Algebra Universalis, **11** (1980) 220–243 Birkhäuser Verlag, Basel

Some questions about Boolean algebras

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Very recently there has been much progress on some fundamental set-theoretic problems concerning Boolean algebras. The purpose of this article is to indicate some problems still left open, put in perspective by what has been shown recently. We have made no attempt to completely cover the field with these questions, but hope that for the problems mentioned here the picture we give is fairly complete. To some extent this is a survey of recent set-theoretical results on Boolean algebras. In particular, part of the information we give here answers questions from earlier informal versions of this paper and has been included so as to make clear what no longer is an open problem.

We are grateful to R. Bonnet, S. Koppelberg, K. Kunen, R. Laver, R. McKenzie, P. Nyikos, S. Shelah and M. Weese for comments on earlier versions of this article.

1. Automorphism groups

Anderson [A] proved a topological theorem which implies that the denumerable free BA and any σ-complete homogeneous BA have simple automorphism groups.

QUESTION 1. Does every homogeneous BA have a simple automorphism group? In particular, does every uncountable free BA have a simple automorphism group?*

We call a topological space X *homogeneous* if for any two points $x, y \in X$ there is an autohomeomorphism f of X such that $fx = y$. Obviously the Stone space ${}^{\kappa}2$

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[1] Research supported in part by US NSF grant MCS 78-09484.

[2] The author gratefully acknowledges partial support by the Forschungsinstitut für Mathematik, ETH-Zürich

[3] Research supported in part by US NSF grant.

Presented by R. S. Pierce. Received February 13, 1979. Accepted for Publication in final form May 29, 1979.

* See note 10 added in proof.

of the free BA of power κ is homogeneous. It is also easily checked that the interval algebra on the ordered set R of real numbers has homogeneous Stone space; more generally, if O is a complete ordered set which is homogeneous (given any two points $\neq 0, 1$ there is an order automorphism taking one to the other) and reversible (isomorphic to its dual), then the interval algebra on O has homogeneous Stone space. A free product of BA's with homogeneous Stone spaces also has a homogeneous Stone space. On the other hand, it is known (see Comfort [C1]) that an infinite compact F-space is not homogeneous (X is an F-space if every co-zero set of X is C^*-embedded). Since the Stone space of a BA A is an F-space iff A satisfies the countable separation property (CSP) (that is, if $B \cup C \subseteq A$, B and C are countable, and $b \wedge c = 0$ for all $b \in B$, $c \in C$, then there is an $a \in A$ with $b \leq a$ and $c \wedge a = 0$ for all $b \in B, c \in C$), a CSP BA has non-homogeneous Stone space. In particular, any homomorphic image of a countably complete BA has non-homogeneous Stone space. Further relevant non-homogeneity results were obtained by van Douwen [vD2]; e.g. if $\mathcal{S}A$, the Stone space of A, has power $> 2^{\pi A}$ (where πA is the smallest cardinality of a dense subset of A), then $\mathcal{S}A$ is non-homogeneous. These results shed light on the following question.

QUESTION 2 (*Kunen*). Can every BA be embedded in a BA which has homogeneous Stone space?

Some related questions have recently been answered. Van Douwen [vD4] has constructed a non-homogeneous BA with homogeneous Stone space. Call a BA A *weakly homogeneous* if for any two non-zero elements $a, b \in A$ there is an automorphism f of A such that $fa \wedge b \neq 0$. R. Solovay (unpublished) has shown that a complete BA is weakly homogeneous iff it is a power of a complete homogeneous BA; an algebraic proof will appear in Koppelberg [Ko5].

We turn to questions concerning the size of automorphism groups. For an arbitrary BA A, $|\mathrm{Aut}\, A|$ can take on any infinite cardinal as a value (see McKenzie, Monk [McM]). This is not true of complete BA's: S. Koppelberg [Ko5] has shown that $|\mathrm{Aut}\, A|^{\aleph_0} = |\mathrm{Aut}\, A|$ if A is complete and $|\mathrm{Aut}\, A|$ is infinite. From results mentioned below concerning rigid complete BA's it follows by McKenzie, Monk [McM] that if $\kappa \geq \aleph_0$ then there is a complete BA A with $|\mathrm{Aut}\, A| = \kappa^{\aleph_0}$. We do not know whether Koppelberg's result above extends to various plausible classes containing the class of all complete BA's, e.g. to σ-complete BA's or to CSP BA's.

QUESTION 3. If A is a CSP BA with $|\mathrm{Aut}\, A|$ infinite, is $|\mathrm{Aut}\, A|^{\aleph_0} = |\mathrm{Aut}\, A|$?

The possible relations between $|A|$ and $|\mathrm{Aut}\, A|$ for an arbitrary BA A are almost completely known. McKenzie and Monk [McM] showed that for any $\kappa \geq 2^{\aleph_0}$ there is a BA A with $|\mathrm{Aut}\, A| = \aleph_0$ and $|A| = \kappa$, while, assuming MA, $|\mathrm{Aut}\, A| = \aleph_0$ implies that $|A| \geq 2^{\aleph_0}$. Van Douwen [vD5] has shown Con (ZFC) $\rightarrow$ Con (ZFC + $2^{\aleph_0} = \aleph_2$ + $\exists$BA B with $|B| = \aleph_1$ and $|\mathrm{Aut}\, B| = \aleph_0$). In [McM] it is also shown that if $\aleph_0 < \kappa \leq \lambda$ then there is a BA of power λ with automorphism group of power κ. Clearly the BA of finite and cofinite subsets of λ has power λ with automorphism group of power 2^λ. It is also easy to see that if $\aleph_0 < \lambda < 2^\mu < 2^\lambda$, then there is a BA of power λ with automorphism group of power 2^μ. The following question, however, remains open in general.

QUESTION 4. If $\aleph_0 < \lambda < \kappa < 2^\lambda$, is there a BA of power λ with automorphism group of power κ?*

The relationship between $|A|$ and $|\mathrm{Aut}\, A|$ for a complete BA A, or a σ-complete BA A, or a CSP BA A is even less known, although partial information can be obtained from results on rigid BA's using methods in [McM].

Rubin [R2] proved that if B does not contain non-zero rigid factors and C is a complete BA without non-zero rigid factors, then $\mathrm{Aut}\,(B) \cong \mathrm{Aut}\,(C)$ iff $C \cong \bar{B}$ (the completion of B) and for every $f \in \mathrm{Aut}\,(\bar{B})$, $f \restriction B \in \mathrm{Aut}\,(B)$.

QUESTION 5. Do there exist two totally different BA's A and B (i.e., without isomorphic non-trivial factors), each having no rigid factors, with $\mathrm{Aut}\, A \cong \mathrm{Aut}\, B$? Could A and B be chosen to be weakly homogeneous?

Some work has been done on reconstruction of BA's from their groups of automorphisms. A class K of BA's is called *faithful* whenever for all $B_1, B_2 \in K$ such that $\mathrm{Aut}\, B_1 \cong \mathrm{Aut}\, B_2$ we have $B_1 \cong B_2$. McKenzie [Mc] proved that the class of countable BA's that have a non–zero maximal atomic element is faithful. (Note that $\mathrm{Aut}\, B_1 \cong \mathrm{Aut}\, B_2$ when B_1 is countable atomless and B_2 is countable with exactly one atom.) McKenzie [Mc] and also independently Shelah showed that the existence of a maximal atomic element is necessary.

A BA B is 1-*homogeneous* if for every $a, b \in B$ which have the same elementary type in B there is an $f \in \mathrm{Aut}\, B$ such that $fa = b$. Rubin [R1] proved that the class of 1-homogeneous BA's B which have a maximal non-zero atomic element non-isomorphic to $\mathbf{S}\omega$ is faithful. The exclusion of atomless BA's and $\mathbf{S}\omega$ is of course necessary. However, it seems plausible that with the exception of some trivialities the class of 1-homogeneous BA's is faithful. To be on the safe side we formulate the question in the following way.

* See note 1 added in proof.

QUESTION 6. Is the class of non-atomless $\aleph_0$-saturated 1-homogeneous BA's faithful?

Remarks. (a) See [R1]. (b) Rubin [R2] proved that the class of complete BA's without rigid factors is faithful.

