Math. Ann. 216, 5–10 (1975)

On the Automorphism Groups of Denumerable Boolean Algebras

J. Donald Monk

It is well-known that the automorphism group of any denumerable Boolean algebra has the power of the continuum; for completeness we prove this fact below. The problem naturally arises to determine the structure of these groups, or at least to determine which of them are isomorphic. In this note we make a modest beginning on these problems. We describe the automorphism groups of several important denumerable Boolean algebras, and we show that some important Boolean notions can be distinguished from each other group-theoretically. Conversations with R. Baer and M. Guillaume were stimulating for this work.

By Anderson [1], the automorphism group F of the denumerable atomless Boolean algebra (BA) $\mathfrak{A}$ is simple. The denumerable $BA\,\mathfrak{B}$ with exactly one atom obviously has the same automorphism group F. We conjecture that with the exception of the pair $\mathfrak{A}$, $\mathfrak{B}$, any two non-isomorphic denumerable BA's have non-isomorphic automorphism groups.

For any $BA\,\mathfrak{A}$, we denote by $\mathrm{Aut}\,\mathfrak{A}$ the automorphism group of $\mathfrak{A}$. Given a $BA\,\mathfrak{A}$, any finite permutation of the atoms of $\mathfrak{A}$ can be extended to an automorphism of $\mathfrak{A}$ which leaves the atomless elements of $\mathfrak{A}$ pointwise fixed; we let $\mathrm{Fin}\,\mathfrak{A}$ be the collection of all such automorphisms. Clearly $\mathrm{Fin}\,\mathfrak{A} \lhd \mathrm{Aut}\,\mathfrak{A}$. The even finite permutations of the atoms also clearly induce a normal subgroup of $\mathrm{Aut}\,\mathfrak{A}$, denoted by $\mathrm{Alt}\,\mathfrak{A}$. Thus $\mathrm{Alt}\,\mathfrak{A} \lhd \mathrm{Fin}\,\mathfrak{A} \lhd \mathrm{Aut}\,\mathfrak{A}$. One more normal subgroup of $\mathrm{Aut}\,\mathfrak{A}$ will play a prominent role in what follows. To define it, for any $BA\,\mathfrak{A}$ let $I_{\mathfrak{A}}$ be the ideal of $\mathfrak{A}$ generated by the atoms of $\mathfrak{A}$. Now every automorphism f of $\mathfrak{A}$ induces an automorphism $\pi_{\mathfrak{A}} f = \pi f$ of $\mathfrak{A}/I_{\mathfrak{A}}$ with the defining property that $(\pi f)[a] = [fa]$ for all $a \in A$. Clearly $\pi_{\mathfrak{A}}$ is a homomorphism of $\mathrm{Aut}\,\mathfrak{A}$ into $\mathrm{Aut}(\mathfrak{A}/I_{\mathfrak{A}})$. The kernel of $\pi_{\mathfrak{A}}$, denoted by $\ker \pi_{\mathfrak{A}}$, will now be briefly investigated. Clearly $\mathrm{Alt}\,\mathfrak{A} \lhd \mathrm{Fin}\,\mathfrak{A} \lhd \ker \pi_{\mathfrak{A}} \lhd \mathrm{Aut}\,\mathfrak{A}$. Thus $\ker \pi_{\mathfrak{A}}$ is non-trivial iff $\mathfrak{A}$ has at least two atoms.

