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Cardinality and cofinality of homomorphs of products of Boolean algebras

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If A is an infinite homomorphic image of a σ-complete BA, then $|A|^{\omega}=|A|$ (see [SK1]). In a letter to one of the authors from H. Andréka and I. Németi the following related question was raised: if $\langle A_i : i \in I\rangle$ is a system of BA's and B is a homomorphic image of $\prod_{i\in A} A_i$ such that $\forall i \in I(|A_i|<|B|)$ and B is infinite, is $|B|^{\omega}=|B|$? We answer this question in the following way: yes if $|I|<$ first uncountable measurable cardinal, no in general for larger I. The proof of the affirmative part uses a result on cofinality by McKenzie which is also given here. Recall that for an infinite BA A, the cofinality of A, cf A, is the smallest infinite cardinal κ such that there is a strictly increasing sequence $\langle B_\alpha : \alpha \in \kappa\rangle$ of subalgebras of A with union A. McKenzie's result is that if each A_i, $i \in I$, has cofinality $\geq \omega_1$, then so does $\prod_{i\in I} A_i$, provided that $|I|<$ first uncountable measurable cardinal. See [SK2] for a systematic account of the cofinality of Boolean algebras. To conclude the paper we give an application of the positive cardinality result mentioned above; this application is due to Andréka and Németi.

NOTATION. Onto functions are indicated by $\twoheadrightarrow$; one–one and onto functions by $\rightarrowtail\!\!\!\!\twoheadrightarrow$. We use $f[X]$ for the f-image of X. If $J \subseteq I$, χ_J is the characteristic function of J: $\chi_J \in {}^I2$ and $\chi_J i=1$ if $i \in J$, $=0$ if $i \notin J$. For any cardinal κ, $\kappa^{+0}=\kappa$, $\kappa^{+(\alpha+1)}=(\kappa^{+\alpha})^{+}$, $\kappa^{+\lambda}=\bigcup_{\alpha<\lambda}\kappa^{+\alpha}$ for λ limit. We use *measurable* to include the assumption *uncountable.*

For BA's $\langle C_n : n \in \omega\rangle$ and B, we write $C_n \uparrow B$ to mean that $\langle C_n : n \in \omega\rangle$ is a strictly increasing sequence of subalgebras of B with union B. A BA A is *weakly countably complete*, wcc, if for all countable $X, Y \subseteq A$, if $X \leq Y$ then there is an $a \in A$ with $X \leq a \leq Y$. The following non-trivial result on wcc algebras will be needed below; see [SK1, 2]. If A is an infinite wcc, then $|A|^{\omega}=|A|$ and cf $A=\omega_1$. Also note that every complete BA is wcc and the class of wcc algebras is closed under taking homomorphs. We use $P(I, A, f, B)$ to abbreviate the condition: $\langle A_i : i \in I\rangle$ is a system of BA's, $|I|<$ first measurable cardinal, and $f:\prod_{i\in I} A_i \twoheadrightarrow B$.

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LEMMA 1. *Assume that* $P(I, A, f, B)$, $\forall i \in I$ $(f\chi_{\{i\}} \neq 1)$, *and* $f \restriction {}^I2 : {}^I2 \twoheadrightarrow 2$. *Then* B *is wcc.*

Proof. Let $F = \{J \subseteq I : f\chi_J = 1\}$. Then F is a non-principal countably incomplete ultrafilter on I. Hence the ultraproduct $C = \prod_{i \in I} A_i/F$ is ω_1-saturated and hence wcc. Clearly there is a homomorphism $f' : C \twoheadrightarrow B$ such that $f'[x] = fx$ for every $x \in \prod_{i \in I} A_i$. Thus B is wcc.

LEMMA 2. *Assume that* $P(I, A, f, B)$, $\forall i \in I(|B \restriction f\chi_{\{i\}}| < |B|)$, *and* B *is infinite. Then* $|B| \geq 2^\omega$.

Proof. Suppose on the contrary that $|B| < 2^\omega$. Say $f \restriction {}^I2 : {}^I2 \twoheadrightarrow C$. Thus C is wcc and $|C| < 2^\omega$, so C is finite. Let c be an atom of C such that $|B \restriction c| = |B|$, and let $g : B \twoheadrightarrow B \restriction c$ be the natural homomorphism. Then $P(I, A, g \circ f, B \restriction c)$, $\forall i \in I$ $(gf\chi_{\{i\}} \neq 1)$, and $(g \circ f) \restriction {}^I2 : {}^I2 \twoheadrightarrow 2$, so by Lemma 1, $B \restriction c$ is wcc, which contradicts $\omega \leq |B \restriction c| < 2^\omega$.

