CONNECTIONS BETWEEN COMBINATORIAL THEORY AND ALGEBRAIC LOGIC

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1. INTRODUCTION

Algebraic logic has arisen as a subdiscipline of algebra mirroring constructions and theorems of mathematical logic. It is similar in this respect to such fields as algebraic topology and algebraic geometry, where the main constructions and theorems are algebraic in nature but the main intuitions underlying them are respectively topological and geometric. The main intuitions underlying algebraic logic are, of course, those of formal logic. We shall describe in this first section the intuitive background of algebraic logic, and state some of the central definitions and results in this area. In later sections we give some constructions which arrive at the fundamental algebras of algebraic logic from entirely different sources, in fact from certain configurations which play a basic role in combinatorial theory. These connections between algebraic logic and combinatorial theory are rather unexpected (at least to this author). Their implications and deeper causes have not been fully explored.

It is our intent to give enough details in the constructions and theorems below so that a reader unversed in algebraic logic can follow the exposition and check those proofs which are presented. We only give proofs for results which are new, however.

Let $\mathcal{L}$ be any first-order language. Thus $\mathcal{L}$ has an infinite sequence of individual variables $v_0, v_1, \cdots$, logical constants—say $\neg, \rightarrow, \forall, =, -$ and non-logical constants—say a system $\langle R_i : i \in I \rangle$ of relation symbols, where R_i is of rank $\rho_i < \omega$ for each $i \in I$ (ω is the set of all non-negative integers). We assume as known the usual syntactic notions defined in terms of $\mathcal{L}$, e.g., the notions of a formula, a sentence (formula without free occurrences of variables), the conjunction $\varphi \wedge \psi$ of formulas of $\mathcal{L}$, the notion of a formal proof from a set of sentences, etc. Given a set Γ of sentences, we may call two formulas φ and ψ *equivalent under* Γ, in symbols $\varphi \equiv_\Gamma \psi$, provided that the biconditional $\varphi \leftrightarrow \psi$ is provable from Γ. The relation $\equiv_\Gamma$ is in fact an equivalence relation on the set of formulas. If we let A_Γ denote the set of all equivalence classes under Γ, we find that algebraic operations can be introduced on A_Γ which reflect the syntactic operations of building formulas:

$$
\begin{aligned}
[\varphi]_\Gamma + [\psi]_\Gamma &= [\varphi \vee \psi]_\Gamma, \\
[\varphi]_\Gamma \cdot [\psi]_\Gamma &= [\varphi \wedge \psi]_\Gamma, \\
-[\varphi]_\Gamma &= [\neg \varphi]_\Gamma, \\
c_i[\varphi]_\Gamma &= [\exists v_i \varphi]_\Gamma, \\
d_{ij} &= [v_i = v_j]_\Gamma.
\end{aligned}
$$

Here $[\varphi]_\Gamma$ is the equivalence class of φ under Γ. Note that d_{ij} is a 0-ary operation on A_Γ, i.e., an element of A_Γ. The algebra $\mathfrak{A}_\Gamma = \langle A_\Gamma, +, \cdot, -, c_i, d_{ij} \rangle_{i,j \in \omega}$ thus associated with $\mathcal{L}$ and Γ is one of the fundamental algebras studied in algebraic logic. It turns out that most of the constructions and theorems of logic can be algebraically reflected using these algebras $\mathfrak{A}_\Gamma$. For example, Γ is complete and consistent iff $\mathfrak{A}_\Gamma$ is simple; the theorem that any consistent theory can be extended to a complete and consistent theory is mirrored by the theorem that any algebra $\mathfrak{A}_\Gamma$ with $|A_\Gamma| > 1$ has a simple homomorphic image.

The notion of a cylindric algebra is obtained from these algebras $\mathfrak{A}_\Gamma$ by a process of abstraction. Let α be any ordinal. A *cylindric algebra of dimension* α, for brevity a CA_α, is an algebraic structure $\mathfrak{A} = \langle A, +, \cdot, -, c_i, d_{ij}\rangle_{i,j<\alpha}$ satisfying the following conditions for all $i, j, k < \alpha$ and all $x, y \in A$:

(C_0) $\langle A, +, \cdot, -\rangle$ is a Boolean algebra,
(C_1) $c_i 0 = 0$,
(C_2) $x \leqq c_i x$,
(C_3) $c_i(x \cdot c_i y) = c_i x \cdot c_i y$,
(C_4) $c_i c_j x = c_j c_i x$,
(C_5) $d_{ii} = 1$,
(C_6) if $j \neq i, k$, then $c_j(d_{ij} \cdot d_{jk}) = d_{ik}$,
(C_7) if $i \neq j$, then $c_i(d_{ij} \cdot x) \cdot c_i(d_{ij} \cdot -x) = 0$.

The abstraction process is so familiar in modern mathematics that we do not have to describe its advantages. That this abstraction is sound is established by the following logical representation theorem, which is not very difficult to prove.

THEOREM 1.1: *For any algebra* $\mathfrak{A}$ *similar to* CA_ω*'s the following two conditions are equivalent:*

(i) $\mathfrak{A} \cong \mathfrak{A}_\Gamma$ *for some* Γ;
(ii) $\mathfrak{A}$ *is a* CA_ω *such that* $|\{i: c_i x \neq x\}| < \omega$ *for all* $x \in A$.

Having at hand the abstract notion of a cylindric algebra, investigations in algebraic logic can proceed in two conceptually different, but actually closely related, ways. First, one can investigate the algebraic meaning of constructions and results of logic. Second, one can consider cylindric algebras as objects of investigation in their own right and discuss questions which naturally arise independently of any connection with logic. In this article we shall be concerned with problems of the second kind, specifically with certain methods of constructing CA's. These problems are, however, motivated by a central problem of the first kind, the set-theoretic representation problem. To describe this latter problem we need to discuss some basic concepts of model theory.

The most important concepts which are studied in mathematical logic concern the notion of a model. Let $\mathfrak{L}$ be a first-order language, as described above. A *model over* $\mathfrak{L}$ is a structure of the form $\mathfrak{A} = \langle A, R_i \rangle_{i \in I}$, where $A \neq 0$ and R_i is of rank ρ_i for each $i \in I$. We assume as known the notion of a sequence $x \in {}^{\omega}A$ *satisfying* a formula φ of $\mathfrak{L}$ in $\mathfrak{A}$, and such derivative notions as φ being *true* in $\mathfrak{A}$, $\mathfrak{A}$ being a *model* of a set Γ of sentences, etc. For each formula φ we set

$$\tilde{\varphi}^{(\mathfrak{A})} = \{x \in {}^{\omega}A : x \text{ satisfies } \varphi \text{ in } \mathfrak{A}\} = \tilde{\varphi} \text{ (if } \mathfrak{A} \text{ is understood).}$$

Thus $\tilde{\varphi}^{(\mathfrak{A})}$ is a point-set in the ω-dimensional space ${}^{\omega}A$. Certain set-theoretic operations similar to the classical operations of descriptive set theory can be introduced corresponding to the basic syntactic operations:

$$\tilde{\varphi} \cup \tilde{\psi} = \widetilde{\varphi \vee \psi},$$

$$\tilde{\varphi} \cap \tilde{\psi} = \widetilde{\varphi \wedge \psi},$$

$${}^{\omega}A \sim \tilde{\varphi} = \widetilde{\neg \varphi},$$

$$C_i\tilde{\varphi} = \widetilde{\exists v_i \varphi} = \text{cylinder obtained by moving } \tilde{\varphi} \text{ parallel to the } i\text{-axis},$$

$$D_{ij} = \widetilde{v_i = v_j}.$$

The collection $\{\tilde{\varphi}^{\mathfrak{A}} : \varphi \text{ a formula of } \mathfrak{L}\}$ forms a $\boldsymbol{CA}_{\omega}$ under these operations. Again we make an abstraction from this notion to obtain a more general set-theoretic object. A *cylindric set algebra of dimension α with base U*, for short a $\boldsymbol{Cs}_{\alpha}^{U}$ or a $\boldsymbol{Cs}_{\alpha}$, is an algebraic structure

$$\mathfrak{A} = \langle A, \cup, \cap, \sim, C_i, D_{ij} \rangle_{i,j<\alpha}$$

such that A is a field of subsets of ${}^{\alpha}U$ closed under each C_i and with each D_{ij} as a member, where

$$D_{ij} = \{x \in {}^{\alpha}U : x_i = x_j\},$$

and for each $X \subseteq {}^{\alpha}U$,

$$C_iX = \{x \in {}^{\alpha}U : (\alpha \sim \{i\}) \restriction x = (\alpha \sim \{i\}) \restriction y \quad \text{for some} \quad y \in X\}.$$

In terms of this concept a purely algebraic form of the completeness theorem can be stated:

THEOREM 1.2: *If* $\mathfrak{A}$ *satisfies the condition* 1.1(ii), *then* $\mathfrak{A}$ *is homomorphic to a* Cs_ω^U *for some* $U \neq 0$.

Proofs of 1.2 are somewhat deeper than those of 1.1.

The set-theoretic representation problem is the vaguely posed problem concerning possible improvements of 1.2. To make the problem more precise, let R_α be the class of all CA_α's isomorphic to a subdirect product of Cs_α's; members of R_α are called *representable* CA_α's. In elementary terms the definition of R_α runs as follows: $\mathfrak{A}$ is representable iff for every non-zero $x \in A$ there is a homomorphism h from $\mathfrak{A}$ onto a Cs_α such that $hx \neq 0$. A very elementary argument shows that Theorem 1.2 is equivalent to the statement that every $\mathfrak{A}$ satisfying 1.1(ii) is an R_α. A more precise form of the representation problem is to describe properties of the class R_α, and give a useful characterization of it in abstract terms.

Some of the basic facts concerning R_α are as follows. For $\alpha \leqq 1$, $CA_\alpha = R_\alpha$ and hence the representation problem is essentially solved. Next, $CA_2 \neq R_2$, but R_2 can be characterized by $(C_0) - (C_7)$ together with the two equations

$$c_0[x \cdot y \cdot c_1(x \cdot -y)] \cdot -c_1(c_0x \cdot -d_{01}) = 0,$$
$$c_1[x \cdot y \cdot c_0(x \cdot -y)] \cdot -c_0(c_1x \cdot -d_{01}) = 0.$$

For each $\alpha > 2$, $CA_\alpha \neq R_\alpha$. The class R_α is always a variety, but for $\alpha \geqq 3$ it is not finitely based, and for $\alpha \geqq \omega$ it cannot even be characterized by a certain natural kind of finite schema. For $\alpha \geqq \omega$, every simple CA_α is representable. Theorem 1.2 gives a fundamental property of representable CA_ω's. The last property of R_α's which we will state will play a small role in section 4; its formulation requires a new concept. Let α and β be ordinals with $\alpha \leqq \beta$. Let

$$\mathfrak{A} = \langle A, +, \cdot, -, c_i, d_{ij} \rangle_{i,j<\alpha}$$

be a CA_α, and let

$$\mathfrak{B} = \langle B, +', \cdot', -', c_i', d_{ij}' \rangle_{i,j<\beta}$$

be a CA_β. We say that $\mathfrak{A}$ *can be neatly embedded in* $\mathfrak{B}$ if there is an isomorphism f of $\mathfrak{A}$ into $\langle B, +', \cdot', -', c'_i, d'_{ij}\rangle_{i,j<\alpha}$ such that $c'_i fx = fx$ for all $x \in A$ and all $i \in \beta \sim \alpha$; f is called a *neat embedding of* $\mathfrak{A}$ *into* $\mathfrak{B}$.

THEOREM 1.3: *For any* α *the following two conditions are equivalent:*

(i) $\mathfrak{A} \in \boldsymbol{R}_\alpha$;
(ii) $\mathfrak{A}$ *can be neatly embedded in a* $CA_{\alpha+\omega}$.

This completes our introduction to algebraic logic. A comprehensive treatment of the algebraic theory of cylindric algebras can be found in Henkin, Monk, Tarski [6]. The closely related theory of polyadic algebras is treated in Halmos [5].

