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OMITTING TYPES ALGEBRAICALLY(1)

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The purpose of this note is to give algebraic formulations and proofs for the consistency property theorem (Smullyan, Makkai, Keisler) and the omitting types theorem (Henkin, Orey, Grzegorczyk, Mostowski, Ryll-Nardzewski). For formulations of these theorems see Keisler [3]. We follow the notation of Henkin, Monk, Tarski [2].

Let $\mathfrak{A}$ be an Lf_ω. A *constant* of $\mathfrak{A}$ is an element $k \in A$ such that $\Delta k \subseteq \{0\}$, $c_0 k = 1$, and $k \cdot s_1^0 k \leq d_{01}$. Constants have been treated in various guises in the literature ; see e.g. Halmos [1] and Pinter [4]. The version we are using is due to Henkin, but its development and applications have not yet appeared in print. Given a constant k and an $i \in \omega$, we define

$$s_k^i x = c_i (s_i^0 k.x)$$

for all $x \in A$. Then s_k^i is an endomorphism of $\mathfrak{A}$. This fact is essential in the detailed proofs of the results below. A *representation pair* is a pair $(\mathfrak{A}, K)$ such that $\mathfrak{A}$ is a denumerable Lf_ω and K is a denumerable set of constants of $\mathfrak{A}$. A *consistency family* over $(\mathfrak{A}, K)$ is a family S of subsets of $Zd\,\mathfrak{A}$ such that for any $F \in S$ the following conditions hold, for any $x, y \in A$ and any $k, \ell, m \in K$:

(1) $x \notin F$ or $-x \notin F$;

(2) if $x \in F$ and $x \leq y$, then $F \cup \{y\} \in S$;

(3) if $x + y \in F$, then $F \cup \{x\} \in S$ or $F \cup \{y\} \in S$;

(4) for any $i \in \omega$, if $c_i x \in F$, then there is a $k \in K$ such that $F \cup \{s_k^i x\} \in S$;

(5) $1 \in F$;

(6) if $c_0(k.\ell) \in F$ and $c_0(\ell.m) \in F$, then $c_0(k.m) \in F$;

(7) if $1 \in G \subseteq F$, then $G \in S$.

Given a consistency family S over $(\mathfrak{A}, K)$, a function $f : S \to S$ is *admissible* over $(\mathfrak{A}, K)$ provided that $F \subseteq fF$ for every $F \in S$.

Now we can give an algebraic version of the model existence theorem.

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Theorem 1. Let $(\mathfrak{A}, K)$ be a representation pair, S a consistency family over $(\mathfrak{A}, K)$, and $\langle f_i : i \in \omega \rangle$ a system of functions from S into S admissible over $(\mathfrak{A}, K)$. Then for any $F \in S$, there is a cylindric set algebra $\mathfrak{B}$ of dimension ω and with base U and there is a homomorphism g of $\mathfrak{A}$ into $\mathfrak{B}$ such that the following conditions hold :

(i) $U = \{[k]_E : k \in K\}$ for some equivalence relation E on K ;

(ii) $g^*F = \{{}^{\omega}U\}$;

(iii) for each $i \in \omega$ there is a $G \in S$ such that $F \subseteq f_iG \subseteq g^{-1*}\{1\}$;

(iv) for each $k \in K$, we have $gk = \{u \in {}^{\omega}U : u_0 = [k]\}$.

Proof. It is easy to construct a set $M \subseteq Zd\mathfrak{A}$ such that the following conditions hold for all $x, y \in A$ and $k, \ell, m \in K$:

(8) $x \notin M$ or $-x \notin M$;

(9) if $x \in M$ and $x \leq y$, then $y \in M$;

(10) if $x + y \in M$, then $x \in M$ or $y \in M$;

(11) if $c_i x \in M$, then $s^i_k x \in M$ for some $k \in K$;

(12) $1 \in K$;

(13) if $c_0(k.\ell) \in M$ and $c_0(\ell.m) \in M$, then $c_0(k.m) \in M$;

(14) for every $i < \omega$ there is a $G \in S$ such that $F \subseteq f_iG \subseteq K$.