LEMMA 3. *If* $P(I, A, f, B)$, *then there is no sequence* $\langle C_n : n \in \omega \rangle$ *such that* $C_n \uparrow B$, $f \restriction {}^I2 : {}^I2 \twoheadrightarrow C_0$, *and* $\forall i \in I$ $\exists n \in \omega (B \restriction f\chi_{\{i\}} \subseteq C_n)$.

Proof. For brevity put $A' = \prod_{i \in I} A_i$. Suppose there is such a sequence. Let $C'_n = \{x \in A' : fx \in C_n\}$ for all $n \in \omega$. Then $C'_n \uparrow A'$, ${}^I2 \subseteq C'_0$, and $\forall i \in I$ $\exists n \in \omega(A' \restriction \chi_{\{i\}} \subseteq C'_n)$. Thus we can forget about f and B.

Let $K = \{M \subseteq I : \exists n \in \omega (A' \restriction \chi_M \subseteq C'_n)\}$. Thus K is an ideal in $\mathcal{P}I$ containing all singletons. For every $a \in A$ let na be minimum such that $a \in C'_{na}$, and for every $M \in K$ let pM be minimum such that $A' \restriction \chi_M \subseteq C'_{pM}$. Now we claim

(1) if $\langle Ji : i \in \omega \rangle$ is a system of pairwise disjoint subsets of I, $y \in {}^\omega A'$, and $0 < y_i \leq \chi_{Ji}$ for each $i \in \omega$, then $\langle ny_i : i \in \omega \rangle$ is bounded.

For, assume otherwise. Let $z \in A'$ with $z \restriction Ji = y_i \restriction Ji$ for all $i \in \omega$. Choose $i \in \omega$ with $nz < ny_i$. Now $z \cdot \chi_{Ji} = y_i$ and $z, \chi_{Ji} \in C'_{nz}$ while $y_i \notin C'_{nz}$, contradiction.

We need two corollaries of (1):

(2) $\mathcal{P}I/K$ is finite.

For, otherwise there are pairwise disjoint subsets Ji, $i \in \omega$, of I such that $\forall i \in \omega$ $(Ji \notin K)$, and (1) is easily contradicted.

(3) $\exists n \in \omega$ $\forall M \in K$ $(A' \restriction \chi_M \subseteq C'_n)$

For, otherwise there is an $M \in {}^{\omega}K$ such that $\langle pMm : m \in \omega \rangle$ is strictly increasing. Let $Ji = Mi \setminus \bigcup_{q<i} Mq$ for every $i \in \omega$. Then the Ji are pairwise disjoint, and $pJi = pMi$ for all $i \in \omega$. Hence there is $y \in {}^{\omega}A'$ such that $y_i \in (A' \upharpoonright \chi_{J(i+1)}) \setminus C'_{pJi}$ for all $i \in \omega$, and (1) is contradicted.

By (2) and (3) we may assume that K is maximal and $A' \upharpoonright \chi_M \subseteq C'_0$ for all $M \in K$. Let $F = \{M : I \setminus M \in K\}$. Thus F is an ultrafilter on I, and we consider the ultraproduct A'/F. Let $C''_m = \{x/F : x \in C_m\}$. We claim $C''_m \uparrow (A'/F)$. For, take $x \in C'_{m+1} \setminus C'_m$; we claim $x/F \in C_{m+1} \setminus C''_m$. Suppose to the contrary that $x/F = y/F$ for some $y \in C'_m$. Let $M = \{i \in I : x_i \neq y_i\}$. Then $M \in K$. Since $x = x \cdot \chi_M + x \cdot \chi_{I/M} = x \cdot \chi_M + y \cdot \chi_{I \setminus M}$ and $A' \upharpoonright \chi_M \subseteq C'_0$ while $y, \chi_{I \setminus M} \in C'_m$, we get $x \in C'_m$, contradiction.

Thus $\mathrm{cf}\,(A'/F) = \omega$. Now (for the first time) we use the assumption that $|I| <$ first measurable cardinal. By it, F is countably incomplete, hence A'/F is ω_1-saturated hence wcc, contradiction.

