On free subalgebras of complete Boolean algebras

By

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Call a complete Boolean algebra (CBA) $\mathfrak{A}$ *semifree* if $\mathfrak{A}$ has a free subalgebra (with respect to finite meets and joins) of power $|A|$. Keslyakov [3] and, independently, S. Koppelberg [4], showed that many infinite complete Boolean algebras are semifree. We extend their results, showing in particular that under GCH any infinite CBA is semifree. Actually the GCH in full form is not needed, only the special hypothesis H that there is no singular cardinal $\mathfrak{m}$ satisfying the conditions: $\mathfrak{m}$ is not a strong limit cardinal, $cf\,\mathfrak{m}$ is weakly inaccessible, $\mathfrak{m}$ is not a power of smaller cardinals. ($\mathfrak{m}$ is a strong limit cardinal iff $\mathfrak{n} < \mathfrak{m}$ implies $2^{\mathfrak{n}} < \mathfrak{m}$.) We do not know whether H is really necessary for the result. Throughout the article we assume knowledge of [3].

The main results are based on a lemma which is a generalization of Lemma 1 of VII.2 of Vladimirov [5]. First, some notation. Let $\mathfrak{A}$ be a CBA and X, $Y \subseteq A$. We say that X is *dense* in Y if for every non-zero $y \in Y$ there is a non-zero $x \in X$ such that $x \leqq y$. Also, we set

$$X^{\pi} = \{\Pi Z : Z \subseteq X\},$$
$$X^{\Sigma} = \{\Sigma Z : Z \subseteq X\}.$$

Lemma 1. *Let $\mathfrak{A}$ be a CBA, and B a subset of A such that for every non-zero $b \in B$ we have $A \restriction b \neq \{x \in B^{\pi\Sigma} : x \leqq b\}$. Then there is a $z \in A$ such that for every non-zero $x \in B$ we have $x \cdot z \neq 0 \neq x \cdot -z$.*

Proof. Let $C = \{c \in A : c \neq 0 \text{ and } A \restriction c \cap B^{\pi} \subseteq \{0\}\}$.

(1) C is dense in B.

For, let $0 \neq b \in B$. Choose $y \in A \restriction b \sim B^{\pi\Sigma}$ and set $y' = \Sigma\{x \in B^{\pi} : x \leqq y\}$. Thus $y' \in B^{\pi\Sigma}$, so $y' < y$. Hence $0 \neq y \cdot -y' \leqq b$. We claim that $y \cdot -y' \in C$. Indeed, suppose that $x \in (A \restriction y \cdot -y') \cap B^{\pi}$. Thus $x \leqq -y'$, but also $x \leqq y$ and $x \in B^{\pi}$, so $x \leqq y'$. Hence $x = 0$, so (1) holds.

Now for any $c \in C$ let $fc = \Pi\{x \in B : c \leqq x\}$.

(2) Range (f) is dense in B.

For, let $0 \neq b \in B$. By (1), choose $c \in C$ with $c \leqq b$. Then $0 \neq c \leqq fc \leqq b$.

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Next, by Zorn's lemma let V be a maximal pairwise disjoint subset of Range (f). For each $v \in V$ choose $c_v \in C$ such that $f c_v = v$. Let $z = \sum_{v \in V} c_v$.

Now suppose $0 \neq x \in B$. By maximality of V and (2), there is a $v \in V$ with $x \cdot v \neq 0$. If $x \cdot c_v = 0$, then $c_v \leqq -x$ and hence $v = f c_v \leqq -x$, which is impossible. Thus $x \cdot c_v \neq 0$, whence $x \cdot z \neq 0$. If $x \cdot v \cdot - c_v = 0$, then $x \cdot v \leqq c_v$; but $x \cdot v \in B^{\pi}$, so our definition of C gives $x \cdot v = 0$, contradiction. Hence $x \cdot v \cdot - c_v \neq 0$. Clearly, $v \cdot z = c_v$ since $c_v \leqq v$, so $v \cdot - c_v \leqq -z$ and hence $x \cdot - z \neq 0$.