Now we let $E = \{(k, \ell) \in K \times K : c_0(k.\ell) \in M\}$. It is easily seen that E is an equivalence relation on K. We set $U = K/E$. Now we are ready to define the homomorphism g. For any $x \in A$, let

$$gx = \{u \in {}^{\omega}U : \text{there is a } w \in {}^{\omega}C \text{ such that } w_i \in u_i \text{ for all } i \in \omega \text{ and}$$
$$c_{(\Delta x)}(\Pi_{i \in \Delta x} s^0_i w_i \,.\, x) \in M\} .$$

The desired conditions of the theorem are now easily checked.

For the application of this result to proving an algebraic version of the omitting types theorem we need the following preliminary result, which is almost of a general algebraic character.

Theorem 2. Let $\mathfrak{A}$ be a non-discrete Lf_ω . Then there is a representation pair $(\mathfrak{B}, K)$ with the following properties :

(i) $\mathfrak{A} \subseteq \mathfrak{B}$, and in fact $\mathfrak{B}$ is generated by $A \cup K$;

(ii) if $\mathfrak{C}$ is any CA_ω , f is a homomorphism of $\mathfrak{A}$ into $\mathfrak{C}$, and g is a mapping of K into the set of constants of $\mathfrak{C}$, then there is a homomorphism t of $\mathfrak{B}$ into $\mathfrak{C}$ such that $f \cup g \subseteq t$;

(iii) if f is a homomorphism of $\mathfrak{B}$ into the cylindric set algebra $\mathfrak{C}$ of all subsets of ${}^{\omega}U$, $a \in Sg(A \cup T)$ with $T \subseteq K$, $k \in K \sim T$, and $fc_i a = {}^{\omega}U$, then there is a homomorphism g of $\mathfrak{B}$ into $\mathfrak{C}$ such that $Sg(A \cup T) \upharpoonright g = Sg(A \cup T) \upharpoonright f$ and $gs^i_k a = 1$;

(iv) any element of B can be written in the form

$$c_{(\Gamma)}(\Pi_{i \in \Gamma}\, s_i^0 k_i \cdot a)$$

for some $a \in A$, some finite $\Gamma \subseteq \omega$, and some $k \in {}^{\Gamma}K$.

Now let $\mathfrak{A}$ be an Lf_ω, $n \in \omega$, and $N \subseteq Nr_n \mathfrak{A}$. A homomorphism f from $\mathfrak{A}$ into a Cs_ω *admits* N provided that $\bigcap_{x \in N} fx \neq 0$; otherwise we say that f *omits* N. Now we prove our algebraic version of the omitting types theorem.

Theorem 3. Let $\mathfrak{A}$ be a countable Lf_ω, $n \in \omega$, and $N \subseteq Nr_n \mathfrak{A}$. Then the following are equivalent :

(i) there is a homomorphism f of $\mathfrak{A}$ into a cylindric set algebra with a countable base such that f omits N ;

(ii) there is a homomorphism f of $\mathfrak{A}$ into an $Lf_\omega\, \mathfrak{B}$ with $|B| > 1$ such that for any non-zero $x \in Nr_n \mathfrak{B}$ there is a $y \in N$ such that $x.{-fy} \neq 0$.

Proof. (i) $\Rightarrow$ (ii). Assume (i), where $\mathfrak{B}$ is the range of f. Thus $|B| > 1$. Let $0 \neq x \in Nr_n \mathfrak{B}$, and assume that $x.{-fy} = 0$ for all $y \in N$. Then $x \subseteq \bigcap_{y \in N} fy = 0$, so $x = 0$, contradiction.

(ii) $\Rightarrow$ (i). Assume (ii). If $\mathfrak{B}$ is discrete, let g be a homomorphism of $\mathfrak{B}$ onto a cylindric set algebra $\mathfrak{C}$ with a one-element base such that $g(-fy) \neq 0$ for some $y \in N$; g exists by (ii). Thus $gfy = 0$, so $g \circ f$ omits N.

So, assume that $\mathfrak{B}$ is non-discrete. Let (D, K) be a representation pair formed from $\mathfrak{B}$ as indicated in Theorem 2. Let S consist of all $F \subseteq ZdD$ satisfying the following conditions :

(15) $1 \in F$;

(16) F is finite ;

(17) there is a homomorphism g of D into a cylindric set algebra with non-empty base such that $g^*F = \{1\}$.

It is easily checked that S is a consistency family over (D, K) ; part (iii) of Theorem 2 can be used to check (4). The following additional condition holds :

(18) if $F \in S$ and $k \in {}^nK$, then there is a $b \in N$ such that

$F \cup \{-c_{(n)}(\Pi_{i<n}\, s_i^0 k_i \cdot fb)\} \in S$.