2. QUASIGROUPS AND CA_3'S

A *quasigroup* is an algebra $\mathfrak{A} = \langle A, \cdot\rangle$ such that for any $a, b \in A$ there is a unique x such that $x \cdot a = b$, and also a unique y such that $a \cdot y = b$. Quasigroups are essentially the same thing as latin squares; the latter form one of the main objects of study in combinatorial theory. For our present purposes, quasigroups are more convenient to deal with than latin squares. A good source of reference for quasigroups is Bruck [1]. Let $\mathfrak{A} = \langle A, \cdot\rangle$ be a quasigroup. We shall consider $\cdot$ as a certain ternary relation on A, in the usual way. If $X \subseteq \cdot$ and $i < 3$, we define

$$c_i X = \{y \in \cdot : y_i = x_i \quad \text{for some} \quad x \in X\}.$$

Further, let $q \in \cdot$. Then for $i, j < 3$ we let

$$d_{ij} = 1 \qquad \text{if} \qquad i = j,$$
$$d_{ij} = c_k\{q\} \qquad \text{if} \qquad \{i, j, k\} = 3.$$

By an $\mathfrak{A}, q - CA_3$ we mean a system $\mathfrak{B} = \langle B, \cup, \cap, \sim, c_i, d_{ij}\rangle_{i,j<3}$ such that $\langle B, \cup, \cap, \sim\rangle$ is a Boolean algebra of subsets of $\cdot$, B is closed under c_i for each $i < 3$, and $d_{ij} \in B$ for all $i, j < 3$.

THEOREM 2.1: *If* $\mathfrak{A} = \langle A, \cdot\rangle$ *is a quasigroup and* $q \in \cdot$, *then any* $\mathfrak{A}, q - CA_3$ *is a* CA_3.

The proof of Theorem 2.1 is routine; let us verify (C_4) as an example. Suppose $\mathfrak{B}$ is an $\mathfrak{A}, q - \boldsymbol{CA}_3$, as above. Let $X \in B$ and assume $i, j < 3$. Obviously we may assume that $i \neq j$. If $X = 0$, clearly $c_i c_j X = 0 = c_j c_i X$. If, on the other hand, $X \neq 0$, we shall establish that $c_i c_j X = \cdot$ (and then by symmetry $c_j c_i X = \cdot = c_i c_j X$). So, let s be any member of $\cdot$. Choose $r \in X$. By the definition of quasigroup, there is then a unique $t \in \cdot$ such that $tj = rj$ and $ti = si$. Thus $t \in c_j X$, so $s \in c_i c_j X$, as desired. (We have actually established that $\mathfrak{B}$ is simple in the algebraic sense, but we do not need this fact below.)

Simple as it is, Theorem 2.1 turns out to have some uses in algebraic logic. Namely, we can use it to give an example of a non-representable $\boldsymbol{CA}_3$. To this end, consider the equation

$$s_1^2 s_0^1 s_2^0 s_1^2 s_0^1 s_2^0 c_2 x = c_2 x, \tag{1}$$

where for brevity we let $s_j^i y = c_i(d_{ij} \cdot y)$ for all $i, j < 3$ and all y. The equation (1) is easily seen to hold in all cylindric set algebras, and hence also in all representable algebras. We shall now give an example of a $\boldsymbol{CA}_3$ in which it fails; thus this $\boldsymbol{CA}_3$ will be non-representable. Let $\mathfrak{A} = \langle A, \cdot \rangle$ be the quasigroup given by the following multiplication table:

$\cdot$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	1	6	3	4	5
3	3	4	5	2	6	1
4	4	5	1	6	2	3
5	5	6	4	1	3	2
6	6	3	2	5	1	4

Let $q = \langle 2, 3, 6 \rangle$ and $X = \{\langle 5, 3, 4 \rangle\}$. Let $\mathfrak{B}$ be the $\mathfrak{A}, q - \boldsymbol{CA}_3$ of all subsets of $\cdot$. It is then easily verified that $\langle 2, 4, 3 \rangle \in s_1^2 s_0^1 s_2^0 s_1^2 s_0^1 s_2^0 c_2 X \sim c_2 X$. Thus, indeed, equation (1) fails to hold in $\mathfrak{B}$.

There are many natural questions that one can ask concerning our construction of a $\boldsymbol{CA}_3$ from a quasigroup. First, we can ask for a characterization of $\boldsymbol{CA}_3$'s isomorphic to an $\mathfrak{A}, q - \boldsymbol{CA}_3$. This question, although unresolved, is somewhat indefinite; a more definite form is as follows.

PROBLEM 1: *Let $\mathfrak{B}$ be a $\boldsymbol{CA}_3$ such that $c_0c_1x = c_0c_2x = c_1c_2x = 1$ for all non-zero $x \in B$. Is there a quasigroup $\mathfrak{A} = \langle A, \cdot \rangle$ and a $q \in \cdot$ such that $\mathfrak{B}$ is isomorphic to an $\mathfrak{A}, q - \boldsymbol{CA}_3$?*

Another natural question concerns relationships between algebraic properties of quasigroups and properties of associated $\boldsymbol{CA}_3$'s. The following result is of interest in this connection. A *loop* is a quasigroup $\mathfrak{A} = \langle A, \cdot \rangle$ having an identity element, i.e., having an element i such that $i \cdot a = a \cdot i = a$ for all $a \in A$. Thus the quasigroup given in the above table is actually a loop, but it is not a group.

THEOREM 2.2: *Let $\mathfrak{A} = \langle A, \cdot \rangle$ be a quasigroup.*

(i) *If $\mathfrak{A}$ is a group, $q \in \cdot$, and $\mathfrak{B}$ is any $\mathfrak{A}, q - \boldsymbol{CA}_3$, then $\mathfrak{B}$ is representable.*

(ii) *If $\mathfrak{A}$ is a loop but not a group, i is the identity of $\mathfrak{A}$, and $q = \langle i, i, i \rangle$, then the $\mathfrak{A}, q - \boldsymbol{CA}_3$ of all subsets of $\cdot$ is not representable.*

Proof: (i) Assume that $\mathfrak{A}$ is a group, let $q \in \cdot$, and let $\mathfrak{B}$ be an $\mathfrak{A}, q - \boldsymbol{CA}_3$. We shall assign to each $g \in \cdot$ a subset Fg of ${}^3A$:

$$Fg = \{x \in {}^3A : g_0 = q_0 x_1 x_2^{-1}, g_1 = x_2 x_0^{-1} q_1\}.$$

Clearly if $x \in Fg$, then g is uniquely determined by x. Thus

(1) $\quad g, h \in \cdot$ and $g \neq h$ imply that $Fg \cap Fh = 0$.

Also, obviously for any $x \in {}^3A$ there is a $g \in \cdot$ such that $x \in Fg$, so

(2) $$\bigcup_{g \in \cdot} Fg = {}^3A.$$

Finally,

(3) $$\text{for any} \quad g \in \cdot \quad \text{we have} \quad Fg \neq 0.$$

In fact, given $g \in \cdot$ we can let $x = \langle q_1 g_1^{-1}, q_0^{-1} g_0, i \rangle$, where i is the identity of $\mathfrak{A}$; clearly $x \in Fg$.

For any $b \in B$ we now let

$$Gb = \bigcup_{g \in b} Fg.$$

Using (1)–(3) it is easy to check that G is an isomorphism of the Boolean part of $\mathfrak{B}$ onto a field of subsets of ${}^3A$. It is straightforward to check that actually G is an isomorphism of $\mathfrak{B}$ itself onto a cylindric set algebra; for illustration we check that $Gc_0b \subseteq C_0Gb$ for any $b \in B$. Assume that $b \in B$ and $x \in Gc_0b$. Say $x \in Fg$ where $g \in c_0b$. Then choose $h \in b$ such that $g_0 = h_0$. Let $x_0' = q_1 h_1^{-1} x_2$, $y = \langle x_0', x_1, x_2 \rangle$. Now $h_0 = g_0 = q_0 x_1 x_2^{-1}$ since $x \in Fg$, and

$$x_2 x_0'^{-1} q_1 = x_2 x_2^{-1} h_1 q_1^{-1} q_1 = h_1.$$

It follows that $y \in Fh$. Since $h \in b$, thus $y \in Gb$, so $x \in C_0Gb$, as desired.

To prove (ii), suppose $\mathfrak{A}$ is a loop but is not a group. Thus $\cdot$ is not associative, so there exist elements $a, b, c \in A$ with $a \cdot (b \cdot c) \neq (a \cdot b) \cdot c$. Let i be the identity of $\mathfrak{A}$, set $q = \langle i, i, i \rangle$, and let $\mathfrak{B}$ be the $\mathfrak{A}, q - \boldsymbol{CA}_3$ of all subsets of $\cdot$. Now consider the following equation:

(4) $$c_2\{c_0[c_2(c_0x \cdot c_1y) \cdot d_{02}] \cdot c_1z\}$$
$$= c_2[c_1(c_2\{c_0[c_2(c_1y \cdot d_{12}) \cdot d_{02}] \cdot c_1z\} \cdot d_{12}) \cdot c_0x].$$

It is easily verified that (4) holds in every cylindric set algebra and hence in every $\boldsymbol{R}_3$. However, (4) does not hold in $\mathfrak{B}$, and thus $\mathfrak{B}$ is not representable. In fact, if we let $x = \{\langle a, i, a \rangle\}$, $y = \{\langle i, b, b \rangle\}$, and $z = \{\langle i, c, c \rangle\}$, then it is easily verified that (4) fails.

COROLLARY 2.3: *If $\mathfrak{A}$ is a loop, i is the identity of $\mathfrak{A}$, $q = \langle i, i, i \rangle$, and $\mathfrak{B}$ is the $\mathfrak{A}, q - \boldsymbol{CA}_3$ of all subsets of $\cdot$, then a necessary and sufficient condition for $\mathfrak{A}$ to be a group is that $\mathfrak{B}$ is representable.*

Theorem 2.2 suggests the following variant of Problem 1:

PROBLEM 2: *Let $\mathfrak{B}$ be an R_3 such that $c_0c_1x = c_0c_2x = c_1c_2x = 1$ for all non-zero $x \in B$. Is there a group $\mathfrak{A} = \langle A, \cdot \rangle$ and a $q \in \cdot$ such that $\mathfrak{B}$ is isomorphic to an $\mathfrak{A}, q - CA_3$?*

It may be that methods of McKenzie [10] can be used to settle this question.

3. PROJECTIVE GEOMETRIES AND CA_3'S

The construction we shall now describe was first carried out for relation algebras by Jónsson [8] and Lyndon [9]. Surprisingly, we do not know any connections between the present construction and that of section 2, although projective planes are essentially just complete systems of mutually orthogonal latin squares. By a projective geometry we understand a system $\langle \mathcal{P}, \mathcal{L} \rangle$ such that $\mathcal{P}$ is a non-empty set (of "points"), $\mathcal{L}$ is a non-empty collection of subsets (called "lines") of $\mathcal{P}$, and:

(G_1) each line contains at least four points;
(G_2) each pair of distinct points p and q lies on a unique line $\overline{pq}$;
(G_3) if p, q, r, and s are distinct points and $\overline{pq}$ and $\overline{rs}$ have a common point, then $\overline{pr}$ and $\overline{qs}$ have a common point (see Figure 1).

We shall assume a knowledge of elementary projective geometry; see, e.g., Seidenberg [16].

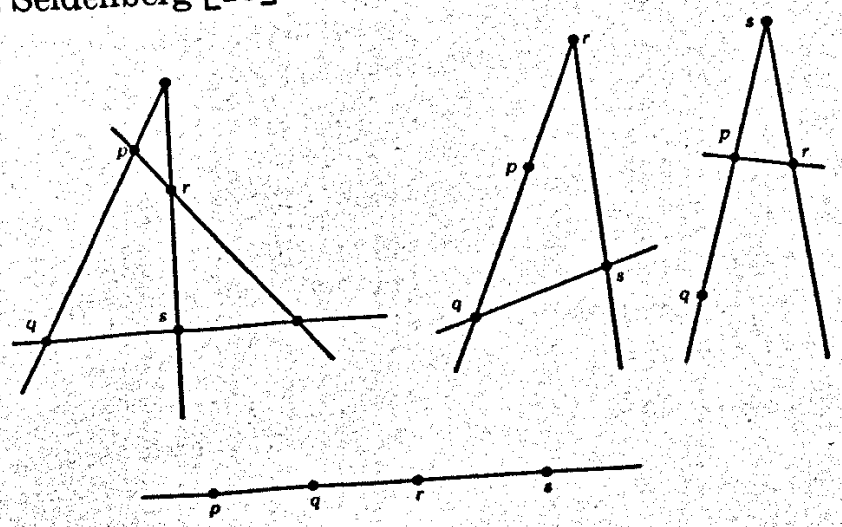

FIG. 1

If R is an equivalence relation on 3, let $R' = \{(i,j): i,j < 3$ and $i \not R j\}$. (Here $3 = \{0, 1, 2\}$.) Let $\mathfrak{G} = \langle \mathcal{P}, \mathcal{L} \rangle$ be a projective geometry; we define $\mathfrak{U}\mathfrak{G}$ to be the collection of all pairs $\langle R, f \rangle$ satisfying the following conditions:

(1) R is an equivalence relation on 3;
(2) f maps R' into $\mathcal{P}$;
(3) for $(i,j) \in R', fij = fji$;
(4) if $i \not R j, i \not R k$, but $j R k$, then $fij = fik$;
(5) if R = identity on 3, then either $f01 = f02 = f12$ or else $f01, f02, f12$ are distinct collinear points.