2. Rigid BA's

A rigid BA, i.e., one with no non-trivial automorphisms, was first constructed, implicitly, by Kuratowski [Ku] in 1926 (for his rigid space X, βX is easily seen to be a rigid Stone space). This was overlooked later, and the existence of a rigid BA was stated as a problem in the first edition of Birkhoff's *Lattice Theory* (1939). Explicit constructions, solving this problem, were given by Jónsson, Katětov and Rieger in 1951. Since then there have been many different constructions of rigid BA's for various purposes, by Bonnet, van Douwen, Ehrenfeucht, de Groot, Jech, Kannan and Rajagopalan, Loats and Rubin, Lozier, McAloon, McKenzie and Monk, Monk, Shelah, and others. All of these constructions are rather special and not easily described. So we ask the vague question concerning the existence of a "natural" rigid BA. So far the only known candidate for such a BA is given in the following question.

QUESTION 7 (*Balcar*). Let κ be an uncountable regular cardinal and let I be the ideal in $\mathbf{S}\kappa$ consisting of all non-stationary subsets of κ. Is it consistent that $\mathbf{S}\kappa/I$ is rigid?

We state the problem in this form since van Wesep (unpublished) has recently shown $\mathrm{Con}(ZF+\text{axiom of determinacy}+\text{more})\Rightarrow\mathrm{Con}(\mathrm{ZFC}+\mathbf{S}\omega_1/I$ is homogeneous). Also, Magidor (unpublished) proved that Con (ZFC + $\exists$ a measurable cardinal)$\Rightarrow\mathbf{S}\omega_1/I$ is a power of a homogeneous BA). The following equivalent form of this question was raised in van Douwen, Lutzer [vDL]. For any cardinal κ and subset $\Gamma\subseteq\kappa$ let

$$I_\Gamma^\kappa=\{\Delta\subseteq\Gamma\colon \text{there is a closed } E\subseteq\Gamma \text{ with } |E|=\kappa \text{ and } \Delta\cap E=0\}$$

(here κ has the order topology, Γ the relativized topology). The question equivalent to Question 6 is:

Let κ be an uncountable regular cardinal, and let S, T be stationary subsets of κ. Is it consistent that

$$\mathbf{S}S/I_S^\kappa\cong\mathbf{S}T/I_T^\kappa \quad\text{iff}\quad S\equiv T(\mathrm{mod}\ I_\kappa^\kappa)?$$

Shelah, modifying and improving his earlier work [S4], has recently shown that for each κ with $\kappa^{\aleph_0} = \kappa$ there are 2^κ non-isomorphic rigid complete BA's of power κ.

A stronger version of rigidity for complete BA's has been studied in the literature. A complete BA A is called *simple* if it is atomless but has no proper atomless complete subalgebra. Any such algebra is rigid. Jech [J] showed that if $V = L$ and κ is uncountable, regular and not weakly compact, then there is a simple complete BA of power κ.

QUESTION 8. Do simple complete BA's exist in ZFC?

Jech also showed that if κ is weakly compact then there are no simple complete BA's of power κ and, under GCH, if κ is singular then there is no simple complete BA of power κ.

QUESTION 9. Show in ZFC that simple complete BA's of singular cardinalities do not exist.

For further discussion of simple complete BA's see Balcar, Štěpánek [BŠ] and Koppelberg [Ko4].

For arbitrary BA's one can consider various strengthenings of the notion of rigidity:

(1) A is *mono-rigid* if every one-to-one endomorphism is the identity.
(2) A is *onto-rigid* if every onto endomorphism is the identity.
(3) A is *bi-rigid* if it is both mono-and onto-rigid.
(4) A is *Bonnet-rigid* if for every BA B, every one-to-one homomorphism f from A into B, and every homomorphism g from A onto B we have $f = g$.
(5) A is *endo-rigid* if for every endomorphism f of A the quotient algebra A/I_f is finite, where I_f is the ideal of A generated by

$$\{a \in A : fa = 0\} \cup \{a \in A : \forall b \leq a (fb = b)\}.$$

(In Shelah [S5] it is shown that this condition is equivalent to having only definable endomorphisms in a certain sense. Generalizations of the notion endo-rigid have been considered by R. Bonnet.)
(6) Two ideals I_0 and I_1 of A are *complementary* if $I_0 \cap I_1 = \{0\}$ while $I_0 \cup I_1$ generates a maximal ideal of A. We say that A is *indecomposable* if A does not have any pair of non-principal complementary ideals. It is easily checked that any complete BA is indecomposable. We call A *Shelah-rigid* if it is endo-rigid and indecomposable.

(7) A has *few endomorphisms* if A has $|A|$ endomorphisms.

(8) A is *absolutely rigid* if A is Bonnet-rigid, Shelah-rigid, and has few endomorphisms.

Remarks. 1) If B is infinite then it has a non-trivial onto order-preserving function.

2) The following relationships among the above notions are easily established:

$$\begin{tikzcd}
 & \text{Shelah-rigid} \arrow[r, Rightarrow] & \text{endo-rigid} \arrow[dr, Rightarrow] & & \text{mono-rigid} \arrow[dr, Rightarrow] & \\
\text{absolutely rigid} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & & \text{bi-rigid} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \text{rigid} \\
 & \text{Bonnet-rigid} \arrow[urr, Rightarrow] & & & \text{onto-rigid} \arrow[ur, Rightarrow] &
\end{tikzcd}$$

3) Shelah (unpublished), using $\diamondsuit_{\aleph_1}$, showed there is a Bonnet-rigid, Shelah-rigid BA of power $\aleph_1$.

4) Bonnet [Bo2] showed that there is a Bonnet-rigid BA of power $2^{\aleph_0}$ (using a suggestion of Shelah to eliminate all extra set-theoretical hypotheses). Shelah (unpublished) showed that they exist in every regular cardinal $>\aleph_0$.* None of the algebras constructed are Shelah-rigid.

5) Shelah [S5] using $\diamondsuit_{\aleph_1}$, and later (unpublished) using CH only, constructed a Shelah-rigid BA of power $\aleph_1$. Monk [M1], using suggestions of Shelah, constructed in ZFC a Shelah-rigid BA of power $2^{\aleph_0}$. This algebra is not Bonnet-rigid.

6) Shelah (unpublished) has shown that any endo-rigid BA has at least $2^{\aleph_1}$ ultrafilters. Hence there is no endo-rigid BA A with $|A|$ endomorphisms if $|A|<2^{\aleph_1}$.**

7) A construction in Koppelberg [Ko2] implicitly shows that if MA and $|A|<2^{\aleph_0}$ then A is not endo-rigid.**

8) Shelah and Rubin have shown that the BA constructed by Rubin in [R3] has just $\aleph_1$ endomorphisms (it is also of power $\aleph_1$). The Bonnet-rigid BA's constructed by Bonnet in [Bo2] also have few endomorphisms.

9) An interval algebra is never endo-rigid; hence the algebras of Bonnet are not endo-rigid.

10) Monk [M1] showed under MA that if $|A|<2^{\aleph_0}$ and A has a denumerable dense atomless subalgebra then A is decomposable.

11) Monk [M1] and Nyikos (unpublished) independently showed that if $A(X)$ is the free extension of A on X and X is uncountable then $A(X)$ is indecomposable. Hence the assumption in 10) that A has a denumerable dense atomless subalgebra cannot be eliminated.

12) Loats and Rubin [LR] showed that for each $\kappa>\aleph_0$ there are 2^κ isomorphism types of onto rigid BA's of power κ; these are interval algebras and hence are not endo-rigid, by 9).

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* See note 2 added in proof.

** See note 11 added in proof.

13) Every mono-rigid interval algebra is also Bonnet-rigid.

14) Shelah (unpublished) has shown that for every regular $\kappa > \aleph_0$, as well as many singular κ, there is a mono-rigid BA of power κ. Monk [M1] showed that there is a mono-rigid BA of power 2^κ if $2^\kappa = \kappa^{\aleph_0}$ (e.g. $\kappa = \beth_0$, $\kappa = \beth_\omega$). It was independently shown in Todorčević [*T*] that for each regular $\kappa > \aleph_0$ there is a mono-rigid BA of power κ.

15) van Douwen (unpublished) has shown that onto-rigid $\not\Rightarrow$ mono-rigid.

16) Katetov's original example is rigid but not mono-rigid.

17) Balcar and Franěk have shown that every infinite complete BA A has a free subalgebra B with $|A| = |B|$. It follows easily that no infinite complete BA is onto-rigid.

