CYLINDRIC ALGEBRAS AND RELATED STRUCTURES

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The authors are happy to have this opportunity to describe a field in which they have worked in close collaboration with Alfred Tarski for many years. As will be seen, Tarski was the founder of this field, and obtained many of the basic results in it. We shall present here a brief account of the development of cylindric algebras and related structures, followed by the presentation of a modest new result relevant to the basic representation problem for cylindric algebras. The paper is entirely selfcontained; the first part is purely expository, while the second part uses only the apparatus described in the first. We do assume a modest acquaintance with logic, set theory, general algebra, and Boolean algebra.

1. Survey. From the beginning of its modern development, logic has had an algebraic aspect. In fact, symbolic logic may be said to have begun with Boole [2], whose work is largely algebraic in nature. The algebraic structures implicitly considered by Boole are now called *Boolean algebras*. As is well known, they stand in a close relationship to sentential logic, as well as to the calculus of classes. We shall not be concerned here, however, with the theory of Boolean algebras themselves; we want to describe certain multidimensional Boolean algebras closely related to predicate logic. We shall bear in mind, however, the logical and class calculus aspects of Boolean algebras and look for analogous relationships in the higher dimensional cases.

The first abstract algebraic theory of a substantial portion of predicate logic is the theory of *relation algebras*, initiated in Tarski [31]. This was a natural extension of earlier directions of Tarski's work. An algebraic approach to metamathematics is in fact evident in Tarski's calculus of systems (Tarski [28], [29]), in his topological models for intuitionistic and modal logic (Tarski [30], McKinsey and Tarski [19]), as well as in Kuratowski and Tarski [13], which is the most direct

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precursor of the theory of cylindric algebras. Relation algebras are an abstraction from the calculus of relations first studied by C. S. Peirce and carried to a high degree of development in E. Schröder [27]. Here one considers a nonempty domain U, and binary relations over U, i.e., subsets of the Cartesian square ${}^2U$. Upon such relations one can perform the usual Boolean operations: union $\cup$, intersection $\cap$, and complementation $\sim$ with respect to ${}^2U$. Two further operations specific to the theory of binary relations have been intensively studied: the *converse* R^{-1} of a relation $R \subseteq {}^2U$, defined by

$$R^{-1} = \{\langle x, y\rangle : \langle y, x\rangle \in R\},$$

and the *relative product* $R \mid S$ of relations $R, S \subseteq {}^2U$ defined by

$$R \mid S = \{\langle x, z\rangle : \text{there is a } y \text{ such that } \langle x, y\rangle \in R \text{ and } \langle y, z\rangle \in S\}.$$

Finally, three special relations play a role in the theory: the empty relation 0, the universal relation ${}^2U$, and the identity relation $U \upharpoonright \mathrm{Id}$, consisting of all pairs $\langle u, u\rangle$ with $u \in U$. A *relation set algebra* is a structure

$$\mathfrak{A} = \langle A, \cup, \cap, \sim, 0, {}^2U, |, {}^{-1}, U \upharpoonright \mathrm{Id}\rangle,$$

where A is a collection of relations over U closed under the indicated operations and having as members the three special relations above. A large portion of the work of Peirce and Schröder consists in the discovery of equations which hold in all relation set algebras. Schröder also investigated the solvability of equations in a detailed fashion. The abstraction which Tarski performed in [31] was to select a small finite number of the equations holding in all relation set algebras and define a *relation algebra* to be any algebra

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, ;, \smile, 1'\rangle,$$

similar to relation set algebras, in which the selected equations hold. Many valid equations, in fact all of those in [27] as far as is known, are derivable from the selected few, and hence hold in all relation algebras. It is natural to ask whether *all* valid equations can be derived. This question is closely related to the representation question discussed below, and, like that question, has a negative answer. Analogously to the theory of Boolean algebras, it is also natural to ask about the connections of relation algebras with logic on the one hand, and with the calculus of relations on the other hand. The answer to the first question is elementary; roughly speaking, relation algebras correspond to predicate logic in which only three individual variables are allowed and all predicates are at most binary. This result can be made plausible by observing that all of the operations in a relation set algebra can be defined using three variables, and that actually the definition of relative product does require three variables. The second question can be posed as a representation problem: Is every relation algebra isomorphic to a subdirect product of relation set algebras? The reason for formulating the problem in this way rather than asking for isomorphism with a single relation set algebra will become evident in the discussion of cylindric algebras. Unfortunately this second question has a negative answer, as shown in Lyndon [16]. One can then consider the class RRA of all *representable* relation algebras: those which *are* isomorphic to a subdirect product of relation set algebras. In Tarski [33] it is shown that RRA is an equational class, while in Monk [21] it is shown that RRA is not finitely axiomatizable. Some further developments in the theory of relation algebras indicate the depth of the theory. Jónsson [10] indicated a connection with projective geometry which was generalized in Lyndon [17]. With any projective geometry G, Lyndon associated a relation algebra $\mathfrak{A}_G$. In case G is finite, G is a hyperplane in a higher dimensional space if and only if $\mathfrak{A}_G \in$ RRA. This result is essential in the proof in [21] that RRA is not finitely axiomatizable. Close connections between group theory and the theory of relation algebras are indicated in McKenzie [20]. Finally, we may mention that the equational theory of relation algebras was shown to be undecidable by Tarski; see Tarski [32].

As we have indicated, relation algebras correspond to a restricted portion of predicate logic, in which, in particular, only three variables are used. Historically, the next step in the development of a full algebraic version of predicate logic was the *projective algebras* of Everett and Ulam [5]; see also McKinsey [18]. These algebras are again abstracted from the calculus of binary relations, but the basic operations are here the projection operations upon the coordinate axes, thinking of the relations as subsets of the "plane" ${}^2U$. Such operations are not as powerful as the relative product operation. Furthermore, the logical counterpart of projective algebras is a logic with only two variables. Thus the passage from relation algebras to projective algebras does not seem to be a generalization in the direction of a more comprehensive algebraic version of logic. But in fact the basic notions of projective algebras immediately suggest a generalization to arbitrary dimensions. Thus they may be viewed as precursors to cylindric algebras, to which we now turn.

The concept of cylindric algebras was invented and initial work done by Tarski in collaboration with his students Louise Chin (Lim) and Frederick Thompson. We can best understand this concept by relating it to predicate logic. Let $\mathscr{L}$ be an arbitrary language of predicate logic, and let $\Phi\mu_{\mathscr{L}}$ be the set of formulas of $\mathscr{L}$. We assume that $\mathscr{L}$ has a simple infinite sequence $v_0, v_1, \ldots$ of individual variables. We consider the usual syntactic operations $\vee$, $\wedge$, $\neg$, $\exists_{v_\kappa}$ of forming the disjunction, conjunction, negation, and existential quantification over v_κ, of formulas. Additionally, we assume that $\mathscr{L}$ is provided with a falsehood symbol **F** and a truth symbol **T**. Now we consider the *formula algebra* of $\mathscr{L}$, $\mathfrak{Fm}_{\mathscr{L}}$, defined as

$$\mathfrak{Fm}_{\mathscr{L}} = \langle \Phi\mu_{\mathscr{L}}, \vee, \wedge, \neg, \mathbf{F}, \mathbf{T}, \exists_{v_\kappa}, v_\kappa = v_\lambda \rangle_{\kappa,\lambda<\omega}.$$

Let Γ be any set of sentences of $\mathscr{L}$. We define a binary relation $\equiv_\Gamma$ on $\Phi\mu_{\mathscr{L}}$ by setting $\varphi \equiv_\Gamma \psi$ iff the biconditional $\varphi \leftrightarrow \psi$ is a consequence of Γ, i.e., holds in every model of Γ, or is derivable from Γ. It is easily seen that $\equiv_\Gamma$ is an equivalence relation on $\Phi\mu_{\mathscr{L}}$, and in fact is a congruence relation on $\mathfrak{Fm}_{\mathscr{L}}$. The quotient algebra $\mathfrak{Fm}_{\mathscr{L}}/\equiv_\Gamma$ is an example of a cylindric algebra; we call it the *Tarski algebra* associated with $\mathscr{L}$ and Γ. We select certain equations which hold in all algebras $\mathfrak{Fm}_{\mathscr{L}}/\equiv_\Gamma$ and take them as axioms for abstract cylindric algebras. Precisely speaking, for any ordinal α a *cylindric algebra of dimension* α, for brevity a CA_α, is an algebra $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, c_\kappa, d_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha}$ satisfying the following conditions for all $\kappa, \lambda, \mu < \alpha$ and $x, y \in A$:

(C_0) $\langle A, +, \cdot, -, 0, 1\rangle$ is a Boolean algebra, c_κ maps A into A, and $d_{\kappa\lambda} \in A$;

(C_1) $c_\kappa 0 = 0$;

(C_2) $x \leqq c_\kappa x$;

(C_3) $c_\kappa(x \cdot c_\kappa y) = c_\kappa x \cdot c_\kappa y$;

(C_4) $c_\kappa c_\lambda x = c_\lambda c_\kappa x$;

(C_5) $d_{\kappa\kappa} = 1$;

(C_6) if $\kappa \neq \lambda, \mu$ then $d_{\lambda\mu} = c_\kappa(d_{\lambda\kappa} \cdot d_{\kappa\mu})$;

(C_7) if $\kappa \neq \lambda$, then $c_\kappa(d_{\kappa\lambda} \cdot x) \cdot c_\kappa(d_{\kappa\lambda} \cdot -x) = 0$.

