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Some cardinal functions on algebras II

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In [2], knowledge of which is assumed here, the relationships between the cardinals $|A|$, $|\mathrm{Aut}\,\mathfrak{A}|$, $|\mathrm{Sub}\,\mathfrak{A}|$, and $|\mathrm{Con}\,\mathfrak{A}|$ were completely described, assuming that $|A|\geqslant\aleph_0$ and that the GCH holds. The purpose here is to do the same thing for the cardinals $|A|$, $|\mathrm{Aut}\,\mathfrak{A}|$, $|\mathrm{End}\,\mathfrak{A}|$, and $|\mathrm{Sub}\,\mathfrak{A}|$. The results here are easy adaptations from [2] and Gould, and Platt [1] except for Lemma 7. The construction in Lemma 7 is a generalization of an example by Ralph McKenzie of a denumerable algebra $\mathfrak{A}$ with $|\mathrm{Aut}\,\mathfrak{A}|=1$, $|\mathrm{End}\,\mathfrak{A}|=2^{\aleph_0}$, and $|\mathrm{Sub}\,\mathfrak{A}|=\aleph_0$; another denumerable algebra with these properties was constucted by James S. Johnson.

First we consider the case of 'large' $\mathfrak{A}$, i.e., algebras $\mathfrak{A}$ with $|\mathrm{End}\,\mathfrak{A}|\leqslant|A|$. In this case we shall just generalize slightly a construction of Gould and Platt [1].

LEMMA 1. *Let $\mathfrak{M}$ be a monoid in which every element is either right cancellative or a right zero. Let $\mathfrak{m}\geqslant|M|+\aleph_0$. Then there is an algebra $\mathfrak{A}$ such that $|A|=\mathfrak{m}$, $\mathrm{End}\,\mathfrak{A}$ contains $\mathfrak{M}$ as a submonoid, $|\mathrm{Sub}\,\mathfrak{A}|=2$, and the subalgebra generated by the empty set has power $\mathfrak{m}$.*

Proof. We modify slightly the proof of sufficiency in Theorem 3 of [1]. We may assume that $(S_0\cup S_1)\cap\mathfrak{m}=0$. Note that $|M|\leqslant|S_0\cup S_1|\leqslant|M|+\aleph_0$. Let $A=S_0\cup S_1\cup\mathfrak{m}$. Thus $|A|=\mathfrak{m}$. We extend f_2^y to A by setting $f_2^y\alpha=\alpha$ for all $\alpha<\mathfrak{m}$. Let $\mathfrak{A}=\langle A, a, f_2^y\rangle_{a\in S_0\cup\mathfrak{m};\,y,z\in S_1}$. The desired properties of $\mathfrak{A}$ are easily checked.

Now we can apply the proofs of Theorems 1 and 2 of [1] to obtain the following result:

THEOREM 2. *Let $\mathfrak{M}$ be a monoid in which every element is either right cancellative or a right zero. Assume that $\mathfrak{m}\geqslant|M|+\aleph_0$. Let $\mathfrak{L}$ be an algebraic lattice with at least two elements, with $|\mathrm{Cmp}\,L|\leqslant\mathfrak{m}$. Then there is an algebra $\mathfrak{A}$ such that $|A|=\mathfrak{m}$, $\mathrm{End}\,\mathfrak{A}\cong\mathfrak{M}$, and $\mathrm{Sub}\,\mathfrak{A}\cong\mathfrak{L}$.*

Next, we treat the case in which both $\mathrm{End}\,\mathfrak{A}$ and $\mathrm{Aut}\,\mathfrak{A}$ are big, while $\mathrm{Sub}\,\mathfrak{A}$ is small. Note by Lemma 2 of [2], valid also for $\mathrm{End}\,\mathfrak{A}$ in place of $\mathrm{Aut}\,\mathfrak{A}$, that if $|\mathrm{End}\,\mathfrak{A}|>|A|$ then in $\mathrm{Sub}\,\mathfrak{A}$ the unit element is not a sum of $<\mathfrak{m}$ compact elements, where $\mathfrak{m}$ is the least cardinal such that $|A|^{\mathfrak{m}}>|A|$.

