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CARDINAL FUNCTIONS ON BOOLEAN ALGEBRAS

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Résumé

Nous esquissons ici la théorie des applications à valeurs cardinales définies sur les algèbres booléennes. Nous étudions par exemple les applications hA = min {|B| : B est une image homomorphe infinie de A}, la longueur length A = sup {|X| : X est une chaîne de A}, et le nombre des ultrafiltres de A, Ult A . Nous décrivons complètement en détails 24 telles applications, et nous en mentionnons 16 autres. Nous pensons que l'étude de ces applications cardinales fournit un cadre commode pour analyser la structure des algèbres booléennes et que c'est un bon terrain d'expérimentation pour les méthodes récentes de démonstration en théorie des ensembles. Beaucoup de ces applications -au moins 13- ont été étudiées de façon approfondie sous leurs formes topologiques duales ; l'étude de ces applications pour les algèbres booléennes est un cas particulier et important de ces recherches. Ainsi bien des problèmes que nous énonçons ont une forme topologique qui n'est résolue ni pour les espaces compacts les plus généraux, ni pour les espaces réguliers.

La plupart des résultats que nous énonçons ne sont pas nouveaux ; les démonstrations que nous en donnons sont simples, et font souvent partie de ce qui ex--iste de plus classique. En général, nous n'avons pas fait l'effort de découvrir le premier auteur de chaque résultat mais nous l'indiquons dans certains cas. Nous mentionnons 66 problèmes non résolus.*

Nous commencons par considérer quelques applications arithmétiques. la cellularité de A, cell A, est sup {|C| : C ⊂ A, ou C est formé d'éléments deux à deux disjoints}. Cette application a été étudiée de façon détaillée par COMFORT et NEGREPONTIS (1982). Les applications longueur et profondeur respectivement notées length A et depth A = sup {|C| : C ⊂ A, où C est une chaîne bien ordonnée} ont été étudiées Mc KENZIE et MONK (1982). L'application inc A = sup {|M| : M ⊂ A où les éléments de M sont deux à deux incomparables} peut s'exprimer en fonction du nombre d'arbres contenus dans A. La densité est la même chose que le π-poids de l'espace booléen. La ramification est une application bien connue liée à la cellularité.

Les applications algébriques que nous considérons sont les suivantes :
- le nombre de sous-algèbres ;
- l'irréductibilité (irr A = sup {|X| : aucune partie propre de X n'engendre X});
- la profondeur des sous-algèbres ou fonction depth de A = sup {κ : il existe une suite strictement décroissante B_α pour $\alpha < \kappa$ de sous-algèbres de A} ;
- la longueur des sous-algèbres (définie partiellement) ;
- l'indépendance (sup {|B| : B est une sous-algèbre libre de A)};
- hA comme définie ci-dessus ;
- le nombre d'endomorphismes ;
- le nombre d'automorphismes ;

Nous considérons pour terminer quelques applications qui ont, au moins

* La lettre s, lorsqu'elle suit le mot problème, indique que la résolution de celui-ci fait probablement intervenir des résultats d'indépendance de théorie des ensembles.

implicitement, une forme topologique :

le nombre d'idéaux ; l'étalement ("spread") ; le degré héréditaire de Lindelöf ; la densité héréditaire ; la longueur des idéaux ; le nombre d'ultrafiltres ; la densité topologique ; le π-caractère ; le caractère ; l'étroitesse ("tightness").

Abstract

We give a survey of the most important cardinal functions on Boolean algebras. In addition to known results about these functions and relationships between them, we formulate many open problems.

We survey cardinal number valued functions defined on Boolean algebras. Examples of such functions are $hA = \min\{|B|: B$ is an infinite homomorphic image of $A\}$, length $A = \sup\{|X|: X$ is a chain in $A\}$, and $|\mathrm{Ult}A|$ = number of ultrafilters on A. Altogether we describe in some detail 24 such functions and mention in passing some 16 more. We believe that the study of these cardinal functions gives a convenient framework for analyzing the structure of Boolean algebras and is a good proving ground for recent set-theoretical methods. Many of our functions--at least 13--have been extensively studied in their topological dual form; their study for Boolean algebras forms an important special case of these investigations. Thus many of the problems we state have a topological form that is open even for arbitrary compact spaces or regular spaces.

Most of the results we state are not new; the proofs we do give here are simple, and are mostly in the folklore. We have not, in general, tried to trace the first person to prove each result, but we give such credits in some cases.

We mention 66 open problems. The superscript ^{s} on a problem means that its solution probably involves set-theoretical independence results. Since some problems have evidently not been considered before, some of them may be rather easy. And because of the scope of the survey, we may have overlooked some work on these topics.

This paper overlaps two other surveys: van Douwen, Monk, Rubin [80] and Arhangelskii [78].

We are grateful to E.K. van Douwen, P. Nyikos and S. Todorčević for useful comments.

The various cardinal functions we consider can be divided into three sorts: "small" ones, whose values are bounded (for example, h, defined above, since $hA \leq 2^\omega$ for any BA A); "widely varying" ones, whose values can vary from ω to cardinals close to A (for example, length, defined above); and "big" ones, whose values are always close to $|A|$ (for example, $|A| \leq |\mathrm{Ult}A| \leq 2^{|A|}$). This classification is heuristically useful. For example, it seems most interesting to investigate the relationships between those cardinal functions which are widely varying. Also, for them it is of most interest to determine their behaviour under algebraic operations such as subalgebras, homomorphisms, etc. Of course it is also possible to classify the functions by comparing them with each other in general, rather than only with cardinality. Thus cellularity and independence are not widely varying with respect to each other.

Another appropriate question concerning many of these functions is <u>attainment</u>. If kA is defined as the supremum of cardinals κ satisfying some property $P(\kappa,A)$, we can ask for kA a limit cardinal whether $P(kA,A)$ itself holds. For such a function k it is frequently appropriate to consider a related function k^sA, defined as the least κ such that $P(\kappa,A)$ fails to hold. Thus $k^sA = (kA)^+$ if kA is a successor cardinal, or a limit cardinal with kA attained, while $k^sA = kA$ for kA a limit cardinal not attained.

There are several other cardinal functions related to a given cardinal function k :

$(H^+k)A = \sup\{kB\colon B \text{ an infinite homomorphic image of } A\}$

$(H^-k)A = \min\{kB\colon B \text{ an infinite homomorphic image of } A\}$

$(S^+k)A = \sup\{kB\colon B \text{ an infinite subalgebra of } A\}$

$(S^-k)A = \min\{kB\colon B \text{ an infinite subalgebra of } A\}$

We discuss these, when non-trivial, for many of our functions k.

With each function k, we could consider two spectrum functions:

$k\text{-hs}(A) = \{kB\colon B \text{ an infinite homomorphic image of } A\}$
$k\text{-ss}(A) = \{kB\colon B \text{ an infinite subalgebra of } A\}$

To shorten this survey we consider only one of the many possibilities, $|\,| - \text{hs}(A) = \{|B|\colon A \twoheadrightarrow B,\ B \text{ infinite}\}$, in section 12.

Another related notion will not be discussed here, but we mention it as a good topic for investiation. Given a property P of subsets of a BA A, we say that A has P-κ-λ-caliber if among any κ elements of A there is a subset of power λ satisfying P. This notion has been studied for pairwise intersecting sets (see Comfort, Negrepontis [82]) and for independence (see Monk [83]).

At the end of the paper we give a rather messy diagram showing the known relationships between our main cardinal functions.

CONTENTS

Notation

Our set-theoretical notation is in general standard. $f: A \twoheadrightarrow B$ means that f maps onto B ; $f : A \rightarrowtail B$ means that f is one-one. $f[X]$ is the f-image of X , $f[X] = \{fx : x \in X\}$. For any infinite cardinal κ , $\mathrm{ded}\kappa = \sup\{\lambda :$ there is a linearly ordered set of size λ with a dense subset of size $\kappa\}$. $[a]$ is the equivalence class of a under some (implicitly understood) equivalence relation. We let $\beth_0\kappa = \kappa$, $\beth_{n+1}\kappa = 2^{\beth_n\kappa}$. $\mathcal{P}X$ is the set of all subsets of X . $[X]^\kappa$ is the set of all subsets of X of size κ ; $[X]^{<\kappa}$, those of size $<\kappa$. For MA_κ and MA , see Kunen [80]. $\lambda^{<\kappa} = \sum_{\mu<\kappa}\lambda^\mu$. <u>Tree</u> has the usual meaning. A <u>pseudo-tree</u> is a partially ordered set P such that for all $x \in P$, $\{y : y < x\}$ is linearly ordered. SH_κ is the hypothesis that there is no κ-Souslin tree.

For BA's, $\rightarrow$, $\twoheadrightarrow$, $\rightarrowtail$ usually mean homomorphisms. For $X \subseteq A$ we let $X^+ = X \setminus \{0\}$. $\mathrm{At}A$ is the collection of atoms of A . $A \restriction a = \{x \in A : x \leq a\}$. $\mathrm{Sg}X$ is the subalgebra generated by X ; $\mathrm{Ig}X$ the ideal generated by X , and $\mathrm{Fg}X$ the filter generated by X . s^A is the Stone isomorphism: $s^A a = \{F \in \mathrm{Ult}A: a \in F\}$, where $\mathrm{Ult}A$ is the collection of ultrafilters on A . For any BA A , $\overline{A}$ or A^- is its completion. For any set X , $\mathrm{finco}X$ is the BA of finite and cofinite subsets of X . fip stands for <u>finite intersection property</u>. wcc means <u>weakly countably complete</u>: A satisfies wcc iff $\forall X,Y \in [A]^{<\omega}$ such that $\forall x \in X\ \forall y \in Y\ (x \cdot y = 0)$, there is an $a \in A$ such that $\forall x \in X\ \forall y \in Y\ (x \leq a$ and $a \cdot y = 0)$. $\mathrm{Fr}\kappa$ is the free BA on κ free generators x_α, $\alpha < \kappa$. If L is a linear order with first element, $\mathrm{intalg}L$ is the BA of subsets of L generated by all sets $[a,b)$ and $[a,-) = \{x : a \leq x\}$. If T is a tree, $\mathrm{treealg}T$ is the BA of subsets of T generated by all sets $T \restriction t$, where $T \restriction t = \{s \in T : t \leq s\}$. For T a pseudo-tree, $\mathrm{ptree}T$ is defined similarly. Note that homomorphisms on $\mathrm{intalg}L$ are in one-one correspondence with convex equivalence relations on $L \cup \{\infty\}$. If A is hereditarily atomic, its <u>cardinal sequence</u> is the sequence of cardinalities of the set of atoms in its natural sequence of quotients A/I_α $(I_0 = \{0\}$, $I_{\alpha+1} = \{a : [a]$ is a finite sum of atoms of $A/I_\alpha\}$, $I_\lambda = \bigcup_{\alpha<\lambda} I_\alpha$ for λ limit). $\Pi_{i\in I}A_i$ is the direct product of the A_i's; $\Pi^w_{i\in I}A_i$ consists of those $f \in \Pi_{i\in I}A_i$ which are 0 except at finitely many places, and complements of such f , $*_{i\in I}A_i$ their free product; $A \times B$ and $A * B$ are used for only two factors. $B *_A C$ is the free product of B and C with amalgamated subalgebra A . If k is any cardinal function, A is k-homogeneous if $kA = k(A \restriction a)$ for all $0 \neq a \in A$.

A. Arithmetical Functions

We discuss functions associated with the arithmetic of Boolean algebras: disjointness, chains, incomparability, etc.

1. Cellularity

We define the cellularity of A to be

$$\mathrm{cell}A = \sup\{|C| : C \subseteq A,\ C \text{ a family of pairwise disjoint elements of } A\}$$

This notion is extensively studied, along with related notions not treated here, in Comfort, Negrepontis [82]. Note that this is a widely varying function. By Erdös, Tarski [43], cellA is always attained for singular cardinals, but there are counterexamples for weakly inaccessible cardinals. Thus $\mathrm{cell}^s A = (\mathrm{cell}A)^+$ unless cellA is weakly inaccessible and not attained, in which case $\mathrm{cell}^s A = \mathrm{cell}A$. A satisfies the κ-cc (κ-chain condition) if $|X| < \kappa$ for every family X of pairwise disjoint elements of A.