If it is necessary to distinguish between operations in several algebras, we write $+^{(\mathfrak{A})}, c_\kappa^{(\mathfrak{A})}, d_{\kappa\lambda}^{(\mathfrak{A})}, +^{(\mathfrak{B})}, c_\kappa^{(\mathfrak{B})}$, etc.

The algebras $\mathfrak{Fm}_{\mathscr{L}}/\equiv_\Gamma$ present two peculiarities which have been abstracted from in presenting the general definition of a CA_α. One is their dimension, ω. The other is the following condition of *locally finite dimension*:

(C_8) For any $x \in A$ there are only finitely many $\kappa < \alpha$ such that $c_\kappa x \neq x$.

This is of course true in $\mathfrak{Fm}_{\mathscr{L}}/\equiv_\Gamma$, since any element of $\mathfrak{Fm}_{\mathscr{L}}/\equiv_\Gamma$ is an equivalence class $[\varphi]$ of a formula φ, and if v_κ is not among the finitely many variables occurring in φ we then have $c_\kappa[\varphi] = [\exists_{v_\kappa}\varphi] = [\varphi]$, since then $\exists_{v_\kappa}\varphi \leftrightarrow \varphi$ is logically valid. Thus each algebra $\mathfrak{Fm}_{\mathscr{L}}/\equiv_\Gamma$ is a CA_ω of locally finite dimension. Conversely, it has been shown that each locally finite dimensional CA_ω is isomorphic to some algebra $\mathfrak{Fm}_{\mathscr{L}}/\equiv_\Gamma$. This indicates the close connection of cylindric algebras with predicate logic.

The abstract theory of cylindric algebras is exhaustively treated in Henkin, Monk and Tarski [8]. Equations holding in all CA_α's are derived there, and the operations c_κ and elements $d_{\kappa\lambda}$ are generalized. The algebraic theory of CA_α's—ideals, products, free algebras, etc.—is carefully developed.

We now want to turn, however, to the representation theory of cylindric algebras. We are concerned here with the calculus of many-placed relations, analogous to the consideration of fields of sets with regard to Boolean algebras, and relation set algebras with regard to relation algebras. Let U be a nonempty set. Subsets of ${}^\alpha U$ may be thought of as α-ary relations, and we may perform on them the usual Boolean operations $\cup, \cap, \sim$ (complementation $\sim$ being performed with respect to ${}^\alpha U$). Now for each $\kappa < \alpha$ we introduce a unary operation $C_\kappa^{({}^\alpha U)}$, or simply C_κ, as follows. For any $R \subseteq {}^\alpha U$,

$$C_\kappa^{({}^\alpha U)} R = \{x \in {}^\alpha U\colon \text{there is a } y \in R \text{ with } x_\lambda = y_\lambda \text{ for all } \lambda < \alpha \text{ with } \lambda \neq \kappa\}.$$

Also we consider special sets $D_{\kappa\lambda}^{({}^\alpha U)}$ for any $\kappa, \lambda < \alpha$:

$$D_{\kappa\lambda}^{({}^\alpha U)} = \{x \in {}^\alpha U\colon x_\kappa = x_\lambda\}.$$

Now a *cylindric set algebra of dimension* α *with base* U *and unit set* ${}^{\alpha}U$, for brevity a $\mathbf{Cs}_{\alpha}$, is a structure

$$\mathfrak{A} = \langle A, \cup, \cap, \sim, 0, {}^{\alpha}U, C_{\kappa}^{({}^{\alpha}U)}, D_{\kappa\lambda}^{({}^{\alpha}U)}\rangle_{\kappa,\lambda<\alpha},$$

where A is closed under the indicated operations and has as elements the indicated special subsets of ${}^{\alpha}U$. It is easily checked that each $\mathbf{Cs}_{\alpha}$ is indeed a $\mathbf{CA}_{\alpha}$. The basic notions can be conveniently indicated in the case $\alpha = 3$; see Figure 1.

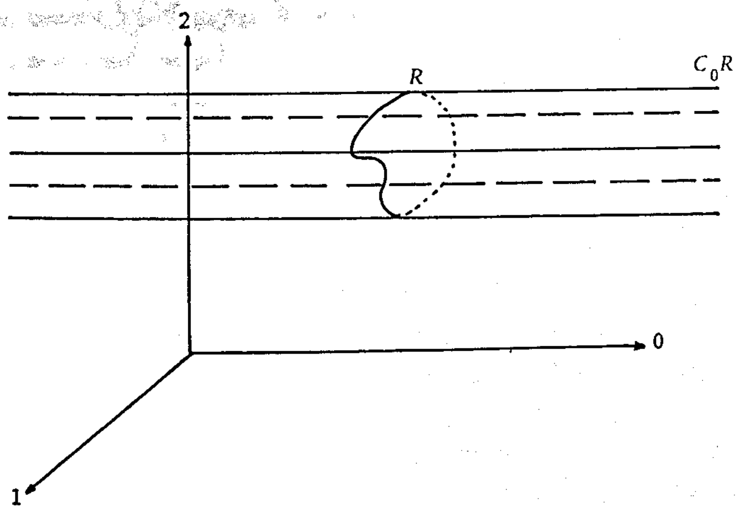

FIGURE 1

A subset $R \subseteq {}^{3}U$ can then be pictured as a point set in 3-space, and C_0R is the cylinder generated by moving R parallel to the 0-axis. Analogous considerations apply to C_1R and C_2R. Hence the operations C_{κ} may be called *cylindrifications*; this also shows the origin of the name *cylindric algebra*. Also, D_{01} consists of all points equidistant from the 0- and 1-axis, and is thus a diagonal plane. Similarly D_{02} and D_{12} are diagonal planes; and of course $D_{10} = D_{01}$, $D_{20} = D_{02}$, and $D_{21} = D_{12}$. We have $D_{\kappa\kappa} = {}^{3}U$ for any $\kappa < 3$; ${}^{3}U$ may be considered a degenerate diagonal plane. The elements $D_{\kappa\lambda}$ are sometimes called *diagonal elements*.

The connection of $\mathbf{Cs}_{\alpha}$'s with the Tarski algebras $\mathfrak{Fm}_{\mathscr{L}}/\equiv_{\Gamma}$ can be seen as follows. Let $\mathscr{L}$ be a first-order language, Γ a set of sentences of $\mathscr{L}$, and $\mathfrak{M} = \langle U, R_i\rangle_{i\in I}$ a model of Γ. For each formula φ of $\mathscr{L}$ let $\mathrm{S}_{\mathfrak{M}}\varphi$ be the collection of all $x \in {}^{\omega}U$ which satisfy φ in $\mathfrak{M}$. Then it is easily seen that $\mathrm{S}_{\mathfrak{M}}$ induces a homomorphism $\mathrm{S}_{\mathfrak{M}}$ from $\mathfrak{Fm}_{\mathscr{L}}/\equiv_{\Gamma}$ onto a $\mathbf{Cs}_{\omega}$ with base U. In particular, for any formula φ of $\mathscr{L}$ and any $\kappa < \omega$, we have

$$\mathrm{S}_{\mathfrak{M}}[\exists_{v_{\kappa}}\varphi] = C_{\kappa}\mathrm{S}_{\mathfrak{M}}[\varphi],$$

which indicates the close connection between cylindrifications and existential qualifications. Similarly, $\mathrm{S}_{\mathfrak{M}}[v_{\kappa} = v_{\lambda}] = D_{\kappa\lambda}$.