Lemma 1.1. *Let D be an infinite set of atoms in a $BA\,\mathfrak{A}$ such that for each $a \in A$, $\{x \in D : x \leqq a\}$ is either finite or else its complement in D is finite. Then any permutation of D can be extended to an automorphism of $\mathfrak{A}$ which is the identity on $\{a \in A : \forall x \in D(x \cdot a = 0)\}$.*

Proof. Let I be the set of all elements $a \in A$ such that $\{x \in D : x \leqq a\}$ is finite. Clearly I is a maximal ideal of $\mathfrak{A}$. For each $a \in I$ there exist a unique finite subset S_a of D and a unique $b_a \in A$ such that $\forall x \in D(x \cdot b_a = 0)$ and $a = \Sigma S_a + b_a$. For each such a and each permutation f of D, let $f^+ a = \Sigma \{fx : x \in S_a\} + b_a$. It is easily verified that f^+ is an automorphism of I (considered as a partially ordered set). As is well-known, f^+ can then be extended to an automorphism of $\mathfrak{A}$ in a unique fashion.

Theorem 1. *If $\mathfrak{A}$ is a denumerable BA with infinitely many atoms, then* $\ker \pi_{\mathfrak{A}}$ *contains a copy of* $\operatorname{Sym}\omega$.

Proof. Let I be a maximal ideal of $\mathfrak{A}$ such that $x \in I$ whenever $a \leqq x$ for only finitely many atoms a. We construct a set D of atoms of $\mathfrak{A}$ such that

(∗) D is infinite, and for each $a \in I$, $\{x \in D : x \leqq a\}$ is finite.

To do this, write $I = \{a_i : i \in \omega\}$. We define a sequence $d_0, d_1, \dots$ of atoms of $\mathfrak{A}$ by recursion. Suppose d_i has been defined for all $i < m$. Since I is a proper ideal, there are infinitely many atoms $\leqq -a_0 \cdot -a_1 \cdot \dots \cdot -a_{m-1}$; we choose d_m to be such an atom different from each d_i with $i < m$. Clearly $D = \{d_i : i \in \omega\}$ satisfies (∗). Now an application of Lemma 1.1 gives the desired result.

A more general result than Theorem 1 was shown in [4]. If a $BA\,\mathfrak{A}$ has a finite number m of atoms, clearly $\operatorname{Fin}\mathfrak{A} = \ker \pi_{\mathfrak{A}} \cong \operatorname{Sym} m$.

Corollary 1.1. *If $\mathfrak{A}$ is a denumerable BA, then* $|\operatorname{Aut}\mathfrak{A}| = \exp\aleph_0$.

Proof. If $\mathfrak{A}$ has infinitely many atoms, this follows from Theorem 1. If $\mathfrak{A}$ has only finitely many atoms, clearly F is a subgroup of $\operatorname{Aut}\mathfrak{A}$, and obviously F has power $\exp\aleph_0$.

Now we begin the consideration of particular BA's and properties of BA's.

Theorem 2. *For any denumerable $BA\,\mathfrak{A}$ the following conditions are equivalent:*

(i) *$\mathfrak{A}$ has at most one atom;*

(ii) $\operatorname{Aut}\mathfrak{A}$ *is simple.*

Proof. We have already observed that (i)⇒(ii). Now if $\mathfrak{A}$ has more than one atom, then $\operatorname{Fin}\mathfrak{A}$ is non-trivial, and by Theorem 1 it is proper. Thus $\operatorname{Aut}\mathfrak{A}$ is not simple.

Corollary 2.1. *If $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$ are denumerable BA's, $\mathfrak{A}$ is atomless, $\mathfrak{B}$ has exactly one atom, and* $\operatorname{Aut}\mathfrak{A} \cong \operatorname{Aut}\mathfrak{C}$, *then* $\mathfrak{C} \cong \mathfrak{A}$ *or* $\mathfrak{C} \cong \mathfrak{B}$.

The following useful proposition is immediate from [4]:

Proposition 1. *If $\mathfrak{A}$ is a BA such that the sum a of all atoms in $\mathfrak{A}$ exists, then* $\operatorname{Aut}\mathfrak{A} \cong \operatorname{Aut}(\mathfrak{A} \upharpoonright a) \times \operatorname{Aut}(\mathfrak{A} \upharpoonright -a)$.