Now for $i, j < 3$ and $X \subseteq \mathfrak{U}\mathfrak{G}$ we let $3 = \{i, k, l\}$ and

$$c_i X = \{\langle R, f \rangle \in \mathfrak{U}\mathfrak{G}: \text{there is an } \langle S, g \rangle \in X \text{ such that } R \cap {}^2(3 \sim \{i\}) = S \cap {}^2(3 \sim \{i\}) \text{ and } fkl = gkl \text{ if } kR'l\};$$

$$d_{ij} = \{\langle R, f \rangle \in \mathfrak{U}\mathfrak{G}: i R j\}.$$

A $\mathfrak{G} - \boldsymbol{CA}_3$ is a structure $\langle A, \cup, \cap, \sim, c_i, d_{ij} \rangle_{i,j<3}$ such that A is a field of subsets of $\mathfrak{U}\mathfrak{G}$ closed under c_i for $i < 3$ and with $d_{ij} \in 3$ for $i, j < 3$. We denote by $\mathfrak{A}_{\mathfrak{G}}$ the $\mathfrak{G} - \boldsymbol{CA}_3$ of all subsets of $\mathfrak{U}\mathfrak{G}$.

A short description of the motivation behind this construction may be helpful. Intuitively speaking, we imagine $\mathfrak{G}$ embedded as a hyperplane in a space $\mathfrak{H}$ of one higher dimension; $\langle R, f \rangle \in \mathfrak{U}\mathfrak{G}$, with R = identity on 3, amounts to an abstract selection of three distinct points x_0, x_1, x_2 in $\mathfrak{H} \sim \mathfrak{G}$ such that $\overline{x_i x_j}$ intersects $\mathfrak{G}$ at fij for $i, j < 3$, $i \neq j$. This accounts, for example, for the strange condition (5) since, by (G_3), if the lines $\overline{x_0 x_1}, \overline{x_0 x_2}, \overline{x_1 x_2}$ are distinct, then they intersect $\mathfrak{G}$ in three distinct collinear points. The cylindrifications are restrictions of the natural set-theoretic cylindrifications. This intuitive description will become clearer during the proofs of the theorems below. For brevity we shall restrict ourselves mostly to the algebras $\mathfrak{A}_{\mathfrak{G}}$ instead of the more general $\mathfrak{G} - \boldsymbol{CA}_3$'s.

THEOREM 3.1: *If $\mathfrak{G}$ is a projective geometry, then $\mathfrak{A}_{\mathfrak{G}}$ is a $\boldsymbol{CA}_3$.*

We shall not give a detailed proof of 3.1, since it is straightforward and very close to the proof of Lemma 2.1 in Monk [12].

To illustrate the proof we shall establish (C_4). Let $\mathfrak{G} = \langle \mathcal{P}, \mathcal{L} \rangle$. Suppose $i, j < 3$ and $X \subseteq \mathfrak{U}\mathfrak{G}$. If $i = j$, obviously $c_i c_j X = c_j c_i X$. Hence assume $i \neq j$. Clearly $c_i c_j 0 = 0 = c_j c_i 0$. So, assume $X \neq 0$. Then, as in the proof of 2.1, we shall establish that $c_i c_j X = \mathfrak{U}\mathfrak{G}$; thus ($C_4$) follows by symmetry. Let, then, $\langle R, f \rangle \in \mathfrak{U}\mathfrak{G}$ be arbitrary. Choose $\langle S, g \rangle \in X$. Choose k so that $3 = \{i, j, k\}$. If $jR'k$ or $iS'k$ let

$$T = (R \cap {}^2\{j, k\}) \cup (S \cap {}^2\{i, k\}),$$

while if $j\,R\,k$ and $i\,S\,k$ let $T = {}^2 3$. Obviously in either case T is an equivalence relation on 3. If $T = {}^2 3$, let $h = 0$. In case $T \neq {}^2 3$, we have $i\,T'\,j$. If moreover $j\,T\,k$ and $i\,T'\,k$ we let $hij = hji = hik = hki = gik$; if $j\,T'\,k$ and $i\,T\,k$ let $hij = hji = hjk = hkj = fjk$. Finally, if $j\,T'\,k$ and $i\,T'\,k$, let $hjk = hkj = fjk$ and $hik = hki = gik$; further, if $fjk = gik$ let $hji = hij = fjk$, while if $fjk \neq gik$ let $hij = hji$ be a third point on the line $\overline{(fjk)(gik)}$. Clearly then $\langle T, h \rangle \in \mathfrak{U}\mathfrak{G}$, $\langle R, f \rangle \in c_i\{\langle T, h \rangle\}$, and $\langle T, h \rangle \in c_j\{\langle S, g \rangle\}$. Thus $\langle R, f \rangle \in c_i c_j X$, as desired. (Again we have actually established that $\mathfrak{A}_{\mathfrak{G}}$ is simple.)

Once more we shall be interested in the relationship between the representability of $\mathfrak{A}_{\mathfrak{G}}$ and properties of the geometry $\mathfrak{G}$.

THEOREM 3.2: *If $\mathfrak{G}$ is a hyperplane in a space $\mathfrak{G}'$ of one higher dimension, then $\mathfrak{A}_{\mathfrak{G}}$ is representable.*

We shall only outline the proof of 3.2, since it is similar to the proof of Lemma 2.2 in Monk [12]. Let $\mathfrak{G} = \langle \mathcal{P}, \mathcal{L} \rangle$ and $\mathfrak{G}' = \langle \mathcal{P}', \mathcal{L}' \rangle$. Set $U = \mathcal{P}' \sim \mathcal{P}$, and for any $x \in {}^3U$ set

$$R_x = \{(i, j) : i, j < 3, x_i = x_j\}.$$

Obviously R_x is an equivalence relation on 3. We define f_x mapping R'_x into $\mathcal{P}$ by setting, for any $(i, j) \in R'_x$, $f_x ij = \overline{x_i x_j} \cdot \mathcal{P}$. Finally, for any $X \in A_{\mathfrak{G}}$ we set

$$FX = \{x \in {}^3U : \langle R_x, f_x \rangle \in X\}.$$

It is now very straightforward to verify that F is an isomorphism from $\mathfrak{A}_{\mathfrak{G}}$ onto a cylindric set algebra of subsets of ${}^3U$. To illustrate, we shall check that F preserves c_i. Let $3 = \{i, j, k\}$.

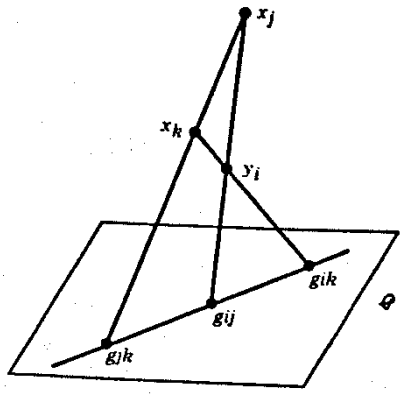

FIG. 2

First, suppose that $X \in A_{\mathfrak{G}}$ and $x \in Fc_iX$. Thus $\langle R_x, f_x \rangle \in c_iX$; choose, then, $\langle S, g \rangle \in X$ such that $R_x \cap {}^2(3 \sim \{i\}) = S \cap {}^2(3 \sim \{i\})$ and $f_x kj = gkj$ if $kR'_x j$. We will find $y \in {}^3U$ such that $(3 \sim \{i\}) \restriction x = (3 \sim \{i\}) \restriction y$ and $\langle R_y, f_y \rangle = \langle S, g \rangle$; this will prove the inclusion $Fc_iX \subseteq C_iFX$. The desired property of y is obvious from its definition in each of the following cases. Of course we let $y_j = x_j$, $y_k = x_k$.

Case 1. $S = {}^23$. Then $j\, R_x\, k$, so $x_j = x_k$. Let $y_i = x_j$.

Case 2. $j\, S\, k$, $i\, S'\, j$. Let y_i be a third point on the line $\overline{x_j g_{ij}}$.

Case 3. $j\, S'\, k$, $i\, S\, j$. (The case $j\, S'\, k$, $i\, S\, k$ is treated similarly.) Let $y_i = x_j$.

Case 4. $S =$ identity on 3, and $g01 = g02 = g12$. Let y_i be a point on the line $\overline{(gjk)x_j}$ different from gjk, x_j, x_k.

Case 5. $S =$ identity on 3, and $g01$, $g02$, $g12$ are distinct collinear points. Let $y_i = \overline{x_k(gik)} \cdot \overline{x_j(gij)}$ (see Figure 2).

Thus the inclusion $Fc_iX \subseteq C_iFX$ is established.

Now suppose that $x \in C_iFX$. Say $y \in FX$ and $(3 \sim \{i\}) \restriction x = (3 \sim \{i\}) \restriction y$. Thus $\langle R_y, f_y \rangle \in X$. Clearly also $\langle R_x, f_x \rangle \in c_i\{\langle R_y, f_y \rangle\} \subseteq c_iX$. Thus $x \in Fc_iX$, as desired.

Some special cases of Theorem 3.2 are worth mention. If $\mathfrak{G}$ has dimension 3 or more, then $\mathfrak{G}$ can always be embedded as a hyperplane in a space of one higher dimension, and hence $\mathfrak{A}_{\mathfrak{G}}$ is always representable. If $\mathfrak{G}$ has dimension 2, then $\mathfrak{G}$ can be so embedded iff $\mathfrak{G}$ is Desarguesian. Thus $\mathfrak{A}_{\mathfrak{G}}$ is representable whenever $\mathfrak{G}$ is a

Desarguesian projective plane. Finally, the seemingly trivial case of dimension one is of great importance. In this case, $\mathfrak{G}$ is merely a set with at least 4 points, and to say that $\mathfrak{G}$ can be embedded as a hyperplane in a space of one higher dimension is just to say that $\mathfrak{G}$ is a line in some projective plane. If $\mathfrak{G}$ is infinite, then this is always true, and hence by Theorem 3.2 $\mathfrak{A}_{\mathfrak{G}}$ is representable. If $\mathfrak{G}$ is finite, then the exact determination of when $\mathfrak{G}$ is a line in a projective plane is unresolved. If $\mathfrak{G}$ has $p^n + 1$ elements for some prime p and some $n > 0$, then $\mathfrak{G}$ is a line in a projective plane. But by a celebrated theorem of Bruck and Ryser (see [2]), there are infinitely many $\mathfrak{G}$ which cannot be a line in a projective plane.

We now wish to consider the converse of Theorem 3.2. We call a complete $\boldsymbol{CA}_\alpha$ $\mathfrak{A}$ *completely representable* if for every non-zero $a \in A$ there is a homomorphism h from $\mathfrak{A}$ into a $\boldsymbol{Cs}_\alpha$ such that $ha \neq 0$ and h carries arbitrary sums (joins) into unions. It is easily verified that a finite $\boldsymbol{CA}_\alpha$ is representable if and only if it is completely representable. Thus for finite $\mathfrak{G}$, the following theorem is an exact converse of Theorem 3.2.

