Remark. Some observations indicate that Lemma 3 can be proved in a much sharper form than ours (see Sublemma 3.3), more precisely, in right hand sides of $2°$ (a) (b) and (c) one may place coefficients increasing unlimitedly with n_2 (the constans given in Lemma 3 are of course, sufficient for getting a contradiction.

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ON AUTOMORPHISM GROUPS OF BOOLEAN ALGEBRAS

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It has been known for a long time that not every group is isomorphic to the automorphism group of some Boolean algebra (BA, for short). For example, de Groot and McDowell [3] showed that for any BA $\mathfrak{A}$ the automorphism group of $\mathfrak{A}$ either consists just of those automorphisms induced by finite permutations of the atoms of $\mathfrak{A}$, or else contains the direct sum of $\aleph_1$ copies of C_2. (We include rigid BA's under the first case.) Thus the problem arises to characterize in some convenient form the automorphism groups of BA's. We address ourselves to this question in Section 1 of this paper. We give there two representation theorems for complete BA's which in principle reduce the characterization problem for complete BA's to two narrower classes of BA's – homogeneous BA's, and those with no rigid or homogeneous factors. In Section 2 we are concerned with rigid BA's. Lozier [9] has shown that for any $\mathfrak{m} \geqslant \aleph_0$ there is a rigid BA of power $2^{\mathfrak{m}}$. We show that there is also one of each strong limit power. de Groot [2] showed that there are $2^{2^{\aleph_0}}$ rigid BA's of

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power $2^{\aleph_0}$. We prove under GCH that for any regular $\mathfrak{m} \geqslant \aleph_0$ there are $\mathfrak{m}^{++}$ rigid BA's of power $\mathfrak{m}^+$. (This result was obtained earlier by Ehrenfeucht, but his construction remains unpublished.) In section 3 we relate the cardinalities of a BA and its automorphism group. The main results there have as consequences that there is a BA of power $2^{\aleph_0}$ with denumerable automorphism group; and under GCH, if $\aleph_0 \leqslant \mathfrak{n} \leqslant \mathfrak{m}^+ > \aleph_1$ then there is a BA of power $\mathfrak{m}$ with automorphism group of power $\mathfrak{n}$.

Our results are actually somewhat stronger than indicated in this brief description, and we give several results related to the above, as well as the statement of 12 open problems.

The notation in the paper is standard, with the following exceptions. The symbols $\rightarrowtail$, $\twoheadrightarrow$, $\rightarrowtail\!\!\!\!\twoheadrightarrow$ indicate one-one, onto, and one-one onto mappings respectively; between BA's they implicitly represent homomorphisms, and between topological spaces – continuous maps. We let $\exp \mathfrak{m} = 2^{\mathfrak{m}}$. The set of all subsets of I (of power $< \mathfrak{n}$) is denoted by SI (respectively $S_{<\mathfrak{n}}I$). $\operatorname{Sym} I$ is the group of all permutations of I, while $\operatorname{Sym}(\mathfrak{m}, \mathfrak{n})$ is the group of all permutations of $\mathfrak{m}$ with support of power $< \mathfrak{n}$. We denote by ${}^{\mathfrak{m}}C_2^w$ the direct sum of $\mathfrak{m}$ copies of C_2.

If $\mathfrak{m}$ is an infinite cardinal, then by $\mathrm{MA}_{\mathfrak{m}}$ we mean the statement that if P is a partially ordered set satisfying the countable antichain condition, and if $\mathfrak{F}$ is a collection of open dense subsets of P with $|\mathfrak{F}| \leqslant \mathfrak{m}$, then an $\mathfrak{F}$-generic set over P exists (cf. Martin, Solovay [10]). Martin's axiom (MA) is the statement $\forall \mathfrak{m} < \exp \aleph_0\ \mathrm{MA}_{\mathfrak{m}}$. It is known that $\mathrm{MA}_{\aleph_0}$ holds. Furthermore, CH $\to$ MA. Finally, Con (ZFC) $\to$ $\to$ Con (ZFC + $\exp \aleph_0 > \aleph_1$ + MA): see Solovay, Tennenbaum [17].

A BA is a structure $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1\rangle$ satisfying the usual axioms. $\mathfrak{A}$ is called *non-trivial* if $|A| > 1$. German capitals denote BA's, and the corresponding Roman letters denote their universes. If $\mathfrak{m}$ is an infinite cardinal, then $\mathfrak{A}$ is $\mathfrak{m}$*-complete* if $\sum X$ exists for each $X \subseteq A$ with $|X| < \mathfrak{m}$. If $\mathfrak{A}$ is a BA and $a \in A$, then $\mathfrak{A} \restriction a$ is the principal ideal generated by a, considered as a BA. For any BA $\mathfrak{A}$, $c\mathfrak{A}$ (the *cellularity* of $\mathfrak{A}$) is the least cardinal greater than all cardinalities of families

of pairwise disjoint elements of $\mathfrak{A}$. A *partition of unity* of a BA $\mathfrak{A}$ is a function $a \in {}^IA$ such that $\sum_{i\in I} a_i = 1$ and $a_i \cdot a_j = 0$ for $i, j \in I$ and $i \neq j$. If I is finite or $\mathfrak{A}$ is complete, such a partition gives rise to an isomorphism $f\colon \mathfrak{A} \rightarrowtail\!\!\!\!\twoheadrightarrow \mathrm{P}_{i\in I}\, \mathfrak{A} \restriction a_i$ defined by $(fx)_i = x \cdot a_i$ for all $x \in A$ and $i \in I$. Two BA's $\mathfrak{A}$ and $\mathfrak{B}$ are called *totally different* iff whenever $0 \neq x \in A$ and $0 \neq y \in B$ we have $\mathfrak{A} \restriction x \not\cong \mathfrak{B} \restriction y$. Next, suppose $\langle \mathfrak{A}_i\colon i \in I\rangle$ is a system of BA's. For $i \in I$ and $x \in A_i$ we define $\delta_i x \in \mathrm{P}_{j\in I} A_j$ by setting $(\delta_i x)_j = x$ if $j = i$ and $(\delta_i x)_j = 0$ if $j \neq i$. For any BA $\mathfrak{A}$, $\operatorname{At}\mathfrak{A}$ is the collection of atoms of $\mathfrak{A}$. We take the *Stone space* of $\mathfrak{A}$ to be the collection of ultrafilters of $\mathfrak{A}$ in the usual way. The automorphism group of $\mathfrak{A}$ is denoted by $\operatorname{Aut}\mathfrak{A}$. We say that $\mathfrak{A}$ is *rigid* if $|\operatorname{Aut}\mathfrak{A}| = 1$. $\mathfrak{A}$ is *homogeneous* if $\mathfrak{A} \cong \mathfrak{A} \restriction a$ whenever $0 \neq a \in A$. An element $a \in A$ is rigid (homogeneous) if $\mathfrak{A} \restriction a$ is rigid (resp. homogeneous). Elements $a, b \in A$ are *isomorphic* if $\mathfrak{A} \restriction a \cong \mathfrak{A} \restriction b$. Similar transfers of terminology from $\mathfrak{A} \restriction a$ to a itself will be made later without explicit mention. $\operatorname{Sg} X$ is the *subalgebra generated* by X.

1. PRODUCTS OF BA'S

In this section we discuss products of BA's, in particular, rigid or homogeneous BA's, always with the automorphism groups in mind. The following lemma and proof, of a general algebraic nature, are well-known.

Lemma 1.1. *If* $\langle \mathfrak{A}_i\colon i \in I\rangle$ *is a system of similar algebras, then* $\mathrm{P}_{i\in I} \operatorname{Aut}\mathfrak{A}_i \rightarrowtail \operatorname{Aut} \mathrm{P}_{i\in I}\, \mathfrak{A}_i$.

Proof. For each $\sigma \in \mathrm{P}_{i\in I} \operatorname{Aut}\mathfrak{A}_i$, each $x \in \mathrm{P}_{i\in I} A_i$, and each $i \in I$, let $(f\sigma)_x i = \sigma_i x_i$. It is easily verified that f is the desired isomorphism into.

In general the isomorphism in the proof of 1.1 is not onto; e.g., 1.11 below. But for BA's there is an important case where it is:

Theorem 1.2. *If* $\langle \mathfrak{A}_i\colon i \in I\rangle$ *is a system of pairwise totally different BA's, then* $\mathrm{P}_{i\in I} \operatorname{Aut}\mathfrak{A}_i \cong \operatorname{Aut} \mathrm{P}_{i\in I}\, \mathfrak{A}_i$.

Proof. It suffices to show that the function f defined in the proof of 1.1 is onto. Let $\varphi \in \text{Aut} \, \mathsf{P}_{i \in I} \mathfrak{A}_i$. We wish to define $\sigma \in \mathsf{P}_{i \in I} \text{Aut} \, \mathfrak{A}_i$ so that $f\sigma = \varphi$. We need two preliminary statements.

(1) if $i, j \in I$, $i \neq j$, and $x \in A_i$, then $(\varphi \delta_i x)_j = 0$.

For, assume otherwise: say $i \neq j$, $(\varphi \delta_i x)_j \neq 0$. Let $z = \varphi^{-1}(\delta_j(\varphi \delta_i x)_j)$. Since $\delta_j(\varphi \delta_i x)_j \leqslant \varphi \delta_i x$, we have $z \leqslant \delta_i x$. Thus $z_k = 0$ for $k \neq i$. Now φ induces an isomorphism of $\mathfrak{A}_i \restriction z_i$ onto $\mathfrak{A}_j \restriction (\varphi \delta_i x)_j$. This contradicts our assumption that $\mathfrak{A}_i$ and $\mathfrak{A}_j$ lack common non-trivial factors. Hence (1) holds. Next,

(2) if $i \in I$, then $\varphi \delta_i 1 = \delta_i 1$.

For, by (1) write $\varphi \delta_i 1 = \delta_i x$. If $x \neq 1$, then by (1) for φ^{-1}, say $\varphi^{-1} \delta_i(-x) = \delta_i u$. Thus $u \neq 0$, but $\delta_i u = \delta_i u \cdot \delta_i 1 = \varphi^{-1} \delta_i(-x) \cdot$ $\cdot \, \varphi^{-1} \delta_i x = \varphi^{-1}(\delta_i(-x) \cdot \delta_i x) = 0$, contradiction.

Now define $\sigma_i x = (\varphi \delta_i x)_i$ for any $i \in I$ and $x \in A_i$. It is easy to verify using (1) and (2) that $\sigma \in \mathsf{P}_{i \in I} \text{Aut} \, \mathfrak{A}_i$ and that $f\sigma = \varphi$, as desired.

Theorem 1.2 motivates our investigations of this section. By it, to describe $\text{Aut} \, \mathfrak{A}$ it is enough to decompose $\mathfrak{A}$ into a product of (simple, in some sense) BA's pairwise lacking common non-trivial factors. The simplest building blocks from the point of view of automorphism groups are the rigid BA's, which we now investigate.

Corollary 1.3. *If* $\langle \mathfrak{A}_i : i \in I \rangle$ *is a system of pairwise totally different rigid* BA's, *then* $\mathsf{P}_{i \in I} \mathfrak{A}_i$ *is rigid.*

A natural conjecture is that also a subdirect product of pairwise totally different rigid BA's is rigid. This is not generally true; see the remark after 1.32.

Corollary 1.4. *If* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are totally different* BA's *and* $\mathfrak{B}$ *is rigid, then* $\text{Aut} \, \mathfrak{A} \cong \text{Aut} \, (\mathfrak{A} \times \mathfrak{B})$.

The following lemma can be seen by an adaptation of the proof of 1.2.

Lemma 1.5. *If* K *is a set of pairwise totally different* BA's, *and* $0 \neq L$, $M \subseteq K$ *and* $L \neq M$, *then* $\mathsf{P}_{\mathfrak{A} \in L} \mathfrak{A} \not\cong \mathsf{P}_{\mathfrak{A} \in M} \mathfrak{A}$.

The next lemma is well-known.

Lemma 1.6. *For any* BA $\mathfrak{A}$, *the following conditions are equivalent.*

(i) $\mathfrak{A}$ *is not rigid;*

(ii) *there are distinct elements* $x, y \in A$ *such that* $\mathfrak{A} \restriction x \cong \mathfrak{A} \restriction y$;

(iii) *there are disjoint non-zero elements* $x, y \in A$ *such that* $\mathfrak{A} \restriction x \cong$ $\cong \mathfrak{A} \restriction y$.

Proof. (i) → (ii). Let f be a non-identity automorphism of $\mathfrak{A}$; say $x \neq fx$. Clearly $f \restriction (A \restriction x) \colon \mathfrak{A} \restriction x \rightarrowtail\!\!\!\rightarrow \mathfrak{A} \restriction fx$, as desired.

(ii) → (iii). Let $g \colon \mathfrak{A} \restriction x \rightarrowtail\!\!\!\rightarrow \mathfrak{A} \restriction y$ with $x \neq y$. Say $x \nleqslant y$. Let $u = x \cdot -y$ and $v = gu$. Clearly u and v are nonzero, $u \cdot v = 0$, and $g \restriction (A \restriction u) \colon \mathfrak{A} \restriction u \to \mathfrak{A} \restriction v$.

(iii) → (i). Clearly $\mathfrak{A} \cong (\mathfrak{A} \restriction x) \times (\mathfrak{A} \restriction y) \times (\mathfrak{A} \restriction (-x \cdot -y)) \cong$ $\cong (\mathfrak{A} \restriction x) \times (\mathfrak{A} \restriction x) \times (\mathfrak{A} \restriction (-x \cdot -y))$, so (i) follows.

Corollary 1.7. *If* $\mathfrak{A}$ *is a rigid* BA, *then for every* $x \in A$, $\mathfrak{A} \restriction x$ *is rigid.*

Corollary 1.8. *If* $\mathfrak{A}$ *is a rigid* BA *and* X *is a collection of pairwise disjoint elements of* $\mathfrak{A}$, *then* $\{\mathfrak{A} \restriction x \colon x \in X\}$ *is a collection of pairwise totally different rigid* BA's.

Now we turn to the consideration of products where there are common non-trivial factors. In contrast to 1.3, any such product has many automorphisms, as is seen in 1.9. Our following results through 1.12 constitute an extension of some remarks in Rieger [13] p. 214, where there are, however, some erroneous statements.

Lemma 1.9. *For any infinite* BA $\mathfrak{A}$, $\mathrm{Aut}(\mathfrak{A} \times \mathfrak{A})$ *contains* ${}^{|A|}C_2^w$.

Proof. For each $a \in A$, let f_a be the automorphism $g^{-1}hg$, where $g\colon \mathfrak{A} \times \mathfrak{A} \rightarrowtail\!\!\!\twoheadrightarrow (\mathfrak{A} \restriction a) \times (\mathfrak{A} \restriction -a) \times (\mathfrak{A} \restriction a) \times (\mathfrak{A} \restriction -a)$ is natural, and h interchanges first and third coordinates but leaves the second and fourth fixed. Thus for any $(x, y) \in A \times A$, $f_a(x, y) = (x \cdot -a + y \cdot a,\ x \cdot a + y \cdot -a)$. If $a, b \in A$ and $a \neq b$, say $a \cdot -b \neq 0$, then $f_b(a, 0) = (a \cdot -b,\ a \cdot b)$, while $f_a(a, 0) = (0, a)$, so $f_a \neq f_b$. Clearly each f_a, $a \neq 0$, has order 2; and clearly $f_a \circ f_b = f_b \circ f_a$ for any $a, b \in A$.