From this lemma we obtain our theorem on cofinalities:

THEOREM A. *If $|I|$ is less than the first measurable cardinal, $f : \prod_{i \in I} A_i \twoheadrightarrow B$, $\forall i \in I$ $(\mathrm{cf}\,(B \upharpoonright f\chi_{\{i\}}) \geq \aleph_1)$, and B is infinite, then $\mathrm{cf}\, B \geq \aleph_1$.*

Proof. Clearly we may assume that I is infinite. Suppose on the contrary $C_n \uparrow B$. Obviously $P(I, A, f, B)$. Clearly, then, we may assume that the conditions on $\langle C_n : n \in \omega \rangle$ formulated in Lemma 3 hold, which contradicts that lemma.

COROLLARY. *If $|I|$ is less than the first measurable cardinal and $\mathrm{cf}\, A_i \geq \aleph_1$ for all $i \in I$, then $\mathrm{cf} \prod_{i \in I} A_i \geq \aleph_1$. (If I is infinite, then $\mathrm{cf} \prod_{i \in I} A_i = \aleph_1$, using* [SK2]).

We do not know whether the condition that $|I| <$ first measurable cardinal is needed in the corollary (a problem stated in [vDMR]); by Theorem D below, it *is* needed in the theorem.

Our first cardinality theorem is as follows.

THEOREM B. *If $|I|$ is less than the first measurable cardinal, $f : \prod_{i \in I} A_i \twoheadrightarrow B$, $\forall i \in I$ $(|B \upharpoonright f\chi_{\{i\}}| < |B|)$, and $|B| \geq \omega$, then $|B|^{\omega} = |B|$.*

Proof. Assume on the contrary that $|B|^{\omega} > |B|$. By Lemma 2, $|B| > 2^{\omega}$. Let μ be minimum such that $\mu^{\omega} > |B|$. Thus $\mu \leq |B|$ and $\forall \nu < \mu (\nu^{\omega} < \mu)$, so $\mathrm{cf}\, \mu = \omega$. Also note that $\mu > 2^{\omega}$. Say $2^{\omega} < \nu_n \uparrow \mu$ for $n \in \omega$, with $\nu_n^{\omega} = \nu_n$. Let $K = \{J \subseteq I : |B \upharpoonright f\chi_J| < \mu\}$. Thus K is an ideal in $\mathscr{P}I$. Now we claim

(1) $\mathscr{P}I/K$ is finite.

For, suppose not. Then there are pairwise disjoint subsets Ji, $i \in \omega$, of I such that $Ji \notin K$ for all $i \in \omega$. Let $g : B \to \prod_{i\in\omega} B \upharpoonright f\chi_{Ji}$ be the natural homomorphism. Then g is onto. In fact, let $x \in \prod_{i\in\omega} B \upharpoonright f\chi_{Ji}$. Say $x_i = fy_i$ with $y_i \leq \chi_{Ji}$ for every $i \in \omega$. Let $z \in \prod_{i\in I} A_i$ be such that $z \upharpoonright Ji = y_i \upharpoonright Ji$ for all $i \in \omega$. Clearly $gfz = x$. So, indeed g is onto. Hence

$$\mu^{\omega} \leq \left|\prod_{i\in\omega} B \upharpoonright f\chi_{Ji}\right| \leq |B|,$$

a contradiction. Thus (1) holds, and so we may assume that K is a maximal ideal.

(2) $\{i\} \in K$ for all $i \in I$.

For, if $\{i\} \notin K$, then $|B \upharpoonright f\chi_{\{i\}}| = |B|$ since K is maximal, contradiction.

Now K is countably incomplete, so let $\langle Mi : i \in \omega \rangle$ be a partition of I such that $Mi \in K$ for all $i \in \omega$. Thus $\prod_{i\in I} A_i \cong \prod_{i\in\omega} \prod_{j\in Mi} A_j$ and $\forall i \in \omega(|B \upharpoonright f\chi_{Mi}| < |B|)$, so we may assume that $I = \omega$.

Next, let $L = \{x \in \prod_{i\in\omega} A_i : |B \upharpoonright fx| < \mu\}$.

(3) $\prod_{i\in\omega} A_i/L$ is finite.