If $A \subseteq B$, clearly $\mathrm{cell}A \leq \mathrm{cell}B$. The cellularity can differ arbitrarily between A and B. If $A \twoheadrightarrow B$, there is no relationship, in general, between cellA and cellB.

Clearly $\mathrm{cell}\Pi_{i\in I}A_i = |I| \cup \sup_{i\in I}\mathrm{cell}A_i$; similarly for weak products. The cellularity of an ultraproduct depends on saturation properties. Thus if F is any countably incomplete ultrafilter on a set I, then $\mathrm{cell}(\Pi_{i\in I}A_i/F) \geq \omega_1$ if all A_i are infinite. Assuming MA we have $\mathrm{cell}(\Pi_{i\in I}A_i/F) \geq 2^\omega$ for any such F; see, e.g. McKenzie, Monk [82] 1.5.5. For any infinite set I there is a $|I|^+$-good countably incomplete ultrafilter F on I, and hence $\mathrm{cell}(\Pi_{i\in I}A_i/F) \geq |I|^+$ if all A_i are infinite. On the other hand, for any infinite set I there is an ultrafilter F on I such that $\mathrm{cell}(\Pi_{i\in I}A_i/F) \geq 2^{|I|}$ for any system $\langle A_i : i\in I\rangle$ of infinite BA's; see McKenzie, Monk [82] 1.5.4. We mention here the following vague question.

<u>Problem 1</u>. If F is a $|I|^+$-good countably incomplete ultrafilter on I, under what conditions does $\Pi_{i\in I}A_i/F$ have a pairwise disjoint set of power $2^{|I|}$?

The behavior of chain conditions under free products has been extensively studied. We survey these results. First we take the free product of two BA's; the case of a free product of infinitely many BA's reduces to the finite case by a result mentioned below, and the finite case, of course, reduces by induction to the case of two BA's. The partition relations $(2^\kappa)^+ \to (\kappa^+)^2_\kappa$, $(2^\kappa)^+ \to ((2^\kappa)^+, \kappa^+)^2$, $\kappa^+ \to (\kappa^+, \mathrm{cf}\kappa)^2$ for κ strong limit, give the following:

(1) $\mathrm{cell}(A*B) \leq 2^{\mathrm{cell}A\cdot\mathrm{cell}B}$;

(2) $\mathrm{cell}A \leq 2^\kappa$ and $\mathrm{cell}B \leq \kappa \Rightarrow \mathrm{cell}(A*B) \leq 2^\kappa$;

(3) if κ is strong limit, $\mathrm{cell}A \leq \kappa$, and $\mathrm{cell}B < \mathrm{cf}\kappa$, then $\mathrm{cell}(A*B) \leq \kappa$.

Results (1) and (2) can be found in Kurepa [62]. Thus under GCH we have $\mathrm{cell}A\cdot\mathrm{cell}B \leq \mathrm{cell}(A*B) \leq (\mathrm{cell}A\cdot\mathrm{cell}B)^+$; $\mathrm{cell}(A*B) = \mathrm{cell}A$ if $\mathrm{cell}A > \mathrm{cell}B$ and cellA is a successor cardinal; $\mathrm{cell}(A*B) = \mathrm{cell}A$ if cellA is a limit cardinal and $cf(cellA) > cellB$. Thus even under GCH two cases are not covered by (1)-(3): cellA limit with $cf(cellA) \leq cellB < cellA$, and $cellA = cellB$. The first case appears to be open:

Problem 2. If κ is strong limit singular, $cellA = \kappa$, and $cf\kappa \leq cellB < \kappa$, is $cell(A*B) = \kappa$?

The second case has been extensively studied. The first result we mention is due to Kurepa [50]:

(4) Suppose T is a normal κ^+-Souslin tree, and let $A = treealgT$. Then $cellA = \kappa$ and $cell(A*B) = \kappa^+$.

Galvin and Laver (see Galvin [80]) proved

(5) if $2^\kappa = \kappa^+$, then there are BA's A and B with $cellA = cellB = \kappa$ and $cell(A*B) = \kappa^+$.

Todorčević [∞] proved

(6) There are arbitrarily large cardinals κ such that there exist BA's A,B which satisfy κ-cc while A*B does not; this holds in particular if $\kappa \leq 2^\omega$ and $cf\kappa = cf2^\omega$.

Fleissner [78] showed

(7) If κ Cohen reals are added to M, then in M[G] there are ccc BA's A and B with $cell(A*B) \geq \kappa$. Consequently, if M satisfies CH and κ,λ,μ are cardinals of M with κ regular and $\omega \leq \lambda,\mu \leq \kappa$, then in M[G] there are BA's A and B with $cellA = \lambda$, $cellB = \mu$, and $cell(A*B) = \kappa = 2^\omega$.

A folklore result, proved independently by many people, is

(8) $(MA + 2^\omega > \omega_1)$ If A and B satisfy ccc, so does A*B.

Finally we note that $cell(*_{i\in I}A_i) = \sup\{cell(*_{i\in F}A_i) : F \in [I]^{<\omega}\}$; see, e.g. Juhász [80], p. 107. The derived function H^+cell is the cardinal function <u>spread</u>, discussed below.

2. <u>Length</u>

We let $lengthA = \sup\{|C| : C \subseteq A, C \text{ a chain}\}$. This notion is extensively studied in McKenzie, Monk [82]. It is a widely varying function. A discussion of maximal chains in BA's can be found in Jakubik [58], Day [70] and S. Koppelberg [∞]. If A has length κ and $cf\kappa = \omega$, then lengthA is attained; if κ is limit with $cf\kappa > \omega$, then there are counterexamples. It is not completely clear what happens to length under direct product. We have

$$\max(ded|I|, \sup_{i\in I} lengthA_i) \leq length\Pi_{i\in A}A_i \leq \Pi_{i\in I} lengthA_i,$$

where the second inequality can be equality, and, consistently, so can the first.

Problem 3. Does $length\Pi_{i\in I}A_i$ depend only on $|I|$ and $\langle lengthA_i : i \in I\rangle$?

The length of ultraproducts has not been investigated.

Problem 4. Discuss the length of ultraproducts.

There is also a problem concerning the length of free products, although much is known. The hardest theorem in McKenzie, Monk [82] is that if $cf\kappa > \omega$, A has no chain of power $cf\kappa$, and B has no chain of power κ, then $A*B$ has no chain of power κ. Hence $\text{length}*_{i\in I}A_i = \sup_{i\in I}\text{length}A_i$. However, the above theorem is not best possible. In fact, the following additional results are known:

(1) Suppose $\lambda < cf\kappa \leq \mu < \kappa$, $\mu \leq \text{ded}\lambda$, $\text{ded}\lambda$ attained if $\mu = \text{ded}\lambda$. Then there exist A,B with $\mu = \text{length}A$ attained, $\text{length}B = \kappa$ not attained, and $\text{length}(A*B)$ not attained.

(2) Suppose $\forall\lambda < cf\kappa\ (2^\lambda \leq cf\kappa)$, $cf\kappa < \text{length}A \leq \kappa$, and $\text{length}B = \kappa$. Then $A*B$ has a chain of size κ.

(3) If $cf\kappa < \kappa$, $cf\kappa$ is weakly compact, $cf\kappa = \text{length}A$ attained, and $\text{length}B = \kappa$, then $A*B$ has a chain of size κ.

(4) If $cf\kappa < \kappa$, A has a chain of size $cf\kappa$ having $cf\kappa$ pairwise disjoint intervals each with at least two elements, and $\text{length}B = \kappa$, then $A*B$ has a chain of size κ.

A very simple form of the questions left open by the abvove results is as follows.

Problem 5. Let A be the interval algebra of a Souslin line, B a BA with $\aleph_{\omega_1} = \text{length}B$ not attained. Is $\text{length}(A*B)$ attained?

Note that $\text{length}A \leq 2^{\text{cell}A}$ by the Erdös, Rado theorem. It would be natural to conjecture that $\forall\kappa\forall\lambda[\kappa < \lambda < 2^\kappa \Rightarrow \exists \text{BA } A\ (\text{cell}A = \kappa \text{ and } \text{length}A = \lambda)]$; this would follow from its special case $\forall\kappa\ \exists\text{BA } A\ (\text{cell}A = \kappa \text{ and } \text{length}A = 2^\kappa)$. This is true under GCH. But as Mitchell [72] showed (see also Juhász [71]) if $M \models \text{GCH}$ and G adds $\aleph_{\omega_1}$ Cohen reals, then in $M[G]$ every BA of length $\geq 2^{\omega_1}$ attained has cellularity $\geq 2^\omega$, and $2^\omega = \aleph_{\omega_1}$, $2^{\omega_1} = \aleph_{\omega_1+1}$. Thus the above fails for $\kappa = \omega_1$.

The function $H^+\text{length}$ does not coincide with any of our cardinal functions.

Problem 6. Is it consistent to have a BA A such that $\omega < \text{length}A < |A|$ and A has no infinite homomorphic image of power $< |A|$?

Some results of W. Just are relevant to this problem.

Problems 3-5 are essentially stated in McKenzie, Monk [82].

3. Depth

By definition, $\text{depth}A = \sup\{|C|: C \text{ is a well-ordered chain in } A\}$. Again, this is a notion extensively studied in McKenzie, Monk [82], and is a widely varying function.

If A has depth κ and $cf\kappa = \omega$, then $\text{depth}A$ is attained; if κ is limit with $cf\kappa > \omega$, then there are counterexamples. $\text{depth}\prod_{i\in I}A_i = |I| \cup \sup_{i\in I}\text{depth}A_i$. Concerning the depth of ultraproducts, there are several interesting facts, and one problem. For any κ there is a non-principal ultrafilter F on κ such that $\Pi_{\alpha<\kappa}A_\alpha/F$ has a chain of type 2^κ for every system $\langle A_\alpha : \alpha < \kappa\rangle$ of infinite BA's. On the other hand, Laver has shown that in a model of Woodin there is a uniform ultrafilter F on ω_1 such that $|{}^{\omega_1}\omega/F| = \omega_1$; hence, of course, $\text{depth}({}^{\omega_1}A/F) = \omega_1$ for every denumerable BA . By the Erdös, Rado theorem, if $\text{depth}A \leq \lambda$, $\kappa \leq \lambda$, and F is any ultrafilter on κ , then $\text{depth}({}^\kappa A/F) \leq 2^\lambda$.

Problem 7. If A is a BA with no chains of type λ and F is an ultrafilter on κ , does it follow that ${}^\kappa A/F$ has no chain of type $\max((2^\kappa)^+,\lambda)$?

The answer to problem 7 is (consistently) no if we replace "ultrafilter" by "filter"; see McKenzie, Monk [82].

If $\text{cf}\kappa > \omega$, A has no chain of type $\text{cf}\kappa$, and B has no chain of type κ , then $A*B$ has no chain of type κ ; but if A has a chain of type $\text{cf}\kappa$ and $\text{depth}B = \kappa$, then $A*B$ has a chain of type κ .

If κ is infinite and regular, $|A| = \kappa$, and there is a homomorphism from A onto a subalgebra of κ containing all subsets of power $< \kappa$, then there exist BA's $B,C \supseteq A$ satisfying κ^+-cc such that $\text{depth}(B*_AC) \geq \kappa^+$. For any infinite BA A in a model M of GCH , there is an extension M' of M with algebras $B,C \supseteq A$ such that $\text{depth}(B*_AC) > \max(\text{depth}B,\text{depth}C)$ (this is a result of Shelah found in McKenzie, Monk [82]).

Problem 8. In ZFC is it true that for every infinite BA A there exist $B,C \supseteq A$ with $\text{depth}(B*_AC) > \max(\text{depth}B,\text{depth}C)$?

Problem 9. For every infinite BA A , is there a cardinal κ such that if $B,C \supseteq A$ and $|B|,|C| \geq \kappa$, then $\text{depth}(B*_AC) = \max(\text{depth}B,\text{depth}C)$?

We clearly have $\text{depth}A \leq \text{length}A \leq 2^{\text{depth}A}$. $(H^+\text{depth})A$ coincides with the tightness of A ; see below. Problems 7-9 are in McKenzie, Monk [82] .