In case $\mathfrak{A}$ is rigid, the automorphisms given in the proof of 1.9 constitute all automorphism of $\mathfrak{A} \times \mathfrak{A}$. (This was first noticed in de Groot [2].) In fact, let k be any automorphism of $\mathfrak{A} \times \mathfrak{A}$. Say $k(1, 0) = (a, b)$. Say $k(c, 0) = (a, 0)$. By 1.6 it is clear that $c = a$. Thus $k(a, 0) = (a, 0)$ and $k(-a, 0) = (0, b)$, so $b = -a$. Then clearly $k(0, 1) = (-a, a)$, $k(0, a) = (0, a)$, $k(0, -a) = (-a, 0)$. Hence, easily, $k = f_{-a}$.

Generalizing these considerations, we get a kind of characterization of $\mathrm{Aut}({}^I\mathfrak{A})$, $\mathfrak{A}$ rigid.

Theorem 1.10. *Let* $\mathfrak{A}$ *be a rigid* BA, I *a set with at least two elements. Assume that* $\mathfrak{A}$ *is* $|I|^+$*-complete. Then every automorphism of* ${}^I\mathfrak{A}$ *has the form* f_a, *where* $a \in {}^{I \times I}A$, $\forall i \in I(\langle a_{ij}\colon j \in I\rangle$ *is a partition of unity*), $\forall j \in I(\langle a_{ij}\colon i \in I\rangle$ *is a partition of unity*), *and for all* $x \in {}^IA$ *and all* $j \in I$, $(f_a x)_j = \sum_{i \in I} x_i \cdot a_{ij}$ (*and each such* f_a *is an automorphism of* ${}^I\mathfrak{A}$).

Proof. First we show that each f_a is an automorphism of ${}^I\mathfrak{A}$. Clearly $x \leq y \to f_a x \leq f_a y$. Assume that $f_a x \leq f_a y$. Then $x_i = \sum_{j \in I} x_i \cdot a_{ij} = \sum_{j \in I} (f_a x)_j \cdot a_{ij} \leq y_i$. Thus $x \leq y$. It follows also that f is one-one. To show that f is onto, let $y \in {}^IA$. Let $x_i = \sum_{j \in I} y_j \cdot a_{ij}$ for all $i \in I$. Then for any $j \in I$, $(f_a x)_j = \sum_{i \in I} x_i \cdot a_{ij} = \sum_{i \in I} y_j \cdot a_{ij} = y_j$, so $f_a x = y$. Hence f_a is an automorphism of ${}^I\mathfrak{A}$.

Now let $g \in \mathrm{Aut}({}^I\mathfrak{A})$. For $i, j \in I$, let $a_{ij} = (g\delta_i 1)_j$. Clearly

(1) for any $j \in I$, $\langle a_{ij}\colon i \in I\rangle$ is a partition of unity;

(2) if $i, j, k \in I$ and $j \neq k$, then $a_{ij} \cdot a_{ik} = 0$.

In fact, choose x such that $g\delta_i x = \delta_j(g\delta_i 1)_j$. By 1.6 it follows that $x = (g\delta_i 1)_j$. Thus $g\delta_i(\delta g_i 1)_j = \delta_j(g\delta_i 1)_j$, and similarly $g\delta_i(g\delta_i 1)_k = \delta_k(g\delta_i 1)_k$. Since $\delta_j(g\delta_i 1)_j \cdot \delta_k(g\delta_i 1)_k = 0$, it follows that $(g\delta_i 1)_j \cdot (g\delta_i 1)_k = 0$.

(3) for any $i \in I$, $\sum_{j \in I} a_{ij} = 1$.

This is true because, as was just shown, $g\delta_i(g\delta_i 1)_j = \delta_j(g\delta_i 1)_j$ for every $j \in I$. Hence $g\delta_i 1 = \sum_{j \in I} \delta_j(g\delta_i 1)_j = \sum_{j \in I} g\delta_i(g\delta_i 1)_j = g \sum_{j \in I} \delta_i(g\delta_i 1)_j$, and (3) follows.

(4) for any $x \in A$ and $i, j \in I$, $(g\delta_i x)_j = x \cdot (g\delta_i 1)_j$.

In fact, choose u so that $g\delta_i u = \delta_j(x \cdot (g\delta_i 1)_j)$. By 1.6 $u = x \cdot (g\delta_i 1)_j$. Thus $g\delta_i(x \cdot (g\delta_i 1)_j) = \delta_j(x \cdot (g\delta_i 1)_j)$, so

$$(g\delta_i x)_j = \Big(g \sum_{k \in I} \delta_i(x \cdot a_{ik})\Big)_j = \sum_{k \in I} (g\delta_i(x \cdot a_{ik}))_j =$$

$$= \sum_{k \in I} (\delta_k(x \cdot a_{ik}))_j = x \cdot a_{ij}\,.$$

Thus (4) holds. Finally, if $x \in {}^IA$ and $j \in I$, then

$$(gx)_j = \Big(g \sum_{i \in I} \delta_i x_i\Big)_j = \sum_{i \in I} (g\delta_i x_i)_j = \sum_{i \in I} x_i \cdot a_{ij} = (f_a x)_j\,.$$

So $g = f_a$, as desired.

Corollary 1.11. *If* $\mathfrak{A}$ *is an infinite rigid* BA *and* $2 \leq m < \omega$, *then* $|\mathrm{Aut}({}^m\mathfrak{A})| = |A|$.

For $\mathfrak{A}$ rigid and $2 \leq m < \omega$ we can give a different description of $\mathrm{Aut}({}^m\mathfrak{A})$ from 1.10:

Theorem 1.12. *Let* $\mathfrak{A}$ *be a rigid* BA, X *its Stone space. Let* m *be a positive integer greater than* 1. *Then* $\mathrm{Aut}({}^m\mathfrak{A})$ *is isomorphic to the* *subgroup of the full Cartesian power* ${}^X\mathrm{Sym}\, m$ *consisting of all continuous maps of* X *into* $\mathrm{Sym}\, m$, *the latter with the discrete topology.*

Proof. Let $g \in \mathrm{Aut}\,({}^m\mathfrak{A})$. By 1.10 and its proof, write $g = f_a$, where $a_{ij} = (g\delta_i 1)_j$ for all $i, j < m$. For $i < m$ and $\mathfrak{F} \in X$, let $(Fg)_{\mathfrak{F}} i$ be the $j < m$ such that $a_{ij} \in \mathfrak{F}$; j exists and is unique since $\langle a_{ij} : j \in I\rangle$ is a partition of unity. Now $(Fg)_{\mathfrak{F}}$ is one-one. For, suppose $(Fg)_{\mathfrak{F}} i = (Fg)_{\mathfrak{F}} k = j$ with $i \neq k$. Thus $(g\delta_i 1)_j \cdot (g\delta_k 1)_j \in \mathfrak{F}$; but this contradicts $\delta_i 1 \cdot \delta_k 1 = 0$. Hence $(Fg)_{\mathfrak{F}} \in \mathrm{Sym}\, m$.

Now we show that Fg is a continuous map of X into $\mathrm{Sym}\, m$; to do this is suffices to take any $\sigma \in \mathrm{Sym}\, m$ and show that $(Fg)^{-1}\{\sigma\}$ is open in X. Since $(Fg)^{-1}\{\sigma\} = \{\mathfrak{F} : \forall i < m (g\delta_i 1)_{\sigma i} \in \mathfrak{F}\}$, this is clear.

To show that F is a homomorphism, let $g, h \in \mathrm{Aut}\,({}^m\mathfrak{A})$, $\mathfrak{F} \in X$, and $i < m$. Say $(Fg)_{\mathfrak{F}} i = j$ and $(Fh)_{\mathfrak{F}} j = k$. Thus $(g\delta_i 1)_j \in \mathfrak{F}$ and $(h\delta_j 1)_k \in \mathfrak{F}$. Hence, using (4) in the proof of 1.10,

$$(g\delta_i 1)_j \cdot (h\delta_j 1)_k = (h\delta_j (g\delta_i 1)_j)_k \leqslant (hg\delta_i 1)_k ,$$

so $(hg\delta_i 1)_k \in \mathfrak{F}$ and hence $(Fhg)_{\mathfrak{F}} i = k$. Thus $Fhg = Fh \cdot Fg$ as desired.

Now F is one-one. For, assume that $g \in \mathrm{Aut}\,({}^m\mathfrak{A})$ is not the identity. Then by (4) in the proof of 1.10 we easily infer that $g\delta_i 1 \neq \delta_i 1$ for some $i < m$. Thus $(g\delta_i 1)_i \neq 1$, so there is an ultrafilter $\mathfrak{F}$ on $\mathfrak{A}$ with $(g\delta_i 1)_i \notin \mathfrak{F}$. Hence $(Fg)_{\mathfrak{F}} i \neq i$. Thus Fg is not the identity.

Finally, we must show that the range of F includes all continuous maps h of X into $\mathrm{Sym}\, m$. For any $\sigma \in \mathrm{Sym}\, m$, $h^{-1}\{\sigma\}$ is a closed-open subset of X, say $h^{-1}\{\sigma\} = \{\mathfrak{F} : b_\sigma \in \mathfrak{F}\}$. For $i, j < m$ let $a_{ij} = \sum_{\sigma i = j} b_\sigma$. Then

(1) if $i, j, k < m$ and $j \neq k$, then $a_{ij} \cdot a_{ik} = 0$, and $a_{ji} \cdot a_{ki} = 0$.

For, if $\sigma i = j$ and $\tau i = k$, then $h^{-1}\{\sigma\} \cap h^{-1}\{\tau\} = 0$, hence there is no $\mathfrak{F}$ with $b_\sigma, b_\tau \in \mathfrak{F}$, hence $b_\sigma \cdot b_\tau = 0$. Hence (1) holds.

(2) If $i < m$, then $\sum_{j<m} a_{ij} = 1 = \sum_{j<m} a_{ji}$.

For, if $\sum_{j<m} a_{ij} \neq 1$, then there is an $\mathfrak{F} \in X$ with $\sum_{j<m} a_{ij} \notin \mathfrak{F}$. Say $h_{\mathfrak{F}} = \tau$. Then $b_\tau \in \mathfrak{F}$ and so $\sum_{j<m} \sum_{\sigma i = j} b_\sigma \in \mathfrak{F}$, contradiction.

Thus a satisfies the conditions of 1.10. We claim that $Ff_a = h$. For, let $\mathfrak{F} \in X$, $i < m$, and $(Ff_a)_{\mathfrak{F}} i = j$. Thus $a_{ij} \in \mathfrak{F}$. Say $b_\sigma \in \mathfrak{F}$ where $\sigma i \in j$. Then $h_{\mathfrak{F}} = \sigma$, so $h_{\mathfrak{F}} i = j$. Thus, indeed $Ff_a = h$. This completes the proof.

By $\mathrm{Inv}\,(\mathfrak{A})$, where $\mathfrak{A}$ is a BA, we denote the subalgebra of $\mathfrak{A}$ constituted by all elements left fixed by every automorphism of $\mathfrak{A}$. (An equivalent condition on the element $x \in A$ is that x and $-x$ be totally different.)

Lemma 1.13. *Let* $\mathfrak{A}$ *be rigid and* I *be nonempty. Then* $\mathrm{Inv}\,({}^I\mathfrak{A}) \cong \mathfrak{A}$; *in fact,* $\mathrm{Inv}\,({}^I\mathfrak{A})$ *is the diagonal subalgebra constituted by all constant mappings from* I *into* $\mathfrak{A}$.

Proof. For each $a \in A$, let c_a be the member of ${}^IA$ such that $c_a i = a$ for all $i \in I$. Then for any $a \in A$,

(1) for any automorphism f of ${}^I\mathfrak{A}$, $fc_a = c_a$.

For, suppose $fc_a \neq c_a$, f an automorphism of ${}^I\mathfrak{A}$. Thus there is an $i \in I$ with $(fc_a)i \neq a$. Hence $(fc_a)i \nleqslant a$ or $a \nleqslant (fc_a)i$. Assume $(fc_a)i \nleqslant a$, and let $b = (fc_a)i \cdot -a$. Thus $\delta_i b \leqslant fc_a$, so $f^{-1}(\delta_i b) \leqslant c_a$. Since $b \neq 0$, there is a $j \in I$ such that $d = (f^{-1}(\delta_i b))_j \neq 0$. Thus $d \leqslant a$, $\delta_j d \leqslant f^{-1}(\delta_i b)$, and hence $f\delta_j d \leqslant \delta_i b \leqslant \delta_i(-a)$. Hence a and $-a$ are not totally different, contradicting $\mathfrak{A}$ being rigid. A similar contradiction is reached if $a \nleqslant (fc_a)i$. Hence (1) holds.

Now suppose that $x \in {}^IA$, and x is not a constant mapping from I into A. Say $i, j \in I$ and $x_i \nleqslant x_j$. Let $a = x_i \cdot -x_j$. Then $a \neq 0$, $\delta_i a \leqslant x$, and $\delta_j a \leqslant -x$, so x and $-x$ are not totally different. Hence $x \notin \mathrm{Inv}\,({}^I\mathfrak{A})$. This completes the proof.

Theorem 1.14. *Assume that* ${}^I\mathfrak{A} \cong {}^J\mathfrak{B}$ *where* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are rigid and* $I \neq 0 \neq J$. *Then* $\mathfrak{A} \cong \mathfrak{B}$; *If* I *is finite and* $\mathfrak{A}$ *is non-trivial, then* $|I| = |J|$.

Proof. It is immediate from 1.13 that the assumptions imply that $\mathfrak{A} \cong \mathfrak{B}$. Suppose also that I is finite and $\mathfrak{A}$ is non-trivial. Then ${}^I\mathfrak{A}$ has a system $\langle x^j : j \in J\rangle$ of pairwise disjoint, non-zero, isomorphic elements. We will show that this entails $|J| \leqslant |I|$. The same argument then puts $|I| \leqslant |J|$.

To obtain a contradiction, suppose that ${}^I\mathfrak{A}$ has a system $\langle x^k : 0 \leqslant k \leqslant |I|\rangle$ of pairwise disjoint, non-zero, isomorphic elements. For any fixed $m \leqslant |I|$, we can convert such a system into a similar system $\langle \bar{x}^k : 0 \leqslant k \leqslant |I|\rangle$ with $\bar{x}^k \leqslant x^k$ (for all k) and $\bar{x}^m \leqslant \delta_i 1$ for some $i \in I$. Hence in $|I| + 1$ steps we can construct such a system $\langle \bar{x}^k : k \leqslant |I|\rangle$ with the property that for each $m \leqslant |I|$, there is a (unique) $i_m \in I$ with $\bar{x}^m \leqslant \delta_{i_m} 1$. Since $\mathfrak{A}$ is rigid, $\delta_i 1$ does not contain two disjoint, non-zero, isomorphic elements; hence the function $\langle i_m : m \leqslant |I|\rangle$ is one-to-one. This is impossible, because of the cardinalities involved.