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LEMMA 3. *Let* $\mathfrak{m} \geqslant \aleph_0$, *and let* $\mathfrak{n}$ *be the least cardinal such that* $\mathfrak{m}^{\mathfrak{n}} > \mathfrak{m}$. *Then there is an algebra* $\mathfrak{A}$ *such that* $|A| = \mathfrak{m}$, $|\mathrm{Sub}\,\mathfrak{A}| = \mathfrak{n}$, *and* $|\mathrm{Aut}\,\mathfrak{A}| = |\mathrm{End}\,\mathfrak{A}| = \mathfrak{m}^{\mathfrak{n}}$.

Proof. We use exactly the algebra of Lemma 3 of [2]. We must determine the endomorphisms of this algebra. Obviously each map r_α, $\alpha < \mathfrak{n}$, is an endomorphism of $\mathfrak{A}$. We claim that any endomorphism ψ of $\mathfrak{A}$ which is not an automorphism has the form $\varphi_x \circ r_\alpha$, and hence $|\mathrm{End}\,\mathfrak{A}| = \mathfrak{m}^{\mathfrak{n}}$. To prove this, first note

(1) $x = r_\alpha x$ iff $\mathrm{Dmn}\, x \leqslant \alpha$.

Now for any x, say with $\mathrm{Dmn}\, x = \alpha$, we have $\psi x = \psi r_\alpha x = r_\alpha \psi x$, so by (1), $\mathrm{Dmn}\, \psi x \leqslant \alpha$. Thus

(2) $\mathrm{Dmn}\, \psi x \leqslant \mathrm{Dmn}\, x$ for all $x \in A$.

As in [2] one sees that

(3) if $\mathrm{Dmn}\, \psi x = \mathrm{Dmn}\, x$ for all $x \in A$, then ψ is an automorphism.

By (2) and (3), fix $x \in A$ such that $\mathrm{Dmn}\, \psi x < \mathrm{Dmn}\, x$. Let $\alpha = \mathrm{Dmn}\, \psi x$. Now

(4) if $\mathrm{Dmn}\, x \leqslant \mathrm{Dmn}\, y$ then $\mathrm{Dmn}\, \psi y = \alpha$.

For, write $\mathrm{Dmn}\, x = \beta$. Then for some z, $x = t_z r_\beta y$, so $\psi x = t_z r_\beta \psi_y$, so $\mathrm{Dmn}\, \psi y = \alpha$. Also,

(5) if $\alpha \leqslant \mathrm{Dmn}\, y < \mathrm{Dmn}\, x$, then $\mathrm{Dmn}\, \psi y = \alpha$.

For, write $y = t_z r_\beta x$, where $\beta = \mathrm{Dmn}\, y$. Then $\psi y = t_z r_\beta \psi x$, so $\mathrm{Dmn}\, \psi y = \alpha$.

Now let u have domain α, with $u_\beta =$ identity for all $\beta < \alpha$. Thus by (5), $\mathrm{Dmn}\, \psi u = \alpha$. Let $y = \psi u \cup \langle \text{identity} : \alpha \leqslant \beta < \mathfrak{n} \rangle$. We claim that $\psi = \varphi_y \circ r_\alpha$. Let $z \in A$ be arbitrary. We distinguish two cases.

Case 1. $\alpha \leqslant \mathrm{Dmn}\, z$. Then $\mathrm{Dmn}\, \psi z = \alpha$ by (4), (5), so, with $r_\alpha z \subseteq w \in B$,

$$\psi z = r_\alpha \psi z = \psi r_\alpha z = \psi t_w u = t_w \psi u = \varphi_y r_\alpha z.$$

Case 2. $\mathrm{Dmn}\, z < \alpha$. Say $\mathrm{Dmn}\, z = \beta$. Then with $z \subseteq w \in B$,

$$\psi z = \psi t_w r_\beta u = t_w r_\beta \psi u = \varphi_y r_\alpha z.$$

This completes the proof.

THEOREM 4. *Assume that* $\mathfrak{m} \geqslant \aleph_0$, *and let* $\mathfrak{n}$ *be the least cardinal such that* $\mathfrak{m}^{\mathfrak{n}} > \mathfrak{m}$. *Assume that* $\mathfrak{n} \leqslant \mathfrak{p} \leqslant \mathfrak{m}$. *Then there is an algebra* $\mathfrak{A}$ *such that* $|A| = \mathfrak{m}$, $|\mathrm{Sub}\,\mathfrak{A}| = \mathfrak{p}$, *and* $|\mathrm{Aut}\,\mathfrak{A}| = |\mathrm{End}\,\mathfrak{A}| = \mathfrak{m}^{\mathfrak{n}}$.

