Remark. Some observations indicate that Lemma 3 can be proved in a much sharper form than ours (see Sublemma 3.3), more precisely, in right hand sides of $2^n$ (a) (b) and (c) one may place coefficients increasing unlimitedly with $n$, (the constants given in Lemma 3 are of course, sufficient for getting a contradiction.

REFERENCES


ON AUTOMORPHISM GROUPS OF BOOLEAN ALGEBRAS

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It has been known for a long time that not every group is isomorphic to the automorphism group of some Boolean algebra (BA, for short). For example, de Groot and McDowell [3] showed that for any BA $\mathcal{W}$ the automorphism group of $\mathcal{W}$ either consists just of those automorphisms induced by finite permutations of the atoms of $\mathcal{W}$, or else contains the direct sum of $N_\infty$ copies of $C_2$. (We include rigid BA's under the first case.) Thus the problem arises to characterize in some convenient form the automorphism groups of BA's. We address ourselves to this question in Section 1 of this paper. We give there two representation theorems for complete BA's which in principle reduce the characterization problem for complete BA's to two narrower classes of BA's — homogeneous BA's, and those with no rigid or homogeneous factors. In Section 2 we are concerned with rigid BA's. Lozier [9] has shown that for any $m \geq N_\infty$ there is a rigid BA of power $2^m$. We show that there is also one of each strong limit power. de Groot [2] showed that there are $2^{2^{2^\kappa}}$ rigid BA's of

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power $2^\aleph_0$. We prove under GCH that for any regular $m > \aleph_0$, there are $m^+$ rigid BA’s of power $m^+$. (This result was obtained earlier by Ehrenfeucht, but his construction remains unpublished.) In section 3 we relate the cardinalities of a BA and its automorphism group. The main results there have as consequences that there is a BA of power $2^{\aleph_0}$ with denumerable automorphism group; and under GCH, if $\aleph_0 < n < m^+ > \aleph_n$, then there is a BA of power $m$ with automorphism group of power $n$.

Our results are actually somewhat stronger than indicated in this brief description, and we give several results related to the above, as well as the statement of 12 open problems.

The notation in the paper is standard, with the following exceptions. The symbols $\rightarrow$, $\otimes$, $\rightarrow$ indicate one-one, onto, and one-one onto mappings respectively; between BA’s they implicitly represent homomorphisms, and between topological spaces – continuous maps. We let $\exp m = 2^m$. The set of all subsets of $I$ (of power $< n$) is denoted by $\mathcal{S}_I$ (respectively $\mathcal{S}_{< n} I$). Sym $I$ is the group of all permutations of $I$, while Sym $(m, n)$ is the group of all permutations of $m$ with support of power $< n$. We denote by $\mathcal{S}_C^m$ the direct sum of $m$ copies of $\mathcal{C}$.

If $m$ is an infinite cardinal, then by $\text{MA}_m$ we mean the statement that if $P$ is a partially ordered set satisfying the countable antichain condition, and if $\mathcal{B}$ is a collection of open dense subsets of $P$ with $|\mathcal{B}| < m$, then an $\mathcal{B}$-generic set over $P$ exists (cf. Martin, Solovay [10]). Martin’s axiom (MA) is the statement $\forall m < \exp \aleph_0 \mathcal{B}_m$. It is known that $\text{MA}_\aleph_0$ holds. Furthermore, CH = MA. Finally, Con (ZFC) $\rightarrow$ Con (ZFC $+$ exp $\aleph_0 > \aleph_1 + \text{MA}$); see Solovay, Tennenbaum [17].

A BA is a structure $\frak{U} = (\mathbb{A}, \cdot, \cdot, 0, 1)$ satisfying the usual axioms. $\frak{U}$ is called non-trivial if $|\mathbb{A}| > 1$. German capitals denote BA’s, and corresponding Roman letters denote their universes. If $m$ is an infinite cardinal, then $\frak{U}$ is $m$-complete if $\Sigma \mathbb{A}$ exists for each $X \in \mathbb{A}$ with $|X| < m$. If $\frak{U}$ is a BA and $a \in \mathbb{A}$, then $\frak{U} \upharpoonright a$ is the principal ideal generated by $a$, considered as a BA. For any BA $\frak{U}$, $\frak{U}$ (the cardinality of $\frak{U}$) is the least cardinal greater than all cardinalities of families of pairwise disjoint elements of $\frak{U}$. A partition of unity of a BA $\frak{A}$ is a function $a \in \mathbb{A}$ such that $\sum_{i \in I} a_i = 1$ and $\partial a_i = 0$ for $i \in I$ and $i \neq f$. If $I$ is finite or $\frak{A}$ is complete, such a partition gives rise to an isomorphism $f : \frak{A} \rightarrow \bigoplus_{i \in I} \frak{A}_i$ defined by $(\partial f_i) = x \cdot a_i$ for all $x \in A$ and $i \in I$. Two BA’s $\frak{U}$ and $\frak{V}$ are called totally different iff whenever $0 \neq x \in A$ and $0 \neq y \in B$ we have $\frak{U} \upharpoonright x \neq \frak{V} \upharpoonright y$. Next, suppose $(\frak{U}, I)$ is a system of BA’s. For $i \in I$ and $x \in A_i$ we define $\delta_i x \in \prod_{i \in I} A_i$ by setting $(\delta_i x)_i = x$ if $i = i$ and $(\delta_i x)_i = 0$ if $i \neq i$. For any BA $\frak{U}$, $\text{At} \frak{U}$ is the collection of atoms of $\frak{U}$. We take the Stone space of $\frak{U}$ to be the collection of ultrafilters of $\frak{U}$ in the usual way. The automorphism group of $\frak{U}$ is denoted by $\text{Aut} \frak{U}$. We say that $\frak{U}$ is rigid if $|\text{Aut} \frak{U}| = 1$.

$\frak{U}$ is homogeneous if $\frak{U} \cong \frak{U} \upharpoonright a$ whenever $0 \neq a \in A$. An element $a \in A$ is rigid (homogeneous) if $\frak{U} \upharpoonright a$ is rigid (resp. homogeneous). Elements $a, b \in A$ are isomorphic if $\frak{U} \upharpoonright a \cong \frak{U} \upharpoonright b$. Similar transfers of terminology from $\frak{U} \upharpoonright a$ to $a$ itself will be made later without explicit mention. $\text{Sg} X$ is the subalgebra generated by $X$.

1. PRODUCTS OF BA’S

In this section we discuss products of BA’s, in particular, rigid or homogeneous BA’s, always with the automorphism groups in mind. The following lemma and proof, of a general algebraic nature, are well-known.

Lemma 1.1. If $(\frak{U}_i ; i \in I)$ is a system of similar algebras, then $\prod_{i \in I} \text{Aut} \frak{U}_i \rightarrow \text{Aut} \bigoplus_{i \in I} \frak{U}_i$.

Proof. For each $i \in I$, let $\phi_i : \prod_{i \in I} \text{Aut} \frak{U}_i \rightarrow \text{Aut} \frak{U}_i$.

In general the isomorphism in the proof of 1.1 is not onto; e.g., 1.11 below. But for BA’s there is an important case where it is:

Theorem 1.2. If $(\frak{U}_i ; i \in I)$ is a system of pairwise totally different BA’s, then $\prod_{i \in I} \text{Aut} \frak{U}_i \cong \text{Aut} \bigoplus_{i \in I} \frak{U}_i$. 

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Proof. It suffices to show that the function \( f \) defined in the proof of 1.1 is onto. Let \( \varphi \in \text{Aut}_{\text{P}} \mathfrak{W} \). We wish to define \( \sigma \in \text{P} \text{Aut}_{\text{I}} \mathfrak{W} \) so that \( f \sigma = \varphi \). We need two preliminary statements.

(1) If \( i, j \in I \), \( i \neq j \), and \( x \in A_j \), then \( \varphi(i, x) = 0 \).

For, assume otherwise: say \( i \neq j \), \( \varphi(i, x) \neq 0 \). Let \( z = \varphi^{-1}(\delta(i, j, x)) \). Since \( \delta(i, j, x) \neq \varphi(i, x) \), we have \( z \neq \delta(i, j, x) \). Thus \( z_k = 0 \) for \( k \neq j \). Now \( \varphi \) induces an isomorphism of \( \mathfrak{W} \) onto \( \mathfrak{W} \) \( \varphi(i, x) \). This contradicts our assumption that \( \mathfrak{W} \) and \( \mathfrak{W} \) lack common non-trivial factors. Hence (1) holds. Next,

(2) If \( i \in I \), then \( \varphi(i) = 0 \).

For, by (1) write \( \varphi(i) = \delta(i, x) \). If \( x \neq 1 \), then by (1) for \( \varphi^{-1} \), say \( \varphi^{-1}(\delta(i, x)) = \delta(i, u) \). Thus \( u \neq 0 \), but \( \delta(i, u) = \delta(i, x) \). Thus \( \varphi(i, x) = 0 \), contradiction.

Now define \( \sigma \varphi(i, x) = \varphi(i, x) \), for any \( i \in I \) and \( x \in A_j \). It is easy to verify using (1) and (2) that \( \sigma \in \text{P} \text{Aut}_{\text{I}} \mathfrak{W} \) and that \( f \sigma = \varphi \), as desired.

Theorem 1.2 motivates our investigations of this section. By it, to describe \( \text{Aut} \mathfrak{W} \) it is enough to decompose \( \mathfrak{W} \) into a product of (simple, in some sense) \( \mathfrak{W} \)'s pairwise lacking common non-trivial factors. The simplest building blocks from the point of view of automorphism groups are the rigid \( \mathfrak{W} \)'s, which we now investigate.

Corollary 1.3. If \( \{ \mathfrak{W}_i : i \in I \} \) is a system of pairwise totally different rigid \( \mathfrak{W} \)'s, then \( \text{P} \text{Aut}_{\text{I}} \mathfrak{W} \) is rigid.

A natural conjecture is that also a subdirect product of pairwise totally different rigid \( \mathfrak{W} \)'s is rigid. This is not generally true; see the remark after 1.32.

Corollary 1.4. If \( \mathfrak{W} \) and \( \mathfrak{Z} \) are totally different \( \mathfrak{W} \)'s and \( \mathfrak{Z} \) is rigid, then \( \text{Aut} \mathfrak{W} \cong \text{Aut} (\mathfrak{W} \times \mathfrak{Z}) \).

The following lemma can be seen by an adaptation of the proof of 1.2.

Lemma 1.5. If \( K \) is a set of pairwise totally different \( \mathfrak{W} \)'s, and \( 0 \neq L, M \subseteq K \) and \( L \neq M \), then \( \text{P} \text{Aut}_{\text{K}} \mathfrak{W} \neq \text{P} \text{Aut}_{\text{M}} \mathfrak{W} \).

The next lemma is well-known.

Lemma 1.6. For any \( \text{Aut} \mathfrak{W} \), the following conditions are equivalent.

(i) \( \mathfrak{W} \) is not rigid;

(ii) there are distinct elements \( x, y \in A \) such that \( \mathfrak{W} \mid x \neq \mathfrak{W} \mid y \);

(iii) there are disjoint non-zero elements \( x, y \in A \) such that \( \mathfrak{W} \mid x = \mathfrak{W} \mid y \).

Proof. (i) \( \Rightarrow \) (ii). Let \( f \) be a non-identity automorphism of \( \mathfrak{W} \); say \( x \neq fx \). Clearly \( f (\mathfrak{W} \mid x) = \mathfrak{W} \mid x \neq \mathfrak{W} \mid y \), as desired.

(ii) \( \Rightarrow \) (iii). Let \( f \mid x : \mathfrak{W} \mid x \to \mathfrak{W} \mid y \) with \( x \neq y \). Say \( x \in U \). Let \( u = x \cdot y \) and \( v = y \cdot u \). Clearly \( u \) and \( v \) are of zero, \( u \cdot v = 0 \), and \( g (\mathfrak{W} \mid x) = \mathfrak{W} \mid x \neq \mathfrak{W} \mid y \).

(iii) \( \Rightarrow \) (i). Clearly \( \mathfrak{W} = (\mathfrak{W} \mid x \cdot y) \times (\mathfrak{W} \mid x (\mathfrak{W} \mid (x \cdot y))) \cong (\mathfrak{W} \mid x \cdot y) \times (\mathfrak{W} \mid x \cdot (\mathfrak{W} \mid (x \cdot y))) \), so (i) follows.

Corollary 1.7. If \( \mathfrak{W} \) is a rigid \( \mathfrak{W} \), then for every \( x \in A, \mathfrak{W} \mid x \) is rigid.

Corollary 1.8. If \( \mathfrak{W} \) is a rigid \( \mathfrak{W} \) and \( X \) is a collection of pairwise disjoint elements of \( \mathfrak{W} \), then \( \mathfrak{W} \mid x : x \in X \) is a collection of pairwise totally different rigid \( \mathfrak{W} \)'s.

Now we turn to the consideration of products where there are common non-trivial factors. In contrast to 1.3, any such product has many automorphisms, as is seen in 1.9. Our following results through 1.12 constitute an extension of some remarks in Rieger [13] p. 214, where there are, however, some erroneous statements.
Lemma 1.9. For any infinite BA $\mathfrak{B}$, $\text{Aut}(\mathfrak{B} \times \mathfrak{B})$ contains $\mathfrak{M} \times \mathfrak{N}$.

