SOME PROBLEMS IN ALGEBRAIC LOGIC [1]

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The purpose of this short article is to give the author's view of the most important problems in algebraic logic at this time. We shall understand algebraic logic in a rather loose sense to encompass the relationships between kinds of quantifier logics on the one hand and certain related kinds of algebraic structures on the other hand. Thus we consider only algebraic structures which exceed the apparatus of Boolean algebras of similar structures. We shall not be concerned with the philosophical motivation for the study of algebraic logic. The survey is also not intended to be comprehensive , but is limited to the author's interests ; in particular, we shall not discuss the directions exemplified in the important work Rasiowa [26] , or in the many works on monadic algebras. Besides mentioning problems, we shall also briefly describe a few results, one or two of them new, to put these problems in some kind of perspective. If no literature references are given for these results, they are to be assumed to be unpublished. We divide the problems into seven categories, in rough descending order of their relevance to ordinary logic ; it will be evident that these categories are not mutually exclusive. They are : (1) logical results proved with algebraic methods, (2) algebraic formulation and proof of known logical results, (3) the algebraic structure of theories, (4) algebras for non-classical logics, (5) metamathematical questions concerning algebras of logic, (6) mathematical questions concerning algebras of logic, (7) reformulations of algebraic versions of logic.

The problems are of two sorts : important but somewhat vague ones, which we shall not dignify with numbers, and more definite ones, which are enumerated.

1. APPLICATIONS OF ALGEBRAIC LOGIC TO LOGIC

Under this heading we understand the proof of logical results with the (at least initially) essential use of certain algebras of logic. The existence of such results should certainly be regarded as strong evidence for the value of algebraic logic. Unfortunately, such results are still scarce.The only clear cut examples known to the author concern first-order logic with finitely many variables ; see Henkin [7,8] , Johnson [12] , Monk [22] . It is, e.g., known for such logics that there is no finite schema axiomatizing in a natural way the logical validities ; and the only proof of this fact available at present uses deep results in the theories of cylindric and polyadic algebras. There are other concerns in logic where one would

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naturally anticipate applications of algebraic logic. We have in mind some topics whose algebraic formulations are rather simple and clear, such as $\aleph_0$-categoricity, definitional equivalence of theories, the logic $L_{\omega_1\omega}$, and Craig's theorem.

2. ALGEBRAIC FORMULATION AND PROOF OF KNOWN LOGICAL RESULTS

This is a topic in which a great deal of work has been done ; there are algebraic analyses of the completeness theorem, the Löwenheim-Skolem theorem, Craig's theorem, and Feferman-Vaught generalized products. These algebraic proofs have not contributed very much that is new to logic, but many people feel that some light has been shed upon the logical results by the algebraizations. A natural problem which has evaded a completely satisfactory solution is :

Problem 1. Give an algebraic proof for Gödel's incompleteness theorem.

There remain many results which have not been subjected to algebraization. We may mention two-cardinal theorems, Morley's categoricity theorem, and various decision procedures, in particular those associated with Rabin's use of tree automata. Many algebraizations are rather straightforward, but still do reveal unexpected aspects of logic.

3. THE ALGEBRAIC STRUCTURE OF THEORIES

Let Γ be a theory in first-order language $\mathscr{L}$. Set $\equiv_\Gamma = \{(\varphi, \psi) : \varphi$ and ψ are formulas of $\mathscr{L}$ and $\Gamma \models \varphi \leftrightarrow \psi\}$ Thus $\equiv_\Gamma$ is an equivalence relation on the set $\mathrm{Fmla}_{\mathscr{L}}$ of formulas of $\mathscr{L}$. The set F_Γ of equivalence classes under $\equiv_\Gamma$ can be made into an algebra $\mathscr{F}_\Gamma$ such that $[\varphi] + [\psi] = [\varphi \wedge \psi]$ $[\varphi] \cdot [\psi] = [\varphi \wedge \psi]$ $-[\varphi] = [\neg\varphi]$, $0 = [\varphi \wedge \neg\varphi]$, $1 = [\varphi \vee \neg\varphi]$ $d_{ij} = [v_i = v_j]$, and $c_i[\varphi] = [\exists v_i \varphi]$. Thus $\mathscr{F}_\Gamma = \langle \mathrm{Fmla}_{\mathscr{L}} / \equiv_\Gamma, +, \cdot, -, 0, 1, c_i, d_{ij} \rangle_{i,j < \omega}$. In trying to fully describe a theory Γ it is natural to attempt an algebraic description of the algebra $\mathscr{F}_\Gamma$. This has been done fully in only a very few cases. In [10] it is essentially done completely for the theory of finitely many unary relation symbols. Myers has recently given necessary and sufficient conditions for $\mathscr{F}_\Gamma$ to be isomorphic to $\mathscr{F}_\Delta$ when Γ and Δ are the purely logical validities of given languages. Perhaps the most interesting problem here is

Problem 2. Characterize algebraically the algebras $\mathscr{F}_\Gamma$, Γ the purely logical validites of a given first-order language.

Hanf has recently solved Problem 2 for the Boolean algebra of sentences in a countable language. The characterization problem for $\mathscr{F}_\Gamma$ is open for most common theories Γ. This is the case even for relatively simple, decidable theories ; for example, we have

Problem 3. Characterize algebraically $\mathscr{F}_\Gamma$, where Γ is the theory of one equivalence relation.