Proof. Let $\mathfrak{C}$ be formed as in Lemma 4 of [2], except without the operation h. It is clear that $|C| = \mathfrak{m}$, $|\mathrm{Sub}\,\mathfrak{C}| = \mathfrak{p}$, and $|\mathrm{Aut}\,\mathfrak{C}| = \mathfrak{m}^{\mathfrak{n}}$. It remains only to check that $|\mathrm{End}\,\mathfrak{C}| = \mathfrak{m}^{\mathfrak{n}}$. If $\psi \in \mathrm{End}\,\mathfrak{A}$ and $\alpha < \mathfrak{p}$, define $\chi_{\psi\alpha} x = \psi x$ if $x \in A$, and $\chi_{\psi\alpha} b_\beta = b_{\alpha \cap \beta}$. Then $\chi_{\psi\alpha} \in \mathrm{End}\,\mathfrak{C}$. To prove this, first note

(1) $\psi b_0 = b_0$.

For, $\psi b_0 = \psi r_0 b_0 = r_0 \psi b_0 = 0 = b_0$.

Obviously $\chi_{\psi\alpha}$ preserves the operations f_i^+. Also we have

$$\chi_{\psi\alpha} S_\beta b_\gamma = b_{\alpha \cap \beta \cap \gamma} = S_\beta \chi_{\psi\alpha} b_\gamma,$$

so $\chi_{\psi\alpha} \in \operatorname{End}\mathfrak{C}$. If $\psi \in \operatorname{End}\mathfrak{A}$, then $\psi^+ x = \psi x$, $\psi^+ b_\alpha = b_\alpha$ for $b_\alpha \in B$ defines an endomorphism ψ^+ of $\mathfrak{C}$.

Now let $\varphi \in \operatorname{End}\mathfrak{C}$; we claim that φ has the form ψ^+ or $\chi_{\psi\alpha}$. To prove this, first note

(2) $\forall x \in A\, (\varphi x \in A)$.

For, suppose $x \in A$ and $\varphi x = b_\alpha$. Then $b_0 = S_0 b_\alpha = S_0 \varphi x = \varphi S_0 x = \varphi x = b_\alpha$, so $\varphi x = b_\alpha = b_0 \in A$. Also,

(3) $\varphi b_0 = b_0$.

For, $\varphi b_0 = \varphi r_0 b_0 = r_0 \varphi b_0 = 0 = b_0$ by (2). Next,

(4) $\forall x \in B\, (\varphi x \in B \cup \{b_0\})$.

For, suppose $\varphi b_\alpha = x \in A$ with $\alpha > 0$. Then $x = S_0 x = S_0 \varphi b_\alpha = \varphi S_0 b_\alpha = \varphi b_0 = b_0$.

By (2), (3), (4) it is clear that if $\varphi \restriction B$ is the identity then φ is ψ^+ for some $\psi \in \operatorname{End}\mathfrak{A}$. So, assume $\varphi b_\alpha = b_\beta$ where $b_\alpha, b_\beta \in B \cup \{b_0\}$ and $\alpha \neq \beta$. If $\alpha < \beta$, then $b_\beta = \varphi b_\alpha = \varphi S_\alpha b_\alpha = S_\alpha \varphi b_\alpha = S_\alpha b_\beta = b_\alpha$, contradiction. Thus $\beta < \alpha$. If $\alpha \leqslant \gamma$, then

$$b_\beta = \varphi b_\alpha = \varphi S_\alpha b_\gamma = S_\alpha \varphi b_\gamma,$$

so $\varphi b_\gamma = b_\beta$. If $\beta \leqslant \gamma < \alpha$, then

$$\varphi b_\gamma = \varphi S_\gamma b_\alpha = S_\gamma \varphi b_\alpha = S_\gamma b_\beta = b_\beta.$$

Finally, if $\gamma < \beta$, then

$$\varphi b_\gamma = \varphi S_\gamma b_\beta = S_\gamma \varphi b_\beta = S_\gamma b_\beta = b_\gamma.$$

Thus φ has the form $\chi_{\psi\beta}$.

Now we take up the construction of an algebra $\mathfrak{A}$ with $\operatorname{Aut}\mathfrak{A}$ small and $\operatorname{End}\mathfrak{A}$, $\operatorname{Sub}\mathfrak{A}$ big. The construction depends on the following lemma.