A *cardinal function* is a function assigning to each CBA a cardinal number. We shall be concerned with three such functions: $|A|$ is the cardinality of A; $c\mathfrak{A}$ is the cellularity of $\mathfrak{A}$, i.e., $\sup\{|B| : B \subseteq A,\ B$ consists of pairwise disjoint elements$\}$; $\tau\mathfrak{A}$ is the minimum cardinality of a complete generating set of $\mathfrak{A}$. Given any cardinal function k, a CBA $\mathfrak{A}$ is *k-homogeneous* provided that $k(\mathfrak{A} \upharpoonright a) = k\mathfrak{A}$ for every non-zero $a \in A$.

Lemma 2. *If $\mathfrak{A}$ is $\|$-homogeneous and $|A|$ is a strong limit cardinal, then $\mathfrak{A}$ is semifree.*

Proof. If $B \subseteq A$ and $|B| < |A|$, then $|B^{\pi\Sigma}| \leqq \exp\exp|B| < |A|$, so a simple inductive argument using Lemma 1 gives the desired independent subset of $\mathfrak{A}$ of power $|A|$.

Theorem 3. *If $|A|$ is a strong limit cardinal, then $\mathfrak{A}$ is semifree.*

Proof. Write $\mathfrak{A} \cong \prod_{i \in I} \mathfrak{B}_i$, where each $\mathfrak{B}_i$ is $\|$-homogeneous. By Lemma 2 we may assume that $|B_i| < |A|$ for each $i \in I$. Since $|A|$ is a strong limit cardinal, it is also clear that $\sup_{i \in I} |B_i| = |A|$. But this clearly contradicts Konig's Lemma.

By virtue of Theorem 4 of [3], it now follows that, under GCH, every infinite CBA is semifree.

Lemma 4. *If $\mathfrak{A}$ is τ-homogeneous and $c\mathfrak{A} = |A|$, then $\mathfrak{A}$ is semifree.*

Proof. Since $c\mathfrak{A} = |A|$, it is clear that $c\mathfrak{A}$ is not attained. By Theorem 3 we may assume that $|A|$ is not strong limit. Thus $|A| \leqq 2^{\mathfrak{m}}$ for some $\mathfrak{m} < |A|$. But $|A| \leqq \tau\mathfrak{A}^{c\mathfrak{A}}$, so $|A| = \tau\mathfrak{A}^{\mathfrak{m}}$ and hence $\mathfrak{A}$ is semifree by [3].

Lemma 5. *If $\mathfrak{A}$ is τ-homogeneous and $|A|$ is a power of smaller cardinals, then $\mathfrak{A}$ is semifree.*

Proof. Let $|A| = \mathfrak{m}^{\mathfrak{n}}$, where $\mathfrak{m}, \mathfrak{n} < |A|$. By Lemma 4 and [3] we may make the following assumptions: $c\mathfrak{A} < |A|$, $|A|$ is a limit cardinal, $c\mathfrak{A}$ is not attained, $\tau\mathfrak{A} < |A|$, and $\tau\mathfrak{A}^{\mathfrak{p}} < |A|$ whenever $\mathfrak{p} < c\mathfrak{A}$. Hence $|A| = \tau\mathfrak{A}^{c\mathfrak{A}}$, and so $c\mathfrak{A} = cf|A|$. Choose $p < c\mathfrak{A}$ so that $\mathfrak{m} \leqq \tau\mathfrak{A}^{\mathfrak{p}}$. Thus $|A| \leqq \tau\mathfrak{A}^{\mathfrak{n}\cdot\mathfrak{p}}$, so $c\mathfrak{A} \leqq \mathfrak{n}$. But this contradicts Bachmann [1], p. 154, since $|A| = \mathfrak{m}^{\mathfrak{n}}$.

Theorem 6. (H) *Every infinite CBA is semifree.*

Proof. It suffices to prove this for an infinite τ-homogeneous BA $\mathfrak{A}$. By [3] and the results here we may assume that $|A|$ is a limit cardinal but not strong limit, $\tau\mathfrak{A} < |A|$, $c\mathfrak{A} < |A|$ and $c\mathfrak{A}$ is not attained, and $|A|$ is not a power of smaller cardinals. Thus $|A| = \tau\mathfrak{A}^{c\mathfrak{A}}$, and $\tau\mathfrak{A}^{\mathfrak{m}} < |A|$ for each $\mathfrak{m} < c\mathfrak{A}$. It follows that $c\mathfrak{A} = cf|A|$. Since $c\mathfrak{A}$ is not attained, it is weakly inaccessible by Erdös, Tarski [2]. All this contradicts H.

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