Proof. For each $a \in A$, let $f_a$ be the automorphism $g \mapsto h_g$, where $g : \mathfrak{B} \times \mathfrak{B} \mapsto (\mathfrak{B} \times \mathfrak{B} \times \mathfrak{B}) - (\mathfrak{B} \times \mathfrak{B} \times \mathfrak{B})$ is natural, and $h$ interchanges first and third coordinates but leaves the second and fourth fixed. Thus for any $(x, y) \in A \times A$, $f_a(x, y) = (x \cdot a + y \cdot a, x \cdot a + y \cdot a)$. If $a, b \in A$ and $a \neq b$, then $f_a(a, b) = (a \cdot b, b \cdot a)$, while $f_a(0, 0) = (0, 0)$, so $f_a = f_b$. Clearly each $f_a$ is an automorphism of $\mathfrak{B}$.

In case $\mathfrak{B}$ is rigid, the automorphisms given in the proof of 1.9 constitute all automorphisms of $\mathfrak{B} \times \mathfrak{B}$. (This was first noticed in D. G. Costa [2].) In fact, let $k$ be any automorphism of $\mathfrak{B} \times \mathfrak{B}$. Say $k(1, 0) = (a, b)$. Say $k(1, 0) = (a, b)$. By 1.6 it is clear that $c = a$. Thus $k(0, 0) = (a, 0)$. Then $k(0, 0) = (a, 0)$ and $k(-a, 0) = (a, 0)$. Thus $k(0, 0) = (-a, a)$, and $k(1, 0) = (a, a)$. Hence, $k = f_{-a}$.

Generalizing these considerations, we get a kind of characterization of $\text{Aut}(\mathfrak{B} \times \mathfrak{B})$, $\mathfrak{B}$ rigid.

Theorem 1.10. Let $\mathfrak{B}$ be a rigid BA, $I$ a set with at least two elements. Assume that $\mathfrak{B}$ is $|I|$-complete. Then every automorphism of $\mathfrak{B}$ has the form $f_{\alpha}$, where $\alpha \in 2^I \setminus I$, $\mathfrak{B} = \{\alpha \in 2^I : \text{I is a partition of unity}\}$, and for all $x \in A$ and all $\alpha \in 2^I$, $(f_{\alpha}(x)) = \sum_{\alpha \in 2^I} x \cdot a_\alpha$ (and each such $f_{\alpha}$ is an automorphism of $\mathfrak{B}$).

Proof. First we show that each $f_{\alpha}$ is an automorphism of $\mathfrak{B}$. Clearly $x < y \Rightarrow f_{\alpha} x < f_{\alpha} y$, and assume that $f_{\alpha} x < f_{\alpha} y$. Then $x = \sum_{\alpha \in 2^I} x \cdot a_\alpha = \sum_{\alpha \in 2^I} (f_{\alpha}(x)) \cdot a_\alpha < y$. Thus $x \neq y$. It follows also that $f$ is one-one. To show that $f$ is onto, let $y \in A$. Let $x = \sum_{\alpha \in 2^I} x \cdot a_\alpha$ for all $\alpha \in I$. Then for any $\alpha \in I$, $f_{\alpha}(x) = \sum_{\alpha \in 2^I} x \cdot a_\alpha = \sum_{\alpha \in 2^I} y \cdot a_\alpha = y$, so $f_{\alpha} x = y$. Hence $f_{\alpha}$ is an automorphism of $\mathfrak{B}$.

Now let $g \in 2^I$. For $i, j \in I$, let $a_{ij} = (g_i, j)$. Clearly

1. for any $i \in I$, $(a_{ij} : i \in I)$ is a partition of unity;
2. if $i, j, k \in I$ and $i \neq k$, then $a_{ij} a_{jk} = 0$.

In fact, choose $x$ such that $g_i x = \delta_{ij}(g_i, j)$. By 1.6 it follows that $x = \delta_{ik}(g_i, k)$. Thus $g_i(g_k, j) = \delta_{ik}(g_i, k)$, and similarly $g_j(g_k, j) = \delta_{jk}(g_j, k)$. Since $\delta_{ij}(g_i, j), \delta_{jk}(g_j, k) = 0$, it follows that $(g_i, j) \cdot (g_j, k) = 0$.

(3) for any $i \in I$, $\sum_{j \in I} a_{ij} = 1$.

This is true because, as was just shown, $g_i(g_k, j) = \delta_{ij}(g_i, j)$ for every $j \in I$. Hence $g_i(1) = \sum_{j \in I} g_i(g_k, j) = \sum_{j \in I} \delta_{ij}(g_i, j) = g_i(1)$, and (3) follows.

(4) For any $x \in A$ and $i, j \in I$, $(g_i x)(j) = x \cdot (g_i, j)$.

In fact, choose $u$ so that $g_i u = \delta_{ij}(g_i, j)$. By 1.6 $u = x \cdot (g_i, j)$. Thus $g_i(x \cdot (g_i, j)) = \delta_{ij}(x \cdot (g_i, j))$, so

$g_i(x) = \sum_{j \in I} \delta_{ij}(x \cdot a_{ij}) = \sum_{j \in I} g_i(x \cdot a_{ij}) = \sum_{j \in I} \delta_{ij}(x \cdot a_{ij}) = x \cdot a_{ij}$.

Thus (4) holds. Finally, if $x \in A$ and $i \in I$, then

$g_i(x) = \sum_{j \in I} \delta_{ij}(x) = \sum_{j \in I} g_i(x) = \sum_{j \in I} x \cdot a_{ij} = (f_{\alpha}(x))$.

So $g = f_{\alpha}$, as desired.

Corollary 1.11. If $\mathfrak{B}$ is an infinite rigid BA and $2 < m < \omega$, then $|\text{Aut}(\mathfrak{B} \times \mathfrak{B})| = |A|$.

For $\mathfrak{B}$ rigid and $2 < m < \omega$ we can give a different description of $\text{Aut}(\mathfrak{B} \times \mathfrak{B})$ from 1.10.

Theorem 1.12. Let $\mathfrak{B}$ be a rigid BA, $X$ its Stone space, Let $m$ be a positive integer greater than 1. Then $\text{Aut}(\mathfrak{B} \times \mathfrak{B})$ is isomorphic to the
subgroup of the full Cartesian power \( \mathfrak{X} \) Sym \( m \) consisting of all continuous maps of \( X \) into \( \text{Sy} \) Sym \( m \), the latter with the discrete topology.

Proof. Let \( g \in \text{Aut}(\mathfrak{X}) \). By 1.10 and its proof, write \( g = f_g \), where \( a_{ij} = (g_{ij}) \), for all \( i, j < m \). For \( i < m \) and \( \beta \in X \), let \( (Fg)\beta \) be the \( j < m \) such that \( a_{ij} \in \beta \); \( f \) exists and is unique since \( a_{ij} \in I \) is a partition of unity. Now \( (Fg)\beta \) is one-one. For, suppose \( (Fg)\beta k = (Fg)\beta l \) for \( i \neq k \). Thus \( (g_{ik}) \cdot (g_{kl}) \in \beta \); but this contradicts \( g_{ik}, g_{kl} = 0 \). Hence \( (Fg)\beta \in \text{Sym} \) \( m \).

Now we show that \( Fg \) is a continuous map of \( X \) into \( \text{Sym} \) \( m \), to do this is sufficient to take any \( \alpha \in \text{Sym} \) \( m \) and show that \( (Fg)^{-1}(\alpha) \) is open in \( X \). Since \( (Fg)^{-1}(\alpha) = \{ \beta : \forall i < m \exists j(g_{ij})_\beta \in \alpha \} \) this is clear.

To show that \( F \) is a homomorphism, let \( g, h \in \text{Aut}(\mathfrak{X}) \), \( \beta \in X \), and \( i < m \). Say \( (Fg)\beta i = j \) and \( (Fh)\beta i = k \). Thus \( (g_{ij})_\beta \in \beta \) and \( (h_{ik})_\beta \in \beta \). Hence, using (4) in the proof of 1.10,

\[
(g_{ij})_\beta \cdot (h_{ik})_\beta = (h_{ik}(g_{ij} - h_{ik}))_\beta \in \beta,
\]

so \( (g_{ij})_\beta \in \beta \) and hence \( (Fg)\beta i = k \). Thus \( Fgh = Fg \cdot Fh \) as desired.

Now \( F \) is one-one. For, assume that \( g \in \text{Aut}(\mathfrak{X}) \) is not the identity. Then by (4) in the proof of 1.10 we easily infer that \( g_{ij} \neq 0 \) for some \( i < m \). Thus \( (g_{ij})_\beta \neq 0 \), so there is a ultrafilter \( \mathfrak{F} \) on \( \mathfrak{X} \) with \( (g_{ij})_\beta \notin \mathfrak{F} \). Hence \( (Fg)\beta i \notin \mathfrak{F} \). Thus \( Fg \) is not the identity.

Finally, we must show that the range of \( F \) includes all continuous maps \( h \) of \( X \) into \( \text{Sym} \) \( m \). For any \( \alpha \in \text{Sym} \) \( m \), \( h^{-1}(\alpha) \) is a closed-open subset of \( X \), say \( h^{-1}(\alpha) = (\beta : \beta \notin \mathfrak{F} \in \mathfrak{F}) \). For \( i < m \), let \( c_i = \sum_{j<i} b_{ij} \). Then

(1) if \( i, j < m, f \neq f_k \), then \( a_{ij} \cdot d_{ik} = 0 \), and \( a_{ij} \cdot d_{jk} = 0 \).

For, if \( c_i = f \) and \( n = k \), then \( h^{-1}(\alpha) \) for \( h^{-1}(c_i) \neq 0 \), hence there is no \( \beta \) with \( b_{ij} \in \beta \), \( b_{ik} \in \beta \), \( b_{jk} \in \beta \). Hence (1) holds.

(2) if \( i < m \), then \( \sum_{j<i} a_{ij} = 1 - \sum_{j<i} b_{ij} \).

For, if \( \sum_{j<i} a_{ij} = 1 \), then there is an \( \beta \in X \) with \( \sum_{j<i} a_{ij} \notin \beta \). Say \( h_{\beta} = 1 \). Then \( b_{ij} \in \beta \) and so \( \sum_{j<i} b_{ij} \notin \beta \), contradiction.

Thus \( a \) satisfies the conditions of 1.10. We claim that \( Fh = h \). For, let \( \beta \in X, l < m \), and \( (Fh)\beta l = i \). Thus \( a_{il} \notin \beta \). Say \( b_{il} \in \beta \) where \( a_{il} \notin \beta \). Then \( h_{\beta} = 1 \), so \( Fh \beta = i \). Thus, indeed \( Fh = h \). This completes the proof.

By \( \text{Inv}(\mathfrak{W}) \), where \( \mathfrak{W} \) is a BA, we denote the subalgebra of \( \mathfrak{W} \) constituted by all elements left fixed by every automorphism of \( \mathfrak{W} \). (An equivalent condition on the element \( x \in A \) is that \( x \) and \( -x \) be totally different.)

Lemma 1.13. Let \( \mathfrak{W} \) be rigid and \( I \) be nonempty. Then \( \text{Inv}(\mathfrak{W}) \) = \( \mathfrak{W} \); in fact, \( \text{Inv}(\mathfrak{W}) \) is the diagonal subalgebra constituted by all constant mappings from \( I \) into \( \mathfrak{W} \).

Proof. For each \( a \in A \), let \( c_a = \text{the member of } I, A \) such that \( c_a = a \) for all \( i \in I \). Then for any \( a \in A \),

(1) for any automorphism \( f \) of \( \mathfrak{W} \), \( f_{c_a} = c_a \).

For, suppose \( f_{c_a} = c_{f_a} \), \( f \) an automorphism of \( \mathfrak{W} \). Thus there is an \( i \in I \) with \( (f_{c_a})_i \neq a \). Hence \( (f_{c_a})_i \neq a \) or \( f_{c_a} \notin (c_a)_i \). Assume \( (f_{c_a})_i \neq a \), and let \( b = (c_a)_i \cdot -a \). Thus \( b_{ij} < f_{c_a} \), so \( f_{c_a} \notin (f_{c_a})_i \). Since \( b \neq 0 \), there is a \( j \in I \) such that \( d = (f_{c_a})_i < f_{c_a} \). Thus \( d < a \), \( b_{ij} \cdot f_{c_a} < f_{c_a} \cdot f_{c_a} \), and hence \( f_{c_a} \notin (c_a)_i \). Hence \( a \) and \( -a \) are not totally different, contradicting \( \mathfrak{W} \) being rigid. A similar contradiction is reached if \( a \notin (f_{c_a})_i \). Hence (1) holds.

Now suppose that \( x \in \mathfrak{W} \), and \( x \) is not a constant mapping from \( I \) into \( A \). Say \( i, f \in I \) and \( x_i = x_f \). Let \( a = x_i \cdot -x_f \). Then \( a \neq 0 \), \( \delta < x \), and \( \delta < -x \), so \( x \) and \( -x \) are not totally different. Hence \( x \notin \text{Inv}(\mathfrak{W}) \). This completes the proof.