THEOREM 3.3: *If $\mathfrak{A}_{\mathfrak{G}}$ is completely representable, then $\mathfrak{G}$ is a hyperplane in a space $\mathfrak{G}'$ of one higher dimension.*

Again, we shall not give a complete proof of Theorem 3.3; cf. the proof of Lemma 2.3 in [12], and the proof of Theorem 1 of [9]. We shall just define $\mathfrak{G}'$, and verify (G_3) for it under the assumption that (G_1) and (G_2) hold. Let H be a complete homomorphism from $\mathfrak{A}_{\mathfrak{G}}$ onto a $\boldsymbol{Cs}_3$ of subsets of ${}^3U$ such that $H\mathfrak{U}\mathfrak{G} \neq 0$. We may assume that $U \cap \mathcal{P} = 0$, where $\mathfrak{G} = \langle \mathcal{P}, \mathcal{L} \rangle$. Let $\mathfrak{G}' = \langle \mathcal{P}', \mathcal{L}' \rangle$, where $\mathcal{P}' = \mathcal{P} \cup U$, and $\mathcal{L}'$ consists of the lines in $\mathcal{L}$ together with all sets of the form

$$L(p, u) = \{p, u\} \cup \{v \in U: \langle v, u, u \rangle \in H\{\langle R, f \rangle\}\},$$

where $p \in \mathcal{P}$, $u \in U$, $R = \{(0,0), (1,1), (2,2), (1,2), (2,1)\}$, and $f01 = f10 = f02 = f20 = p$. First note

(1) if $0 \neq X \in A_{\mathfrak{G}}$, then $HX \neq 0$.

In fact, $c_0c_1X = \mathfrak{U}\mathfrak{G}$ by the proof of Theorem 3.1, so $Hc_0c_1X = C_0C_1HX = {}^3U$. Since $C_0C_10 = 0$, it follows that $HX \neq 0$.

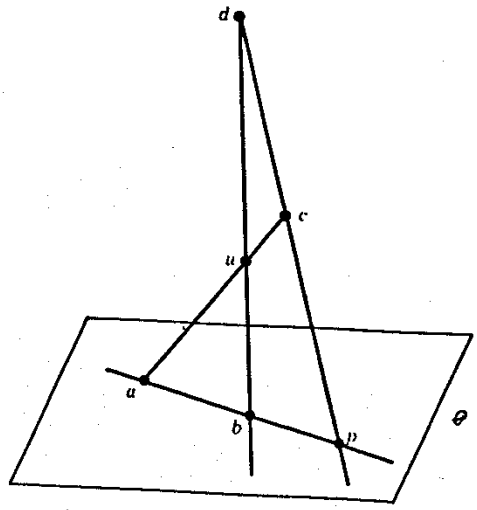

FIG. 3

Now we turn to the proof of (G_3), assuming that (G_1) and (G_2) hold. Let a, b, c, d be distinct points in $\mathcal{P}'$ such that $\overline{ab}$ and $\overline{cd}$ have a common point; we are to show that $\overline{ac}$ and $\overline{bd}$ have a common point. We may assume that $\overline{ab} \neq \overline{cd}$, and that $\overline{ab} \cdot \overline{cd}$ is different from a, b, c, d. If $\overline{ab}$ and $\overline{cd}$ are in $\mathcal{L}$, the desired conclusion follows since $\mathfrak{G}$ is a geometry.

Now suppose $\overline{ab}$ is in $\mathcal{L}$, while $\overline{cd}$ is not in $\mathcal{L}$ (the case $\overline{ab}$ not in $\mathcal{L}$, $\overline{cd}$ in $\mathcal{L}$ is similar). Clearly then $a, b \in \mathcal{P}$ while $c, d \in U$. Let $p = \overline{ab} \cdot \overline{cd}$. Thus $c \in L(p, d)$, so $\langle c, d, d\rangle \in H\{\langle R, f\rangle\}$, where $\langle R, f\rangle$ is as above. Now let S be the identity on 3, and let $g01 = g10 = p$, $g02 = g20 = a$, $g12 = g21 = b$. Note that $\{\langle R, f\rangle\} \subseteq c_2\{\langle S, g\rangle\}$; hence $H\{\langle R, f\rangle\} \subseteq C_2H\{\langle S, g\rangle\}$. Thus, since $\langle c, d, d\rangle \in H\{\langle R, f\rangle\}$, it follows that there is a $u \in U$ such that $\langle c, d, u\rangle \in H\{\langle S, g\rangle\}$. (See Figure 3.) Next, let

$$h01 = h10 = h02 = h20 = a;$$
$$k01 = k10 = k02 = k20 = b.$$

Clearly

$$c_1\{\langle S, g\rangle\} \cap d_{12} = \{\langle R, h\rangle\},$$
$$c_2(c_0\{\langle S, g\rangle\} \cap d_{02}) \cap d_{12} = \{\langle R, k\rangle\};$$

since $\langle c, u, u\rangle \in C_1H\{\langle S, g\rangle\} \cap D_{12}$ and

$$\langle u, d, d\rangle \in C_2(C_0H\{\langle S, g\rangle\} \cap D_{02}) \cap D_{12},$$

it follows that $\langle c, u, u\rangle \in H\{\langle R, h\rangle\}$ and $\langle u, d, d\rangle \in H\{\langle R, k\rangle\}$. Hence $u \in L(a, c)$ and $d \in L(b, u)$. Therefore by (G_2) u is a common point of $\overline{ac}$ and $\overline{bd}$.

It remains to treat the case in which neither $\overline{ab}$ nor $\overline{cd}$ is in $\mathfrak{L}$. Here we will also consider several subcases. First suppose that $a \in \mathcal{P}$ and $c \in \mathcal{P}$ (the case $b \in \mathcal{P}$ and $d \in \mathcal{P}$ is similar); see Figure 4. Define

$$f01 = f02 = f10 = f20 = a;$$

$$g01 = g02 = g10 = g20 = c.$$

Let $\overline{ab} \cdot \overline{cd} = u$. Clearly $u \in U$, and $\langle u, b, b\rangle \in H\{\langle R, f\rangle\}$, $\langle u, d, d\rangle \in H\{\langle R, g\rangle\}$. Hence

$$\langle u, b, d\rangle \in C_2H\{\langle R, f\rangle\} \cap C_1H\{\langle R, g\rangle\}$$
$$= H(c_2\{\langle R, f\rangle\} \cap c_1\{\langle R, g\rangle\}).$$

Since H is completely additive, it follows that there is an $\langle S, h\rangle \in c_2\{\langle R, f\rangle\} \cap c_1\{\langle R, g\rangle\}$ such that $\langle u, b, d\rangle \in H\{\langle S, h\rangle\}$. Here S is the identity on 3, $h01 = a$, and $h02 = c$. Let $p = h12$. Then a, c, p are collinear since $\langle S, h\rangle \in \mathfrak{U}\mathfrak{G}$. Furthermore, if $k01 = k02 = k10 = k20 = p$, then

$$\{\langle R, k\rangle\} = c_1(c_0\{\langle S, h\rangle\} \cap d_{01}) \cap d_{12}.$$

Clearly $\langle b, d, d\rangle \in C_1(C_0H\{\langle S, h\rangle\} \cap D_{01}) \cap D_{12}$, so $\langle b, d, d\rangle \in H\{\langle R, k\rangle\}$. Thus $p = \overline{ac} \cdot \overline{bd}$, as desired.

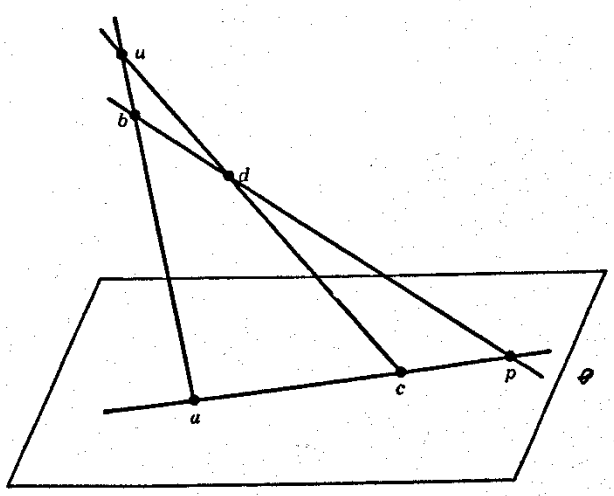

FIG. 4

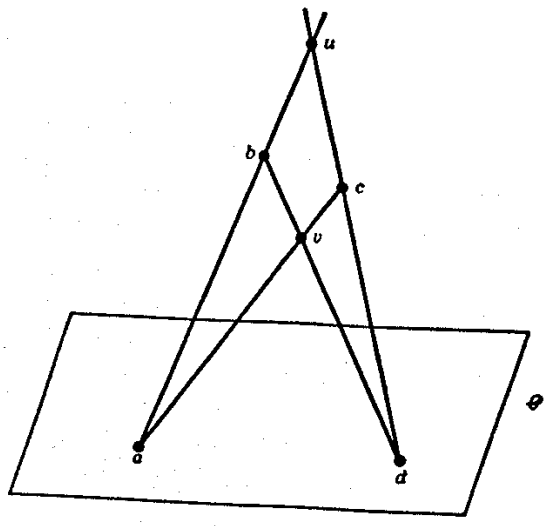

FIG. 5

Second, suppose that $a \in \mathcal{P}$ and $d \in \mathcal{P}$ (the case $b \in \mathcal{P}$ and $c \in \mathcal{P}$ is similar). (See Figure 5.) Let $u = \overline{ab} \cdot \overline{cd}$. Thus $u \in U$. Hence $\langle u, b, b\rangle \in H\{\langle R, f\rangle\}$ with $f01 = a$ and $\langle u, c, c\rangle \in H\{\langle R, g\rangle\}$ with $g01 = d$. Now let $x = c_0(c_2\{\langle R, f\rangle\} \cap c_1\{\langle R, g\rangle\})$. Then the following is easily established:

(1) $$d_{01} \cap c_1[d_{12} \cap c_2(d_{02} \cap x)] = x \cap d_{01}.$$

Now clearly $\langle b, b, c\rangle \in H(x \cap d_{01})$, so by (1), $\langle b, c, b\rangle \in Hx$. Hence there is a $v \in U$ such that $\langle v, c, b\rangle \in C_2H\{\langle R, f\rangle\}$ and $\langle v, c, b\rangle \in C_1H\{\langle R, g\rangle\}$. Obviously then v is a common point of $\overline{ac}$ and $\overline{bd}$, as desired.

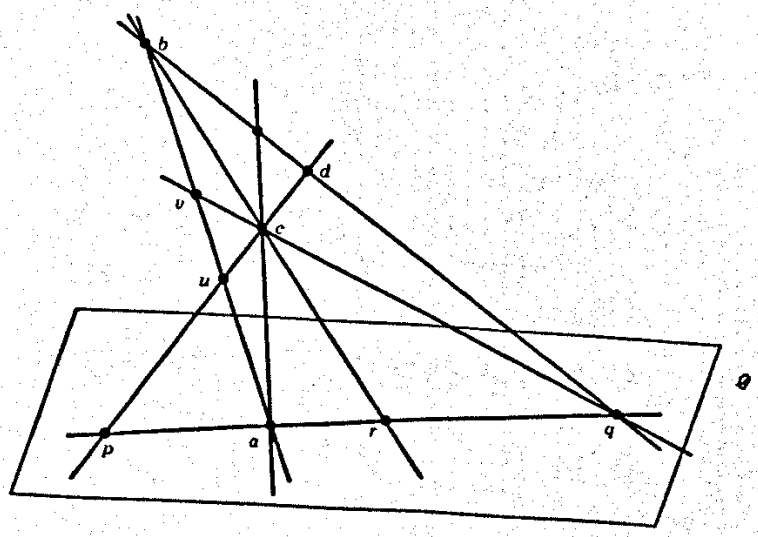

FIG. 6

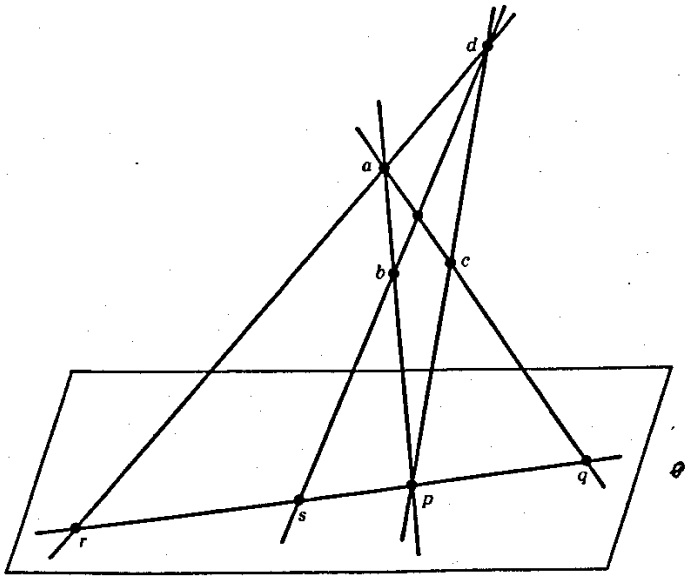

FIG. 7

We have now taken care of all cases in which two or more of the points a, b, c, d are in $\mathcal{P}$. Now suppose just one of them is in $\mathcal{P}$, say $a \in \mathcal{P}$. (See Figure 6.) Say $c, d \in L(p, c)$. Let $\overline{ab} \cdot \overline{cd} = u$. Then $\overline{pd} \cdot \overline{ab} = u$ also. Since both $a, p \in \mathcal{P}$, we apply a previous case to see that $\overline{pa}$ and $\overline{bd}$ intersect in some point q of $\mathcal{P}$. Thus $\overline{bq} \cdot \overline{cp} = d$; both $p, q \in \mathcal{P}$, so by a previous case $\overline{bc}$ and $\overline{pq}$ intersect in some point $r \in \mathcal{P}$. Thus $\overline{aq} \cdot \overline{bc} = r$, and $a, q \in \mathcal{P}$, so $\overline{ab} \cdot \overline{qc} = v$ for some v. Hence $\overline{ac} \cdot \overline{bq}$ exists; since $\overline{bq} = \overline{bd}$, the desired result follows.