It is also easy to see that, if ${}^I\mathfrak{A} \cong {}^J\mathfrak{A}$ ($\mathfrak{A}$ not necessarily rigid), $\mathfrak{A}$ is non-trivial, $\aleph_0 \leqslant |I| \leqslant |J|$, and $|J| \geqslant \mathrm{c}\mathfrak{A}$ (in particular, if $|J| > |A|$), then $|I| = |J|$. (Use the fact that $\mathrm{c}\mathfrak{A}$ is regular.) However, we do not know whether the conclusion $|I| = |J|$ in 1.14 is generally valid.

Problem 1. If ${}^I\mathfrak{A} \cong {}^J\mathfrak{A}$ where $\mathfrak{A}$ is rigid and non-trivial, is $|I| = |J|$?

We now turn to the consideration of $\operatorname{Aut}\mathfrak{A}$ for more general $\mathfrak{A}$. The non-triviality of the center of $\operatorname{Aut}\mathfrak{A}$ is related to the existence of rigid factors of $\mathfrak{A}$ in a special fashion explicated below.

Theorem 1.15. *For any* BA $\mathfrak{A}$, *the center of* $\operatorname{Aut}\mathfrak{A}$ *is a 2-group.*

Proof. Let f be a non-trivial automorphism of $\mathfrak{A}$ of order > 2; we shall show that f is not in the center of $\operatorname{Aut}\mathfrak{A}$. First we claim.

(1) There is a non-zero $x \in A$ such that x, fx, and f^2x are pairwise disjoint.

For, as in the proof of 1.6 we can find $0 \neq y \in A$ so that $y \cdot f^2 y = 0$. It follows that $y \neq fy$. We can let $x = y \cdot -fy$ or $x = -y \cdot fy$ depending on which is non-zero, and (1) follows.

Let h be the natural isomorphism $\mathfrak{A} \rightarrowtail\!\!\!\!\twoheadrightarrow (\mathfrak{A} \upharpoonright x) \times (\mathfrak{A} \upharpoonright fx) \times \times (\mathfrak{A} \upharpoonright f^2x) \times (\mathfrak{A} \upharpoonright -x \cdot -fx \cdot -f^2x)$. Let k be the automorphism of $\operatorname{Rng} h$ which takes (a, b, c, d) to $(f^{-2}c, b, f^2a, d)$ for all $a \leqslant x$, $b \leqslant fx$, $c \leqslant f^2x$, $d \leqslant -x \cdot -fx \cdot -f^2x$, and let $g = h^{-1} \circ k \circ h$. Then $gfx = fx$ and $fgx = f^3x$ so $gfx \cdot fgx = 0$ and hence $g \circ f \neq f \circ g$, as desired.

Theorem 1.16. *For any* BA $\mathfrak{A}$ *the following conditions are equivalent.*

(i) $\operatorname{Aut}\mathfrak{A}$ *has a non-trivial center;*

(ii) *There is a non-trivial rigid* BA $\mathfrak{B}$ *and a* BA $\mathfrak{C}$ *such that* $\mathfrak{B}$ *and* $\mathfrak{C}$ *are totally different and* $\mathfrak{A} \cong \mathfrak{B} \times \mathfrak{B} \times \mathfrak{C}$.

Proof. (i) $\rightarrow$ (ii). Let σ be a member of the center of $\operatorname{Aut}\mathfrak{A}$ different from the identity. By the proof of 1.6, there is a non-zero $x \in A$ with $x \cdot \sigma x = 0$. Let f be the natural isomorphism $\mathfrak{A} \rightarrowtail\!\!\!\!\twoheadrightarrow (\mathfrak{A} \upharpoonright x) \times \times (\mathfrak{A} \upharpoonright \sigma x) \times (\mathfrak{A} \upharpoonright -x \cdot -\sigma x)$. Let $\mathfrak{B} = \mathfrak{A} \upharpoonright x$, $\mathfrak{C} = \mathfrak{A} \upharpoonright -x \cdot -\sigma x$. Thus $\mathfrak{B}$ is non-trivial and $\mathfrak{A} \cong \mathfrak{B} \times \mathfrak{B} \times \mathfrak{C}$. If $\mathfrak{B}$ is not rigid, let τ be a non-identity automorphism of $\mathfrak{B}$, say $y \cdot \tau y = 0$ with $y \neq 0$. Let ρ be the automorphism of $(\mathfrak{A} \upharpoonright x) \times (\mathfrak{A} \upharpoonright \sigma x) \times (\mathfrak{A} \upharpoonright -x \cdot -\sigma x)$ acting like τ on the first coordinate and like the identity elsewhere. Then $\sigma f^{-1}\rho f y = \sigma\tau y$ and $f^{-1}\rho f \sigma y = \sigma y$; $\sigma y \cdot \sigma\tau y = 0$ and $y \neq 0$, so $\sigma f^{-1}\rho f \neq f^{-1}\rho f \sigma$, contradiction. Thus $\mathfrak{B}$ is rigid. Also, suppose $\mathfrak{B}$ and $\mathfrak{C}$ have a common non-trivial factor; say $u \leqslant x$, $v \leqslant -x \cdot -\sigma x$ and $g\colon \mathfrak{A} \upharpoonright u \rightarrowtail\!\!\!\!\twoheadrightarrow \mathfrak{A} \upharpoonright v$, with $u \neq 0$. Let h be the natural isomorphism $\mathfrak{A} \rightarrowtail\!\!\!\!\twoheadrightarrow (\mathfrak{A} \upharpoonright u) \times (\mathfrak{A} \upharpoonright \sigma u) \times (\mathfrak{A} \upharpoonright v) \times (\mathfrak{A} \upharpoonright -u \cdot -\sigma u \cdot -v)$. Let k act like g on $\mathfrak{A} \upharpoonright u$, like g^{-1} on $\mathfrak{A} \upharpoonright v$, and like the identity on the second and fourth coordinates. Then $\sigma h^{-1}khu = \sigma v \leqslant -\sigma x$; $h^{-1}kh\sigma u = \sigma u \leqslant \sigma x$, so again $\sigma h^{-1}kh \neq h^{-1}kh\sigma$, contradiction. Hence $\mathfrak{B}$ and $\mathfrak{C}$ have no common non-trivial factors.

(ii) $\rightarrow$ (i). By 1.9, $\operatorname{Aut}(\mathfrak{B} \times \mathfrak{B})$ is non-trivial. Hence it suffices to show that each member g of $\operatorname{Aut}(\mathfrak{B} \times \mathfrak{B})$ induces an automorphism in the center of $\mathfrak{B} \times \mathfrak{B} \times \mathfrak{C}$. Let $g'(x, y, z) = (g(x, y), z)$ for any $x, y \in B$

and $z \in C$. Let h be any automorphism of $\mathfrak{B} \times \mathfrak{B} \times \mathfrak{C}$. By 1.2 we may write $h(x, y, z) = (k(x, y), lz)$ for all $x, y \in B$ and $z \in C$, where $k \in \operatorname{Aut}(\mathfrak{B} \times \mathfrak{B})$ and $l \in \operatorname{Aut} \mathfrak{C}$. By the remark after 1.9, g and k commute. Hence g' and h commute, as desired.

Corollary. 1.17. *If $\mathfrak{A}$ has no non-trivial rigid factors, then* $\operatorname{Aut} \mathfrak{A}$ *is centerless.*

There are other BA's $\mathfrak{A}$ with $\operatorname{Aut} \mathfrak{A}$ centerless, e.g., $\mathfrak{A} = {}^I\mathfrak{B}$ where $\mathfrak{B}$ is non-trivial and rigid and $3 \leqslant |I|$, as is easily seen from 1.10.

For our further results in this section we restrict attention to complete BA's. In this connection the following problem naturally occurs.

Problem 2. Characterize the automorphism groups of complete BA's among the automorphism groups of arbitrary BA's.

We may remark that the two classes of groups mentioned here do not coincide. For example, there is a BA $\mathfrak{A}$ with $\operatorname{Aut} \mathfrak{A}$ denumerable (see 3.2), but this is never the case for a complete BA, by 1.9.

The following theorem is easy to prove, using Zorn's lemma.

Theorem 1.18. *For any complete* BA $\mathfrak{A}$ *there is a unique element $a \in A$ such that $\mathfrak{A} \upharpoonright a$ is a product of rigid* BA*'s and $\mathfrak{A} \upharpoonright -a$ has no non-trivial rigid factors.*

Thus the determination of $\operatorname{Aut} \mathfrak{A}$, $\mathfrak{A}$ complete, reduces to two special cases: $\mathfrak{A}$ is a product of rigid BA's, or $\mathfrak{A}$ has no non-trivial rigid factors. We now consider the first case. Note that if $\mathfrak{A}$ is a product of rigid BA's, then the collection of rigid elements of $\mathfrak{A}$ is dense in $\mathfrak{A}$. If $\mathfrak{A}$ is complete, then the converse holds.

Lemma 1.19. *Let $\mathfrak{A}$ be a complete* BA *with at least one non-trivial rigid factor. Then there is a nonempty collection C of rigid, pairwise disjoint and isomorphic, non-zero elements of $\mathfrak{A}$ such that $\sum C$ and $-\sum C$ are totally different.*

Proof. Let y be a non-zero rigid element of $\mathfrak{A}$. By Zorn's lemma

let D be a maximal family of pairwise disjoint elements of $\mathfrak{A}$ each isomorphic to y, with $y \in D$. For each $u \in D$ let $f_u : \mathfrak{A} \upharpoonright y \rightarrowtail\!\!\!\!\twoheadrightarrow \mathfrak{A} \upharpoonright u$, where f_y is the identity. Let E be a maximal collection of pairwise disjoint elements $(d, e) \in (\mathfrak{A} \upharpoonright y) \times (\mathfrak{A} \upharpoonright - \sum D)$ such that $\mathfrak{A} \upharpoonright d \cong \mathfrak{A} \upharpoonright e$. For each $(d, e) \in E$ let $g_{de} : \mathfrak{A} \upharpoonright d \rightarrowtail\!\!\!\!\twoheadrightarrow \mathfrak{A} \upharpoonright e$. Let $z = \sum_{(d,e) \in E} d$, and set $x = y \cdot -z$. Note that $x \neq 0$ since D is maximal; and x is rigid since $x \leqslant y$. Let $C = \{f_u x : u \in D\}$. Thus C is a collection of pairwise disjoint elements of $\mathfrak{A}$ each isomorphic to x, and $x \in C$. To complete the proof it suffices to derive a contradiction from the assumptions $0 \neq v \leqslant \sum C$, $w \leqslant - \sum C$, and $h : \mathfrak{A} \upharpoonright v \rightarrowtail\!\!\!\!\twoheadrightarrow \mathfrak{A} \upharpoonright w$. Choose $u \in D$ such that $v \cdot f_u x \neq 0$. Now there are three cases.

Case 1. $\exists t \in D(h(v \cdot f_u x) \cdot t \neq 0)$. Thus $s = h(v \cdot f_u x) \cdot f_t z \neq 0$. Clearly $f_t^{-1} s$ and $f_u^{-1} h^{-1} s$ are isomorphic pairwise disjoint non-zero subelements of y, contradiction.

Case 2. $\forall t \in D(h(v \cdot f_u x) \cdot t = 0)$, but $\exists (d, e) \in E(h(v \cdot f_u x) \cdot e \neq 0)$. Again $g_{de}^{-1}(h(v \cdot f_u x) \cdot e)$ and $f_u^{-1} h^{-1}(h(v \cdot f_u x) \cdot e)$ are isomorphic pairwise disjoint non-zero subelements of y, contradiction.

Case 3. $h(v \cdot f_u x) \leqslant (- \sum D) \cdot - \sum_{(d,e) \in E} e$. This contradicts the maximality of E.

The proof is complete.

Theorem 1.20. *Let $\mathfrak{A}$ be a complete* BA *in which the rigid elements are dense. There exists a strictly increasing seguence $\langle \mathfrak{m}_\alpha : \alpha < \beta \rangle$ of non-zero cardinals, and a system $\langle \mathfrak{B}_\alpha : \alpha < \beta \rangle$ of non-trivial, pairwise totally different rigid* BA*'s, such that* $\mathfrak{A} \cong \mathrm{P}_{\alpha < \beta} {}^{\mathfrak{m}_\alpha}\mathfrak{B}_\alpha$.

Proof. An easy transfinite construction, using 1.19. See the proof of the next theorem.

The question whether the representation given by Theorem 1.20 is unique, is equivalent to Problem 1. We cannot prove it. However, we can obtain uniqueness by imposing an additional condition.

Theorem 1.21. *Let* $\mathfrak{A}$ *be a complete* BA *in which the rigid elements are dense. There are unique sequences* $\langle \mathfrak{m}_\alpha : \alpha < \beta \rangle$ *(strictly increasing,* $\mathfrak{m}_0 > 0$*) and* $\langle \mathfrak{B}_\alpha : \alpha < \beta \rangle$ *(non-trivial, pairwise totally different, rigid* BA*'s) satisfying.*

(i) $\mathfrak{A} \cong \mathop{\mathsf{P}}_{\alpha<\beta} {}^{\mathfrak{m}_\alpha}\mathfrak{B}_\alpha$;

(ii) *for each* α, *every representation* ${}^{\mathfrak{m}_\alpha}\mathfrak{B}_\alpha \cong {}^{\mathfrak{m}}\mathfrak{C} \times \mathfrak{D}$ *where* $\mathfrak{m} > 0$ *and* $\mathfrak{C}$ *is a non-trivial rigid algebra totally different from* $\mathfrak{D}$, *has* $\mathfrak{m}_\alpha \leqslant \mathfrak{m}$.

Proof. If $\mathfrak{A}$ is trivial, we must put $\beta = 0$. Let us prove the existence, assuming that $\mathfrak{A}$ is non-trivial.

Let $\mathfrak{m}_0$ be the least cardinal $\mathfrak{m}$ such that there exists a collection C as in 1.19 with $|C| = \mathfrak{m}$. Choose as C^0 any collection C as in 1.19 with $|C| = \mathfrak{m}_0$. If C^δ, $\delta < \gamma$, have been chosen so that the ΣC^δ are pairwise disjoint and each $|C^\delta| = \mathfrak{m}_0$, and if in $\mathfrak{A} \upharpoonright -\big(\sum_{\delta<\gamma} \Sigma C^\delta\big)$ there is such a collection C of power $\mathfrak{m}_0$, let C^γ be one such. In this way, we obtain a sequence $\langle C^\delta : \delta < \gamma_0 \rangle$ such that the ΣC^δ are pairwise disjoint, every C^δ satisfies 1.19 for $\mathfrak{A}$, $|C^\delta| = \mathfrak{m}_0$, and $\mathfrak{A} \upharpoonright - \big(\sum_{\delta<\gamma_0} \Sigma C^\delta\big)$ has no such collection C with $|C| \leqslant \mathfrak{m}_0$. Since $\mathfrak{A}$ is complete, the pairwise disjoint set $\bigcup_{\delta<\gamma_0} C^\delta$ gives us a decomposition

$$\mathfrak{A} \cong \mathop{\mathsf{P}}_{\delta<\gamma_0} {}^{\mathfrak{m}_0}\mathfrak{C}_\delta \times \mathfrak{D} \cong {}^{\mathfrak{m}_0}\mathfrak{C} \times \mathfrak{D}\,.$$

We can state a general proposition proved by the above argument.

