since, by 1.2.5 (whose proof used only (C_0)–(C_3)), $y \leqq \mathsf{c}_\kappa x$ iff $y \cdot \mathsf{c}_\kappa x \neq 0$ iff $\mathsf{c}_\kappa y \cdot x \neq 0$ iff $x \leqq \mathsf{c}_\kappa y$, each T_κ is also symmetric, finishing the proof.

LEMMA 5.5.2. *Let $\mathfrak{A}$ be similar to CA_2's and satisfy* (C_0)–(C_3) *and* (C_7). *Assume that $\mathfrak{B}$ satisfies* 5.5.1(i),(ii). *Then*

(*) $T_0 \cap {}^2E_{01} = T_1 \cap {}^2E_{01} \subseteq Id$.

PROOF. By symmetry it suffices to show that $T_0 \cap {}^2E_{01} \subseteq Id$. Suppose on the contrary that $a, b \in E_{01}$, $a \neq b$, and aT_0b. By 5.5.1(ii) choose $x \in A'$ such that $a \in x$ and $b \notin x$. Then $a \in \mathsf{c}_0(\mathsf{d}_{01} \cdot x) \cdot \mathsf{c}_0(\mathsf{d}_{01} \cdot -x)$, contradicting the fact that $\mathfrak{A}$ (hence $\mathfrak{A}'$) satisfies (C_7).

LEMMA 5.5.3. *Let $\mathfrak{B} = \langle B, T_\kappa, E_{\kappa\lambda}\rangle_{\kappa,\lambda<2}$* be similar to Ca_2's with T_0, T_1 equivalence

relations on B, $T_0 \mathbf{n}^2 E_{01} = T_1 \mathbf{n}^2 E_{01} \subseteq Id$, and $E_{00} = E_{11} = B$, $E_{01} = E_{10}$.

Then there is a relational structure $\mathfrak{B}' = \langle B', T'_\kappa, E'_{\kappa\lambda} \rangle_{\kappa,\lambda<2}$ *such that* $\mathfrak{Cm}\mathfrak{B} \cong | \subseteq \mathfrak{Cm}\mathfrak{B}'$ *and the following conditions hold*:

(i) T'_0 *and* T'_1 *are equivalence relations on* B'.

(ii) $T'_0 \mathbf{n}^2 E'_{01} = T'_1 \mathbf{n}^2 E'_{01} \subseteq Id$.

(iii) $T'_0 \mathbf{n} T'_1 \subseteq Id$.

(iv) $E'_{00} = E'_{11} = B'$; $E'_{01} = E'_{10}$.

PROOF. Let $\kappa \geqq |B|^+ + \omega$ be arbitrary, and set $B' = B \times \kappa$. Now we claim that there is an equivalence relation R on B' with the following two properties:

(1) For all $\xi \in \kappa$ and $a, b \in B$, ($\langle a,\xi \rangle R \langle b,\xi \rangle$ iff $a = b$)

(2) For all $\xi \in \kappa$ and $a, b \in B$, there is a unique $\eta \in \kappa$ such that $\langle a,\xi \rangle R \langle b,\eta \rangle$.

To see this, let $\langle a_\xi : \xi < \kappa \rangle$ list the elements of B, each repeated κ times. For each $\xi < \kappa$ let f_ξ be a one–one map of B into κ such that $f_\xi a_\xi = min(\kappa \sim \{f_\eta a_\xi : \eta < \xi\})$ and $f_\xi b \in \kappa \sim \{f_\eta b : \eta < \xi\}$ for all $b \in B$. Then $\langle f_\xi : \xi < \kappa \rangle$ forms a partition of B'. With R the associated equivalence relation, (1), (2) are easily checked.

Now we define the relations of the structure $\mathfrak{B}'$:

$$T'_0 = \{\langle \langle a,\xi \rangle, \langle b,\xi \rangle \rangle : a T_0 b,\ \xi < \kappa\};$$
$$T'_1 = \{\langle \langle a,\xi \rangle, \langle b,\eta \rangle \rangle : a T_1 b,\ \langle a,\xi \rangle R \langle b,\eta \rangle\};$$
$$E'_{01} = E'_{10} = \{\langle a,\xi \rangle : a \in E_{01},\ \xi < \kappa\};$$
$$E'_{00} = E'_{11} = B'.$$

This defines $\mathfrak{B}'$. It is routine to check (i)–(iv). The desired $f \in Ism(\mathfrak{Cm}\mathfrak{B}, \mathfrak{Cm}\mathfrak{B}')$ is defined by

$$f = \langle \{\langle a,\xi \rangle : a \in x,\ \xi \in \kappa\} : x \subseteq B \rangle.$$

Again it is routine to check the desired properties of f.

The next lemma is of more general interest, so we call it a theorem. Compare once more with 2.7.40, and see also the concluding remarks of section 2.7.

THEOREM 5.5.4. *Let* $\mathfrak{A}$ *be a structure similar to* Ca_2*'s, where* $\mathfrak{A} = \langle A, T_\kappa, E_{\kappa\lambda} \rangle_{\kappa,\lambda<2}$. *Then the following conditions are sufficient for* $\mathfrak{Cm}\mathfrak{A}$ *to be isomorphic to a* Crs_α:

(a) T_0 *and* T_1 *are equivalence relations on* A.

(b) $T_0 \mathbf{n}^2 E_{01} = T_1 \mathbf{n}^2 E_{01} \subseteq Id$.

(c) $T_0 \mathbf{n} T_1 \subseteq Id$.

(d) $E_{01} = E_{10}$ *and* $E_{00} = E_{11} = A$.

PROOF. Define $M = (A/T_0) \sim \{b/T_0 : b \in E_{01}\}$ and let U be any set with at least $|A/T_1| + |M|$ elements. Let f be a one–one function from A/T_1 into U and g a one–one function from M into $U \sim Rgf$. Define h from A/T_0 into U by the stipulations that $g \subseteq h$, and $h(b/T_0) = f(b/T_1)$ if $b \in E_{01}$; this is justified by (b). Thus

h is one–one. Then by (c) we see that for any $a \in A$, $f(a/T_1) = h(a/T_0)$ iff $a \in E_{01}$. Now define k mapping A into ${}^2U$ by setting $ka = \langle f(a/T_1), h(a/T_0)\rangle$. Then for any $a,b \in A$ we have: aT_0b iff $h(a/T_0) = h(b/T_0)$, and aT_1b iff $f(a/T_1) = f(b/T_1)$. It is now easy to check that k^* is an isomorphism of $\mathfrak{Em}\mathfrak{A}$ onto $\mathfrak{Gb}(k^*A)$, as desired.

Now we are ready for the above mentioned theorem of Henkin and Resek.

THEOREM 5.5.5. $\mathsf{SCr}_2 = \mathsf{ICrs}_2$; *moreover,* SCr_2 *is characterized by the postulates* (C_0)–(C_3), (C_5), (C_7), *along with the equation* $\mathsf{d}_{01} = \mathsf{d}_{10}$.

PROOF. Let K be the variety determined by (C_0)–(C_3), (C_5), (C_7), and $\mathsf{d}_{01} = \mathsf{d}_{10}$. Clearly $\mathsf{ICrs}_2 \subseteq \mathsf{SCr}_2 \subseteq \mathsf{K}$. The inclusion $\mathsf{K} \subseteq \mathsf{ICrs}_2$ follows from 5.5.1–5.5.4.

REMARKS 5.5.6. Theorem 5.5.5 has as an immediate consequence the inclusion $\mathsf{CA}_2 \subseteq \mathsf{ICrs}_2$, proved in section 3.2 as Theorem 3.2.61. Henkin and Resek [75'] improved 5.5.5 by also showing that $\mathsf{SCr}_2 = \mathsf{Cr}_2$. Andréka and Németi have shown that $\mathsf{Crs}_2 \subseteq \mathsf{IRICs}_2$; hence $\mathsf{Cr}_2 = \mathsf{IRICs}_2$.

Now we turn to the case $\alpha > 2$. Henkin and Resek [75'] also showed:

(A) For $\alpha > 2$, the class Cr_α is not equational; in fact, $\mathsf{SCr}_\alpha \neq \mathsf{Cr}_\alpha$.

A similar situation holds for the class RICs_α, for $\alpha = 2$ as well as $\alpha > 2$, as has been shown by Andréka and Németi:

EXAMPLE 5.5.7. (See Andréka, Németi [81'], p.155.) For each $\alpha \geq 2$ there is an $\mathfrak{A} \in \mathsf{Crs}_\alpha \sim \mathsf{RICs}_\alpha$. For, let $\mathfrak{B}$ be the full Cs_α with base 4. Set $f = \langle 0,1,0,0,\ldots\rangle$, $g = \langle 1,2,0,0,\ldots\rangle$, $h = \langle 2,3,0,0,\ldots\rangle$, $V = \{f,g,h\}$, $\mathfrak{C} = \mathfrak{Rl}_V\mathfrak{B}$, $\mathfrak{A} = \mathfrak{Sg}^{\mathfrak{C}}\{f\}$. Thus $\mathfrak{A} \in \mathsf{Crs}_\alpha$, and we now show that $\mathfrak{A} \notin \mathsf{RICs}_\alpha$. Note that $A = \{0, V, \{f\}, \{g,h\}\}$. Suppose that $\mathfrak{A} = \mathfrak{Rl}_V\mathfrak{D}$, where $\mathfrak{D}$ is a Cs_α. Then

$$\{g\} = \{g,h\} \cap \mathsf{C}_1(\mathsf{D}_{01} \cap \mathsf{C}_0\{f\}) \in D,$$

hence $\{g\} \in Rl_V\mathfrak{D} = \mathfrak{A}$, contradiction.

The analogy between (A) and Example 5.5.7 is not as close as it first appears. In fact, $\mathsf{ICr}_\alpha = \mathsf{Cr}_\alpha$ but $\mathsf{IRICs}_\alpha \neq \mathsf{RICs}_\alpha$. Thus the most natural question associated with (A) is whether IRICs_α is an equational class, or whether it is closed under S. But the proof of (A) actually gives the following result, answering both questions negatively:

(B) For each $\alpha > 2$ there is an $\mathfrak{A} \in \mathsf{Crs}_\alpha \sim \mathsf{Cr}_\alpha$.

Next, recall from 2.2.8 that $\mathsf{PCr}_\alpha = \mathsf{Cr}_\alpha$ and $\mathsf{UpCr}_\alpha = \mathsf{Cr}_\alpha$. Hence $\mathsf{UpSCr}_\alpha = \mathsf{SCr}_\alpha$. Thus SCr_α is a universal class. Actually, Resek has shown that SCr_α is always an equational class. This is an unpublished result which we now present with a simple proof due to Don Pigozzi. We note first that the definition of an *ideal* in a Cr_α can

be taken just as for CA_α's, and the correspondence with congruence relations follows as for CA_α's; see 2.3.5–2.3.7.

THEOREM 5.5.8. SCr_α *is an equational class for any ordinal* α.

PROOF. From the above remarks we see that it is sufficient to show that $\mathsf{HSCr}_\alpha = \mathsf{SCr}_\alpha$, and that to do this it suffices to take any CA_α $\mathfrak{A}$, $a \in A$, $\mathfrak{B} \subseteq \mathfrak{Rl}_a\mathfrak{A}$, I an ideal of $\mathfrak{B}$, and show that $\mathfrak{B}/I$ can be isomorphically embedded in a Cr_α. Now if J is the ideal of $\mathfrak{Rl}_a A$ generated by I then, as is easily seen,

$$J = \{x \in Rl_a\mathfrak{A} : x \leqq y \text{ for some } y \in I\}.$$

It follows that $J \cap B = I$. Hence $\mathfrak{B}/I$ can be isomorphically embedded in $\mathfrak{Rl}_a\mathfrak{A}/J$. Thus we may assume that $\mathfrak{B} = \mathfrak{Rl}_a\mathfrak{A}$ in what follows.

Let $\mathfrak{C} = \mathfrak{Em}\mathfrak{A}$. We shall find $c \in C$ such that $\mathfrak{Rl}_a\mathfrak{A}/I$ can be isomorphically embedded in $\mathfrak{Rl}_c\mathfrak{C}$. We shall use $+, \cdot$, etc. for operations of $\mathfrak{A}$ and $\mathfrak{C}$, not of $\mathfrak{Rl}_a\mathfrak{A}$ and $\mathfrak{Rl}_c\mathfrak{C}$. Let $e = \Sigma^{\mathfrak{C}} I$. Then for each $\kappa < \alpha$ we have $\mathsf{c}_\kappa x \cdot a \in I$ for all $x \in I$, and hence

$$\begin{aligned} \mathsf{c}_\kappa e \cdot a &= (\mathsf{c}_\kappa \Sigma^{\mathfrak{C}} I) \cdot a = (\Sigma_{x \in I} \mathsf{c}_\kappa x) \cdot a \\ &= \Sigma_{x \in I} \mathsf{c}_\kappa x \cdot a \leqq \Sigma I = e. \end{aligned}$$

Hence $a \cdot -e \cdot \mathsf{c}_\kappa e = 0$, so $\mathsf{c}_\kappa(a \cdot -e) \cdot e = 0$. Thus

(1) $\mathsf{c}_\kappa(a \cdot -e) \cdot a = a \cdot -e$

for every $\kappa < \alpha$. We set $c = a \cdot -e$. We now define a homomorphism h from $\mathfrak{Rl}_a\mathfrak{A}$ into $\mathfrak{Rl}_c\mathfrak{C}$ such that $(h^{-1})^* 0 = I$, thus finishing the proof. Namely, we let $h = \langle x \cdot c : x \in \mathfrak{Rl}_a\mathfrak{A} \rangle$. Clearly $h \in Hom(\mathfrak{Bl}\mathfrak{Rl}_a\mathfrak{A}, \mathfrak{Bl}\mathfrak{Rl}_c\mathfrak{C})$, and $h(\mathsf{d}_{\kappa\lambda} \cdot a) = \mathsf{d}_{\kappa\lambda} \cdot c$ for all $\kappa, \lambda < \alpha$. To show that h preserves cylindrifications, suppose that $\kappa < \alpha$ and $x \in \mathfrak{Rl}_a\mathfrak{A}$. Then

$$\begin{aligned} h(\mathsf{c}_\kappa x \cdot a) &= \mathsf{c}_\kappa x \cdot a \cdot -e = \mathsf{c}_\kappa x \cdot \mathsf{c}_\kappa(a \cdot -e) \cdot a \cdot -e \\ &= \mathsf{c}_\kappa(x \cdot \mathsf{c}_\kappa(a \cdot -e)) \cdot a \cdot -e \\ &= \mathsf{c}_\kappa(x \cdot a \cdot -e) \cdot a \cdot -e \qquad \text{by (1)} \\ &= \mathsf{c}_\kappa h x \cdot c, \end{aligned}$$

as desired. So, h is a homomorphism.