4. Incomparability

$\text{inc}A = \sup\{|M| : M$ is a set of pairwise incomparable elements of $A\}$. As will be detailed below, this is a large function. We let pie abbreviate set of pairwise incomparable elements. We first indicate an equivalent way of defining inc .

Theorem 4.1. For any infinite BA A we have $\text{inc}A = \sup\{|T| : T$ is a tree, $T \subseteq A\}$.

Note that for a tree $T \subseteq A$ we do not insist that $s + t = 1$ for s and t incomparable in T ; see section 6 .

Proof. Since any pie is a trivial tree with only roots, the $\leq$ part of the equality is clear. To show = , assume that A has no pie of size κ , κ regular; we show that A has no tree of size κ . Suppose that T is a tree in A of power κ . By Baumgartner, Komjath [81], A has a dense subset D of power $< \kappa$. Now each level of T is a pie , so has power $< \kappa$. Hence T has at least κ levels. Let T' be a subset of T of power κ consisting exclusively of elements of successor level. For each $d \in D$, let

$$M_d = \{t \in T' : \text{if } s \text{ is the immediate predecessor of } t \text{, then } d \leq t\cdot -s\} .$$

Thus $T' = \bigcup_{d\in D} M_d$, so there is a $d \in D$ with $|M_d| = \kappa$. But M_d is clearly a pie , contradiction.

If $A \subseteq B$ or $B \twoheadrightarrow A$, clearly $\mathrm{inc}A \leq \mathrm{inc}B$. If $A \subseteq B$, then $\mathrm{inc}(A\times B) \geq |A|$. In fact, $\{(a,-a) : a \in A\}$ is a pie in $A\times B$. Hence if A is cardinality-homogeneous and has no pie of size $|A|$, then A is rigid. We have $\mathrm{inc}(A*B) = \max(|A|,|B|)$ if $|A|,|B| \geq 4$, since $A*C \cong A\times A$ if $|C| = 4$.

Clearly $\mathrm{cell}A \leq \mathrm{inc}A$ and $|A| \leq 2^{\mathrm{inc}A}$ using the Erdös, Rado theorem. Some deep results and problems are found in connection with trying to construct a BA A with no pie of size $|A|$; we call such a BA narrow. Bonnet and Shelah (Bonnet [∞]) have shown in ZFC that there is a narrow BA of power $\mathrm{cf}(2^{\omega})$. Bonnet [∞] has shown assuming GCH that there is a narrow BA of power κ^{+} . By a theorem of Arhangelskiĭ [71], if $|A|$ is singular strong limit, then A is not narrow. Baumgartner, Komjath and Shelah in Shelah [83] have shown that if A has no pie of size λ, then it has a dense suset of size $< \lambda$. So if $|A|$ is strong limit, then A is not narrow. Even stronger results are known in which A also does not have big chains. Call A concentrated if A has no pie and no chain of size $|A|$. Shelah has shown under GCH that for each $\lambda \geq \omega$ there is a concentrated BA of size λ^{+} , and for each $\lambda \geq \omega$ with $\lambda \neq \omega_1$ there is a λ-complete concentrated BA of size λ^{+} .

Problem 10. (CH) Is there a concentrated σ-BA of size ω_2 ?

Rubin [83] has shown that if B is a subalgebra of an interval algebra and $|B|$ is regular, then B is not concentrated.

Problem 11. If B is a subalgebra of an interval algebra and $|B|$ is singular, can B be concentrated?

Note by the above remarks that the answer to Problem 11 is no for $|B|$ strong limit singular.

Shelah [80] and independently van Wesep have shown that it is consistent to have 2^{ω} arbitrarily large and to have a BA A of size 2^{ω} with countable length and incomparability. On the other hand, Baumgartner [80] has shown the following consistent: MA + $2^{\omega} = \omega_2$ + "every uncountable BA has an uncountable pie." Shelah has generalized this, showing that it is consistent to have the continuum arbitrarily large.

Problem 12^{S}. Is it consistent that every BA of power ω_2 has a pie of size ω_2 ?

Shelah [83] has shown that for any singular λ with $\mathrm{cf}\lambda > \omega$ it is consistent that there is a BA A with $\mathrm{inc}A = \lambda$ not attained. Milner and Pouzet (unpublished) have shown that if $\mathrm{inc}A = \lambda$, with $\omega = \mathrm{cf}\lambda$, then $\mathrm{inc}A$ is attained. Todorčević has shown that if 2^{ω} is weakly inaccessible, then there is a BA of size 2^{ω} with incomparability 2^{ω} not attained.

Problem 13. Is it true that for every weakly but not strongly inaccessible cardinal κ there is a BA with incomparability κ not attained?

Todorčević has shown that if $\mathrm{cf}\lambda < \lambda$ and A is a BA with a tree of size λ , then A has a pie of size λ .

We have $(H^{+}\mathrm{inc})A = \mathrm{inc}A$ for all A . Note that $H^{-}\mathrm{inc}$ is in general non-trivial; for example, $(H^{-}\mathrm{inc})A = 2^{\omega}$ if A is wcc. This gives rise to the following question.

Problem 14. Under any set-theoretical assumptions, is there a BA A with $|A| > \omega$, $\mathrm{inc}A = \omega$, and $|B| > \omega$ for every homomorphic image B of A?

Problems 10, 11, 13 are mentioned in van Douwen, Monk, Rubin [80]. A function related to incomparability has been considered by P. Nyikos:

$$\text{h-cof}(A) = \min\{\kappa : \text{every subset of } A \text{ has a cofinal subset of power} \leq \kappa\}.$$

One can show that $\mathrm{inc}A \leq \text{h-cof}(A)$ and

$$\text{h-cof}(A) = \sup\{|T| : T \subseteq A,\ T \text{ well-founded}\}.$$

The algebra constructed by Shelah [81] assuming CH is of power ω_1 but has h-cofinality ω. An algebra A constructed under $\diamondsuit_{\omega_1}$ by Baumgartner, Komjath [81] has $|A| = \omega_1$, $\mathrm{inc}A = \omega$, while $\{a \in A : |A \restriction a| \leq \omega\} = I$ is a maximal ideal. Clearly I is not countably generated, so $\text{h-cof}(A) = \omega_1$.

Problem 15. Can one construct in ZFC a BA A with $\mathrm{inc}A < \text{h-cof}(A)$?

5. Algebraic Density

$\pi A = \min\{|D| : D \text{ dense in } A\}$. Again this is a large function. This is the same as the π-weight of $\mathrm{ult}A$, but we shall usually call it algebraic density. It has been extensively studied in topology. First we discuss its behavior under algebraic operations, then its relationships to the previous functions introduced.

Suppose $A \twoheadrightarrow B$. Then there is no generally valid relationship between πA and πB. To get $\pi A < \pi B$, let $A = \mathscr{P}\kappa$. Thus $\pi A = \kappa$. Let C be a free subalgebra of A of power 2^{κ}. Extend $\mathrm{Id} : C \to C$ to a homomorphism $f : A \twoheadrightarrow B \subseteq C$. Clearly $\pi B = 2^{\kappa}$. To get $\pi B < \pi A$, take $\mathrm{Fr}(2^{\kappa}) \twoheadrightarrow \mathscr{P}\kappa$. One can still ask, however, for what pairs κ, λ it is true that every BA A of power λ has a homomorphic image B with $\pi B = \kappa$. Assuming GCH we can answer this question fully (Corollary 5.5).

Theorem 5.1. Suppose L is a dense linear order, D is dense in L, and $f : \mathrm{intalg}L \twoheadrightarrow B$. Then $\pi B \leq |D|$.

We omit the easy proof.

Corollary 5.2. If $\kappa \leq \lambda < \mathrm{ded}^{s}\kappa$, then there is a BA A of power λ such that for every B, $A \twoheadrightarrow B$ implies $\pi B \leq \kappa$.

Theorem 5.3. If $|A| \geq (2^{\kappa})^{+}$ and $\lambda \leq \kappa^{+}$, then A has a homomorphic image B with $\pi B = \lambda$.

Proof. We use here some results on s = spread; see below. By a theorem of Arhangelskiĭ and Sapirovskiĭ (see Juhász [80], p. 56) we have $s(\mathrm{Ult}A) \geq \kappa^{+}$. Now it is easy to see that

$$s(\mathrm{Ult}A) = \sup\{\mathrm{cell}C : A \twoheadrightarrow C\}.$$

Hence choose C so that $A \twoheadrightarrow C$ and $\mathrm{cell}C = \kappa^{+}$. Let $X \in [C]^{\lambda}$ with X pairwise disjoint, and let $D = \mathrm{Sg}^{(C)}X$. Then extend $\mathrm{id} : D \to D$ to $g : C \twoheadrightarrow B \subseteq \overline{D}$. Thus B is as desired.

<u>Theorem 5.4</u>. If κ is singular strong limit and $|A| \geq \kappa$, then for any $\lambda \leq \kappa$ A has a homomorphic image B with $\pi B = \lambda$.

<u>Proof</u>. As for 5.3, using also Juhász [80], 4.2.

<u>Corollary 5.5</u>. (GCH) Let $\omega \leq \lambda \leq \kappa$. If $\lambda < \kappa$, then every BA A of power κ has a homomorphic image $\overline{B}$ with $\pi B = \lambda$. If $\lambda = \kappa$ is a limit cardinal, then A itself has π-weight λ. If $\kappa = \mu^+$, then there is a BA of power κ with no homomorphic image of π-weight κ.

<u>Problem 16</u>. Describe completely the behavior of π-weight under homomorphisms without GCH. In particular, is it consistent that there is a BA A of power $\mathrm{ded}^s\omega_1$ such that for every B, $A \quad B$ implies $\pi B \leq \omega_1$?

Next, suppose $A \subseteq B$. Again, there is no generally valid relationship between πA and πB, as easy examples show. The problem of specification of πA arises as in the case of homomorphisms.

<u>Theorem 5.6</u>. If A has a subalgebra B such that $\pi B = \lambda$, then A has a homomorphic image C such that $\pi C = \lambda$.

<u>Proof</u>. Let $f : A \twoheadrightarrow C \subseteq \overline{B}$ extending the identity on B. Clearly $\pi C \leq \lambda$. Suppose D is dense in C, $|D| < \lambda$. For all $0 \neq d \in D$ choose $0 \neq x_d \in \overline{B}$ with $x_d \leq d$. Then $\{x_d : d \in D\}$ is dense in B, contradiction.

As noted by van Douwen, the converse of 5.6 does not hold in general. In fact, let $\mathcal{G} \subseteq [\omega]^\omega$ consist of almost disjoint sets, $|\mathcal{G}| = \omega_1$, and let $A = \mathrm{Sg}([\omega]^{<\omega})$. It is easily checked that A has a homomorphic image C with $\pi C = \omega_1$ But no subalgebra B with $\pi B = \lambda$.

<u>Corollary 5.7</u>. If $\kappa \leq \lambda \leq \mathrm{ded}^s\kappa$, then there is a BA A of power λ such that for every B, $B \subseteq A$ implies $\pi B \leq \kappa$.

<u>Theorem 5.8</u>. If $|A| \geq (2^\kappa)^+$ and $\lambda \leq \kappa^+$, then A has a subalgebra B with $\pi B = \lambda$.

<u>Proof</u>. If A has a family of pairwise disjoint elements of power κ^+, let $B = \mathrm{Sg}C$, C a family of pairwise disjoint elements of power λ. Otherwise, A has $(2^\kappa)^+$ independent elements, and we can let B be generated by λ of them.

<u>Corollary 5.9</u>. If κ is singular strong limit and $|A| \leq \kappa$, then for any $\lambda \leq \kappa$ A has a subalgebra B with $\pi B = \lambda$.

<u>Corollary 5.10</u>. (GCH) Let $\omega \leq \lambda \leq \kappa$. If $\lambda < \kappa$, then every BA A of power κ has a subalgebra B with $\pi B = \lambda$. If $\lambda = \kappa$ is a limit cardinal, then A itself has π-weight λ. If $\kappa = \mu^+$, then there is a BA of power κ with no subalgebra of π-weight κ.

<u>Problem 17</u>. Describe completely the behavior of π-weight under subalgebras without GCH. In particular, is it consistent that there is a BA A of power $\mathrm{ded}^s\omega_1$ such that for every $B \subseteq A$, $\pi B \leq \omega_1$?