(1) Let $\mathfrak{G}$ be a complete BA with a non-zero rigid element. There exists a decomposition $\mathfrak{G} \cong {}^{\mathfrak{m}}\mathfrak{C} \times \mathfrak{D}$ in which

(a) $\mathfrak{C}$ is non-trivial, rigid, and totally different from $\mathfrak{D}$;

(b) $\mathfrak{m} > 0$ is minimal for all decompositions of $\mathfrak{G}$ satisfying (a);

(c) in every decomposition $\mathfrak{D} \cong {}^{\mathfrak{n}}\mathfrak{C} \times \mathfrak{F}$ satisfying condition (a), one has $\mathfrak{n} > \mathfrak{m}$.

Moreover, (obviously) for ${}^{\mathfrak{m}}\mathfrak{C}$ as for $\mathfrak{G}$, the cardinal number $\mathfrak{m}$ is minimal. [We need this for condition 1.21. (ii).] It will follow from the remaining argument below that $\mathfrak{C}$ and $\mathfrak{D}$ are uniquely determined by $\mathfrak{G}$, and in fact that ${}^{\mathfrak{m}}\mathfrak{C}$ and $\mathfrak{D}$ are isomorphic with unique relativized algebras of $\mathfrak{G}$.

To continue with the proof, we choose $\mathfrak{B}_0, \mathfrak{A}^0$ so that statement (1) is true with $\mathfrak{m}, \mathfrak{G}, \mathfrak{C}, \mathfrak{D}$ replaced by $\mathfrak{m}_0, \mathfrak{A}, \mathfrak{B}_0, \mathfrak{A}^0$ and in fact, say, ${}^{\mathfrak{m}_0}\mathfrak{B}_0 \cong \mathfrak{A} \upharpoonright b_0$ and $\mathfrak{A}^0 = \mathfrak{A} \upharpoonright -b_0$.

If $\mathfrak{A}_0$ is non-trivial, we can apply (1) to $\mathfrak{G} = \mathfrak{G}_0$. Thus an easy transfinite construction, which uses the completeness of $\mathfrak{A}^0$ in passing over limit ordinals, produces a partition of unity in $\mathfrak{A}$, $1 = \sum_{\alpha<\beta} b_\alpha$, where $\mathfrak{A} \upharpoonright b_\alpha \cong {}^{\mathfrak{m}_\alpha}\mathfrak{B}_\alpha$; and the decomposition $\mathfrak{A} \cong \mathop{\mathsf{P}}_{\alpha<\beta} {}^{\mathfrak{m}_\alpha}\mathfrak{B}_\alpha$ satisfies all the conditions of Theorem 1.21.

For uniqueness, suppose that also $\mathfrak{A} \cong \mathop{\mathsf{P}}_{\alpha<\beta_1} {}^{\mathfrak{n}_\alpha}\mathfrak{C}_\alpha$ satisfying our conditions, say

$$f\colon \mathop{\mathsf{P}}_{\alpha<\beta} {}^{\mathfrak{m}_\alpha}\mathfrak{B}_\alpha \rightarrowtail\!\!\!\!\rightarrow \mathop{\mathsf{P}}_{\alpha<\beta_1} {}^{\mathfrak{n}_\alpha}\mathfrak{C}_\alpha\,.$$

It suffices, by symmetry, to take any $\sigma < \beta$ and find $\xi < \beta_1$ such that $\mathfrak{m}_\sigma = \mathfrak{n}_\xi$ and $\mathfrak{B}_\sigma \cong \mathfrak{C}_\xi$.

We first remark that

$$\operatorname{Inv}\Big(\mathop{\mathsf{P}}_{\alpha<\beta} {}^{\mathfrak{m}_\alpha}\mathfrak{B}_\alpha\Big) = \mathop{\mathsf{P}}_{\alpha<\beta} \operatorname{Inv}\big({}^{\mathfrak{m}_\alpha}\mathfrak{B}_\alpha\big), \quad \text{and}$$

(2)

$$f\Big(\mathop{\mathsf{P}}_{\alpha<\beta} \operatorname{Inv}\ {}^{\mathfrak{m}_\alpha}\mathfrak{B}_\alpha\Big)\Big) = \operatorname{Inv}\Big(\mathop{\mathsf{P}}_{\alpha<\beta_1} {}^{\mathfrak{n}_\alpha}\mathfrak{C}_\alpha\Big) = \mathop{\mathsf{P}}_{\alpha<\beta_1} \operatorname{Inv}\big({}^{\mathfrak{n}_\alpha}\mathfrak{C}_\alpha\big)\,.$$

(See Definition 1.1 and Lemma 1.13.)

Now let $\gamma < \beta_1$ be such that $(f\delta_\sigma 1)_\gamma \neq 0$. Then also $(f^{-1}\delta_\gamma 1)_\sigma \neq 0$. We conclude, using (2), that

$$^{\mathfrak{m}_\sigma}\mathfrak{B}_\sigma \cong \Big(\mathop{\mathsf{P}}_{\alpha<\beta_1} {}^{\mathfrak{n}_\alpha}\mathfrak{C}_\alpha\Big) \upharpoonright f\delta_\sigma 1 \cong {}^{\mathfrak{n}_\gamma}\mathfrak{C} \times \mathfrak{D}\,, \qquad \text{and}$$

$$^{\mathfrak{n}_\gamma}\mathfrak{C}_\gamma \cong \Big(\mathop{\mathsf{P}}_{\alpha<\beta} {}^{\mathfrak{m}_\alpha}\mathfrak{B}_\alpha\Big) \upharpoonright f^{-1}\delta_\gamma 1 \cong {}^{\mathfrak{m}_\sigma}\mathfrak{B} \times \mathfrak{E}\,,$$

where $\mathfrak{C}$ is non-trivial, rigid, and totally different from $\mathfrak{D}\,(\mathfrak{C} = \mathfrak{C}_\gamma \upharpoonright$ $\upharpoonright ((f\delta_\sigma 1)_\gamma)_0)$; and likewise $\mathfrak{B}, \mathfrak{E}$. Consequently, by 1.21 (ii), we have $\mathfrak{m}_\sigma \leqslant \mathfrak{n}_\gamma$; and conversely, so $\mathfrak{m}_\sigma = \mathfrak{n}_\gamma$. Moreover, since $\langle \mathfrak{n}_\alpha : \alpha < \beta_1 \rangle$ is one-to-one, at most one γ exists with $\mathfrak{m}_\sigma = \mathfrak{n}_\gamma$, or with $(f\delta_\sigma 1)_\gamma \neq 0$. We conclude that $f\delta_\sigma 1 \leqslant \delta_\gamma 1$ for the unique such γ; and conversely so $f\delta_\sigma 1 = \delta_\gamma 1$. We have γ satisfying $\mathfrak{m}_\sigma = \mathfrak{n}_\gamma$ and $^{\mathfrak{m}_\sigma}\mathfrak{B}_\sigma \cong {}^{\mathfrak{n}_\gamma}\mathfrak{C}_\gamma$. By 1.14, we also have $\mathfrak{B}_\sigma \cong \mathfrak{C}_\gamma$. That completes the proof.

Theorems 1.21, 1.2 and 1.10 yield a structure theorem for the automorphism group of a complete BA in which the rigid elements are dense. The theorem is obvious and we will not bother to formulate it. We can make some further comments on these algebras.

Corollary 1.22. *Let* $\mathfrak{A}$ *be complete and the rigid elements dense in* $\mathfrak{A}$. *Let* $\mathfrak{A} \cong \mathop{\mathsf{P}}_{\alpha<\beta} {}^{\mathfrak{m}_\alpha}\mathfrak{B}_\alpha$ *be any decomposition of* $\mathfrak{A}$ *as product of powers of pairwise totally different rigid algebras. Then* $\mathrm{Inv}(\mathfrak{A})$ *is isomorphic to* $\mathop{\mathsf{P}}_{\alpha<\beta} \mathfrak{B}_\alpha$, $\mathrm{Inv}(\mathfrak{A})$ *is rigid and, in fact, is isomorphic to* $\mathfrak{A} \upharpoonright a$ *where* a *is any maximal rigid element of* $\mathfrak{A}$.

Proof. We observed the truth of the first statement in proving 1.21. For the second, observe first that Zorn's lemma ensures the existence of maximal rigid elements in any complete BA. Let a be any such element. For each $x \leqslant a$, put

$$fx = \sum_{h \in \mathrm{Aut}\,\mathfrak{A}} hx\,;$$

and for each $u \in \mathrm{Inv}(\mathfrak{A})$, put

$$gu = u \cdot a\,.$$

Using the completeness of $\mathfrak{A}$ and the density of its set of rigid elements,

one shows easily that f and g are one-to-one, inverse functions between $\mathfrak{A} \upharpoonright a$ and $\mathrm{Inv}(\mathfrak{A})$, and that $x \leqslant y$ iff $fx \leqslant fy$ whenever $x, y \leqslant a$. Hence, $f\colon \mathfrak{A} \upharpoonright a \rightarrowtail\!\!\!\twoheadrightarrow \mathrm{Inv}(\mathfrak{A})$.

Corollary 1.23. *Let* $\mathfrak{A}$ *be any complete* BA, *and write* $\mathfrak{A} \cong \mathfrak{C} \times$ $\times \mathop{\mathsf{P}}_{\alpha<\beta} {}^{\mathfrak{m}_\alpha}\mathfrak{B}_\alpha$, *where* $\mathfrak{C}$ *has no rigid factors, and* $\mathop{\mathsf{P}}_{\alpha<\beta} {}^{\mathfrak{m}_\alpha}\mathfrak{B}_\alpha$ *is as in* 1.20. *Then* $\mathrm{Aut}\,\mathfrak{A}$ *has a non-trivial center iff* $\mathfrak{m}_\alpha = 2$ *for some* $\alpha < \beta$. *If* $\alpha < \beta$ *and* $\mathfrak{m}_\alpha = 2$, *then the center of* $\mathrm{Aut}\,\mathfrak{A}$ *is isomorphic to* $\mathrm{Aut}\,({}^2\mathfrak{B}_\alpha)$.

Corollary 1.24. *Any complete* BA $\mathfrak{A}$ *can be isomorphically represented in the form* $\mathfrak{B} \times \mathfrak{B} \times \mathfrak{C}$ *where* $\mathfrak{B}$ *is rigid,* $\mathfrak{B}$ *and* $\mathfrak{C}$ *are totally different, and* $\mathrm{Aut}\,\mathfrak{C}$ *is centerless. The center of* $\mathrm{Aut}\,\mathfrak{A}$ *is a direct factor of* $\mathrm{Aut}\,\mathfrak{A}$.

The proofs are straightforward.

We can make a similar analysis of products of homogeneous BA's; the proofs are actually easier.

Theorem 1.25. *For any complete* BA $\mathfrak{A}$ *there is a unique element* $a \in A$ *such that* $\mathfrak{A} \upharpoonright a$ *is a product of homogeneous* BA's *and* $\mathfrak{A} \upharpoonright -a$ *has no non-trivial homogeneous factors.*

If $\mathfrak{A}$ is a product of homogeneous BA's, then the collection of homogeneous elements of $\mathfrak{A}$ is dense in $\mathfrak{A}$; if $\mathfrak{A}$ is complete, the converse holds.

Theorem 1.26. *Let* $\mathfrak{A}$ *be a complete homogeneous* BA, *and let* I *be an index set with* $|I| < \mathrm{c}\mathfrak{A}$. *Then* ${}^I\mathfrak{A} \cong \mathfrak{A}$.

Theorem 1.27. *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be non-trivial, complete and homogeneous* BA's *with* ${}^I\mathfrak{A} \cong {}^J\mathfrak{B}$, where $|I| \geqslant \mathrm{c}\mathfrak{A}$. *Then* $|I| = |J|$ *and* $\mathfrak{A} \cong \mathfrak{B}$.

Proof. Obvious by the remark following 1.14.

Theorem 1.28. *Let* $\mathfrak{A}$ *be a complete* BA *in which the homogeneous elements are dense. Then there is a non-decreasing sequence* $\langle \mathfrak{m}_\alpha : \alpha < \beta \rangle$

of non-zero cardinals and a system $\langle \mathfrak{B}_\alpha : \alpha < \beta \rangle$ *of pairwise totally different, non-trivial, homogeneous* BA*'s such that* $\forall \alpha < \beta (\mathfrak{m}_\alpha > 1 \to \mathfrak{m}_\alpha \geqslant c\mathfrak{B}_\alpha)$ *and* $\mathfrak{A} \cong \mathrm{P}_{\alpha<\beta} {}^{\mathfrak{m}_\alpha}\mathfrak{B}_\alpha$. *This representation is unique in the sense that if also* $\mathfrak{A} \cong \mathrm{P}_{\alpha<\gamma} {}^{\mathfrak{n}_\alpha}\mathfrak{C}_\alpha$ *with similar conditions, then* $\beta = \gamma$, $\mathfrak{m} = \mathfrak{n}$, *and for each* $\alpha < \gamma$ *there is a permutation* σ *of* $\{\epsilon : \mathfrak{m}_\epsilon = \mathfrak{m}_\alpha\}$ *such that* $\mathfrak{B}_\epsilon \cong \mathfrak{C}_{\sigma\epsilon}$ *for each* ϵ *with* $\mathfrak{m}_\epsilon = \mathfrak{m}_\alpha$.

Proof. Given any homogeneous element a of $\mathfrak{A}$, by Zorn's lemma let C be a maximal set of pairwise disjoint elements of A each isomorphic to a, with $a \in C$. Then $\sum C$ and $-\sum C$ have no common non-trivial factors. This construction, along with 1.26, makes the existence of the representation as indicated obvious. Now suppose $f: \mathrm{P}_{\alpha<\beta} {}^{\mathfrak{m}_\alpha}\mathfrak{B}_\alpha \rightarrowtail\!\!\!\twoheadrightarrow \mathrm{P}_{\alpha<\gamma} {}^{\mathfrak{n}_\alpha}\mathfrak{C}_\alpha$ with conditions as in the theorem. It suffices by 1.27 to take any $\alpha < \beta$ and find $\epsilon < \gamma$ such that ${}^{\mathfrak{m}_\alpha}\mathfrak{B}_\alpha \cong {}^{\mathfrak{n}_\alpha}\mathfrak{C}_\epsilon$. Given $\alpha < \beta$, there is exactly one $\epsilon < \gamma$ with $(f\delta_\alpha 1)_\epsilon \neq 0$. For, if $\epsilon, \theta < \gamma$, and $(f\delta_\alpha 1)_\epsilon \neq 0 \neq (f\delta_\alpha 1)_\theta$, say $((f\delta_\alpha 1)_\epsilon)_\mu \neq 0 \neq ((f\delta_\alpha 1)_\theta)_\nu$ with $\mu < \mathfrak{n}_\epsilon$, $\nu < \mathfrak{n}_\theta$. Clearly then $\mathfrak{B}_\alpha \cong \mathfrak{C}_\epsilon$ and $\mathfrak{B}_\alpha \cong \mathfrak{C}_\theta$, so $\epsilon = \theta$. Thus $f\delta_\alpha 1 = \delta_\epsilon u$ for some u. So ${}^{\mathfrak{m}_\alpha}\mathfrak{B}_\alpha$ is a factor of some ${}^{\mathfrak{n}_\epsilon}\mathfrak{C}_\epsilon$. By symmetry the latter is a factor of some ${}^{\mathfrak{m}_\rho}\mathfrak{B}_\rho$; hence $\alpha = \rho$. It follows that ${}^{\mathfrak{m}_\alpha}\mathfrak{B}_\alpha \cong {}^{\mathfrak{n}_\epsilon}\mathfrak{C}_\epsilon$, as desired.