Now if $x \in I$, then $x \leqq e$ and so $hx = x \cdot c = 0$. On the other hand, if $x \in \mathfrak{Rl}_a\mathfrak{A}$ and $hx = 0$, then $x \leqq e = \Sigma I$; by Theorem 2.7.5, $x \leqq \Sigma K$ for some finite $K \subseteq I$: so, $x \in I$. Hence $(h^{-1})^* 0 = I$, finishing the proof.

The analogous result to 5.5.8 for the class ICrs_α is also true; but before proceeding to that, it is apropos to show that for $\alpha > 2$ the classes in question are distinct:

EXAMPLE 5.5.9. For every $\alpha \geqq 3$ there is an $\mathfrak{A} \in \mathsf{CA}_\alpha \sim \mathsf{ICrs}_\alpha$. Since $\mathsf{CA}_\alpha \subseteq \mathsf{SCr}_\alpha$, this

shows that $\mathsf{SCr}_\alpha \neq \mathsf{ICrs}_\alpha$ for all $\alpha \geqq 3$. For $\mathfrak{A}$ we take a CA_α in which the equation

(1) ${}_2\mathsf{s}(0,1)\mathsf{c}_2 x = {}_2\mathsf{s}(1,0)\mathsf{c}_2 x$

fails to hold; see 3.2.71. We now show that every $\mathfrak{B} \in \mathsf{CA}_\alpha \cap \mathsf{ICrs}_\alpha$ *does* satisfy the equation (1). Thus suppose that $\mathfrak{B}$ is a $\mathsf{Crs}_\alpha \cap \mathsf{CA}_\alpha$, say with unit element V and base U. Suppose that $x \in B$. It suffices to show that ${}_2\mathsf{s}(0,1)\mathsf{c}_2 x \subseteq {}_2\mathsf{s}(1,0)\mathsf{c}_2 x$. Suppose that $f \in {}_2\mathsf{s}(0,1)\mathsf{c}_2 x$. Now ${}_2\mathsf{s}(0,1)\mathsf{c}_2 x = \mathsf{s}_0^2\mathsf{s}_1^0\mathsf{s}_2^1\mathsf{c}_2 x$. Say $f = \langle a,b,c,\ldots\rangle$. We thus easily get that $\langle b,a,d,\ldots\rangle \in x$ for some $d \in U$. Using the laws $\mathsf{c}_2(\mathsf{d}_{02}\cdot\mathsf{c}_2 y) = \mathsf{c}_2 y$, $\mathsf{c}_0(\mathsf{d}_{01}\cdot\mathsf{c}_0 y) = \mathsf{c}_0 y$, $\mathsf{c}_1(\mathsf{d}_{12}\cdot\mathsf{c}_1 y) = \mathsf{c}_1 y$, which are valid in any CA_α, we then get successively $\langle b,a,b,\ldots\rangle \in \mathsf{d}_{02}\cdot\mathsf{c}_2 x$, $\langle a,a,b,\ldots\rangle \in \mathsf{d}_{01}\cdot\mathsf{s}_2^0\mathsf{c}_2 x$, $\langle a,b,b,\ldots\rangle \in \mathsf{d}_{12}\cdot\mathsf{s}_0^1\mathsf{s}_2^0\mathsf{c}_2 x$, $f \in \mathsf{s}_1^2\mathsf{s}_0^1\mathsf{s}_2^0\mathsf{c}_2 x = {}_2\mathsf{s}(1,0)\mathsf{c}_2 x$, as desired.

The fact that ICrs_α is also a variety is a result of Németi [79'],[81']. In the proof we need the following fact, obtained by repeating the proof of 2.3.26, for Crs's: If $\mathfrak{A}$ is a Crs_α, $a \in A$, and $\mathsf{c}_\kappa a = a$ for all $\kappa < \alpha$, then the function $\langle x\cdot a : x \in A\rangle$ is a homomorphism of $\mathfrak{A}$ into some Crs_α.

THEOREM 5.5.10. *For $\alpha \geqq 2$, ICrs_α is an equational class.*

PROOF. It is easily checked that ICrs_α is closed under S and P. Hence it suffices to show that it is closed under H also. To do this it suffices to take any $\mathfrak{A} \in \mathsf{Crs}_\alpha$ and any ideal L of $\mathfrak{A}$ and show that $\mathfrak{A}/L$ is isomorphic to a Crs_α. And to do this it is enough to take any $x \in A \sim L$ and find a homomorphism h from $\mathfrak{A}$ into a Crs_α such that $hx \neq 0$ and $h^* L = \{0\}$.

Clearly there is an ultrafilter F on L such that $\{y \in L : z \leqq y\} \in F$ for all $z \in L$. Let U be the base of $\mathfrak{A}$, and set $X = {}^L U/\bar{F}$. Now $x \notin L$, so for every $z \in L$ we have $x \not\leqq z$, and so we can choose $p_z \in x \sim z$. Set $r = \langle \langle p_z \kappa : z \in L\rangle : \kappa < \alpha\rangle$. Thus $r \in {}^\alpha({}^L U)$. Now let c be an (F,U,α)–choice function such that $c(\kappa, r_\kappa/\bar{F}) = r_\kappa$ for all $\kappa < \alpha$ (see definition 3.1.89). For each $y \in A$ let $\bar{y} = \langle y : z \in L\rangle$. Finally, we set

$$fy = Rep(\bar{y}/\bar{F})$$

for any $y \in A$, where $Rep = Rep(F,U,\alpha,\langle A : s \in L\rangle, c)$. By Lemma 3.1.90, f is a homomorphism from $\mathfrak{A}$ into some Crs_α.

Let $t = \langle r_\kappa/\bar{F} : \kappa < \alpha\rangle$. We claim that $t \in fx$. In fact, by 3.1.89 this is true iff

$$\{z \in L : (c^+ t)_z \in x\} \in F.$$

Since $(c^+ t)_z = \langle c(\kappa, t_\kappa)_z : \kappa < \alpha\rangle = \langle r_\kappa z : \kappa < \alpha\rangle = \langle p_z \kappa : \kappa < \alpha\rangle = p_z$, this *is* true.

Now we set $V = f1$ and

(1) $W = \bigcup\{\mathsf{C}_{\kappa 0}^{[V]} \ldots \mathsf{C}_{\kappa\iota}^{[V]}\{t\} : \kappa 0, \ldots, \kappa\nu < \alpha\}$.

Let $\mathfrak{B} = \mathfrak{Cb}\, V$. Thus $\mathsf{C}_\kappa^{[V]} W = W$ for all $\kappa < \alpha$, and hence by the remark preceding this

theorem,

$$h = \langle fy \cap W : y \in A \rangle$$

is a homomorphism of $\mathfrak{A}$ onto a Crs_α with unit element W. Since $t \in W$ we have $t \in hx$ and so $hx \neq 0$. Now we show that $hz = 0$ for an arbitrary $z \in L$, finishing the proof. Assume that $\kappa 0, \ldots, \kappa\nu < \alpha$; we want to show that $fz \cap \mathsf{C}_{\kappa 0} \ldots \mathsf{C}_{\kappa\iota}\{t\} = 0$. Using 1.2.5, which is clearly valid for ICrs_α also, we see that it suffices to show that

(2) $\{t\} \cap \mathsf{C}_{\kappa\iota} \ldots \mathsf{C}_{\kappa 0} fz = 0$.

Since f is a homomorphism, we thus want to show that $t \notin f(\mathsf{c}_{\kappa\iota} \ldots \mathsf{c}_{\kappa 0} z)$. Let $w = \mathsf{c}_{\kappa\iota} \ldots \mathsf{c}_{\kappa 0} z$. Thus $w \in L$, so $\{v \in L : w \leqq v\} \in F$, hence $\{v \in L : p_v \notin w\} \in F$, so $\{v \in L : (c^+t)_v \notin w\} \in F$ since $(c^+t)_v = p_v$ for all $v \in L$, as was pointed out above prior to (1). Thus $t \notin fw$, as desired.

Explicit equational axioms for the class SCr_α have been given by Resek [75']. She also showed that SCr_α is not finitely axiomatizable for $\alpha \geqq 3$. We shall give a proof for this due to Andréka and Németi. Their construction also shows that ICrs_α is not finitely axiomatizable for $\alpha \geqq 3$.

THEOREM 5.5.11. *For each $\alpha \geqq 3$ there is a system $\langle \mathfrak{A}_m : m \in \omega \rangle$ of algebras similar to CA_α's such that $\mathfrak{A}_m \notin \mathsf{ISCr}_\alpha$ for all $m \in \omega$, but $\mathsf{P}_{m \in \omega} \mathfrak{A}_m / \bar{F} \in \mathsf{ICrs}_\alpha$ for any non-principal ultrafilter F on ω.*

PROOF. For each $m \in \omega$ and $\kappa, \lambda < \alpha$ we set

$$\begin{aligned}
V_m &= \{f \in {}^\alpha\omega : \text{for some } n \leqq m \text{ we have } f0 = 2n+1, \\
&\qquad f1 \in \{2n, 2n+2\}, \text{ and } f\kappa = 0 \text{ for all } \kappa \in \alpha \sim 2\}, \\
\mathsf{d}^m_{\kappa\lambda} &= \mathsf{d}^m_{\lambda\kappa}, \quad \mathsf{d}^m_{\kappa\kappa} = V_m, \\
\mathsf{d}^m_{0\kappa} &= 0 \text{ if } 0 < \kappa < \alpha, \\
\mathsf{d}^m_{\kappa\lambda} &= V_m \text{ if } 2 \leqq \kappa < \lambda, \\
\mathsf{d}^m_{1\kappa} &= \{\langle 1,0,0,\ldots\rangle, \langle 2m+1, 2m+2, 0, 0, \ldots\rangle\} \text{ if } 1 < \kappa, \\
\mathfrak{A}_m &= \langle Sb\, V_m, \cup, \cap, {}_W\sim, 0, V_m, \mathsf{C}^{[W]}_\kappa, \mathsf{d}^m_{\kappa\lambda} \rangle_{\kappa,\lambda<\alpha},
\end{aligned}$$

with $W = V_m$. First we claim:

(1) $\mathfrak{A}_m \notin \mathsf{ISCr}_\alpha$ for all $m \in \omega$.

To prove this we consider the following inequality which holds in all members of ISCr_α; see 2.2.5(iv) and 2.2.6:

$$(\mathsf{c}_0 \mathsf{c}_1)^{m+1}(\mathsf{d}_{12} \cdot x) \cdot \mathsf{d}_{12} \leqq \mathsf{c}_0 x.$$

Taking $x = \{\langle 1,0,0,\ldots\rangle\}$, we see that $\langle 2m+1, 2m+2, 0, 0, \ldots\rangle$ is in the left side of this inequality but not in the right side.

Now suppose that F is any non-principal ultrafilter on ω. We set $\mathfrak{B} = \mathsf{P}_{m \in \omega} \mathfrak{A}_m / \bar{F}$

and $\mathfrak{C} = \mathsf{P}_{m\in\omega}\langle m+1,<\rangle/\bar{F}$. If u is an atom of $\mathfrak{A}_m$, say $u = \{\langle 2n+1,\ldots\rangle\}$ with $n \leq m$; then we set $intu = n$. Note that $\mathfrak{B}$ is atomic. If $x/\bar{F}$ is an atom of $\mathfrak{B}$, we say that x/F is of *type* 1 if $\{m\in\omega$: for some n, $x_m = \{\langle 2n+1,2n,0,0,\ldots\rangle\}\} \in F$; otherwise $x/\bar{F}$ is of *type* 2. Note that if $x/\bar{F}$ is of type 2, then $\{m\in\omega$: for some n, $x_m = \{\langle 2n+1,2n+2,0,0,\ldots\rangle\}\} \in F$. It is well-known that $|C| = 2^\omega$; see Chang, Keisler [73**]. Let $\bar{0} = \langle 0 : m\in\omega\rangle/\bar{F}$, $\bar{\infty} = \langle m : m\in\omega\rangle/\bar{F}$ Now $\mathfrak{C}$ is a linearly ordered structure with least element $\bar{0}$ and greatest element $\bar{\infty}$; every element except $\bar{\infty}$ has an immediate successor, and every element except $\bar{0}$ has an immediate predecessor. Therefore the order type of $\mathfrak{C}$ consists of ω followed by 2^ω copies of $\mathbb{Z}$ in some order not of interest in this proof, followed by ω^*. For any atom $x/\bar{F}$ of $\mathfrak{B}$ we set $int(x/\bar{F}) = \langle int(x_m) : m\in\omega\rangle/\bar{F}$; clearly this is a well-defined function from $At\mathfrak{B}$ into C. We call two elements a,b of C *equivalent* if $a = b+n$ for some integer n, i.e., a is the nth successor of b for n positive, the nth predecessor of b for n non-positive. Then atoms $x/\bar{F}$ and $y/\bar{F}$ of $\mathfrak{B}$ are *equivalent* if $int(x/\bar{F})$ and $int(y/\bar{F})$ are. Let $u_m = \{\langle 1,0,0,\ldots\rangle\}$ and $v_m = \{\langle 2m+1,2m+2,0,0,\ldots\rangle\}$ for all $m\in\omega$.

Now we indicate some rules for calculating cylindrifications in $\mathfrak{B}$:

(2) For any $i\in\{1,2\}$ and any $n\in\mathsf{P}_{m\in\omega}(m+1)$ there is at most one atom $x/\bar{F}$ of type i with $int(x/\bar{F}) = n/\bar{F}$.

(3) $\mathsf{c}_0(u/\bar{F}) = u/\bar{F}$ and $\mathsf{c}_0(v/\bar{F}) = v/\bar{F}$.

(4) If $int(x/\bar{F}) = n/\bar{F}$, then $\mathsf{c}_1(x/\bar{F}) = x/\bar{F} + y/\bar{F}$, where $y/\bar{F}$ is the second atom with $int(y/\bar{F}) = n/\bar{F}$.

(5) If $x/\bar{F} \neq u/\bar{F}$ and $x/\bar{F}$ is of type 1, then $\mathsf{c}_0(x/\bar{F}) = x/\bar{F} + y/\bar{F}$, where $y/\bar{F}$ is the atom of type 2 with $int(y/\bar{F}) = int(x/\bar{F})-1$.

(6) *If* $x/\bar{F} \neq v/\bar{F}$ and $x/\bar{F}$ is of type 2, then $\mathsf{c}_0(x/\bar{F}) = x/\bar{F} + y/\bar{F}$, where $y/\bar{F}$ is the atom of type 1 with $int(y/\bar{F}) = int(x/\bar{F})+1$.

These rules are all easy to check.

Next we define G mapping $At\mathfrak{B}$ into ${}^\alpha C$ by defining its restriction to each equivalence class k under the above equivalence relation.