Clearly $\pi(\Pi_{i \in I} A_i) = \bigcup_{i \in I} \pi A_i$, and similarly for weak products. It is also clear that for any ultrafilter F on I, $\pi(\prod_{i \in I} A_i/F) \leq \prod_{i \in I} \pi A_i/F$, but we do not know if this is best possible.

Problem 18. For F an ultrafilter on I, is $\pi(\prod_{i \in I} A_i/F) = \prod_{i \in I} \pi A_i/F$?

It is well-known that $\pi(*_{i \in I} A_i) = |I| \cup \bigcup_{i \in I} \pi A_i$; see Juhász [80], 5.3 a).

Clearly $\mathrm{cell}A \leq \pi A$. Baumgartner and Komjath [81] showed that $\pi A \leq \mathrm{inc}A$, while Shelah [83] gave the stronger result that for any infinite cardinal κ, if $\pi A \geq \kappa$, then A has an irredundant pie of power κ (see section 8). Another easy relationship between our functions is given in

Theorem 5.11. $|A| \leq \pi A^{\mathrm{cell}A}$.

Proof. Let D be dense in A, $|D| = \pi$. Then $\forall a \in A\ \exists X \subseteq D\ (|X| \leq \mathrm{cell}A$ and $a = \sum X)$. This proves the theorem.

Finally we note that $\pi A \leq (H^+\pi)A \leq |A|$, with $<$ possible in both cases. Clearly $(H^-\pi)A = \omega$ for any infinite BA A. Similar observations hold for $S^+\pi$ and $S^-\pi$.

6. Ramification

A ramification system in A is a subset R of A such that

$$\forall x,y \in R\ (x \leq y \text{ or } y \leq x \text{ or } x \cdot y = 0),\quad 0 \notin R,$$
$$\forall x \in R\ (\{y \in R : x \leq y\} \text{ is inversely well-ordered});$$

see, e.g., Horn, Tarski [48]. We set

$$\mathrm{ram}A = \sup\{|R| : R \text{ a ramification system in } A\}.$$

This is a widely varying function. One can prove that for any tree T the following two conditions are equivalent:

(i) T is inversely isomorphic to a ramification system in A ;

(ii) treealg$T$ is embeddable in A.

Thus we have the equivalent definition:

$$\mathrm{ram}A = \sup\{|B| : B \text{ is a tree algebra and } B \text{ can be embedded in } A\}.$$

ram$A$ is closely related to cell$A$: $\mathrm{cell}A \leq \mathrm{ram}A \leq \mathrm{cell}^s A$. Furthermore, if $\mathrm{ram}A = \mathrm{cell}^s A = (\mathrm{cell}A)^+$, then A has a ramification system R whose inverse is a $\mathrm{cell}^s A$-Souslin tree. Conversely, if T is a κ^+-Souslin tree, then $\mathrm{ram}\,\mathrm{treealg}T = |T| = \kappa^+$ and $\mathrm{cell}^s\mathrm{treealg}T = (\mathrm{cell}\,\mathrm{treealg}T)^+ = \kappa^+$. Thus the following problem has a purely set-theoretical character: (it is equivalent to $\forall$ non-limit $\kappa(SH_\kappa)$):

Problem 19. Is it consistent that $\forall A(\mathrm{ram}A = \mathrm{cell}A)$?

Another relation between our cardinal functions is that $\mathrm{ram}A \leq \pi A$. In fact, otherwise $\mathrm{cell}A \leq \pi A < \mathrm{ram}A$. Say D is dense in A, $0 \notin D$, $|D| = \pi A$. For all $d \in D$, let $M_d = \{y \in R : d \leq y\}$, where R is a ramification system in A. Then $R = \bigcup_{d \in D} M_d$, so there is a $d \in D$ with $|M_d| > \mathrm{cell}A$. But M_d is an inversely well-ordered chain, contradiction.

We also note that ramA and lengthA are, in general, not comparable. In fact, in intalg $\mathbb{R}$ there is no ramification system of power ω_1. For, suppose T is such a system. Let T' consist of all nodes of T of successor level. For each $r \in Q$ let

$$L_r = \{t \in T' : \text{if } s \text{ is the predecessor of } t, \text{ then } r \in s \setminus t\}$$

Thus $T' = \bigcup_{r \in Q} L_r$, so there is an $r \in Q$ with L_r uncountable. But the members of L_r are pairwise disjoint, contradiction. Conversely, if T is an Aronszajn tree, then length(intalgT) = ω by a theorem of Brenner, Monk [83].

It would be natural to define a new cardinal function using the notion of a pseudo-tree. This does not lead to an essentially new notion, however. In fact, Kurepa [77] showed that if T is a pseudo-tree of regular size κ with no chains of size κ, then T contains a tree of size κ. Thus

$$\sup\{|T| : T \subseteq A, \ T \text{ a pseudo-tree}\} = \text{lengthA} \cdot \text{incA};$$
$$\sup\{|B| : B \text{ is a pseudo-tree algebra}, \ B \subseteq A\} = \text{lengthA} \cdot \text{ramA}.$$

The above result of Kurepa does not extend to singular κ, as he essentially observed in the same paper. On the other hand, Todorčević observed that by adding $\aleph_{\omega_1}$ Cohen reals to a model of GCH one can get a BA B such that $|B| = \aleph_{\omega_1}$, B has a pseudo-tree of size $\aleph_{\omega_1}$, but no tree or pie of size $\aleph_2$ and no chain of size $\aleph_{\omega_1}$.

Finally, note that spreadA $\leq$ $(H^+\text{ram})$A ; the possibility of equality is related to Problem 19. $(H^-\text{ram})$A is always ω, although it is not completely trivial to see this. Let A be an arbitrary infinite BA. Then $A \twoheadrightarrow B$ for some BA A with $\text{finco}\,\omega \subseteq B \subseteq \omega$. Suppose R is an uncountable ramification system in B. Now $R = \bigcup_{n \in \omega}\{x \in R : n \in x\}$, so there is an $n \in \omega$ for which $C = \{x \in R : n \in x\}$ is uncountable. But then C is a well-ordered chain, contradiction.

B. Algebraic functions

We now survey cardinal functions having to do with algebraic aspects of BA's: subalgebras, automorphisms, and homomorphisms.

7. Subalgebras

We let SubA be the set of all subalgebras of A. Clearly $|A| \leq |\text{SubA}| \leq 2^{|A|}$. The following topological equivalent of the subalgebra relation is well-known.

Theorem 7.1. Let A be a BA. If B is a subalgebra of A, set

$$\approx_B = \{(G,F) : F,G \in \text{UltA} \text{ and } F \cap B = G \cap B\}.$$

Then $\approx_B$ is an equivalence relation on UltA, and if $F \not\approx_B G$, then there is a closed-open subset U such that $F \in U$ and $G \notin U$, and U is a union of $\approx_B$-classes.

Conversely, if $\equiv$ is an equivalence relation on UltA such that $\forall F,G \in$ UltA $(F \not\equiv G \Rightarrow \exists$ closed open U with $F \in U$, $G \notin U$ and U a union of $\equiv$-classes), let $C_\equiv = \{a \in A : sa \text{ is a union of } \equiv\text{-classes}\}$. Then $C_\equiv$ is a sub-subalgebra of A, and $\approx_{C_\equiv} = \equiv$. Furthermore, if B is any subalgebra of A, then $C_{\underset{B}{\approx}} = B$.

Shelah [79], generalizing Rubin [83], showed assuming $V = L$ that for every regular $\kappa \geq \omega$ there is a BA A with $|A| = |\mathrm{Sub}A| = \kappa^+$. In section 8 we note that if κ is a strong limit cardinal and $|A| = \kappa$, then $|\mathrm{Sub}A| = 2^\kappa$. These two facts are essentially all that is known about $|\mathrm{Sub}A|$. In particular, the following questions are open.

<u>Problem 20</u>. Is $|\mathrm{Sub}A|$ always a power of 2 ?

<u>Problem 21</u>. Can one prove in ZFC that there is a BA A with $|A| = |\mathrm{Sub}A| \geq \omega$?

<u>Problem 22^s</u>. For κ singular is it consistent that there is a BA A with $|A| = \kappa$ and $|\mathrm{Sub}A| < 2^\kappa$?

If $A \subseteq B$ or $B \twoheadrightarrow A$, then $|\mathrm{Sub}A| \leq |\mathrm{Sub}B|$.

8. <u>Irredundance</u>

A subset X of A is <u>irredundant</u> if $\forall x \in X\ (x \notin \mathrm{Sg}(X \setminus \{x\}))$. We let $\mathrm{irr}A = \sup\{|X| : X \text{ irredundant}\}$. This is a large function. Shelah [79], generalizing Rubin [83], showed assuming $V = L$ that for every regular $\kappa > \omega$ there is a BA A with $|A| = \kappa^+$ and $\mathrm{irr}A = \kappa$. Devlin [73] showed that if κ is real-valued measurable, then every algebra with countably many operations and with κ elements has an irredundant subset of power κ. Shelah [83] showed that if $\pi A \geq \kappa$, then A has an irredundant pie of power κ. In particular, $\pi A \leq \mathrm{irr}A$, and if $|A|$ is strong limit, then $\mathrm{irr}A = |A|$. Shelah [80a] has shown that it is consistent to have $2^\omega > \aleph_{\omega_1}$ and every algebra of power $\aleph_{\omega_1}$ with $< \aleph_{\omega_1}$ operations has an irredundant set of power $\aleph_1$.

<u>Problem 23</u>. Can one prove in ZFC that there is a BA A with $\mathrm{irr}A < |A|$?

<u>Problem 24^s</u>. Is it consistent that $\omega_1 < 2^\omega$ and there is a BA of power 2^ω with no uncountable irredundant set?

Shelah [81] showed there is a concentrated BA A of power $\aleph_1$ with irredundance $\aleph_1$, assuming CH. Rubin (unpublished) showed that it is consistent to have a BA A with $\mathrm{irr}A = \omega$, $|A| = \mathrm{inc}A = \omega_1$. It is clear that $\mathrm{irr}A = |A|$ for A an interval algebra. Hence $\mathrm{length}A \leq \mathrm{irr}A$ for any BA A. From the normal form theorem of Brenner, Monk [83] it follows that $\mathrm{irr}A = |A|$ for A a tree algebra, so $\mathrm{ram}A \leq \mathrm{irr}A$ for any BA A. Note also that $2^{\mathrm{irr}A} \leq |\mathrm{Sub}A|$.

<u>Problem 25</u>. Is $2^{\mathrm{irr}A} = |\mathrm{Sub}A|$?

Problems 23 and 24 are mentioned in van Douwen, Monk, Rubin [80].

9. <u>Subalgebra depth</u>

$\mathrm{sdepth}A = \sup\{\kappa :$ there is a strictly decreasing system $\langle B_\alpha : \alpha < \kappa\rangle$ of subalgebras of $A\}$. Again this is a large function. First we give some equivalents of this definition.

Theorem 9.1. For any BA A and any infinite regular cardinal κ the following conditions are equivalent:

(i) there is a strictly decreasing sequence $\langle B_\alpha : \alpha < \kappa\rangle$ of subalgebras of A ;

(ii) there is a sequence $\langle b_\alpha : \alpha < \kappa\rangle$ of elements of A such that $\forall\alpha < \kappa(b_\alpha \notin \mathrm{Sg}\{b_\beta : \alpha < \beta\})$;

(iii) there is a sequence $\langle b_\alpha : \alpha < \kappa\rangle$ of elements of A such that $\forall\alpha < \kappa(b_\alpha \notin \mathrm{Sg}\{b_\beta : \alpha < \beta\}$ and $b_\alpha \notin \mathrm{Sg}\{b_\beta : \beta < \alpha\})$.

Proof. (i) $\Rightarrow$ (ii) . Choose $b_\alpha \in B_\alpha \setminus B_{\alpha+1}$ for all $\alpha < \kappa$. (ii) $\Rightarrow$ (iii). Define $\langle \gamma_\xi : \xi < \kappa\rangle$ by induction: $\gamma_\xi \in \kappa$, $\gamma_\xi > \sup\{\beta : b_\beta \in \mathrm{Sg}\{b_{\gamma_\eta} : \eta < \xi\}\}$. Then $\langle b_{\gamma_\xi} : \xi < \kappa\rangle$ is as desired. (iii) $\Rightarrow$ (i) : Let $B = \mathrm{Sg}\{b_\beta : \beta \geq \alpha\}$ for all $\alpha < \kappa$.