18) Shelah (unpublished) has shown under GCH that for every infinite regular $\lambda \neq \aleph_2$ there is a $(<\lambda)$-complete onto-rigid BA of power λ.

19) Shelah (unpublished) has shown that for every regular $\lambda \geq \aleph_0$ there is a complete mono-rigid BA of power $\lambda^{\aleph_0}$.

These results lead to the following questions.

QUESTION 10. Is there an absolutely rigid BA?

Note by 6) that such an algebra must have power at least $2^{\aleph_1}$.

QUESTION 11. Under CH, or in ZFC, is there a Bonnet-rigid, Shelah-rigid BA?

QUESTION 12. Do Shelah-rigid BA's exist in all cardinalities $\geq 2^{\aleph_0}$?

QUESTION 13. Is it consistent to have a Shelah-rigid BA of power $< 2^{\aleph_0}$?*

QUESTION 14. Is there (under any set-theoretic assumption) a Shelah rigid BA with few endomorphisms?

Again note by 6) that such an algebra must have at least $2^{\aleph_1}$ elements.

QUESTION 15. In what cardinalities do there exist endo-rigid BA's?

QUESTION 16. Under any set-theoretical assumptions, in what limit cardinalities do there exist Bonnet-rigid BA's?**

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* See note 3 added in proof.

** See note 2 added in proof.

QUESTION 17. Do mono-rigid BA's exist in all singular cardinalities?

See Remark 14) concerning this question.

QUESTION 18. Under GCH, is there a σ-complete onto-rigid BA of power $\aleph_2$?

See Remark 18).

3. Hopfian BA's

A BA is *Hopfian* if every onto endomorphism is one-to-one; it is *dual Hopfian* if every one-to-one endomorphism is onto. These concepts were studied by Loats [L], who showed: (1) if $\kappa < 2^{\aleph_0}$ and MA_κ, then no BA of power κ with infinitely many atoms is Hopfian or dual-Hopfian; (2) there are no denumerable Hopfian or dual-Hopfian BA's; (3) there is an atomic Hopfian BA of power $2^{\aleph_0}$ with denumerable automorphism group. The algebra B in van Douwen [vD4] is Hopfian, while $|B| = |\text{Aut } B| = 2^{\aleph_0}$, $\mathcal{S}B$ is homogeneous, and B is atomless. Some of the above questions and results concerning rigid BA's are relevant to these notions also.

QUESTION 19. Assume $\aleph_0 \le \lambda \le 2^\kappa$, $\kappa > \aleph_0$. Is there a Hopfian BA of power κ with automorphism group of power λ?

4. Chains, antichains, irredundant sets and subalgebras

A BA is λ-*narrow* if it does not contain an antichain (i.e. a set of pairwise incomparable elements) of power λ; it is λ-*short* if it does not contain a chain of power λ; and it is λ-*concentrated* if it is λ-narrow and λ-short. We call A simply *narrow* or *concentrated*, if it is $|A|$-narrow, resp. $|A|$-concentrated.

A model of the form (B, P), where B is a BA and P is a subset of B which generates B, is called a *configuration*. Let $L = (B_0, P)$ be a configuration, and let A be a subset of a BA B. We say that A *admits* L if there is an $A' \subseteq A$ such that $L \cong (\text{cl}(A'), A')$. (Here $\text{cl}(A')$ denotes the subalgebra of A generated by A'.) For B a BA and L a configuration we say that B is L-*admitting* if every subset of B of cardinality $|B|$ admits L.

EXAMPLE. Let $L_0 = (B_0, \{a, b, c\})$ be the configuration in which $c \ne a \ne b \ne c = a \wedge b \ne 0$ and $a \vee b \ne 1$. A BA B of power λ is L_0-admitting iff for every subset A of B of power λ there are three distinct elements $a, b, c \in A$ such that $a \wedge b = c$ and $a \vee b \ne 1$. Note that if B is L_0-admitting, then B is $|B|$-concentrated.

B is λ-*redundant* if every subset of B of power λ is redundant. (A subset A of B is *redundant* if for some $a \in A$, $a \in \mathrm{cl}\,(A - \{a\})$). Note that if B is L_0-admitting then B is $|B|$-redundant.

A subset A of B is *nowhere dense* (nwd) if for every $n > 0$ and pairwise disjoint $a, b_1, \ldots, b_4 \in B$ such that $b_1, \ldots, b_n \neq 0$ there are $c_1, c_2 \in B$ such that $a \leq c_1 \leq c_2 \leq a \vee \bigvee_{i=1}^{n} b_i$, for every $i = 1, \ldots, n$: $(c_2 - c_1) \wedge b_i \neq 0$, and for every $b \in B$: $c_1 \leq b \leq c_2$ implies $b \notin A$. B is *strongly concentrated* (SC) if every nwd subset of B has cardinality $< |B|$.

Rubin [R3] divided the finite configurations into two classes:

$$\mathfrak{L}_1 = \left\{ (B_0, \{a_1, \ldots, a_k\}) \colon \bigwedge_{i=1}^{k} a_i \neq 0,\ \bigvee_{i=1}^{k} a_i \neq 1, \text{ and for every } 1 \leq j \leq k,\ a_j \notin \mathrm{cl}\,\{a_1, \ldots, a_{j-1}\} \right\},$$

and $\mathfrak{L}_2$ is the class of all finite configurations not belonging to $\mathfrak{L}_1$. Clearly if B is infinite and $L \in \mathfrak{L}_2$, then B is not L-admitting.

The following implications hold for every BA B.

$(P_{SC})B$ is SC$\Rightarrow(P_{\mathfrak{L}_1})$ for every $L \in \mathfrak{L}_1$, B is L-admitting

$$\Rightarrow (P_{L_0})B \text{ is } L_0\text{-admitting} \begin{cases} \Rightarrow (P_C)B \text{ is } |B|\text{-concentrated} \\ \Rightarrow (P_R)B \text{ is } |B|\text{-redundant} \end{cases}$$

The existence of BA's with the above properties was established under the following set-theoretic assumptions (mentioned in chronological order).

(1) (Bonnet [Bo1]) ($2^{\aleph_0}$ is regular). There is a $2^{\aleph_0}$-narrow BA of power $2^{\aleph_0}$.

(2) (Rubin [R3] ($\diamondsuit_{\aleph_1}$). There is an SC BA of power $\aleph_1$.

(3) (Shelah [S1] and, independently, van Wesep [vW]). It is consistent with ZFC that $\aleph_1 < 2^{\aleph_0}$ (in fact $2^{\aleph_0}$ can be arbitrary) and there is an $\aleph_1$-concentrated BA of power $2^{\aleph_0}$.

(4) (Shelah [S2]) (CH). There is an $\aleph_1$-concentrated BA of power $\aleph_1$.

(5) *Remarks.* (a) Berney and Nyikos [BN] proved (1) assuming CH, independently of Bonnet.

(b) (2), (3) and (4) were preceded by a result of Baumgartner and Komjath (independently) [BK] that $\diamondsuit_{\aleph_1}$ implies "there is an $\aleph_1$-concentrated BA of power $\aleph_1$."

(c) In fact by Shelah's omitting type theorem [S3], (2) extends to λ^+ whenever (A) λ is strongly inaccessible or $\diamondsuit_\lambda$ holds, and (B) $\diamondsuit_{\{\alpha < \lambda^+ : cf\alpha = \lambda\}}$ holds.

(6) The necessity of certain set-theoretic assumptions was proved by Baumgartner [B1], who showed that it is consistent with ZFC that every uncountable BA contains an uncountable antichain.

(7) Improving somewhat Bonnet's proofs, Shelah (unpublished) showed there are narrow BA's of power cf $2^{\aleph_0}$ and $2^{\aleph_0}$.

(8) Devlin [D] proved that if κ is real-valued measurable then every algebra with countably many operations and κ elements has a κ-irredundant subset of power κ. Since Con (ZFC+there is a measurable cardinal)$\Leftrightarrow$Con (ZFC+$2^{\aleph_0}$ is real-valued measurable), by Solovay, it follows that Con (ZFC+there is a measurable cardinal) $\Rightarrow$ Con (ZFC+every BA of power $2^{\aleph_0}$ has a $2^{\aleph_0}$-irredundant set of power $2^{\aleph_0}$).

(9) Shelah [S7] has shown that it is consistent to have $2^{\aleph_0} > \aleph_{\omega_1}$ and every algebra of power $\aleph_{\omega_1}$ with $< \aleph_{\omega_1}$ operations has an irredundant set of power $\aleph_1$.