The question naturally arises whether every cylindric algebra is isomorphic to a cylindric set algebra; we may call this a *representation problem*. In this form, however, the representation problem has an easy negative answer. In fact, first suppose that $\alpha < \omega$. If $\mathfrak{A}$ is a $\mathbf{Cs}_\alpha$ with base U and $0 \neq R \subseteq {}^\alpha U$, then ${}^\alpha U = C_0 \cdots C_{\alpha-1} R$, as is easily verified. Thus any $\mathbf{CA}_\alpha$ isomorphic to a $\mathbf{Cs}_\alpha$ identically satisfies the condition

$$x \neq 0 \rightarrow \mathbf{c}_0 \cdots \mathbf{c}_{\alpha-1} x = 1.$$

However, $\mathfrak{A} \times \mathfrak{A}$ fails to satisfy this condition, since $\langle 1, 0\rangle \neq \langle 0, 0\rangle$ but

$$c_0 \cdots c_{\alpha-1}\langle 1, 0\rangle = \langle 1, 0\rangle \neq \langle 1, 1\rangle.$$

Thus $\mathfrak{A} \times \mathfrak{A}$ is not isomorphic to a $\mathbf{Cs}_\alpha$. On the other hand, since $\mathbf{CA}_\alpha$ is defined by a set of equations, it is obvious that $\mathfrak{A} \times \mathfrak{A} \in \mathbf{CA}_\alpha$. Thus $\mathfrak{A} \times \mathfrak{A}$ is a $\mathbf{CA}_\alpha$ not isomorphic to any $\mathbf{Cs}_\alpha$. In case $\alpha \geqq \omega$, a different construction is needed to find such a $\mathbf{CA}_\alpha$: Let $\mathfrak{A}$ be the $\mathbf{Cs}_\alpha$ of all subsets of ${}^\alpha 2$, and let $I = 2^{2^{|\alpha|}}$. Then ${}^I\mathfrak{A}$ (the Ith direct power of $\mathfrak{A}$) has cardinality $> 2^{2^{|\alpha|}}$. Suppose ${}^I\mathfrak{A}$ is isomorphic to a $\mathbf{Cs}_\alpha$ $\mathfrak{B}$; say that $\mathfrak{B}$ has base U. Now the equation

$$\mathbf{d}_{01} + \mathbf{d}_{02} + \mathbf{d}_{12} = 1$$

holds in $\mathfrak{A}$, hence also in ${}^I\mathfrak{A}$ and in $\mathfrak{B}$. But this implies that $|U| \leqq 2$, and hence $|{}^I A| = |B| \leqq 2^{2^{|\alpha|}}$, contradicting the fact noted above that $|{}^I A| > 2^{2^{|\alpha|}}$. Thus for $\alpha \geqq \omega$, ${}^I\mathfrak{A}$ is a $\mathbf{CA}_\alpha$ not isomorphic to any $\mathbf{Cs}_\alpha$. Because of these two examples, we now reformulate the representation problem, in a standard algebraic fashion. A $\mathbf{CA}_\alpha$ $\mathfrak{A}$ is said to be *representable*, in symbols $\mathfrak{A} \in \mathbf{R}_\alpha$, if $\mathfrak{A}$ is isomorphic to a subdirect product of $\mathbf{Cs}_\alpha$'s. Unfortunately, even in this form it turns out that for each $\alpha \geqq 2$ there is a nonrepresentable $\mathbf{CA}_\alpha$. Many properties of the class $\mathbf{R}_\alpha$ are known, however. We have $\mathbf{R}_0 = \mathbf{CA}_0$ and $\mathbf{R}_1 = \mathbf{CA}_1$, almost trivially. As Henkin has shown, the class $\mathbf{R}_2$ can be characterized by the addition of two simple equations to the conditions (C_0)–(C_7). Tarski showed that $\mathbf{R}_\alpha$ is always an equational class, while Monk showed in [22] that $\mathbf{R}_\alpha$ is not finitely axiomatizable for $\alpha \geqq 3$. One of the earliest results in the theory of cylindric algebras, due to Tarski, is that any locally finite dimensional $\mathbf{CA}_\alpha$ of infinite dimension is representable. This may be considered as an algebraic version of the completeness theorem for predicate logic. Various characterizations of the class $\mathbf{R}_\alpha$ are known. One of the most useful is the following, due to Henkin:

The following conditions are equivalent:
(i) $\mathfrak{A} \in \mathbf{R}_\alpha$,
(ii) *there is a* $\mathbf{CA}_{\alpha+\omega}$ $\mathfrak{B}$ *such that* $\mathfrak{A}$ *is a subalgebra of*

$$\langle B, +, \cdot, -, 0, 1, c_\kappa^{(\mathfrak{B})}, d_{\kappa\lambda}^{(\mathfrak{B})}\rangle_{\kappa,\lambda<\alpha}$$

and $c_{\alpha+\kappa}^{(\mathfrak{B})} x = x$ *for all* $x \in A$ *and all* $\kappa < \omega$.

To finish this brief survey of the theory of cylindric algebras, we mention a few metalogical results concerning them. Based upon his work on relation algebras, Tarski has shown that the equational theory of $\mathbf{CA}_\alpha$'s is undecidable for any $\alpha \geqq 4$. The equational theories of $\mathbf{CA}_0$'s and $\mathbf{CA}_1$'s are trivially decidable, and

Henkin has shown that the equational theory of CA_2's is decidable. The first-order theory of CA_α's, for any fixed $\alpha \geqq 2$, is undecidable.

We shall return to the representation problem for cylindric algebras in the next section. To conclude this section we want to mention briefly some other versions of algebraic logic. Subsequent to Tarski's first work concerning cylindric algebras, which was mainly reported in abstracts, several other algebraic versions of predicate logic have been introduced. The most well-developed of these is the theory of *polyadic algebras* of Halmos; his work on them is collected in [6]. Roughly speaking, they differ from cylindric algebras in having as primitive notions cylindrifications on infinite subsets of α and other operations S_τ for $\tau \in {}^\alpha\alpha$, corresponding to substitution in formulas. Daigneault and LeBlanc have further developed the theory of polyadic algebras, devoting most of their efforts to formulating logical theorems in algebraic form and providing them with purely algebraic proofs. This program has been carried through, for example, for Beth's theorem, Feferman-Vaught generalized products, and, partly, for Gödel's incompleteness theorem. See Daigneault [3], [4] and LeBlanc [15]. Extensive work in algebraic logic has been done by Craig. Some of his versions of algebraic logic derive from taking satisfaction by finite sequences as basic in defining set algebras, rather than satisfaction by infinite sequences, as is implicitly done above; see, e.g., Monk [23]. Among other work in algebraic logic we may mention the many-sorted relation algebras of Bernays [1], algebraic versions of higher-order logic in LeBlanc [14] and Venne [35], the general notion of relation algebra in Šaĭn [26], as well as the Boolean algebras with operators developed in Jónsson and Tarski [11], [12], Henkin [7], and Monk [24], which provide a general framework for many of these versions. For a comprehensive bibliography of algebraic logic, see [8].

2. A representation theorem. The main result we want to establish is as follows. *For infinite α, any subdirect product of Cs_α's each having an infinite base is isomorphic to a* Cs_α. This is formulated as Theorem 2 below.

As we have seen in §1, the hypothesis $\alpha \geqq \omega$ is necessary here, and also the hypothesis that each base is infinite cannot be dropped. From Theorem 2 we see that under these two hypotheses the reformulation of the representation problem made in §1 is not necessary. We shall establish a result somewhat stronger than Theorem 2. To formulate it, and lemmas needed in its proof, we need some new notions.