By 1.28 the automorphism group of a complete BA $\mathfrak{A}$ in which the homogeneous elements are dense is a direct product of the groups Aut $({}^{\mathfrak{m}}\mathfrak{B})$, $\mathfrak{B}$ homogeneous. For $\mathfrak{m} = 1$ it is known that Aut $\mathfrak{A}$ is simple when $\mathfrak{A}$ is σ-complete and homogeneous (see Anderson [1]); but the structure of Aut $\mathfrak{A}$ is not fully known.

Problem 3. Describe the automorphism groups of (complete) homogenous BA's.

Concerning Aut $({}^{\mathfrak{m}}\mathfrak{B})$ in general we have the following not very satisfactory characterization, the proof of which is straightforward

Theorem 1.29. *Let* $\mathfrak{B}$ *be a complete* BA, I *any set. Let* M *be the set of all triples* $\langle a, b, g\rangle$ *satisfying the following conditions.*

(i) $a: I \times I \to B$, *and* $\forall i \in I (\langle a_{ij} : j \in I\rangle$ *is a partition of unity in* $\mathfrak{B})$;

(ii) $b: I \times I \to B$, *and* $\forall j \in I (\langle b_{ij} : i \in I\rangle$ *is a partition of unity in* $\mathfrak{B})$;

(iii) $\forall i, j \in I \; g_{ij} : \mathfrak{B} \restriction a_{ij} \rightarrowtail\!\!\!\twoheadrightarrow \mathfrak{B} \restriction b_{ij}$.

For $\langle a, b, g\rangle \in M$, *define* $f = f_{abg} : {}^I B \rightarrowtail\!\!\!\twoheadrightarrow {}^I B$ *by setting*

$$(fx)_j = \sum_{i \in I} g_{ij}(x_i \cdot a_{ij})$$

for all $x \in {}^I B$ *and all* $j \in I$.

Then Aut $({}^I\mathfrak{B}) = \{f_{abg} : \langle a, b, g\rangle \in M\}$.

By our results concerning products of rigid BA's and products of homogeneous BA's, the problem concerning the structure of Aut $\mathfrak{A}$ for $\mathfrak{A}$ complete reduces to that problem for $\mathfrak{A}$ of special kinds – powers of rigid BA's, powers of homogeneous BA's, and complete BA's with no rigid or homogeneous factors. We have discussed Aut $\mathfrak{A}$ for the first two special kinds of $\mathfrak{A}$. We have no results concerning Aut $\mathfrak{A}$ for the third kind; the following problems remain open.

Problem 4. Does there exist a non-trivial (complete) BA without non-trivial rigid or homogeneous factors?

Problem 5. Describe Aut $\mathfrak{A}$ where $\mathfrak{A}$ is a complete BA with no rigid factors and no homogeneous factors.

We close this section by showing that any BA can be embedded in a rigid BA. McAloon [11] states without proof the stronger result that any BA can be embedded in a rigid complete BA.

A BA $\mathfrak{A}$ is *cardinality-homogeneous* if $|A| = |A \restriction x|$ for all non-zero $x \in A$.

Lemma 1.30. *Let* $\mathfrak{m}$ *be an infinite cardinal. If* $\mathfrak{A}$ *is a rigid* BA *of power* $\exp\exp\exp \mathfrak{m}$, *then* $\mathfrak{A}$ *has a cardinality-homogeneous factor of power* $> \mathfrak{m}$.

Proof. By Zorn's lemma let B be a maximal collection of peirwise disjoint non-zero cardinality-homogeneous elements of A. Clearly $\sum B = 1$. If the conclusion of the lemma fails, then $|B| > \exp \mathfrak{m}$ since $\mathfrak{A} \subseteq \mathrm{P}_{b\in B}\, \mathfrak{A} \upharpoonright b$. But there are only $\exp \mathfrak{m}$ isomorphism types of BA's of power $\leqslant \mathfrak{m}$, so two elements of B are isomorphic, contradicting $\mathfrak{A}$ rigid.

Lemma 1.31. *For any cardinal* $\mathfrak{m}$ *there is a family of* $\mathfrak{m}$ *pairwise totally different* BA's.

Proof. By Lozier [9] there are arbitrarily large rigid BA's. Hence by Lemma 1.30 there are arbitrarily large cardinality-homogeneous rigid BA's. But clearly any two cardinality-homogeneous BA's of different power are totally different, so the lemma follows.

Theorem 1.32. *Any* BA *can be embedded in a rigid* BA.

Proof. Any BA $\mathfrak{A}$ can be embedded in a product $\mathrm{P}_{i\in I}\, \mathfrak{B}_i$, where $|B_i| = 2$ for each $i \in I$. If $\langle \mathfrak{C}_i : i \in I\rangle$ is a family of non-trivial pairwise totally different rigid BA's, then $\mathrm{P}_{i\in I}\, \mathfrak{B}_i \subseteq \mathrm{P}_{i\in I}\, \mathfrak{C}_i$, and the latter is rigid by 1.3.

We can also use Lemma 1.30 to show that a subdirect product of pairwise totally different rigid BA's is not necessarily rigid. To this end, let $\mathfrak{p}$, mapping ordinals into cardinals be defined recursively by:

$$\mathfrak{p}_0 = \aleph_0 \ ;$$

$$\mathfrak{p}_{\alpha+1} = \exp \exp \exp \mathfrak{p}_\alpha \ ;$$

$$\mathfrak{p}_\lambda = \bigcup_{\alpha<\lambda} \mathfrak{p}_\alpha \quad \text{for } \lambda \text{ limit} .$$

Now let $\mathfrak{m}$ be a fixed point of $\mathfrak{p}$, i.e., $\mathfrak{p}_\mathfrak{m} = \mathfrak{m}$. For each $\alpha < \mathfrak{m}$, let $\mathfrak{A}_\alpha$ be a rigid BA such that $\mathfrak{p}_\alpha < |A_\alpha| \leqslant \mathfrak{p}_{\alpha+1}$ and $\mathfrak{A}_\alpha$ is cardinality-homogeneous; $\mathfrak{A}_\alpha$ exists by 1.30. Thus $\langle \mathfrak{A}_\alpha : \alpha < \mathfrak{m}\rangle$ is a system of pairwise totally different rigid BA's. Let $\mathfrak{B}$ be the free BA on a set of

$\mathfrak{m}$ free generators, and let $\langle b_\alpha : \alpha < \mathfrak{m}\rangle$ be an enumeration of the non-zero elements of B. Now $|A_\alpha| < \mathfrak{m}$ for all $\alpha < \mathfrak{m}$, so there is a homomorphism f_α of $\mathfrak{B}$ onto $\mathfrak{A}_\alpha$ such that $f_\alpha b_\alpha \neq 0$. Thus the system $\langle f_\alpha : \alpha < \mathfrak{m}\rangle$ induces an isomorphism of $\mathfrak{B}$ onto a subdirect product of the $\mathfrak{A}_\alpha$'s. Obviously $\mathfrak{B}$ is not rigid.

2. RIGID BA'S

As mentioned in the introduction, we shall be concerned in this section with the existence of rigid BA'S. First we shall modify the construction of Lozier [9] and use the modification to construct rigid BA's of singular powers. We could use here 1.30 instead of 2.1, but 2.1 is perhaps interesting in itself.

Lemma 2.1. *For each infinite cardinal* $\mathfrak{m}$ *there is a rigid* BA $\mathfrak{A}$ *of power* $\exp(\mathfrak{m}^+)$ *such that for every* $a \in A$ *with* $0 \neq a$, $\mathfrak{m} \leqslant |A \upharpoonright a|$

Proof. We modify the construction in [9] as follows. Let $X = X^{\mathfrak{m}^+}$, $A = A_{\mathfrak{m}^+}$. We claim that there is an injection $\varphi\colon X \to A$ such that $\varphi(\beta,\gamma) > \beta$ for all $(\beta,\gamma) \in X$, and such that $\varphi(\beta,\gamma)$ always has the form $\mathfrak{m}\cdot\delta + 1$ (ordinal operations), with $\delta \neq 0$. For, write $A = \bigcup_{\alpha<\mathfrak{m}^+} B_\alpha$, where the B_α are pairwise disjoint sets of cardinality $\mathfrak{m}^+$. For each $\alpha < \mathfrak{m}^+$ let $B'_\alpha = \{\mathfrak{m}\cdot\delta + 1 : \delta \in B_\alpha, \delta \neq 0\}$. Clearly the B'_α are pairwise disjoint sets of cardinality $\mathfrak{m}^+$. The desired φ is obtained by letting $\varphi \upharpoonright X_\alpha$ for $\alpha < \mathfrak{m}^+$ be any injection into $B'_\alpha \sim [0,\alpha]$. Clearly this modification of φ leaves the rest of the construction in [9] valid. Now

(1) if $\alpha \in \mathrm{Rng}\,\varphi$ and $\beta < \omega^\alpha$, then $|[\beta,\omega^\alpha)| = \mathfrak{m}$. For, write $\alpha = \mathfrak{m}\cdot\delta + 1$, where $\delta \neq 0$. Then $\omega^\alpha = \omega^{\mathfrak{m}\cdot\delta}\cdot\omega = \mathfrak{m}^\delta\cdot\omega$. Say $\beta < \mathfrak{m}^\delta\cdot n$, $n \in \omega$. Then $\beta < \mathfrak{m}^\delta\cdot n + \epsilon < \omega^\alpha$ for each $\epsilon < \mathfrak{m}$, and (1) follows.

Let $\mathfrak{A}$ be the BA of closed-open subsets of X. By [9], $\mathfrak{A}$ is rigid and $|A| = \exp(\mathfrak{m}^+)$. Suppose that $0 \neq a \in A$; we want to prove that $\mathfrak{m} \leqslant |A \upharpoonright a|$. First note that

(2) if $x \in X$ and $Y \in I_X$, then $\langle x, Y\rangle$ is closed-open.

For, $\langle x, Y\rangle$ is open by definition of the topology on X. Suppose $y \notin \langle x, Y\rangle$. If $y \not\leqslant x$, then $y \in \langle y, \{x\}\rangle$ and $\langle y, \{x\}\rangle \cap \langle x, Y\rangle = 0$. If $y \leqslant x$, then $y \in \langle y, 0\rangle$ and $\langle y, 0\rangle \cap \langle x, Y\rangle = 0$. Thus (2) holds.

By (2), we may assume that $a = \langle x, Y\rangle$ for some $x \in X$, $Y \in I_X$. Note that $x \in \langle x, Y\rangle$ since $a \neq 0$. Now at most one member of Y is in $X_{\varphi x}$. Choose $\beta < \omega^{\varphi x}$ so that $\{(\varphi x, \gamma): \beta \leqslant \gamma < \omega^{\varphi x}\} \cap Y = 0$. Then $\{\langle(\varphi x, \gamma), Y\rangle: \beta \leqslant \gamma < \omega^{\varphi x}\}$ is, by (1) and (2), a set of $\mathfrak{m}$ distinct members of $A \upharpoonright a$. This completes the proof.

Theorem 2.2. *If* $\mathfrak{m}$ *is an uncountable strong limit cardinal, then there is a rigid* BA *of power* $\mathfrak{m}$.

Proof. Let $\langle \mathfrak{n}_\alpha : \alpha < \operatorname{cf} \mathfrak{m}\rangle$ be a strictly increasing sequence of infinite cardinals $< \mathfrak{m}$ such that $\bigcup_{\alpha < \operatorname{cf}\mathfrak{m}} \mathfrak{n}_\alpha = \mathfrak{m}$. Now we define $\langle \mathfrak{p}_\alpha : \alpha \leqslant \operatorname{cf}\mathfrak{m}\rangle$.

$$\mathfrak{p}_0 = \mathfrak{n}_0 .$$

$$\mathfrak{p}_\alpha = \bigcup_{\beta<\alpha} \mathfrak{p}_\beta \quad \text{if } \alpha \text{ is a limit ordinal } \leqslant \operatorname{cf}\mathfrak{m} ,$$

$$\mathfrak{p}_{\alpha+1} = (\exp(\mathfrak{p}_\alpha^+))^+ \cup \mathfrak{n}_{\alpha+1} \quad \text{if } \alpha < \operatorname{cf}\mathfrak{m} .$$

Clearly $\langle \mathfrak{p}_\alpha : \alpha < \operatorname{cf}\mathfrak{m}\rangle$ is a strictly increasing sequence of cardinals $< \mathfrak{m}$ such that $\bigcup_{\alpha<\operatorname{cf}\mathfrak{m}} \mathfrak{p}_\alpha = \mathfrak{m}$. For each $\alpha < \operatorname{cf}\mathfrak{m}$ let $\mathfrak{A}_\alpha$ be a rigid BA such that, for every $a \in A$ with $a \neq 0$, $\mathfrak{p}_\alpha \leqslant |A \upharpoonright a| \leqslant \exp(\mathfrak{p}_\alpha^+)$; $\mathfrak{A}_\alpha$ exists by 2.1. Now we define $\langle \mathfrak{B}_\alpha : \alpha \leqslant \operatorname{cf}\mathfrak{m}\rangle$ and $\langle I_\alpha : \alpha \leqslant \operatorname{cf}\mathfrak{m}\rangle$ by recursion so that $\mathfrak{B}_\alpha$ is a BA, I_α is a proper ideal of $\mathfrak{B}_\alpha$, and

(1_α) if $\beta \leqslant \alpha \leqslant \operatorname{cf}\mathfrak{m}$ then $\mathfrak{B}_\beta \subseteq \mathfrak{B}_\alpha$ and $I_\beta \subseteq I_\alpha$.

Let $\mathfrak{B}_0 = \mathfrak{A}_0$, and let $I_0 = \{0\}$. If γ is a limit ordinal $\leqslant \operatorname{cf}\mathfrak{m}$ and $\mathfrak{B}_\alpha$, I_α have been defined for all $\alpha < \gamma$ so that (1_α) holds, let $\mathfrak{B}_\gamma = \bigcup_{\alpha<\gamma} \mathfrak{B}_\alpha$ and $I_\gamma = \bigcup_{\alpha<\gamma} I_\alpha$. Clearly (1_γ) holds. Now suppose that $\mathfrak{B}_\alpha$ and I_α have been defined so that (1_α) holds, where $\alpha < \operatorname{cf}\mathfrak{m}$. Let J_α be a maximal ideal of $\mathfrak{B}_\alpha$ such that $I_\alpha \subseteq J_\alpha$. We define $f_\alpha : B_\alpha \to B_\alpha \times A_{\alpha+1}$

by setting $f_\alpha x = (x, x/J_\alpha)$ for each $x \in B_\alpha$. Here x/J_α is treated as the 0 or 1 of $\mathfrak{A}_{\alpha+1}$. Clearly f_α is an isomorphism of $\mathfrak{B}_\alpha$ into $\mathfrak{B}_\alpha \times \mathfrak{A}_{\alpha+1}$. Let g_α be an isomorphism of $\mathfrak{B}_\alpha \times \mathfrak{A}_{\alpha+1}$ onto a BA $\mathfrak{B}_{\alpha+1} \supseteq \mathfrak{B}_\alpha$ such that $g_\alpha f_\alpha x = x$ for all $x \in B_\alpha$. Let $I_{\alpha+1} = \{g_\alpha(b, 0): b \in B_\alpha\}$. Clearly $I_{\alpha+1}$ is a proper ideal of $\mathfrak{B}_{\alpha+1}$ and $I_\alpha \supseteq I_{\alpha+1}$. Thus $(1_{\alpha+1})$ holds, and our construction is finished. Now for each $\alpha \leqslant \operatorname{cf}\mathfrak{m}$ the following hold:

(2_α) $\mathfrak{B}_\alpha$ is rigid;

(3_α) for every $x \in B_\alpha \sim I_\alpha$, $|B_\alpha \upharpoonright x| \geqslant \mathfrak{p}_\alpha$;

(4_α) for every $x \in B_\alpha$, $|B_\alpha \upharpoonright x| \leqslant \exp(\mathfrak{p}_\alpha^+)$;

(5_α) for every $\beta < \alpha$ and all $x \in I_\beta$, $B_\alpha \upharpoonright x \subseteq B_\beta$;

(6_α) for every $\beta < \alpha$, $I_\alpha \cap B_\beta$ is a maximal ideal of $\mathfrak{B}_\beta$.