Certain implications between the above properties have been shown not to hold.

(10) (Shelah) (CH). There is an $\aleph_1$-concentrated BA of power $\aleph_1$ which is not $\aleph_1$-redundant. (This is the BA in (4) above.)

(11) (Shelah, unpublished) ($\diamondsuit_{\aleph_1}$). There is a BA B of power $\aleph_1$ such that B has $P_{\mathfrak{L}_1}$ but B is not SC.

(12) (Shelah, unpublished) (MA+$\aleph_1 < 2^{\aleph_0}$). There are no SC BA's of power $\aleph_1$. In fact, there is no BA B of power $\aleph_1$ with the property that B has a countable set of atoms and for every dense ideal $I \subseteq B$, B/I is countable.

(13) Rubin (unpublished) has shown that it is consistent to have a BA A of power $\aleph_1$ which is redundant but not concentrated.

(14) Rubin (unpublished) has shown that it is consistent to have a BA of power $\aleph_1$ which is redundant and concentrated, but is not L-admitting for any $L \in \mathfrak{L}_1$.

(15) *Remarks.* In an attempt to strengthen Baumgartner's result (6) we may ask whether every uncountable BA has either (1) an uncountable subset of pairwise disjoint elements, or (2) an uncountable chain, or (3) an uncountable independent subset (or whether it is consistent that this be true); the general question for an arbitrary cardinal in place of $\aleph_1$ was raised by R. Lawson. Concerning this conjecture we make the following remarks.

(a) Let $B \subseteq \mathbf{S}\omega$ be the BA generated by $\{\{n\}: n < \omega\} \cup \{A_i: i < \aleph_1\}$, where the A_1 are infinite pairwise almost disjoint sets. Then B fails to satisfy (1), (2), (3).

(b) A concentrated BA fails to have (1), (2), or (3).

(c) Monk has proved in ZFC that for any limit ordinal δ, if B_α is the BA of finite and cofinite subsets of $\beth_\alpha$ for each $\alpha < \delta$ and $B = \prod_{\alpha<\delta} B_\alpha$ (the full direct product), then every chain in B has power $\leq \beth_1$, every independent subset of B has power $\leq \beth_3$, and every subset of B consisting of pairwise disjoint elements has power $\leq \beth_\delta$. This answers the above question for $\prod_{\alpha<\delta} \beth_\alpha$.

(d) Improving results of Efimov [E3], Shelah [S1] proved that if $|B| = (2^{\aleph_0})^+$

and B satisfies the $\aleph_1$-chain condition (i.e., every set of pairwise disjoint elements has power $<\aleph_1$), then B has an independent subset of power $(2^{\aleph_0})^+$. (In fact Shelah gives the best possible results dealing with the κ-chain condition, for any κ.)*

(16) Baumgartner noted that if B does not contain uncountable anti-chains and $|B \restriction a|>\aleph_0$ for all non-zero $a \in B$, then B is rigid.

The above state of affairs gives rise to the following questions.

QUESTION 20. Does CH imply the existence of an SC, or $\mathfrak{L}_1$-admitting, or L_0-admitting, or $\aleph_1$-redundant BA of power $\aleph_1$?

QUESTION 21. Does ZFC imply the existence of an $\aleph_1$-redundant BA of power $\aleph_1$?

QUESTION 22. Does ZFC imply the existence of a $2^{\aleph_0}$-concentrated or an $\mathfrak{L}_1$-admitting or an SC BA of power $2^{\aleph_0}$?

Along the lines of (13), (14) we ask

QUESTION 23. Is there a model of ZFC in which there is a concentrated BA of power $\aleph_1$, but no strongly concentrated BA of power $\aleph_1$?

Along the lines of Shelah [S1] (see (3)) we ask

QUESTION 24. Is it consistent with ZFC that $\aleph_1 < 2^{\aleph_0}$ and there is a BA of power $2^{\aleph_0}$ which is $\aleph_1$-redundant?

Remark. Shelah (see [R3]) showed that if $|B|>\aleph_1$ then B contains an uncountable set with no three distinct elements $a, b, a \wedge b$. See also (19) below.

The known results for higher cardinals are less detailed. Let us survey them.

(17) (Bonnet [Bo1] ($\kappa^+=2^\kappa$). There is a κ^+-narrow BA of power κ^+.

(18) (Shelah) SC BA's of power λ^+ exist under appropriate set-theoretic assumptions. (See remark (5c) above.)

(19) (McKenzie, unpublished). The Boolean algebra generated by any maximal irredundant subset of B is dense in B. Hence if $|B|$ is strong limit then B is not $|B|$-redundant.

(20) (Arhangel'skiĭ [Ar]). If X is a topological space of pointwise countable type, in particular, if X is compact then the weight of X is $\leq 2^{\text{spread of } X}$. Since

* See note 4 added in proof.

spread is attained at singular strong limit cardinals for Hausdorff spaces, it follows easily that if $|B|$ is singular strong limit then B is not $|B|$-narrow. Trivially the same is true when $|B|$ is weakly compact. However, by Shelah's theorem (3) this result does not extend to arbitrary singular cardinals. E.g., in (3) make $\aleph_\omega < 2^{\aleph_0}$.

(21) (Baumgartner [BK]). If λ is regular and B is λ-narrow, then B has a dense subset of power $<\lambda$; hence λ is not strongly inaccessible. Thus for $|B|$ strongly inaccessible, B is not $|B|$-narrow. If B is cardinality homogeneous, i.e., $|B| = |B \restriction b|$ for all $0 \neq b \in b$, and $|B|$-narrow, then B is rigid.

(22) (Shelah, unpublished.) For any λ with cf $\lambda > \aleph_0$ it is consistent that there is a BA B which is λ-narrow but not μ-narrow for any $\mu < \lambda$.

(23) (Shelah, unpublished.) If B has no dense subset of power $<\lambda$ then B has an irredundant subset of power λ consisting of pairwise incomparable elements.

(24) (Shelah, unpublished) (GCH). For each $\lambda \geq \aleph_0$ there is a λ^+-concentrated BA of power λ^+; moreover, every BA of power $\leq \lambda$ can be embedded in it, and is a homomorphic image of it.

(25) (Shelah, unpublished) (GCH). For each $\lambda \geq \aleph_0$ with $\lambda \neq \aleph_1$ there is a $(<\lambda)$-complete concentrated BA of power λ^+.

(26) (Rubin [R3]). If B is a subalgebra of an interval algebra and $|B|$ is regular, then B is not concentrated.

QUESTION 25. If λ is singular with cf $\lambda = \omega$, or if λ is weakly inaccessible but not strongly inaccessible, is there a λ-narrow BA which fails to be κ-narrow for all $\kappa < \lambda$?

Note that this question related to well-known problems concerning attainment of spread in topological spaces; also see (22).

QUESTION 26. Does Baumgartner's independence result (6) generalize to cardinals $>\aleph_1$?

QUESTION 27. Is Rubin's theorem (25) true without assuming that $|B|$ is regular? (Note that one has to deal with the case when $|B|$ is singular but not strong limit.)

QUESTION 28. Under GCH is there a σ-complete concentrated BA of power $\aleph_2$?

We turn now to subalgebras

An infinite BA B is *almost Jónsson* (AJ) if for every subset C of B either (1)

$|C|<|B|$ or (2) there is $D \subseteq B$ such that $|D|<|B|$ and $C \cup D$ generates B. (Note that a BA cannot be a Jónsson algebra.)

An infinite BA B is *packed* if for every two subalgebras B_1 and B_2 of G, if $|B_1|=|B_2|=|B|$, then $|B_1 \cap B_2|=|B|$.

Let λ be an infinite cardinal. B is λ-*like* if $|B|=\lambda$ and $\{a \in B: |\{b \in B: b \leq a\}|<\lambda\}$ is a prime ideal in B.

Note that if B is AJ or packed then B is $|B|$-like, and if B is *AJ*, then B is $|B|$-redundant.

(27) (Rubin [R3]) ($\diamondsuit_{\aleph_1}$). There is a packed AJ BA of power $\aleph_1$. This result also transfers to λ^+ under the same set-theoretic assumptions of Remark (5) (c).

(28) *Remark.* (27) was preceded by a theorem of Baumgartner and Komjath [BK] who constructed assuming $\diamondsuit_{\aleph_1}$ and $\aleph_1$-like $\aleph_1$-concentrated BA.