To check (18), several small facts about constants are needed ($i,j < \omega$ and k a constant) :

(19) $\quad s_i^0 k \cdot s_j^0 k = s_i^0 k \cdot d_{ij}$;

(20) $\quad s_k^i x = s_k^j s_j^i x \quad$ if $j \notin \Delta x$;

(21) $\quad s_k^i x = s_k^j s_j^i c_j x \quad$ if $j \notin \Delta s_k^i x$, $j \neq i$.

Now assume the hypotheses of (18). By Theorem 2 (iv) we may write

$$\Pi F = c_{(\Delta)}(\Pi_{i \in \Delta}\, s_i^0 \ell_i \cdot d),$$

where $\ell \in {}^{\Delta}K$, Δ is finite, and $d \in B$. By (19) we may assume that ℓ is one-one. Define $E = \{(i,j) : i, j < n \text{ and } k_i = k_j\}$. Thus E is an equivalence relation. Let Γ have one member from each equivalence class. Now using (20) we may assume that $\Delta \subseteq \omega \sim n$: then using (21) and the fact that $\Delta \cap F = 0$ we may assume that if $\ell_i = k_j$ for some i,j, then $i = j \in \Gamma$. In summary, by (19) we can then write

$$\Pi F = c_{(n)} c_{(\Delta)} (\Pi_{i \in \Delta} s^0_i \ell_i \cdot d)$$

where $d_{ij} \leq d$ whenever iEj, ℓ is one-one, and if $\ell_i \in$ range k then $\ell_i = k_i$ and $i \in \Gamma$. Furthermore, we can assume that $\Delta d \subseteq n \cup \Delta$. Now choose g by (17). Then $gd \neq 0$, so $0 \neq c_{(\Delta \sim n)} d \in Nr_n \mathfrak{B}$. From (ii) of our theorem it follows that there is a $y \in N$ such that $c_{(\Delta \sim n)} d \cdot -fy \neq 0$. Let h be a homomorphism of $\mathfrak{B}$ into a cylindric set algebra such that $h(c_{(\Delta \sim n)} d \cdot -fy) \neq 0$. Thus

$$h\, c_{(n)} (c_{(\Delta \sim n)} d \cdot -fy) = 1.$$

Hence by Theorem 2 (ii) there is a $t \in {}^{\Delta \cup n}K$ and an extension u of $h \cup t$ such that $k \cup \ell \subseteq t$ and

$$u\, c_{(n)} (\Pi_{i \in n} s^0_i t_i \cdot c_{(\Delta \sim n)} d \cdot -fy) = 1,$$

$$u\, c_{(n \cup \Delta)} (\Pi_{i \in \Delta \cup n} s^0_i t_i \cdot d) = 1.$$

It is then clear that $u \Pi F = 1$ and $u(-c_{(n)} (\Pi_{i < n} s^0_i k_i \cdot fy)) = 1$.
From this, (18) easily follows.

The set ${}^nK$ is denumerable, and we enumerate it $\langle k_i : i < \omega \rangle$. For each $i \in \omega$ define $r_i : S \to S$ by setting for any $F \in S$

$$r_i F = F \cup \{-c_{(n)} (\Pi_{j < n} s^0_j k_{ij} \cdot fb)\},$$

where $b \in B$ is chosen minimal in some well-ordering of B so that the above set is in S. Now we apply Theorem 1, and obtain a cylindric set algebra $\mathfrak{C}$ with base $\{[k] : k \in K\}$ and a homomorphism p from $\mathfrak{T}$ into $\mathfrak{C}$ such that for each $i \in \omega$ there is a $G \in S$ with $\{1\} \subseteq r_i G \subseteq p^{-1*}\{1\}$; and satisfying Theorem 1 (iv). We claim that $p \circ f$ omits N. For, let $w \in {}^{\omega}K$. Say $\langle w_j : j < n \rangle = k_i$. Then there is a $G \in S$ with $r_i G \subseteq p^{-1*}\{1\}$, and hence there is a $b \in N$ such that

$$p(-c_{(n)} (\Pi_{j < n} s^0_j w_j \cdot fb)) = 1.$$

Using Theorem 1 (iv), we infer that $\langle [w_j] : j < \omega \rangle \notin pfb$. Thus $p \circ f$ omits N, and the proof is complete.

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