Theorem 3. *Let m be an integer > 1. For any denumerable $BA\,\mathfrak{A}$ the following conditions are equivalent:*

(i) *$\mathfrak{A}$ has exactly m atoms;*

(ii) $\operatorname{Aut}\mathfrak{A} \cong \operatorname{Sym} m \times F$.

Proof. The implication (i)⇒(ii) is clear from Proposition 1. Now assume that (ii) holds. Then by Theorem 2, $\mathfrak{A}$ has at least two atoms. Suppose $\mathfrak{A}$ has infinitely many atoms. By (ii), there exist normal subgroups G, H of $\operatorname{Aut}\mathfrak{A}$ with $G \cap H = \{e\}$, $GH = \operatorname{Aut}\mathfrak{A}$, $G \cong \operatorname{Sym} m$, and $H \cong F$. Choose $f \in \operatorname{Fin}\mathfrak{A}$ of order $> m!$. Write $f = gh$ with $g \in G$, $h \in H$. Since every element of $\operatorname{Sym} m$ has order dividing $m!$, we have $f^{m!} = h^{m!}$. Thus $\operatorname{Fin}\mathfrak{A} \cap H \neq \{e\}$. Since H is simple, $H \subsetneq \operatorname{Fin}\mathfrak{A}$. This is impossible, since $|H| = \exp\aleph_0$ while $|\operatorname{Fin}\mathfrak{A}| = \aleph_0$. Thus $\mathfrak{A}$ has a finite number n of atoms. By Proposition 1, $\operatorname{Aut}\mathfrak{A} \cong \operatorname{Sym} n \times F$. Let G and H be as above, and let G' and H' be normal subgroups of $\operatorname{Aut}\mathfrak{A}$ such that $G' \cap H' = \{e\}$, $G'H' = \operatorname{Aut}\mathfrak{A}$, $G' \cong \operatorname{Sym} n$, and $H' \cong F$. Now H' has an element x of infinite order, and we can write $x = yz$ with $y \in G$, $z \in H$. Since y has finite order, it follows that $x^t = z^t$ for some positive t. Thus $H \cap H' \neq \{e\}$. Since both H and H' are simple, it follows that $H = H'$. Hence $G \cong G'$ and $m = n$.

Corollary 3.1. *If* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are denumerable BA's,* $\mathfrak{A}$ *has a finite number* > 1 *of atoms, and* $\mathrm{Aut}\,\mathfrak{A} \cong \mathrm{Aut}\,\mathfrak{B}$, *then* $\mathfrak{A} \cong \mathfrak{B}$.

Theorem 4. *For any denumerable BA* $\mathfrak{A}$ *the following conditions are equivalent:*

(i) $\mathfrak{A}$ *is atomic;*

(ii) $\mathrm{Aut}\,\mathfrak{A}$ *has a smallest non-trivial normal subgroup, all of whose elements have finite order.*

Proof. (i)⇒(ii). Just as for the infinite symmetric groups, one easily shows that $\mathrm{Alt}\,\mathfrak{A}$ is the smallest non-trivial normal subgroup of $\mathfrak{A}$ when $\mathfrak{A}$ is atomic. (ii)⇒(i). Assume that $\mathfrak{A}$ is not atomic. If $\mathfrak{A}$ has at most one atom, then $\mathrm{Aut}\,\mathfrak{A} \cong F$, which is a simple group having elements of infinite order, so (ii) fails. Suppose $\mathfrak{A}$ has at least two atoms. Let $N = \{f \in \mathrm{Aut}\,\mathfrak{A}\colon f \text{ fixes all atoms}\}$. Clearly N is a non-trivial normal subgroup of $\mathfrak{A}$. Since $\mathrm{Fin}\,\mathfrak{A} \cap N = \{e\}$, (ii) fails.

Next we turn to the BA of finite and cofinite subsets of ω. The following lemma may be of independent interest; it follows easily from Pierce [6], but for completeness we prove it here.