QUESTION 29. (a) Do AJ or packed BA's of power $\aleph_1$ or any other power exist assuming ZFC, CH or GCH?*

(b) Does the fact that B is packed imply that B is $|B|$-redundant? Does B $|B|$-like and $|B|$-redundant imply that B is packed?

(c) Construct a BA (under any set-theoretic assumption) which is packed and not AJ, and vice versa.

(29) (Bonnet [Bo3]) (GCH). There is a family of $\aleph_2$ Bonnet-rigid BA's of power $\aleph_1$, such that no two of them have a common uncountable subalgebra or homomorphic image. (An assumption somewhat weaker than GCH is actually all that is required.

(30) *Remark.* Kunen has pointed out that Shelah's result (15) (c) implies that certain improvements of (29) are impossible. Namely, under GCH it is true that, given any three BA's of power $\aleph_2$, at least two of them have a common uncountable subalgebra; but there are two BA's of power $\aleph_2$ without a common uncountable subalgebra.

The following question seems to us to be related to the topics of this section See [R3].

QUESTION 30. Let T be the set of all sentences in the Magidor–Malitz language L^2 which are true in every BA (in the $\aleph_1$-interpretation). Does it follow from ZFC that T is undecidable?

(31) M. Weese [W1] proved that the L^1-theory of BA's in the λ interpretation is decidable, for any λ.

(32) Rubin [R3] showed that, assuming CH, T is undecidable.

* See note 5 added in proof.

5. Free algebras

Various results are known concerning the freeness of subalgebras of free BA's.

(1) (Efimov [E2], Monk, [M3]). Every uncountable subset of a free BA has an uncountable independent subset. (*Independent* means freely generates the subalgebra it generates.)

(2) No uncountable subalgebra of a free BA has a countable dense subalgebra.

(3) (Rubin, unpublished). Let X be a family of $\aleph_1$ independent subsets of ω, and let A be the BA of subsets of ω generated by X together with the finite subsets of ω. Then every uncountable subset of A has an uncountable independent subset, but A cannot be embedded in a free BA.

(4) (Efimov and Kuznetzov [EK]). Every uncountable free BA A has $2^{|A|}$ isomorphism types of dense subalgebras, each with a factor isomorphic to A.

(5) Pašenkov [P] provides a BA A of any power 2^κ, $\kappa \geq \aleph_1$, which has homogeneous Stone space, can be embedded in a free BA, but is not itself free. Van Douwen (unpublished) showed that A is homogeneous as well.

(6) Van Douwen (unpublished) has shown that for $\kappa > \omega$ the free BA on κ generators has 2^κ totally different dense rigid subalgebras of cardinality κ (totally different means no non-trivial isomorphic factors).

(7) A BA A is *projective* if it is a retract of a free BA, i.e., if A is a subalgebra of a free BA B such that there is a homomorphism f from B onto A with $fa = a$ for all $a \in A$. There is no nice characterisation of projective BA's; see Koppelberg [Ko6], [Ko7]. The rigid algebras in 6) are not projective, and the same applies to Pašenkov's algebras.

(8) Engelking [En] has shown that every projective BA satisfies the following condition:

(*) for every two ideals I, I' with $I \cap I' = \{0\}$ there are σ-generated ideals J, J' with $I \subseteq J$, $I' \subseteq J'$ and $J \cap J' = \{0\}$.

(9) Balcar and Franěk (unpublished) have shown that every infinite complete BA B has a free subalgebra A with $|A| = |B|$. Earlier partial results were obtained by Keslyakov, Koppelberg and Monk.

(10) Shelah (unpublished), using the methods of [S6], has shown that the class of free algebras is first-order characterizable in the language of BA's with quantification over endomorphisms.

QUESTION 31. Does every uncountable projective BA have a free factor?*

* See note 6 added in proof.

QUESTION 32. Does there exist a rigid projective BA?**

QUESTION 33. Is a BA projective iff it is a subalgebra of a free BA and satisfies 8(*) above?*

QUESTION 34. Is it true that the class of subalgebras of free BA's is an elementary class in the language which permits quantification over ideals, subalgebras and endomorphisms?

6. Interval algebras and retractiveness

If $\langle L, <\rangle$ is a linear ordering, let $B(L)$ be the interval algebra created from L. A BA B is *retractive* if for every ideal I in B there is a subalgebra B' of B such that $|b/I \cap B'| = 1$ for every $b \in B'$.

(1) S. Koppelberg (unpublished) has shown that the following conditions are equivalent:(a) $\operatorname{cf} \kappa > \aleph_0$, (b) if $P \subseteq B(L)$ and $|P| = \kappa$, then there is a $Q \subseteq P$ and $n \in \omega$ such that $|Q| = |P|$ and every n-element subset of Q is dependent.

Rotman [Ro] investigated the relationship between retractive BA's and interval algebras. He also raised the question whether a free product of retractive BA's is retractive.

Let B_0 denote the BA of finite and cofinite subsets of ω. Results 2(a), (b), (d) appear in [R3].

(2) (a) (Rubin). If B is embeddable in an interval algebra, then B is retractive. (b) (Rubin). If B is an SC BA of power $\aleph_1$, then B is retractive, but every subalgebra of B which is embeddable in an interval algebra is countable. (c) The concentrated BA B of power $\aleph_1$ constructed under CH by Shelah [S2] has the property that $|B/I| \leq \aleph_0$ for every dense ideal I. Thus B also satisfies the conclusion of (b). (d) (Rubin). If $\langle S, <\rangle$ is a Suslin ordering (i.e., every family of pairwise disjoint non-trivial open intervals is countable, S is dense, and S is not embeddable in $\mathfrak{R}$), then $B_0 * B(S)$ is retractive ($*$ denotes free product). (e) (Laver, Rubin) (MA). There is a subset L of $\mathfrak{R}$ of power $2^{\aleph_0}$ such that $B_0 * B(L)$ is retractive. (f) (Rotman [Ro]). If $|B_1| \geq \aleph_0$ and $|B_2| \geq \aleph_1$, then $B_1 * B_2$ is not embeddable in an interval algebra. (g) (van Douwen). If A and B are infinite BA's and $A * B$ is retractive, then $\mathcal{S}A$ and $\mathcal{S}B$ have countable spread. In particular (using Arhangel'skiĭ's result above), if $|B_1| > 2^{\aleph_0}$ or $|B_2| > 2^{\aleph_0}$ then $B_1 * B_2$ is not retractive. (h) (van Douwen). If B contains an uncountable independent set, then B is

* See note 8 added in proof.

** See note 7 added in proof.

not retractive. (i) (van Douwen). If $\langle S, <\rangle$ is a Souslin line, then $B(S)*B(S)$ is not retractive. (j) (van Douwen). $B*B*B$ is retractive iff B is countable. (k) It follows from a result of Nyikos that $MA+\neg CH$ implies $B*B$ retractive iff B is countable. (l) Nyikos observed that "all $\aleph_1$-dense sets of reals are isomorphic" implies that for all uncountable $L_1, L_2 \subseteq R$ the algebra $B(L_1)*B(L_2)$ is not retractive. The hypothesis is consistent with $MA+\neg CH$ by Baumgartner [B2], and its negation is consistent with $MA+\neg CH$ by Shelah [S8].

For some more details see [R3] and [Ro]. These facts give rise to several vague questions. Is there a nice characterization of the class of all retractive BA's? (We suspect that no such characterization exists.) Is there a nice condition on B_1 and B_2 which is equivalent to the fact that $B_1 * B_2$ is retractive? In particular, is there such a condition when B_1 and B_2 are interval algebras, or when B_1 is countable, etc.?

Here are some definite questions.

QUESTION 35. Construct in ZFC a retractive BA not embeddable in an interval algebra.*

QUESTION 36. Is it true that a subalgebra of a retractive BA is retractive?

Note that a weak direct product of retractive BA's is retractive.

QUESTION 37. Construct (under any set-theoretic assumptions) a retractive BA B such that B is not embeddable in an interval algebra, but for every $b \in B-\{0\}$, $|\{a \in B: a \leq b\}| > 2^{\aleph_0}$.

QUESTION 38. Are there two uncountable BA's whose free product is retractive?