Theorem 1.14. Assume that \( \mathfrak{U} = \mathfrak{W} \) where \( \mathfrak{W} \) and \( \mathfrak{U} \) are rigid and \( I = 0 \neq f \). Then \( \mathfrak{U} \cong \mathfrak{W} \); if \( I \) is finite and \( \mathfrak{U} \) is non-trivial, then \( |I| = |J| \).
Proof. It is immediate from 1.13 that the assumptions imply that $\mathfrak{U} = \mathfrak{W}$. Suppose also that $1$ is finite and $\mathfrak{U}$ is non-trivial. Then $\mathfrak{U}$ has a system $(x^i; i \in I)$ of pairwise disjoint, non-zero, isomorphic elements. We will show that this entails $|I| \leq |J|$. The same argument then puts $|I| < |J|$. To obtain a contradiction, suppose that $\mathfrak{U}$ has a system $(x^i; 0 < k < |I|)$ of pairwise disjoint, non-zero, isomorphic elements. For any fixed $m \in |I|$, we can convert such a system into a similar system $(x^i; 0 < k < |I|)$ with $x^i < x^k$ (for all $k$) and $x^m < x^k$ for some $i \in I$. Hence in $|I| + 1$ steps we can construct such a system $(x_i; k \in |I|)$ with the property that for each $m < |I|$, there is a (unique) $i_m \in I$ with $2^m < x_{i_m}$. Since $\mathfrak{U}$ is rigid, $i_m$ does not contain two disjoint, non-zero, isomorphic elements; hence the function $(i_m; m < |I|)$ is one-to-one. This is impossible, because of the cardinalities involved.

It is also easy to see that, if $\mathfrak{U} \Rightarrow \mathfrak{W}$ (not necessarily rigid), $\mathfrak{W}$ is non-trivial, $N_0 < |I| < |J|$, and $|J| > |K|$ (in particular, if $|J| > |A|$), then $|I| = |J|$. (Use the fact that $\mathfrak{U}$ is regular.) However, we do not know whether the conclusion $|I| = |J|$ in 1.14 is generally valid.

Problem 1. If $\mathfrak{U} \Rightarrow \mathfrak{W}$ where $\mathfrak{U}$ is rigid and non-trivial, is $|I| = |J|$?

We now turn to the consideration of $\text{Aut} \mathfrak{U}$ for more general $\mathfrak{U}$. The non-triviality of the center of $\text{Aut} \mathfrak{U}$ is related to the existence of rigid factors of $\mathfrak{U}$ in a special fashion explicated below.

Theorem 1.15. For any BA $\mathfrak{B}$, the center of $\text{Aut} \mathfrak{B}$ is a 2-group.

Proof. Let $f$ be a non-trivial automorphism of $\mathfrak{U}$ of order $> 2$; we shall show that $f$ is not in the center of $\text{Aut} \mathfrak{U}$. First we claim.

(1) There is a non-zero $x \in A$ such that $x, f_2 x$ are pairwise disjoint.

For, as in the proof of 1.6 we can find $0 \neq y \in A$ so that $y \cdot f^2 y = 0$. It follows that $y \neq f_2 y$. We can let $x = y \cdot f_2 y$ or $x = -y \cdot f_2 y$ depending on which is non-zero, and (1) follows.

Let $h$ be the natural isomorphism $\mathfrak{B} \Rightarrow (\mathfrak{U} \cup \mathfrak{W}) \times (\mathfrak{U} \cup \mathfrak{W}) \times (\mathfrak{U} \cup \mathfrak{W}) \times (\mathfrak{U} \cup \mathfrak{W}) \times (\mathfrak{U} \cup \mathfrak{W})$. Let $\lambda_1$ be the automorphism of $\text{Rng} \mathfrak{B}$ which takes $(a, b, c, d)$ to $(f^2 a, f^2 b, c, d, d)$ for all $a \leq x, b \leq x, c \leq x, d \leq x$, and let $g = h^{-1} a k h$. Then $g f x = f x$ and $g f x = f^2 x$ so $g f x \cdot g f x = 0$ and hence $g f = f g = f g$, as desired.

Theorem 1.16. For any BA $\mathfrak{B}$, the following conditions are equivalent.

(i) $\text{Aut} \mathfrak{B}$ has a non-trivial center.

(ii) There is a non-trivial rigid BA $\mathfrak{B}$ and a BA $\mathfrak{C}$ such that $\mathfrak{B}$ and $\mathfrak{C}$ are totally different and $\mathfrak{B} \otimes \mathfrak{B} \otimes \mathfrak{C}$.

Proof. (i) $\Rightarrow$ (ii). Let $a$ be a member of the center of $\text{Aut} \mathfrak{B}$ different from the identity. By the proof of 1.6, there is a non-zero $x \in A$ with $x \cdot a x = 0$. Let $f$ be the natural isomorphism $\mathfrak{B} \Rightarrow (\mathfrak{U} \cup \mathfrak{W}) \times (\mathfrak{U} \cup \mathfrak{W}) \times (\mathfrak{U} \cup \mathfrak{W}) \times (\mathfrak{U} \cup \mathfrak{W}) \times (\mathfrak{U} \cup \mathfrak{W})$. Let $\mathfrak{B} = \mathfrak{U} \cup \mathfrak{W} \cup \mathfrak{C} \cup \mathfrak{D}$. If $\mathfrak{B}$ is not rigid, let $r$ be the non-identity automorphism of $\mathfrak{B}$, say $r x = 0$ with $y \neq 0$. Let $r$ be the automorphism of $(\mathfrak{U} \cup \mathfrak{W}) \times (\mathfrak{U} \cup \mathfrak{W}) \times (\mathfrak{U} \cup \mathfrak{W})$ acting like $r$ on the first coordinate and like the identity elsewhere. Then $o_f^{-1} g f = o_r$ and $f^{-1} g f = o_r$; $o_r \cdot o_r = 0$ and $y \neq 0$, so $o_f^{-1} g f + f^{-1} g f a$, contradiction. Thus $\mathfrak{B}$ is rigid. Also, suppose $\mathfrak{B}$ and $\mathfrak{C}$ have a common non-trivial factor; say $u \leq x, v \leq x \cdot a x$ and $g: \mathfrak{B} \otimes \mathfrak{C} \Rightarrow \mathfrak{B} \otimes \mathfrak{C}$, with $u \neq 0$. Let $h$ be the natural isomorphism $\mathfrak{B} \Rightarrow (\mathfrak{U} \cup \mathfrak{W}) \times (\mathfrak{U} \cup \mathfrak{W}) \times (\mathfrak{U} \cup \mathfrak{W}) \times (\mathfrak{U} \cup \mathfrak{W}) \times (\mathfrak{U} \cup \mathfrak{W})$. Let $k$ act like $g$ on $\mathfrak{U} \cup \mathfrak{W}$, like $g^{-1}$ on $\mathfrak{B} \otimes \mathfrak{C}$, and like the identity on the second and fourth coordinates. Then $h^{-1} k h u = u \neq 0$, $h^{-1} k h u = u \neq 0$, so again $o_f^{-1} k h \neq h^{-1} k h u$, contradiction. Hence $\mathfrak{B}$ and $\mathfrak{C}$ have no common non-trivial factors.

(ii) $\Rightarrow$ (i). By 1.9, $\text{Aut}(\mathfrak{B} \otimes \mathfrak{C})$ is non-trivial. Hence it suffices to show that each member $g$ of $\text{Aut}(\mathfrak{B} \otimes \mathfrak{C})$ induces an automorphism in the center of $\mathfrak{B} \otimes \mathfrak{B} \otimes \mathfrak{C}$. Let $g(x, y, z) = (g(x, y), z)$ for any $x, y \in B$. 

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Corollary. 1.17. If $\mathcal{W}$ has no non-trivial rigid factors, then $\text{Aut}\mathcal{W}$ is centerless.

There are other BA's $\mathcal{W}$ with $\text{Aut}\mathcal{W}$ centerless, e.g., $\mathcal{W} = \mathcal{W}_0$ where $\mathcal{W}_0$ is non-trivial and rigid and $3 < \xi_1$, as is easily seen from 1.10.

For our further results in this section we restrict attention to complete BA's. In this connection the following problem naturally occurs.

Problem 2. Characterize the automorphism groups of complete BA's among the automorphism groups of arbitrary BA's.

We may remark that the two classes of groups mentioned here do not coincide. For example, there is a BA $\mathcal{W}$ with $\text{Aut}\mathcal{W}$ denumerable (see 3.2), but this is never the case for a complete BA, by 1.9.

The following theorem is easy to prove, using Zorn's lemma.

Theorem 1.18. For any complete BA $\mathcal{W}$ there is a unique element $a \in A$ such that $\mathcal{W} \vdash a$ is a product of rigid BA's and $\mathcal{W} \vdash a$ has no non-trivial rigid factors. Thus the determination of $\text{Aut}\mathcal{W}$, $\mathcal{W}$ complete, reduces to two special cases: $\mathcal{W}$ is a product of rigid BA's, or $\mathcal{W}$ has no non-trivial rigid factors. We now consider the first case. Note that if $\mathcal{W}$ is a product of rigid BA's, then the collection of rigid elements of $\mathcal{W}$ is dense in $\mathcal{W}$. If $\mathcal{W}$ is complete, then the converse holds.

Lemma 1.19. Let $\mathcal{W}$ be a complete BA with at least one non-trivial rigid factor. Then there is a nonempty collection $\mathcal{C}$ of rigid, pairwise disjoint and isomorphic, non-zero elements of $\mathcal{W}$ such that $\sum \mathcal{C}$ and $\sum \mathcal{C}^*$ are totally different.

Proof. Let $y$ be a non-zero rigid element of $\mathcal{W}$. By Zorn's lemma let $D$ be a maximal family of pairwise disjoint elements of $\mathcal{W}$ each isomorphic to $y$, with $y \in D$. For each $u \in D$ let $f_u : \mathcal{W} \vdash y \sim u \vdash y$. where $f_u$ is the identity. Let $E$ be a maximal collection of pairwise disjoint elements $(d, e) \in (\mathcal{W} \vdash y) \times (\mathcal{W} \vdash y - \sum D)$ such that $\mathcal{W} \vdash d \sim \mathcal{W} \vdash e$. For each $(d, e) \in E$ let $g_{de} : \mathcal{W} \vdash d \sim \mathcal{W} \vdash e$. Let $\zeta = \sum (d, e) \in E$ and set $z = y - z$. Note that $z \neq 0$ since $D$ is maximal, and $x$ is rigid since $x \leq y$. Let $C = \{f_x : u \in D\}$. Thus $C$ is a collection of pairwise disjoint elements of $\mathcal{W}$ each isomorphic to $x$, and $x \in C$. To complete the proof it suffices to derive a contradiction from the assumptions $0 \neq y \leq \sum C$, $w \leq \sum C$, and $h : \mathcal{W} \vdash y \sim \mathcal{W} \vdash w$. Choose $u \in D$ such that $r \cdot f_u x \neq 0$. Now there are three cases.

Case 1. $\exists t \in D(h(r \cdot f_u x) \cdot t \neq 0)$. Thus $s = h(r \cdot f_u x) \cdot f_u t \neq 0$. Clearly $f_u^{-1} s$ and $f_u^{-1} h^{-1} s$ are isomorphic pairwise disjoint non-zero subelements of $y$, contradiction.

Case 2. $\forall t \in D(h(r \cdot f_u x) \cdot t = 0)$, but $\exists (d, e) \in E(h(r \cdot f_u x) \cdot e \neq 0)$. Again $g_{de}^{-1}(h(r \cdot f_u x) \cdot e)$ and $f_u^{-1} h^{-1}(h(r \cdot f_u x) \cdot e)$ are isomorphic pairwise disjoint non-zero subelements of $y$, contradiction.

Case 3. $h(r \cdot f_u x) \leq (\sum D) \cdot \sum (d, e) \in E$. This contradicts the maximality of $E$.

The proof is complete.

Theorem 1.20. Let $\mathcal{W}$ be a complete BA in which the rigid elements are dense. There exists a strictly increasing sequence $(\mathcal{W}_n)_{n \in \mathcal{N}}$ of non-zero cardinals, and a system $(\mathcal{W}_n : \alpha \leq \beta)$ of non-trivial, pairwise totally different rigid BA's, such that $\mathcal{W} = \bigcup_{n \geq 0} \mathcal{W}_n$.

Proof. An easy transfinite construction, using 1.19. See the proof of the next theorem.

The question whether the representation given by Theorem 1.20 is unique, is equivalent to Problem 1. We cannot prove it. However, we can obtain uniqueness by imposing an additional condition.
Theorem 1.21. Let $\mathfrak{A}$ be a complete BA in which the rigid elements are dense. There are unique sequences $(\varepsilon_{\alpha}; \alpha < \beta)$ (strictly increasing, $m_{\varepsilon_{\alpha}} > 0$) and $(\mathfrak{D}_{\varepsilon_{\alpha}}; \alpha < \beta)$ (non-trivial, pairwise totally different, rigid BA's) satisfying:

(i) $\mathfrak{A} \cong \bigcup_{\alpha < \beta} m_{\varepsilon_{\alpha}} \mathfrak{D}_{\varepsilon_{\alpha}}$;

(ii) for each $\alpha$, every representation $m_{\varepsilon_{\alpha}} \mathfrak{D}_{\varepsilon_{\alpha}} \cong m_{\varepsilon_{\alpha}} \times \mathfrak{D}$ where $m_{\varepsilon_{\alpha}} > 0$ and $\mathfrak{D}$ is a non-trivial rigid algebra totally different from $\mathfrak{D}$, has $m_{\varepsilon_{\alpha}} < m$.