Lemma 5.1. *If* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are denumerable atomic BA's and the algebras* $\mathfrak{A}/I_{\mathfrak{A}}$ *and* $\mathfrak{B}/I_{\mathfrak{B}}$ *are atomless, then* $\mathfrak{A} \cong \mathfrak{B}$.

Proof. We shall apply the following theorem of Vaught, whose proof can be found in Hanf [3]:

(*) If R is a symmetric relation among countable BA's satisfying the following three conditions:

(1) $\mathfrak{A} R \mathfrak{B}$ if $\mathfrak{A}$ and $\mathfrak{B}$ are finite and isomorphic;

(2) if $\mathfrak{A} R \mathfrak{B}$, then $\forall a \in A\, \exists b \in B[(\mathfrak{A} \upharpoonright a)\, R(\mathfrak{B} \upharpoonright b)$ and $(\mathfrak{A} \upharpoonright -a)\, R(\mathfrak{B} \upharpoonright -b)]$;

(3) $\mathfrak{A} R \mathfrak{B}$ implies $|A| = |B|$;

then $\mathfrak{A} R \mathfrak{B}$ implies $\mathfrak{A} \cong \mathfrak{B}$.

Now let $\mathfrak{A} R \mathfrak{B}$ mean that $\mathfrak{A}$ and $\mathfrak{B}$ are finite and isomorphic, or else that they both satisfy the conditions of the lemma. It is easily checked that the hypotheses of (*) hold. Hence the lemma follows by (*).

Theorem 5. *For any denumerable BA* $\mathfrak{A}$ *the following conditions are equivalent:*

(i) $\mathfrak{A}$ *is isomorphic to the BA of finite and cofinite subsets of* ω;

(ii) $\mathrm{Aut}\,\mathfrak{A} \cong \mathrm{Sym}\,\omega$;

(iii) $\mathrm{Aut}\,\mathfrak{A}$ *has exactly three non-trivial normal subgroups, and all of them are infinite;*

(iv) $\ker \pi_{\mathfrak{A}} = \mathrm{Aut}\,\mathfrak{A}$.

Proof. It is obvious that (i) implies (ii). By Schreier and Ulam [7], (ii) implies (iii). The implication (iii)⇒(iv) is clear by Theorems 1–3.

Now assume that (i) fails to hold; we shall show that (iv) fails. If $\mathfrak{A}$ has only finitely many atoms, (iv) fails since $\ker \pi_{\mathfrak{A}}$ is then finite. If $\mathfrak{A}$ has an atomless element, then in $\mathrm{Aut}\,\mathfrak{A}$ there is a non-trivial automorphism f fixing all atoms, so $f \notin \ker \pi_{\mathfrak{A}}$. Thus we may assume that $\mathfrak{A}$ is atomic. We now distinguish three cases.

Case 1. $\mathfrak{A}/I_{\mathfrak{A}}$ is atomless. Then by Lemma 5.1, $\mathfrak{A}$ has an ordered basis X of type $1 + \omega \cdot \eta$. We shall assume that the elements of X are 0 and all pairs (m, r)

with $m \in \omega$, $r \in Q$ (the rationals), with ordering by second differences. The mapping f of X into X such that $f0=0$ and $f(m,r)=(m,r+1)$ for all $m \in \omega$, $r \in Q$, is obviously an order automorphism of X, which hence extends to an automorphism f^+ of $\mathfrak{A}$. Clearly $f^+ \notin \ker \pi_{\mathfrak{A}}$, so (iv) fails.