From this theorem it is clear that $\mathrm{irr}A \leq \mathrm{sdepth}A$. If A is strongly concentrated in the sense of van Douwen, Monk, Rubin [80], then $\mathrm{sdepth}A < |A|$; hence under $V = L$ for every regular $\kappa \geq \omega$ there is a BA A of power κ^+ with $\mathrm{irr}A = \mathrm{sdepth}A = \kappa < |A| = \kappa^+$. In fact, suppose $\langle c_\alpha : \alpha < \lambda\rangle$ is a sequence as in 9.1 (ii) , where $\lambda = |A|$; we shall get a contradiction. Let $a, b_1, \ldots, b_n$, $n > 0$ be as in the definition of somewhere dense, applied to $\{c_\alpha : \alpha < \lambda\}$. Choose α minimum such that $a \leq c_\alpha \leq a + b_1 + \ldots + b_n$ and $\forall i(c_\alpha \cdot b_i < b_i)$ $\gamma < \lambda$. Then choose $\beta < \lambda$ such that $c_\alpha < c_\beta \leq a + b_1 + \ldots + b_n$ and $\forall i(c_\beta \cdot b_i < b_i)$ and $\gamma < \lambda$ such that $\forall i(c_\gamma \cdot b_i < b_i)$ and $c_\beta \cdot c_\gamma = c$. Clearly $\alpha < \beta, \gamma$ by the choice of α , contradiction.

Problem 26. $\mathrm{irr}A = \mathrm{sdepth}A$?

10. Subalgebra length

$\mathrm{slength}A = \sup\{|\mathcal{C}| : \mathcal{C}$ is a set of subalgebras of A simply ordered by $\subseteq\}$. This is a large function. Thus $\mathrm{sdepth}A \leq \mathrm{slength}A$. Now, as shown by Kurepa [57], for any $\kappa \geq \omega$ we have $\mathrm{ded}\kappa = \sup\{\lambda : \mathcal{P}\kappa$ has a chain of size $\lambda\}$. Now if $X \subseteq A$ is irredundant, X infinite, and if $\mathcal{C} \subseteq X$ is linearly ordered by $\subseteq$, then $\langle \mathrm{Sg}Y : Y \in \mathcal{C}\rangle$ is an isomorphism from $\mathcal{C}$ into $\mathrm{Sub}A$. Thus $\mathrm{slength}A \geq \mathrm{ded}|X|$, hence

$$\sup\{\mathrm{ded}\kappa : \kappa < \mathrm{irr}^s A\} \leq \mathrm{slength}A .$$

Thus the most natural question concerning this cardinal function is:

Problem 27. $\mathrm{slength}A = |\mathrm{Sub}A|$?

11. Independence

A subset X of A is <u>independent</u> if X freely generates the subalgebra it generates, or equivalently, if $\prod Y \cdot \prod_{z \in Z} -z \neq 0$ for any two disjoint finite subsets Y and Z of X . Set $\mathrm{ind}A = \sup\{|X| : X \subseteq A$, X independent$\}$. This is a widely varying function. This notion has been widely studied and a detailed survey can be found in Monk [83].

For each limit cardinal κ, there is a BA A with independence κ not attained. Independence in subalgebras was characterized by Sapirovskiĭ [80]; if $A \subseteq B$, then

$$\mathrm{ind}B = \mathrm{ind}A \cup \sup\{\mathrm{ind}(B/FgF) : F \in \mathrm{Ult}A\} .$$

Concerning homomorphisms, clearly $\mathrm{ind}B \leq \mathrm{ind}A$ if $A \twoheadrightarrow B$. If A has an independent set of power λ with $\lambda^{\omega} = \lambda$, then $A \twoheadrightarrow B$ for some B with $\mathrm{ind}B = \lambda$, and one can specify whether indB is attained or not under some mild conditions; moreover, this result is in a sense best possible.

Using a construction of T. Cramer [74], for any $\kappa \geq 2^{\omega}$ there is a hereditarily atomic BA A (which thus has independence ω) such that $\mathrm{ind}({}^{\omega}A) = \kappa$. T. Carlson has shown that if $\langle A_i : i \in I\rangle$ is a system of interval algebras, and $|A_i| \geq 2$ for all $i \in I$, then

$$\beth_1|I| \leq \mathrm{ind}(\Pi_{i\in I} A_i) \leq \beth_2|I| .$$

<u>Problem 28</u>. If $\langle A_i : i \in I\rangle$ is a system of interval algebras with $|A_i| \geq 2$ for all $i \in I$, is $\mathrm{ind}(\Pi_{i\in I}A_i) = \beth_1|I|$?

Sapirovskiĭ [80] showed that if neither A nor B has an independent subset of power κ, then neither does $A*B$. Hence

$$\mathrm{ind}(*_{i\in I}A_i) = |I| \cup \sup_{i\in I}\mathrm{ind}(A_i) .$$

Monk [83] showed that if A is wcc, then $(\mathrm{ind}A)^{\omega} = \mathrm{ind}A$. Balcar and Franěk [83] showed that any complete BA A has an independent subset of power $|A|$.

The relationship between cell and ind is not fully known, although some strong results have been obtained; in particular, the situation is clear if $V = L$. Shelah [80] showed the following.

(1) Let $\kappa = \mathrm{cell}^s A$. Suppose either that λ is regular and $\forall\mu < \lambda\ (\mu^{<\kappa} < \lambda)$, or $\lambda = \kappa$ is weakly compact. Then among any λ elements of A there are λ independent elements.

Assuming $V = L$, (1) is best possible for λ regular. Most of the examples needed to show this are easy, or well-known, but the following example of Argyros [82] is more involved:

(2) If μ is singular strong limit, $2^{\mu} = \mu^{+}$, $(\mathrm{cf}\mu)^{\kappa} = \mathrm{cf}\mu$ for all $\kappa < \mathrm{cf}\mu$, then there is a BA A of power μ^{+} satisfying the $(\mathrm{cf}\mu)^{+}$-cc with no independent set of power μ^{+}

Without $V = L$ there are several open problems.

<u>Problem 29</u>. (In ZFC) If κ is strongly inaccessible but not weakly compact, is there a BA of power κ satisfying the κ-cc with no independent set of power κ ?

<u>Problem 30</u>. (In ZFC) Assume $\rho < \nu < 2^{\rho} < \lambda \leq 2^{\nu}$, with λ regular. Is there a BA of power λ satisfying the ν^{+}-cc with no independent set of power λ ?

<u>Problem 31</u>. (In ZFC) Assume $\mathrm{cf}\mu < \mu < \lambda \leq \mu^{\mathrm{cf}\mu}$ and $\forall\rho < \mu\ (\rho^{\mathrm{cf}\mu} < \mu)$, with λ regular. Is there a BA of power λ satisfying the $(\mathrm{cf}\mu)^{+}$-cc with no independent set of power λ ?

For λ singular, we first note the following result of Shelah [80]:

(3) Let κ be regular, and let λ be singular with $\forall\mu < \lambda\ (\mu^{<\kappa} < \lambda)$. Then the following conditions are equivalent:

(a) every BA of power λ satisfying the κ-cc has an independent set of power λ.

(b) for all A, if A satisfies κ-cc, then $\forall a \in {}^{cf\lambda}A$ $\exists\Gamma \in [cf\lambda]^{cf\lambda}\ \forall F \in [\Gamma]^{<\omega}\ (\Pi_{\alpha\in F} x_\alpha \neq 0)$.

The property (b) has been widely studied; see Comfort, Negrepontis [82]. The initial cardinality condition in (3) is trivial under GCH (true if $\kappa < \lambda$, false if $\kappa > \lambda$). Without GCH, its falsity leads to the following question.

Problem 32. (In ZFC) Suppose that κ is uncountable and weakly inaccessible, $2^\nu < \lambda$ for all $\nu < \kappa$, and $2^{<\kappa} = \lambda$ is singular. Is there a BA of power λ satisfying κ-cc with no independent set of size λ?

If κ is regular, λ is singular, and $\forall\mu < \lambda\ (\mu^{<\kappa} < \lambda)$, then (3) applies, and we are concerned with (b). Condition (b) is fully resolved under $V = L$. Without $V = L$ the following problem arises. We say that A has precaliber κ if $\forall a \in {}^{\kappa}A\ \ \exists\Gamma \in [\kappa]^\kappa(\{a_\alpha : \alpha \in \Gamma\}$ has fip).

Problem 33. (In ZFC) If κ and λ are regular, $\kappa < \lambda$, $\lambda > 2^\omega$, and $\mu^\nu \geq \lambda$ for some $\mu < \lambda$ and $\nu < \kappa$, is there a BA satisfying κ-cc without precaliber λ?

Problems 28-33 are mentioned in Monk [83]. In connection with these problems, see also Todorčević [∞].

12. Homomorphism type and spectrum

Let $hA = \min\{|B| : A \twoheadrightarrow B,\ |B| \geq \omega\}$ for A infinite. Also let $hsA = \{|B| : A \twoheadrightarrow B,\ |B| \geq \omega\}$. This is a different kind of function from the preceding ones; its values are sets of cardinals. hsA always has a largest element $|A|$ and a smallest one hA. We are concerned with the possibilities for hA and hsA. We mention some known results about H :

(1) (S. Koppelberg [75]) $hA = 2^\omega$ for A infinite wcc.

(2) (S. Koppelberg [77]) MA and $\omega \leq |A| < 2^\omega \Rightarrow hA = \omega$.

(3) (S. Koppelberg [77]) If A can be embedded in a free BA or in a interval algebra, then $hA = \omega$.

(4) (W. Just, unpublished) Con(2^ω is arbitrarily large and for every $\lambda \leq 2^\omega$ with $\omega < cf\lambda$ there is a BA A with $|A| = hA = \lambda$).

Problem 34. If A has an irredundant set of generators, is $hA = \omega$?

Problem 35s. Is it consistent to have a BA A with $|A| = hA = \lambda$, $\omega < \lambda < 2^\omega$, and $cf\lambda = \omega$?

Now we turn to hs. If A is free, then $hsA = [\omega, |A|]$. If A is complete, then by Balcar, Franek [82], $hsA = [\omega, |A|] \cap \{\kappa : \kappa^\omega = \kappa\}$. Concerning interval algebras, we have:

Theorem 12.1. if L is a linear ordring and $\langle a_\alpha : \alpha < \kappa\rangle$ is strictly increasing in L, where κ is an infinite cardinal, then $[\omega,\kappa] \subseteq hs(\text{intalg}L)$.

Proof. It suffices to show that $\kappa \in \mathrm{hs}(\mathrm{intalg}L)$. Define $x \equiv y$ iff $x, y \in L$ and $\forall\alpha < \kappa[(a_\alpha < x$ iff $a_\alpha < y)$ and $(x < a_\alpha$ iff $y < a_\alpha)]$. Then $\equiv$ is a convex equivalence relation on L with κ classes, and the theorem follows.

From Theorem 12.1 it follows that $\mathrm{hs}(\mathrm{intalg}\kappa) = [\omega,\kappa]$ for any infinite cardinal κ. Moreover, by the Erdös, Rado theorem, if $|L| \geq (2^\kappa)^+$, then $[\omega,\kappa^+] \subseteq \mathrm{hs}(\mathrm{intalg}L)$.

Theorem 12.2. $\mathrm{hs}(\mathrm{intalg}\,\mathbb{R}) = \{\omega, 2^\omega\}$.

Proof. Suppose $f : \mathrm{intalg}\,\mathbb{R} \twoheadrightarrow A$, where $|A| > \omega$. Then f is determined by a convex equivalence relation E on $\mathbb{R}$ with $|\mathbb{R}/E| = |A|$. Now $L' = \bigcup\{k : k$ is an E-class, $|k| > 1\}$ is Borel, so $L'' = \mathbb{R}\setminus L'$ is also. Clearly $|L''| = |A|$. Hence $|A| = 2^\omega$ by the Alexandroff, Hausdorff theorem.