Let U be a nonempty set and $V \subseteq {}^\alpha U$. For any $R \subseteq V$ and $\kappa < \alpha$, we define $C_\kappa^{(V)}R = C_\kappa^{({}^\alpha U)}R \cap V$. For κ, $\lambda < \alpha$, we set $D_{\kappa\lambda}^{(V)} = D_{\kappa\lambda}^{({}^\alpha U)} \cap V$. By a **Gcs**$_\alpha$ *with unit set* V we mean a structure

$$\mathfrak{A} = \langle A, \cup, \cap, \sim, 0, V, C_\kappa^{(V)}, D_{\kappa\lambda}^{(V)} \rangle_{\kappa,\lambda<\alpha}$$

such that $\langle A, \cup, \cap, \sim, 0, V\rangle$ is a Boolean algebra of subsets of V, A is closed under each operation $C_\kappa^{(V)}$, and $D_{\kappa\lambda}^{(V)} \in A$ for any κ, $\lambda < \alpha$; its *base* is U. It should be remarked that not every **Gcs**$_\alpha$ is a CA_α. Furthermore, many **Gcs**$_\alpha$'s are known which *are* $\mathbf{CA}_\alpha$'s, but are nonrepresentable. We shall not go into the theory of general $\mathbf{Gcs}_\alpha$'s here.

Again, let U be a nonempty set. For $p, q \in {}^\alpha U$ we define $p \approx q$ to mean that $\{\kappa < \alpha : p_\kappa \neq q_\kappa\}$ is finite. The relation $\approx$ is clearly in equivalence relation on ${}^\alpha U$. The equivalence class of an element $p \in {}^\alpha U$ is denoted by ${}^\alpha U^{(p)}$ and is called a *weak Cartesian space*. A $\mathbf{Gcs}_\alpha$ whose unit set is a weak Cartesian space is called a *weak cylindric set algebra of dimension* α; the class of all such is denoted by $\mathbf{WCs}_\alpha$. It is easy to verify that every $\mathbf{WCs}_\alpha$ is a $\mathbf{CA}_\alpha$.

The stronger version of Theorem 2, referred to above, states that: *For infinite* α, *any subdirect product of* $\mathbf{WCs}_\alpha$*'s each having an infinite base is isomorphic to a* $\mathbf{Cs}_\alpha$. This is formulated as Theorem 1 below. Theorem 2 follows readily from it because each $\mathbf{Cs}_\alpha$ is a subdirect product of $\mathbf{WCs}_\alpha$'s (Lemma 13 below).

The idea of the proof of Theorem 1 is as follows. We first show that any $\mathbf{WCs}_\alpha$ with infinite base is isomorphic to a $\mathbf{WCs}_\alpha$ having a base of cardinality larger than any preassigned cardinal number (Lemma 4). Then, given any set of $\mathbf{WCs}_\alpha$'s each having an infinite base, we can choose a single set U of sufficiently high cardinality so that each of the given $\mathbf{WCs}_\alpha$'s can be represented isomorphically as a $\mathbf{WCs}_\alpha$ with base U, in such a way that the representatives of any two distinct given $\mathbf{WCs}_\alpha$'s have disjoint unit sets. These steps are accomplished by algebraic versions of the upward and downward Löwenheim-Skolem theorems (Lemmas 3 and 5). Finally, we show that a product of $\mathbf{WCs}_\alpha$'s, each having the same base U, but having pairwise disjoint unit sets, is isomorphic to a $\mathbf{Cs}_\alpha$ with base U. (See Lemmas 10–12.)

In our final result (Theorem 3), we obtain a weaker form of representation for subdirect products of $\mathbf{WCs}_\alpha$'s having finite bases of the same cardinality.

LEMMA 1. *Let* $\mathfrak{A}$ *be a* $\mathbf{WCs}_\alpha$ *with unit set* ${}^\alpha U^{(p)}$, *and let* f *be a one-one function mapping* U *onto a set* W. *Then* $\mathfrak{A}$ *is isomorphic to a* $\mathbf{WCs}_\alpha$ *with unit set* ${}^\alpha W^{(f\circ p)}$.

PROOF. For each $X \in A$ let $\bar{f}X = \{f \circ x : x \in X\}$. It is easily checked that $\bar{f}$ is the desired isomorphism.

LEMMA 2. *Let* U *be a nonempty set,* $p \in {}^\alpha U$, *and* $\mathfrak{A}$ *the* $\mathbf{WCs}_\alpha$ *of all subsets of* ${}^\alpha U^{(p)}$. *Then* $\mathfrak{A}$ *is subdirectly indecomposable. In fact, if* f *is any homomorphism on* $\mathfrak{A}$ *which is not one-one, then* $f\{p\} = 0$.

PROOF. Suppose f is a homomorphism on $\mathfrak{A}$ which is not one-one. Say $fX = 0$, where $X \neq 0$. Choose $q \in X$. Then $\{\kappa : p_\kappa \neq q_\kappa\}$ is finite, say $\{\kappa : p_\kappa \neq q_\kappa\} = \{\kappa_0, \ldots, \kappa_{\lambda-1}\}$ with $\kappa_0 < \cdots < \kappa_{\lambda-1}$. Clearly then $p \in c_{\kappa_0} \cdots c_{\kappa_{\lambda-1}} X$, and hence

$$f\{p\} \leqq f c_{\kappa_0} \cdots c_{\kappa_{\lambda-1}} X = c_{\kappa_0} \cdots c_{\kappa_{\lambda-1}} fX = c_{\kappa_0} \cdots c_{\kappa_{\lambda-1}} 0 = 0.$$

Thus $f\{p\} = 0$, as desired.

The following lemma may be considered as an algebraic version of the fundamental theorem on ultraproducts, or rather the specialization of that theorem to ultrapowers, which is all we need in this paper.

LEMMA 3. *Let $\mathfrak{A}$ be a* $\mathbf{WCs}_\alpha$ *with unit set* ${}^\alpha U^{(p)}$. *Let I be any set, and F an ultrafilter over I; set* $W = {}^I U/F$. *Assume that* $s' \in {}^\alpha({}^I U)$ *and that* $s_\kappa = s'_\kappa/F$ *for any* $\kappa < \alpha$. *Let f be a choice function for W, i.e.,* $fx \in x$ *for each* $x \in W$, *and for each* $i \in I$ *let* pr_i *be the function mapping* ${}^I U$ *into U such that* $pr_i y = yi$ *for any* $y \in {}^I U$. *Define u mapping* ${}^\alpha W$ *into* ${}^\alpha({}^I U)$ *by setting, for any* $x \in {}^\alpha W$ *and any* $\kappa < \alpha$,

$$(1)\qquad \begin{aligned}(ux)_\kappa &= s'_\kappa \quad \text{if } x_\kappa = s_\kappa,\\ &= fx_\kappa \quad \text{otherwise.}\end{aligned}$$

Now assume that

$$(2)\qquad \text{for any } x \in {}^\alpha W^{(s)} \text{ and } i \in I \text{ we have } pr_i \circ ux \in {}^\alpha U^{(p)}.$$

Then there is a homomorphism g of ${}^I\mathfrak{A}/F$ *into a* $\mathbf{WCs}_\alpha$ *with unit set* ${}^\alpha W^{(s)}$ *such that, for any* $a \in {}^I A$,

$$(3)\qquad g(a/F) = \{x \in {}^\alpha W^{(s)}: \{i \in I: pr_i \circ ux \in a_i\} \in F\}.$$

PROOF. Let $V = {}^\alpha W^{(s)}$, $X = {}^\alpha U^{(p)}$. It is easily seen that there is a function g satisfying (3), and it is obvious that g preserves $+$. Furthermore, using (2) it is easy to check that g preserves $-$. Now suppose $x \in g(d_{\kappa\lambda}/F)$. Thus $x \in V$ and $\{i \in I: pr_i \circ ux \in D^{(X)}_{\kappa\lambda}\} \in F$. Since $\{i \in I: pr_i \circ ux \in D^{(X)}_{\kappa\lambda}\} \subseteq \{i \in I: (ux)_\kappa i = (ux)_\lambda i\}$, it follows easily that $(ux)_\kappa/F = (ux)_\lambda/F$. But $(ux)_\mu/F = x_\mu$ for any $\mu < \alpha$, by (1). Thus $x_\kappa = x_\lambda$, so $g(d_{\kappa\lambda}/F) \subseteq D^{(V)}_{\kappa\lambda}$. The converse is analogously established, using (2).