Case 2. $\mathfrak{A}/I_{\mathfrak{A}}$ has exactly one atom $[a]$. Since (i) fails, $|\mathfrak{A}/I_{\mathfrak{A}}| > 2$ and hence $[-a]$ is an atomless element of $\mathfrak{A}/I_{\mathfrak{A}}$. With $\mathfrak{B} = \mathfrak{A} \upharpoonright -a$ it is clear that $\mathfrak{B}$ is denumerable and atomic and that $\mathfrak{B}/I_{\mathfrak{B}}$ is atomless. Hence by Case 1 for $\mathfrak{B}$, there is an $f \in \operatorname{Aut}\mathfrak{B} \sim \ker \pi_{\mathfrak{B}}$. Clearly f extends to an automorphism of $\mathfrak{A}$ which is not in $\ker \pi_{\mathfrak{A}}$.

Case 3. $\mathfrak{A}/I_{\mathfrak{A}}$ has at least two atoms, $[a]$ and $[b]$. We may assume that a and b have no atoms in common, and thus that $a \cdot b = 0$. Clearly $\mathfrak{A} \upharpoonright a$ and $\mathfrak{A} \upharpoonright b$ are both isomorphic to the BA of finite and cofinite subsets of ω, and hence to each other. Hence there is an automorphism f of $\mathfrak{A}$ which interchanges a and b. Clearly $f \notin \ker \pi_{\mathfrak{A}}$.

Corollary 5.1. *If $\mathfrak{A}$ and $\mathfrak{B}$ are denumerable BA's, $\mathfrak{A}$ the BA of finite and cofinite subsets of ω, and $\operatorname{Aut}\mathfrak{A} \cong \operatorname{Aut}\mathfrak{B}$, then $\mathfrak{A} \cong \mathfrak{B}$.*

Corollary 5.2. *The BA of finite and cofinite subsets of ω is the only denumerable BA with automorphism group isomorphic to $\operatorname{Sym}\omega$.*

This corollary answers a question of R. Baer.

Our final two results are intended to hint at how complicated the automorphism groups of denumerable BA's can be. For the first result we need to discuss further the ideal $I_{\mathfrak{A}}$. Given any $BA\,\mathfrak{A}$, we now define a transfinite sequence $J_{\mathfrak{A}}^0, J_{\mathfrak{A}}^1, \ldots, J_{\mathfrak{A}}^\alpha, \ldots$ where α ranges through all ordinals; the definition is by recursion. Let $J_{\mathfrak{A}}^0 = \{0\}$, the trivial ideal. If α is a limit ordinal, set $J_{\mathfrak{A}}^\alpha = \bigcup_{\beta<\alpha} J_{\mathfrak{A}}^\beta$. Finally, if $J_{\mathfrak{A}}^\beta$ has been defined, let

$$J_{\mathfrak{A}}^{\beta+1} = \{x \in A : [x] \quad \text{is a finite sum of atoms in the algebra} \quad \mathfrak{A}/J_{\mathfrak{A}}^\beta\}.$$

Clearly $J_{\mathfrak{A}}^1 = I_{\mathfrak{A}}$, $J_{\mathfrak{A}}^2$ is the preimage of $I_{\mathfrak{B}}$ under the natural homomorphism $\mathfrak{A} \to \mathfrak{A}/I_{\mathfrak{A}} = \mathfrak{B}$, etc. This is a well-known construction. For each ordinal α let $\sigma_{\mathfrak{A}}^\alpha = \sigma^\alpha$ be the natural homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}/I_{\mathfrak{A}}^\alpha$. Every automorphism f of $\mathfrak{A}$ induces an automorphism $\pi_{\mathfrak{A}}^\alpha f = \pi^\alpha f$ of $\mathfrak{A}/I_{\mathfrak{A}}^\alpha$ with the defining property that $(\pi_{\mathfrak{A}}^\alpha f)\sigma_{\mathfrak{A}}^\alpha x = \sigma_{\mathfrak{A}}^\alpha f x$. Again $\pi_{\mathfrak{A}}^\alpha$ is a homomorphism of $\operatorname{Aut}\mathfrak{A}$ into $\operatorname{Aut}(\mathfrak{A}/I_{\mathfrak{A}}^\alpha)$. Clearly $\ker \pi_{\mathfrak{A}}^\alpha \subseteq \ker \pi_{\mathfrak{A}}^\beta$ if $\alpha \leqq \beta$. Note that $\pi_{\mathfrak{A}}^1 = \pi_{\mathfrak{A}}$.