7. The countable separation property

(See section 1 for the definition of the countable separation property.) Some important facts concerning CSP are as follows. Every σ-complete BA has CSP, and every homomorphic image of a CSP BA has CSP. S. Koppelberg [Ko1] showed that if A has CSP then $|A|^{\aleph_0} = |A|$. A. Louveau [Lo] showed under CH that if A is a CSP BA of power $2^{\aleph_0}$, then A is a homomorphic image of a complete BA. Van Douwen and van Mill [vDvM] showed under $MA+2^{\aleph_0}=\aleph_2$ that there is a CSP BA of power $2^{\aleph_0}$ that is not a homomorphic image of any countably complete BA; hence under this assumption, for each $\kappa \geq 2^{\aleph_0}$ with

* See note 12 added in proof.

$\kappa^{\aleph_0} = \kappa$ there is such a CSP BA of power κ. Kunen, Shelah, Balcar and Simon showed that the countable-cocountable BA on $\kappa > \aleph_0$ is a homomorphic image of the completion of the free BA on κ generators. Other results on CSP algebras are found in van Douwen [vD2].

QUESTION 39. Is it true in ZFC that there is a CSP BA which is not a homomorphic image of a σ-complete BA?

QUESTION 40. Is every σ-complete BA a homomorphic image of a complete BA?

8. Cofinality of BA's

For any infinite BA A, let $\operatorname{cf} A$ (the *cofinality* of A) be the least infinite cardinal κ such that there is a strictly increasing sequence of type κ of subalgebras of A with union A. This notion has been extensively studied by S. Koppelberg [Ko2], where it is shown (1) $\operatorname{cf} A$ is regular $\leq 2^{\aleph_0}$ for all A; (2) $\operatorname{cf} A = \aleph_1$ for every CSP algebra A; (3) $\operatorname{cf} A \leq \aleph_1$ if A has an uncountable free subalgebra; (4) $\operatorname{cf} A = \aleph_0$ if A has no infinite free subalgebra (i.e., if A is hereditarily atomic (= superatomic)); (5) $\operatorname{cf} A = \aleph_0$ if MA and $|A| < 2^{\aleph_0}$. M. Hušek (unpublished) has shown that every infinite compact Hausdorff space has either a nontrivial convergent sequence or a nontrivial convergent ω_1-net; hence $\operatorname{cf} A \leq \aleph_1$ for any BA A.*

There is an interesting question concerning cofinality of products. Let A_i be a BA with $\operatorname{cf} A_i \geq \aleph_1$ for all $i \in I$, where I is an infinite index set. By (3) above, $\operatorname{cf} \prod_{i \in I} A_i \leq \aleph_1$. Loats [L] showed that $\operatorname{cf} \prod_{i \in I} A_i = \aleph_1$ if $I = \omega$, and more generally McKenzie (unpublished) showed that if $\operatorname{cf} \prod_{i \in I} A_i = \aleph_0$, then there is an uncountable measurable cardinal $\leq |I|$.

QUESTION 41. If $\operatorname{cf} A_i \geq \aleph_1$ for all $i \in I$, I infinite, is $\operatorname{cf} \prod_{i \in I} A_i = \aleph_1$?

There are three more cardinal functions closely related to the function cf (the first two, and some of the comments below, are due to van Douwen):

altitude:	$a(B) = \min\{\kappa \geq \aleph_0$: some ultrafilter on B is a union of a strictly increasing chain of filters of type $\kappa\}$.
pseudoaltitude:	$pa(B) = \min\{\kappa \geq \aleph_0$: some ultrafilter in some homomorphic image of B is κ-generated but is not $(<\kappa)$-generated$\}$.

* See note 9 added in proof.

homomorphism type: $h(B) = \min\{|C|: C$ is an infinite homomorphic image of $B\}$.

The following facts are easily established using Koppelberg [Ko2].
(6) $\mathrm{cf}\,(B) \le a(B) \le pa(B) \le h(B) \le 2^{\aleph_0}$; (7) if B has an uncountable free subalgebra then $a(B) \le \aleph_1$; (8) if B is CSP then $\mathrm{cf}\,(B) = a(B) = pa(B) = \aleph_1 \le 2^{\aleph_0} = h(B)$; (9) there is a BA B with $\mathrm{cf}\,(B) = \aleph_0$, $a(B) = pa(B) = \aleph_1$, $h(B) = 2^{\aleph_0}$; (10) MA and $|A| < 2^{\aleph_0}$ imply $h(A) = \aleph_0$; (11) if $a(B) = \aleph_0$, then $pa(B) = \aleph_0$. By the result of Hušek mentioned above we have $a(B) \le \aleph_1$ for every infinite BA B.*

QUESTION 42. Is $pa(B)$ regular?

QUESTION 43. If $\aleph_1 < \kappa \le 2^{\aleph_0}$ is there a BA with $pa(B) = \kappa$ $(h(B) = \kappa)$?

QUESTION 44. Is $a(B) = pa(B)$?

9. Almost-disjoint BA's

For $\kappa \le \lambda^+$, let $\mathbf{S}'_{<\kappa}\lambda$ be the BA of subsets of λ of power $<\kappa$ and their complements in λ. Thus for $\kappa = \lambda^+$, $\mathbf{S}'_{<\kappa}\lambda$ is the BA of all subsets of λ. Let $\mathbf{S}_{<\kappa}\lambda$ be the collection of all subsets of λ of power $<\kappa$. We raise the general question concerning possible isomorphisms among the algebras $\mathbf{S}'_{<\kappa}\lambda/\mathbf{S}_{<\mu}\lambda$.

QUESTION 45. Does $\mathbf{S}'_{<\kappa}\lambda/\mathbf{S}_{<\mu}\lambda \cong \mathbf{S}'_{<\kappa'}\lambda'/\mathbf{S}_{<\mu'}\lambda'$ imply $\kappa = \kappa'$, $\lambda = \lambda'$ and $\mu = \mu'$?

To fix the ideas let us take $\kappa = \lambda^+$ and either $\mu = \aleph_0$ or $\mu = \lambda$. First consider the algebras $\mathbf{S}\lambda/\mathbf{S}_{<\aleph_0}\lambda$, which have been extensively studied in their topological form. They are, of course, all non-isomorphic assuming GCH. R. Frankiewicz [F] has shown that if $\mathbf{S}\omega/\mathbf{S}_{<\omega} \not\cong \mathbf{S}\omega_1/\mathbf{S}_{<\omega}\omega_1$, then $\mathbf{S}\lambda/\mathbf{S}_{<\omega}\lambda \not\cong \mathbf{S}\kappa/\mathbf{S}_{<\omega}\kappa$ for all distinct κ, λ. He also showed that $\mathbf{S}\omega/\mathbf{S}_{<\omega}\omega \not\cong \mathbf{S}\omega_1/\mathbf{S}_{<\omega}\omega_1$ under MA. Van Douwen (see Comfort [C2]) showed that $\mathbf{S}\omega/\mathbf{S}_{<\omega}\omega \not\cong \mathbf{S}\omega_1/\mathbf{S}_{<\omega}\omega_1$ iff $\mathbf{S}\omega_1/\mathbf{S}_{<\omega}\omega_1$ is *not* homogeneous (note that $\mathbf{S}\omega/\mathbf{S}_{<\omega}\omega$ *is* homogeneous). Balcar and Frankiewicz [BF] have shown that if $\kappa, \lambda > \aleph_0$ and $\kappa \ne \lambda$ then $\mathbf{S}\kappa/\mathbf{S}_{<\omega}\kappa \not\cong \mathbf{S}\lambda/\mathbf{S}_{<\omega}\lambda$.

QUESTION 46. Show that $\mathbf{S}\omega/\mathbf{S}_{<\omega}\omega \not\cong \mathbf{S}\omega_1/\mathbf{S}_{<\omega}\omega_1$ in ZFC.

Now consider the algebra $\mathbf{S}\lambda/\mathbf{S}_{<\lambda}\lambda$. It is easily seen to be $(<\mathrm{cf}\,\lambda)$-complete, but not $(\le \mathrm{cf}\,\lambda)$-complete. Hence $\mathbf{S}\kappa/\mathbf{S}_{<\kappa}\kappa \not\cong \mathbf{S}\lambda/\mathbf{S}_{<\lambda}\lambda$ if $\mathrm{cf}\,\kappa \ne \mathrm{cf}\,\lambda$.

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* See note 9 added in proof.