Case 1. $u/\bar{F} \in k$. Then we set

$$G(u/\bar{F}) = \langle \bar{0}+1,\bar{0},\bar{0},\ldots\rangle.$$

Suppose that $u/\bar{F} \neq x/\bar{F} \in k$. Write $int(x/\bar{F}) = \bar{0}+n$. We set

$$G(x/\bar{F}) = \begin{cases} \langle \bar{0}+2n+1,\bar{0}+2n,\bar{0},\bar{0},\ldots\rangle \text{ if } x/\bar{F} \text{ is of type 1,} \\ \langle \bar{0}+2n+1,\bar{0}+2n+2,\bar{0},\bar{0},\ldots\rangle \text{ otherwise.} \end{cases}$$

Case 2. $v/\bar{F} \in k$. Then we set

$$G(v/\bar{F}) = \langle \bar{\infty}-1,\bar{\infty},\bar{\infty},\ldots\rangle.$$

Suppose that $v/\bar{F} \neq x/\bar{F} \in k$. Write $int(x/\bar{F}) = \bar{\infty}-n$. We set

$$G(x/\bar{F}) = \begin{cases} \langle \bar{\infty}-(2n+1),\bar{\infty}-2n,\bar{\infty},\bar{\infty},\ldots\rangle \text{ if } x/\bar{F} \text{ is of type 2,} \\ \langle \bar{\infty}-(2n+1),\bar{\infty}-(2n+2),\bar{\infty},\bar{\infty},\ldots\rangle \text{ otherwise.} \end{cases}$$

Case 3. $u/\bar{F}, v/\bar{F} \notin k$. Let a be any member of the equivalence class of $int(x/\bar{F})$, with $x/\bar{F}$ any member of k. Now for any $y/\bar{F} \in k$, write $int(y/\bar{F}) = a+n$, where $n \in \mathbf{Z}$. We set

$$G(y/\bar{F}) = \begin{cases} \langle a+2n+1, a+2n, \bar{0}, \bar{0}, \ldots \rangle \text{ if } y/\bar{F} \text{ is of type 1,} \\ \langle a+2n+1, a+2n+2, \bar{0}, \bar{0}, \ldots \rangle \text{ otherwise.} \end{cases}$$

We define H mapping B into $Sb({}^{\alpha}C)$ by setting, for any $b \in B$,

$$Hb = \{Ga : b \geqq a \in At\mathfrak{B}\}.$$

Further, let $Z = H1$. Then it is routine to check that G is an isomorphism of $\mathfrak{B}$ onto a Crs_α with unit element Z, finishing the proof.

COROLLARY 5.5.12. *Let* $\alpha \geqq 3$. *Then the classes* SCr_α *and* Crs_α *are not finitely axiomatizable.*

Of course, for $\alpha \geqq \omega$ we do not *expect* SCr_α or Crs_α to be finitely axiomatizable. We return to the notion finite schema axiomatizable introduced in section 4.1. The theorems 5.5.13 and 5.5.14 which follow are due to Andréka and Németi (unpublished).

THEOREM 5.5.13. *If* $\alpha \geqq \omega$, *then* ICrs_α *is not axiomatizable by a finite schema of equations.*

PROOF. Assume otherwise. Then by Definition 4.1.4(iii) (which clearly can be extended to an arbitrary class K of algebras similar to CA_α's), there is a finite set $E \subseteq Ev^{\omega}$ such that

(1) $\mathsf{ICrs}_\alpha = \mathsf{Md}\{\gamma^+\sigma = \gamma^+\tau : \sigma = \tau \in E,\ \gamma \in {}^{\omega}\alpha,\ \gamma \text{ one-one}\}$.

Now for any term σ of $\mathcal{L}_\alpha$ let σ' be obtained from σ by the following process:

$\mathbf{c}_\kappa$ is deleted if $\kappa \in \alpha \sim 3$;
$\mathbf{d}_{\kappa 0}$ and $\mathbf{d}_{0\kappa}$ are replaced by $\mathbf{d}_{01}$ if $\kappa \in \alpha \sim 1$;
$\mathbf{d}_{\kappa 1}$ and $\mathbf{d}_{1\kappa}$ are replaced by $\mathbf{d}_{12}$ if $\kappa \in \alpha \sim 2$;
$\mathbf{d}_{\kappa\lambda}$ is replaced by 1 if $\kappa, \lambda \in \alpha \sim 2$.

Let Γ be the following set of equations:

$$\{\mathbf{c}_\kappa x = x : 2 \leqq \kappa < \alpha\} \cup \{\mathbf{d}_{0\kappa} = \mathbf{d}_{01} : \kappa \in \alpha \sim 1\}$$
$$\cup \{\mathbf{d}_{1\kappa} = \mathbf{d}_{12} : \kappa \in \alpha \sim 2\} \cup \{\mathbf{d}_{\kappa\lambda} = 1 : \kappa, \lambda \in \alpha \sim 2\}.$$

Now we consider the algebras $\mathfrak{A}_m$, $m \in \omega$, as defined in the proof of Theorem 5.5.11. Clearly each algebra $\mathfrak{A}_m$ is a model of Γ, and hence so is $\mathsf{P}_{m \in \omega}\mathfrak{A}_m/\bar{F}$ (F as in that proof too). If $\sigma = \tau$ is an equation in $\mathcal{L}_\alpha$, clearly it holds in $\mathfrak{A}_m$ iff $\sigma' = \tau'$ does, for

any $m \in \omega$. The same is true of $\mathsf{P}_{m\in\omega}\mathfrak{A}_m/\bar{F}$. But $\{(\gamma^+\sigma)' = (\gamma^+\tau)' : \sigma = \tau \in E,\ \gamma \in {}^\omega\alpha,\ \gamma$ one–one$\}$ is clearly finite; since each of its members holds in $\mathsf{P}_{m\in\omega}\mathfrak{A}_m/\bar{F}$, there is an $m \in \omega$ such that each one holds in $\mathfrak{A}_m$. Thus by (1), $\mathfrak{A}_m \in \mathsf{ICrs}_\alpha$, contradiction.

The same proof yields:

THEOREM 5.5.14. *If $\alpha \geqq \omega$, then SCr_α is not axiomatizable by a finite schema.*

As our last major result of this section we show that for $\alpha \geqq \omega$, ICrs_α is axiomatizable by a countable schema; this result is also due to Andréka and Németi. It follows on rather general grounds from 5.5.10 and the following result (Proposition 8(ii) in Németi [81']):

THEOREM 5.5.15. *Let $1 < \alpha \leqq \beta$ and let $\rho \in {}^\alpha\beta$ be a one–one function. Then $\mathsf{ICrs}_\alpha = \mathsf{Rd}^\rho \mathsf{ICrs}_\beta$.*

PROOF. For the inclusion $\subseteq$, let $\mathfrak{A} \in \mathsf{Crs}_\alpha$, say with base U. Choose $u \notin U$ and set $W = U \cup \{u\}$. Now we define a function f mapping ${}^\alpha U$ into ${}^\beta W$ by setting, for all $s \in {}^\alpha U$ and all $\kappa < \beta$,

$$(fs)\kappa = \begin{cases} s\lambda \text{ if } \rho\lambda = \kappa, \\ u \text{ otherwise.} \end{cases}$$

Let $\mathfrak{C} = \mathfrak{Cb} f^* 1^{\mathfrak{A}}$, as a Crs_β. Then it is easy to check that f is one–one and that f^* is an isomorphism from $\mathfrak{A}$ into $\mathfrak{Rb}^\rho\mathfrak{C}$. Now let $B = \{f^*x : x \in A\}$. Then $B \in Su\mathfrak{C}$. This follows from the following easily verified facts, where $x \in A$ and $\kappa, \lambda < \beta$:

$$\begin{aligned} &B \text{ is closed under } +, -, ; \\ &\mathsf{c}_{\rho\mu} f^* x = f^* \mathsf{c}_\mu x \text{ if } \mu < \alpha; \\ &\mathsf{c}_\kappa f^* x = f^* x \text{ if } \kappa \notin Rg\rho; \\ &\mathsf{d}_{\rho\mu,\rho\nu} = f^* \mathsf{d}_{\mu\nu} \text{ if } \mu, \nu < \alpha; \\ &\mathsf{d}_{\rho\mu,\kappa} = 0 \text{ if } \mu < \alpha,\ \kappa \notin Rg\rho; \\ &\mathsf{d}_{\kappa\lambda} = 1 \text{ if } \kappa, \lambda \notin Rg\rho. \end{aligned}$$

Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{C}$ with universe B. Clearly f^* is an isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$, as desired.

For $\supseteq$, let $\mathfrak{A} \in \mathsf{Crs}_\beta$, say with base U. We define a function f mapping ${}^\beta U$ into ${}^\alpha W$ (for some W) by setting, for all $s \in {}^\beta U$ and all $\kappa < \alpha$,

$$(fs)_\kappa = \langle s\rho\kappa, (\beta \sim Rg\rho) \upharpoonright s \rangle.$$

It is straightforward to check that f^* is an isomorphism of $\mathfrak{Rb}^\rho\mathfrak{A}$ onto some Crs_α.

THEOREM 5.5.16 *Let $\alpha \geqq \omega$. ICrs_α is axiomatizable by a countable schema.*

PROOF. Let $E = \{\sigma = \tau : \sigma = \tau$ is an equation of $\mathcal{L}_\omega$ which holds in every $\mathsf{Crs}_\omega\}$. We claim:

(1) $\mathsf{ICrs}_\alpha = \mathsf{Md}\{\gamma^+\sigma = \gamma^+\tau : \sigma = \tau \in E,\ \gamma \in {}^\omega\alpha,\ \gamma \text{ one-one}\}$,

which will finish the proof. First let $\mathfrak{A} \in \mathsf{Crs}_\alpha$, $\sigma = \tau \in E$, $\gamma \in {}^\omega\alpha$, γ one-one; we are to show that $\gamma^+\sigma = \gamma^+\tau$ holds identically in $\mathfrak{A}$. By Theorem 5.5.15, $\mathfrak{Rb}^\gamma\mathfrak{A} \in \mathsf{ICrs}_\omega$, and so $\sigma = \tau$ holds in $\mathfrak{Rb}^\gamma\mathfrak{A}$. Hence $\gamma^+\sigma = \gamma^+\tau$ holds in $\mathfrak{A}$.

Conversely, let $\mathfrak{A}$ be a member of the right-hand side of (1). To show that $\mathfrak{A} \in \mathsf{ICrs}_\alpha$ it suffices by Theorem 5.5.10 to take any equation $\sigma = \tau$ holding in ICrs_α and show that it holds in $\mathfrak{A}$. Choose a one-one $\gamma \in {}^\omega\alpha$ such that if $\mathbf{c}_\kappa$ or $\mathbf{d}_{\kappa\lambda}$ occurs in $\sigma = \tau$ then $\kappa, \lambda \in Rg\gamma$. Now $\mathfrak{Rb}^\gamma\mathfrak{A} \in \mathsf{ICrs}_\omega$, since $\mathfrak{A}$ is in the right-hand side of (1) and hence $\mathfrak{Rb}^\gamma\mathfrak{A}$ is a model of E (we use Theorem 5.5.10 again). By Theorem 5.5.15, choose a Crs_α $\mathfrak{B}$ and an isomorphism f from $\mathfrak{Rb}^\gamma\mathfrak{A}$ onto $\mathfrak{Rb}^\gamma\mathfrak{B}$. There are terms σ', τ' in $\mathcal{L}_\omega$ such that $\gamma^+\sigma' = \sigma$ and $\gamma^+\tau = \tau$. Since $\sigma = \tau$ holds in $\mathfrak{B}$, it follows that $\sigma' = \tau'$ holds in $\mathfrak{Rb}^\gamma\mathfrak{B}$, hence in $\mathfrak{Rb}^\gamma\mathfrak{A}$, and thus $\sigma = \tau$ holds in $\mathfrak{A}$, as desired.

5.6 ABSTRACT ALGEBRAIC LOGIC AND MORE ALGEBRAIC LOGICS

In this last section of the book we want to survey other algebraic treatments of logic. The main part of the section will be concerned with a general kind of algebraic logic, while the rest will be devoted to brief descriptions of various specific algebraic logics not discussed in the earlier parts of this book.

Abstract algebraic logic is an algebraic version of abstract (or "soft") model theory. It is concerned with very general algebraic structures which correspond to very general languages. We begin our discussion by describing some general languages, general algebraic logics, and their connections. Then we specialize the discussion, first to finitary one-sorted algebraic logics, and then to varieties definable by schemes of equations. All of this direction in algebraic logic has been developed mainly by Andréka and Németi; it is summarized in Andréka, Németi, Sain [84'].

General languages and algebraic logics

A *general language* is a triple $\langle Syn, \mathsf{K}, Sem\rangle$ such that Syn is a function with $Syn_i \neq 0$ for all $i \in DoSyn$, K is a class, and Sem is a function with domain $\bigcup_{i \in DoSyn} Syn_i \times \mathsf{K}$. For this notion and the development below, see Andréka, Sain [81'] (other references can be found there). Syn describes *syntactic categories* (like terms, formulas), K is a class of *models* (in some sense), and Sem defines *semantic notions* in the models. The following examples, which will be developed further as we proceed, illustrate these ideas.

EXAMPLE 5.6.1. Let Λ be a language in the sense of section 4.3. The associated general language is as follows. Syn has domain 1, with $Syn_0 = \Phi\mu^\Lambda$, the set of all formulas of Λ. K is the class of all possible models of Λ. For any $\varphi \in \Phi\mu^\Lambda$ and any $\mathfrak{A} \in \mathsf{K}$, $Sem(\varphi, \mathfrak{A}) = \varphi^{\mathfrak{A}}$, the set of all $a \in {}^\alpha A$ which satisfy φ in $\mathfrak{A}$.

EXAMPLE 5.6.2. Let κ and λ be infinite cardinals, and Λ an ordinary language in the sense of section 4.3, with a sequence $\langle v_\xi : \xi < \alpha\rangle$ of individual variables, $\alpha \geqq \kappa, \lambda$. Now we take Syn to have domain 1, and Syn_0 to be the set of all $L_{\kappa\lambda}$-formulas in Λ, i.e., we allow conjunctions of sets of formulas of size $<\kappa$, and simultaneous universal quantification of a set of $<\lambda$ variables. K and Sem are defined analogously to example 5.6.1. $L_{\infty\lambda}$, $L_{\kappa\infty}$, $L_{\omega\omega}$ are treated similarly.