LEMMA 5. *For any $\mathfrak{m} \geqslant \aleph_0$ there is an algebra $\mathfrak{A}$ such that $|A| = \mathfrak{m}$, $|\operatorname{End}\mathfrak{A}| = 2^{\mathfrak{m}}$, $|\operatorname{Aut}\mathfrak{A}| = 1$, and $|\operatorname{Sub}\mathfrak{A}| = 2^{\mathfrak{m}}$.*

Proof. Let $\mathfrak{A} = \langle \mathfrak{m}, \cap \rangle$. Then, as is easily seen,

(1) for any $X \subseteq \mathfrak{m}$, X is a subalgebra of $\mathfrak{A}$;

(2) $f \in \operatorname{End}\mathfrak{A}$ iff $\forall \alpha, \beta \in \mathfrak{m}\, (\alpha \leqslant \beta \Rightarrow f\alpha \leqslant f\beta)$

From (2) it follows that any automorphism of $\mathfrak{A}$ is an order-preserving map of $\mathfrak{m}$ onto $\mathfrak{m}$; hence only the identity is an automorphism of $\mathfrak{A}$.

Now we exhibit $2^{\mathfrak{m}}$ endomorphisms of $\mathfrak{A}$. For any $f \in 2^{\mathfrak{m}}$ let $g_f : \mathfrak{m} \to \mathfrak{m}$ be defined as follows. Given $\gamma < \mathfrak{m}$, write $\gamma = \omega \cdot \alpha + m$ with $\alpha < \mathfrak{m}$, $m < \omega$. Set $g_f \gamma = \omega \cdot \alpha + f\alpha$. Then g_f is an endomorphism of $\mathfrak{A}$. For, suppose that $\gamma \leqslant \delta < \mathfrak{m}$. Write $\gamma = \omega \cdot \alpha + m$, $\delta = \omega \cdot \beta + n$. If $\alpha < \beta$, then $g_f \gamma < g_f \delta$. If $\alpha = \beta$, then $g_f \gamma = g_f \delta$. Hence by (2), $g_f \in \operatorname{End}\mathfrak{A}$. Obviously $g_f \neq g_{f'}$ if $f \neq f'$. This completes the proof.

THEOREM 6. *If $\mathfrak{G}$ is a group with $|G| \leqslant \mathfrak{m}$, then there is an algebra $\mathfrak{A}$ with $|A| = \mathfrak{m}$, $\operatorname{Aut}\mathfrak{A} \cong \mathfrak{G}$, and $|\operatorname{Sub}\mathfrak{A}| = |\operatorname{End}\mathfrak{A}| = 2^{\mathfrak{m}}$.*

Proof. Let $\mathfrak{A}$ be as in Lemma 5, and let $\mathfrak{B}$ be an algebra with $|B|=\mathfrak{m}$ and $\operatorname{Aut}\mathfrak{B}\cong\mathfrak{G}$. We may assume that $A\cap B=0$ and $(A\cup B)\cap 2=0$. Let $C=A\cup B\cup 2$. For f_i a fundamental operation of $\mathfrak{A}$ we define f_i^+ on C:

$$f_i^+(x_0,\dots,x_{m-1})=\begin{cases} f_i(x_0,\dots,x_{m-1}) & \text{if all } x_i\in A,\\ x_0 & \text{otherwise}.\end{cases}$$

Similarly the operations g_j of $\mathfrak{B}$ are extended to operations g_j^+ on C. The algebra $\mathfrak{C}$ is to consist of all operations f_i^+, g_j^+, the distinguished elements 0, 1, and the following unary operation k:

$$kx=\begin{cases}0 & x\in A\cup\{0\},\\ 1 & x\in B\cup\{1\}.\end{cases}$$

Clearly

(1) if $X\in\operatorname{Sub}\mathfrak{A}$, then $X\cup\{0,1\}\in\operatorname{Sub}\mathfrak{C}$.

Thus $|\operatorname{Sub}\mathfrak{C}|=2^{\mathfrak{m}}$.

Next, if h is an endomorphism of $\mathfrak{A}$, define for any $x\in C$

$$h^+x=\begin{cases}hx & \text{if } x\in A,\\ x & \text{if } x\in B\cup\{0,1\}.\end{cases}$$

It is easily checked that h^+ is an endomorphism of $\mathfrak{C}$. Thus $|\operatorname{End}\mathfrak{C}|=2^{\mathfrak{m}}$.