Lemma 6.1. *If $\mathfrak{A}$ is a denumerable hereditarily atomic BA and α is an ordinal for which $|\mathfrak{A}/I_{\mathfrak{A}}^\alpha| > 2$, then $\ker \pi_{\mathfrak{A}}^\alpha \subset \ker \pi_{\mathfrak{A}}^{\alpha+1}$.*

Proof. Under the hypothesis, there are elements $x, y \in A$ for which $\sigma^\alpha x$ and $\sigma^\alpha y$ are distinct atoms of $\mathfrak{A}/I_{\mathfrak{A}}^\alpha$. We may assume that $x \cdot y = 0$. In the terminology of Day [2], clearly $\mathfrak{A} \upharpoonright x$ and $\mathfrak{A} \upharpoonright y$ have the same cardinal sequence. Hence by [2] they are isomorphic, so there is an automorphism f of $\mathfrak{A}$ which interchanges x and y and leaves elements $\leqq -x \cdot -y$ fixed. Clearly $f \in \ker \pi^{\alpha+1} \sim \ker \pi^\alpha$.

Theorem 6. *For each countable ordinal $\alpha \geqq 4$ there is a countable $BA\,\mathfrak{A}$ such that the normal subgroup lattice of $\mathfrak{A}$ contains a chain of order type α.*

Proof. Choose β such that $\alpha = 4 + \beta$. If $\beta = 0$, we may take for $\mathfrak{A}$ the BA of finite and cofinite subsets of ω. Assume $\beta > 0$. Choose a countable hereditarily atomic $BA\,\mathfrak{A}$ whose cardinal sequence is of type $\beta+1$ with last term 2 (for example, $\mathfrak{A}$ can be chosen as a BA with an ordered basis of type $\omega^\beta\cdot 2$). Thus by Lemma 6.1, the normal subgroup lattice of $\mathfrak{A}$ contains the chain of length $\geqq\alpha$

$$\{e\}\lhd \operatorname{Alt}\mathfrak{A}\lhd \operatorname{Fin}\mathfrak{A}\lhd \ker\pi^1\lhd\cdots\lhd\ker\pi^\beta\lhd\operatorname{Aut}\mathfrak{A}$$

with all inclusions proper.

For our final result we exhibit the automorphism groups for a class of countable BA's which forms a first natural extension of the BA of finite and cofinite subsets of ω. Note that if $\mathfrak{A}$ is a BA of subsets of ω, then any automorphism f of $\mathfrak{A}$ has the form $\forall a\in A(fa=g^*a)$ for some permutation g of ω, where g^*a is the g-image of the set a. We denote by $\operatorname{Aut}^*\mathfrak{A}$ all the permutations of ω which induce automorphisms of $\mathfrak{A}$. Obviously $\operatorname{Aut}^*\mathfrak{A}$ consists of all those permutations g of ω such that $\forall a\in A(g^*a\in A)$.

Theorem 7. *Let $\mathfrak{P}$ be a partition of ω into infinite sets, with $|\mathfrak{P}|>1$. Let $\mathfrak{A}$ be the subalgebra of the BA of all subsets of ω generated by $\mathfrak{P}\cup\{F:F\subseteqq\omega,\ F$ finite$\}$. Let $M=\{(a,b,c,d,e)\}:a,b,$ and d are functions with domain $\mathfrak{P}$, $\forall\alpha\in\mathfrak{P}(a\alpha$ and $b\alpha$ are cofinite subsets of $\alpha)$, $c\in\operatorname{Sym}\mathfrak{P}$, $\forall\alpha\in\mathfrak{P}(d\alpha:a\alpha\rightarrowtail\!\!\!\twoheadrightarrow bc\alpha)$, and*

$$e:\left\{\bigcup_{\alpha\in\mathfrak{P}}(\alpha\sim a\alpha)\rightarrowtail\!\!\!\twoheadrightarrow\bigcup_{\alpha\in\mathfrak{P}}(\alpha\sim b\alpha)\right\}.$$