EXAMPLE 5.6.3. Let $\Lambda = \langle \omega, \mathbf{R}, \rho, \mathbf{S}, \sigma\rangle$ be an ordinary language with function symbols. That is, $Do\mathbf{R}$ is some cardinal β and ρ maps β into ω, and similarly for $\mathbf{S}$ and σ; each $\mathbf{R}_i$ is treated as a relation symbol of rank ρ_i and each $\mathbf{S}_i$ as a function symbol of rank σ_i. To define the associated general language, let Syn have domain 3

with $Syn_0 = \{v_i : i \in \omega\}$, $Syn_1 = \{\tau : \tau$ is a term of $\Lambda\}$, $Syn_2 = \Phi\mu^\Lambda$. $\mathbf{K}$ is the class of all possible models of Λ, and if $i \in \omega$, τ is a term, φ a formula, and $\mathfrak{A} \in \mathbf{K}$, then

$$Sem(v_i, \mathfrak{A}) = \{\langle a, a_i\rangle : a \in {}^\omega A\},$$
$$Sem(\tau, \mathfrak{A}) = \{\langle a, \tau^{\mathfrak{A}} a\rangle : a \in {}^\omega A\},$$
$$Sem(\varphi, \mathfrak{A}) = \varphi^{\mathfrak{A}}.$$

To algebraize general languages we need the notion of a general context–free grammar. For any set X we let

$$X^* = \bigcup_{\alpha \text{ an ordinal}} {}^\alpha X.$$

We may consider X^* as the set of all words (of possibly infinite length) on X; the notation is intended to suggest a generalization of the usual notation X^* for the set of all finite sequences of members of X. There should be no confusion with our entirely different use of * (see Part I, p.28). A *general CF grammar* is a triple $\mathfrak{G} = \langle A, B, R\rangle$ such that A and B are disjoint non–empty sets and $R \subseteq A \times (A \cup B)^*$. It is conventional, because of the intuitive meaning of these notions, to write $x \to y$ in place of $\langle x, y\rangle$ if $\langle x, y\rangle \in R$. A is a set of *syntactic categories*, B a set of *basic symbols*, R the class of *rules*. We define extended relations $\Rightarrow_\alpha$ for every ordinal α as follows. $x \Rightarrow_0 x$ for any $x \in A$. For $\alpha > 0$ we define $x \Rightarrow_\alpha y$ iff $x \in A$ and there is a rule $x \to z$ with the following property: let $\Gamma = \{\kappa \in Doz : z_\kappa \in A\}$. Then there is a $w \in {}^\Gamma((A \cup B)^*)$ such that for each $\kappa \in \Gamma$ there is a $\beta < \alpha$ with $z_\kappa \Rightarrow_\beta w_\kappa$, and y is obtained from z by simultaneously replacing z_κ by w_κ for every $\kappa \in \Gamma$. Then we write $x \Rightarrow y$ iff $x \Rightarrow_\alpha y$ for some α. In case all rules are finitary – that is, with second terms of finite length – this definition can be given a simpler form, which we leave to the reader.

EXAMPLE 5.6.4 (Continuing 5.6.1). Let $\Lambda = \langle \alpha, \mathbf{R}, \rho\rangle$ be a language in the sense of section 4.3. Consider the general *CF* grammar $\langle \{formula\}, Rg\mathbf{R} \cup \{v_\kappa : \kappa < \alpha\} \cup \{\mathbf{A}, \mathbf{K}, \mathbf{N}, \mathbf{Q}, \mathbf{E}, \mathbf{T}, \mathbf{F}\}, R\rangle$, where R is given as follows:

(1) $formula \to \langle \mathbf{T}\rangle$
(2) $formula \to \langle \mathbf{F}\rangle$
(3) $formula \to \langle \mathbf{R}_\xi\rangle^\frown(v \circ \kappa)$ $\quad (\xi \in Do\mathbf{R},\ \kappa \in {}^{\rho\xi}\alpha)$
(4) $formula \to \langle \mathbf{E}, v_\kappa, v_\lambda\rangle$ $\quad (\kappa, \lambda < \alpha)$
(5) $formula \to \langle \mathbf{A}, formula, formula\rangle$
(6) $formula \to \langle \mathbf{K}, formula, formula\rangle$
(7) $formula \to \langle \mathbf{N}, formula\rangle$
(8) $formula \to \langle \mathbf{Q}, v_\kappa, formula\rangle$ $\quad (\kappa < \alpha)$.

Then $\Phi\mu^\Lambda = \{\varphi : formula$ is not in the range of the sequence φ, and $formula \Rightarrow \varphi\}$.

EXAMPLE 5.6.5 (Continuing 5.6.2). We assume that $\Lambda = \langle \alpha, \mathbf{R}, \rho\rangle$ just as in section 4.3, but we take $\alpha \geqq \kappa, \lambda$ and use the $L_{\kappa\lambda}$ formation rules. This means that we just replace (5),(6),(8) in 5.6.4 by the following rules:

(5′) $formula \to \langle \mathbf{A}\rangle^\frown\langle formula : \xi < \beta\rangle$ $\quad$ for $0 < \beta < \kappa$

(6') $\mathit{formula} \to \langle \mathbf{K} \rangle^\frown \langle \mathit{formula} : \xi < \beta \rangle$ for $0 < \beta < \kappa$
(8') $\mathit{formula} \to \langle \mathbf{Q} \rangle^\frown a^\frown \langle \mathit{formula} \rangle$, where $a \in {}^\beta\{v_\xi : \xi < \alpha\}$ for some $0 < \beta < \lambda$.

Then again we have the same description of the $L_{\kappa\lambda}$–formulas.

EXAMPLE 5.6.6 (Continuing 5.6.3). The general *CF* grammar this time is $\langle \{\mathit{variable}, \mathit{term}, \mathit{formula}\}, Rg\mathbf{R} \cup Rg\mathbf{S} \cup \{v, |, \mathbf{A}, \mathbf{K}, \mathbf{N}, \mathbf{Q}, \mathbf{E}, \mathbf{T}, \mathbf{F}\}, R\rangle$, where R is given as follows:

$$\begin{aligned}
&\mathit{variable} \to \langle v \rangle \\
&\mathit{variable} \to \langle \mathit{variable}, | \rangle \\
&\mathit{term} \to \langle \mathbf{S}_\eta \rangle^\frown \langle \mathit{term} : \kappa < \sigma\eta \rangle \qquad (\eta \in Do\mathbf{S}) \\
&\mathit{term} \to \mathit{variable} \\
&\mathit{formula} \to \langle \mathbf{E}, \mathit{term}, \mathit{term} \rangle \\
&\mathit{formula} \to \langle \mathbf{R}_\xi \rangle^\frown \langle \mathit{term} : \kappa < \rho\xi \rangle \ (\xi \in Do\mathbf{R}) \\
&\mathit{formula} \to \langle \mathbf{Q}, \mathit{variable}, \mathit{formula} \rangle
\end{aligned}$$

plus rules (5)–(7) of 5.6.4. Then, with B the second term of our general *CF* grammar, we have $\{v_i : i \in \omega\} = \{\varphi \in B^* : \mathit{variable} \Rightarrow \varphi\}$, $T\mu^\Lambda = \{\varphi \in B^* : \mathit{term} \Rightarrow \varphi\}$, and $\Phi\mu^\Lambda = \{\varphi \in B^* : \mathit{formula} \Rightarrow \varphi\}$. Here for each $i \in \omega$, v_i is considered to be the sequence consisting of v followed by i $|$'s.

Now let $\mathcal{L} = \langle Syn, \mathsf{K}, Sem\rangle$ be a general language. We say that $\mathcal{L}$ is *syntactically well–presented by a* general *CF* grammar $\mathfrak{G} = \langle A, B, R\rangle$ provided that $A = DoSyn$ and for every $a \in A$ we have $Syn_a = \{\varphi \in B^* : a \Rightarrow \varphi\}$. Thus we have shown that our three canonical examples are syntactically well–presented.

We want to define also an appropriate notion *semantically well–presented.* To this end we need, first of all, the notion of a *many–sorted algebra.* Let S be a non–empty set (of *sorts*). An *S–type* is a function g from some set I into $S^*{\sim}1$. A *many–sorted algebra of type g* is a structure $\mathfrak{A} = \langle A, f_i\rangle_{i \in I}$ such that A is a function with domain S with $A_s \neq 0$ for all $s \in S$, and if $i \in I$ then f_i maps $\mathsf{P}_{\kappa \in Do(gi)\sim 1} A_{(gi)\kappa}$ into $A_{(gi)0}$. (If $Do(gi) = 1$, then merely $f_i \in A_{(gi)0}$.) Now let $\mathfrak{G} = \langle A, B, R\rangle$ be a general *CF* grammar. We associate with $\mathfrak{G}$ the set A of sorts, an A–type g, and a many–sorted algebra $\mathfrak{F}_{\mathfrak{G}}$ as follows. Each rule in R can be written uniquely in the form

(*) $a_0 \to u_0^\frown \langle a_1 \rangle^\frown u_1^\frown \ldots {}^\frown \langle a_\kappa \rangle^\frown u_\kappa^\frown \ldots \ (\kappa < \alpha)$

for some α, where $a_\kappa \in A$ and $u_\kappa \in B^*$ for each $\kappa < \alpha$. Let g have domain R, and if $r \in R$ has the form (*), we let $g_r = \langle a_0, a_1, \ldots, a_\kappa, \ldots\rangle_{\kappa < \alpha}$. Next, for each $b \in A$ we let $F_b = \{x \in B^* : b \Rightarrow x\}$. For each $r \in R$, if r has the form (*) and $y_\kappa \in F_{a\kappa}$ for each $\kappa \in \alpha{\sim}1$, we let

$$f_r\langle y_\kappa : \kappa \in \alpha{\sim}1\rangle = u_0^\frown y_1^\frown u_1^\frown \ldots {}^\frown y_\kappa^\frown u_\kappa^\frown \ldots \ (\kappa < \alpha).$$

Thus $f_r\langle y_\kappa : \kappa \in \alpha{\sim}1\rangle \in F_{a0}$. Let $\mathfrak{F}_{\mathfrak{G}} = \langle F, f_r\rangle_{r \in R}$. So, $\mathfrak{F}_{\mathfrak{G}}$ is a many–sorted algebra of type g.

EXAMPLE 5.6.7 (Continuing 5.6.1, 5.6.4). The algebra $\mathfrak{F}_{\mathfrak{G}}$ is one–sorted, and it is in fact

$$\langle \Phi\mu^{\Lambda}, \mathbf{v}, \mathbf{\wedge}, \mathbf{\neg}, \mathbf{F}, \mathbf{T}, \mathbf{\exists} v_{\kappa}, \varphi \rangle_{\kappa<\alpha, \varphi \text{ atomic}}.$$

This is almost the same as $\mathfrak{F}\mathrm{m}^{\Lambda}$ (see section 4.3).

EXAMPLE 5.6.8 (Continuing 5.6.2, 5.6.5). The algebra $\mathfrak{F}_{\mathfrak{G}}$, still one–sorted, is

$$\langle \Xi, \mathbf{V}_{\eta}, \mathbf{\Lambda}_{\eta}, \mathbf{\neg}, \mathbf{F}, \mathbf{T}, \mathbf{\exists}_{a}, \varphi \rangle,$$

where the indices range as follows: $\eta<\kappa$, $a \in {}^{\beta}\{v_{\kappa} : \kappa<\alpha\}$, $\beta<\lambda$, φ atomic, and Ξ is the set of all $L_{\kappa\lambda}$–formulas of Λ. Of course $\mathbf{V}_{\eta}\langle \varphi_{\nu} : \nu<\eta \rangle$ is the disjunction of the formulas φ_{ν}, $\nu<\eta$; similarly for $\mathbf{\Lambda}_{\eta}$. For $\kappa=\omega$ this comes close to the absolutely free algebra suitable for polyadic equality algebras.

EXAMPLE 5.6.9 (Continuing 5.6.3, 5.6.6). This time the algebra $\mathfrak{F}_{\mathfrak{G}}$ is 3–sorted:

$$\langle \{v_{\kappa} : \kappa \in \omega\}, T\mu^{\Lambda}, \Phi\mu^{\Lambda}, v, \vartheta, \iota, \mathbf{S}_{\eta}, Eq, \mathbf{R}_{\xi}, \mathbf{v}, \mathbf{\wedge}, \mathbf{\neg}, \mathbf{F}, \mathbf{T}, \mathbf{\exists} \rangle_{\eta \in Do\mathbf{S}, \xi \in Do\mathbf{R}},$$

where $\vartheta x = x|$ for any variable x, $\iota x = x$ for any variable x, and $Eq(\sigma,\tau) = \sigma \equiv \tau$ for any terms σ,τ.

Returning to our general situation, we say that $\mathcal{L} = \langle Syn, \mathsf{K}, Sem \rangle$ is ***semantically well–presented by*** $\mathfrak{G}$ provided that it is syntactically well–presented by $\mathfrak{G}$ and for every $\mathfrak{A} \in \mathsf{K}$, the relation

$$h_{\mathfrak{A}} = \bigcup_{a \in A} \langle Sem(\varphi, \mathfrak{A}) : a \Rightarrow \varphi,\ \varphi \in B^{*} \rangle$$

is actually a function which is a homomorphism on $\mathfrak{F}_{\mathfrak{G}}$. Each of our three examples is semantically well–presented, as is easily seen.

The general algebraic logic corresponding to $\mathcal{L}$ in this situation is the pair $\langle \mathfrak{F}_{\mathfrak{G}}, \{h_{\mathfrak{A}} : \mathfrak{A} \in \mathsf{K}\} \rangle$ ($\{h_{\mathfrak{A}} : \mathfrak{A} \in \mathsf{K}\}$ is a proper class in general).

In Andréka, Németi, Sain [84'] it is shown how, even at this very general level, algebraic and logical properties have interesting interconnections. By considering classes of general languages, one can also motivate, in our examples 5.6.1, 5.6.4, 5.6.7, the final step leading to cylindric algebras – distinguishing equality formulas while ignoring other atomic formulas. Roughly speaking, the idea is that objects common to all languages in a class of general languages are exactly those that deserve to be called purely logical notions and hence should correspond to basic operations in an associated algebra.

By simplifying the situation slightly we will be able to give here a concrete instance of the interplay of logical and algebraic notions at this level of abstraction.

Algebraic logics

Considering the above situation when there is only one sort and all the rules in

the grammar are finitary, the associated general algebraic logic is a pair consisting of an algebra $\mathfrak{F}$ (in the usual general algebraic sense) and a class of homomorphisms on $\mathfrak{F}$. Moreover, in important examples, like 5.6.1 (continued in 5.6.4, 5.6.7), the algebra $\mathfrak{F}_{\mathfrak{G}}$ is absolutely free, although this is not generally the case. (Consider, for example, a grammar $\langle\{formula\},\{a\},R\rangle$ with rules $formula \to \langle a, formula\rangle$, $formula \to a$, and $formula \to \langle formula, a\rangle$, and build a semantically well-presented language on it.)