Now if h is an automorphism of $\mathfrak{B}$, define for any $x\in C$

$$h'x=\begin{cases}hx & \text{if } x\in B,\\ x & \text{if } x\in A\cup\{0,1\}.\end{cases}$$

Then h' is an automorphism of $\mathfrak{C}$. Finally we must show that any automorphism l of $\mathfrak{C}$ has the form h'. Clearly $l0=0$ and $l1=1$. Suppose $la=b$ with $a\in A$, $b\in B$. Then

$$1=kb=kla=ka=0=0,$$

a contradiction. Thus $l^*A\subseteq A$. Similarly $l^*B\subseteq B$, so $l^*A=A$ and $l^*B=B$. Now $l\restriction A\in\operatorname{Aut}\mathfrak{A}$, so $l\restriction A=$ identity on A. Thus l has the form h'.

Our final aim is to produce an algebra $\mathfrak{A}$ in which $\operatorname{Aut}\mathfrak{A}$ and $\operatorname{Sub}\mathfrak{A}$ are small while $\operatorname{End}\mathfrak{A}$ is big.

LEMMA 7. *Let $\mathfrak{m}\geqslant\aleph_0$, and let $\mathfrak{n}$ be minimal such that $\mathfrak{m}^{\mathfrak{n}}>\mathfrak{m}$. Then there is an algebra $\mathfrak{A}$ such that $|A|=\mathfrak{m}$, $|\operatorname{Aut}\mathfrak{A}|=1$, $|\operatorname{End}\mathfrak{A}|=\mathfrak{m}^{\mathfrak{n}}$, and $|\operatorname{Sub}\mathfrak{A}|=\mathfrak{n}$.*

Proof. Let $Y=\bigcup_{\alpha<\mathfrak{n}}{}^{\alpha}\mathfrak{m}$, let X be a set disjoint from Y with $|X|=\mathfrak{n}$, and let $A=X\cup Y$. For each $\alpha<\mathfrak{n}$ we introduce an operation f_α on A. Let $X=\{x_\alpha:\alpha<\mathfrak{n}\}$ with x one-one. For all $\alpha,\beta<\mathfrak{n}$ let $f_\alpha x_\beta=x_{\alpha\cap\beta}$. For any $y\in Y$ let $f_\alpha y=y\restriction\alpha$. In addition, for any $y \in Y$ introduce the constant operation g_y on A with value y. Let $\mathfrak{A} = \langle A, f_\alpha, g_y \rangle_{\alpha < \mathfrak{n}, y \in Y}$. The subalgebras of $\mathfrak{A}$ are the empty set and all subsets of A of the form $\{x_\beta : \beta < \alpha\} \cup Y$, where $\alpha \leqslant \mathfrak{n}$. Thus $|\mathrm{Sub}\, \mathfrak{A}| = \mathfrak{n}$. It is easily seen that $\mathfrak{A}$ has only the identity automorphism. Also, it is easily checked that the following are the only non-identity endomorphisms of $\mathfrak{A}$. For $z \in {}^{\mathfrak{n}}\mathfrak{m}$, the following operation h_z on $A: h_z x_\alpha = z \restriction \alpha$, h_z the identity on Y. For $z \in {}^{\mathfrak{n}}\mathfrak{m}$, $\alpha < \mathfrak{n}$, an operation $h'_{z\alpha}$ on $A: h'_{z\alpha} x_\beta = z \restriction \alpha \cap \beta$, $h'_{z\alpha}$ the identity on Y. Finally, for any $\alpha < \mathfrak{n}$ an operation h''_α on $A: h''_\alpha x_\beta = x_{\alpha \cap \beta}$, h''_α the identity on Y.

LEMMA 8. *Let $\mathfrak{m} \geqslant \aleph_0$, and let $\mathfrak{n}$ be minimal such that $\mathfrak{m}^{\mathfrak{n}} > \mathfrak{m}$. Let $\mathfrak{n} \leqslant \mathfrak{p} \leqslant \mathfrak{m}$; then there is an algebra $\mathfrak{A}$ such that $|A| = \mathfrak{m}$, $|\mathrm{Sub}\, \mathfrak{A}| = \mathfrak{p}$, $|\mathrm{Aut}\, \mathfrak{A}| = 1$, and $|\mathrm{End}\, \mathfrak{A}| = \mathfrak{m}^{\mathfrak{n}}$.*

Proof. The proof of Lemma 4 of [2], without the operation h, easily gives the desired result (see the proof of Theorem 4 above).