Then

$$\operatorname{Aut}^*\mathfrak{A}=\left\{\bigcup_{\alpha\in\mathfrak{P}}d\alpha\cup e:(a,b,c,d,e)\in M\right\}.$$

Proof. Let $\mathfrak{B}$ be the BA of finite and cofinite subsets of ω. Let $N=\{x:x:\mathfrak{P}\to B$ and either $\{\alpha:x\alpha\neq 0\}$ is finite or $\{\alpha:x\alpha\neq\omega\}$ is finite$\}$. Then, as is easily checked,

(1) $A=\left\{\bigcup_{\alpha\in\mathfrak{P}}x\alpha\cap\alpha:x\in N\right\}.$

Thus for any $a\subseteqq\omega$, $a\in A$ iff the following two conditions hold:

(2) $\forall\alpha\in\mathfrak{P}(a\cap\alpha$ is a finite or cofinite subset of $\alpha)$.

(3) $\{\alpha:\alpha\in\mathfrak{P},\ a\cap\alpha\neq 0\}$ is finite or $\{\alpha:\alpha\in\mathfrak{P},\ a\cap\alpha\neq\alpha\}$ is finite.

Now let $(a,b,c,d,e)\in M$ and let $f=\bigcup_{\alpha\in\mathfrak{P}}d\cup e$; we show that $f\in\operatorname{Aut}^*\mathfrak{A}$. To this end, take any $x\in A$; we show that $f^*x\in A$. For any $\alpha\in\mathfrak{P}$, $f^*x\cap\alpha$ has the form $(dc^{-1}\alpha)^*(x\cap c^{-1}\alpha)\cup y$ for some $y\subseteqq d\sim b\alpha$. Since $x\cap c^{-1}\alpha$ is a finite or cofinite subset of $c^{-1}\alpha$ by (2), and $Dmn\,dc^{-1}\alpha=ac^{-1}\alpha$ is a cofinite subset of $c^{-1}\alpha$, it follows that $f^*x\cap\alpha$ is a finite or cofinite subset of α. Thus (2) holds for f^*x. To check (3) for f^*x, we consider two cases [by (3) for x].

Case 1. $\{\alpha:x\cap\alpha\neq 0\}=\Gamma$ is finite. Then the set $c^*\Gamma\cup\left\{\alpha:e^*\bigcup_{\beta\in\Gamma}(\beta\sim a\beta)\cap\alpha\neq 0\right\}$ if finite; if α is not in this set, then

$$\begin{aligned}f^*x\cap\alpha&=(dc^{-1}\alpha)^*(x\cap c^{-1}\alpha)\cup(e^*x\cap\alpha)\\&\subseteqq e^*\bigcup_{\beta\in\Gamma}(\beta\sim a\beta)\cap\alpha\quad(\text{since}\quad c^{-1}\alpha\notin\Gamma)\\&=0\,.\end{aligned}$$

Thus $\{\alpha:f^*x\cap\alpha\neq 0\}$ is finite.

Case 2. $\{\alpha:x\cap\alpha\neq\alpha\}=\Gamma$ is finite. Similar to Case 1.

Thus (3) holds for f^*a, so $f^*a \in A$. Since $(a, b, c, d, e) \in M$ and $f = \bigcup_{\alpha \in \mathfrak{P}} d\alpha \cup e$ imply $(b, a, c^{-1}, d^{-1}, e^{-1}) \in M$ and $f^{-1} = \bigcup_{\alpha \in \mathfrak{P}} d'\alpha \cup e^{-1}$, where $d'\alpha = (d\alpha)^{-1}$ for all $\alpha \in \mathfrak{P}$, it follows that $f^*a \in A$ implies $a \in A$. We have now proved the inclusion $\supseteq$ in the statement of the theorem.