This motivates the following definition. An *algebraic logic* is a pair $\langle\mathfrak{F},\mathcal{H}\rangle$ such that $\mathfrak{F}$ is an absolutely free algebra of some type and $\mathcal{H} \subseteq Ho\mathfrak{F}$. This notion is investigated in Andréka, Gergely, Németi [77']; see also Andréka, Németi [75'] for a briefer introduction. To illustrate this notion we shall develop it far enough to prove a result of Németi [83'] which gives an algebraic equivalent of Beth's theorem. (This is a solution of a problem stated in Part I, p.357.) To motivate the following definitions, keep in mind the special case roughly corresponding to ordinary logic: $\mathfrak{F}$ is an absolutely free algebra similar to CA_ω's, and $\mathcal{H}$ is the class of all homomorphisms of $\mathfrak{F}$ into $\mathsf{Cs}_\omega^{reg} \cap \mathsf{Lf}_\omega$'s.

Let t be any (ordinary, one-sorted, algebraic) similarity type, fixed for the rest of this discussion, and let K be a class of algebras of type t. Given an arbitrary set X, we think of the elements of X as *non-logical constants*. $\mathfrak{Fr}_X$ is the absolutely free algebra of type t with X as its set of free generators. We then consider the algebraic logic $\mathcal{A}_X = \langle\mathfrak{Fr}_X, \bigcup_{\mathfrak{A}\in\mathsf{K}} Hom(\mathfrak{Fr}_X,\mathfrak{A})\rangle$. Elements of $\mathfrak{Fr}_X$ are called *X-formulas*. An *X-K-model is a pair* $\langle\mathfrak{A},h\rangle$ with $\mathfrak{A}\in\mathsf{K}$ and $h\in Hom(\mathfrak{Fr}_X,\mathfrak{A})$. Given *X-formulas* φ,ψ and an X-K-model $\langle\mathfrak{A},h\rangle$, we write $\langle\mathfrak{A},h\rangle\models\varphi\equiv\psi$ provided that $h\varphi = h\psi$. An *X-theory* is a set $\Sigma\subseteq {}^2Fr_X$. (We think of Σ as "saying" of each $\langle\varphi,\psi\rangle\in\Sigma$ that φ is equivalent to ψ.) Then an X-K-model $\langle\mathfrak{A},h\rangle$ is a *model* of Σ provided that $h\varphi = h\psi$ for each $\langle\varphi,\psi\rangle\in\Sigma$. Now assume that $x\in X$. We say that Σ *defines x implicitly* provided that if $\langle\mathfrak{A},h\rangle$, $\langle\mathfrak{A},k\rangle$ are models of Σ such that $(X\sim\{x\})\restriction h\subseteq k$, then $hx = kx$ (and hence $h = k$). We write $\Sigma\models\varphi\equiv\psi$ iff every model $\langle\mathfrak{A},h\rangle$ of Σ is such that $h\varphi = h\psi$. Then Σ *defines x explicitly* provided there is a $\varphi\in Sg^{\mathfrak{Fr}X}(X\sim\{x\})$ with $\Sigma\models\varphi\equiv x$. Finally, we say that t,K has the *Beth property* provided that for every set X, every *X-theory* Σ, and every $x\in X$, if Σ defines x implicitly, then Σ defines x explicitly. This is a purely logical notion, in a general sense.

Now we formulate a purely algebraic notion, involving only the class K. Let L be any class of algebras of type t. An *L-epimorphism* is a homomorphism $h\in Hom(\mathfrak{A},\mathfrak{B})$ with $\mathfrak{A},\mathfrak{B}\in\mathsf{L}$ such that for all homomorphisms $k,l\in Hom(\mathfrak{B},\mathfrak{C})$ with $\mathfrak{C}\in\mathsf{L}$, if $k\circ h = l\circ h$ then $k = l$. We call a homomorphism $h\in Hom(\mathfrak{A},\mathfrak{B})$ *almost onto* $\mathfrak{B}$ if there is an element $b\in B$ such that $Rgh\cup\{b\}$ generates $\mathfrak{B}$. The theorem of Németi [83'] is as follows.

THEOREM 5.6.10. *Let* $\mathsf{K} = \mathsf{SPK}$. *Then* t,K *has the Beth property iff every almost onto* K*-epimorphism is onto.*

PROOF. $\Rightarrow$. Suppose that $h\in Hom(\mathfrak{A},\mathfrak{B})$ is an almost onto K-epimorphism. Say $b\in B$ and $Sg(Rgh\cup\{b\}) = B$. Choose a set X and an element $x\in X$ such that there is a one-one function g from $X\sim\{x\}$ onto A. Let $f\in Hom(\mathfrak{Fr}_X,\mathfrak{B})$ be such that $(X\sim\{x\})\restriction f\subseteq h\circ g$, while $fx = b$. Let Σ be the X-theory $f|f^{-1}$. Now Σ defines x implicitly. For, suppose that $\langle\mathfrak{C},k\rangle$ and $\langle\mathfrak{C},l\rangle$ are models of Σ such that $(X\sim\{x\})\restriction k\subseteq l$. Then there are homomorphisms $k',l'\in Hom(\mathfrak{B},\mathfrak{C})$ such that $k'f\varphi = k\varphi$ and $l'f\varphi = l\varphi$ for any $\varphi\in Fr_X$. Then $k'\circ h = l'\circ h$, since any element of A has the form

gy for some $y \in X \sim \{x\}$, and

$$k'hgy = k'fy = ky = ly = l'fy = l'hgy.$$

Thus since h is a K–epimorphism we infer that $k' = l'$. Hence $kx = k'fx = l'fx = lx$, showing that Σ defines x implicitly. So, by the Beth property choose $\varphi \in Sg(X \sim \{x\})$ such that $\Sigma \models \varphi \equiv x$. Now $\langle \mathfrak{B}, f \rangle$ is a model of Σ, so $b = fx = f\varphi \in Rgh$, and so h is onto.

$\Leftarrow$. Suppose that Σ is an X–theory, $x \in X$, Σ defines x implicitly, but Σ does not define x explicitly. Let $I = Sg^{\mathfrak{A}}(X \sim \{x\})$, with $\mathfrak{A} = \mathfrak{Fr}_X$. Then for each $\varphi \in I$ there is a model $\langle \mathfrak{A}_\varphi, h_\varphi \rangle$ of Σ such that $h_\varphi \varphi \neq h_\varphi x$. Recall from 0.4.22 that $\mathfrak{Fr}_X\mathsf{K} \in \mathsf{K}$, since $\mathsf{SPK} = \mathsf{K}$. Now there is a homomorphism $f \in Hom(\mathfrak{Fr}_X\mathsf{K}, \mathsf{P}_{\varphi \in I}\mathfrak{A}_\varphi)$ such that $f(\psi / Cr_X\mathsf{K}) = \langle h_\varphi \psi : \varphi \in I \rangle$ for every $\psi \in Fr_X$. Let $Y = \{y / Cr_X\mathsf{K} : y \in X \sim \{x\}\}$, and let $\mathfrak{B} = \mathfrak{Cg}\, Y$. We claim:

(1) $B1f \in Hom(\mathfrak{B}, f^*\mathfrak{Fr}_X\mathsf{K})$ is a K–epimorphism which is almost onto.

For, suppose $k, l \in Hom(f^*\mathfrak{Fr}_X\mathsf{K}, \mathfrak{C})$, where $\mathfrak{C} \in \mathsf{K}$, and $k \circ (B1f) = l \circ (B1f)$. Let $n = \langle \varphi / Cr_X\mathsf{K} : \varphi \in Fr_X \rangle$ be the natural homomorphism from $\mathfrak{Fr}_X$ onto $\mathfrak{Fr}_X\mathsf{K}$. Then $\langle \mathfrak{C}, k \circ f \circ n \rangle$ and $\langle \mathfrak{C}, l \circ f \circ n \rangle$ are models of Σ, as is easily checked. Moreover, if $y \in X \sim \{x\}$ we have $kfny = k(B1f)ny = l(B1f)ny = lfny$. Therefore, since Σ defines x implicitly, $kfnx = lfnx$. Hence $k = l$, as desired: $B1f$ is a K–epimorphism. Since $f^*B \cup \{fnx\}$ generates $f^*\mathfrak{Fr}_X\mathsf{K}$, $B1f$ is almost onto.

Now (1) and the hypothesis yield that $B1f$ maps onto $f^*\mathfrak{Fr}_X\mathsf{K}$. Hence there is a $\varphi \in I$ such that $f(\varphi / Cr_X\mathsf{K}) = f(x / Cr_X\mathsf{K})$. Hence by the definition of f, $h_\varphi \varphi = f(\varphi / Cr_X\mathsf{K})_\varphi = f(x / Cr_X\mathsf{K})_\varphi = h_\varphi x$, contradicting the choice of h_φ.

Theorem 5.6.10 is generalized in Németi [83'], and several applications are given.

Systems of varieties

We now specialize the idea of an algebraic logic even further; but the notion will still be general enough to encompass very many specific algebraic logics. The notions and results here can be found in Andréka, Németi [80a']. We give only a sampling of their results. We generalize an aspect of cylindric algebras emphasized in section 4.1: our important classes CA_α and IGs_α are definable by countable schemas of equations. The indices κ, λ in the fundamental operations c_κ, $\mathsf{d}_{\kappa\lambda}$ vary according to a schema, and the system $\langle \mathsf{IGs}_\alpha : \alpha \geqq \omega \rangle$ (for example) is completely determined by a schema involving only IGs_ω. The following definition expresses which of the fundamental operations are to be determined by schemas.

DEFINITION 5.6.11.(i) *A* **type schema** *is a quadruple* $t = \langle T, \delta, \rho, \mathsf{c} \rangle$ *such that* T *is a set,* δ *maps* T *into* ω, ρ *maps* T *into* ω, $\mathsf{c} \in T$, *and* $\delta\mathsf{c} = \rho\mathsf{c} = 1$.

(ii) *A type schema* t *as in* (i) *defines a similarity type* t_α *for each* α *as follows. The domain* T_α *of* t_α *is*

$$T_\alpha = \{\langle f, \kappa_0, \ldots, \kappa_{\delta f - 1} \rangle : f \in T,\ \kappa \in {}^{\delta f}\alpha\}.$$

For each $\langle f,\kappa_0,\ldots,\kappa_{\delta f-1}\rangle \in T_\alpha$ *we set* $t_\alpha\langle f,\kappa_0,\ldots,\kappa_{\delta f-1}\rangle = \rho f$.

(iii) *A system* $\langle \mathsf{K}_\alpha : \alpha \geqq \omega\rangle$ *of classes of algebras is of* **type schema** t *if for each* $\alpha \geqq \omega$, K_α *is a class of algebras of type* t_α. Alg_α *is the class of* **all** *algebras of type* t_α. *Thus cylindric algebras fall under this definition with* $T = \{+,\cdot,-,0,1,\mathsf{d},\mathsf{c}\}$ *and* δ,ρ *the obvious functions.*

Note that every system $\langle K_\alpha : \alpha \geqq \omega\rangle$ given by a schema has unary operations c_κ, $\kappa<\alpha$. These special operations will be used to define local finite–dimension and related concepts below. Since most logics presently considered are extensions or modifications of first–order logic, this aspect of 5.6.11 does not seem very restrictive – it corresponds very roughly to this fact.

DEFINITION 5.6.12. *Let* t *be a type schema as in* 5.6.11. (i) *With each* α *we associate a language* $\mathcal{L}^t_\alpha$ *of type* t_α : *for each* $f \in T$ *and* $\kappa \in {}^{\delta f}\alpha$ *we have a function symbol* $\mathbf{f}_{\kappa 0,\ldots,\kappa(\delta f-1)}$ *of rank* ρf.

(ii) *Let* $\xi \in {}^\beta\alpha$. *We associate with each term* τ *of* $\mathcal{L}^t_\beta$ *a term* $\xi^+\tau$ *of* $\mathcal{L}^t_\alpha$ (*cf.*4.1.4). *For any* $\kappa<\omega$, $\xi^+ v_\kappa = v_\kappa$. *If* $f \in T$, $\kappa \in {}^{\delta f}\alpha$, *and* $\sigma_0,\ldots,\sigma_{\rho f-1}$ *are terms of* $\mathcal{L}^t_\beta$, *then*

$$\xi^+\mathbf{f}_{\kappa 0,\ \ldots,\kappa(\delta f-1)}\sigma_0\cdots\sigma_{\rho f-1} = \mathbf{f}_{\xi\kappa 0,\ldots,\xi\kappa(\delta f-1)}\xi^+\sigma_0\cdots\xi^+\sigma_{\rho f-1}.$$

Then we associate with each equation $\sigma = \tau$ *of* $\mathcal{L}^t_\beta$ *the equation* $\xi^+\sigma = \xi^+\tau$ *of* $\mathcal{L}^t_\alpha$.

(iii) *Let* E *be a set of equations of* $\mathcal{L}^t_\omega$. *A system* $\langle \mathsf{K}_\alpha : \alpha \geqq \omega\rangle$ *of type schema* t *is* **definable by** E *if for every* $\alpha \geqq \omega$ *we have*

$$\mathsf{K}_\alpha = \mathsf{Md}\{\xi^+ e : e \in E,\ \xi \in {}^\omega\alpha,\ \xi \text{ one–one}\}.$$

Then we say that $\langle \mathsf{K}_\alpha : \alpha \geqq \omega\rangle$ *is* **definable by the schema** $\langle t,E\rangle$.

We first note that this notion can be given a purely algebraic formulation.

THEOREM 5.6.13. *Let* t *be a type schema,* $\mathsf{K} = \langle \mathsf{K}_\alpha : \alpha \geqq \omega\rangle$ *a system of varieties of type schema* t. *Then the following conditions are equivalent:*

(i) K *is definable by a schema.*

(ii) *If* $\omega \leqq \alpha \leqq \beta$ *and* $\sigma \in {}^\alpha\beta$ *is one–one, then* $\mathsf{K}_\alpha = \mathsf{HSPRd}^\sigma\mathsf{K}_\beta$.

PROOF. First observe that $\mathsf{Rd}^\sigma\mathsf{K}_\beta$ is defined in a natural way, extending the definition for CA_α's in section 2.6.