Hence we may assume that none of a, b, c, d are in $\mathcal{P}$. Suppose $\overline{ab} \cdot \overline{cd} = p \in \mathcal{P}$. (See Figure 7.) Let $\overline{ac}$ intersect $\mathcal{P}$ at q. Then $\overline{aq} \cdot \overline{dp} = c$, so by a previous case $\overline{ad} \cdot \overline{pq} = r \in \mathcal{P}$ for some r. Also, $\overline{rd} \cdot \overline{pb} = a$, so $\overline{pr} \cdot \overline{bd} = s \in \mathcal{P}$ for some s. Finally, $\overline{da} \cdot \overline{sq} = r$, so $\overline{ds} \cdot \overline{aq} = \overline{bd} \cdot \overline{ac}$ exists, as desired. Finally, suppose $\overline{ab} \cdot \overline{cd} = u \notin \mathcal{P}$. (See Figure 8.) Let $\overline{ab} \cdot \mathcal{P} = p$, $\overline{cd} \cdot \mathcal{P} = q$. Now $\overline{pa} \cdot \overline{qc} = u$, so $\overline{pq} \cdot \overline{ac} = r \in \mathcal{P}$ for some r. Also, $\overline{pb} \cdot \overline{qd} = u$, so $\overline{pq} \cdot \overline{bd} = s \in \mathcal{P}$ for some s. Next, $\overline{pa} \cdot \overline{qd} = u$, so $\overline{pq} \cdot \overline{ad} = t \in \mathcal{P}$ for some t. $\overline{ad} \cdot \overline{rs} = t$, so $\overline{ar} \cdot \overline{ds} = \overline{ac} \cdot \overline{bd}$ exists, as desired. This completes the proof.

Theorems 3.2 and 3.3 can be used together to yield an important result concerning the class $\boldsymbol{R}_3$ of representable three-dimensional cylindric algebras. Namely, let $\boldsymbol{K}$ be an infinite collection of finite

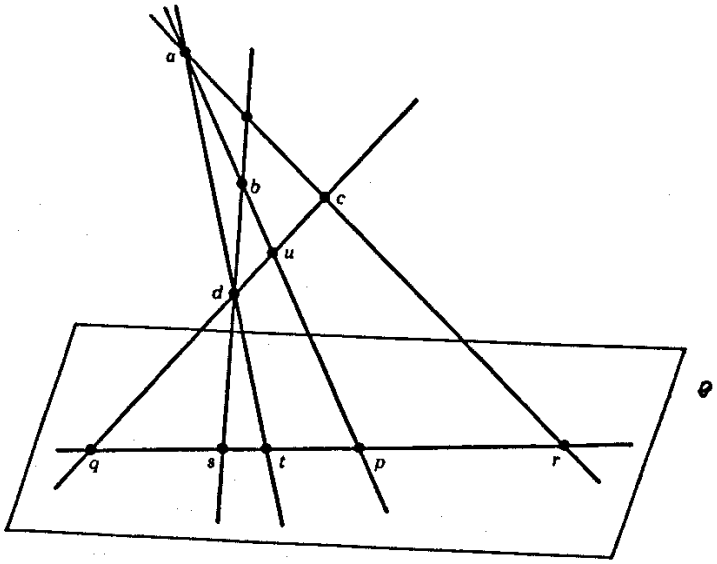

FIG. 8

one-dimensional projective geometries none of which can be embedded as a line in a plane; as mentioned above, such exist by [2]. Let F be a non-principal ultrafilter on K. Then the following conditions hold:

(1) for each $\mathfrak{G} \in K$, $\mathfrak{A}_{\mathfrak{G}}$ is non-representable (by Theorem 3.3);
(2) the ultraproduct $\mathfrak{B} = P_{\mathfrak{G} \in K} \mathfrak{A}_{\mathfrak{G}}/F$ can be isomorphically embedded in $\mathfrak{A}_{\mathfrak{K}}$ for some infinite one-dimensional geometry $\mathfrak{K}$, and hence $\mathfrak{B}$ is representable.

From (1) and (2) it follows that R_3 cannot be characterized by any finite set of first-order axioms. This is a special case of a more general result which can be established using results of the next section. For a proof of (2) see [12]; cf. also Monk [11] and McKenzie [10].

For the purpose of comparison with results in the next section we shall conclude this section with a purely combinatorial theorem of a simple nature. Let $S^2U = \{X : X \subseteq U, |X| = 2\}$.

THEOREM 3.4: *Let $\mathfrak{G} = \langle \mathcal{P}, \mathcal{L} \rangle$ be a one-dimensional projective geometry. Then the following conditions are equivalent:*

(i) *$\mathfrak{G}$ is a line in some projective plane;*
(ii) *there is a non-empty set U and a partition $\langle T_p : p \in \mathcal{P} \rangle$ of S^2U such that*

(a) *for all distinct* $u, v, w \in U$ *and all* $p \in \mathcal{P}$, *if* $\{u, v\} \in T_p$ *and* $\{v, w\} \in T_p$ *then* $\{u, w\} \in T_p$;

(b) *for all distinct* $u, v \in U$ *and all* $p, q, r \in \mathcal{P}$ *with* $|\{p, q, r\}| \neq 2$, *if* $\{u, v\} \in T_p$ *then there exists* $w \in U \sim \{u, v\}$ *such that* $\{u, w\} \in T_q$ *and* $\{w, v\} \in T_r$.

Proof: (i) $\Rightarrow$ (ii). Let $\mathfrak{G}$ be a line in a projective plane $\mathfrak{G}' = \langle \mathcal{P}', \mathcal{L}' \rangle$. Let $U = \mathcal{P}' \sim \mathcal{P}$, and for each $p \in \mathcal{P}$ let

$$T_p = \{\{u, v\} : u, v \in U, u \neq v, \quad \text{and} \quad \overline{uv} \cdot \mathcal{P} = p\}.$$

Clearly $\langle T_p : p \in \mathcal{P} \rangle$ is the desired partition of S^2U satisfying (a) and (b).

(ii) $\Rightarrow$ (i). Assume (ii). Let $\mathcal{P}' = \mathcal{P} \cup U$, where we assume that $\mathcal{P} \cap U = 0$. Let the lines of $\mathfrak{G}'$ be $\mathcal{P}$ together with all sets of the form

$$L(p, u) = \{v : \{u, v\} \in T_p\} \cup \{p, u\},$$

where $u \in U$ and $p \in \mathcal{P}$.

Since $|\mathcal{P}| \geqq 4$ by (G_1), using (b) we easily infer that

(1) any line of $\mathfrak{G}'$ has at least four points.

It is also clear from the definition of $L(p, u)$ that

(2) any two points of $\mathfrak{G}'$ lie on at least one line of $\mathfrak{G}'$.

(3) if $v \in V \cap L(p, u)$ with $u \in U$, then $L(p, u) = L(p, v)$.

To prove (3), first let $w \in L(p, u)$. We may assume that $u \neq v$. Thus, by hypothesis of (3), $\{u, v\} \in T_p$. If $w = u$, then obviously $w \in L(p, v)$. Also $w = v$ trivially yields $w \in L(p, v)$. Assume that $w \neq u$ and $w \neq v$. Then $\{u, w\} \in T_p$, so by (a) $\{v, w\} \in T_p$, and $w \in L(p, v)$ again. Therefore $L(p, u) \subseteq L(p, v)$. The converse is proved similarly.

Using (3) it is easy to check that

(4) any two distinct points of $\mathfrak{G}'$ lie on at most one line of $\mathfrak{G}'$.

To check that any two distinct lines intersect it suffices to take the distinct lines of the forms $L(p, u)$ and $L(q, v)$ with $p \neq q$ and

$u \neq v$. Choose $r \in \mathcal{P}$ with $\{u, v\} \in T_r$. If $r = p$, then $v \in L(p, u)$ and hence $L(p, u)$ and $L(q, v)$ intersect. Thus we may assume that $r \neq p$ and, similarly, that $r \neq q$. Then by (b) choose $w \in U \sim \{u, v\}$ so that $\{u, w\} \in T_p$ and $\{w, v\} \in T_q$. Thus $w \in L(p, u) \cap L(q, v)$, as desired. Thus

(5) any two distinct lines of $\mathfrak{G}'$ intersect.

From (1), (2), (4), (5) we see that $\mathfrak{G}'$ is a projective plane, as desired.

4. GRAPHS AND CA_α'S

The construction of section 3 can be slightly modified to yield a connection between certain graphs and CA_α's (now with $3 \leqq \alpha$). Throughout this section we assume that α is arbitrary but fixed, $3 \leqq \alpha$. If R is an equivalence relation on α, we let $R' = \{(i, j) : i, j < \alpha \text{ and } i \not R j\}$. A *selection set* for R is a subset T of α such that $i\, R'\, j$ whenever $i, j \in T$ and $i \neq j$. Now for any $\beta, \gamma < \omega$ we define $\mathfrak{C}(\alpha, \beta, \gamma)$ to be the collection of all pairs $\langle R, f \rangle$ satisfying the following conditions:

(1) R is an equivalence relation on α;
(2) f maps R' into γ;
(3) for $(i, j) \in R'$, $fij = fji$;
(4) if $i\, R'\, j$, $i\, R'\, k$, but $j\, R\, k$, then $fij = fik$;
(5) if T is a selection set for R and $|T| = \beta$, then

$$|\{fij : i, j \in T, i \neq j\}| \neq 1.$$

If $\langle R, f \rangle \in \mathfrak{C}(\alpha, \beta, \gamma)$, then f can be considered as an edge coloring of the complete graph on α/R vertices using γ colors, such that no subcomplete-graph on β vertices is monochromatic (edges all of the same color). This will play an implicit role later. Now we construct a CA_α from $\mathfrak{C}(\alpha, \beta, \gamma)$ analogously to section 3. For

$i, j < \alpha$ and $X \subseteq \mathcal{C}(\alpha, \beta, \gamma)$ we set

$$c_i X = \{\langle R, f\rangle \in \mathcal{C}(\alpha, \beta, \gamma) : \text{there is an } \langle S, g\rangle \in X \text{ such that } R \cap {}^2(\alpha \sim \{i\}) = S \cap {}^2(3 \sim \{i\}) \text{ and } fkl = gkl \text{ whenever } k\, R'\, l \text{ and } k, l \neq i\};$$

$$d_{ij} = \{\langle R, f\rangle \in \mathcal{C}(\alpha, \beta, \gamma) : i\, R\, j\}.$$

Finally, we let $A_{\alpha\beta\gamma}$ be the collection of all subsets of $\mathcal{C}(\alpha, \beta, \gamma)$ and

$$\mathfrak{A}_{\alpha\beta\gamma} = \langle A_{\alpha\beta\gamma}, \cup, \cap, \sim, c_i, d_{ij}\rangle_{i,j<\alpha}.$$

The algebras $\mathfrak{A}_{\alpha 3\gamma}$ have been discussed at some length in Monk [13], some of the results of which will be generalized here. (Different generalizations have appeared in Demaree [3], Johnson [7] and Monk [14].) First of all, in fact, the following two results have identical proofs with those of Theorem 1.1 and 1.2 of [13]:

THEOREM 4.1: *If $\gamma \geqq \alpha - 1$ and $\beta \geqq 3$, then $\mathfrak{A}_{\alpha\beta\gamma}$ is a CA_α.*

THEOREM 4.2: *If $3 \leqq \alpha \leqq \delta$, $\beta \geqq 3$, $\gamma \geqq \delta - 1$, then $\mathfrak{A}_{\alpha\beta\gamma}$ is neatly embeddable in $\mathfrak{A}_{\delta\beta\gamma}$.*

By virtue of Theorem 1.3, we may interpret the conclusion of Theorem 4.2 as saying that $\mathfrak{A}_{\alpha\beta\gamma}$ is "approximately" representable. If $\delta = \alpha + \omega$, then $\mathfrak{A}_{\alpha\beta\gamma}$ really is representable. We shall need the following supplement to 4.1; general algebraically it expresses the fact that $\mathfrak{A}_{\alpha\beta\gamma}$ is simple (see [6]).