Proof. If $\mathfrak{A}$ is trivial, we must put $\beta = 0$. Let us prove the existence, assuming that $\mathfrak{A}$ is non-trivial.

Let $m_{\varepsilon_{\alpha}}$ be the least cardinal $m$ such that there exists a collection $C$ as in 1.19 with $|C| = m$. Choose as $C^{0}$ any collection $C$ as in 1.19 with $|C| = m_{\varepsilon_{\alpha}}$. If $C^{\beta}$, $\delta < \gamma$, have been chosen so that the $C^{\delta}$ are pairwise disjoint and each $|C^{\delta}| = m_{\varepsilon_{\delta}}$ and if in $\mathfrak{A} = \bigcup_{\delta < \gamma} \Sigma C^{\delta}$ there is such a collection $C$ of power $m_{\varepsilon_{\gamma}}$, let $C^{\gamma}$ be one such. In this way, we obtain a sequence $C^{\delta}$, $\delta < \gamma$, such that the $\Sigma C^{\delta}$ are pairwise disjoint, every $C^{\delta}$ satisfies 1.19 for $\mathfrak{A}$, $|C^{\delta}| = m_{\varepsilon_{\delta}}$, and $\mathfrak{A} = \bigcup_{\delta < \gamma} \Sigma C^{\delta}$ has no such collection $C$ with $|C| < m_{\varepsilon_{\gamma}}$. Since $\mathfrak{A}$ is complete, the pairwise disjoint set $\bigcup_{\delta < \gamma} C^{\delta}$ gives us a decomposition

$$\mathfrak{A} \cong \bigcup_{\delta < \gamma} m_{\varepsilon_{\delta}} \mathfrak{D}_{\varepsilon_{\delta}} \times \mathfrak{D} = m_{\varepsilon_{\gamma}} \times \mathfrak{D}.$$ 

We can state a general proposition proved by the above argument.

(1) Let $\mathfrak{B}$ be a complete BA with a non-zero rigid element. There exists a decomposition $\mathfrak{B} = m_{\varepsilon_{\gamma}} \times \mathfrak{D}$ in which

(a) $\mathfrak{D}$ is non-trivial, rigid, and totally different from $\mathfrak{D}$;

(b) $m_{\varepsilon_{\gamma}} > 0$ is minimal for all decompositions of $\mathfrak{B}$ satisfying (a);

(c) in every decomposition $\mathfrak{B} = m_{\varepsilon_{\gamma}} \times \mathfrak{D}$ satisfying condition (a), one has $\varepsilon_{\gamma} > m$.

Moreover, (obviously) for $m_{\varepsilon_{\gamma}}$ as for $\mathfrak{A}$, the cardinal number $m$ is minimal. [We need this for condition 1.21 (ii).] It will follow from the remaining argument below that $\mathfrak{D}$ and $\mathfrak{D}$ are uniquely determined by $\mathfrak{A}$, and in fact that $m_{\varepsilon_{\gamma}} \mathfrak{D}$ and $\mathfrak{D}$ are isomorphic with unique relativized algebras of $\mathfrak{A}$.

To continue with the proof, we choose $\mathfrak{A}_{\delta}, \mathfrak{A}^{\delta}$ so that statement (1) is true with $m, \mathfrak{A}, \mathfrak{D}$ replaced by $m_{\varepsilon_{\delta}}, \mathfrak{A}_{\delta}, \mathfrak{A}^{\delta}$ and in fact, say, $m_{\varepsilon_{\delta}} \mathfrak{D} = \mathfrak{A}_{\delta} \times \mathfrak{A}^{\delta}$, $\mathfrak{A}^{\delta} = \mathfrak{A}^{\delta}$, where $\mathfrak{A}^{\delta} = m_{\varepsilon_{\delta}} \mathfrak{D}_{\varepsilon_{\delta}}$, and the decomposition $\mathfrak{A} = \bigcup_{\alpha < \delta} m_{\varepsilon_{\alpha}} \mathfrak{D}_{\varepsilon_{\alpha}}$ satisfies all the conditions of Theorem 1.21.

For uniqueness, suppose that also $\mathfrak{A} = \bigcup_{\alpha < \delta} m_{\varepsilon_{\alpha}} \mathfrak{D}_{\varepsilon_{\alpha}}$ satisfying our conditions, say

$$f: \bigcup_{\alpha < \beta} m_{\varepsilon_{\alpha}} \mathfrak{D}_{\varepsilon_{\alpha}} \rightarrow \bigcup_{\alpha < \delta} m_{\varepsilon_{\alpha}} \mathfrak{D}_{\varepsilon_{\alpha}}.$$ 

It suffices, by symmetry, to take any $\alpha < \beta$ and find $\xi < \beta_{1}$ such that $m_{\varepsilon_{\alpha}} = m_{\varepsilon_{\xi}}$, and $\mathfrak{D}_{\varepsilon_{\alpha}} \cong \mathfrak{D}_{\varepsilon_{\xi}}$.

We first remark that

$$\text{In}(\bigcup_{\alpha < \delta} m_{\varepsilon_{\alpha}} \mathfrak{D}_{\varepsilon_{\alpha}}) = \bigcup_{\alpha < \delta} \text{In}(m_{\varepsilon_{\alpha}} \mathfrak{D}_{\varepsilon_{\alpha}}),$$

and

$$f(\text{In}(\bigcup_{\alpha < \delta} m_{\varepsilon_{\alpha}} \mathfrak{D}_{\varepsilon_{\alpha}})) = \bigcup_{\alpha < \delta_{1}} \text{In}(m_{\varepsilon_{\alpha}} \mathfrak{D}_{\varepsilon_{\alpha}}) = \bigcup_{\alpha < \delta_{1}} \text{In}(m_{\varepsilon_{\alpha}} \mathfrak{D}_{\varepsilon_{\alpha}}).$$

(See Definition 1.1 and Lemma 1.13.)

Now let $\gamma < \beta_{1}$ be such that $\gamma < \delta_{1}$, $\gamma \neq 0$. Then also $(f^{-1} \delta_{1})\gamma \neq 0$. We conclude, using (2), that
$M \times E = \{ \sum_{a \in f_1}^n a \circ \beta \} / f \delta_1 \times E \times E$.

and

$M \times E = \{ \sum_{a \in f_2}^n a \circ \beta \} / f \delta_1 \times E \times E$.

where $E$ is non-trivial, rigid, and totally different from $X(E = X)$; and likewise $E, E$. Consequently, by 1.21 (iii), we have $m_1 \leq n_1$, and conversely so $m_1 = n_1$. Moreover, since $(m_1 ; \alpha \leq \beta)$ is one-to-one, at most one $\gamma$ exists with $m_1 = n_1$, or with $(f \delta_1 \times E \times E$ = 0. We conclude that $f \delta_1 \times E \times E$ = 0.1. We have $\gamma$ satisfying $m_1 = n_1$, and $M \times E = \{ \sum_{a \in f_2}^n a \circ \beta \} / f \delta_1 \times E \times E$. By 1.14, we also have $B_2 \times E = \{ \sum_{a \in f_2}^n a \circ \beta \} / f \delta_1 \times E \times E$. That completes the proof.

Theorems 1.21, 1.2 and 1.10 yield a structure theorem for the automorphism group of a complete BA in which the rigid elements are dense. The theorem is obvious and we will not bother to formulate it. We can make some further comments on these algebras.

**Corollary 1.22.** Let $A$ be complete and the rigid elements dense in $A$. Let $E = \sum_{a \in f_2}^n a \circ \beta$ be any decomposition of $A$ as product of powers of pairwise totally different rigid algebras. Then Inv $A$ is isomorphic to $\sum_{a \in f_2}^n a \circ \beta$. Inv $A$ is rigid and, in fact, is isomorphic to $A \times a$ where $a$ is any maximal rigid element of $A$.

**Proof.** We observed the truth of the first statement in proving 1.21. For the second, observe first that Zorn’s lemma ensures the existence of maximal rigid elements in any complete BA. Let $a$ be any such element. For each $x \in A$, put

$f \nu \sum_{a \in f_2}^n a \circ \beta$.

and for each $u \in Inv A$, put

$g(u) = u \cdot a$.

Using the completeness of $A$ and the density of its set of rigid elements,
of non-zero cardinals and a system \( \mathfrak{B} : \alpha < \beta \) of pairwise totally different, non-trivial, homogeneous BA's such that \( \forall \alpha < \beta \left( m_{\alpha} > 1 = m_{\alpha} > c \mathfrak{B}_{\alpha} \right) \) and \( \mathfrak{B}_{\alpha} : \mathfrak{B}_{\alpha} = \mathfrak{B}_{\alpha} \). This representation is unique in the sense that if also \( \mathfrak{B} : \mathfrak{B} = \mathfrak{B} \) with similar conditions, then \( \beta = \gamma, m = n, \) and for each \( \alpha < \gamma \) there is a permutation \( \alpha \) of \( \left( m_{\alpha} = m_{\alpha} \right) \) such that \( \mathfrak{B}_{\alpha} = \mathfrak{B}_{\alpha} \) for each \( \alpha \) with \( m_{\alpha} = m_{\alpha} \).

Proof. Given any homogeneous element \( a \) of \( \mathfrak{B} \), by Zorn's lemma let \( C \) be a maximal set of pairwise disjoint elements of \( A \) each isomorphic to \( a \), with \( a \in C \). Then \( \sum C \) and \( \sum C \) have no common non-trivial factors. This construction, along with 1.26, makes the existence of the representation as indicated obvious. Now suppose \( f : \sum \mathfrak{B} \rightarrow \sum \mathfrak{B} \) with conditions as in the theorem. It suffices by 1.27 to take any \( \alpha < \beta \) and find \( e < \gamma \) such that \( \mathfrak{B}_{\alpha} = \mathfrak{B}_{\beta} \). Given \( \alpha < \beta \), there is exactly one \( e < \gamma \) with \( \left( \mathfrak{B}_{\alpha}, 1 \right) \neq 0 \). For, if \( e, \theta < \gamma \), and \( \left( \mathfrak{B}_{\alpha}, 1 \right) \neq 0 \), \( \left( \mathfrak{B}_{\beta}, 1 \right) \neq 0 \), say \( \left( \mathfrak{B}_{\alpha}, 1 \right) \neq 0 \) with \( \mu < n_{\alpha}, \nu < n_{\beta} \). Clearly then \( \mathfrak{B}_{\alpha} = \mathfrak{B}_{\beta} \) and \( \mathfrak{B}_{\alpha} = \mathfrak{B}_{\beta} \), so \( e = \theta \). Thus \( \beta_{\alpha} = 1 = \beta_{\beta} \) for some \( u \). So \( \mathfrak{B}_{\alpha} = \mathfrak{B}_{\beta} \) is a factor of some \( \mathfrak{B}_{\alpha} \); by symmetry the latter is a factor of some \( \mathfrak{B}_{\beta} \); hence \( \alpha = \beta \). It follows that \( \mathfrak{B}_{\alpha} = \mathfrak{B}_{\beta} \) as desired.

By 1.28 the automorphism group of a complete BA \( \mathfrak{B} \) in which the homogeneous elements are dense is a direct product of the groups \( \operatorname{Aut}(\mathfrak{B}) \), \( \mathfrak{B} \) homogeneous. For \( m = 1 \) it is known that \( \operatorname{Aut}(\mathfrak{B}) \) is simple when \( \mathfrak{B} \) is \( - \)-complete and homogeneous (see Anderson [1]); but the structure of \( \operatorname{Aut}(\mathfrak{B}) \) is not fully known.

Problem 3. Describe the automorphism groups of (complete) homogeneous BA's.

Concerning \( \operatorname{Aut}(\mathfrak{B}) \) in general we have the following not very satisfactory characterization, the proof of which is straightforward.

Theorem 1.29. Let \( \mathfrak{B} \) be a complete BA, \( I \) any set. Let \( \mathfrak{M} \)
the set of all triples \( (a, b, g) \) satisfying the following conditions.

(i) \( a : I \rightarrow \mathfrak{B}, \) and \( \forall i \in I(a_{i} : I \rightarrow \mathfrak{B}) \) is a partition of unity in \( \mathfrak{B} \);
(ii) \( b : I \rightarrow \mathfrak{B}, \) and \( \forall i \in I(b_{i} : I \rightarrow \mathfrak{B}) \) is a partition of unity in \( \mathfrak{B} \);
(iii) \( \forall i, j \in I, g_{i} : I \rightarrow \mathfrak{B} \) with \( g_{i} b_{j} = g_{i} b_{j} \).