Now let $f \in \mathrm{Aut}^* \mathfrak{A}$. Then

(4) for every $\alpha \in \mathfrak{P}$ there is a unique $\beta \in \mathfrak{P}$ such that $f^*\alpha \cap \beta$ is a cofinite subset of β.

In fact, first suppose that all sets $f^*\alpha \cap \beta$ are finite. Then by (3) for $f^*\alpha$, $\{\beta : f^*\alpha \cap \beta \neq 0\}$ is finite. But this implies that $f^*\alpha$ is finite, contradiction. Thus by (2), at least one set $f^*\alpha \cap \beta$ is a cofinite subset of β. Suppose there are two such, say $f^*\alpha \cap \beta_0$ and $f^*\alpha \cap \beta_1$, with $\beta_0 \neq \beta_1$. Then $f^{-1*}(f^*\alpha \cap \beta_0)$ and $f^{-1*}(f^*\alpha \cap \beta_1)$ are infinite disjoint subsets of α and both are in A, contradicting (2). Thus (4) holds.

By (4), let c be the function such that $f^*\alpha \cap c\alpha$ is a cofinite subset of $c\alpha$ for each $\alpha \in \mathfrak{P}$. By considering the action of f^{-1} it is easily seen that c is a permutation of $\mathfrak{P}$.

Now for each $\alpha \in \mathfrak{P}$ let $a\alpha = f^{-1*}(f^*\alpha \cap c\alpha) = \alpha \cap f^{-1*}c\alpha$. Since $f^*\alpha \cap c\alpha$ is a cofinite subset of $c\alpha$, it is infinite, and hence by (4) for f^{-1}, $\alpha \cap f^{-1*}c\alpha$ is a cofinite subset of α. For each $\alpha \in \mathfrak{P}$ let $b\alpha = f^*c^{-1}\alpha \cap \alpha$. By (4), $b\alpha$ is a cofinite subset of α. For $\alpha \in \mathfrak{P}$ let $d\alpha = f \upharpoonright a\alpha$; clearly $d\alpha : a\alpha \rightarrowtail\!\!\!\!\twoheadrightarrow b c\alpha$. Finally, let $e = f \sim \bigcup_{\alpha \in \mathfrak{P}} d\alpha$. It is clear that $(a, b, c, d, e) \in M$ and $f = \bigcup_{\alpha \in \mathfrak{P}} d\alpha \cup e$.

References

1. Anderson, R. D.: The algebraic simplicity of certain groups of homomorphisms. Amer. J. Math. **80**, 955—963 (1958)
2. Day, G. W.: Superatomic Boolean algebras. Pacific J. Math. **23**, 479—489 (1967)
3. Hanf, W.: Primitive Boolean algebras. Proc. Tarski Symp. AMS 75—90, 1974
4. McKenzie, R., Monk, J. D.: On automorphism groups of Boolean algebras. Proc. Erdös Symp. 951—988, 1973
5. Mostowski, A., Tarski, A.: Boolesche Ringe mit geordneter Basis. Fund. Math. **32**, 69—86 (1939)
6. Pierce, R. S.: Existence and uniqueness theorems for extensions of zero-dimensional compact metric spaces. Trans. Amer. Math. Soc. **148**, 1—21 (1970)
7. Schreier, J., Ulam, S.: Über die Permutationsgruppe der natürlichen Zahlenfolge. Stud. Math. **4**, 134—141 (1933)

J. D. Monk
Department of Mathematics
University of Colorado
Boulder, Col. 80302, USA

(Received July 17, 1974)

Note added in proof. The main conjecture has been refuted by McKenzie and Shelah, independently.