QUESTION 47. Show in ZFC that $\mathbf{S}\kappa/\mathbf{S}_{<\kappa}\kappa \not\cong \mathbf{S}\lambda/\mathbf{S}_{<\lambda}\lambda$ for $\operatorname{cf}\kappa = \operatorname{cf}\lambda$ but $\kappa \neq \lambda$; in particular, that $\mathbf{S}\omega/\mathbf{S}_{<\omega}\omega \not\cong \mathbf{S}\aleph_\omega/\mathbf{S}_{<\aleph_\omega}\aleph_\omega$.

We would also like to mention a problem of Erdös concerning possible isomorphism of certain algebras $\mathbf{S}\omega/I$. First it is perhaps relevant to note that there are $2^{2^{\aleph_0}}$ isomorphism types of algebras of the form $\mathbf{S}\omega/I$, where I is an ideal including $\mathbf{S}_{<\omega}\omega$. In fact, any complete BA of power $2^{\aleph_0}$ is isomorphic to some such algebra, so this is a consequence of [MS]. Also, for background we note that if

$$I_0 = \left\{A \subseteq \omega : \sum_{n \in A} \frac{1}{n+1} < \infty\right\},$$

then I_0 is an ideal including $\mathbf{S}_{<\omega}\omega$ and $\mathbf{S}\omega/I_0 \cong \mathbf{S}\omega/\mathbf{S}_{<\omega}\omega$ under CH (a fact noticed by Erdös and Monk). now consider the following two ideals: I_1 is the ideal of $A \subseteq \omega$ of density 0, and I_2 is the ideal of $A \subseteq \omega$ of logarithmic density 0. (A has *density* 0 if

$$\lim \frac{|\{a \in A : a \leq n\}|}{n} = 0,$$

while A has *logarithmic density* 0 if

$$\lim \left(\sum_{a \in A, a \leq n} \frac{1}{a+1}\right) \cdot \frac{1}{\log n} = 0.)$$

It is easily seen that both $\mathbf{S}\omega/I_1$ and $\mathbf{S}\omega/I_2$ are non-isomorphic to $\mathbf{S}\omega/\mathbf{S}_{<\omega}\omega$.

QUESTION 48 (*Erdös*). (CH) Is $\mathbf{S}\omega/I_1 \cong \mathbf{S}\omega I_2$?

10. Cardinal sequences

For any BA A, let $I(A)$ be the ideal generated by the atoms of A. Now we iterate the formation of $I(A)$:

$I_A^0 = \{0\}$;

having defined I_A^α to be an ideal of A, we let

$I_A^{\alpha+1}$ = the ideal of A generated by
$I_A^\alpha \cup \{a \in A : a/I_A^\alpha$ is an atom of $A/I_A^\alpha\}$;

and for λ a limit ordinal, with I_A^α defined for all $\alpha<\lambda$, we let

$$I_A^\lambda=\bigcup_{\alpha<\lambda} I_A^\alpha.$$

We let $\delta(A)$ be the least α such that $I_A^\alpha=I_A^{\alpha+1}$. The *cardinal sequence of* A is the sequence $\langle\kappa_\alpha:\alpha<\delta(A)\rangle$, where for each $\alpha<\delta(A)$, κ_α is the number of atoms of A/I_A^α. The central question here is:

QUESTION 49. Describe, in cardinality terms, the possible cardinal sequences of Boolean algebras.

We describe some partial results obtained on this problem. If $I_A^{\delta(A)}=A$, then $\delta(A)$ is a successor ordinal $\gamma+1$, the last term of the cardinal sequence is finite and non-zero, and A is superatomic. If $I_A^{\delta(A)}\neq A$, then A is not superatomic, and the last term of the cardinal sequence, if it exists, is non-zero. Lagrange [La] completely solved question 49 for a countable sequence $\kappa=\langle\kappa_\alpha:\alpha<\delta\rangle$, $\delta<\omega_1$:

(1) κ is the cardinal sequence of some non-trivial superatomic BA iff

(a) $\delta=\gamma+1$ for some γ,
(b) $0<\kappa_\gamma<\omega$,
(c) if $\alpha<\gamma$, then $\kappa_\alpha\geq\aleph_0$,
(d) $\alpha<\beta<\gamma$ implies $\kappa_\beta\leq\kappa_\alpha$.

(2) κ is the cardinal sequence of some non-superatomic BA iff

(a) if $\delta=\gamma+1$ for some γ, then $\kappa_\gamma\neq 0$ and $\kappa_\alpha\geq\aleph_0$ for all $\alpha<\gamma$,
(b) if δ is a limit ordinal, then $\kappa_\alpha\geq\aleph_0$ for all $\alpha<\delta$,
(c) $\alpha<\beta<\delta$ implies $\kappa_\beta\leq\kappa_\alpha^{\aleph_0}$.

Weese (unpublished) has extended (1) to give a full description of cardinal sequences of length $<\omega_2$ of superatomic BA's.

Added in proof (September 1979)

1. Shelah has shown that if λ is a strong limit of cofinality $\aleph_0$ and $|\mathrm{Aut}\, B|>|B|=\lambda$, then $|\mathrm{Aut}\, B|=2^\lambda$; and he has shown that if V is a model of GCH and $\lambda^{\Delta}=\lambda$, $\lambda<\mu$ in V, and if V' is formed from V by adding μ generic subsets of λ by λ-closed conditions, then in V', if B is a BA with $|B|<\mu$ and $|\mathrm{Aut}\, B|>|B|^+$, then $|\mathrm{Aut}\, B|\geq\mu$.

2. Bonnet-rigid BA's of regular uncountable powers were also constructed independently by Todorčevič [T], who also showed assuming $V=L$ that they exist in all singular cardinals.
3. Shelah suggests the following possibility of answering Question 13. Start say with $V=L$ and force a Shelah-rigid BA by conditions which are countable BA's together with some countably many types to be omitted. This results in a Shelah-rigid generic BA. Then blow up the continuum by adding $\aleph_2$ Sacks reals. This should not harm the Shelah-rigidity of the generic BA.
4. Another possible way to generalize Baumgartner's result (6) in section 4 is: is it consistent with ZFC that every uncounttable BA has either an uncountable chain or a non c.c.c. quotient?
5. Rubin, assuming $MA+\aleph_1<2^{\aleph_0}$, showed that there is no packed BA of power $\aleph_1$.
6. Efimov (Sov. Math. Dokl. 10 (1969), 776–779) has shown that for each singular κ there is a projective BA of cardinality κ that has no free factor; this BA is not rigid.
7. A recent result of Ščepin easily implies that any projective BA A with $|A|$ regular has a free factor of power $|A|$, and hence is not rigid.
8. Ščepin has shown that the answer to Question 33 is yes for BA's of power $\aleph_1$, but no in general.
9. Hušek has withdrawn his claim about convergent sequences or ω_1-nets, so it remains open whether cfA$\leq\aleph_1$ or $a(A)\leq\aleph_1$ in general.
10. The second part of this question has been answered affirmatively by S. Kemmerich and M. M. Richter (unpublished).
11. The results 6) and 7) concerning endo-rigid BA's follow respectively from results of S. P. Franklin (Proc. A.M.S. 21 (1969), 597–599) and Malyhin and Sapirovskii (Sov. Math. Dokl. 14 (1973), 1946–1751).
12. Question 35 has been answered; Gurevich, Magidor and Rubin have made the desired construction.

REFERENCES

[A] R. D. Anderson, *The algebraic simplicity of certain groups of homeomorphisms*, Amer. J. Math. *80* (1958), 955–963.

[Ar] A. V. Arhangel'skiĭ, *On bicompacta hereditarily satisfying Suslin's condition, tightness and free sequences*, Sov. Math. Dokl. *12* (1971), 1253–1257.

[BŠ] B. Balcar and P. Štěpánek, *Boolean matrices, subalgebras and automorphisms of complete Boolean algebras*, Fund. Math. *96* (1977), 211–223.

[B1] J. E. Baumgartner, *Chains and antichains in $P(\omega)$*.

[B2] J. E. Baumgartner, *All $\aleph_1$-dense sets of reals can be isomorphic*, Fund. Math. *79* (1973), 101–106.

[BF] B. Balcar and R. Frankiewicz, *To distinguish topologically the spaces m^**. II, Bull. de l'Acad. Pol. des Sci. *26* (1978), 521–523.

[BK] J. E. BAUMGARTNER and P. KOMJATH, *Boolean algebras in which every chain and every antichain is countable.*

[Bo1] R. BONNET, *Super-rigid Boolean algebras, narrow Boolean algebras.* Submitted to Israel J. Math.