If A is hereditarily atomic and infinite, then $hA = \omega$. In fact, let $[a]$ be an atom of A/IgAtA; then $A \twoheadrightarrow A \restriction a \cong \mathrm{finco}\kappa \twoheadrightarrow \mathrm{finco}\omega$ for some κ. Also, it can be shown that if A is hereditarily atomic, $\omega \leq \kappa \leq A$, then $[\kappa,\kappa^{<\kappa}] \cap \mathrm{hs}A \neq 0$. On the other hand, Juhász, Nagy and Weiss [79] constructed under $V = L$ a BA A of power $\aleph_{\omega+1}$ with $\aleph_\omega \notin \mathrm{hs}A$. van Douwen [$\infty$] constructed an hereditarily atomic BA A of power 2^ω with $\mathrm{hs}A = \{\omega, 2^\omega\}$. van Douwen asked the following question.

Problem 36. (In ZFC) If A is hereditarily atomic and $|A| > \kappa = \kappa^\omega$, does A have a homomorphic image of size κ?

Problem 37[s]. Con($\forall A$(A hereditarily atomic and infinite $\Rightarrow \mathrm{hs}A = [\omega, |A|]$))?

We note some other easy facts about hs :

(5) If $\omega \leq \kappa \leq |A|$, then $\mathrm{hs}A \cap [\kappa, 2^\kappa] \neq 0$.

(6) If A has a free subalgebra of power $\kappa \geq \omega$, then $\mathrm{hs}A \cap [\kappa,\kappa^\omega] \neq 0$.

(7) $\mathrm{hs}(A\times B) = \mathrm{hs}(A*B) = \mathrm{hs}A \cup \mathrm{hs}B$.

(8) If $\omega \leq \kappa < 2^\omega$, then there is a BA A such that $\mathrm{hs}A = [\omega,\kappa] \cup \{2^\omega\}$.

For, we take $A = \mathrm{Fr}\kappa \times \mathcal{P}\omega$.

Theorem 12.3. (CH) If there is a BA A such that $\mathrm{hs}A = \{\omega, \omega_2\}$, then there is a Kurepa family.

Proof. By fact (6) above, A has no uncountable independent subset. Hence A does not satisfy ccc. Hence A has a homomorphic image B such that $\mathrm{finco}\omega_1 \subseteq B \subseteq \mathcal{P}\omega_1$. Clearly, still $\mathrm{hs}B = \{\omega,\omega_2\}$. If Γ is any countable subset of ω_1, then $b \mapsto b \cap \Gamma$ for $b \in B$ is a homomorphism, and hence $\{b \cap \Gamma : b \in B\}$ is countable. Thus B is a Kurepa family.

It is consistent with CH that there is no Kurepa family, hence no BA as in 12.3.

Problem 38[s]. Is it consistent with CH that there is a BA A such that $\mathrm{hs}A = \{\omega,\omega_2\}$?

13. Endomorphisms

EndA is the set of all endomorphisms of A . Since clearly $|\mathrm{UltA}| < |\mathrm{EndA}|$, we are dealing here with a "large" cardinal function, and the most interesting question is to construct BA's A with $|\mathrm{EndA}|$ small.

Theorem 13.1. Suppose L is a complete dense linear ordering of power $\lambda \geq \omega$, and D is a dense subset of L of power κ , where $\lambda^\kappa = \lambda$. Then $|\mathrm{intalgL}| = |\mathrm{End(intalgL)}| = \lambda$.

Proof. $\mathrm{Ult(intalgA)} = X$ is a linearly ordered space with a dense subspace of power κ and $|X| = \kappa$. Hence there are at most $\lambda^\kappa = \lambda$ continuous functions from X into X , as desired.

Recall that if μ is any infinite cardinal, and ν is minimum such that $\mu^\nu > \mu$, then there is a complete linear ordering of power μ^ν with a dense subset of power μ . Thus:

Corollary 13.2. If μ is an infinite cardinal and $\forall\nu < \mu(\mu^\nu = \mu)$, there is a BA A such that $|A| = |\mathrm{EndA}| = 2^\mu$.

Corollary 13.3. $|\mathrm{End}(\mathrm{intalg}\,\mathbb{R})| = 2^\omega$.

Corollary 13.4. (GCH) If κ is infinite and regular, then there is a BA A such that $|A| = |\mathrm{EndA}| = \kappa^+$.

Problem 39. (GCH) For λ a limit cardinal or the successor of a singular cardinal, is there a BA A such that $|A| = |\mathrm{EndA}| = \lambda$?

It is easy to see that if $|A| \geq \omega_1$, then $|\mathrm{EndA}| \geq 2^\omega$. Thus the assumption $\omega_1 < 2^\omega$ implies that there is no A with $|A| = |\mathrm{EndA}| = \omega_1$.

Problem 40. In ZFC can one show that there are arbitrarily large κ for which there is a BA A with $|A| = |\mathrm{EndA}| = \kappa$?

Problem 41. Under any set-theoretical assumptions, if $\lambda < \kappa < 2^\lambda$, is there a BA A with $|A| = \lambda$ and $|\mathrm{EndA}| = \kappa$?

Problem 42. Is $|\mathrm{EndA}| \leq |\mathrm{SubA}|$?

The derived functions $H^+|\mathrm{End}|$, $H^-|\mathrm{End}|$, $S^+|\mathrm{End}|$, $S^-|\mathrm{End}|$ appear to be unrelated to $|\mathrm{End}|$; they have not been investigated.

14. Automorphisms

AutA is the group of automorphisms of A . This is a widely varying function, not in general related to most of our other functions. There are many papers studying this group. There are infinite BA's A with $|\mathrm{AutA}| = 1$ (A is then called rigid). If AutA is finite, then it is isomorphic to a finite symmetric group. McKenzie, Monk [73] showed that for any $\kappa \geq 2^\omega$ there is a BA A with $|A| = \kappa$ and $|\mathrm{AutA}| = \omega$; assuming MA , $|\mathrm{AutA}| = \omega$ implies $|A| \geq 2^\omega$. van Douwen [80] showed $\mathrm{Con}(\mathrm{ZFC} + 2^\omega = \omega_2 + \exists \mathrm{BA}\ A(|A| = \omega_1 , |\mathrm{AutA}| = \omega))$. In McKenzie, Monk [73] it is also shown that if $\omega < \kappa \leq \lambda$, then there is a BA A with $|A| = \lambda$ and $|\mathrm{AutA}| = \kappa$. For any λ there is a BA A with $|A| = \lambda$

and $|\mathrm{Aut}A| = 2^\lambda$. In case $\aleph_0 < \lambda < \kappa < 2^\lambda$ it is not completely clear when there is a BA A with $|A| = \lambda$ and $|\mathrm{Aut}A| = \kappa$. Some consistency results have been obtained by Roitman and Shelah.

Problem 43[s]. Describe fully when it is possible to have a BA A with $|A| = \lambda$, $|\mathrm{Aut}A| = \kappa$, for $\aleph_0 < \lambda < \kappa < 2^\lambda$.

Consideration of the functions H^+ and H^- associated with $|\mathrm{Aut}A|$ gives rise to some natural questions:

Problem 44. Is there a rigid BA A such that every infinite atomless homomorphic image of A is rigid?

Problem 45. Is there a BA A with $|\mathrm{Aut}A| = \omega$ such that A has no infinite rigid homomorphic image?

The functions $s^+|\mathrm{Aut}|$ and $s^-|\mathrm{Aut}|$ can vary widely.

C. Topological functions: open sets or ideals

Next we deal with functions that at least implicitly are of a topological nature. The first ones concern open sets.

15. Ideals

$\mathrm{Id}A = \{I : I$ is an ideal of $A\}$. Again we are dealing with a "large" function, since $|\mathrm{Ult}A| \le |\mathrm{Id}A|$. We note also the following easy theorem of Loats [77]:

Theorem 15.1. $|\mathrm{Id}A| \le |\mathrm{Sub}A|$.

Proof. Let $X = \{I : I$ is a proper non-maximal ideal of $A\}$. For all $I \in X$ let $fI = I \cup -I$; it is easily checked that f is one-one. To finish the proof it suffices to show that $|\mathrm{Ult}A| \le |X|$. Fix $F \in \mathrm{Ult}A$. For any $G \in \mathrm{Ult}A \setminus \{F\}$ let $gG = F \cap G$. Thus g maps $\mathrm{Ult}A \setminus \{F\}$ into X, so it suffices to show that g is one-one, which is easy.

The construction of Shelah [79], generalizing Rubin [83], yields assuming $V = L$ for every regular $\kappa \ge \omega$ a BA A with $|A| = |\mathrm{Id}A| = \kappa^+$. Since there is a one-one correspondence between ideals of A and open sets in $\mathrm{Ult}A$, the problem concerning what $|\mathrm{Id}A|$ can be is related to the well-known topological problem concerning the number of open sets. Thus from Juhász [80], 4.5,4.7 we know that $|\mathrm{Id}A|$ is never strong limit singular, and that it is likely that $|\mathrm{Id}A|^\omega = |\mathrm{Id}A|$ always holds (it does under GCH).

Problem 46. (In ZFC) Is $|\mathrm{Id}A|^\omega = |\mathrm{Id}A|$?

Problem 47. If $\omega < \kappa < \lambda < 2^\kappa$, is there a BA A with $|A| = \kappa$ and $|\mathrm{Id}A| = \lambda$?

Various possibilities in Problem 47 are excluded by known results on the number of open sets. Note that $|\mathrm{Id}A| = 2^\omega$ whenever $|A| = \omega$. Note that $|\mathrm{Id}(\mathrm{intalg}\,\mathbb{R})| = 2^\omega$, $|\mathrm{Sub}(\mathrm{intalg}\,\mathbb{R})| = 2^{2^\omega}$, $|\mathrm{End}(\mathrm{intalg}\,\mathbb{R})| = 2^\omega$.

Problem 48. Is $|\mathrm{Id}A| \le |\mathrm{End}A|$ for all A, or $|\mathrm{End}A| \le |\mathrm{Id}A|$ always?

16. Spread

We let spreadA = $\sup\{|X| : X$ is a minimal set of generators of $IgX\}$. There are several equivalents of this notion. For one of them, we call X ideal-independent if $\forall m \in \omega\setminus\{0\}\ \forall x \in {}^mX$ (x one-one $\Rightarrow x_0 \not\leq x_1 + \ldots + x_{m-1}$).

Theorem 16.1. For any infinite BA A we have spreadA = $\sup\{|X| : X$ is ideal independent$\}$ = $\sup\{|D| : D \subseteq$ UltA, D is discrete$\}$ = $\sup\{|AtB| : A \twoheadrightarrow B$, B atomic$\}$ = $\sup\{\text{cell}B : A \twoheadrightarrow B\}$.

Proof. Let the cardinals in question be $\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5$. Clearly X is a minimal set of generators of IgX iff X is ideal independent, so $\kappa_1 = \kappa_2$. To show $\kappa_1 \leq \kappa_5$, let X be ideal independent; we find a homomorphic image B of A with cell$B \geq |I|$. Let $J = Ig\{a\cdot b : a, b \in X, a \neq b\}$, and let $B = A/J$. It suffices to show that $a \notin J$ for all $a \in X$. Assuming otherwise, we then have

$$a \leq b_0\cdot c_0 + \ldots + b_{m-1}\cdot c_{m-1}$$

where $a, b_0, c_0, \ldots, b_{m-1}, c_{m-1} \in X$ and $b_i \neq c_i$ for all $i < m$. Hence without loss of generality, say $\forall i < m\ (a \neq b_i)$. Then $a \leq b_0 + \ldots + b_{m-1}$, contradiction. So $\kappa_1 \leq \kappa_5$.

For $\kappa_5 \leq \kappa_4$, suppose $A \twoheadrightarrow B$ and D is a system of pairwise disjoint elements of B; we find an atomic homomorphic image C of A with $|D| = |AtC|$. Let $C = Sg^B D$ and by the Sikorski extension theorem let $f : B \to \overline{C}$ extend the identity on C. Then range(f) is as desired. So $\kappa_5 \leq \kappa_4$.