We prove these statements by induction on α. They are clear for $\alpha = 0$. Now suppose they hold for α. By (4_α) and the choice of $\mathfrak{A}_{\alpha+1}$, $\mathfrak{B}_\alpha$ and $\mathfrak{A}_{\alpha+1}$ have no common non-trivial factors. Since $\mathfrak{B}_\alpha$ is rigid by (2_α), we infer from 1.3 that $\mathfrak{B}_\alpha \times \mathfrak{A}_{\alpha+1}$ and $\mathfrak{B}_{\alpha+1}$ are rigid. Thus $(2_{\alpha+1})$ holds. Let $x \in B_{\alpha+1} \sim I_{\alpha+1}$; say $x = g_\alpha(b, a)$ where $a \neq 0$; then $|B_{\alpha+1} \upharpoonright x| \geqslant |A_{\alpha+1} \upharpoonright a| \geqslant \mathfrak{p}_{\alpha+1}$, so $(3_{\alpha+1})$ holds. Next if $b \in B_\alpha$ and $a \in A_{\alpha+1}$, then

$$|(B_\alpha \times A_{\alpha+1}) \upharpoonright (b, a)| = |B_\alpha \upharpoonright b| \cdot |A_{\alpha+1} \upharpoonright a| \leqslant \exp(\mathfrak{p}_{\alpha+1}^+)$$

by (4_α) and the choice of $\mathfrak{A}_{\alpha+1}$. Therefore $(4_{\alpha+1})$ holds. It suffices to check $(5_{\alpha+1})$ when $\beta = \alpha$. Suppose $x \in I_\alpha$ and $y \in B_{\alpha+1} \upharpoonright x$. Thus $y \in B_{\alpha+1}$ and $y \leqslant x$. Say $y = g_\alpha(b, a)$ with $b \in B_\alpha$ and $a \in A_{\alpha+1}$. Now $g_\alpha(b, a) = y \leqslant x = g_\alpha f_\alpha x = g_\alpha(x, 0)$, so $b \leqslant x$ and $a = 0$. Thus $b \in I_\alpha$; and $b = g_\alpha f_\alpha b = g_\alpha(b, 0) = y$, so $y \in I_\alpha \subseteq B_\alpha$. Thus $(5_{\alpha+1})$ holds. To check $(6_{\alpha+1})$, it again suffices to take the case $\beta = \alpha$. Let $x \in B_\alpha$. If $x \in J_\alpha$, then $x = g_\alpha f_\alpha x = g_\alpha(x, 0) \in I_{\alpha+1}$; and if $-x \in J_\alpha$, then $-x \in I_{\alpha+1}$. Hence $(6_{\alpha+1})$ holds.

Now suppose that γ is a limit ordinal $\leqslant \operatorname{cf}\mathfrak{m}$, and (2_α) - (6_α) hold for all $\alpha < \gamma$. To check (2_γ), suppose on the contrary that $\mathfrak{B}_\gamma$ is not rigid. Say h is an automorphism of $\mathfrak{B}_\gamma$ and x an element of $\mathfrak{B}_\gamma$ such that $hx \neq x$. Choose $\alpha < \gamma$ such that x, $hx \in B_\alpha$. Since $hx \neq x$ iff $h(-x) \neq -x$, by $(6_{\alpha+1})$ we may assume that $X \in I_{\alpha+1}$. Then

$$(7) \qquad B_\gamma \upharpoonright x = B_\alpha \upharpoonright x .$$

In fact, since (5_δ) holds for all $\delta < \gamma$ it is clear that $B_\gamma \upharpoonright x = B_{\alpha+1} \upharpoonright x$. Now suppose $y \in B_{\alpha+1} \upharpoonright x$. Thus $y \in B_{\alpha+1}$ and $y \leqslant x$. Say $y = g_\alpha(b, a)$, $x = g_\alpha(b', 0)$, with $b, b' \in B_\alpha$, $a \in A_{\alpha+1}$. Now $x \in B_\alpha$, so $x = g_\alpha f_\alpha x = g_\alpha(x, x/J_\alpha)$. Hence $b' = x \in J_\alpha$. Since $y \leqslant x$, it follows that $b \leqslant b'$ and $a = 0$. Hence $b \in J_\alpha$. So $b = g_\alpha f_\alpha b = g_\alpha(b, 0) = y \in B_\alpha$. Hence (7) holds. Now it follows that $hx \in I_{\alpha+1}$. For, if $hx \notin I_{\alpha+1}$, then

$$|B_\gamma \upharpoonright hx| \geqslant |B_{\alpha+1} \upharpoonright hx| \geqslant \mathfrak{p}_{\alpha+1} \quad \text{by} \quad (3_{\alpha+1})$$

while by (7) and (4_α), $|B_\gamma \upharpoonright x| = |B_\alpha \upharpoonright x| \leqslant \exp(\mathfrak{p}_\alpha^+) < \mathfrak{p}_{\alpha+1}$. This is a contradiction, since h maps $B_\gamma \upharpoonright x$ one-one onto $B_\gamma \upharpoonright hx$. Thus, indeed, $hx \in I_{\alpha+1}$. But then by (5_δ) for $\delta < \gamma$ we have $B_\gamma \upharpoonright x = B_{\alpha+1} \upharpoonright x$ and $B_\gamma \upharpoonright hx = B_{\alpha+1} \upharpoonright hx$, and $(2_{\alpha+1})$ is contradicted. Hence (2_γ) holds after all. The other conditions (3_γ) - (6_γ) are easily checked.

In particular, $\mathfrak{B}_{\mathrm{cf}\,\mathfrak{m}}$ is rigid. Since (3_α) and (4_α) hold for all $\alpha < \mathrm{cf}\,\mathfrak{m}$, it is clear that $|B_{\mathrm{cf}\,\mathfrak{m}}| = \mathfrak{m}$. This completes the proof.

Corollary 2.3 (GCH). *For every uncountable cardinal* $\mathfrak{m}$, *there is a rigid* BA *of power* $\mathfrak{m}$.

Thus under the assumption of GCH, the cardinalities of rigid BA's are known: 1, 2, and all uncountable cardinals.

In unpublished work, B. Balcar and P. Štěpánek have proved that the existence of a rigid BA of power $\aleph_1$ is consistent with the negation of CH. Very recently, S. Shelah has shown that in fact Corollary 2.3 holds without GCH.

Now we turn to de Groot's theorem. Our extension of his theorem will follow the same general lines as his construction, except that

instead of working on the real line we use the $\mathfrak{m}$-metric spaces of Sikorski [15]. Before we begin this construction we would like to mention two problems. De Groot claims to have constructed (1) BA's with no non-trivial onto endomorphisms or one-one endomorphisms and (2) $\exp\exp\aleph_0$ BA's with no non-trivial homomorphisms onto each other. We have not been able to reconstruct these proofs, and the following problems are hence open so far as we know:

Problem 6. Is there an infinite BA with no non-trivial one-one endomorphism?

Problem 7. For which infinite cardinals $\mathfrak{m}$ do there exist BA's of power $\mathfrak{m}$ with no non-trivial onto endomorphisms?

Recall that Rieger [13] has shown the existence of BA's with no non-trivial onto endomorphisms; the cardinalities of his examples are rather large.

Returning to de Groot's construction, we begin with the following lemma, essentially obtained from [9]:

Lemma 2.4. *If* Y *and* Z *are completely regular Hausdorff spaces,* $\forall y \in Y(Y \sim \{y\}$ *is not* C^**-embedded in* $Y)$, $\forall z \in Z(Z \sim \{z\}$ *is not* C^**-embedded in* $Z)$, *and* $f\colon \beta Y \rightarrowtail\!\!\!\!\twoheadrightarrow \beta Z$, *then* $f\colon Y \rightarrowtail\!\!\!\!\twoheadrightarrow Z$.

Proof. By symmetry it is enough to show that $f^*Y \subseteq Z$. Suppose, to the contrary, that $y \in Y$ and $fy \notin Z$. We shall obtain a contradiction by showing that $Y \sim \{y\}$ is C^*-embedded in Y. Let $g \in C^*(Y \sim \{y\})$. From Gillman, Jerison [5] 9N.1 we know that $Y \sim \{y\}$ is C^*-embedded in $\beta Y \sim \{y\}$. Hence let $g^+ \in C^*(\beta Y \sim \{y\})$ be an extension of g. Now $Z \subseteq f^*(\beta Y \sim \{y\})$, so $f^*(\beta Y \sim \{y\})$ is C^*-embedded in βZ. Hence let $h \in C^*\beta Z$ be an extension of $g^+ \circ f^{-1} \upharpoonright f^*(\beta Y \sim \{y\})$. Then $h \circ f \upharpoonright Y$ is the desired extension of g.

Lemma 2.5. *Let* Y *be a Hausdorff space. Assume that* $y \in Y$ *and there is an infinite cardinal* $\mathfrak{m}$ *and a sequence* $\langle U_\alpha\colon \alpha < \mathfrak{m}\rangle$ *of closed-open subsets of* Y *such that:*

(i) *if* $\alpha < \beta < \mathfrak{m}$ *then* $U_\alpha \supset U_\beta$;

(ii) *if* $\lambda < \mathfrak{m}$ *is a limit ordinal, then* $U_\lambda = \bigcap_{\alpha<\lambda} U_\alpha$;

(iii) $\{U_\alpha: \alpha < \mathfrak{m}\}$ *is a neighborhood base for* y.

Then $Y \sim \{y\}$ *is not* C^**-embedded in* Y.

Proof. We may assume that $U_0 = Y$. Clearly then for every $x \in Y \sim \{y\}$ there is a unique $\alpha = \alpha_x$ such that $x \in U_\alpha \sim U_{\alpha+1}$. Now for any $x \in Y \sim \{y\}$ let $fx = 1$ if α_x is even, $fx = 0$ if α_x is odd. Clearly $f \in C^*(Y \sim \{y\})$. Suppose f extends to $f^+ \in C^*Y$. We may assume that $f^+y = 0$. Let V be a neighborhood of y such that $0 \notin f^{+*}V$. Say $U_\alpha \subseteq V$. Choose β odd, $\alpha < \beta < \mathfrak{m}$. Then $0 \in f^{+*}U_\beta \subseteq f^{+*}V$, contradiction.

For some of the following, see Sikorski [15]. A $G_\mathfrak{m}$*-set* is a set which is the intersection of $\leqslant \mathfrak{m}$ open sets. If X is a space of weight $\leqslant \mathfrak{m}$, then X has $\leqslant \exp \mathfrak{m}$ open sets, also $\leqslant \exp \mathfrak{m}$ $G_\mathfrak{m}$-sets, and it has a dense subset of power $\leqslant \mathfrak{m}$; and any subspace of X has weight $\leqslant \mathfrak{m}$. We shall consider below the notion of an $\mathfrak{m}$-metric space from [15]. We will always take the value group to be the additive group of some ordered field. Recall that for such an ordered field there is a strictly decreasing sequence $\langle \epsilon_\alpha : \alpha < \mathfrak{m} \rangle$ of positive elements coinitial in the set of all positive elements.

If X and Y are $\mathfrak{m}$-metric spaces with values in an ordered field B, $A \subseteq X$, and $f\colon A \to Y$ is continuous, we let

$$A_f = \{p \in X\colon \text{ for every positive } \epsilon \in B \text{ there is a positive } \delta \in B \text{ such that for all } x, y \in S_\delta p \cap A \text{ we have } \rho(fx, fy) < \epsilon\}.$$

Here $S_\delta p = \{z \in X\colon \rho(z, p) < \delta\}$. Clearly $A \subseteq A_f$. The following three lemmas are proved just as for the (classical) metric spaces.

Lemma 2.6. A_f *is a* $G_\mathfrak{m}$*-set.*

Lemma 2.7. *In any* $\mathfrak{m}$*-metric space, any closed set* F *is a* $G_\mathfrak{m}$*-set.*

Lemma 2.8. *Let* X *and* Y *be* $\mathfrak{m}$*-metric spaces,* Y *complete, values of* X *and* Y *in* B, $A \subseteq X$, $f\colon A \to Y$ *continuous. Then* f *can be extended continuously to the* $G_\mathfrak{m}$*-set* $A_f \cap \mathrm{cl}\, A$.

For some of the notation in the next lemma see [1]. A *continuous* $\mathfrak{n}$*-displacement* is a continuous function which is a displacement of order $\mathfrak{n}$.

Lemma 2.9. *Let* M *be a complete* $\mathfrak{m}$*-metric space, with* $|M| = \exp \mathfrak{m}$ *and with the weight of* $M \leqslant \mathfrak{m}$. *Let* $\langle K_\alpha : \alpha < \exp \mathfrak{m} \rangle$ *be a family of subsets of* M, *each* K_α *of power* $\exp \mathfrak{m}$. *Then there is a family* $\langle F_\alpha : \alpha < \exp \exp \mathfrak{m} \rangle$ *of subsets of* M *such that*

(i) $|F_\alpha \sim F_\beta| = \exp \mathfrak{m}$ if $\alpha \neq \beta$;

(ii) *no* F_α *admits any continuous* $(\exp \mathfrak{m})$*-displacement into itself or another* F_β;

(iii) $|F_\alpha \cap K_\beta| = \exp \mathfrak{m} = |K_\beta \sim F_\alpha|$ *for all* $\alpha < \exp \exp \mathfrak{m}$, $\beta < \exp \mathfrak{m}$.

The proof of this lemma is a straightforward generalization of the proof of Theorem 1 of [2]. At the appropriate place in this proof, Lemma 2.8 is used.

Lemma 2.10. *Let* X *be a Hausdorff space,* $|X| = \mathfrak{n}$. Suppose $Y, Z \subseteq X$ *and* $f\colon Y \rightarrowtail\!\!\!\!\twoheadrightarrow Z$ *is continuous,* f *not the identity on* Y. *Assume that for every* $y \in Y$, *each neighborhood of* y *contains* $\mathfrak{n}$ *points of* Y. *Then* f *is an* $\mathfrak{n}$*-displacement.*

Proof. Choose $a \in Y$, $b \in Z$ with $a \neq b$ and $fa = b$. Let U and V be disjoint open neighborhoods of a and b respectively. Then $W = U \cap f^{-1*}V$ is an open neighborhood of a, hence $|W| = \mathfrak{n}$, and $W \cap f^*W \subseteq U \cap V = 0$, as desired.