[Bo2] R. BONNET, *Very strongly rigid Boolean algebras, continuum discrete set condition, countable antichain condition.* Submitted to Alg. Univ.

[Bo3] R. BONNET, *Algèbres m-complètes, m-super-rigides; algèbres completes et ∞-super-rigides.*

[BN] E. S. BERNEY and P. J. NYIKOS, *The length and breadth of Boolean algebras.*

[C1] W. W. COMFORT, *Ultrafilters: some old and some new results*, Bull. A.M.S. *83* (1977), 417–455.

[C2] W. W. COMFORT, *Compactifications: recent results from several countries*, Top. Proc. *2* (1977), 61–87.

[vD1] E. K. VAN DOUWEN, *Cardinal functions on compact F-spaces and on weakly countably complete Boolean algebras.*

[vD2] E. K. VAN DOUWEN, *Nonhomogeneity of products of preimages and π-weight*, Proc. A. M. S. *69* (1978), 183–192.

[vD3] E. K. VAN DOUWEN, *Doctoral Dissertation.*

[vD4] E. K. VAN DOUWEN, *A compact space with a measure that knows which sets are homeomorphic.*

[vD5] E. K. VAN DOUWEN, *A consistent very small Boolean algebra with countable automorphism group.*

[vDL] E. K. VAN DOUWEN and D. J. LUTZER, *On the classification of stationary sets*, Mich. Math. J. *26* (1979) 41–64.

[vDvM] E. K. VAN DOUWEN and J. VAN MILL, *Subspaces of basically disconnected spaces, or quotients of countably complete Boolean algebras*, Trans. A.M.S. (to appear).

[D] K. J. DEVLIN, *Some weak versions of large cardinal axioms*, Ann. Math. Logic *4* (1973), 291–326.

[E1] B. EFIMOV, *Dyadic bicompacta*, Trudy Mosk. Mat. *14* (1965), 211–247.

[E2] B. EFIMOV, *Mappings and embeddings of dyadic spaces*, Mat. Sb. *103* (1977), 52–68.

[E3] B. EFIMOV, *Extremally disconnected compact spaces and absolutes*, Trudy Mosk. Mat. *23* (1970), 235–276.

[EK] B. EFIMOV and V. KUZNETZOV, *On topological types of dyadic spaces*, Dokl. A.N. S.S.S.R. *195* 20–23.

[En] R. ENGELKING, *Cartesian products and dyadic spaces*, Fund. Math. *57* (1965), 287–304.

[F] R. FRANKIEWICZ, *To distinguish topologically the space m^**, Bull. Acad. Sci. Pol. *25* (1977). 891–893.

[J] T. J. JECH, *Simple complete Boolean algebras*, Israel J. Math. *18* (1974), 1–10.

[K] S. KESLYAKOV, *Free subalgebras of complete Boolean algebras and spaces of continuous functions*, Sib. Mat. Zh. *14* (1973), 569–581.

[Ko1] S. KOPPELBERG, *Homomorphic images of σ-complete Boolean algebras*, Proc. A.M.S. *51* (1975), 171–175.

[Ko2] S. KOPPELBERG, *Boolean algebras as unions of chains of subalgebras*, Alg. Univ. *7* (1977), 195–204.

[Ko3] S. KOPPELBERG, *Unabhängige Teilmengen von vollständigen Booleschen Algebren*, Habilitationsschrift, Univ. Bonn. 1973.

[Ko4] S. KOPPELBERG, *Cofinalities of complete Boolenana algebras*, Arch. f. Math. Logik u. Grundl., to appear.

[Ko5] S. KOPPELBERG, *A lattice structure on the isomorphism types of complete Boolean algebras.*

[Ko6] S. KOPPELBERG, *A problem of Halmos on projective Boolean algebras*, Colloq. Math. *25* (1972), 191–200.

[Ko7] S. KOPPELBERG, *Some classes of projective Boolean algebras*, Math. Ann. *201* (1973), 283–300.

[KR1] V. KANNAN and M. RAJAGOPALAN, *On rigidity and groups of homeomorphisms*, Proc. Third Top. Symp. Prag., Gen. Top. and its appl. to analysis and algebra (1973), 231–234.

[KR2] V. KANNAN and M. RAJAGOPALAN, *Constructions and applications of rigid spaces, III*, Canad. J. Math. *30* (1978) 926–932.

[Ku] C. KURATOWSKI, *Sur la puissance de l'ensemble des "nombres de dimensions" de M. Frechet*, Fund. Math. *8* (1926), 201–208.

[L] J. LOATS, *On endomorphism semigroups of Boolean algebras and other problems*, Ph.D. Thesis, Univ. of Colo. 1977.

[LR] J. LOATS, and M. RUBIN, *Boolean algebras without nontrivial onto endomorphisms exist in every uncountable cardinality*, to appear in Proc. A.M.S.

[La] R. LAGRANGE, *Concerning the cardinal sequence of a Boolean algebra*, Alg. Univ. *7* (1977), 307–313.

[Lo] A. LOUVEAU, *Caractérisation des sous-espaces compacts de βN*, Bull. Sci. Math. *97* (1973), 259–263.

[Mc] R. MCKENZIE, *Automorphism groups of denumerable Boolean algebras*, Can. J. Math. *29* (1977), 466–471.

[McM] R. MCKENZIE and J. D. MONK, *On automorphism groups of Boolean algebras*, Erdös symposium (1973), 951–988.

[M1] J. D. MONK, *A Boolean algebra with only definable endomorphisms*, submitted to Israel J. Math.

[M2] J. D. MONK, *On free subalgebras of complete Boolean algebras*, Arch. der Math. *29* (1977), 113–115.

[M3] J. D. MONK, *Independent subsets of subsets of free Boolean algebras*, Notices A.M.S. *25* (1978), A-3, 78T-A7.

[MR] J. D. MONK and W. RASSBACH, *The number of rigid Boolean algebras*, Alg. Univ. *9* (1979), 207–210.

[MS] J. D. MONK and R. M. SOLOVAY, *On the number of complete Boolean algebras*, Alg. Univ. *2* (1972), 365–368.

[P] V. PAŠENKOV, *Extensions of compact spaces*, Sov. Math. *15* (1974), 43–47.

[R1] M. RUBIN, *On the automorphism groups of homogeneous and saturated Boolean algebras*, to appear, Alg. Univ.

[R2] M. RUBIN, *On the reconstruction of Boolean algebras from their automorphism groups.*

[R3] M. RUBIN, *A Boolean algebra with few subalgebras, interval algebras and retractiveness*, submitted to Trans. A.M.S.

[Ro] B. ROTMAN, *Boolean algebras with ordered bases*, Fund. Math. *75* (1972), 187–197.

[S1] S. SHELAH, *Remarks on Boolean algebras*, submitted to Alg. Univ.

[S2] S. SHELAH, In preparation.

[S3] S. SHELAH, *Models with second-order properties* III, *omitting types in $L(Q)$*, to appear in Arch. für math. Logik und Grundl. der Math.

[S4] S. SHELAH, *Why there are many nonisomorphic models of unsuperstable theories*, Int. Congr. Math. Vancouver (1974), 259–264.

[S5] S. SHELAH, *Boolean algebras with few endomorphisms*, Proc. Am. Math. Soc. *74* (1979), 135–142.

[S6] S. SHELAH, *Interpreting set theory in the endomorphism semigroups of free algebras in a category*, Ann. Sci. Univ. Clermont *13* (1976), 1–13.

[S7] S. SHELAH, *Independence of strong partition relations for small cardinals.*

[S8] S. SHELAH, *Isomorphism of $\aleph_1$-dense subsets of reals*, Notices A.M.S. *26* (1979), 79T-E8, p. A-224.

[ŠB] P. ŠTĚPÁNEK and B. BALCAR, *Embedding theorems for Boolean algebras and consistency results on ordinal definable sets*, J. Symb. Logic *42* (1977), 64–76.

[T] S. TODORČEVIĆ, *Rigid Boolean algebras*, Publ. l'Inst. Math. *25* (1979), 217–222.

[W] M. WEESE, *The decidability of the theory of Boolean algebras with cardinality quantifiers*, Bull. Acad. Sci. Pol. *25* (1977), 93–97.

[vW] R. VAN WESEP, *Large Boolean algebras with no uncountable chains or antichains.*

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