$\kappa_4 \leq \kappa_3$: let B be an atomic homomorphic image of A; we find a discrete subset of UltA with $|AtB|$ elements. Say $f : A \twoheadrightarrow B$. For every atom b of B let $F_b = \{a \in A : b \leq fa\}$. Clearly F_b is an ultrafilter on A, and $\{F_b : b \in AtB\}$ is discrete.

Finally, for $\kappa_3 \leq \kappa_1$ let $D \subseteq$ UltA be discrete. Thus for every $F \in D$ choose $a_F \in A$ so that $sa_F \cap D = \{F\}$. Then $\langle a_F : F \in D\rangle$ is one-one and $\{a_F : F \in D\}$ is ideal independent, as desired.

Note that all of the equivalent definitions given in 16.1 involve sups, and thus give rise to attainment problems. The proof of 16.1 shows, however, that all these sups are attained or not attained simultaneously. Now by Juhász [80], 4.2, spread is attained for singular strong limit cardinals, for singular cardinals of cofinality ω, and for weakly compact cardinals. If κ is strongly inaccessible and not weakly compact, and L is a κ-Souslin line, then spread(intalgL) = κ not attained.

Problem 49s. Is it consistent to have a BA A such that spreadA is singular and not attained?

A result of Sapirovskii [76] implies that $|A| \leq 2^{\text{spread}A}$. Note also that indA $\leq$ spreadA $\leq$ incA and cellA $\leq$ spreadA.

17. Ideal generation

igA = $\min\{\kappa$: every non-principal ideal of A can be generated by $\leq \kappa$ elements$\}$. This is a large function. There are numerous equivalents of this notion, given in the next theorem. In particular, this theorem shows that igA is the same as the hereditary Lindelöf degree of UltA, a well-known function.

Theorem 17.1. For A infinite we have igA = sup{κ : there is a strictly increasing sequence of ideals of type κ} = sup{κ : there is a strictly increasing sequence of filters of type κ} = sup{κ : there is a strictly increasing sequence of open sets in UltA of type κ} = sup{κ : there is a strictly decreasing sequence of closed sets in UltA of type κ} = sup{κ : there is a right-separated sequence in UltA of type κ} (see Juhász [80]) = sup{LS : S ⊆ UltA} (see Juhász [80]) = min{κ : every S ⊆ UltA has the property that any open cover has a subcover of power ≤ κ} = min{κ : every closed set is the intersection of ≤ κ open sets} = min{κ : every open set is a union of ≤ κ closed sets} = sup{κ : $\exists \langle x_\alpha : \alpha < \kappa \rangle \in {}^{\kappa}A \; \forall \alpha < \kappa \; \forall \Gamma \in [\alpha]^{<\omega} (x_\alpha \nleq \sum_{\beta \in \Gamma} x_\beta)$}.

From this theorem it is clear that spreadA ≤ igA. The one-point compactification of the Kunen line, constructed under CH, gives a BA A with igA = ω_1 and spreadA = ω; see Juhász, Kunen, Rudin [76].

Problem 50. (In ZFC) Is there a BA A with spreadA < igA ?

Note that ig(intalg ℝ) = ω, while inc(intalg ℝ) = 2^ω. On the other hand, incA = ω and igA = ω_1 for an algebra A constructed by Baumgartner, Komjath [81] under $\diamondsuit$.

Problem 51. (In ZFC) Is there a BA A with incA < igA ?

Also note that if L is a Souslin line obtained in the usual way (using branches) from a normal Souslin tree, then |ram(intalgL)| = ω_1 while ig(intalgL) = ω. Since cellA ≤ igA for all A, we see from the discussion in section 6 that it is not possible to get in ZFC an algebra A with igA < ramA.

18. Ideal depth

hdA = sup{κ : there is a strictly decreasing sequence of ideals of A of order type κ}. This large function, naturally called ideal depth, coincides with the hereditary density of the Stone space, by the following theorem.

Theorem 18.1. For A infinite we have hdA = sup{κ : there is a strictly decreasing sequence of filters of type κ} = sup{κ : there is a strictly decreasing sequence of open sets in UltA of type κ} = sup{κ : there is a strictly increasing sequence of closed sets in UltA of type κ} = sup{κ : there is a left-separated sequence in UltA of type κ} (see Juhász [80]) = sup{density S : S ⊆ UltA} = min{κ : every S ⊆ UltA has a dense subset of power ≤ κ} = sup{density UltB : A B} = sup{πB : A B} = sup{κ : $\exists \langle x_\alpha : \alpha < \kappa \rangle \in {}^{\kappa}A \; \forall \alpha < \kappa \; \forall \Gamma \in [\kappa \setminus (\alpha+1)]^{<\omega} (x_\alpha \nleq \sum_{\beta \in \Gamma} x_\beta)$}.

The proof of this theorem follows from duality theory and Juhász [80].

Clearly spreadA ≤ hdA. Also, hdA ≤ incA; if $A \twoheadrightarrow B$ then πB ≤ incB ≤ incA by section 5, so hdA ≤ incA by 18.1.

Also note that hdA ≤ sdepthA by the proof of 15.1. It is also clear that πA ≤ hdA. Again the one-point compactification of the Kunen line (Juhász, Kunen, Rudin [76]) gives under CH a BA A with spreadA = ω and hdA ≥ ω_1. Szentmiklossy [80] showed that under MA + ¬CH there is no such A.

Problem 52. (In ZFC) Is there a BA A such that spreadA < hdA ?

$\mathcal{P}\kappa$ has ideal depth 2^κ by 18.1 (since it has spread 2^κ) but has algebraic density κ. intalg ℝ has ideal depth ω, but subalgebra depth and incomparability 2^ω. Sapirovskiĭ [74] showed $hdA \le (spreadA)^+$. Juhász [71] showed under MA + $\neg$CH that $igA = \omega \Rightarrow hdA = \omega$.

19. Ideal length

We set $ilengthA = \sup\{|X| : X$ is a chain under $\subseteq$ of ideals of $A\}$; this is ideal length. Clearly igA, hdA, $lengthA \le ilengthA$, which is thus a large function. Note that $ilengthA \le slengthA$ by the proof of 15.1. It is possible to have $igA < ilengthA$, $hdA < ilengthA$, and $ilengthA < slengthA$. Also note that it is consistent that there is a BA A with $ilengthA < |IdA|$: let A be the BA of finite and cofinite subsets of ω_1. Then $ilengthA = ded\omega_1$, while $|idA| = 2^{\omega_1}$. By Mitchell [72] it is consistent to have $ded\,\omega_1 < 2^{\omega_1}$. Ideal length has been studied in a general topological setting by by Ginsburg [∞].

Problem 53. (In ZFC) Is there a BA A with $ilengthA < |IdA|$?

D. Topological functions: points or ultrafilters

Our final group of functions are concerned with ultrafilters, i.e., points in UltA.

20. Ultrafilters

It is well-known that $|A| \le |UltA|$. Clearly also $|UltA| \le |IdA|$, $|EndA|$, $2^{|A|}$. Concerning the possible relationships between $|A|$ and $|UltA|$ we mention some well-known or easy facts:

(1) For A infinite, $|UltA| = \omega$ or $|UltA| \ge 2^\omega$.

(2) If $\lambda < \kappa < 2^\lambda < 2^\kappa$, then there is a BA A with $|A| = \kappa$ and $|UltA| = 2^\lambda$.

(3) Con(ZFC + BA A such that $|A| = \aleph_\omega$, $|UltA| = \aleph_{\omega+1}$, $2^{\aleph_1} = \aleph_{\omega+2}$).

(4) (See Tall [80] ; the result is due to Kunen.) Con($\exists$ inaccessible) $\Rightarrow$ Con($\forall A(|A| = \omega_1 \Rightarrow |UltA| \in \{\omega_1, 2^{\omega_1}\}) + 2^\omega = \omega_1 + 2^{\omega_1} > \omega_2$).

Problem 54. Describe the possibilities for $|A|$ and $|UltA|$.

The BA A constructed by Fedorcuk [75] using $\diamondsuit$ is such that $cellA = indA = \omega$, $|A| = \omega_1$, $|UltA| = 2^{\omega_1}$.

Problem 55. (In ZFC). Is there a BA A with $|UltA| > 2^{cellA \cdot indA}$?

By an easy argument, under GCH, if $\kappa^{++} \le |UltA|$ then A has a homomorphic image B with $|UltB| = \kappa^+$ or κ^{++}. If A is infinite and wcc, then $|UltB| \ge \beth_{2^\omega}$ for every infinite homomorphic image B of A ; also $|UltB|^\omega = |UltB|$ in this case; see, e.g., van Douwen [81].

Problem 56. (GCH) Is there a BA A with $|UltA| > \omega_2$ such that A has no homomorphic image B with $|UltB| = \omega_2$?

Problem 57. (GCH) Is there a BA A with $|\mathrm{Ult}A| > \omega_2$ such that A has no subalgebra B with $|\mathrm{Ult}B| = \omega_2$?

Problem 58. ($MA + \neg CH$) $tA = \mathrm{cell}A = \omega \Rightarrow |\mathrm{Ult}A| \leq 2^{\omega}$?

21. Topological density

$dA = \min\{\kappa : \mathrm{Ult}A$ has a dense subset of power $\kappa\}$. This a large function. A less topological form for dA is:

Theorem 21.1. $dA = \min\{\kappa : A$ is isomorphic to a field of subsets of $\kappa\}$.

Proof. $\leq$: Suppose $A \subseteq \mathcal{P}\kappa$. For each $\alpha < \kappa$ let $F_\alpha = \{a \in A : \alpha \in a\}$. Thus F_α is an ultrafilter on A , and $\{F_\alpha : \alpha < \kappa\}$ is clearly dense in $\mathrm{Ult}A$.

$\geq$. Let D be dense in $\mathrm{Ult}A$. For each $a \in A$ let $fa = \{F \in D : a \in F\}$. Clearly $F : A \rightarrowtail \mathcal{P}D$.

Theorem 21.2. $\mathrm{ram}A \leq dA$.

Proof. Obviously $\mathrm{cell}A \leq dA$; hence if the theorem is false, we have $\mathrm{cell}A = dA$ and $\mathrm{ram}A = (\mathrm{cell}A)^+$ by section 6. Say $A \subseteq \mathcal{P}\lambda$, $\lambda = dA$, and R is a ramification system in A of power λ^+. For all $\alpha < \lambda$ let $F_\alpha = \{x \in R : \alpha \in x\}$. Then F_α is a well-ordered chain and $\mathrm{cell}A = \lambda$, so $|F_\alpha| \leq \lambda$. But $R = \bigcup_{\alpha<\lambda} F_\alpha$, contradiction.

Clearly $dA \leq \pi A$. Sapirovskiǐ [74] has shown that $dA \leq (\mathrm{spread}A)^+$. It is easy to see that $|A| \leq 2^{dA}$. Clearly $\mathrm{ram}A < dA$ in general; $dA < \pi A$ for A free, in general. We have $\mathrm{spread}(\mathrm{intalg}\,\mathbb{R}) = d(\mathrm{intalg}\,\mathbb{R}) = \omega$. Under CH there is a BA A with $\mathrm{spread}A = \omega$ and $dA = \omega_1$ (the Kunen line).

Problem 59. (In ZFC) Is there a BA A with $\mathrm{spread}A < dA$?

Recall from 18.1 that $(H^+d)A = hdA$. Arhangelskiǐ [70] proved that $dA \leq 2^{tA \cdot \mathrm{cell}A}$. Malyhin and Sapirovskiǐ [73] showed that $MA + \neg CH$ implies that $tA, \mathrm{cell}A = \omega \Rightarrow dA = \omega$; under the same assumption Hajnal and Juhász [71] showed $\mathrm{cell}A = \omega$, $\pi A < 2^{\omega} \Rightarrow dA = \omega$. Sapirovskiǐ (see Arhangelskiǐ [78]) showed $\mathrm{spread}A \leq \kappa$, $tA < \kappa \Rightarrow dA \leq \kappa$.

Problem 60^s. $\mathrm{Con}(dA < tA \cdot \mathrm{cell}A$ for all $A)$?

Problem 61^s. $\mathrm{Con}(\forall A\, \forall\kappa(dA < 2^{\kappa}$, $tA < 2^{\kappa}$, $\mathrm{cell}A < \kappa \Rightarrow dA < \kappa)$?