To show that g preserves c_κ, first note, by (1),

$$(4)\qquad \text{if } x, y \in V,\ \kappa < \alpha \text{ and } x_\kappa = y_\kappa, \text{ then } (ux)_\kappa = (uy)_\kappa.$$

Now assume that $x \in g(c_\kappa a/F)$. Thus

$$(5)\qquad x \in V \text{ and } J \in F, \text{ where } J = \{i \in I: pr_i \circ ux \in C^{(X)}_\kappa a_i\}.$$

Choose $t \in {}^I X$ such that for any $i \in J$ we have $(pr_i \circ ux)\lambda = t_i\lambda$ for all $\lambda \in \alpha \sim \{\kappa\}$, and $t_i \in a_i$. Define t' mapping α into ${}^I U$ by setting, for any $\lambda < \alpha$ and $i \in I$, $t'_\lambda i = t_i\lambda$. Finally, for any $\lambda < \alpha$ let $t''_\lambda = t'_\lambda/F$. Thus $t'' \in {}^\alpha W$. Furthermore,

$$(6)\qquad t''_\lambda = x_\lambda \quad \text{whenever } \lambda \in \alpha \sim \{\kappa\}.$$

Indeed, for $\lambda \in \alpha \sim \{\kappa\}$ and $i \in J$, we have

$$(ux)_\lambda i = (pr_i \circ ux)_\lambda = t_i\lambda = t'_\lambda i.$$

Since $J \in F$ by (5), it follows that $(ux)_\lambda/F = t'_\lambda/F$, i.e., $x_\lambda = t''_\lambda$. Thus (6) holds. Since $x \in V$ by (5), we also have

$$(7)\qquad t'' \in V.$$

Now let $K = \{i \in I: (ut'')_\kappa i = t'_\kappa i\}$. Since by (1) we have $(ut'')_\kappa \in t''_\kappa = t'_\kappa/F$, it follows that $K \in F$. Thus, by (5),

$$(8)\qquad J \cap K \in F.$$

Now

(9) $$J \cap K \subseteq \{i \in I : pr_i \circ ut'' = t_i\}.$$

Indeed, let $i \in J \cap K$,

$$(pr_i \circ ut'')\kappa = (ut'')_\kappa i = t'_\kappa i = t_i\kappa;$$

if $\lambda \in \alpha \sim \{\kappa\}$, then, using (4) and (6),

$$(pr_i \circ ut'')\lambda = (ut'')_\lambda i = (ux)_\lambda i = (pr_i \circ ux)_\lambda = t_i\lambda.$$

Thus (9) holds. It now follows that $x \in C_\kappa^{(V)} g(a/F)$. In fact, $x \in V$ by (5), and $x_\lambda = t''_\lambda$ whenever $\lambda \in \alpha \sim \{\kappa\}$ by (6). Furthermore, $t'' \in V$ by (7), and by (8) and (9), $\{i \in I : pr_i \circ ut'' \in a_i\} \in F$, $t'' \in g(a/F)$. Thus we have shown that $g(c_\kappa a/F) \subseteq C_\kappa^{(V)} g(a/F)$.

Conversely, suppose that $x \in C_\kappa^{(V)} g(a/F)$. Thus $x \in V$, and there is a $y \in g(a/F)$ such that $x_\lambda = y_\lambda$ for any $\lambda \in \alpha \sim \{\kappa\}$. Thus by (3), $y \in V$ and $L \in F$, where $L = \{i \in I : pr_i \circ uy \in a_i\}$. Now using (4) we easily see that if $i \in I$ and $\lambda \in \alpha \sim \{\kappa\}$ then $(pr_i \circ ux)\lambda = (pr_i \circ uy)\lambda$. Thus by (2) we have $L \subseteq \{i \in I : pr_i \circ ux \in C_\kappa^{(X)} a_i\}$. Hence $x \in g(c_\kappa a/F)$, as desired.

The next lemma is an algebraic version of the upward Löwenheim-Skolem-Tarski theorem. Its proof is analogous to the usual ultraproduct proof of that theorem.

LEMMA 4. *Let $\mathfrak{A}$ be a* WCs_α *with an infinite base, and let* $\mathfrak{m}$ *be any cardinal. Then $\mathfrak{A}$ is isomorphic to a* WCs_α *with a base of cardinality* $\geqq \mathfrak{m}$.

PROOF We may assume that $\mathfrak{A}$ is the WCs_α of all subsets of ${}^\alpha U^{(p)}$, where U is infinite. Choose an index set I and an ultrafilter F over I such that $|{}^I U/F| \geqq \mathfrak{m}$. Set $W = {}^I U/F$. For each $\kappa < \alpha$ let s'_κ be the constant map of I onto $\{p_\kappa\}$, i.e., $(s'_\kappa)i = p_\kappa$ for every $i \in I$. Now let s, f, pr, and u be as in the statement of Lemma 3. We now verify condition (2) of Lemma 3. Let $x \in {}^\alpha W^{(s)}$. Set $\Gamma = \{\kappa : x_\kappa \neq s_\kappa\}$. Thus Γ is finite. If $\kappa \in \alpha \sim \Gamma$ and $i \in I$, then

$$(pr_i \circ ux)_\kappa = (ux)_\kappa i = (s'_\kappa)i = p_\kappa;$$

thus $pr_i \circ ux \in {}^\alpha U^{(p)}$ for any $i \in I$, so (2) holds. Thus we obtain a homomorphism g as indicated in Lemma 3. Now let h be the natural isomorphism of $\mathfrak{A}$ into ${}^I\mathfrak{A}/F$: For any $a \in A$, ha is $h'a/F$, where $(h'a)i = a$ for all $i \in I$. Note that $s \in gh\{p\}$. In fact, $us = s'$, so $pr_i \circ us = p$ for any $i \in I$. Also, $h\{p\} = h'\{p\}/F$, and $(h'\{p\})i = \{p\}$ for any $i \in I$. Thus $pr_i \circ us \in (h'\{p\})i$ for any $i \in I$. Thus, indeed, $s \in gh\{p\}$. Hence $gh\{p\} \neq 0$. By Lemma 2, $g \circ h$ is one-one. Hence $g \circ h$ is the desired isomorphism.

Next, we shall prove an algebraic version of the downward Löwenheim-Skolem theorem. The proof is patterned after the proof of the logical theorem given in Tarski and Vaught [34]. For any function p, $\mathrm{Ra}\, p$ is the *range* of p. If p is a function, i is in the domain of p, and a is arbitrary, we denote by p_a^i the function $(p \sim \{(i, p_i)\}) \cup \{(i, a)\}$.

LEMMA 5. *Let* $\mathfrak{m}$ *be an infinite cardinal such that* $\mathfrak{m}^{|\alpha|} = \mathfrak{m}$. *Suppose that* $\mathfrak{A}$ *is a* $\mathbf{WCs}_\alpha$ *with unit set* ${}^\alpha U^{(p)}$, *where* $|A| \leqq \mathfrak{m} \leqq |U|$. *Then there is a subset* W *of* U *with* $|W| = \mathfrak{m}$ *and* $\mathrm{Ra}\, p \subseteq W$ *such that* $\mathfrak{A}$ *is isomorphic to a* $\mathbf{WCs}_\alpha$ *with unit set* ${}^\alpha W^{(p)}$.