For \( (a, b) \in \mathfrak{M}, \) define \( f = f_{a, b} : I \rightarrow \mathfrak{B} \) by setting

\[ f(x) = \sum_{i \in I} f_{a, b}(x, a_{i}) \]

for all \( x \in \mathfrak{B} \) and all \( i \in I \).

Then \( \operatorname{Aut}(\mathfrak{B}) = \{ f_{a, b} : (a, b) \in \mathfrak{M} \} \).

By our results concerning products of rigid BA's and products of homogeneous BA's, the problem concerning the structure of \( \operatorname{Aut}(\mathfrak{B}) \) for \( \mathfrak{B} \) complete reduces to that problem for \( \mathfrak{B} \) of special kinds -- powers of rigid BA's, powers of homogeneous BA's, and complete BA's with no rigid or homogeneous factors. We have discussed \( \operatorname{Aut}(\mathfrak{B}) \) for the first two special kinds of \( \mathfrak{B} \). We have no results concerning \( \operatorname{Aut}(\mathfrak{B}) \) for the third kind; the following problems remain open.

Problem 4. Does there exist a non-trivial (complete) BA without non-trivial rigid or homogeneous factors?

Problem 5. Describe \( \operatorname{Aut}(\mathfrak{B}) \) where \( \mathfrak{B} \) is a complete BA with no rigid factors and no homogeneous factors.

We close this section by showing that any BA can be embedded in a rigid BA. McAlloon [11] states without proof the stronger result that any BA can be embedded in a rigid complete BA.

A BA \( \mathfrak{B} \) is cardinality-homogeneous if \( \left| \mathfrak{B} \right| = \left| \mathfrak{B} \times A \right| \) for all non-zero \( x \in A \).

Lemma 1.30. Let \( \mathfrak{M} \) be an infinite cardinal. If \( \mathfrak{B} \) is a rigid BA of power \( \mathfrak{M} \), then \( \mathfrak{B} \) has a cardinality-homogeneous factor of power \( > \mathfrak{M} \).
Proof. By Zorn's lemma let $B$ be a maximal collection of pairwise disjoint non-zero cardinality-homogeneous elements of $A$. Clearly $\sum B = 1$. If the conclusion of the lemma fails, then $|B| > \exp m$ since $\aleph \subseteq \bigcap_{\beta \in B} \exists \gamma \in \beta$. But there are only $\exp m$ isomorphism types of BA's of power $< m$, so two elements of $B$ are isomorphic, contradicting $\mathfrak{W}$ rigid.

Lemma 1.31. For any cardinal $m$ there is a family of $m$ pairwise totally different BA's.

Proof. By L. G. [9] there are arbitrarily large rigid BA's. Hence by Lemma 1.30 there are arbitrarily large cardinality-homogeneous rigid BA's. But clearly any two cardinality-homogeneous BA's of different power are totally different, so the lemma follows.

Theorem 1.32. Any BA can be embedded in a rigid BA.

Proof. Any BA $\mathfrak{W}$ can be embedded in a product $\prod I$, where $|B_i| = 2$ for each $i \in I$. If $(\mathfrak{W}_i : i \in I)$ is a family of non-trivial pairwise totally different rigid BA's, then $\prod I \subseteq \prod I$, and the latter is rigid by 1.3.

We can also use Lemma 1.30 to show that a subdirect product of pairwise totally different rigid BA's is not necessarily rigid. To this end, let $\mathfrak{W}$ mapping ordinals into cardinals be defined recursively by:

$$p_0 = \aleph_0,$$

$$p_{\alpha+1} = \exp \exp \exp p_\alpha,$$

$$p_\lambda = \bigcup_{\alpha < \lambda} p_\alpha \text{ for limit } \lambda.$$

Now let $m$ be a fixed point of $\mathfrak{W}$, i.e., $p_m = m$. For each $\alpha < m$, let $\mathfrak{W}_\alpha$ be a rigid BA such that $p_\alpha < |A_\alpha| < p_{\alpha+1}$, and $\mathfrak{W}_\alpha$ is cardinality-homogeneous; $\mathfrak{W}_\alpha$ exists by 1.30. Thus $(\mathfrak{W}_\alpha : \alpha < m)$ is a system of pairwise totally different rigid BA's. Let $\mathfrak{W}$ be the free BA on a set of $m$ free generators, and let $(B_\alpha : \alpha < m)$ be an enumeration of the non-zero elements of $B$. Now $|A_\alpha| < m$ for all $\alpha < m$, so there is a homomorphism $f_\alpha$ of $\mathfrak{W}$ onto $\mathfrak{W}_\alpha$ such that $f_\alpha B_\alpha \neq 0$. Thus the system $(\mathfrak{W}_\alpha : \alpha < m)$ induces an isomorphism of $\mathfrak{W}$ onto a subdirect product of the $\mathfrak{W}_\alpha$'s. Obviously $\mathfrak{W}$ is not rigid.

2. RIGID BA'S

As mentioned in the introduction, we shall be concerned in this section with the existence of rigid BA's. First we shall modify the construction of L. G. [9] and use the modification to construct rigid BA's of singular powers. We could use here 1.30 instead of 2.1, but 2.1 is perhaps interesting in itself.

Lemma 2.1. For each infinite cardinal $m$ there is a rigid BA $\mathfrak{W}$ of power $\exp (m^\omega)$ such that for every $u \in A$ with $0 \neq a, u < |A| + 1$.

Proof. We modify the construction in [9] as follows. Let $X = X^{m^\omega}$, $A = A^{m^\omega}$. We claim that there is a structure $\mathfrak{W} = X - A$ such that $\varphi(\beta, \gamma) > \beta$ for all $(\beta, \gamma) \in X$, and such that $\varphi(\beta, \gamma)$ always has the form $m \cdot \delta + 1$ (ordinal operations), with $\delta \neq 0$. For, let $A = \bigcup \mathfrak{W}_\alpha$, where the $\mathfrak{W}_\alpha$ are pairwise disjoint sets of cardinality $m^\omega$. For each $\alpha < m^\omega$ let $\mathfrak{W}_\alpha = (m \cdot \delta + 1, \delta \in B_\alpha, \delta \neq 0)$. Clearly the $\mathfrak{W}_\alpha$ are pairwise disjoint sets of cardinality $m^\omega$. The desired $\mathfrak{W}$ is obtained by letting $\varphi(\beta, \gamma)$ be any injection into $\mathfrak{W}_\alpha$. Clearly this modification of $\mathfrak{W}$ leaves the rest of the construction in [9] valid. Now

(1) if $\alpha \in \text{Reg} \varphi$ and $\beta < \omega^\alpha$, then $|\beta, \omega^\alpha| = m$. For, write $\alpha = m \cdot \delta + 1$, where $\delta \neq 0$. Then $\omega^\alpha = \omega^m \cdot \omega = \omega^\delta \cdot \omega$. Say $\beta < m^\delta \cdot \omega$, $\beta < \omega^\delta$. Then $\beta < m^\delta \cdot \omega$. Hence $\beta < m^\delta \cdot n + \epsilon < \omega^\delta$ for each $\epsilon < m$, and (1) follows.

Let $\mathfrak{W}$ be the BA of closed-open subsets of $X$. By [9], $\mathfrak{W}$ is rigid and $|\mathfrak{W}| = \exp (m^\omega)$. Suppose that $0 \neq u \in A$. We want to prove that $\mathfrak{W}$. First note that.....
(2) if \( x \in X \) and \( Y \subseteq I_x \), then \( (x, Y) \) is closed-open.

For, \( (x, Y) \) being open by definition of the topology on \( X \). Suppose
\[ y \not\in x \; \text{and} \; (y, y) \in (x, Y) \] \( y \not\in x \), then \( y \in (y, y) \) and \( (y, (y)) \cap (x, Y) = \emptyset \). If
\( y \not\in x \), then \( y \in (y, y) \) and \( (y, (y)) \cap (x, Y) = \emptyset \). Thus (2) holds.

By (2), we may assume that \( x = (x, Y) \) for some \( x \in X, Y \subseteq I_x \).

Note that \( x \subseteq (x, Y) \) since \( a \neq 0 \). Now at most one member of \( Y \) is in \( X_{\beta} \). Choose \( \beta < \omega^\alpha \) so that \((\bar{\alpha}, \beta) \subseteq \omega^\alpha \) and \( \beta \not\subseteq x \). Then
\( (\bar{\alpha}, \beta, Y) \) is a proper ideal of \( \mathfrak{A} \). Thus (1) and (3), a set of \( m \) distinct members of \( \mathfrak{A} \). This completes the proof.

**Theorem 2.2.** If \( m \) is an uncountable strongly limit cardinal, then there is a rigid BA of power \( m \).

**Proof.** Let \( \langle \sigma_n : \alpha < \text{cf } m \rangle \) be a strictly increasing sequence of infinite cardinals \( \langle \alpha < \text{cf } m \rangle \). Now we define \( \langle \sigma_n : \alpha < \text{cf } m \rangle \):
\[
\sigma_0 = \text{cf } m,
\sigma_{n+1} = \bigcup_{\alpha < \text{cf } m} \sigma_\beta \text{ if } \alpha \text{ is a limit ordinal } < \text{cf } m,
\sigma_{n+1} = (\exp(\sigma_n))^\alpha \bigcup_{\alpha < n+1} \sigma_{n+1} \text{ if } \alpha < \text{cf } m.
\]

Clearly \( \langle \sigma_n : \alpha < \text{cf } m \rangle \) is a strictly increasing sequence of cardinals \( \langle \alpha < \text{cf } m \rangle \) such that \( \sigma_{\text{cf } m} = \text{cf } m \). For each \( \alpha < \text{cf } m \) let \( \mathfrak{A}_\alpha \) be a rigid BA such that, for every \( a \in A \) with \( a \neq 0 \), \( \mathfrak{A}_\alpha \) is rigid by 2.1. Now we define \( \langle \mathfrak{A}_\alpha : \alpha < \text{cf } m \rangle \) and \( \langle I_\alpha : \alpha < \text{cf } m \rangle \) by recursion so that \( \mathfrak{A}_\alpha \) is a BA \( I_\alpha \) is a proper ideal of \( \mathfrak{A}_\alpha \), and \((I_\alpha) \) if \( \beta < \alpha < \text{cf } m \) then \( \mathfrak{A}_\beta \leq \mathfrak{A}_\alpha \) and \( I_\beta \subseteq I_\alpha \).

Let \( \mathfrak{A}_0 = \mathfrak{A}_\alpha \), and let \( I_0 = \{0\} \). If \( \gamma \) is a limit ordinal \( < \text{cf } m \) and \( \mathfrak{A}_\alpha \) have been defined for all \( \alpha < \gamma \) so that (1) holds, let \( \mathfrak{A}_\gamma = \bigcup_{\alpha < \gamma} \mathfrak{A}_\alpha \) and \( I_\gamma = \bigcup_{\alpha < \gamma} I_\alpha \). Clearly (1) holds. Now suppose that \( \mathfrak{A}_\alpha \) and \( I_\alpha \) have been defined so that (1) holds, where \( \alpha < \text{cf } m \). Let \( J_\alpha \) be a maximal ideal of \( \mathfrak{A}_\alpha \) such that \( I_\alpha \subseteq J_\alpha \). We define \( f_\alpha : \mathfrak{A}_\alpha \to \mathfrak{A}_\alpha \times \mathfrak{A}_{\alpha+1} \) by setting \( f_\alpha(x) = (x, x/I_\alpha) \) for each \( x \in B_\alpha \). Here \( x/I_\alpha \) is treated as the 0 or 1 of \( \mathfrak{A}_{\alpha+1} \). Clearly \( f_\alpha \) is an isomorphism of \( \mathfrak{A}_\alpha \) into \( \mathfrak{A}_\alpha \times \mathfrak{A}_{\alpha+1} \). Let \( g_\alpha \) be an isomorphism of \( \mathfrak{A}_\alpha \times \mathfrak{A}_{\alpha+1} \) onto a BA \( \mathfrak{A}_{\alpha+1} \times \mathfrak{A}_{\alpha+1} \) such that \( g_\alpha(x, x/I_\alpha) = x \) for all \( x \in B_\alpha \). Let \( J_{\alpha+1} = (g_\alpha, \beta, 0) : \beta \in B_\alpha \). Clearly \( J_{\alpha+1} \) is a proper ideal of \( \mathfrak{A}_{\alpha+1} \) and \( I_{\alpha+1} \subseteq J_{\alpha+1} \). Thus (1) holds, and our construction is finished. Now for each \( \alpha < \text{cf } m \) the following hold:

- \( (a) \) \( \mathfrak{A}_\alpha \) is rigid;
- \( (b) \) \( \mathfrak{A}_\alpha \) is rigid for every \( x \in B_\alpha \); \( I_\alpha \subseteq x \geq \mathfrak{A}_\alpha \);
- \( (c) \) \( \mathfrak{A}_\alpha \) is rigid for every \( x \in B_\alpha \); \( I_\alpha \subseteq x \geq \exp(\mathfrak{A}_\alpha) \);
- \( (d) \) \( \mathfrak{A}_\alpha \) is rigid for every \( \beta < \alpha \) and all \( x \in I_\alpha \); \( B_\alpha \subseteq x \geq \mathfrak{A}_\alpha \);
- \( (e) \) \( \mathfrak{A}_\alpha \) is rigid for every \( \beta < \alpha \) and all \( x \in I_\alpha \); \( B_\alpha \subseteq x \geq \mathfrak{A}_\alpha \);