Problem 62. (In ZFC) Is there a BA A with $\mathrm{cell}A = \omega$, $dA > \omega$ and $|\mathrm{Ult}A| < 2^{\omega}$?

The interval algebra A on a Souslin line has $\mathrm{cell}A = \omega$, $dA = \omega_1$, $|A| = \omega_1$. On the other hand, by the above result of Hajnal and Juhász we have, under $MA + \neg CH$, $|A| = \omega_1$ and $\mathrm{cell}A = \omega \Rightarrow dA = \omega$.

22. Ultrafilter density

For any ultrafilter F on A let

$$\pi\chi F = \min\{|X| : \forall a \in F\ \exists x \in X^+(x \leq a)\}.$$

The ultrafilter density of A is

$$\pi\chi A = \sup\{\pi\chi F : F \in \mathrm{Ult}A\}$$

and the lower ultrafilter density of A is

$$\pi\chi_0 A = \min\{\pi\chi F : F \in \mathrm{Ult}A\} .$$

Both of these are widely varying functions. Mainly we shall consider ultrafilter density. We mention about $\pi\chi_0$ only the useful result of Sapirovskiĭ that $\mathrm{ind}A = (H^+\pi\chi_0)A$; see Sapirovskiĭ [80].

Sapirovskiĭ [75] showed that $\pi\chi A \leq tA$. Sapirovskiĭ [74] showed $|A| \leq \pi\chi A^{\mathrm{cell}A}$. From various results it follows that $\pi A = \pi\chi A \cdot dA$. Note that for κ uncountable and regular, $\pi\chi(\mathrm{intalg}\kappa) = \kappa$.

23. Character

Let $\chi A = \min\{\kappa :$ every ultrafilter on A can be generated by $\leq \kappa$ elements$\}$. This is a large function. Note that $\chi A \leq \mathrm{ig}A$. The famous theorem of Arhangelskiĭ [69] implies that $|\mathrm{Ult}A| \leq 2^{\chi A}$. One can have $\chi A < \mathrm{ig}A$. For example, if A is the Alexandroff duplicate of $\mathrm{Fr}\kappa$, then $2^\kappa = \mathrm{cell}A \leq \mathrm{ig}A$, while $\chi A = \kappa$. The Kunen line (under CH) gives a BA with character ω_1 and spread ω . Szentmiklossy [80] showed under MA + $\neg$CH that spreadA = ω implies $\chi A = \omega$.

Problem 63. (In ZFC) Is there a BA A with $\chi A > \mathrm{spread}A$?

It is known that $|A| \leq \chi A^{\mathrm{cell}A}$.

Problem 64. $\mathrm{ig}A = \chi A \cdot \mathrm{spread}A$?

Baumgartner and Komjath [81] assuming $\diamondsuit$ constructed a BA A with $\mathrm{inc}A = \omega$ and $\chi A = \omega_1$.

Problem 65. $\chi A < \mathrm{sdepth}A$?

Problem 66. (In ZFC) Is there a BA A with $\chi A = \omega$ and $\mathrm{cell}A \neq dA$?

24. Tightness

Our last function is tightness. For any ultrafilter F on A we set

$$tF = \min\{\kappa : \text{if } Y \subseteq \mathrm{Ult}A \text{ and } F \subseteq Y \text{ then } \exists Z \in [Y]^{<\kappa}(F \subseteq \bigcup Z)\}.$$
$$tA = \sup\{tF : F \in \mathrm{Ult}A\} .$$

This is a widely varying function. There is a useful equivalence of Arhangelskiĭ [71] in terms of free sequences. A sequence $\langle F_\xi : \xi < \alpha\rangle$ in UltA is free if for every $\xi < \alpha$ there is no $G \subseteq \mathrm{Ult}A$ with

$$G \subseteq (\bigcup_{\eta<\xi} F_\eta) \cap (\bigcup_{\xi\leq\eta<\alpha} F_\eta)$$

Then $tA = \sup\{|\alpha| :$ there is a free sequence of length $\alpha\}$. There is also a more algebraic version of the definition:

<u>Theorem 24.1</u>. $tA = \sup\{|\alpha| : \exists \langle a_\xi : \xi < \alpha\rangle \in {}^\alpha A\ \exists Y \subseteq \mathrm{Ult}A\ \langle sa_\xi \cap Y : \xi < \alpha\rangle$ is strictly increasing$\} = \sup\{\mathrm{depth}(A/I) : I$ an ideal in $A\}$.

<u>Proof</u>. Let the three cardinals be $\kappa_1, \kappa_2, \kappa_3$. $\kappa_1 \leq \kappa_2$: Let $\langle F_\xi : \xi < \alpha\rangle$ be any free sequence. For all $\xi < \alpha$ there is an $a_\xi \in A$ such that $\{F_\eta : \eta < \xi\}^- \subseteq sa_\xi$ and $sa_\xi \cap \{F_\eta : \eta < \alpha\} = 0$. Let $Y = \{F_\xi : \xi < \alpha\}$.

$\kappa_2 \leq \kappa_1$: Let $\langle a_\xi : \xi < \alpha\rangle$ and Y be as indicated. For all $\xi < \alpha$ choose $F_\xi \in sa_{\xi+1} \cap Y \setminus sa_\xi$. It is easily checked that $\langle F_\xi : \xi < \alpha\rangle$ is a free sequence.

$\kappa_2 \leq \kappa_3$: Let $\langle a_\xi : \xi < \alpha\rangle \in {}^\alpha A$ and Y be as indicated. Let $I = \{x \in A : Y \subseteq s(-x)\}$. Clearly I is an ideal in A. Suppose $\xi < \eta < \alpha$. Then $s(a_\xi \cdot -a_\eta) \cap Y = 0$, so $a_\xi \cdot -a_\eta \in I$. Hence $[a_\xi] \leq [a_\eta]$. Choose $F \in sa_\eta \cap Y \setminus sa_\xi$. Then $F \in s(a_\eta \cdot -a_\xi) \cap Y$, so $Y \not\subseteq s(-(a_\eta \cdot -a_\xi))$. Thus $a_\eta \cdot -a_\xi \notin I$. Hence $[a_\xi] < [a_\eta]$.

$\kappa_3 \leq \kappa_2$: Let I be an ideal, $\langle [a_\xi] : \xi < \alpha\rangle$ a strictly increasing sequence in A/I. Set $Y = \bigcap_{x \in I} s(-x)$. Suppose $\xi < \eta < \alpha$. If $F \in sa_\xi \cap Y$, then $a_\xi \in F$ and $a_\xi \cdot -a_\eta \in I$, so $-a_\xi + a_\eta \in F$ hence $a_\eta \in F$ and $F \in sa_\eta$. So $sa_\xi \cap Y \subseteq sa_\eta \cap Y$. Now $a_\eta \cdot -a_\xi \notin I$. Let $F = \{x : -x \in I\} \cup \{a_\eta\} \cup \{-a_\xi\}$. Then F has fip, so $F \subseteq G$ for some ultrafilter G. Thus $G \in Y$ and $G \in sa_\eta \setminus sa_\xi$, as desired.

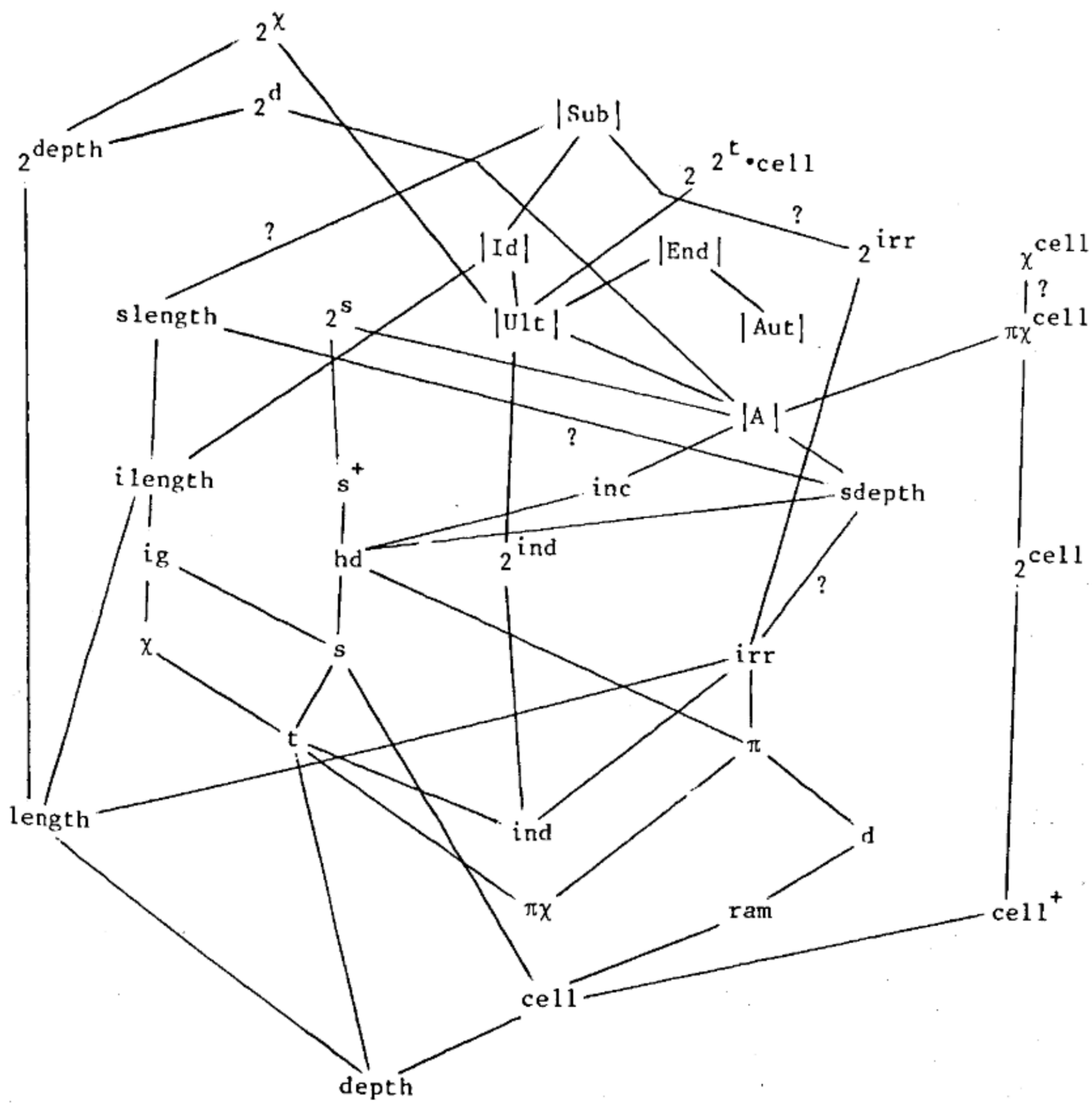
The actual definition of tightness and the free sequence equivalent give rise to attainment problems; the two equivalents in 24.1 have the same attainment properties as the free sequence equivalent. For each limit cardinal κ there is a BA A with tightness κ not attained; for $cf\kappa > \omega$ the same is true for the free sequence definition. If κ is singular, $cf\kappa = \omega$, and A has tightness κ, then A has a free sequence of length κ. These facts are easily proved using the methods of McKenzie, Monk [82].

We have $t(A \times B) = tA \cup tB$ and $t\prod^w_{i \in I} A_i = \sup_{i \in I} tA_i$. On the other hand, for each $\kappa \geq \omega$ there is a BA A with tightness ω such that $t({}^\omega A) = \kappa$; see Monk [83].

It follows from a result of Malyhin [72] that $t{*}_{i \in I} A_i = |I| \cup \sup_{i \in I} tA_i$.

Clearly independence, depth $\leq$ tightness. Sapirovskiĭ [75] has shown that ultrafilter density $\leq$ tightness. Clearly tightness $\leq$ spread, character. Sapirovskiĭ [74] showed $hdA \leq \mathrm{spread}A \cdot (tA)^+$.

We also should mention that $tA = \sup\{\pi\chi B : A \twoheadrightarrow B\}$.



Possible equalities are indicated by ? . In addition, there may be relationships not shown; see the problems.

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