PROOF Let U be given a well-ordering. The hypothesis $\mathfrak{m}^{|\alpha|} = \mathfrak{m}$ implies that $\alpha < \mathfrak{m}$. Hence, also using the assumption that $|A| \leqq \mathfrak{m} \leqq |U|$, there is a subset T_0 of U such that $|T_0| = \mathfrak{m}$, $\mathrm{Ra}\, p \subseteq T_0$, and $X \cap {}^\alpha T_0 \neq 0$ whenever $0 \neq X \in A$. Now suppose that $0 < \lambda \leqq \mathfrak{m}$ and T_κ has been defined for all $\kappa < \lambda$. Let $M = \bigcup_{\kappa<\lambda} T_\kappa$ and let

$$T_\lambda = M \cup \{a \in U: \exists X \in A\ \exists \kappa < \alpha\ \exists u \in {}^\alpha M\ (a \text{ is the first element of } U \text{ such that } u_a^\kappa \in X)\}.$$

Let $W = T_{\mathfrak{m}}$. By transfinite induction it is easily seen that $|T_\kappa| = \mathfrak{m}$ for all $\kappa \leqq \mathfrak{m}$; in particular, $|W| = \mathfrak{m}$. Obviously $\mathrm{Ra}\, p \subseteq W$. Now for any $X \in A$ let $fX = X \cap {}^\alpha W^{(p)}$. Clearly f is a Boolean isomorphism from $\mathfrak{A}$ into the $\mathbf{WCs}_\alpha$ of all subsets of ${}^\alpha W^{(p)}$. It is also clear that f preserves $d_{\kappa\lambda}$. Now let $N = {}^\alpha U^{(p)}$ and $P = {}^\alpha W^{(p)}$. Suppose that $u \in fC_\kappa^{(N)}X$. Thus $u \in C_\kappa^{(N)}X \cap P$, so $u \in P$ and $u_a^\kappa \in X$ for some $a \in U$. Now $\mathfrak{m}^{|\alpha|} = \mathfrak{m}$ implies that $\alpha < \mathrm{cf}\, \mathfrak{m}$, so there is a $\lambda < \mathfrak{m}$ such that $u \in {}^\alpha T_\lambda$. There is then a $b \in T_{\lambda+1}$ such that $u_b^\kappa \in X$. Thus $u_b^\kappa \in fX$, so $u \in C_\kappa^{(P)} fX$. Thus we have shown that $fC_\kappa^{(N)}X \subseteq C_\kappa^{(P)} fX$. The converse is trivial, so the proof is complete.

In the next few lemmas we shall be concerned with the problem of changing the function p appearing as exponent in the unit set ${}^\alpha U^{(p)}$ of a $\mathbf{WCs}_\alpha$. If p is any function, say with domain I, we let $\ker p = \{(i,j): i, j \in I \text{ and } pi = pj\}$.

LEMMA 6. *Let* $\mathfrak{A}$ *be the* $\mathbf{WCs}_\alpha$ *of all subsets of* ${}^\alpha U^{(p)}$, *and* $\mathfrak{B}$ *the* $\mathbf{WCs}_\alpha$ *of all subsets of* ${}^\alpha U^{(q)}$. *Assume that* $\ker p = \ker q$. *Suppose that either* U *is finite, or* $\alpha < |U|$. *Then* $\mathfrak{A} \cong \mathfrak{B}$.

PROOF. Let $f = \{(p_\kappa, q_\kappa): \kappa < \alpha\}$. Then f is a one-one function mapping a subset of U into U. Since U is finite or $\alpha < |U|$, we can extend f to a permutation $\hat{f}$ of U. Now for any subset X of ${}^\alpha U^{(p)}$ we set

$$\hat{f}X = \{\hat{f} \circ u : u \in X\}.$$

It is easily verified that $\hat{f}$ is the desired isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$.

LEMMA 7. *Let* $\mathfrak{A}$ *be a* $\mathbf{WCs}_\alpha$ *with unit set* ${}^\alpha U^{(p)}$, $|U| > 1$, *and let* R *be any equivalence relation on* α *such that* $|\alpha/R| \leqq |U|$. *Then* $\mathfrak{A}$ *is homomorphic to a* $\mathbf{WCs}_\alpha$ *with unit set of the form* ${}^\alpha Y^{(q)}$, *where* $U \subseteq Y$ *and* $\ker q = R$. *Moreover, if* U *is finite we may assume that* $U = Y$.

PROOF. We shall apply Lemma 3. Let $I = \{\Gamma : \Gamma \subseteq \alpha,\ |\Gamma| < \omega\}$. For each $\Gamma \in I$ let $M_\Gamma = \{\Delta \in I : \Gamma \subseteq \Delta\}$, and let F be an ultrafilter over I such that $M_\Gamma \in F$ for all $\Gamma \in I$. Set $W = {}^I U/F$. For each $\Gamma \in I$, let $s''_\Gamma \in {}^\alpha U^{(p)}$ be such that $\ker s''_\Gamma \cap {}^2\Gamma = R \cap {}^2\Gamma$. Define $s' \in {}^\alpha({}^I U)$ by setting, for any $\kappa < \alpha$ and $\Gamma \in I$, $s'_\kappa \Gamma = s''_\Gamma \kappa$. Now let s, f, pr, and u be as in Lemma 3. We proceed to verify condition (2) of

Lemma 3. Let $x \in {}^{\alpha}W^{(s)}$. If $\Gamma \in I$ and $x_\kappa = s_\kappa$, then

$$(pr_\Gamma \circ ux)_\kappa = (ux)_\kappa\Gamma = (us)_\kappa\Gamma = s'_\kappa\Gamma = s''_\Gamma\kappa.$$

Thus $\{\kappa:(pr_\Gamma \circ ux)_\kappa \neq s''_\Gamma\kappa\}$ is finite, so, since $s''_\Gamma \in {}^{\alpha}U^{(p)}$, we also have $pr_\Gamma \circ ux \in {}^{\alpha}U^{(p)}$, as desired. Thus (2) of Lemma 3 holds. Hence from Lemma 3 and the fact that $\mathfrak{A}$ can be isomorphically embedded in ${}^{I}\mathfrak{A}/F$ our lemma follows as soon as we show that $\ker s = R$. In fact, we can then apply Lemma 1 to change W to a set $Y \supseteq U$, and if U is finite we must have $|Y| = |U|$ and hence $Y = U$. To show that $\ker s = R$, take any $\kappa, \lambda < \alpha$. Assume that $\langle \kappa, \lambda \rangle \in R$. For any $\Gamma \in M_{\{\kappa,\lambda\}}$ we then have $s''_\Gamma\kappa = s''_\Gamma\lambda$, hence $s'_\kappa\Gamma = s'_\lambda\Gamma$. Since $M_{\{\kappa,\lambda\}} \in F$, it follows easily that $s_\kappa = s_\lambda$. Similarly, $\langle \kappa, \lambda \rangle \notin R$ implies that $s_\kappa \neq s_\lambda$. Thus $\ker s = R$, and the proof is complete.

Lemma 8. *Let $\mathfrak{A}$ be a* $\mathbf{WCs}_\alpha$ *with unit set ${}^{\alpha}U^{(p)}$, U finite, and let $q \in {}^{\alpha}U$. Then $\mathfrak{A}$ is homomorphic to a* $\mathbf{WCs}_\alpha$ *with unit set ${}^{\alpha}U^{(q)}$.*

Proof. By Lemma 7, $\mathfrak{A}$ is homomorphic to a $\mathbf{WCs}_\alpha$ with unit set ${}^{\alpha}U^{(r)}$, where $\ker r = \ker q$. By Lemma 6 we may assume that $r = q$.

Lemma 9. *If $\mathfrak{A}$ is a* $\mathbf{WCs}_\alpha$ *with an infinite base and $\mathfrak{m}$ is any cardinal such that $|A| \leqq \mathfrak{m}$ and $\mathfrak{m}^{|\alpha|} = \mathfrak{m}$, then $\mathfrak{A}$ is isomorphic to a* $\mathbf{WCs}_\alpha$ *whose base has power $\mathfrak{m}$.*

Proof. By Lemmas 4 and 5.

Lemma 10. *If $\mathfrak{A}$ is a* $\mathbf{WCs}_\alpha$ *with unit set ${}^{\alpha}U^{(p)}$, where U is infinite, $|A| \leqq |U|$, $|\alpha| < |U|$, and $|U|^{|\alpha|} = |U|$, and if $q \in {}^{\alpha}U$, then $\mathfrak{A}$ is homomorphic to a* $\mathbf{WCs}_\alpha$ *with unit set ${}^{\alpha}U^{(q)}$.*

Proof. By Lemma 7, $\mathfrak{A}$ is homomorphic to a $\mathbf{WCs}_\alpha$ $\mathfrak{B}$ with unit set ${}^{\alpha}Y^{(r)}$, where $U \subseteq Y$ and $\ker r = \ker q$. By Lemma 6 we may assume that $r = q$, and by Lemma 5 we may assume that $|U| = |Y|$. Since $|\alpha| < |U|$, there is a one-one function f mapping Y onto U such that $f \circ q = q$. Hence the desired result follows by Lemma 1.

Next, we need to represent products of weak set algebras in a geometric fashion.