Now we turn to the second main result of this section. The result has previously been established by Ehrenfeucht (unpublished).

Theorem 2.11. *Let* $\mathfrak{m}$ *be a regular cardinal such that* $\forall \mathfrak{n} < \mathfrak{m} (\exp \mathfrak{n} \leqslant \mathfrak{m})$. *Then there are exactly* $\exp \exp \mathfrak{m}$ *isomorphism types* *of rigid* BA*'s of power* $\exp \mathfrak{m}$.

Proof. By [2] we may assume that $\mathfrak{m} > \aleph_0$. The space $\mathfrak{D}_{\mathfrak{m}}$ of [15] (denoted by $\mathfrak{D}_\mu$ in [13], where $\mathfrak{m} = \aleph_\mu = \omega_\mu$) is a complete $\mathfrak{m}$-metric space of power $\exp \mathfrak{m}$ and weight $\mathfrak{m}$. $\mathfrak{D}_{\mathfrak{m}}$ is simply ${}^{\mathfrak{m}}2$ with a suitable $\mathfrak{m}$-metric. Let E be the collection of $f \in {}^{\mathfrak{m}}2$ which are not eventually 1, i.e., $E = \{f \in {}^{\mathfrak{m}}2 \colon \forall \alpha < \mathfrak{m} \exists \beta > \alpha (f\beta = 0)\}$. Clearly $|E| = 2^{\mathfrak{m}}$. Let ${}^{\mathfrak{m}}2$ be lexicographically ordered, and let $\langle K_\alpha \colon \alpha < \exp \mathfrak{m} \rangle$ enumerate all closed intervals $[f, g]$, where $f, g \in E$ and $f < g$. For $f, g \in {}^{\mathfrak{m}}2$, $f \neq g$, let η_{fg} be the least $\alpha < \mathfrak{m}$ such that $f\alpha \neq g\alpha$. Note:

(1) $|[f, g]| = \exp \mathfrak{m}$ whenever $f, g \in E$ and $f < g$.

In fact, let $\alpha = \eta_{fg}$. Thus $f\alpha = 0$, $g\alpha = 1$. Now since $f \in E$, there is a $\beta > \alpha$ with $f\beta = 0$. If h is any member of ${}^{\mathfrak{m}}2$ such that $h \restriction \beta = f \restriction \beta$ while $h\beta = 1$, then $f < h < g$. Thus (1) follows. Now apply Lemma 2.9 to get a family $\langle F_\alpha \colon \alpha < \exp \exp \mathfrak{m} \rangle$ with the indicated properties. Next we note:

(2) $[f, g] \cap F_\alpha$ is a closed-open subset of F_α whenever $\alpha < \exp \exp \mathfrak{m}$, $f, g \in E \sim F_\alpha$, and $f < g$.

In fact, to show that $[f, g] \cap F_\alpha$ is open in F_α, let $h \in [f, g] \cap F_\alpha$. Let $\beta = \eta_{fh} \cup \eta_{hg}$. For any $k \in S_{1/\beta} h \cap F_\alpha$, $k \neq h$, we have $\beta < \eta_{hk}$, and hence $k \in [f, g] \cap F_\alpha$. Therefore $[f, g] \cap F_\alpha$ is open in F_α. To show that it is closed, assume that $h \in Cl([f, g] \cap F_\alpha) \cap F_\alpha$. Let $\beta = \eta_{fh} \cup \eta_{hg}$. Choose $k \in S_{1/\beta} h \cap [f, g] \cap F_\alpha$. Then clearly $h \in [f, g] \cap F_\alpha$. Thus (2) holds.

For each $\alpha < \exp \exp \mathfrak{m}$ let $\mathfrak{A}_\alpha$ be the BA of closed-open subsets of F_α.

(3) $|A_\alpha| = \exp \mathfrak{m}$ for each $\alpha < \exp \exp \mathfrak{m}$.

In fact, each F_α has weight $\leqslant \mathfrak{m}$, so $|A_\alpha| \leqslant \exp \mathfrak{m}$. On the other hand $\{[f, g] \cap F_\alpha \colon f, g \in E \sim F_\alpha, f < g\}$ is, by (2), a collection of members of A_α. There are $\exp \mathfrak{m}$ members of $E \sim F_\alpha$. If $(f, g) \neq (h, k)$ as ordered pairs, then $[f, g] \cap F_\alpha \neq [h, k] \cap F_\alpha$. Indeed, say $g < k$. Then

$[f, g] \cap F_\alpha \cap [g, k] = 0$, but $[h, k] \cap F_\alpha \cap [g, k] \neq 0$ by our choice of the F_α's. Other possible situations regarding the intervals $[f, g]$ and $[h, k]$ are treated similarly.

Now we turn to the proof that the $\mathfrak{A}_\alpha$'s are rigid.

(4) if $f \in F_\alpha$, then each neighborhood of f contains $\exp \mathfrak{m}$ points of F_α.

In fact, let U be any neighborhood of f. Say $S_{1/\alpha} f \subseteq U$. Let $g = f \restriction (\alpha + 1) \cup \langle 0 \colon \beta \in \mathfrak{m} \sim (\alpha + 1) \rangle$ and $h = f \restriction (\alpha + 1) \cup \{(\alpha + 1, 1)\} \cup \langle 0 \colon \beta \in \mathfrak{m} \sim (\alpha + 2) \rangle$. Then $g, h \in E$, $g < h$, and $[g, h] \cap F_\alpha \subseteq S_{1/\alpha} f \cap F_\alpha \subseteq U \cap F_\alpha$. Furthermore, $|[g, h] \cap F_\alpha| = \exp \mathfrak{m}$ by our choice of the F_α's. Hence (4) holds. Therefore by (4), Lemma 2.10, and our choice of the F_α's,

(5) no F_α admits any one-one continuous map into another F_β or any non-identity one-one continuous map into itself.

Now by [15] (viii) and (vi), each F_α is a normal topological space with a base of closed-open sets. Hence βF_α is a Stone space. We shall now prepare to apply Lemmas 2.4 and 2.5. Let $f \in F_\alpha$. We shall define a sequence $\langle U_\beta \colon \beta < \mathfrak{m} \rangle$ of closed-open neighborhoods of f in F_α. Let $U_0 = F_\alpha$. For β a limit ordinal $< \mathfrak{m}$, let $U_\beta = \bigcap_{\gamma < \beta} U_\gamma$. Now suppose U_β has been defined. Then $U_\beta \cap S_{1/\beta} f$ is a neighborhood of f; say $S_{1/\gamma} f \subseteq U_\beta \cap S_{1/\beta} f$, $\gamma \geqslant \beta$. Let g and h be members of E with $\gamma < \eta_{fg} \cap \eta_{fh}$ and $\eta_{gh} > \gamma$. By choice of F_α, $|[g, h] \cap F_\alpha| = 2^{\mathfrak{m}}$. Hence we may choose $k \in ([g, h] \cap F_\alpha) \sim \{f\}$. Let $\delta = \eta_{fk}$, and choose $U_{\beta+1}$ to be a closed-open set $\subseteq U_\beta \cap S_{1/\delta} f$. Note that $\gamma < \delta$, and hence $U_{\beta+1} \subseteq S_{1/\beta} f$. Note that $k \in U_\beta \sim U_{\beta+1}$. This completes the construction of $\langle U_\beta \colon \beta < \mathfrak{m} \rangle$. If V is any neighbourhood of f, say $S_{1/\beta} f \subseteq V$; then $U_{\beta+1} \subseteq V$.

Thus the hypothesis of 2.5 is satisfied. By 2.5, 2.4, (5), and duality, the theorem follows.

Of course, Theorem 2.11 leaves several problems open.

Problem 8. Assuming GCH, how many isomorphism types of rigid BA's of power $\mathfrak{m}^+$, $\mathfrak{m}$ singular, are there? In particular, how many are there of power $\aleph_{\omega+1}$?

Problem 9. If $\mathfrak{m}$ is singular, how many rigid BA's of power $\mathfrak{m}$ are there, up to isomorphism?

3. CARDINALITY OF AUTOMORPHISM GROUPS

The general question we consider in this section is the relationship between the cardinalities of $\mathfrak{A}$ and $\operatorname{Aut}\mathfrak{A}$. The case $\operatorname{Aut}\mathfrak{A}$ finite essentially reduces to considerations about rigid BA's because of the result of de Groot and McDowell quoted in the introduction. Thus the question to which we address ourselves is: given $\mathfrak{m}, \mathfrak{n} \geqslant \aleph_0$, is there a BA $\mathfrak{A}$ of power $\mathfrak{m}$ with $|\operatorname{Aut}\mathfrak{A}| = \mathfrak{n}$? Obviously $|\operatorname{Aut}\mathfrak{A}| \leqslant \exp|A|$. This bound can actually be attained. For example, for $\mathfrak{A}$ the BA of finite and cofinite subsets of $\mathfrak{m}$ we clearly have $|A| = \mathfrak{m}$ and $|\operatorname{Aut}\mathfrak{A}| = \exp\mathfrak{m}$.

We begin with our strongest theorem. The other results in this section are corollaries of this theorem or treat special problems suggested by it.

Theorem 3.1. *Let* $\aleph_0 \leqslant \mathfrak{n} \leqslant \mathfrak{m}$. *There exists a* BA $\mathfrak{A} \subseteq \mathfrak{S}\mathfrak{m}$ *with* $|A| = \exp\mathfrak{m}$ *such that* $\mathfrak{S}_{<\mathfrak{n}}\mathfrak{m} \subseteq \mathfrak{A}$ *and* $\operatorname{Aut}\mathfrak{A}$ *is naturally isomorphic to* $\operatorname{Sym}(\mathfrak{m}, \mathfrak{n})$.

Proof. Let $\langle f_\xi : \xi < \exp\mathfrak{m}\rangle$ be an enumeration of $\operatorname{Sym}\mathfrak{m} \sim \operatorname{Sym}(\mathfrak{m}, \mathfrak{n})$. For each $\mathfrak{p} \leqslant \mathfrak{m}$ let $\equiv_\mathfrak{p}$ be the congruence relation associated with the ideal $\mathfrak{S}_{<\mathfrak{p}}\mathfrak{m}$ of $\mathfrak{S}\mathfrak{m}$: $X \equiv_\mathfrak{p} Y$ iff $X \Delta Y$ (the symmetric difference) is in $\mathfrak{S}_{<\mathfrak{p}}\mathfrak{m}$. Now by Hausdorff [6], let $\langle X_\alpha : \alpha < \exp\mathfrak{m}\rangle$ be a system of subsets of $\mathfrak{m}$, independent modulo $\mathfrak{S}_{<\mathfrak{m}}\mathfrak{m}$: thus $\langle X_\alpha/\mathfrak{S}_{<\mathfrak{m}}\mathfrak{m} : \alpha < \exp\mathfrak{m}\rangle$ freely generates a subalgebra of $\mathfrak{S}\mathfrak{m}/\mathfrak{S}_{<\mathfrak{m}}\mathfrak{m}$.

We now construct by transfinite recursion two sequences $\langle Y_\alpha : \alpha < \exp\mathfrak{m}\rangle$ and $\langle U_\alpha : \alpha < \exp\mathfrak{m}\rangle$; each Y_α will be a subset of $\mathfrak{m}$, and each U_α a subset of $\exp\mathfrak{m}$ with $|U_\alpha| \leqslant \aleph_0 + |\alpha|$. We shall ensure that

(1) for each $\beta < \exp\mathfrak{m}$, $\langle Y_\alpha : \alpha \leqslant \beta\rangle \cup \langle X_\alpha : \beta < \alpha < \exp\mathfrak{m}$ and $\alpha \notin U_\beta\rangle$ is independent modulo $\mathfrak{S}_{<\mathfrak{m}}\mathfrak{m}$; and furthermore, for all $\xi \leqslant \beta$, $f_\xi^* Y_\xi \notin \operatorname{Sg}(\{Y_\alpha : \alpha \leqslant \beta\} \cup \mathfrak{S}_{<\mathfrak{n}}\mathfrak{m})$.

Let $\gamma < \exp\mathfrak{m}$ and Y_α, U_α already defined for all $\alpha < \gamma$, so that (1) is true for each $\beta < \gamma$. We determine Y_γ, U_γ as follows. To begin, put $U^\gamma = \bigcup_{\alpha<\gamma} U_\alpha \cup (\gamma + 1)$. Then clearly

(1.1) $\langle Y_\alpha : \alpha < \gamma\rangle \cup \langle X_\alpha : \alpha \in \exp\mathfrak{m} \sim U^\gamma\rangle$ is independent modulo $\mathfrak{S}_{<\mathfrak{m}}\mathfrak{m}$; $|U^\gamma| \leqslant \aleph_0 + |\gamma|$; and for all $\xi < \gamma$, $f_\xi^* Y_\xi \notin \operatorname{Sg}(\{Y_\alpha : \alpha < \gamma\} \cup \cup \mathfrak{S}_{<\mathfrak{n}}\mathfrak{m})$.

Let us prove:

(1.2) Let δ_0, δ_1 be the least two ordinals in $\exp\mathfrak{m} \sim U^\gamma$. One of the fifteen elements different from $\mathfrak{m}$ in the set $\operatorname{Sg}(X_{\delta_0}, X_{\delta_1})$ contains a set $Z = Z_0 \cup Z_1$ (disjoint union) such that $|Z| = \mathfrak{n}$ and $f_\gamma^* Z_0 = Z_1$.

Indeed, since $f_\gamma \notin \operatorname{Sym}(\mathfrak{m}, \mathfrak{n})$, there are disjoint sets C, $D \subseteq \mathfrak{m}$ with $|C| = \mathfrak{n}$ and $f_\gamma^* C = D$ (easily verified). Since $X_{\delta_0}, X_{\delta_1}$ are independent, they generate four nonzero atoms in $\operatorname{Sg}(X_{\delta_0}, X_{\delta_1})$. One of the atoms must include an $\mathfrak{n}$-element set $C' \subseteq C$. One of the atoms must contain an $\mathfrak{n}$-element set $D' \subseteq f_\gamma^* C'$. Then we can put $Z_0 = f_\gamma^{-1*} D'$, $Z_1 = D'$. So (1.2) is proved.

Now we take $Z = Z_0 \cup Z_1$ as in (1.2). Also we let $\bar{X} \in \operatorname{Sg}(X_{\delta_0}, X_{\delta_1})$, $Z \subseteq \bar{X} \neq \mathfrak{m}$. We are going to take, for some $\delta \in \exp\mathfrak{m} \sim U^\gamma \sim \{\delta_0, \delta_1\}$ and some $S \subseteq Z_0$,

(1.3) $Y_\gamma = (X_\delta \sim Z) \cup S$; $\quad U_\gamma = U^\gamma \cup \{\delta, \delta_0, \delta_1\}$.

Of course, we want δ and S so that (1) will be true with $\beta = \gamma$.