(i)$\Rightarrow$(ii). Assume that K is definable by $\langle t,E\rangle$ and the hypotheses of (ii) hold. To prove that $\mathsf{K}_\alpha \supseteq \mathsf{HSPRd}^\sigma\mathsf{K}_\beta$ it suffices to take any $e \in E$, and any $\xi \in {}^\omega\alpha$ with ξ one–one, and show that $\mathsf{Rd}^\sigma\mathsf{K}_\beta \models \xi^+ e$. Now $\mathsf{K}_\beta \models \sigma^+\xi^+ e$, so clearly $\mathsf{Rd}^\sigma\mathsf{K}_\beta \models \xi^+ e$. To prove that $\mathsf{K}_\alpha \subseteq \mathsf{HSPRd}^\sigma\mathsf{K}_\beta$, let $\mathfrak{A} \in \mathsf{K}_\alpha$ be arbitrary. Let $I = \{\Gamma \subseteq \beta : |\Gamma| < \omega\}$. For each $\Gamma \in I$ let η_Γ be a one–one function from Γ into α such that $(\Gamma \cap Rg\sigma)1\eta_\Gamma \subseteq \sigma^{-1}$. Let $\mathfrak{B}_\Gamma = \mathfrak{Rd}^{\eta_\Gamma}\mathfrak{A}$, and let $\mathfrak{C}$ be any algebra of type t_β such that $\mathfrak{Rd}_\Gamma\mathfrak{C}_\Gamma = \mathfrak{B}_\Gamma$. Let F be an ultrafilter on I such that $\{\Delta \in I : \Gamma \subseteq \Delta\} \in F$ for every $\Gamma \in I$, and set $\mathfrak{D} = \mathsf{P}_{\Gamma \in I}\mathfrak{C}_\Gamma/F$. Clearly $\mathfrak{A} \cong |\subseteq \mathfrak{Rd}^\sigma\mathfrak{D}$. Now we show that $\mathfrak{D} \in \mathsf{K}_\beta$. So, let $e \in E$ and let $\xi \in {}^\omega\beta$ be one–one: we want to show that $\mathfrak{D} \models \xi^+ e$. Let Γ be the set of all $\kappa \in \beta$ which occur as indices in operation symbols occurring in $\xi^+ e$. Taking any $\Delta \in I$ such that $\Gamma \subseteq \Delta$, we have $\mathfrak{A} \models \eta^+_\Delta \xi^+ e$ since $\mathfrak{A} \in \mathsf{K}_\alpha$, and hence $\mathfrak{Rd}^{\eta_\Delta}\mathfrak{A} \models \xi^+ e$ and so $\mathfrak{C}_\Delta \models \xi^+ e$. It follows that $\mathfrak{D} \models \xi^+ e$. We have shown that $\mathfrak{D} \in \mathsf{K}_\beta$. Hence $\mathfrak{A} \in \mathsf{ISRd}^\sigma\mathsf{K}_\beta \subseteq \mathsf{HSPRd}^\sigma\mathsf{K}_\beta$, as desired.

(ii)$\Rightarrow$(i). Assume (ii). Say $\mathsf{K}_\omega = \mathsf{Md}E$, E the set of all equations holding in K_ω. We show that K is definable by $\langle t, E\rangle$. *Let* $\alpha \geqq \omega$, and let e be any equation of $\mathcal{L}^t_\alpha$. Then there exist an equation e' of $\mathcal{L}^t_\omega$ and a one–one $\xi \in {}^\omega\alpha$ such that $\xi^+ e' = e$. Now $\mathsf{K}_\alpha \models e$ iff $\mathsf{Rd}^\xi \mathsf{K}_\alpha \models e'$ iff $\mathsf{K}_\omega \models e'$ (by (ii)). Hence $\mathsf{K}_\alpha = \mathsf{Md}\{\xi^+ e : e \in E,\ \xi$ maps ω into α, ξ one–one$\}$, as desired.

Note that in 5.16.13(ii), HSP can be replaced by SI (see the proof).

DEFINITION 5.6.14. *Let* $\langle \mathsf{K}_\alpha : \alpha \geqq \omega\rangle$ *be of type schema* t. (i) *For* $\alpha \geqq \omega$, $\mathfrak{A} \in \mathsf{K}_\alpha$, $a \in A$, *we set*

$$\Delta a = \{\kappa < \alpha : \mathsf{c}_\kappa a \neq a\}.$$

(ii) *For* $\alpha \geqq \omega$ *and* $\mathfrak{A} \in \mathsf{K}_\alpha$, *we say that* $\mathfrak{A}$ *is* **locally finite–dimensional** *if* $|\Delta a| < \omega$ *for all* $a \in A$. Kf_α *is the class of all locally finite–dimensional* $\mathfrak{A} \in \mathsf{K}_\alpha$.

(iii) *For* $\alpha \geqq \omega$ *and* $\mathfrak{A} \in \mathsf{K}_\alpha$, *we say that* $\mathfrak{A}$ *is* **dimension complemented** *if for every finite* $X \subseteq A$ *we have that* $\alpha \sim \bigcup_{b \in X} \Delta b$ *is infinite.* Kc_α *is the class of all dimension complemented* $\mathfrak{A} \in \mathsf{K}_\alpha$.

(iv) *Suppose* $\omega \leqq \alpha \leqq \beta$ *and* $\mathfrak{A} \in \mathsf{Alg}_\beta$. *An element* $\mathfrak{B}$ *of* Alg_α *is a* **neat** α**–subreduct** *of* $\mathfrak{A}$ *if* $\mathfrak{B} \subseteq \mathfrak{Rd}_\alpha \mathfrak{A}$ *and* $\Delta^{\mathfrak{A}} b \subseteq \alpha$ *for all* $b \in B$. Kn_α *is the class of all neat* α*–subreducts of members of* $\mathsf{K}_{\alpha+\omega}$.

(v) *For any class* K *of similar algebras,* RpK *is the class of all algebras isomorphic to reduced products of algebras in* K.

For the next theorem, cf. 2.6.49(i) and 2.6.32.

THEOREM 5.6.15. *Let* $\langle \mathsf{K}_\alpha : \alpha \geqq \omega\rangle$ *be definable by a schema, and let* $\alpha \geqq \omega$. *Then* $\mathsf{Kf}_\alpha \subseteq \mathsf{Kc}_\alpha \subseteq \mathsf{Kn}_\alpha = \mathsf{SRpKn}_\alpha \subseteq \mathsf{K}_\alpha$.

PROOF. Obviously $\mathsf{Kf}_\alpha \subseteq \mathsf{Kc}_\alpha$. Using the schema property, it is also clear that $\mathsf{Kn}_\alpha = \mathsf{SRpKn}_\alpha \subseteq \mathsf{K}_\alpha$. Now suppose that $\mathfrak{A} \in \mathsf{Kc}_\alpha$; we construct $\mathfrak{B} \in \mathsf{K}_{\alpha+\omega}$ so that $\mathfrak{A}$ is a neat α–subreduct of $\mathfrak{B}$ – hence $\mathfrak{B} \in \mathsf{Kn}_\alpha$, as desired.

Let $I = \{\langle \Gamma, X\rangle : \Gamma \subseteq \alpha+\omega,\ X \subseteq A,\ \Gamma$ and X finite$\}$. Let $\langle \Gamma, X\rangle \in I$. Since $\mathfrak{A} \in \mathsf{Kc}_\alpha$, the set $H \overset{\mathsf{def}}{=} \alpha \sim \bigcup_{x \in X} \Delta x$ is infinite. Let $\eta_{\Gamma X}$ be a one–one function from Γ into α such that $\alpha \upharpoonright \eta_{\Gamma X} \subseteq Id$ and $Rg((\Gamma \sim \alpha) \upharpoonright \eta_{\Gamma X}) \subseteq H$. Then let $\mathfrak{B}_{\Gamma X}$ be a member of $\mathsf{Alg}_{\alpha+\omega}$ whose Γ–reduct is $\mathfrak{Rd}^{\eta_{\Gamma X}} \mathfrak{A}$. Let F be the filter on I generated by

$$\{\{\langle \Gamma, X\rangle \in I : \Delta \subseteq \Gamma,\ Y \subseteq X\} : \langle \Delta, Y\rangle \in I\}$$

Then $\mathfrak{A}$ is isomorphic to a neat α–subreduct of $\mathsf{P}_{\iota \in I} \mathfrak{B}_\iota / F$, and $\mathsf{P}_{\iota \in I} \mathfrak{B}_\iota / F \in \mathsf{K}_{\alpha+\omega}$, as desired.

DEFINITION 5.6.16. *Let* $\mathsf{K} = \langle \mathsf{K}_\alpha : \alpha \geqq \omega\rangle$ *be of type schema* t. *We say that* K **satisfies the finiteness generating condition** *if for all* $\alpha \geqq \omega$, $\mathfrak{A} \in \mathsf{K}_\alpha$, *and* $X \subseteq A$, *if* $|\Delta x| < \omega$ *for all* $x \in X$, *then* $\mathfrak{Cg}^{\mathfrak{A}} X \in \mathsf{Kf}_\alpha$.

For the next theorem, cf. 2.6.52.

THEOREM 5.6.17. *Let* $\langle \mathsf{K}_\alpha : \alpha \geqq \omega \rangle$ *be definable by a schema and satisfy the finiteness generating condition. Then for any* $\alpha \geqq \omega$ *we have* $\mathsf{SUpKf}_\alpha = \mathsf{SUpKc}_\alpha = \mathsf{Kn}_\alpha$, *and hence* $\mathsf{HSPKf}_\alpha = \mathsf{HSPKc}_\alpha = \mathsf{HKn}_\alpha$.

PROOF By Theorem 5.6.15 it suffices to show that $\mathsf{Kn}_\alpha \subseteq \mathsf{SUpKf}_\alpha$. Let $\mathfrak{A} \in \mathsf{Kn}_\alpha$, and choose $\mathfrak{B} \in \mathsf{K}_{\alpha+\omega}$ such that $\mathfrak{A}$ is a neat α–subreduct of $\mathfrak{B}$. For every finite $\Gamma \subseteq \alpha$ we shall construct $\mathfrak{C}_\Gamma \in \mathsf{Kf}_\alpha$ such that $\mathfrak{Rb}_\Gamma \mathfrak{A} \subseteq \mathfrak{Rb}_\Gamma \mathfrak{C}_\Gamma$; this easily gives the desired conclusion $\mathfrak{A} \in \mathsf{SUpKf}_\alpha$.

So let Γ be any finite subset of α. For *any* finite subset Δ of α let η_Δ be a one–one function from Δ into $\alpha+\omega$ such that $(\Gamma \cap \Delta) 1 \eta_\Delta \subseteq Id$ and $Rg((\Delta \sim \Gamma) 1 \eta_\Delta) \subseteq (\alpha+\omega) \sim \alpha$. Now let $\mathfrak{D}_\Delta$ be any member of Alg_α such that $\mathfrak{Rb}_\Delta \mathfrak{D}_\Delta = \mathfrak{Rb}^{\eta_\Delta} \mathfrak{B}$. Let F be an ultrafilter on $I \overset{\text{def}}{=} \{\Delta \subseteq \omega : \Delta \text{ finite}\}$ such that $\{\Omega \in I : \Delta \subseteq \Omega\} \in F$ for every $\Delta \in I$. Then Let $\mathfrak{C}_\Gamma = \mathsf{P}_{\Delta \in I} \mathfrak{D}_\Delta / F$. Then $\mathfrak{Rb}_\Gamma \mathfrak{B} \subseteq \mathfrak{Rb}_\Gamma \mathfrak{C}_\Gamma$ and $\mathfrak{C}_\Gamma \in \mathsf{K}_\alpha$. Also, $\Delta^{\mathfrak{C}_\Gamma} a \subseteq \Gamma$ for all $a \in A$, by construction. Let $\mathfrak{C}_\Gamma = \mathfrak{Cg}^{\mathfrak{C}_\Gamma} A$. Then $\mathfrak{Rb}_\Gamma \mathfrak{A} \subseteq \mathfrak{Rb}_\Gamma \mathfrak{C}_\Gamma$, and $\mathfrak{C}_\Gamma \in \mathsf{Kf}_\alpha$ by the generating condition. The proof is complete.

The finiteness generatlng condition is extensively investigated in Andréka, Németi [78'] and in Andréka, Németi, Sain [84']. In particular, CA results are classified according to whether they hold for arbitrary systems of varieties definable by a schema, for those having the finiteness generating condition, or are very specific for cylindric algebras. They make a similar classification for the other algebraic logics treated below.

A survey of some additional algebraic logics

We now turn to the other part of this section, in which we briefly describe some other specific algebraic logics. We shall discuss these in alphabetical order of the inventor of the logic.

1. Bernays [59]: *many–sorted cylindric algebras.* The idea here is to reformulate the notion Lf_ω by having, for each $\kappa \in \omega$, variables of sort κ to denote elements x such that $\Delta x \subseteq \kappa$. This enables one to simplify the primitive notions, too. Specifically there are ω sorts. For each $m \in \omega$ we have variables $v_0^m, v_1^m, \ldots$ of sort m. The non–logical constants are as follows:

$\mathbf{s}_{ij}$ (for $i,j \in \omega$),	permutation symbols, which are one place function symbols;
$\mathbf{c}$	cylindrification, one place function symbol;
$\mathbf{d}$	diagonal, individual constant;
$-$	unary function symbol;
$+, \cdot$	binary function symbols;
$\mathbf{0}, \mathbf{1}$	individual constants.

Terms are formed as follows. Each variable v_i^m is an m–term. Every m–term is an $(m+1)$–term. If σ and τ are m–terms, so are $\sigma + \tau$, $\sigma \cdot \tau$, $-\tau$, and $\mathbf{c}\sigma$. If σ is an

m–term and $l \geq i,j,m$, then $\mathsf{s}_{ij}\sigma$ is an l–term. If σ is a $(k+1)$–term, then $\mathsf{s}_{0,k+1}\mathsf{c}\sigma$ is a k–term. $\mathbf{d}$ is a 2–term. $\mathbf{0}$ and $\mathbf{1}$ are 0–terms. The following are taken as axioms, with arbitrary indices except where restrictions are mentioned.

$$
\begin{array}{lll}
v_0^m + v_1^m = v_1^m + v_0^m, & & v_0^m \cdot v_1^m = v_1^m \cdot v_0^m, \\
v_0^m + v_1^m \cdot v_2^m = (v_0^m + v_1^m) \cdot (v_0^m + v_2^m), & & \\
v_0^m \cdot (v_1^m + v_2^m) = v_0^m \cdot v_1^m + v_0^m \cdot v_2^m, & & \\
v_0^m + \mathbf{0} = v_0^m, & & v_0^m \cdot \mathbf{1} = v_0^m, \\
v_0^m + -v_0^m = \mathbf{1}, & & v_0^m \cdot -v_0^m = \mathbf{0}, \\
\mathsf{s}_{ij}(v_0^m + v_1^m) = \mathsf{s}_{ij}v_0^m + \mathsf{s}_{ij}v_1^m, & & \\
\mathsf{s}_{ij}(-v_0^m) = -\mathsf{s}_{ij}v_0^m, & & \\
\mathsf{s}_{ii}v_0^m = v_0^m, & & \mathsf{s}_{ij}v_0^m = \mathsf{s}_{ji}v_0^m, \\
\mathsf{s}_{ij}\mathsf{s}_{jk}\mathsf{s}_{ki}v_0^k = \mathsf{s}_{jk}v_0^k & \text{if } i \neq j,k, & \\
\mathsf{s}_{ij}v_0^k = v_0^k & \text{if } i,j > k, & \\
\mathsf{c}\mathsf{s}_{0i}v_0^m = \mathsf{s}_{0i}v_0^m & \text{if } i > k, & \\
\mathsf{s}_{ij}\mathsf{c}v_0^m = \mathsf{c}\mathsf{s}_{ij}v_0^m & \text{if } i,j > 0, & \\
\mathsf{s}_{01}(v_0^m \cdot \mathbf{d}) = v_0^m \cdot \mathbf{d}, & & \\
\mathsf{c}\mathbf{d} = \mathbf{1}, & & v_0^m \cdot \mathsf{c}v_0^m = v_0^m, \\
\mathsf{c}(v_0^m \cdot \mathsf{c}v_1^m) = \mathsf{c}v_0^m \cdot \mathsf{c}v_1^m, & & \\
\mathsf{c}(\mathbf{d} \cdot -v_0^m) = -\mathsf{c}(\mathbf{d} \cdot v_0^m), & & \\
\mathsf{c}\mathsf{s}_{01}\mathsf{c}\mathsf{s}_{01}v_0^m = \mathsf{c}\mathsf{s}_{01}\mathsf{c}v_0^m, & & \\
\mathbf{0} \neq \mathbf{1}, & & v_0^1 = \mathbf{0} \vee \mathsf{c}v_0^1 = \mathbf{1}.
\end{array}
$$

The main result in Bernays [59] is that this axiom system is complete with respect to the interpretation initially alluded to. See also Schwartz [79].