Lemma 11. *Assume $\alpha \geqq 2$. Let $V = \bigcup_{i \in I} {}^{\alpha}U_i^{(p_i)}$ each $U_i \neq 0$, where for $i, j \in I$ and $i \neq j$ we have ${}^{\alpha}U_i^{(p_i)} \cap {}^{\alpha}U_j^{(p_j)} = 0$. Further assume that either $U_i = U_j$ whenever $i, j \in I$, or else that $U_i \cap U_j = 0$ whenever $i, j \in I$ and $i \neq j$. Let $\mathfrak{A}$ be the* $\mathbf{Gcs}_\alpha$ *of all subsets of V, and for each $i \in I$ let $\mathfrak{B}_i$ be the* $\mathbf{WCs}_\alpha$ *of all subsets of ${}^{\alpha}U_i^{(p_i)}$. For any $x \in A$ and $i \in I$ let $(fx)_i = x \cap {}^{\alpha}U_i^{(p_i)}$. For any $y \in P_{i \in I}B_i$ let $gy = \bigcup_{i \in I} y_i$. Then f is an isomorphism from $\mathfrak{A}$ onto $P_{i \in I}\mathfrak{B}_i$, and g is its inverse.*

Proof. Clearly f is a Boolean isomorphism from $\mathfrak{A}$ onto $P_{i \in I}\mathfrak{B}_i$, with g as its inverse; furthermore, f clearly preserves $d_{\kappa\lambda}$. Now let $W_i = {}^{\alpha}U_i^{(p_i)}$ for each $i \in I$ and let $u \in (fC_\kappa^{(V)}x)_i$. Thus $u \in C_\kappa^{(V)}x \cap W_i$. Say $v \in x$, where $u_\lambda = v_\lambda$ for all $\lambda \in \alpha \sim \{\kappa\}$. Say $v \in W_j$. If $U_k = U_l$ for all $k, l \in I$, then $U_i = U_j$ and $v \in W_i$, so $i = j$; hence $v \in x \cap W_i$. If $U_k \cap U_l = 0$ whenever $k, l \in I$ and $k \neq l$, then, since $\alpha \geqq 2$, there is a $\lambda \in \alpha \sim \{\kappa\}$, and $u_\lambda \in U_i$, $v_\lambda \in U_j$, $u_\lambda = v_\lambda$, so $i = j$; hence again

$v \in x \cap W_i$. Thus $u \in C_\kappa^{(V)}(x \cap W_i) \cap W_i = C_\kappa^{(W_i)}(fx)_i$. We have shown that $fC_\kappa^{(V)}x \leqq c_\kappa fx$. The converse is trivial, so the proof is complete.

Our last lemma for Theorem 1 is the following simple result of a general algebraic nature.

LEMMA 12. *Let $\langle \mathfrak{A}_j : j \in J \rangle$ be a system of similar algebras. Let I be a subset of J. Assume that for some $i_0 \in I$ and each $j \in J \sim I$ there is a homomorphism f_j from $\mathfrak{A}_{i_0}$ into $\mathfrak{A}_j$. Then $P_{i \in I}\mathfrak{A}_i$ can be isomorphically embedded in $P_{j \in J}\mathfrak{A}_j$.*

PROOF. For any $x \in P_{i \in I}A_i$ and any $j \in J$ we define

$$\begin{aligned}(gx)_j &= x_j \quad \text{if } j \in I,\\ &= f_j x_{i_0} \quad \text{if } j \in J \sim I.\end{aligned}$$

Clearly g is the desired isomorphism.

THEOREM 1. *Assume $\alpha \geqq \omega$. For each $i \in I$ let $\mathfrak{A}_i$ be a* $\mathbf{WCs}_\alpha$ *with unit set ${}^\alpha U_i^{(p_i)}$, each U_i infinite. Then $P_{i \in I}\mathfrak{A}_i$ is isomorphic to a* $\mathbf{Cs}_\alpha$.

PROOF. We may assume that $I \neq 0$. Let $\mathfrak{m}$ be an infinite cardinal such that $|I| \leqq \mathfrak{m}$, $|A_i| \leqq \mathfrak{m}$, and $\mathfrak{m}^{|\alpha|} = \mathfrak{m}$. By Lemma 9, each $\mathfrak{A}_i$ is isomorphic to a $\mathbf{WCs}_\alpha$ $\mathfrak{B}_i$ with unit set ${}^\alpha V_i^{(q_i)}$, where $|V_i| = \mathfrak{m}$. By Lemma 1 we may assume that $V_i \cap V_j = 0$ whenever $i, j \in I$ and $i \neq j$. Let $W = \bigcup_{i \in I} V_i$. Note that $|W| = |V_i|$ for each $i \in I$. Furthermore, $|\alpha| < \mathfrak{m}$ since $\mathfrak{m}^{|\alpha|} = \mathfrak{m}$. Hence for each $i \in I$ there is a one-one function f_i mapping V_i onto W such that $f_i \circ q_i = q_i$. Hence by Lemma 1, $\mathfrak{B}_i$ is isomorphic to a $\mathbf{WCs}_\alpha$ $\mathfrak{C}_i$ with unit set ${}^\alpha W^{(q_i)}$. Since $\operatorname{Ra} q_i \subseteq V_i$ for any $i \in I$, and the V_i's are pairwise disjoint, it follows that ${}^\alpha W^{(q_i)} \cap {}^\alpha W^{(q_j)} = 0$ whenever $i, j \in I$ and $i \neq j$. Now let J be a superset of I, and let q_j be defined for all $j \in J \sim I$ so that the following conditions hold:

(1) ${}^\alpha W = \bigcup_{j \in J} {}^\alpha W^{(q_j)}$,

(2) ${}^\alpha W^{(q_j)} \cap {}^\alpha W^{(q_\kappa)} = 0$ whenever $j, k \in J$ and $j \neq k$.

For each $j \in J \sim I$ let $\mathfrak{C}_j$ be the $\mathbf{WCs}_\alpha$ of all subsets of ${}^\alpha W^{(q_j)}$. By Lemmas 10 and 12, $P_{i \in I}\mathfrak{A}_i$ is isomorphic to a subalgebra of $P_{j \in J}\mathfrak{C}_j$. By Lemma 11, $P_{j \in J}\mathfrak{C}_j$ is isomorphic to a $\mathbf{Cs}_\alpha$ with base W. This completes the proof.

The following lemma is an immediate consequence of Lemma 11 and the definition of the relation $\approx$ given prior to Lemma 1:

LEMMA 13. *If $\mathfrak{A}$ is a* $\mathbf{Cs}_\alpha$ *with base U, then $\mathfrak{A}$ is isomorphic to a subdirect product of* $\mathbf{WCs}_\alpha$*'s all with base U.*

By Theorem 1 and Lemma 13 we have

THEOREM 2. *For $\alpha \geqq \omega$, any subdirect product of* $\mathbf{Cs}_\alpha$*'s each having an infinite base is isomorphic to a* $\mathbf{Cs}_\alpha$.

Now we shall consider products of $\mathbf{WCs}_\alpha$'s with finite bases. As noted in the preceding section, we cannot expect a result like Theorem 1. It is still possible, however, to "coalesce" part of a product in this case.

THEOREM 3. *Let* $1 < \kappa < \omega \leqq \alpha$. *Suppose that, for each* $i \in I$, $\mathfrak{A}_i$ *is a* $\mathbf{WCs}_\alpha$ *with unit set* ${}^\alpha U_i^{(p_i)}$, *where* $|U_i| = \kappa$. *Let* $\mathfrak{m}$ *be the least cardinal number such that for each equivalence relation* R *on* α,

$$|\{i \in I \colon \ker p_i = R\}| \leqq \mathfrak{m} \cdot |\{{}^\alpha\kappa^{(x)} \colon x \in {}^\alpha\kappa, \ker x = R\}|.$$

Then $P_{i \in I}\mathfrak{A}_i$ *is isomorphic to a product of* $\mathfrak{m}$ $\mathbf{Cs}_\alpha$*'s.*[2]

PROOF. For each equivalence relation R on α let $J_R = \{i \in I \colon \ker p_i = R\}$. By assumption we may write $J_R = \bigcup_{\beta < \mathfrak{m}} K_{R\beta}$, where $|K_{R\beta}| \leqq |\{{}^\alpha\kappa^{(x)} \colon x \in {}^\alpha\kappa, \ker x = R\}|$. For each $\beta < \mathfrak{m}$ let $f_{R\beta}$ be a one-one mapping of $K_{R\beta}$ into $\{{}^\alpha\kappa^{(x)} \colon x \in {}^\alpha\kappa, \ker x = R\}$. By Lemma 1 and Lemma 6, for each $\beta < \mathfrak{m}$ and each $i \in K_{R\beta}$, $\mathfrak{A}_i$ is isomorphic to a $\mathbf{WCs}_\alpha$ $\mathfrak{B}_i$ with unit set $f_{R\beta}i$. Now one can argue as in the last part of the proof of Theorem 1 to show that, for each $\beta < \mathfrak{m}$,

$$P\langle \mathfrak{B}_i \colon R \text{ an equivalence relation on } \alpha,\ i \in K_{R\beta}\rangle$$

is isomorphic to a $\mathbf{Cs}_\alpha$ with base κ whenever there exists an $i \in I$ such that $i \in K_{R\beta}$ for some equivalence relation R on α. Hence our theorem follows.