THEOREM 4.3: *If $\alpha < \omega$, $\gamma \geqq \alpha - 1$, $\beta \geqq 3$, and $0 \neq X \subseteq \mathcal{C}(\alpha, \beta, \gamma)$, then $c_0 \cdots c_{\alpha-1} X = \mathcal{C}(\alpha, \beta, \gamma)$.*

Proof: It obviously suffices to prove the following statement:

(6) if $\langle R, f\rangle, \langle S, g\rangle \in \mathcal{C}(\alpha, \beta, \gamma)$ then $\langle R, f\rangle \in c_0 \cdots c_{\alpha-1}\{\langle S, g\rangle\}$.

To prove (6) we define a sequence $\langle\langle T_i, h_i\rangle : i \leqq \alpha\rangle$ of elements of $\mathcal{C}(\alpha, \beta, \gamma)$ in such a way that the following conditions hold for

each $i \leqq \alpha$:

(7) $T_i \cap {}^2(\alpha \sim i) = R \cap {}^2(\alpha \sim i)$ and $T_i \cap {}^2 i = S \cap {}^2 i$;
(8) if $j, k \in \alpha \sim i$ and $j\, R'\, k$, then $h_i jk = fjk$;
(9) if $j, k \in i$ and $j\, S'\, k$, then $h_i jk = gjk$;
(10) if $i > 0$, then $\langle T_{i-1}, h_{i-1} \rangle \in c_{i-1}\{\langle T_i, h_i \rangle\}$.

First we set $\langle T_0, h_0 \rangle = \langle R, f \rangle$; obviously (7)–(10) hold then with $i = 0$. Now suppose that $\langle T_i, h_i \rangle$ has been defined so that (7)–(10) hold. We set

$$T_{i+1} = [T_i \cap^2 (\alpha \sim \{i\})] \cup \{(i, i)\} \cup \{(i, k), (k, i) : k < i, i\, S\, k\}$$
$$\cup \{(i, j), (j, i) : j > i \text{ and there is a } k < i \text{ with } j\, T_i\, k\, S\, i\}.$$

Obviously T_{i+1} is symmetric and reflexive on α. To show that it is transitive it is enough, by symmetry, to consider the following cases:

Case 1. $j\, T_{i+1}\, k\, T_{i+1}\, i$ with $j, k < i$. Then $j\, T_i\, k\, S\, i$ and hence $j\, S\, k\, S\, i$ by (7). Thus $j\, S\, i$, so $j\, T_{i+1}\, i$.

Case 2. $j\, T_{i+1}\, k\, T_{i+1}\, i$ with $j < i$, $k > i$. Then there is an $l < i$ with $j\, T_i\, k\, T_i\, l\, S\, i$; thus $j\, T_i\, l\, S\, i$, so $j\, T_{i+1}\, i$ as in Case 1.

Case 3. $j\, T_{i+1}\, k\, T_{i+1}\, i$ with $j > i$, $k < i$. Then $j\, T_i\, k\, S\, i$, so obviously $j\, T_{i+1}\, i$.

Case 4. $j\, T_{i+1}\, k\, T_{i+1}\, i$ with $j, k > i$. This case is similar to Case 2.

Case 5. $j\, T_{i+1}\, i\, T_{i+1}\, k$ with $j, k < i$. Then $j\, S\, i\, S\, k$, so $j\, S\, k$ and hence $j\, T_i\, k$ by (7), and $j\, T_{i+1}\, k$.

Case 6. $j\, T_{i+1}\, i\, T_{i+1}\, k$ with $j < i$, $k > i$. Then there is an $l < i$ with $j\, S\, i$, $k\, T_i\, l\, Si$. Thus $l\, S\, j$ so $l\, T_i\, j$ by (7), hence $k\, T_i\, j$ and hence $j\, T_{i+1}\, k$.

Case 7. $j\, T_{i+1}\, i\, T_{i+1}\, k$ with $j, k > i$. Then there exist $l, m < i$ with $j\, T_i\, l\, S\, i$ and $k\, T_i\, m\, S\, i$. Thus $l\, S\, m$, so $l\, T_i\, m$ by (7). Hence $j\, T_i\, k$, so $j\, T_{i+1}\, k$.

This establishes that T_{i+1} is transitive, and hence is an equivalence relation on α. Further, (7) for $i+1$ is now obvious. Now if $j, k \in \alpha \sim \{i\}$ and $j\, T'_i\, k$ (hence $j\, T'_{i+1}\, k$), we set $h_{i+1} jk = h_i jk$. If $j < i$ and $j\, T'_{i+1}\, i$ (hence $j\, S'\, i$) we set $h_{i+1} ij = h_{i+1} ji = gji$. Finally, if $j > i$ and $j\, T'_{i+1}\, i$, we consider several cases.

Case 1). There is a $k \neq i$ such that $k\, T_{i+1}\, i$. Then we set $h_{i+1} ij = h_{i+1} ji = h_i jk$; clearly this does not depend on our particular choice of such a k.

Case 2). There is no $k \neq i$ such that $k\, T_{i+1}\, i$, but there is an $l < i$ with $l\, T_{i+1}\, j$. Then we set $h_{i+1} ij = h_{i+1} ji = gil$; again it is easy to check that this does not depend on our particular choice of such an l.

Case 3). There is no $k \neq i$ such that $k\, T_{i+1}\, i$, and there is no $l < i$ with $l\, T_{i+1}\, j$. Let $j_1, \cdots, j_m$ be a sequence of members of $\alpha \sim (i+1)$ satisfying the following conditions:

$j_s\, T'_{i+1}\, j_t \qquad \text{for} \qquad 1 \leqq s < t \leqq m;$

for each s with $1 \leqq s \leqq m$, there is no $l < i$ such that $l\, T_{i+1}\, j_s$; if $u \in \alpha \sim (i+1)$ and there is no $l < i$ such that $l\, T_{i+1}\, u$, then there is an s with $1 \leqq s \leqq m$ such that $u\, T_{i+1}\, j_s$.

Since $m \leqq \alpha - i - 1$ and $\gamma \geqq \alpha - 1$, we may pick distinct elements

$$h_{i+1} ij_1, \cdots, h_{i+1} ij_m \in \gamma \sim \{h_{i+1} it : t < i, i\, T'_{i+1}\, t\}.$$

Then we set $h_{i+1} j_s i = h_{i+1} ij_s$ for all s with $1 \leqq s \leqq m$. Furthermore, if $u \in \alpha \sim (i+1)$, there is no $l < i$ such that $l\, T_{i+1}\, u$, and $u \notin \{j_s : 1 \leqq s \leqq m\}$, then there is a unique s with $1 \leqq s \leqq m$ such that $u\, T_{i+1}\, j_s$, and we set

$$h_{i+1} ui = h_{i+1} iu = h_{i+1} ij_s.$$

In this way each element j as in Case 3 is taken care of. Thus h_{i+1} is now completely defined.

Now if $j, k \in \alpha \sim (i+1)$, then $j, k \in \alpha \sim i$ and hence

$$h_{i+1} jk = h_i jk = fjk \text{ by (8) for } i.$$

Thus (8) holds for $i+1$. Next, if $j, k \in i$ and $j\, S'\, k$, then

$$h_{i+1}jk = h_i jk = gjk;$$

to check that (9) holds we thus need only look at an element $h_{i+1}ij$, where $j < i$ and $i\, S'\, j$. Thus $i\, T'_{i+1}\, j$, so $h_{i+1}ij = gij$, as desired.

We still have to check that $\langle T_{i+1}, h_{i+1}\rangle \in \mathfrak{C}(\alpha, \beta, \gamma)$; it will then be obvious that (10) holds. Clearly here we only need to concern ourselves with conditions (4) and (5). To check condition (4), several cases need to be considered.

Case (1). $i\, T'_{i+1}\, j$, $i\, T'_{i+1}\, k$, $j\, T_{i+1}\, k$, $j, k < i$. This case is obvious since $\langle S, g\rangle \in \mathfrak{C}(\alpha, \beta, \gamma)$.

Case (2). $i\, T'_{i+1}\, j$, $i\, T'_{i+1}\, k$, $j\, T_{i+1}\, k$, $j < i$, $k > i$, and there is an $l < i$ such that $l\, T_{i+1}\, i$. Then $l\, S\, i$ and

$$\begin{aligned} h_{i+1}ik &= h_i lk && \text{by Case 1)} \\ &= h_i lj && \text{since } \langle T_i, h_i\rangle \in \mathfrak{C}(\alpha, \beta, \gamma) \\ &= glj && \text{by (9)} \\ &= gij && \text{since } \langle S, g\rangle \in \mathfrak{C}(\alpha, \beta, \gamma) \\ &= h_{i+1}ij. \end{aligned}$$

Case (3). $i\, T'_{i+1}\, j$, $i\, T'_{i+1}\, k$, $j\, T_{i+1}\, k$, $j < i$, $k > i$, and there is an $l > i$ such that $l\, T_{i+1}\, i$. Then by the definition of T_{i+1} there is a $u < i$ with $l\, T_i\, u\, S\, i$. Thus $u\, T_{i+1}\, i$, and the proof runs as in Case (2).

Case (4). $i\, T'_{i+1}\, j$, $i\, T'_{i+1}\, k$, $j\, T_{i+1}\, k$, $j < i$, $k > i$, and there is no $l \neq i$ such that $l\, T_{i+1}\, i$. Then $h_{i+1}ik = gij = h_{i+1}ij$ by Case 2).

Case (5). $i\, T'_{i+1}\, j$, $i\, T'_{i+1}\, k$, $j\, T_{i+1}\, k$, $j > i$, $k > i$, and there is an $l < i$ such that $l\, T_{i+1}\, i$. Then $l\, S\, i$ and

$$\begin{aligned} h_{i+1}ik &= h_i lk && \text{by Case 1)} \\ &= h_i lj && \text{since } \langle T_i, h_i\rangle \in \mathfrak{C}(\alpha, \beta, \gamma) \\ &= h_{i+1}ij && \text{by Case 1)} \end{aligned}$$

Case (6). $i\,T'_{i+1}\,j,\ i\,T'_{i+1}\,k,\ j\,T_{i+1}\,k,\ j > i,\ k > i$, and there is an $l > i$ such that $l\,T_{i+1}\,i$. This is treated similarly to Case (3).

Case (7). $j\,T'_{i+1}\,i,\ j\,T'_{i+1}\,k,\ i\,T_{i+1}\,k,\ j, k < i$. Then $i\,S\,k$ and

$$h_{i+1}ji = gji = gjk = h_i jk = h_{i+1}jk.$$

Case (8). $j\,T'_{i+1}\,i,\ j\,T'_{i+1}\,k,\ i\,T_{i+1}\,k,\ j < i,\ k > i$. Say $u < i$ and $k\,T_i\,u\,S\,i$. Then

$$h_{i+1}ji = gji = gju = h_i ju = h_i jk = h_{i+1}jk.$$

Case (9). $j\,T'_{i+1}\,i,\ j\,T'_{i+1}\,k,\ i\,T_{i+1}\,k,\ j > i,\ k \neq i$. Then $i\,S\,k$ and $h_{i+1}ji = h_i jk = h_{i+1}jk$.