2. Copeland [55]: *simplified polyadic equality algebras.* This notion was not much developed in Copeland [55], and we shall instead follow Demaree [72′]. Let $\mathbf{Z}$ be the set of integers. We recall some familiar operations on $Sb({}^{\mathbf{Z}}U)$, U any set. Below s is the successor operation on $\mathbf{Z}$, and $x \subseteq {}^{\mathbf{Z}}U$.

$$
\begin{aligned}
\mathbf{C}_0 x &= \{u \in {}^{\mathbf{Z}}U : u_a^0 \in x \text{ for some } a\}. \\
\mathbf{Q} x &= \{u \in {}^{\mathbf{Z}}U : u \in x \text{ and } u_0 = u_1\}. \\
\mathbf{P} x &= \{u \in {}^{\mathbf{Z}}U : u \circ [0/1, 1/0] \in x\}. \\
\mathbf{T} x &= \{u \in {}^{\mathbf{Z}}U : u \circ \mathsf{s} \in x\}.
\end{aligned}
$$

A *Copeland set algebra*, Cpsa, is an algebra

$$\langle A, \cup, \cap, \sim, 0, {}^{\mathbf{Z}}U, \mathbf{C}_0, \mathbf{Q}, \mathbf{P}, \mathbf{T}, \mathbf{T}^{-1} \rangle$$

where A is a collection of subsets of ${}^{\mathbf{Z}}U$. Rcpa is the class of all algebras isomorphic to a subdirect product of Cpsa's, and Cpa the variety determined by all equations holding in all Cpsa's. The main results in Demaree [72′] are: (1) neither SCpsa nor Rcpa are elementary classes; (2) Cpa is not finitely axiomatizable; (3) there is an $\mathfrak{A} \in \mathsf{Cpa} \sim \mathsf{Rcpa}$ which is locally finite–dimensional; (4) within each Cpa one can define

in a natural way a CA_Z (understood in the obvious sense).

3. Craig: *algebras of sets of finite sequences.* Several kinds of algebraic logics corresponding to satisfaction by finite sequences were developed mainly by Craig; see Craig [65],[68],[74a'],[74b'],[74c'] and also Howard [65] and Monk [70]. This development is more ambitious than implied by our title: he attempts to give a justification for the primitives chosen, on the basis of extensive concern with the set–theoretic expressibility of logical notions. We illustrate the specific systems he develops with one case: the *augmented cylindric theory for sets of finite sequences.* The set algebra notion is as follows. For any non–empty set U, let ${}^{<\omega}U$ denote the collection of all finite sequences of members of U. We consider the following operations on $Sb({}^{<\omega}U)$: for any $X \subseteq {}^{<\omega}U$ and $i<\omega$,

$\mathsf{C}_i X = \{f \in {}^{<\omega}U$: there is a $g \in X$ such that $Dof = Dog$ and $fj = gj$ for all $j \in Dof \sim \{i\}\}$.
$\mathsf{Q}X = \{\langle u\rangle^\frown h : u \in U,\ h \in X\}$.
$\mathsf{P}X = \{h \in {}^{<\omega}U : \langle u\rangle^\frown h \in X$ for some $u \in U\}$.
$\mathsf{D} = \{f \in {}^{<\omega}U : 2 \subseteq Dof$ and $f_0 = f_1\}$.

Then a *finite sequence set algebra* is an algebraic structure

$$\mathfrak{A} = \langle A, \mathsf{u}, \mathsf{n}, \sim, 0, {}^{<\omega}U, \mathsf{Q}, \mathsf{P}, \mathsf{D}, \mathsf{C}_i\rangle_{i<\omega}$$

with A closed under the indicated operations. The main result about this notion is a completeness theorem: an equality $\sigma = \tau$ is valid in all finite sequence set algebras iff it is derivable from the following equations, where $i,j<\omega$, $\mathsf{d}_{i,i+1} = \mathsf{d}_{i+1,i} = \mathsf{q}^i\mathsf{d}$, and $\mathsf{d}_{i,i+2} = \mathsf{d}_{i+2,i} = \mathsf{c}_{i+1}(\mathsf{q}^i\mathsf{d}\cdot\mathsf{q}^{i+1}\mathsf{d})$:

(1) Axioms for BA's.
(2) $\mathsf{c}_i 0 = 0$.
(3) $x\cdot\mathsf{c}_i x = x$.
(4) $\mathsf{c}_i(x\cdot\mathsf{c}_i y) = \mathsf{c}_i x\cdot\mathsf{c}_i y$.
(5) $\mathsf{c}_i\mathsf{c}_j x = \mathsf{c}_j\mathsf{c}_i x$.
(6) $\mathsf{q}0 = 0$.
(7) $\mathsf{q}(x+y) = \mathsf{q}x + \mathsf{q}y$.
(8) $\mathsf{q}(x\cdot y) = \mathsf{q}x\cdot\mathsf{q}y$.
(9) $\mathsf{q}\mathsf{c}_i x = \mathsf{c}_{i+1}\mathsf{q}x$.
(10) $\mathsf{c}_i(-\mathsf{q}^{i+1}1\cdot x) = -\mathsf{q}^{i+1}1\cdot x$.
(11) $\mathsf{q}\mathsf{p}x = \mathsf{q}1\cdot\mathsf{c}_0 x$.
(12) $\mathsf{p}\mathsf{q}x = x$.
(13) $\mathsf{c}_0\mathsf{d} = \mathsf{q}^2 1$.
(14) $\mathsf{c}_1\mathsf{d} = \mathsf{q}^2 1$.
(15) $\mathsf{c}_i\mathsf{d} = \mathsf{d}$ if $i \geq 2$.
(16) $\mathsf{d}_{ij}\cdot\mathsf{c}_i(\mathsf{d}_{ij}\cdot x) = \mathsf{d}_{ij}\cdot x$ if $i \neq j$ and $-2 \leq i-j \leq 2$.

4. Freeman [76']: *modal cylindric algebras.* This is an algebraization of modal predicate logic. A *modal cylindric algebra* is a cylindric algebra with an additional unary operation * (corresponding to "possibility") so that the following conditions hold:

(M1) $^*(x+y) = {}^*x + {}^*y$.
(M2) $c_\kappa {}^*c_\kappa x = {}^*c_\kappa x$.
(M3) $d_{\kappa\lambda} \cdot -{}^*-x \leq -{}^*-(d_{\kappa\lambda} \cdot x)$.
(M4) $-d_{\kappa\lambda} \cdot -{}^*-x \leq -{}^*-(-d_{\kappa\lambda} \cdot x)$.

Corresponding to various kinds of modal logic, one adds additional equational axioms. The notion of a modal cylindric set algebra is as follows; it involves Kripke semantics. Let U be a non-empty set, α an ordinal, W a non-empty set, R a binary relation on W, Q a subset of W, and f a function from W into $SbU \sim \{0\}$ such that vRw implies $fv \subseteq fw$. For each $w \in W$ let $\mathfrak{A}_w$ be the Cs_α $\mathfrak{Sb}({}^\alpha fw)$. We make $P_{w \in W} \mathfrak{A}_w$ into a modal cylindric algebra by defining the operation * as follows. Let $x \in P_{w \in W} A_w$ and $w \in W$. Then

$$({}^*x)_w = \begin{cases} {}^\alpha fw \text{ if } w \in Q, \\ \{u \in {}^\alpha fw : \text{there is a } v \text{ such that } wRv \text{ and } u \in x_v\} \text{ if } w \notin Q. \end{cases}$$

Any subalgebra of $P_{w \in W} \mathfrak{A}_w$ is called a *modal cylindric set algebra*. The main result of Freeman [76'] is the representation theorem: any locally finite-dimensional modal cylindric algebra is isomorphic to a subdirect product of modal cylindric set algebras.

Very analogous results were proved later in Georgescu [79'].

5. Georgescu [72'] *Polyadic θ-valued Łukasiewicz algebras.* Instead of a BA, one takes a θ-valued Łukasiewicz algebra. A representation theorem for locally-finite dimensional algebras of infinite dimension is proved.

6. Georgescu [79a'] *Polyadic tense algebras.* Instead of a BA, one takes a tense algebra, which is an algebraic version of tense logic. Again a representation theorem for locally finite infinite dimensional algebras is proved.

7. Georgescu [82a'] *Polyadic interior algebras.* Rather than taking the underlying algebra to be an interior algebra instead of a BA, the idea here is to take a Cs_α with base U, where U has a topology on it; then one can define for each $\kappa < \alpha$ a unary operation I_κ on $Sb({}^\alpha U)$ as follows. For any $X \subseteq {}^\alpha U$,

$$I_\kappa X = \{u \in {}^\alpha U : u_\kappa \in \text{Interior}\{a \in U : u^\kappa_a \in X\}\}.$$

This gives an appropriate notion of a set algebra, with a similarity type extending that of PEA_α's by a new sequence $\langle I_\kappa : \kappa < \alpha \rangle$ of unary operators. Equational axioms are given, and a representation theorem as in 5,6 is proved.

8. Georgescu [82'] *Monotone polyadic algebras.* Extensions of polyadic algebras which algebraize the monotone quantifiers of Makowsky, Tulipani [78**] are introduced, and a representation theorem as in 5,6,7 is proved.

9. Halmos [57] and [62]. *Quasi-polyadic algebras.* These are like polyadic algebras, except that s_τ is allowed only for *finite* transformations, and $c_{(\Gamma)}$ only for finite Γ. Their theory has not been much developed, but they form an interesting stage

between cylindric and polyadic algebras. See also Daigneault, Monk [63] and Andréka, Gergely, Németi [77].

10. Kotas, Piecskowski [67] *Cylindric algebras over a Boolean ring.* Basic definitions and simple facts are given.

11. LeBlanc [62a] *Nonhomogeneous polyadic algebras.* We have here an algebraic version of many–sorted logic. Let α be an ordinal and $\mathcal{P}$ a partition of α. A $\mathcal{P}$–transformation of α is a function $\sigma \in {}^{\alpha}\alpha$ such that $\sigma^*\Gamma \subseteq \Gamma$ for every $\Gamma \in \mathcal{P}$. A *$\mathcal{P}$–sorted polyadic algebra of dimension* α is an algebra

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, \mathsf{c}_{(\Gamma)}, \mathsf{s}_{\tau} \rangle_{\Gamma \subseteq \alpha, \tau \in P},$$

where P is the set of all $\mathcal{P}$–transformations, such that all of the axioms for polyadic algebras hold, restricting to $\tau \in \mathcal{P}$ rather than arbitrary $\tau \in {}^{\alpha}\alpha$. The usual representation theorem for locally finite algebras with each $\Gamma \in \mathcal{P}$ infinite is given, and a reduction to ordinary polyadic algebras is indicated – corresponding to the reduction of many–sorted logic to ordinary logic.

12. Lucas [68]: *Extended cylindric algebras.* The extension is to cylindrification over infinite sets, and infinitary diagonal elements. Specifically, let κ be an infinite cardinal, α any ordinal. A κ–CA_{α} is an algebra

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, \mathsf{c}_{(\Gamma)}, \mathsf{d}_E \rangle_{\Gamma \in M, E \in N},$$

where $M = \{\Gamma \subseteq \alpha : |\Gamma| < \kappa\}$, $N = \{E \subseteq \alpha \times \alpha : E$ is an equivalence relation on α and $|\{\xi < \alpha : \xi/E \neq \{\xi\}\}| < \kappa\}$, such that the following conditions hold:

(E_1) $\langle A, +, \cdot, -, 0, 1 \rangle$ is a BA.
(E_2) $\mathsf{c}_{(\Gamma)} 0 = 0$.
(E_3) $x \leq \mathsf{c}_{(\Gamma)} x$.
(E_4) $\mathsf{c}_{(\Gamma)}(x \cdot \mathsf{c}_{(\Gamma)} y) = \mathsf{c}_{(\Gamma)} x \cdot \mathsf{c}_{(\Gamma)} y$.
(E_5) $\mathsf{c}_{(0)} x = x$.
(E_6) $\mathsf{c}_{(\Gamma \cup \Delta)} x = \mathsf{c}_{(\Gamma)} \mathsf{c}_{(\Delta)} x$.
(E_7) $\mathsf{d}_I = 1$, where $I = \alpha \upharpoonright Id$.
(E_8) $\mathsf{c}_{(\Gamma)} \mathsf{d}_E = \mathsf{d}_F$, where $F = E \cap {}^2(\alpha \sim \Gamma) \cup (\alpha \upharpoonright Id)$.
(E_9) $\mathsf{c}_{(\Gamma)}(x \cdot \mathsf{d}_E) \cdot \mathsf{c}_{(\Gamma)}(-x \cdot \mathsf{d}_E) = 0$, where $|(\xi/E) \sim \Gamma| = 1$ for every $\xi < \alpha$.

Intuitively, d_E denotes $\Pi_{\langle \xi, \eta \rangle \in E} \mathsf{d}_{\xi\eta}$. The results of Lucas [68] are concerned with the relationships of extended cylindric algebras with similarly defined extended polyadic algebras.

13. Pinter [73b'] *Substitution–cylindrification algebras.* These are essentially polyadic algebras with simplified primitives, definitionally equivalent (in an extension of first–order logic) to PA_{α}'s in the locally–finite infinite dimension case.