For, otherwise there are pairwise disjoint elements $x_0, x_1, \ldots$ of $\prod_{i\in\omega} A_i \backslash L$. For each $i \in \omega$ let $y_i = x_i \cdot \chi_{\omega \backslash i}$; then also $y_i \notin L$. Let $h : B \to \prod_{i\in\omega} B \upharpoonright fy_i$ be the natural homomorphism. Again we claim that h is onto. Let $b \in \prod_{i\in\omega} B \upharpoonright fy_i$; say $fa_i = b_i$ with $a_i \leq y_i$ for each $i \in \omega$. Now define $c \in \prod_{i\in\omega} A_i$ by setting $c_i = \sum_{j\leq i} a_j i$ for each $i \in \omega$. Note that $c \cdot y_k = a_k$ for all $k \in \omega$. Hence $hc = b$, as desired. Again, this gives a contradiction, so (3) holds. We may assume that L is a maximal ideal.

For each $n \in \omega$ let

$$C_n = \{b \in B : |B \upharpoonright b| \leq \nu_n \text{ or } |B \upharpoonright -b| \leq \nu_n\}.$$

Then C_n is a subalgebra of B, and $\bigcup_{n\in\omega} C_n = B$ since L is maximal. Furthermore, $C_n \subseteq C_m$ for $n < m$. Next we claim

(4) $\forall n \in \omega(C_n \neq B)$.

For, suppose $C_n = B$. Let T be the ideal in B generated by $\{f\chi_J : |B \upharpoonright f\chi_J| \leq \nu_n\}$. Clearly $|T| \leq 2^{\omega} \cdot \nu_n = \nu_n$, so $|B/T| = |B|$. Let g be the natural homomorphism $B \to B/T$. Then $P(\omega, A, g \circ f, B/T)$, $\forall i \in \omega(gf\chi_{\{i\}} \neq 1)$, and $(g \circ f) \upharpoonright {}^{\omega}2 : {}^{\omega}2 \twoheadrightarrow 2$, so by Lemma 1, B/T is wcc, contradicting $|B|^{\omega} \neq |B|$. So (4) holds.

There is an n such that $f[{}^I2] \subseteq C_n$. Hence we have contradicted Lemma 3.

COROLLARY. *If $|I|$ is less than the first measurable cardinal, B is a homomorphic image of $\prod_{i \in I} A_i$, $|B| \geq \omega$, and $\forall i \in I(|A_i| < B)$, then $|B|^\omega = |B|$.*

Our second theorem on cardinality shows that the assumption $|I| <$ first measurable cardinal is needed in both Theorem B and its corollary.

THEOREM C. (i) *Suppose κ is uncountable and there is a non-principal countably complete ultrafilter on κ. Then there is a system $\langle A_\alpha : \alpha \in \kappa \rangle$ of Boolean algebras and a homomorphism $f : \prod_{\alpha \in \kappa} A_\alpha \to B$ such that $|B| \geq \omega$, $\forall \alpha \in \kappa(|B \restriction f\chi_{\{\alpha\}}| = 1)$, and $|B|^\omega > |B|$.*

(ii) *Suppose $\kappa \geq \omega$ and there is a countably complete $(\kappa, \kappa^{+\omega})$ – regular ultrafilter on some I such that $|I| > \kappa^{+\omega}$. Then there is a system $\langle A_i : i \in I \rangle$ of Boolean algebras and an infinite homomorphic image B of $\prod_{i \in I} A_i$ such that $\forall i \in I(|A_i| < |B|)$ and $|B|^\omega > |B|$.*

Proof. (i) Let A_α be the interval algebra on $\kappa^{+\omega}$ for all $\alpha \in \kappa$, let F be a non-principal countably complete ultrafilter on κ, and set $C = \prod_{\alpha \in \kappa} A_\alpha / F$ with $g : \prod_{\alpha \in \kappa} A_\alpha \twoheadrightarrow C$ the natural homomorphism. Then $|C| \geq \kappa^{+\omega}$ and $\forall \alpha \in \kappa(|C \restriction g\chi_{\{\alpha\}}| = 1)$. Because F is countably complete, C is isomorphic to the interval algebra on the well-ordered set $(\kappa^{+\omega})^\kappa / F$, whose order type is $\geq \kappa^{+\omega}$. Hence there is a homomorphism h from C onto the interval algebra B on $\kappa^{+\omega}$. Clearly B is as desired.

(ii) Let A_i be the interval algebra on κ for all $i \in I$. Then proceed as in (i), using the ultrafilter given in the hypothesis of (ii). The regularity hypothesis assures that κ^I/F has order type $\geq \kappa^{+\omega}$, since $\kappa^{+\omega} \leq |\kappa|^{\kappa^{+\omega}} \leq |\kappa^I/F|$.