Thus (4) has been established for $\langle T_{i+1}, h_{i+1}\rangle$. Turning to (5), let U be a selection set for T_{i+1} with $|\,T\,| = \beta$. If $i \notin U$, then the desired conclusion follows since $\langle T_i, h_i\rangle \in \mathfrak{C}(\alpha, \beta, \gamma)$. If $i \in U$ and there is a $k \neq i$ such that $k\,T_{i+1}\,i$, the desired conclusion again easily follows. Hence suppose $i \in U$ and there is no $k \neq i$ such that $k\,T_{i+1}\,i$. If for each $j \in U \sim \{i\}$ there is a $k < i$ with $k\,T_{i+1}\,j$, then the desired conclusion follows since $\langle S, g\rangle \in \mathfrak{C}(\alpha, \beta, \gamma)$. Finally, Case 3) takes care of the remaining possibility.

Thus $\langle T_{i+1}, h_{i+1}\rangle \in \mathfrak{C}(\alpha, \beta, \gamma)$, and our construction of $\langle\langle T_i, h_i\rangle : i < \alpha\rangle$ is complete. By (10), $\langle R, f\rangle \in c_0 \cdots c_{\alpha-1}\langle T_\alpha, h_\alpha\rangle$, and by (7) and (9), $\langle T_\alpha, h_\alpha\rangle = \langle S, g\rangle$, so the theorem is proved.

COROLLARY 4.4: *If* $\alpha < \omega$, $\gamma \geqq \alpha - 1$, $\beta \geqq 3$, *and* $\mathfrak{A}_{\alpha\beta\gamma}$ *is representable, then* $\mathfrak{A}_{\alpha\beta\gamma}$ *is isomorphic to a cylindric set algebra.*

Proof: If H is a homomorphism from $\mathfrak{A}_{\alpha\beta\gamma}$ onto a Cs_α^U with $U \neq 0$ (and such an H must exist, by assumption), then $H\mathfrak{C}(\alpha, \beta, \gamma) = {}^\alpha U$. Hence by 4.3 $HX \neq 0$ whenever $X \neq 0$. Thus H is an isomorphism.

We now want to give some analogs of Theorem 3.4 for our algebras $\mathfrak{A}_{\alpha\beta\gamma}$; as will be seen, these theorems give connections between combinatorial questions and representation problems which are analogous to the connections given in section 3. Before giving our general result in this direction we first give a special result whose formulation is much simpler. The special result is taken from the unpublished work Demaree [3], and is included

with his permission. Its proof is generalized for 4.6, and should serve to illustrate the more complicated proof.

THEOREM 4.5: *Let $\gamma \geqq 3$. Then the following two conditions are equivalent:*

(i) $\mathfrak{A}_{33\gamma}$ *is representable;*
(ii) *there is a non-empty set U and a partition $\langle T_\delta : \delta < \gamma \rangle$ of S^2U such that*

(a) *no T_δ contains a triangle, i.e., for all distinct $u, v, w \in U$ and all $\delta < \gamma$, if $\{u, v\} \in T_\delta$ and $\{u, w\} \in T_\delta$ then $\{u, w\} \notin T_\delta$;*
(b) *for all distinct $u, v \in U$ and all $\delta, \epsilon, \zeta < \gamma$ with $|\{\delta, \epsilon, \zeta\}| \geqq 2$, if $\{u, v\} \in T_\delta$ then there exists a $w \in U \sim \{u, v\}$ such that $\{u, w\} \in T_\epsilon$ and $\{w, v\} \in T_\zeta$.*

Proof: (i) $\Rightarrow$ (ii). By Corollary 4.4, let F be an isomorphism from $\mathfrak{A}_{33\gamma}$ onto a $\boldsymbol{Cs}^U$, where $U \neq 0$. Let P and R be the equivalence relations on 3 associated respectively with the partitions $\{\{0\}, \{1\}, \{2\}\}$ and $\{\{0\}, \{1, 2\}\}$; and for each $\delta < \gamma$ let f_δ be the mapping of R' into γ such that $f_\delta 01 = \delta$ and (3), (4) hold. Then

(11) if $\quad \langle u, v, w \rangle \in F\{\langle R, f_\delta \rangle\}, \quad$ then $\quad v = w$

and $\quad \langle v, u, u \rangle \in F\{\langle R, f_\delta \rangle\}$.

In fact, since $\langle R, f_\delta \rangle \in d_{12}$ it is clear that $v = w$. Also, it is easily checked that

$$\begin{aligned}\langle v, u, u \rangle &\in D_{12} \cap C_2[D_{02} \cap C_0(D_{01} \cap \sim D_{12} \cap C_1 F\{\langle R, f_\delta \rangle\})] \\ &= F(d_{12} \cap c_2[d_{02} \cap c_0(d_{01} \cap \sim d_{12} \cap c_1\{\langle R, f_\delta \rangle\})]) \\ &= F\{\langle R, f_\delta \rangle\},\end{aligned}$$

as desired in (11). Now for each $\delta < \gamma$ let

$$T_\delta = \{\{u, v\} \in S^2U : \langle u, v, v \rangle \in F\{\langle R, f_\delta \rangle\}\}.$$

Clearly, then, $\langle T_\delta : \delta < \gamma \rangle$ is a partition of S^2U. To check (a) suppose u, v, and w are distinct members of U with $\{u, v\}$, $\{v, w\}$,

$\{u, w\} \in T_\delta$. Thus, by (11),

$$\langle u, v, v\rangle, \langle v, w, w\rangle, \langle u, w, w\rangle \in F\{\langle R, f_\delta\rangle\}.$$

Hence, as is easily seen,

$$\langle u, v, w\rangle \in C_2F\{\langle R, f_\delta\rangle\} \cap C_1F\{\langle R, f_\delta\rangle\} \cap C_0(D_{01} \cap C_1F\{\langle R, f_\delta\rangle\})$$
$$= F[c_2\{\langle R, f_\delta\rangle\} \cap c_1\{\langle R, f_\delta\rangle\} \cap c_0(d_{01} \cap c_1\{\langle R, f_\delta\rangle\})];$$

on the other hand, it is easily checked that

$$c_2\{\langle R, f_\delta\rangle\} \cap c_1\{\langle R, f_\delta\rangle\} \cap c_0(d_{01} \cap c_1\{\langle R, f_\delta\rangle\}) = 0.$$

This contradiction shows that (a) must hold.

To check (b), suppose $u \neq v$, $\delta, \epsilon, \zeta < \gamma$, $|\{\delta, \epsilon, \zeta\}| \geqq 2$, and $\{u, v\} \in T_\delta$. Thus $\langle u, v, v\rangle \in F\{\langle R, f_\delta\rangle\}$. Now let g map P' into γ in such a way that $g01 = \delta$, $g02 = \epsilon$, and $g12 = \zeta$. Thus $\langle P, g\rangle \in \mathfrak{C}(3, 3, \gamma)$. Clearly $\langle R, f_\delta\rangle \in c_2\{\langle P, g\rangle\}$, so

$$\langle u, v, v\rangle \in F\{\langle R, f_\delta\rangle\} \subseteq C_2F\{\langle P, g\rangle\}.$$

Thus we may choose $w \in U$ so that $\langle u, v, w\rangle \in F\{\langle P, g\rangle\}$. Since $c_1\{\langle P, g\rangle\} \cap d_{12} = \{\langle R, f_\epsilon\rangle\}$ and $d_{12} \cap c_1(c_0\{\langle P, g\rangle\} \cap d_{01}) = \{\langle R, f_\zeta\rangle\}$, it follows that $\{u, w\} \in T_\epsilon$ and $\{w, v\} \in T_\zeta$, as desired in (b).

(ii) $\Rightarrow$ (i). Assume (ii). For each $x \in {}^3U$ let

$$R_x = \{(i, j) \in {}^23 : x_i = x_j\},$$

and for any $(i, j) \in R'_x$ let $f_x ij =$ the $\delta < \gamma$ such that $\{x_i, x_j\} \in T_\delta$. By (a), $\langle R_x, f_x\rangle \in \mathfrak{C}(3, 3, \gamma)$. Then for any $X \subseteq \mathfrak{C}(3, 3, \gamma)$ we set

$$FX = \{x \in {}^3U : \langle R_x, f_x\rangle \in X\}.$$

Clearly F is a Boolean homomorphism. To show that F is one-one, suppose that $0 \neq X \subseteq \mathfrak{C}(3, 3, \gamma)$; say $\langle S, g\rangle \in X$. If $S \neq$ identity on 3, obviously $x \in F\{\langle S, g\rangle\} \subseteq FX$ for some x. Suppose $S =$ identity on 3. Choose $\{u, v\} \in T_{g01}$. Then by (b) choose $w \neq u, v$ so that $\{u, w\} \in T_{g02}$ and $\{w, v\} \in T_{g12}$. Thus $\langle u, v, w\rangle \in F\{\langle S, g\rangle\} \subseteq FX$, as desired.

Clearly F preserves d_{ij} for any $i, j < 3$; also it is clear that $C_iFX \subseteq Fc_iX$ whenever $i < 3$ and $X \subseteq \mathfrak{C}(3, 3, \gamma)$. To show that

$Fc_iX \subseteq C_iFX$, suppose that $x \in Fc_iX$. Thus $\langle R_x, f_x \rangle \in c_iX$, so there is an $\langle S, g \rangle \in X$ such that $R_x \cap {}^3(\alpha \sim \{i\}) = S \cap {}^3(\alpha \sim \{i\})$ and $f_x kl = gkl$ if $k R'_x l$ and $k, l \neq i$. Let $3 = \{i, k, l\}$. If $S \neq$ identity on 3, it is easily seen that $x \in C_iFX$. Hence assume that $S =$ identity on 3. By (b), choose $w \in U \sim \{x_k, x_l\}$ so that $\{x_k, w\} \in T_{gik}$ and $\{w, x_l\} \in T_{gil}$. Then with

$$y = \{(k, x_k), (l, x_l), (i, w)\}$$

we have $\langle R_y, f_y \rangle = \langle S, g \rangle$ and hence $x \in C_iFX$.

This completes the proof.

THEOREM 4.6: *If $\alpha < \omega$, $\gamma \geqq \alpha - 1$, and $\alpha \geqq \beta \geqq 3$, then the following two conditions are equivalent:*

(i) *$\mathfrak{A}_{\alpha\beta\gamma}$ is representable;*

(ii) *there is a non-empty set U and a partition $\langle T_\delta : \delta < \gamma \rangle$ of S^2U such that*

(a) *no T_δ contains the complete graph on β vertices, i.e., if $V \subseteq U$ with $|V| = \beta$ then $S^2V \nsubseteq T_\delta$;*

(b) *if $V \subseteq U$, $|V| < \alpha$, $f \in {}^V\gamma$, and there do not exist a $\delta < \gamma$ and a $W \subseteq V$ such that $|W| = \beta - 1$, $fw = \delta$ for all $w \in W$, and $S^2W \subseteq T_\delta$, then there is a $u \in U \sim V$ such that $\{v, u\} \in fv$ for all $v \in V$.*

Proof: (i) $\Rightarrow$ (ii). By Corollary 4.4, let F be an isomorphism from $\mathfrak{A}_{\alpha\beta\gamma}$ onto a $\boldsymbol{Cs}^U$, where $U \neq 0$. Let R be the equivalence relation on α associated with the partition $\{\{0\}, \alpha \sim \{0\}\}$, and for each $\delta < \gamma$ let f_δ be the mapping of R' into γ such that $f_\delta 01 = \delta$. Then, as in the proof of 4.5,

(12) if $\quad \langle u, v, v, \cdots \rangle \in F\{\langle R, f_\delta \rangle\}$

then $\quad \langle v, u, u, \cdots \rangle \in F\{\langle R, f_\delta \rangle\}$.