14. Pinter [73b'] *Substitution algebras.* A *substitution algebra* of dimension α is an algebra $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, \mathsf{s}^{\kappa}_{\lambda} \rangle_{\kappa, \lambda < \alpha}$ such that the following conditions hold:

(S_1) $\langle A,+,\cdot,-,0,1\rangle$ is a BA.
(S_2) $s^\kappa_\lambda(x+y)=s^\kappa_\lambda x+s^\kappa_\lambda y$.
(S_3) $s^\kappa_\lambda -x=-s^\kappa_\lambda x$.
(S_4) $s^\kappa_\kappa x=x$.
(S_5) $s^\mu_\lambda s^\kappa_\mu x=s^\mu_\lambda s^\kappa_\lambda x$.
(S_6) $s^\kappa_\mu s^\kappa_\lambda x=s^\kappa_\lambda x$ if $\kappa\neq\lambda$.
(S_7) $s^\kappa_\lambda s^\mu_\nu x=s^\mu_\nu s^\kappa_\lambda x$ if $\kappa\neq\mu,\nu$ and $\mu\neq\nu$.

It is shown that cylindric algebras are definitionally equivalent (again, in an extension of first–order logic) to substitution algebras which satisfy the following three conditions:

(K_1) For each $x\in A$ and $\kappa<\alpha$, $\{y\in A: y\in Rg\, s^\kappa_\lambda,\ \lambda\neq\kappa,\ y\geqq x\}$ has a least element, denoted by $c_\kappa x$.
(K_2) For all $\kappa,\lambda<\alpha$, $\{x: s^\kappa_\lambda x=1\}$ has a least element.
(K_3) If $\mu\neq\kappa,\lambda$, then $s^\kappa_\lambda c_\mu x=c_\mu s^\kappa_\lambda x$.

See also Preller [70].

15. Pinter [75'] *Generalized–quantifier algebras.* A *GQ* CA_α is an algebra

$$\mathfrak{A}=\langle A,+,\cdot,-,0,1,c_\kappa,d_{\kappa\lambda},q_\kappa\rangle_{\kappa,\lambda<\alpha}$$

such that $\langle A,+,\cdot,-,0,1,c_\kappa,d_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha}$ is a CA_α and the following axioms hold:

(Q_1) $q_\kappa(x+y)=q_\kappa x+q_\kappa y$.
(Q_2) $q_\kappa d_{\kappa\lambda}=0$.
(Q_3) $q_\kappa c_\kappa=c_\kappa$.
(Q_4) $c_\kappa q_\kappa=q_\kappa$.
(Q_5) $q_\kappa c_\lambda x\leqq c_\lambda q_\kappa x+q_\lambda c_\kappa x$.

Cylindric algebras which can be expanded to *GQ* CA_α's are characterized. See also Georgescu [80'].

16. Pratt, Kozen. *Dynamic algebras.* The main reference is Pratt [79']. These algebras arose in computer science, but they are a kind of algebraic logic. A *dynamic algebra* is a two–sorted algebra $\langle A,+,\cdot,-,0,1,B,+',;,*,\Diamond\rangle$ satisfying the following conditions (with $a,b\in A$ and $x,y,z\in B$):

(D_1) $\langle A,+,\cdot,-,0,1\rangle$ is a BA.
(D_2) $\langle B,+',;,*\rangle$ is a one–sorted algebra.
(D_3) $\Diamond$ maps $B\times A$ into A.
(D_4) $x\Diamond 0=0$.
(D_5) $x\Diamond(a+b)=(x\Diamond a)+(x\Diamond b)$.
(D_6) $(x+'y)\Diamond a=(x\Diamond a)+(x\Diamond b)$.
(D_7) $(x;y)\Diamond a=x\Diamond(y\Diamond a)$.

(D_8) $x+y\Diamond(y^*\Diamond z)\leq y^*\Diamond x\leq x+y^*\Diamond(-x\cdot(y\Diamond x))$.

There is an associated set algebra notion, namely an algebra

$$\langle A,\mathbf{u},\mathbf{n},{}_I\sim,0,I,B,\mathbf{u},|,^*,\langle\rangle\rangle$$

such that $\langle A,\mathbf{u},\mathbf{n},{}_I\sim,0,I\rangle$ is a BA of subsets of I, B is a set of binary relations on I closed under $\mathbf{u}$, $|$ (relative product) and *, where $^*R$ is the reflexive, transitive closure of R, and such that $\langle\rangle$ maps $B\times A$ into A where, for any $R\in B$ and $X\in A$, $\langle R,X\rangle = R^{-1}{}^*X$.

The representation theory for dynamic algebras is fairly well-developed; see Andréka, Németi [79'], Kozen [79'], [79a'], [79b'], [80'], and Németi [80'],[81a'],[82'].

17. Schein [70]: *general relation algebras.* This paper (a survey of his earlier work in this area) gives a general schema for arriving at finite dimensional set algebras related to cylindric algebras.

Fix $m\in\omega$; we show how to construct algebras of m-ary relations. Let $\mathbf{K}$ be a class of relational structures (with primitive relations *and* functions). To a language $\mathcal{L}$ appropriate for $\mathbf{K}$ we adjoin m-ary *relation variables* $\mathbf{R}_0,\mathbf{R}_1,\ldots$. A *possible model* of $\mathcal{L}$ is a pair $\langle\mathfrak{M},A\rangle$ such that $\mathfrak{M}\in\mathbf{K}$ and A is a set of m-ary relations on M. A formula φ of $\mathcal{L}$ is *semi-closed* if it has no free individual variables. $\langle\mathfrak{M},A\rangle$ is a *model* of φ if $(\mathfrak{M},R_i)_{i<\omega}\models\varphi$ for every $R\in{}^\omega A$. Now let $t\in{}^I\omega$ be a similarity type (of algebras). This type t does not have to have any relationship to the type of members of $\mathbf{K}$. A system $\langle\varphi_i : i\in I\rangle$ of formulas of $\mathcal{L}$ *conforms to* t provided that the following conditions hold for each $i\in I$:

(1) The free individual variables of φ_i are among $v_0,\ldots,v_{m-1}$.
(2) The relation variables of φ_i are among $\mathbf{R}_0,\ldots,\mathbf{R}_{t_i-1}$.

Furthermore, let $\langle\mathfrak{M},A\rangle$ be a model of a set Γ of semi-closed formulas of $\mathcal{L}$. Then $\langle\varphi_i : i\in I\rangle$ *conforms to* $t,\mathfrak{M},A$ provided that it conforms to t and the following condition holds for each $i\in I$:

(3) For all $R_0,\ldots,R_{t_i-1}\in A$, we have $\{u\in{}^mM : \langle\mathfrak{M},A\rangle\models\varphi_i[u_0,\ldots,u_{m-1},R_0,\ldots,R_{t_i-1}]\}\in A$.

In case this happens, we can define a *general relation algebra* $\mathfrak{R}_{\mathfrak{M}t\varphi A}=\mathfrak{R}$ with universe A and of type t as follows: for any $R_0,\ldots,R_{t_i-1}\in A$,

$$\mathbf{F}_i^{\mathfrak{R}}R_0\ldots R_{t_i-1}=\{u\in{}^mM : \langle\mathfrak{M},A\rangle\models\varphi_i[u_0,\ldots,u_{m-1},R_0,\ldots,R_{t_i-1}]\}.$$

For any $\mathbf{K},\mathcal{L},\Gamma,t,\varphi$, $\mathbf{Rs}_{\mathbf{K}\mathcal{L}\Gamma t\varphi m}$ is the class of all general relation algebras $\mathfrak{R}_{\mathfrak{M}t\varphi A}$ such that $\langle\mathfrak{M},A\rangle$ is a model of Γ and φ conforms to $t,\mathfrak{M},A$.

This completes the basic definitions. We give two examples:

(1) *Ordinary relation algebras.* Here $m=2$. $\mathbf{K}$ is the class of all sets, $\Gamma=0$, $t=\langle 2,2,1,0,0,2,1,0\rangle$, and $\varphi_0,\ldots,\varphi_7$ are the following formulas of $\mathcal{L}$:

φ_0: $\mathbf{R}_0v_0v_1 \vee \mathbf{R}_1v_0v_1$
φ_1: $\mathbf{R}_0v_0v_1 \wedge \mathbf{R}_1v_0v_1$
φ_2: $\neg\mathbf{R}_0v_0v_1$
φ_3: $\mathbf{F}$
φ_4: $\mathbf{T}$
φ_5: $\exists v_2(\mathbf{R}_0v_0v_2 \wedge \mathbf{R}_1v_2v_1)$
φ_6: $\mathbf{R}_0v_1v_0$
φ_7: $v_0 = v_1$

(2) *Cylindric algebras of dimension* $m \in \omega \sim 1$. Again K is the class of all sets, $\Gamma = 0$, t is the type of CA_m's, where in addition to $\varphi_0 - \varphi_4$ above we have the following formula ψ for each $i < m$:

$$\exists v_i \mathbf{R}_0 v_0 \ldots v_{m-1},$$

and the following formula $\chi_{i,j}$ for all $i,j < m$:

$$v_i = v_j.$$

We know by 3.1.70 and 3.1.108 that for $m < \omega$, ICs_m is a universal class. The following theorem generalizes this.

THEOREM 5.6.18. *Let* $m \in \omega$. *If* K *is closed under ultraproducts, then* $\mathsf{IRs}_{\mathsf{K}\mathcal{L}\Gamma t\varphi m}$ *is a universal class.*

Proof. We omit the subscript on Rs. First, IRs is closed under S. For, let $\mathfrak{B} \in \mathsf{Rs}$. Say φ conforms to $t, \mathfrak{M}, A$, $\langle \mathfrak{M}, A \rangle$ a model of Γ, and $\mathfrak{B} = \mathfrak{R}_{\mathfrak{M}t\varphi A}$. Suppose $\mathfrak{C} \subseteq \mathfrak{B}$. Clearly $\langle \mathfrak{M}, C \rangle$ is a model of Γ, φ conforms to $t, \mathfrak{M}, c$, and $\mathfrak{R}_{\mathfrak{M}t\varphi C} = \mathfrak{C}$, as desired. We finish the proof by showing that IRs is closed under ultraproducts. Suppose that $\mathfrak{A}_s \in \mathsf{Rs}$ for each $s \in S$ and F is an ultrafilter on S. Say $\mathfrak{A}_s = \mathfrak{R}_{\mathfrak{M}_s t\varphi A_s}$ for each $s \in S$, where $\langle \mathfrak{M}_s, A_s \rangle$ is a model of Γ and φ conforms to $t, \mathfrak{M}_s, A_s$. Thus $\mathfrak{R} \overset{\text{def}}{=} \mathsf{P}_{s \in S} \mathfrak{M}_s / F$ is a member of K. Now for each $k \in \mathsf{P}_{s \in S} A_s$ let

$$G_k = \{\bar{F}^* \circ a : a \in {}^m\mathsf{P}_{s \in S} M_s \text{ and } \{s \in S : \langle a_i s : i < m \rangle \in k_s\} \in F\}.$$

Then it is easily seen that if $\bar{F}^* \circ a \in G_k$ then $\{s \in S : \langle a_i s : i < m \rangle \in A_s\} \in F$, and if $k/F = l/F$ then $G_k = G_l$. Let $C = \{G_k : k \in \mathsf{P}_{s \in S} A_s\}$. We claim that $\langle \mathfrak{R}, C \rangle$ is a model of Γ, φ conforms to $t, \mathfrak{R}, C$, and $\mathsf{P}_{s \in S} \mathfrak{A}_s / F \cong \mathfrak{R}_{\mathfrak{R}t\varphi C}$. All of these conditions are straightforward, and we omit the proofs.

Schein also showed that in Theorem 5.6.18, if K is recursively axiomatizable, and φ and Γ are recursive, then $\mathsf{IRs}_{\mathsf{K}\mathcal{L}\Gamma t\varphi m}$ is recursively axiomatizable.

18. Schwartz [77'], [80']: *Polyadic many-valued algebras.* Polyadic algebras over an MV algebra rather than a BA are introduced, and a weak representation theorem is proved.

19. Schwartz [80a']: *Cylindric algebras with filter quantifiers.* Cylindric algebras are extended by additional operations $q_{(\Gamma)}$ (Γ a finite subset of α) with axioms algebraically expressing the quantifier "for almost all", and a representation theorem is proved.

20. Schwartz [81']: *Algebraic analysis of Hermes' term logic with choice operator.* This is a rather unusual kind of algebra; a representation theorem is proved.

21. Venne [66b] *Polyadic higher–order algebras.* A representation theorem is proved.

PROBLEMS

PROBLEM 5.1. *Characterize those* $\mathfrak{A} \in \mathsf{DfCA}_\alpha$ *for which there is a unique* $\mathfrak{B} \in \mathsf{CA}_\alpha$ *such that* $\mathfrak{A} = \mathfrak{Df}\mathfrak{B}$.

See Remark 5.1.5.

PROBLEM 5.2. *Is there an equation not involving diagonal elements holding in every* CA_3 *but not in every* Df_3? *Is* $\mathsf{SDfCA}_3 = \mathsf{Df}_3$?

Cf. 5.1.6 and 5.1.9(ii).

Recall that Dr_α is the class of all diagonal–free relativized algebras of dimension α. For the next problem cf. 5.1.14.

PROBLEM 5.3. *Is* Dr_α *an equational class for* $\alpha \geqq 2$? *Is* $\mathsf{SDr}_\alpha = \mathsf{Dr}_\alpha$ *or* $\mathsf{HDr}_\alpha = \mathsf{Dr}_\alpha$?

PROBLEM 5.4. Is IGs_α *finitely axiomatizable over the class of representable* Df_α*'s for* $\alpha \geqq 3$?

PROBLEM 5.5. *For* $3 \leqq \alpha < \omega$, *can every* Df_α *be embedded in* $\mathfrak{Rb}_{df}\mathfrak{A}$ *for some* PA_α $\mathfrak{A}$?

Cf. Remark 5.4.6.

PROBLEM 5.6. *Extend* 5.4.34 *in some form to* PA_α*'s, for* $3 \leqq \alpha < \omega$.

PROBLEM 5.7. *Is there a non–representable* PEA_3 $\mathfrak{A}$ *with* $\mathfrak{Rb}_{ca}\mathfrak{A}$ *representable?*

PROBLEM 5.8. *Let* $3 \leqq \alpha < \omega$. *Is the class of representable* PEA_α*'s finitely axiomatizable over the class of representable* CA_α*'s?*

For these problems cf. Remarks 5.4.40.

I. BIBLIOGRAPHY OF CYLINDRIC ALGEBRAS AND RELATED ALGEBRAIC STRUCTURES

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II. SUPPLEMENTARY BIBLIOGRAPHY

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INDEX OF NAMES AND SUBJECTS

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