We prove these statements by induction on \( \alpha \). They are clear for \( \alpha = 0 \). Now suppose they hold for \( \alpha \). By (4) and the choice of \( \mathfrak{A}_{\alpha+1} \) we have \( \beta \not\subseteq x \). Thus \( (\mathfrak{A}_{\alpha+1} \times \mathfrak{A}_{\alpha+1}) = \exp(\mathfrak{A}_{\alpha+1}) \). Therefore \( (a) \) holds. It suffices to check (5) holds. Suppose \( x \in I_\alpha \) and \( y \in \mathfrak{A}_{\alpha+1} \). Thus \( y \leq x \). Say \( y = g_\alpha(b, a) \) with \( b \in B_\alpha \) and \( a \in \mathfrak{A}_{\alpha+1} \). Now \( g_\alpha(b, a) = y \leq x = g_\alpha(x, 0) \). Thus \( b \leq x \). Thus \( b \in I_\alpha \) and \( b \leq g_\alpha(b, 0) = y \). Thus (5) holds. Now suppose that \( \gamma \) is a limit ordinal \( < \text{cf } m \), and \( \mathfrak{A}_\alpha \) is rigid for all \( \alpha < \gamma \). To check (2), suppose on the contrary that \( \mathfrak{A}_\gamma \) is not rigid. Say \( h \) is an automorphism of \( \mathfrak{A}_\gamma \) and \( x \) an element of \( \mathfrak{A}_\gamma \) such

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that \( h \neq x \). Choose \( \alpha < \gamma \) such that \( x, h \in B_\alpha \). Since \( h \neq x \) iff \( h(-x) \neq -x \), by \((6 \alpha + 1)\) we may assume that \( x \in \mathcal{I}_{\alpha + 1} \). Then

\[
(7) \quad B_\beta \vdash x = B_\alpha \upharpoonright x.
\]

In fact, since \((5 \alpha)\) holds for all \( \delta < \gamma \) it is clear that \( B_\delta \vdash x = B_{\alpha + 1} \upharpoonright x \). Now suppose \( y \in B_{\alpha + 1} \upharpoonright x \). Thus \( y \in B_{\alpha + 1} \) and \( y < x \). Say \( y = g_\beta(b, a), x = g_\beta(b', 0), \) with \( b, b' \in B_\alpha, a \in A_{\alpha + 1} \). Now \( x \in B_\alpha \), so \( x = g_\beta(b, a) \in g_\beta(b', 0) \). Hence \( b' = x \in \mathcal{I}_{\alpha} \). Since \( y < x \), it follows that \( b < b' \) and \( a = 0 \). Hence \( b \in \mathcal{I}_{\alpha} \). So \( b = g_\beta(b, 0) = g_\beta(x, 0) \). Hence \((7)\) holds. Now it follows that \( h \in \mathcal{I}_{\alpha + 1} \). For, if \( h \notin \mathcal{I}_{\alpha + 1} \), then

\[
B_\beta \vdash h \upharpoonright x \neq B_{\alpha + 1} \upharpoonright h \upharpoonright x \upharpoonright B_{\alpha + 1} \upharpoonright x \quad \text{by} \quad (3 \alpha + 1)
\]

while by \((7)\) and \((4 \alpha)\), \( B_\beta \vdash x \in B_{\alpha + 1} \upharpoonright x \in \exp(p_{\alpha + 1}) \in p_{\alpha + 1} \). This is a contradiction, since \( h \) maps \( B_\beta \upharpoonright x \) one-one onto \( B_\beta \upharpoontright h \). Thus, indeed, \( h \in \mathcal{I}_{\alpha + 1} \). But then by \((5 \delta)\) for \( \delta < \gamma \) we have \( B_\delta \vdash x = B_{\alpha + 1} \upharpoonright x \) and \( B_\beta \vdash h \upharpoonright x \in \mathcal{I}_{\alpha + 1} \). Hence \((5 \alpha)\) holds after all. The other conditions \((3 \alpha)\) and \((6 \alpha)\) are easily checked.

In particular, \( B_{\alpha} \upharpoonright m \) is rigid. Since \((5 \alpha)\) and \((6 \alpha)\) hold for all \( \alpha < \text{cf} \, m \), it is clear that \( B_{\alpha} \upharpoonright m = \alpha \). This completes the proof.

**Corollary 2.3 (GCH).** For every uncountable cardinal \( m \), there is a rigid BA of power \( m \).

Thus under the assumption of GCH, the cardinals of rigid BAs are known: 1, 2, and all uncountable cardinals.

In unpublished work, B. Balcar and P. Štěpánek have proved that the existence of a rigid BA of power \( N_\omega \) is consistent with the negation of CH. Very recently, S. Shelah has shown that in fact Corollary 2.3 holds without GCH.

Now we turn to de Groot's theorem. Our extension of his theorem will follow the same general lines as his construction, except that instead of working on the real line we use the \( m \)-metric spaces of Sikorski [15]. Before we begin this construction we would like to mention two problems. De Groot claims to have constructed (1) BA's with no non-trivial onto endomorphisms or one-one endomorphisms and (2) \( \exp \mathcal{N}_m \) BA's with no non-trivial homomorphisms onto each other. We have not been able to reconstruct these proofs, and the following problem are hence open so far as we know:

**Problem 6.** Is there an infinite BA with no non-trivial one-one endomorphism?

**Problem 7.** For which infinite cardinals \( m \) do there exist BA's of power \( m \) with no non-trivial onto endomorphisms? Recall that Rieger [13] has shown the existence of BA's with no non-trivial onto endomorphisms; the cardinals of his examples are rather large.

Returning to de Groot's construction, we begin with the following lemma, essentially obtained from [9]:

**Lemma 2.4.** If \( Y \) and \( Z \) are completely regular Hausdorff spaces, then \( Y \) is not C*-embedded in \( Y \), \( \forall z \in Z \) is not C*-embedded in \( Z \), and \( f: Y \rightarrow Z \) is not C*-embedded in \( Z \), then \( f: Y \rightarrow Z \).

**Proof.** By symmetry it is enough to show that \( f \upharpoonleft Y \subseteq Z \). Suppose, to the contrary, that \( y \in Y \) and \( f \upharpoonleft z \in Z \). We shall obtain a contradiction by showing that \( Y \upharpoonright (y) \) is C*-embedded in \( Y \). Let \( g \in C(Y \upharpoonright (y)) \). From Gillman, Jerison [5] 9.1 we know that \( Y \upharpoonright (y) \) is C*-embedded in \( \beta Y \upharpoonright (y) \). Hence let \( g \in C(\beta Y \upharpoonright (y)) \) be an extension of \( g \). Now \( Z \subseteq f^* \upharpoonleft (\beta Y \upharpoonright (y)) \), so \( f^* \upharpoonleft (\beta Y \upharpoonright (y)) \) is C*-embedded in \( Z \). Hence let \( h \in C(\beta Z) \) be an extension of \( g^* = f^{-1} \circ f^* \circ (\beta Y \upharpoonright (y)) \). Then \( h \neq f \upharpoonleft Y \upharpoonright (y) \) is the desired extension of \( g \).

**Lemma 2.5.** Let \( Y \) be a Hausdorff space. Assume that \( y \in Y \) and there is an infinite cardinal \( m \) and a sequence \( (U_\alpha : \alpha < m) \) of closed-open subsets of \( Y \) such that:
(i) If $a < b < m$ then $U_a \supseteq U_b$;
(ii) If $\lambda < m$ is a limit ordinal, then $U_\lambda = \bigcap_{a < b} U_a$;
(iii) $\{U_a : a < m\}$ is a neighborhood base for $y$.

Then $Y \sim (y)$ is not $C^\ast$-embedded in $Y$.

**Proof.** We may assume that $U_0 = Y$. Clearly then for every $x \in Y \sim (y)$ there is a unique $a = a_x$ such that $x \in U_a \sim U_{a+1}$. Now for any $x \in Y \sim (y)$ let $a_x = 1$ if $a_x$ is even, $a_x = 0$ if $a_x$ is odd. Clearly $f \in C^\ast(Y \sim (y))$. Suppose $f$ extends to $f' \in C^\ast Y$. We may assume that $f' \mid Y = 0$. Let $Y$ be a neighborhood of $y$ such that $0 \not\in f'' Y$. Say $U_a \subseteq Y$. Choose $\beta$ odd, $a < \beta < m$. Then $0 \not\in f'' U_a \subseteq f'' Y$, contradiction.

For some of the following, see Sikorski [15]. A $G^\alpha_n$-set is a set which is the intersection of $< m$ open sets. If $X$ is a space of weight $< m$, then $X$ has $< \exp m$ open sets, also $< \exp m$ is a $G^\alpha_n$-set, and it has a dense subset of power $< m$; and any subspace of $X$ has weight $< m$. We shall consider below the notion of an $m$-metric space from [15]. We will always take the value group to be the additive group of some ordered field. Recall that for such an ordered field there is a strictly decreasing sequence $(\epsilon_n : m)$ of positive elements coinitial in the set of all positive elements.

If $X$ and $Y$ are $m$-metric spaces with values in an ordered field $B$, $A \subseteq X$, and $f : A \rightarrow Y$ is continuous, we let

$$ A_f = \{x \in X : \text{for every positive } \epsilon \in B \text{ there is a positive } \delta \in B \text{ such that for all } x, y \in S_{\rho} \delta \cap A \text{ we have } p(f(x), f(y)) < \epsilon\}.$$ 

Here $S_{\rho} p = \{x \in X : \rho(x, p) < \delta\}$. Clearly $A \subseteq A_f$. The following three lemmas are proved just as for the (classical) metric spaces.

**Lemma 2.6.** $A_f$ is a $G^\alpha_n$-set.

**Lemma 2.7.** In any $m$-metric space, any closed set $F$ is a $G^\alpha_n$-set.

---

**Lemma 2.8.** Let $X$ and $Y$ be $m$-metric spaces, $Y$ complete, values of $X$ and $Y$ in $B$, $A \subseteq X$, $f : A \rightarrow Y$ continuous. Then $f$ can be extended continuously to the $G^\alpha_n$-set $A_f \cap \partial A$.

For some of the notation in the next lemma see [1]. A continuous $n$-displacement is a continuous function which is a displacement of order $n$.

**Lemma 2.9.** Let $M$ be a complete $m$-metric space, with $|M| = \exp m$ and with the weight of $M < m$. Let $(K_a : a < \exp m)$ be a family of subsets of $M$, each $K_a$ of power $\exp m$. Then there is a family $(U_a : a < \exp m)$ of subsets of $M$ such that

(i) $|F_a - F_b| = \exp m$ if $a \neq b$;

(ii) no $F_a$ admits any continuous $(\exp m)$-displacement into itself or another $F_a$;

(iii) $|F_a \cap K_b| = \exp m$ for all $a < \exp m$, $b < \exp m$.

The proof of this lemma is a straightforward generalization of the proof of Theorem 1 of [2]. At the appropriate place in this proof, Lemma 2.8 is used.

**Lemma 2.10.** Let $X$ be a hausdorff space, $|X| = \aleph_n$. Suppose $Y, Z \subseteq X$ and $f : Y \rightarrow Z$ is continuous, $f$ not the identity on $Y$. Assume that for every $y \in Y$, each neighborhood of $y$ contains a points of $Y$. Then $f$ is an $n$-displacement.

**Proof.** Choose $a \in Y, b \in Z$ with $a \neq b$ and $a - b$. Let $U =$ and $V$ be disjoint open neighborhoods of $a$ and $b$, respectively. Then $W =$ and $f^{-1} Y$ is an open neighborhood of $a$, hence $|W| = \aleph_n$, and $W \cap f^{-1} Y = 0$, as desired.

Now we turn to the second main result of this section. The result has previously been established by Eberlein [unpublished].

**Theorem 2.11.** Let $m$ be a regular cardinal such that $\forall \alpha < \aleph_n$, $\exists a < m(\exp \alpha < \aleph_n)$. Then there are exactly $\aleph_n$ isomorphism types.
of rigid BA's of power \( \exp m \).

Proof. By (2) we may assume that \( m > \kappa_{\omega} \). The space \( T_{\kappa_{\omega}} \) of [15] (denoted by \( T_{\omega} \) in [13], where \( m = \kappa_{\omega} = \omega_{2} \)) is a complete \( m \)-metric space of power \( \exp m \) and weight \( m \). \( T_{\omega} \) is simply \( m \)-2 with a suitable \( m \)-metric. Let \( E \) be the collection of \( f \in m \) which are not eventually 1, i.e., \( \bar{E} = \{ f \in m : \forall n < m \exists \beta > 0 \land (\beta \in f) \} \). Clearly \( |E| = 2^{m} \). Let \( m \)-2 be lexicographically ordered, and let \( (K_{\alpha} : \alpha < \exp m) \) enumerate all closed intervals \([f,g] \), where \( f, g \in E \) and \( f < g \). For \( f, g \in m \), \( f \neq g \), let \( \eta_{f,g} \) be the least \( \alpha < m \) such that \( f \alpha \in E \). Note:

(1) \( |[f,g]| = \exp m \) whenever \( f, g \in E \) and \( f < g \).