For a characterization of the Boolean algebra of sentences in this case, see Hanf [6]. For two kinds of theories, the Boolean algebras of sentences are well-known : consistent axiomatizable essentially undecidable theories (the Boolean algebra is denumerable atomless), and consistent complete theories (with two-element Boolean algebra). The full algebra $\mathscr{F}_\Gamma$ is unknown in most cases, however. Some typical problems are :

Problem 4. Characterize algebraically $\mathscr{F}_P$, where P is Peano arithmetic.

Problem 5. Characterize algebraically $\mathscr{F}_N$, where N is the set of all true sentences of arithmetic.

4. ALGEBRAS FOR NON-CLASSICAL LOGICS

By non-classical logic we shall understand any quantifier logic other than ordinary first-order logic with equality in standard formalization. Many of these logics have recived an algebraic treatment. For example, polyadic algebras give an adequate apparatus for algebraizing languages $L_{\kappa\lambda}$, and many-sorted logic was treated algebraically by LeBlanc [13]. Many logics still await an algebraic treatment, however. Perhaps most interesting among these are the Q-quantifier languages (see Bell, Slomson [1] for an exposition of them). The theory of relation algebras may be mentioned here ; this is an old form of algebraic logic first introduced by Tarski [29], and corresponds to a limited version of first-order logic. A couple of open problems here are as follows.

Problem 6. (McKenzie [16]). Let K be the class of all isomorphs of relation set algebras which are the class of all invariant relations under a group of permutations of a set. Is K the same as the class of integral representable relation algebras ?

Problem 7. Develop the theory of relation algebras with an additional operation corresponding to the formation of the transitive closure of a relation.

Several algebraizations of non-classical logic have been initiated, and are awaiting further development ; this applies for example to intuitionistic logic (Monteiro, Varsavsky [24], Monk [18]), modal logic (Freeman [4]), higher-order logic (Venne [30]), and Hilbert's ε-operator (Guillaume [5]).

5. METAMATHEMATICAL QUESTIONS CONCERNING ALGEBRAS OF LOGIC

The most well-developed versions of algebraic logic are cylindric and polyadic algebras. The remaining questions mainly concern the former.

Most of the natural metamathematical questions concerning cylindric algebras (CA's) have been anwered. The class R_α of representable CA_α's is equational. For $\alpha \leq 2$ it is finitely axiomatizable, but not for $\alpha > 2$. A simple axiomatization of it would be very desirable. Other natural classes of CA_α's, like Lf_α, Dc_α, Ss_α, are not even elementary classes. Of course, CA_α itself is a (finitely axiomatizable, if $\alpha < \omega$) equational class. For any $\alpha \geqq 1$ the elementary theory of CA_α's is undecidable (this is due to Tarski for $\alpha \geqq 2$, and, very recently, to M. Rubin for $\alpha = 1$). The equational theory of CA_α's is decidable for $\alpha \leqq 1$ (trivial), also for $\alpha = 2$ (due to Henkin), while for $\alpha \geqq 3$ it is undecidable (Tarski ; Maddux for $\alpha = 3$). A study of equational classes in algebraic logic was begun in Monk [21] (see also Lucas [14, 15]), but has not been brought to any stage of completion. J.S. Johnson has shown that there are $2^{\aleph_0}$ varieties of representable CA_α's for any α with $2 \leqq \alpha \leqq \omega$, in contrast to the results of [21].

6. MATHEMATICAL QUESTIONS CONCERNING ALGEBRAS OF LOGIC

The mathematical theory of CA_α's has been extensively developed (cf. [10]), so it is not surprising that there are a large number of open problems, most of them of a rather technical nature. We state here a few typical ones

Problem 8. Is every group isomorphic to the automorphism group of a CA_α ?

For $\alpha = 0$, Problem 8 has a negative answer ; see McKenzie, Monk [17] for references and results. The answer remains negative for $\alpha = 1$ by an unpublished result of D. Demaree.

Problem 9. Is the completion of a representable CA_α still representable ?

Problem 10. Define $\beta \underset{\alpha}{\to} \gamma$ iff every CA_α which can be neatly embedded in a $CA_{\alpha+\beta}$ can be neatly embedded in a $CA_{\alpha+\gamma}$. Characterize this ternary relation. Do there exist α, β with $3 \leqq \alpha < \omega$ and $\beta < \omega$ such that $\beta \underset{\alpha}{\to} \beta + 1$?

Problem 11. Determine cardinals $\mathfrak{m}$ such that there is a Jonsson CA_α of power $\mathfrak{m}$.

7. REFORMULATIONS OF ALGEBRAS OF LOGIC

We want to mention under this heading several proposed reformulations of the standard algebras of logic which have not been well enough developed to be compared fruitfully with the standard ones : Bernays [2], Everett, Ulam [3], Nolin [25], Rieger [27].

To conclude this survey of problems, we would like to indicate the problems in [19, 10, 23, 9] which have now been solved. Problems 1 and 3 of [19] were solved (negatively) by J.S. Johnson ; see [10], p. 418. The negative solution of Problem 4 of [19] is the main content of Monk [20]. Problem 2 of [19] does not exist, by a typographical error. Problem 1.2 of [10] was solved affirmatively by Sobocinski [28]. Problem 2.8 of [10] was solved negatively by D. Myers in unpublished work. S. Comer has given counterexamples to problems 1 and 2 of [23]. Finally, as mentioned earlier, the very important Problem 7 of [9] has been solved negatively by M. Rubin.

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