THEOREM 9. *Let $\mathfrak{m} \geqslant \aleph_0$, let $\mathfrak{n}$ be minimal such that $\mathfrak{m}^{\mathfrak{n}} > \mathfrak{m}$, and let $\mathfrak{n} \leqslant \mathfrak{p} \leqslant \mathfrak{m}$. Suppose $\mathfrak{G}$ is a group such that $|G| \leqslant \mathfrak{m}$. Then there is an algebra $\mathfrak{A}$ such that $|A| = \mathfrak{m}$, $\mathrm{Aut}\, \mathfrak{A} \cong \mathfrak{G}$, $|\mathrm{End}\, \mathfrak{A}| = \mathfrak{m}^{\mathfrak{n}}$, and $|\mathrm{Sub}\, \mathfrak{A}| = \mathfrak{p}$.*

Proof. Let $\mathfrak{A}$ be as in Lemma 8, and let $\mathfrak{B}$ be an algebra such that $|B| = \mathfrak{m}$, Aut $\mathfrak{B} \cong \mathfrak{G}$, and $\mathrm{Sub}\, \mathfrak{B} = \{0, B\}$. Form $\mathfrak{C}$ by the process of the proof of Theorem 6. Obviously then $\mathrm{Aut}\, \mathfrak{C} \cong \mathfrak{G}$ and $|\mathrm{Sub}\, \mathfrak{C}| = \mathfrak{p}$. The endomorphisms of $\mathfrak{C}$ are as follows. For h an endomorphism of $\mathfrak{A}$, let

$$h^+ x = \begin{cases} hx & \text{if } x \in A, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x \in B \cup \{1\}. \end{cases}$$

Then h^+ is an endomorphism of $\mathfrak{C}$. Similarly define l' for $l \in \mathrm{End}\, \mathfrak{B}$. Finally, for $h \in \mathrm{End}\, \mathfrak{A}$, $l \in \mathrm{End}\, \mathfrak{B}$ define

$$(h, l)^* x = \begin{cases} hx & \text{if } x \in A, \\ x & \text{if } x \in \{0, 1\}, \\ lx & \text{if } x \in B. \end{cases}$$

The maps l' and $(h, l)^*$ are also endomorphisms of $\mathfrak{C}$. To see that any endomorphism of $\mathfrak{C}$ is of one of these forms, the only hard step is to check the following statement:

(1) if $s \in \mathrm{End}\, \mathfrak{C}$ and $sa = 0$ for some $a \in A$, then $sx = x$ for all $x \in A$.

The proof of this statement relies on the particular construction of $\mathfrak{A}$. In fact, let $R = \{(x, y) \in {}^2 A : x = y$ or for some operation t of $\mathfrak{A}$, $tx = y$ or $ty = x\}$. Then (1) follows from the following two easily checked statements:

(2) ${}^2 A$ is the transitive closure of R;

(3) if xRy, $s \in \mathrm{End}\, \mathfrak{C}$, and $sx = 0$, then $sy = 0$.

The main theorem of this note now follows:

THEOREM 10. *Assume* GCH. *Let* $\mathfrak{m}$, $\mathfrak{n}$, $\mathfrak{p}$, $\mathfrak{q}$ *be cardinals such that* $\mathfrak{m} \geq \aleph_0$ *and* $\mathfrak{q} > 1$. *Then the following conditions are equivalent:*

(i) *there is an algebra* $\mathfrak{A}$ *such that* $|A| = \mathfrak{m}$, $|\mathrm{Aut}\,\mathfrak{A}| = \mathfrak{n}$, $|\mathrm{End}\,\mathfrak{A}| = \mathfrak{p}$, *and* $|\mathrm{Sub}\,\mathfrak{A}| = \mathfrak{q}$;

(ii) *one of these conditions holds:*

(1) $1 \leq \mathfrak{n} \leq \mathfrak{p} \leq \mathfrak{m}$ *and* $\mathfrak{q} \leq \mathfrak{m}^+$,

(2) $1 \leq \mathfrak{n} \leq \mathfrak{p} = \mathfrak{m}^+$ *and* $\mathrm{cf}\,\mathfrak{m} \leq \mathfrak{q} \leq \mathfrak{m}^+$.

REFERENCES

[1] M. Gould and C. Platt, *Versatile monoids and versatile categories*, Alg. Univ. *1* (1971), 54–62.
[2] J. D. Monk, *Some cardinal functions on algebras*, Alg. Univ. *5* (1975), 76–81.

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