In fact, let \( \alpha = \eta_{f,g} \). Thus \( f \alpha = 0 \), \( g \alpha = 1 \). Now since \( f \in E \), there is a \( \beta > \alpha \) with \( \beta \alpha = 0 \). If \( h \) is any member of \( m \)-2 such that \( h \beta = f \beta \), while \( h \alpha = 1 \), then \( f < h < g \). Thus (1) follows. Now apply Lemma 2.9 to get a family \( (F_{\alpha} : \alpha < \exp m) \) with the indicated properties. Next we note:

(2) \([f,g] \cap F_{\alpha} \) is a closed-open subset of \( F_{\alpha} \) whenever \( \alpha < \exp m \), \( f, g \in E \setminus F_{\alpha} \), and \( f < g \).

In fact, to show that \([f,g] \cap F_{\alpha} \) is open in \( F_{\alpha} \), let \( h \in [f,g] \cap F_{\alpha} \). Let \( \beta \eta_{f,g} \). For any \( k \in S_{\gamma} \cap h \cap F_{\alpha} \), \( k \beta h \), we have \( \beta < \eta_{f,g} \), and hence \( k \in [f,g] \cap F_{\alpha} \). Therefore \([f,g] \cap F_{\alpha} \) is open in \( F_{\alpha} \). To show that it is closed, assume \( h \in \text{Cl}(f, g) \cap F_{\alpha} \cap F_{\alpha} \). Let \( \beta \eta_{f,g} \). Choose \( k \in S_{\gamma} \cap [f,g] \cap F_{\alpha} \). Then clearly \( h \in [f,g] \cap F_{\alpha} \). Thus (2) holds.

For each \( \alpha < \exp m \) let \( A_{\alpha} \) be the BA of closed-open subsets of \( F_{\alpha} \).

(3) \( |A_{\alpha}| = \exp m \) for each \( \alpha < \exp m \).

In fact, each \( F_{\alpha} \) has weight \( \leq m \), so \( |A_{\alpha}| < \exp m \). On the other hand \([f,g] \cap F_{\alpha} \) is closed-open in \( F_{\alpha} \), as \( \alpha < \exp m \). It follows from (2) that \( [f,g] \cap F_{\alpha} \) is a collection of members of \( A_{\alpha} \). There are \( \exp m \) members of \( E \setminus F_{\alpha} \). If \( f, g \neq (h, k) \) as ordered pairs, then \([f,g] \cap F_{\alpha} \neq [h,k] \cap F_{\alpha} \). Indeed, say \( g < k \). Then

\([f,g] \cap F_{\alpha} \cap [h,k] \cap F_{\alpha} \neq 0 \), but \([h,k] \cap F_{\alpha} \cap [g,k] \cap F_{\alpha} \neq 0 \) by our choice of the \( F_{\alpha} \)'s. Other possible situations regarding the intervals \([f,g] \) and \([h,k] \) are treated similarly.

Now we turn to the proof that the \( A_{\alpha} \)'s are rigid.

(4) If \( f \in F_{\alpha} \), then each neighborhood of \( f \) contains \( \exp m \) points of \( F_{\alpha} \).

In fact, let \( U \) be any neighborhood of \( f \). Say \( S_{\gamma} f \subseteq U \). Let \( g = f \cup (\alpha + 1) \cup (0: \beta \in m \sim (\alpha + 1)) \) and \( h = f \cup (\alpha + 1) \cup (\alpha + 1, 1) \cup (0: \beta \in m \sim (\alpha + 2)) \). Then \( g, h \in E \), \( g < h \), and \( [g,h] \cap F_{\alpha} \subseteq S_{\gamma} F_{\alpha} \subseteq U \cap F_{\alpha} \). Furthermore, \([g,h] \cap F_{\alpha} \) is \( \exp m \) by our choice of the \( F_{\alpha} \)'s. Hence (4) holds. Therefore by (4), Lemma 2.10, and our choice of the \( F_{\alpha} \)'s.

(5) no \( F_{\alpha} \) admits any one-one continuous map into another \( F_{\beta} \) or any non-identity one-one continuous map into itself.

Now by [15] (vii) and (vi), each \( F_{\alpha} \) is a normal topological space with a base of closed-open sets. Hence \( \beta F_{\alpha} \) is a Stone space. We shall now proceed to apply Lemma 2.4 and 2.5. Let \( f \in F_{\alpha} \). We shall define a sequence \((U_{n} : \beta < m)\) of closed-open neighborhoods of \( f \) in \( F_{n} \). Let \( U_{0} = F_{\alpha} \). For \( \beta \) a limit ordinal \( < m \), let \( U_{\beta} = \bigcap U_{\alpha} \). Suppose \( U_{\alpha} \) has been defined. Then \( U_{\beta} \cap S_{\gamma} F_{\alpha} \) is a neighborhood of \( f \); say \( S_{\gamma} f \subseteq U_{\beta} \cap S_{\gamma} F_{\alpha} \). Hence we may choose \( k \in ([g,h] \cap F_{\alpha}) \cap (f) \). Let \( \delta = \eta_{g,h} \), and choose \( U_{\beta+1} \) to be a closed-open set \( \subseteq U_{\beta} \cap S_{\gamma} F_{\alpha} \). Note that \( \gamma < \delta \), and hence \( U_{\beta+1} \subseteq S_{\gamma} F_{\alpha} \). This completes the construction of \((U_{\beta} : \beta < m)\). If \( V \) is any neighborhood of \( f \), say \( S_{\gamma} F_{\alpha} \subseteq V \); then \( U_{m} \subseteq V \).

Thus the hypothesis of 2.5 is satisfied. By 2.5, 2.4, (5), and duality, the theorem follows.

Of course, Theorem 2.11 leaves several problems open.
Problem 8. Assuming GCH, how many isomorphism types of rigid BA's of power $m^+$, $m$ singular, are there? In particular, how many are there of power $N_{\omega+1}$?

Problem 9. If $m$ is singular, how many rigid BA's of power $m$ are there, up to isomorphism?

3. CARDINALITY OF AUTOMORPHISM GROUPS

The general question we consider in this section is the relationship between the cardinalities of $\mathfrak{A}$ and $\text{Aut} \, \mathfrak{A}$. The case $\text{Aut} \, \mathfrak{A}$ finite essentially reduces to considerations about rigid BA's because of the result of de Groot and McDowell quoted in the introduction. Thus the question to which we address ourselves is: given $m$, $n > N_m$, is there a BA $\mathfrak{A}$ of power $m$ with $|\text{Aut} \, \mathfrak{A}| = n$? Obviously $|\text{Aut} \, \mathfrak{A}| < \exp |\mathfrak{A}|$. This bound can actually be attained. For example, for $m$ the BA of finite cofinite subsets of $\mathcal{m}$ we clearly have $|\mathfrak{A}| = m$ and $|\text{Aut} \, \mathfrak{A}| = \exp m$.

We begin with our strongest theorem. The other results in this section are corollaries of this theorem or treat special problems suggested by it.

Theorem 3.1. Let $N_0 < n < m$. There exists a BA $\mathfrak{A} \subseteq \mathcal{m}$ with $|\mathfrak{A}| = \exp m$ such that $\mathfrak{A} \subseteq \mathcal{m}$ and $\text{Aut} \, \mathfrak{A}$ is naturally isomorphic to $\text{Sym} (\mathfrak{m}, n)$.

Proof. Let $(\varphi_n : \alpha < \exp m)$ be an enumeration of $\text{Sym} (\mathfrak{m}, n)$. For each $\varphi_n < \exp m$, let $\varphi_n$ be the congruence relation associated with the ideal $\mathfrak{a}_{\varphi_n}$ of $\mathfrak{m}$: $X \equiv Y \iff X \Delta Y$ (the symmetric difference) is in $\mathfrak{a}_{\varphi_n}$. Now by Hausdorff [6], let $(X_\alpha : \alpha < \exp m)$ be a system of subsets of $m$, independent modulo $\mathfrak{a}_{\exp m}$; thus $(X_\alpha / \mathfrak{a}_{\exp m})_{\alpha < \exp m}$ freely generates a subalgebra of $\mathfrak{m} / \mathfrak{a}_{\exp m}$.

We now construct by transfinite recursion two sequences $(Y_\alpha : \alpha < \exp m)$ and $(U_\alpha : \alpha < \exp m)$; each $U_\alpha$ will be a subset of $m$, and each $U_\alpha$ a subset of $\exp m$ with $|U_\alpha| \leq N_0 + n$. We shall ensure that

(1) for each $\beta < \exp m$, $(Y_\alpha : \alpha < \beta) \cup (X_\alpha : \beta < \alpha < \exp m$ and $\alpha \in \mathfrak{U}_n)$ is independent modulo $\mathfrak{a}_{\exp m}$; and furthermore, for all $\alpha < \beta$, $f_\alpha \mathfrak{Y}_n \notin \text{Sg} ((Y_\alpha : \alpha < \beta) \cup \mathfrak{a}_{\exp m})$.

Let $\gamma < \exp m$ and $Y_\alpha, U_\alpha$ already defined for all $\alpha < \gamma$, so that (1) is true for each $\beta < \gamma$. We determine $Y_\gamma, U_\gamma$ as follows. To begin, put $U_\gamma' = \bigcup U_\alpha \cup (\gamma + 1)$. Then clearly

(1.1) $(Y_\alpha : \alpha < \gamma) \cup (X_\alpha : \alpha \in \exp m \sim U_\gamma')$ is independent modulo $\mathfrak{a}_{\exp m}$; $|U_\gamma| \leq N_0 + 1)$; and for all $\xi \leq \gamma$, $f_\xi Y_\alpha \notin \text{Sg} ((Y_\alpha : \alpha < \gamma) \cup \mathfrak{a}_{\exp m})$.

Let us prove:

(1.2) Let $\delta_0, \delta_1$ be the least two ordinals in $\exp m \sim U_\gamma'$. One of the fifteen elements different from $m$ in the set $\text{Sg} (X_{\delta_0}, X_{\delta_1})$ contains a set $Z = Z_0 \cup Z_1$ (disjoint union) such that $|Z| = n$ and $f_\gamma^* Z_0 \cap Z_1^* = Z_0$.

Indeed, since $f_\gamma \not\in \text{Sym} (m, n)$, there are disjoint sets $C, D \subseteq m$ with $|C| = n$ and $f_\gamma^* C \subseteq D$ (easily verified). Since $X_{\delta_0}, X_{\delta_1}$ are independent, they generate four nonzero atoms in $\text{Sg} (X_{\delta_0}, X_{\delta_1})$. One of the atoms must include an $n$-element set $C \subseteq C$. One of the atoms must contain an $n$-element set $D \subseteq C$. Then we can put $Z_0 = f_\gamma^{-1} D', Z_1 = D'$. So (1.2) is proved.

Now we take $Z = Z_0 \cup Z_1$ as in (1.2). Also we let $X \in \text{Sg} (X_{\delta_0}, X_{\delta_1})$, $Z \subseteq \mathfrak{m} \setminus m$. We are going to take, for some $\delta \in \exp m \sim U_\gamma' \sim (\delta_0, \delta_1)$ and some $S \subseteq Z_1$,

(1.3) $Y_\alpha \equiv X_{\delta} \sim Z \cup S; \quad U_\alpha = U_\gamma' \cup (\delta_0, \delta_1)\beta$.

Of course, we want $\delta$ and $S$ so that (1) will be true with $\beta = \gamma$.

The "independence" assertion in (1) will hold true whatever the choice of $\delta, S$ subject to the above. In words, the argument is simply that $Y_\alpha$ and $X_{\delta_1}$ agree on the non-zero set $\sim X_{\delta_0} \cap \text{Sg} (X_{\delta_0}, X_{\delta_1})$, hence non-trivial relations (modulo $\mathfrak{a}_{\exp m}$) among $(Y_\alpha : \alpha < \gamma) \cup (X_\alpha : \alpha \in \exp m \sim U_\gamma')$, imply non-trivial relations (modulo $\mathfrak{a}_{\exp m}$) among $(Y_\alpha : \alpha < \gamma) \cup (X_\alpha : \alpha \in \exp m \sim U_\gamma')$, but these are ruled out by (1.1). This argument
can be made precise, but we won’t bother.

To assure the second condition of (1) for \( \beta = \gamma \), we first prove

(1.4) Let \( \xi < \gamma \), and \( A, B \in \text{Sg}(Y_2; \alpha \leq \gamma) \). There is at most one \( \delta \in \exp m \sim U_9 - (\delta_0, \delta_1) \) such that for some \( S \subseteq Z_9 \), and for \( Y = (X_9 \sim Z) \cup (B \sim Y) \).