Now for each $\delta < \gamma$ we let

$$T_\delta = \{\{u, v\} \in S^2U : \langle u, v, v, \cdots \rangle \in \{\langle R, f_\delta \rangle\}\}.$$

Clearly, then, $\langle T_\delta : \delta < \gamma \rangle$ is a partition of S^2U. To check (a), suppose $V \subseteq U$, $|V| = \beta$, and $S^2V \subseteq T_\delta$. Thus for any two

distinct $u, v \in V$ we have $\langle u, v, v, \cdots \rangle \in F\{\langle R, f_\delta \rangle\}$. Let $w \in {}^\alpha U$ be such that for some $\epsilon \leqq \alpha$, $\epsilon \upharpoonright w$ is a one-one map onto V. For $\Gamma \subseteq \alpha$, say $\Gamma = \{\lambda_0, \cdots, \lambda_{k-1}\}$, let $c_{(\Gamma)} = c_{\lambda_0} \cdots c_{\lambda_{k-1}}$. Let

$$X = \bigcap_{0<i<\epsilon} c_{(\alpha \sim \{0,1\})}\{\langle R, f_\delta \rangle\}$$
$$\cap \bigcap_{0<i<j<\epsilon} c_0[d_{0i} \cap c_1(d_{1j} \cap c_{(\alpha \sim \{0,1\})}\{\langle R, f_\delta \rangle\}.$$

Then it is easily verified that $X = 0$, but on the other hand $w \in FX$. This contradiction shows that (a) holds.

Now to check (b), we assume its hypothesis. Let $x \in {}^\alpha U$ be such that for some $\epsilon < \alpha$, x maps ϵ one-one onto V. Let $x \in F\{\langle Q, g \rangle\}$. Now we set

$$S = [Q \cap^2 (\alpha \sim \{\alpha - 1\})] \cup \{(\alpha - 1, \alpha - 1)\}.$$

Clearly S is an equivalence relation on α. For $k, l \in \alpha \sim \{\alpha - 1\}$ and $kQ'l$ we let $hkl = gkl$. For $i < \epsilon$ we let $h(\alpha - 1, i) = h(i, \alpha - 1) = fx_i$. Choose $j_1, \cdots, j_m \in (\alpha - 1) \sim \epsilon$ so that

$$j_s R' j_t \quad \text{for} \quad 1 \leqq s < t \leqq m;$$

if $k \in (\alpha - 1) \sim \epsilon$ then there is an s with $1 \leqq s \leqq m$ such that $k\, R\, j_s$.

Now let $h(j_1, \alpha - 1), h(j_2, \alpha - 1), \cdots, h(j_m, \alpha - 1)$ be distinct members of $\gamma \sim \{h(i, \alpha - 1) : i < \epsilon\}$. For $k \in (\alpha - 1) \sim (\epsilon \cup \{j_s : 1 \leqq s \leqq m\})$ let s be such that $1 \leqq s \leqq m$ and $k\, R\, j_s$ (there is only one such s), and let $h(k, \alpha - 1) = h(j_s, \alpha - 1)$. Finally, if $l \in (\alpha - 1) \sim \epsilon$ we let $h(\alpha - 1, l) = h(l, \alpha - 1)$. This completes the definition of h; clearly h maps S' into γ. We claim that $\langle S, h \rangle \in \mathfrak{C}(\alpha, \beta, \gamma)$. Indeed, conditions (1)–(4) are clear from the definitions. Now, in order to check (5), suppose T is a selection set for S with $|T| = \beta$. If $\alpha - 1 \notin T$, the desired conclusion is obvious. Suppose $\alpha - 1 \in T$. If $k \in T$ for some k with $\epsilon \leqq k < \alpha - 1$, then again the desired conclusion is clear. Suppose there is no such k. Assume that $|\{hij : i, j \in T, i \neq j\}| = 1$; say $\{hij : i, j \in T, i \neq j\} = \{\delta\}$. Now from the fact that $x \in F\{\langle Q, g \rangle\}$ it easily follows that for distinct $i, j \in T \sim \{\alpha - 1\}$ we have $\{x_i, x_j\} \in T_{gij} = T_{hij} = T_\delta$. Furthermore, $fx_i = \delta$ for all $i \in T \sim \{\alpha - 1\}$. This contradicts the hypothesis of (b).

Hence our assumption that $|\{hij: i, j \in T, i \neq j\}| = 1$ is false. Thus $\langle S, h\rangle \in \mathfrak{C}(\alpha, \beta, \gamma)$, as stated.

Now clearly $\langle Q, g\rangle \in c_{\alpha-1}\{\langle S, h\rangle\}$; since $x \in F\{\langle Q, g\rangle\}$, we may hence choose $u \in U$ such that $\langle x_0, \cdots, x_{\alpha-2}, u\rangle \in F\{\langle S, h\rangle\}$. Clearly u is as desired in the conclusion of (b).

(ii) $\Rightarrow$ (i). Assume (ii). For each $x \in {}^{\alpha}U$ let

$$R_x = \{(i, j) \in {}^2\alpha: x_i = x_j\},$$

and for any $(i, j) \in R'_x$ let $f_x ij =$ the $\delta < \gamma$ such that $\{x_i, x_j\} \in T_\delta$. By (a), $\langle R_x, f_x\rangle \in \mathfrak{C}(\alpha, \beta, \gamma)$. Now for any $X \subseteq \mathfrak{C}(\alpha, \beta, \gamma)$ we set

$$FX = \{x \in {}^{\alpha}U: \langle R_x, f_x\rangle \in X\}.$$

Clearly F is a Boolean homomorphism. To show that F is one-one, suppose that $0 \neq X \subseteq \mathfrak{C}(\alpha, \beta, \gamma)$; say $\langle S, g\rangle \in X$. Choose $j_1, \cdots, j_m \in \alpha$ satisfying the following conditions:

(13) $\quad j_s S' j_t \quad$ if $\quad 1 \leqq s < t \leqq m,$

(14) if $\quad k \in \alpha, \quad$ then $\quad k S j_s \quad$ for some s with $\quad 1 \leqq s \leqq m.$

We now define $xj_1, \cdots, xj_m$ by recursion. Let xj_1 be any element of U. Now suppose that $1 \leqq s < m$ and $xj_1, \cdots, xj_s$ have been defined so that the following conditions hold:

(15) $\quad xj_1, \cdots, xj_s \quad$ are all distinct;

(16) if $\quad 1 \leqq t < u \leqq s, \quad$ then $\quad \{xj_t, xj_u\} \in T_{gtu}.$

Let $V = \{xj_t: 1 \leqq t \leqq s\}$. Since $s < m \leqq \alpha$, $|V| < \alpha$. For each $xj_t \in V$ let $fxj_t = gj_t j_{s+1}$. Then the hypothesis of (b) holds. In fact, suppose there exist $\delta < \gamma$ and $W \subseteq V$ so that $|W| = \beta - 1$, $fw = \delta$ for all $w \in W$, and $S^2W \subseteq T_\delta$. Say $W = \{xj_t: t \in Z\}$ where $Z \subseteq \{t: 1 \leqq t \leqq s\}$. Thus by (16), $gj_t j_u = \delta$ for distinct $t, u \in Z$, and $gj_t j_{s+1} = \delta$ for any $t \in Z$. This contradicts condition (5) for $\langle S, g\rangle$. Thus, indeed, the hypothesis of (b) holds. By the conclusion of (b), let $xj_{s+1} \in U \sim V$ be such that $\{xj_t, xj_{s+1}\} \in fxj_t$ for all t with $1 \leqq t \leqq s$. Thus (15) and (16) hold for $s + 1$. This completes the construction of $xj_1, \cdots, xj_m$. If $k \in \alpha \sim \{j_1, \cdots, j_m\}$, let s be such that $1 \leqq s \leqq m$ and $k S j_s$ (by (14),

where s is unique by (13)), and set $xk = xj_s$. Thus $x \in {}^{\alpha}U$. Clearly $x \in F\{\langle S, g\rangle\} \subseteq FX$, as desired. Thus F is one-one.

Clearly F preserves d_{ij} for any $i, j < \alpha$; also it is clear that $C_iFX \subseteq Fc_iX$ whenever $i < \alpha$ and $X \subseteq \mathfrak{C}(\alpha, \beta, \gamma)$. To show that $Fc_iX \subseteq C_iFX$, suppose that $x \in Fc_iX$. Thus $\langle R_x, f_x\rangle \in c_iX$, so there is an $\langle S, g\rangle \in X$ such that $R_x \cap {}^3(\alpha \sim \{i\}) = S \cap {}^3(\alpha \sim \{i\})$ and $f_xkl = gkl$ if $k\, R'_x\, l$ and $k, l \neq i$. If $i\, S\, j$ for some $j \neq i$, it is clear that $x \in C_iFX$. Hence assume that $i\, S'\, j$ for all $j \neq i$. Choose $j_1, \cdots, j_m \in \alpha \sim \{i\}$ such that $j_s\, S'\, j_t$ if $1 \leqq s < t \leqq m$;

$$\text{if} \quad k \in \alpha \sim \{i\}, \quad \text{then} \quad k\, S\, j_s \quad \text{for some } s \text{ with } 1 \leqq s \leqq m.$$

Let $V = \{xj_s\colon 1 \leqq s \leqq m\}$. For each s with $1 \leqq s \leqq m$ let $hxj_s = gij_s$. Then the hypothesis of (b) holds, as is easily seen. Applying (b), we get a $u \in U \sim V$ such that $\{xj_s, u\} \in hxj_s$ for each s with $1 \leqq s \leqq m$. Let y be like x except that $y_i = u$. Clearly $R_y = S$ and $f_y = g$. It follows that $x \in C_iFX$, as desired.

The combinatorial condition expressed in 4.6(ii) may be loosely termed a *free decomposition of the complete graph on U into subgraphs each excluding the complete graph on β vertices.* This condition has not been investigated in the literature, as far as this author knows. However, condition 4.6(ii) (a) by itself has been extensively investigated. The basic result here is the following theorem of Ramsey (see Ramsey [15]):

THEOREM 4.7: *Suppose $3 \leqq \beta < \omega$ and $2 \leqq \gamma < \omega$. Then there is an integer $n(\beta, \gamma) \in \omega \sim 1$ with the following property. If U is a set with at least $n(\beta, \gamma)$ elements and if $\langle T_\delta\colon \delta < \gamma\rangle$ is a partition of S^2U, then there exist a $\delta < \gamma$ and a subset V of U with $|V| = \beta$ such that $S^2V \subseteq T_\delta$.*

From this theorem and the proof of 4.6 it follows that if $\mathfrak{A}_{\alpha\beta\gamma}$ is representable, say isomorphic to a $\boldsymbol{Cs}^U$, then $|U| < n(\beta, \gamma)$. Thus a good knowledge of representation of the algebras $\mathfrak{A}_{\alpha\beta\gamma}$ would yield lower bounds for the Ramsey numbers $n(\beta, \gamma)$. We do not have such knowledge yet. However, in [3] it is shown directly that $\mathfrak{A}(3, 3, 2)$ and $\mathfrak{A}(3, 3, 3)$ are representable; but since $n(3, 2) = 6$ and $n(3, 3) = 17$ by [4], no new information on the Ramsey numbers is obtained.

On the other hand, $|A_{\alpha\beta\gamma}|$ forces a lower bound on $|U|$. We shall give such a lower bound for the case $\beta = 3$; for the proof see the proof of Theorem 1.8 of [13].

THEOREM 4.8: *If* $\mathfrak{A}_{\alpha 3\gamma}$ *is isomorphic to a* Cs^U *and* $\alpha < \omega$, $2 \leqq \gamma < \omega$, *then*

$$|S^2U| \geqq (\gamma - 2)^{2\alpha-6}.$$

Now upper bounds for the Ramsey numbers $n(\beta, \gamma)$ are not known in general, but for $\beta = 3$ we have the following result of Greenwood–Gleason [4]:

THEOREM 4.9: $n(3, \gamma) \leqq [\gamma!\, e] + 1$.

COROLLARY 4.10: *If* $\mathfrak{A}(\alpha, 3, \gamma)$ *is representable, then*

$$(\gamma - 2)^{2\alpha-6} \leqq ([\gamma!\, e] + 1)^2 \leqq 9 \cdot \gamma!^2.$$

Corollary 4.10 implies that certain of the algebras $\mathfrak{A}(\alpha, 3, \gamma)$ are nonrepresentable. For example, a simple computation shows that $\mathfrak{A}(13, 3, 13)$ is non-representable. In [13] these methods are refined so as to yield a far-reaching generalization of the result mentioned at the end of section 3. Namely, with the essential use of Theorem 1.3, it is shown there that $\boldsymbol{R}_\alpha$ is not finitely axiomatizable for any $\alpha \geqq 3$.

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