COROLLARY. *Any* $\mathbf{WCs}_\alpha$ *is isomorphic to a* $\mathbf{Cs}_\alpha$.

This corollary follows immediately from Theorems 1 and 3. It is also an immediate consequence of the characterization of representable algebras given in the last section. In fact, let $\mathfrak{C}$ be a $\mathbf{WCs}_\alpha$ with unit set ${}^\alpha U^{(p)}$. To show that $\mathfrak{C}$ is isomorphic to a $\mathbf{Cs}_\alpha$ it suffices to assume that $\mathfrak{C}$ is the $\mathbf{WCs}_\alpha$ of all subsets of ${}^\alpha U^{(p)}$. Let $q \in {}^{\alpha+\omega}U$ be such that the $p \subseteq q$. Let $\mathfrak{B}$ be the $\mathbf{WCs}_{\alpha+\omega}$ of all subsets of ${}^{\alpha+\omega}U^{(q)}$. For any $X \in C$ let $fX = \{u \in {}^{\alpha+\omega}U^{(q)} \colon \alpha \upharpoonleft u \in X\}$. It is easily verified that f is an isomorphism of $\mathfrak{C}$ onto a $\mathbf{CA}_\alpha$ $\mathfrak{A}$ which is related to $\mathfrak{B}$ as in (ii) of the characterization of representable $\mathbf{CA}_\alpha$'s. Hence $\mathfrak{A} \in \mathbf{R}_\alpha$ and $\mathfrak{C} \in \mathbf{R}_\alpha$. Since $\mathfrak{C}$ is subdirectly indecomposable by Lemma 2, $\mathfrak{C}$ *is* isomorphic to a $\mathbf{Cs}_\alpha$. If we based the proof of the Corollary on this characterization, our entire proof of Theorems 1–3 could be rearranged so as to simplify various details. But then our account would not be selfcontained; moreover, the proof of the characterization is itself rather complicated, so that the proof in all would be longer.

One should note with regard to the above results that if $\alpha \geqq \omega$, $\mathfrak{A}$ is a $\mathbf{WCs}_\alpha$ with base U, $\mathfrak{B}$ a $\mathbf{WCs}_\alpha$ with base V, $|U| \neq |V|$, and $|U|$ or $|V|$ is finite, then $\mathfrak{A} \times \mathfrak{B}$ is not isomorphic to a $\mathbf{Cs}_\alpha$. In fact, suppose that $|U| < |V|$, and $|U|$ is finite. Let $\kappa = |U|$. Then $c_0^{(\mathfrak{A})} \cdots c_\kappa^{(\mathfrak{A})} \prod_{\lambda<\mu\leqq\kappa} -d_{\lambda\mu}^{(\mathfrak{A})} = 0$, while $c_0^{(\mathfrak{B})} \cdots c_\kappa^{(\mathfrak{B})} \prod_{\lambda<\mu\leqq\kappa} -d_{\lambda\mu}^{(\mathfrak{B})} = 1$. Hence

$$0 \neq c_0^{(\mathfrak{A}\times\mathfrak{B})} \cdots c_\kappa^{(\mathfrak{A}\times\mathfrak{B})} \prod_{\lambda<\mu\leqq\kappa} -d_{\lambda\mu}^{(\mathfrak{A}\times\mathfrak{B})} \neq 1.$$

But it is easily checked that in any $\mathbf{Cs}_\alpha$ D, either

$$c_0^{(D)} \cdots c_\kappa^{(D)} \prod_{\lambda<\mu\leqq\kappa} -d_{\lambda\mu}^{(D)} = 0 \quad \text{or} \quad c_0^{(D)} \cdots c_\kappa^{(D)} \prod_{\lambda<\mu\leqq\kappa} -d_{\lambda\mu}^{(D)} = 1.$$

Thus $\mathfrak{A} \times \mathfrak{B}$ is not isomorphic to a $\mathbf{Cs}_\alpha$.

[2] See Note added in proof preceding the Bibliography.

3. Open problems. We shall state here what seem to us to be some of the most important open problems in algebraic logic, restricting ourselves, however, to those that can be conveniently formulated on the basis of the definitions given in this paper. The first five are of a programatic nature and are not given an exact formulation.

Problem 1. *Devise an algebraic version of predicate logic in which the class of representable algebras forms a finitely based equational class.*

It has been shown that many versions of algebraic logic fail to satisfy the criterion of Problem 1; see [23] for references.

Problem 2. *Describe the structure of Tarski algebras of well-known theories, such as Peano arithmetic and group theory.*

The program suggested in Problem 2 has been carried through for the theory of real-closed fields. In fact, Tarski's decision method essentially gives a complete description of the Tarski algebra in this case. Concerning Peano arithmetic and group theory, we can say that their Tarski algebras are not simple, in the technical sense, using Gödel's incompleteness theorem to establish this for Peano arithmetic. The Boolean algebra of *sentences* for Peano arithmetic, which is a definable part of the Tarski algebra, is a denumerable atomless Boolean algebra.

Problem 3. *Investigate cylindric algebras corresponding to the language* $L_{\omega_1\omega}$.

The cylindric algebras mentioned in Problem 3 are easy to describe. Such an algebra $\mathfrak{A}$ must be locally finite of infinite dimension and such that $\sum_{i\in I} x_i$ exists whenever $\langle x_i : i\in I\rangle$ is a countable system of elements of A such that

$$\bigcup_{i\in I} \{\kappa : c_\kappa x_i \neq x_i\}$$

is finite. For algebraically investigating $L_{\omega_1\omega}$, cylindric algebras seem more appropriate than polyadic algebras. The latter are, however, more suitable with respect to the general languages $L_{\kappa\lambda}$. For work along the lines of Problem 3 see Preller [25].

Problem 4. *Show that any* **CA** *can be represented in terms of* **Cs**'*s by means of some kinds of operations.*

From the remarks of §1 it is seen that the usual algebraic operations—homomorphisms, direct products, etc.—are inadequate for the purpose of Problem 4. The addition of the operation of *relativization*, i.e., passing from a $\mathbf{Cs}_\alpha$ to a $\mathbf{Gcs}_\alpha$, is also not sufficient for this purpose, but it comes close: Most of the nonrepresentable $\mathbf{CA}_\alpha$'s which have been constructed can be obtained from $\mathbf{Cs}_\alpha$'s by relativization and the usual algebraic operations. Other construction methods are known, mostly unpublished, which may be useful in solving Problem 4.

Problem 5. *Give an equational characterization of* $\mathbf{R}_\alpha$ *with a clear and simple mathematical content.*

Rather complicated equations characterizing $\mathbf{R}_\alpha$ can be found in [**22**].

For our final two problems, cf. the discussion of decision methods for **CA**'s in §1.

Problem 6. *Is the equational theory of* $\mathbf{CA}_3$*'s decidable?*

A solution of Problem 6 may result from an analysis of Jaśkowski [**9**]; unfortunately, detailed proofs were never published for all the results of that paper.

Problem 7. *Is the first-order theory of* $\mathbf{CA}_1$*'s decidable?*

Problem 7 is closely related to the problem whether the theory of a Boolean algebra with a distinguished subalgebra is decidable.

NOTE ADDED IN PROOF. Richard Thompson has pointed out to us that the formulation and proof of Theorem 3 are in error. To the left of $\leq$ in the displayed formula of Theorem 3 there should appear

$$|\{i \in I : \text{for some } q,\ q \approx p_i \text{ and } \ker q = R\}|$$

and then the proof as given must be modified somewhat.

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