For if not, we have say \( Y_1 = (X_9 \sim Z) \cup S_1 \) and \( Y \in \exp m \sim U_9 \sim (\delta_0, \delta_1) \), \( S_2 \subseteq Z_9 \) (for \( i = 1, 2 \)) with \( A \neq B \), and

\[
I_{A}^{*} Y_{1} = (A \cap Y_{1}^{*}) \cup (B \cap Y_{1}^{*}) \quad \text{for} \quad i = 1, 2.
\]

Note that the symmetric difference of the right hand sets (for \( i = 1, 2 \)) in the above formulas includes the set \( \sim Z \cap (A \Delta B) \cap (X_9 \Delta X_{9, b}) \), and

\[
\sim \bar{X} \cap (A \Delta B) \cap (X_9 \Delta X_{9, b}),
\]

which is hence \( \neq 0 \). By 1.1, “independence” module \( \mathfrak{Q}_{m} \), that is impossible unless \( A \cap B \in \text{Sg}(Y_2; \alpha < \gamma) \) is 0. That in turn means \( A = B \) and

\[
I_{A}^{*} Y_{1} = (A \cap Y_{1}^{*}) \cup (A \cap Y_{1}^{*}) = A,
\]

which implies that \( I_{A}^{*} Y_{2} \in \text{Sg}(Y_2; \alpha < \gamma) \cup \mathfrak{Q}_{m} \), contradicting (1.1).

Statement (1.4) is proved.

Now there are at most \( N_0 + \gamma \) triples \( (\xi, A, B) \) as in (1.4), and \( \exp m \sim U_9 - (\delta_0, \delta_1) \) has the power \( m + \gamma \). So we are enabled to choose \( \delta \) as the least member of \( \exp m \sim U_9 - (\delta_0, \delta_1) \) such that there are no \( \xi, A, B, S \) satisfying the formulas of (1.4). Now, however we choose \( S \) and define \( Y_2 \) by (1.3), condition (1) will hold for \( \beta = \gamma \) and all \( \xi < \beta \).

We claim that one of the choices \( S \neq 0 \) or \( S = Z_9 \) will satisfy (1) for \( \beta = \gamma = \xi \). Assuming otherwise, we shall get a contradiction.

So let \( Y_1 = X_9 \sim Z \) and \( Y_2 = (X_9 \sim Z) \cup Z_9 \), and suppose that \( I_{A}^{*} Y_{2} \in \text{Sg}(Y_2; \alpha < \gamma) \cup (Y_2^{*}) \cup \mathfrak{Q}_{m} \cup \) (for \( i = 1, 2 \)). This means that there exist \( A, B \in \text{Sg}(Y_2; \alpha < \gamma) \) (for \( i = 1, 2 \)) such that

(1) \( I_{A}^{*} Y_{1} = (A \cap Y_{1}^{*}) \cup (B \cap Y_{1}^{*}) \).

We intersect both sides of these relations by \( \sim \bar{X} \), noting that

\[
(I_{A}^{*} Y_{1}^{*}) \cap \sim \bar{X} = (I_{A}^{*} Y_{1}^{*}) \cap \sim \bar{X}, \quad Y_1 \cap \sim \bar{X} = Y_1 \cap \sim \bar{X} = X_9 \cap \sim \bar{X},
\]

and (consequently) \( Y_1 \cap \sim \bar{X} = Y_1 \cap \sim \bar{X} = X_9 \cap \sim \bar{X} \).

Hence \( I_{A}^{*} Y_{1}^{*} \in \text{Sg}(Y_2; \alpha < \gamma) \cup \mathfrak{Q}_{m} \cup \) (for \( i = 1, 2 \)).

Since \( \mathfrak{Q}_{m} \) is free module \( \mathfrak{Q}_{m} \), this is actually an equality, and it implies \( A = A, B = B \). [In the free algebra, one can map endomorphically \( \bar{X} \) to 0, \( X_9 \) to \( \omega \), and all \( \mathfrak{Q}_{m} \) to themselves. Then the two sides go to \( A = A \).]

We use that \( \bar{X} \in \text{Sg}(X_{9, b}, X_{9, a}) \) is distinct from \( 0, m \). Likewise \( B = B \).]

Now if we intersect (1) for \( i = 1 \) with \( Z_9 \) we obtain \( 0 = B \cap \sim \bar{X} \).

If we intersect (1) for \( i = 2 \) with \( Z_9 \) we obtain \( 0 = B \cap \sim \bar{X} \). But \( B = B \) and \( 0 = B \cap \sim \bar{X} \), so we have our contradiction.

Having succeeded in our construction, we define \( \mathfrak{Q} \) as the subalgebra of \( \mathfrak{Q}_{m} \), generated by \( (Y_2; \alpha < \exp m) \cup \mathfrak{Q}_{m} \). All properties required by Theorem 3.1 follow trivially from statement (1). Note that \( \mathfrak{Q} \) is a free algebra of power \( \exp m \).

Now we give a few special cases and generalizations of this theorem.

Corollary 3.2. There is a BA of power \( 2^{m_0} \) with automorphism group of power \( N_0 \).

It was first shown by Johnsson (unpublished) that there exists a BA with automorphism group of power \( N_0 \). His algebra has a large cardinality.

Corollary 3.3. If \( m = \aleph_{n} \) for some \( n \) or if \( m \) is an uncountable strong limit cardinal, then there is a BA of power \( m \) with exactly \( N_0 \) automorphisms.

Proof. By [9], 2.2, 1.4, and 3.1.

Corollary 3.4 (GCH). If \( N_0 \leq n < m^{+} \leq N_0 \), then there is a BA of power \( m \) with automorphism group of power \( n \).
Proof. If \( n = m^+ \) we may let \( \mathfrak{A} \) be the BA of finite and cofinite subsets of \( m \). If \( m = n \), we use 2.3 and 1.11. Now assume that \( m > n \).

By 3.1 let \( \mathfrak{A} \) be an atomic BA of power \( n^+ \) with \( \text{Aut} \mathfrak{A} = \text{Sym}(n, R_n) \), and let \( \mathfrak{B} \) be atomless and rigid of power \( m \); then \( \mathfrak{A} \times \mathfrak{B} \) is the desired algebra.

Corollary 3.5. For any \( n > N_0 \) there is a BA \( \mathfrak{A} \) such that 
\[ |\text{Aut} \mathfrak{A}| = n. \]

In the remainder of the paper we give some results showing a few ways in which 3.1-3.5 cannot be improved. We aim first for 3.10, relevant to 3.2.

Theorem 3.6. If \( \mathfrak{A} \) has infinitely many atoms and \( \text{MA}, \text{MA}_1 \) holds, then \( \text{Sym} \omega \) can be isomorphically embedded in \( \text{Aut} \mathfrak{A} \).

Proof. We shall apply Theorem 2.2 of Martin, Solovay [10].

Let \( \alpha : \omega \to \mathfrak{A} \). Let \( F \) be an ultrafilter on \( \omega \) such that \( x \in F \) whenever \( \mathfrak{A} \), \( \mathfrak{B} \) is finite. Set \( B = \{ J \subseteq \omega : \text{for some } x \in F, J = \{ i : a_i < x \} \} \), \( C = \{ J \subseteq \omega : \text{for some } x \in F, J = \{ i : a_i = x \} \} \).

The hypotheses of 2.2 of [9] are easily verified. Hence
\[(1) \text{there is an infinite } D \subseteq \mathfrak{A} \cap \mathfrak{B} \text{ such that for all } x \in A \sim F, (d \in D : d < x) \text{ is finite.} \]

Now by (1) we can write each \( x \in A \sim F \) uniquely in the form \( t_x + \sum d \) where \( \mathfrak{A} \cap D = 0 \) and \( \mathfrak{B}_x \) is a finite subset of \( D \). For any permutation \( f \) of \( D \) and any \( x \in A \sim F \) we set
\[ f^+ x = t_x + \sum_{a \in \mathfrak{B}_x} f d. \]

For \( x \in F \) we set \( f^+ x = f^+ - x \). Then for any \( x, y, A \), \( f^+ (x + y) = f^+ x + f^+ y \). This is easy to see if \( x, y \in A \sim F \) or \( x, y \in F \). Now suppose, say, \( x \in F \) and \( y \in A \). Then
\[-(f^+ x + f^+ y) = -(f^+ - x + f^+ y) = -\left( -t_x - \sum_{a \in \mathfrak{B}_x} fd + t_y + \sum_{a \in \mathfrak{B}_y} fd \right) = -984. \]
There is a homomorphism $f : A/I \to P(\{a\})$ such that $f(x) = x - a$ for all $x \in A$ and $a \in M$.

Clearly $f$ is one-one, so it follows that $|A/I| < \exp m$. Hence $|I| = |A|$. Suppose $I$ has a largest element, $c$. Then $\mathfrak{A} \in c$ has power $|A|$, and is isomorphic to a subalgebra of $\mathfrak{A}$ (as is well-known). If it is not rigid, we cannot obtain, as in the proof of (1), an $a < c$ with $|A/I_a| < m$ and $|\text{Aut}(\mathfrak{A} \upharpoonright a)| > 1$, contradicting the maximality of $M$. Thus $\mathfrak{A} \in c$ is rigid.

Now suppose $I$ has no largest element. Let $B = I \cup \{x : x \neq I\}$. It is easily verified that $B$ is closed under $+$ and $-$, so $B \subseteq \mathfrak{A}$. Obviously $|B| = |A|$. Suppose $B$ is not rigid. By 1.6 (iii) choose disjoint non-zero elements $x, y$ of $B$ with $x \in B \upharpoonright y = B \upharpoonright y$; say $f : B \upharpoonright x \to B \upharpoonright y$. If $x + y \in I$, then $\mathfrak{A} \upharpoonright x + y = \mathfrak{A} \upharpoonright x + y$ is non-rigid and again we easily obtain a contradiction to the maximality of $M$. Assume that $x + y \notin I$. Then $x \cdot y \in I$. Choose $c \in I$ with $c \cdot y < x$. Thus $c \cdot x \neq 0$. Thus $f(I \upharpoonright c \cdot x) : B \upharpoonright c \cdot x \to B \upharpoonright f(c \cdot x)$ has $f(c \cdot x) \notin I$, we again obtain a contradiction. If $f(c \cdot x) \in I$, then $1 + (d - f(c \cdot x)) = 0$. But $d \cdot f(c \cdot x) \notin I$, and we may choose $d \in B \upharpoonright x$ with $d \cdot f(c \cdot x)$, which again yields a contradiction.

Corollary. 3.12. For every $m > \exp \kappa_0$, the following conditions are equivalent:

(i) there is a rigid BA of power $m$;

(ii) there is a BA of power $m$ with denumerable automorphism group.

Proof. (i) $\Rightarrow$ (ii). Let $\mathfrak{A}$ be rigid, $|A| = m$. By 3.1 let $\mathfrak{B}$ be an atomic BA of power $\exp \kappa_0$ with $|\text{Aut} \mathfrak{B}| = \kappa_0$. We may assume that $\mathfrak{B}$ is atomless. $\mathfrak{A} \times \mathfrak{B}$ satisfies (ii).

(iii) $\Rightarrow$ (i). By Katetov [8] we may assume that $m > \exp \kappa_0$. Then (i) follows by 3.11.

We conclude the paper with a result concerning a possible improvement of 3.4.

Lemma 3.13. Assume that $\kappa$ and $\nu$ are infinite cardinals with $\kappa > \exp m > \exp m > \exp \kappa$. Then there is a BA $\mathfrak{A}$ of power $\exp \kappa$ with $|\text{Aut} \mathfrak{A}| > \exp \kappa$.

Proof. Let $C$ be the BA of finite and cofinite subsets of $\kappa$, and let $\mathfrak{B}$ be an atomless rigid BA of power $\exp \kappa$. Then $\mathfrak{A} \times \mathfrak{B}$ satisfies the desired conditions.

Theorem 3.14. Con$(\text{ZFC}) \rightarrow$ Con$(\text{ZFC} + \forall \kappa (\kappa \text{ regular} \rightarrow \exists \text{ BA of power } \kappa^+ \text{ with } \kappa^+ < |\text{Aut} \mathfrak{A}| < \exp \kappa^+))$.

Proof. We shall apply Easton [4]. Let $Fm = m^+$ for every regular cardinal $m$. Then [4] gives the desired result.

From the problems left open in this section, we may mention the following.

Problem 10. Does $|\text{Aut} \mathfrak{A}| = \kappa_0$ imply $|\mathfrak{A}| > \exp \kappa_0$ without MA?

Problem 11. Con$(\text{ZFC})$ $\rightarrow$ Con$(\text{ZFC} + \forall \kappa (\kappa > \exp \kappa \text{ BA of power } \kappa^+ \text{ with } \kappa < |\text{Aut} \mathfrak{A}| < \exp \kappa^+))$.

REFERENCES


1. INTRODUCTION

Let $S$ be an $n$-element set and let $2^S$ represent the collection of subset of $S$. A subset of $2^S$ no member of which contains another will be called a Sperner family. To each Sperner family we can correspond a monotone 0–1 function (or Boolean function) defined on $2^S$, by assigning the value 1 to those members of $2^S$ that are contained in no member of the family. One may also correspond a member of the free distributive lattice on $n$ generators to each Sperner family.

Thus the number of Sperner families represents the number of monotone Boolean functions definable on $2^S$ and the size of the free distributive lattice on $n$ generators as well.

A number of authors, especially Korobkov, Hansel and the present author have obtained upper bounds on this number. The general