The existence of an ultrafilter as in (ii) is guaranteed if κ is strongly compact, but it also follows from much weaker assumptions; see [BK].

Now we show that Theorem A cannot be strengthened by dropping its initial assumption.

THEOREM D. *Suppose κ is uncountable and there is a non-principal countable complete ultrafilter on κ. Then there is a system $\langle A_\alpha : \alpha < \kappa \rangle$ of Boolean algebras and a homomorphism $f : \prod_{\alpha \in \kappa} A_\alpha \twoheadrightarrow B$ such that $\forall \alpha < \kappa(\mathrm{cf}\,(B \restriction f\chi_{\{i\}}) = \aleph_1)$, B is infinite, and* $\mathrm{cf}\, B = \aleph_0$.

Proof. Let C_α be the interval algebra on $\kappa^{+\omega}$ for all $\alpha \in \kappa$, and let F be a non-principal countably complete ultrafilter on κ. For each $\alpha \in \kappa$ let $A_\alpha = C_\alpha \times \mathscr{P}\omega$, and let $f : \prod_{\alpha \in \kappa} A_\alpha \twoheadrightarrow (\prod_{\alpha \in \kappa} C_\alpha / F) \times {}^\kappa\mathscr{P}\omega$ be the natural homomorphism. Note that $f\chi_{\{\alpha\}} = (0, \chi_{\{\alpha\}})$ for each $\alpha \in \kappa$. Hence, using the proof of Theorem C(i), the conclusion is clear.

To conclude the paper we give an application of the Corollary to Theorem B. This application, due to Andréka and Németi, is to the theory of cylindric set algebras, and we use notation and concepts from [HMT].

THEOREM E. *Assume that there is no uncountable measurable cardinal. Suppose that α is an infinite ordinal, κ is a cardinal, $2^{|\alpha|} < \kappa < 2^{2^{|\alpha|}}$, and $\kappa^\omega \neq \kappa$. Then there is an $\mathfrak{A} \in Cs_\alpha^{\text{reg}} \setminus HPWs_\alpha$.*

Proof. By 4.13 of [AN] let $\mathfrak{A} \in {}_2Cs_\alpha^{\text{reg}}$ with $|A| = \kappa$. We claim that $\mathfrak{A} \notin HPWs_\alpha$; assume otherwise. Say $\langle \mathfrak{B}_i : i \in I \rangle \in {}^I Ws_\alpha$ and $f \in \mathrm{Ho}\,(\Pi_{i \in I} \mathfrak{B}_i, \mathfrak{A})$. Let $J = \{i \in I$: the base of $\mathfrak{B}_i$ has exactly two elements$\}$, $\mathfrak{C} = \Pi_{i \in J} \mathfrak{B}_i$, $\mathfrak{D} = \Pi_{i \in I \setminus J} \mathfrak{B}_i$. We identify $\Pi_{i \in I} \mathfrak{B}_i$ with $\mathfrak{C} \times \mathfrak{D}$. For any CA $\mathfrak{E}$ let $\mathfrak{E}'$ be its Boolean reduct. There is an isomorphism g of $\mathfrak{C}'$ into $\mathfrak{C}' \times \mathfrak{D}'$ such that gx has the form $(x, \mathfrak{E})$ for every $x \in \mathfrak{C}'$. Let σ be the term

$$c_0 c_1 - d_{01} \cdot - c_0 c_1 c_2 (-d_{01} \cdot - d_{02} \cdot - d_{12}).$$

Thus $\sigma^{\mathfrak{A}} = 1$ and $\sigma^{\mathfrak{D}} = 0$. Hence

$$\begin{aligned} 0 &= f(0,0) = f(0,0) \cdot \sigma^{\mathfrak{A}} = f((0,0) \cdot \sigma^{\mathfrak{C} \times \mathfrak{D}}) \\ &= f((0,1) \cdot \sigma^{\mathfrak{C} \times \mathfrak{D}}) = f(0,1). \end{aligned}$$

It follows that $f \circ g : \mathfrak{C}' \twoheadrightarrow \mathfrak{A}'$. For any $i \in J$, $|B_i| \leq 2^{|\alpha|}$ since $\mathfrak{B}_i \in Ws_\alpha$. This contradicts the Corollary of Theorem B.

Theorem E shows that it is relatively consistent that Problem 6 of [HMT] has a negative solution.

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