The "independence" assertion in (1) will hold true whatever the choice of δ, S subject to the above. In words, the argument is simply that Y_γ and X_δ agree on the non-zero set $\sim \bar{X} \in \operatorname{Sg}(X_{\delta_0}, X_{\delta_1})$, hence non-trivial relations (modulo $\mathfrak{S}_{<\mathfrak{m}}\mathfrak{m}$) among $\langle Y_\alpha : \alpha \leqslant \gamma\rangle \cup \langle X_\alpha : \alpha \in \exp\mathfrak{m} \sim U_\gamma\rangle$ imply non-trivial relations (modulo $\mathfrak{S}_{<\mathfrak{m}}\mathfrak{m}$) among $\langle Y_\alpha : \alpha < \gamma\rangle \cup \cup \langle X_\alpha : \alpha \in \exp\mathfrak{m} \sim U^\gamma\rangle$ but these are ruled out by (1.1). This argument can be made precise, but we won't bother.

To assure the second condition of (1) for $\beta = \gamma$, we first prove

(1.4) Let $\xi < \gamma$, and $A, B \in \mathrm{Sg}\{Y_\alpha : \alpha < \gamma\}$. There is at most one $\delta \in \exp \mathfrak{m} \sim U^\gamma \sim \{\delta_0, \delta_1\}$ such that for some $S \subseteq Z_0$, and for $Y = (X_\delta \sim Z) \cup S$, we have $f_\xi^* Y_\xi \equiv_{\mathfrak{n}} (A \cap Y) \cup (B \cap \sim Y)$.

For if not, we have say $Y^i = (X_{\delta^i} \sim Z) \cup S^i$ and $\delta^i \in \exp \mathfrak{m} \sim U^\gamma \sim \{\delta_0, \delta_1\}$, $S^i \subseteq Z_0$ (for $i = 1, 2$) with $\delta^1 \neq \delta^2$, and

$$f_\xi^* Y_\xi \equiv_{\mathfrak{n}} (A \cap Y^i) \cup (B \cap \sim Y^i) \qquad \text{for} \qquad i = 1, 2 .$$

Note that the symmetric difference of the right hand sets (for $i = 1, 2$) in the above formulas includes the set $\sim Z \cap (A \,\Delta\, B) \cap (X_{\delta^1} \,\Delta\, X_{\delta^2})$, and hence includes $\sim \bar{X} \cap (A \,\Delta\, B) \cap (X_{\delta^1} \,\Delta\, X_{\delta^2})$, which is hence $\equiv_{\mathfrak{n}} 0$. By 1.1, "independence" modulo $\mathfrak{S}_{<\mathfrak{n}}\mathfrak{m}$, that is impossible unless $A \,\Delta\, B \in \mathrm{Sg}\{Y_\alpha : \alpha < \gamma\}$ is 0. That in turn means $A = B$ and

$$f_\xi^* Y_\xi \equiv_{\mathfrak{n}} (A \cap Y^1) \cup (A \cap \sim Y^1) = A ,$$

which implies that $f_\xi^* Y_\xi \in \mathrm{Sg}(\{Y_\alpha : \alpha < \gamma\} \cup \mathfrak{S}_{<\mathfrak{n}}\mathfrak{m})$, contradicting (1.1). Statement (1.4) is proved.

Now there are at most $\aleph_0 + |\gamma|$ triples $\langle \xi, A, B \rangle$ as in (1.4), and $\exp \mathfrak{m} \sim U^\gamma \sim \{\delta_0, \delta_1\}$ has the power $\exp \mathfrak{m}$. So we are enabled to *choose* δ as the least member of $\exp \mathfrak{m} \sim U^\gamma \sim \{\delta_0, \delta_1\}$ such that there are no ξ, A, B, S satisfying the formulas of (1.4). Now, however we choose S and define Y_γ by (1.3), condition (1) will hold for $\beta = \gamma$ and all $\xi < \beta$. We claim that one of the choices $S = 0$ or $S = Z_0$ will satisfy (1) for $\beta = \gamma = \xi$. Assuming otherwise, we shall get a contradiction.

So let $Y^1 = X_\delta \sim Z$ and $Y^2 = (X_\delta \sim Z) \cup Z_0$, and suppose that $f_\gamma^* Y^i \in \mathrm{Sg}(\{Y_\alpha : \alpha < \gamma\} \cup \{Y^i\} \cup \mathfrak{S}_{<\mathfrak{n}}\mathfrak{m})$ for $i = 1, 2$. This means that there exist $A^i, B^i \in \mathrm{Sg}\{Y_\alpha : \alpha < \gamma\}$ (for $i = 1, 2$) such that

(I) $f_\gamma^* Y^i \equiv_{\mathfrak{n}} (A^i \cap Y^i) \cup (B^i \cap \sim Y^i)$.

We intersect both sides of these relations by $\sim \bar{X}$, noting that

$(f_\gamma^* Y^1) \cap \sim \bar{X} = (f_\gamma^* Y^2) \cap \sim \bar{X}$, $Y^1 \sim \bar{X} = Y^2 \sim \bar{X} = X_\delta \sim \bar{X}$, and (consequently) $\sim Y^1 \cap \sim \bar{X} = \sim Y^2 \cap \sim \bar{X} = \sim X_\delta \cap \sim \bar{X}$; and we get

$$(A^1 \cap X_\delta \sim \bar{X}) \cup (B^1 \cap \sim X_\delta \cap \sim \bar{X}) \equiv_{\mathfrak{n}}$$
$$(A^2 \cap X_\delta \sim \bar{X}) \cup (B^2 \cap \sim X_\delta \cap \sim \bar{X}) .$$

Since $\langle Y_\alpha : \alpha < \gamma \rangle \cup \langle X_\delta, X_{\delta_0}, X_{\delta_1} \rangle$ are free modulo $\mathfrak{S}_{<\mathfrak{m}}\mathfrak{m}$, this is actually an equality, and it implies $A^1 = A^2$, $B^1 = B^2$. [In the free algebra, one can map endomorphically $\bar{X}$ to 0, X_δ to $\mathfrak{m}$ and all Y_α $(\alpha < \gamma)$ to themselves – then the two sides go to $A^1 = A^2$. We use that $\bar{X} \in \mathrm{Sg}(X_{\delta_0}, X_{\delta_1})$ is distinct from $0, \mathfrak{m}$. Likewise $B^1 = B^2$.]

Now if we intersect (I) (for $i = 1$) with Z_1 we obtain $0 \equiv_{\mathfrak{n}} B^1 \cap Z_1$. If we intersect (I) (for $i = 2$) with Z_1 we obtain $Z_1 \equiv_{\mathfrak{n}} B^2 \cap Z_1$. But $B^1 = B^2$ and $0 \not\equiv_{\mathfrak{n}} Z_1$, so we have our contradiction.

Having succeeded in our construction, we define $\mathfrak{A}$ as the subalgebra of $\mathfrak{S}\mathfrak{m}$ generated by $\{Y_\alpha : \alpha < \exp \mathfrak{m}\} \cup \mathfrak{S}_{<\mathfrak{n}}\mathfrak{m}$. All properties required by Theorem 3.1 follow trivially from statement (1). Note that $\mathfrak{A}/\mathfrak{S}_{<\mathfrak{n}}\mathfrak{m}$ is a free algebra of power $\exp \mathfrak{m}$.

Now we give a few special cases and generalizations of this theorem.

Corollary 3.2. *There is a* BA *of power* $2^{\aleph_0}$ *with automorphism group of power* $\aleph_0$.

It was first shown by J o n s s o n (unpublished) that there exists a BA with automorphism group of power $\aleph_0$. His algebra has a large cardinality.

Corollary 3.3. *If* $\mathfrak{m} = 2^{\mathfrak{n}}$ *for some* $\mathfrak{n}$ *or if* $\mathfrak{m}$ *is an uncountable strong limit cardinal, then there is a* BA *of power* $\mathfrak{m}$ *with exactly* $\aleph_0$ *automorphisms.*

Proof. By [9], 2.2, 1.4, and 3.1.

Corollary 3.4 (GCH). *If* $\aleph_0 \leqslant \mathfrak{n} \leqslant \mathfrak{m}^+ > \aleph_1$, *then there is a* BA *of power* $\mathfrak{m}$ *with automorphism group of power* $\mathfrak{n}$.

Proof. If $\mathfrak{n} = \mathfrak{m}^+$ we may let $\mathfrak{A}$ be the BA of finite and cofinite subsets of $\mathfrak{m}$. If $\mathfrak{m} = \mathfrak{n}$, we use 2.3 and 1.11. Now assume that $\mathfrak{m} > \mathfrak{n}$. By 3.1 let $\mathfrak{A}$ be an atomic BA of power $\mathfrak{n}^+$ with $\mathrm{Aut}\,\mathfrak{A} \cong \mathrm{Sym}\,(\mathfrak{n}, \aleph_0)$, and let $\mathfrak{B}$ be atomless and rigid of power $\mathfrak{m}$; then $\mathfrak{A} \times \mathfrak{B}$ is the desired algebra.

Corollary 3.5. *For any* $\mathfrak{n} \geqslant \aleph_0$ *there is a* BA $\mathfrak{A}$ *such that* $|\mathrm{Aut}\,\mathfrak{A}| = \mathfrak{n}$.

In the remainder of the paper we give some results showing a few ways in which 3.1-3.5 cannot be improved. We aim first for 3.10, relevant to 3.2.

Theorem 3.6. *If* $\mathfrak{A}$ *has infinitely many atoms and* $\mathrm{MA}_{|A|}$ *holds, then* $\mathrm{Sym}\,\omega$ *can be isomorphically embedded in* $\mathrm{Aut}\,\mathfrak{A}$.

Proof. We shall apply Theorem 2.2 of Martin, Solovay [10]. Let $a\colon \omega \rightarrowtail \mathrm{At}\,\mathfrak{A}$. Let F be an ultrafilter on $\mathfrak{A}$ such that $x \in F$ whenever $\mathrm{At}\,(-x) \cap \mathrm{Rng}\,a$ is finite. Set $B = \{J \subseteq \omega\colon$ for some $x \in F$, $J = \{i\colon a_i \leqslant x\}\}$ and $C = \{J \subseteq \omega\colon$ for some $x \in F$, $J = \{i\colon a_i \cdot x = 0\}\}$. The hypotheses of 2.2 of [9] are easily verified. Hence

(1) there is an infinite $D \subseteq \mathrm{At}\,\mathfrak{A} \cap \mathrm{Rng}\,a$ such that for all $x \in A \sim F$, $\{d \in D\colon d \leqslant x\}$ is finite.

Now by (1) we can write each $x \in A \sim F$ uniquely in the form $t_x + \sum_{d \in M_x} d$ where $\mathrm{At}\,tx \cap D = 0$ and M_x is a finite subset of D. For any permutation f of D and any $x \in A \sim F$ we set

$$f^+x = t_x + \sum_{d \in M_x} fd\,.$$

For $x \in F$ we set $f^+x = -f^+ - x$. Then for any $x, y \in A$, $f^+(x + y) = f^+x + f^+y$. This is easy to see if $x, y \in A \sim F$ or $x, y \in F$. Now suppose, say, $x \in F$ and $y \in A \sim F$. Then

$$-(f^+x + f^+y) = -(-f^+ - x + f^+y) =$$

$$= -\Big(-t_{-x} \cdot - \sum_{d \in M_{-x}} fd + t_y + \sum_{d \in M_y} fd\Big) =$$

$$= -t_y \cdot - \sum_{d \in M_y} fd \cdot \Big(t_{-x} + \sum_{d \in M_{-x}} fd\Big) =$$

$$= t_{-x} \cdot - t_y + \sum_{d \in M_{-x} \sim M_y} fd = f^+(-x \cdot - y)$$

and hence $f^+(x + y) = f^+x + f^+y$. The rest of the proof that f^+ is the desired isomorphism is easy.

Corollary 3.7. *If* $\mathfrak{A}$ *is a denumerable* BA, *then* $\mathrm{Sym}\,\omega$ *can be isomorphically embedded in* $\mathrm{Aut}\,\mathfrak{A}$.

This leads to the following well-known result.

Corollary 3.8. *If* $\mathfrak{A}$ *is a denumerable* BA, *then* $|\mathrm{Aut}\,\mathfrak{A}| = \exp \aleph_0$.

Corollary 3.9. *If* $\mathfrak{A}$ *has infinitely many atoms,* $|A| < \exp \aleph_0$, *and* MA *holds, then* $\mathrm{Sym}\,\omega$ *can be isomorphically embedded in* $\mathrm{Aut}\,\mathfrak{A}$.

Now recalling the theorem of de Groot and McDowell quoted in the introduction, the following corollary follows.

Corollary 3.10. *If* MA *and* $|\mathrm{Aut}\,\mathfrak{A}| = \aleph_0$, *then* $|A| \geqslant \exp \aleph_0$.

This rules out one possible improvement of 3.2. We can relate some other possibilities of improving 3.2 to our earlier question concerning the possible cardinalities of rigid BA's:

Theorem 3.11. *If* $\mathfrak{m}$ *is an infinite cardinal,* $|\mathrm{Aut}\,\mathfrak{A}| = \mathfrak{m}$, *and* $|A| > \exp \mathfrak{m}$, *then there is a rigid* $\mathfrak{B}$ *with* $|B| = |A|$ *and* $\mathfrak{B} \subseteq \mathfrak{A}$.

Proof. First we claim

(1) There is an $a \in A$ with $|A \upharpoonright a| \leqslant \mathfrak{m}$ and $|\mathrm{Aut}\,(\mathfrak{A} \upharpoonright a)| > 1$.

For, by 1.6 (iii) we can choose disjoint non-zero elements x, y of A with $\mathfrak{A} \upharpoonright x \cong \mathfrak{A} \upharpoonright y$. Since $\mathfrak{A} \cong (\mathfrak{A} \upharpoonright x) \times (\mathfrak{A} \upharpoonright y) \times (\mathfrak{A} \upharpoonright (-x \cdot -y))$, it is clear that the elements of $\mathfrak{A} \upharpoonright x$ induce distinct automorphisms of $\mathfrak{A}$. Hence $|A \upharpoonright x| \leqslant \mathfrak{m}$. Let $a = x + y$; (1) is then clear.

By (1), let M be a maximal set of pairwise disjoint elements $a \in A$ such that $|A \upharpoonright a| \leqslant \mathfrak{m}$ and $|\mathrm{Aut}\,(\mathfrak{A} \upharpoonright a)| > 1$. Clearly M gives rise to

$|M|$ automorphisms of $\mathfrak{A}$, so $|M| \leqslant \mathfrak{m}$. Now let $I = \{x \in A\colon \forall a \in \in M(x \cdot a = 0)\}$. Clearly I is an ideal of $\mathfrak{A}$. There is a homomorphism $f\colon A/I \to \mathrm{P}_{a \in M} (A \upharpoonright a)$ such that $f(x/I)a = x \cdot a$ for all $x \in A$ and $a \in M$. Clearly f is one-one, so it follows that $|A/I| \leqslant \exp \mathfrak{m}$. Hence $|I| = |A|$. Suppose I has a largest element, c. Then $\mathfrak{A} \upharpoonright c$ has power $|A|$, and is isomorphic to a subalgebra of $\mathfrak{A}$ (as is well-known). If it is not rigid we easily obtain, as in the proof of (1), an $a \leqslant c$ with $|A \upharpoonright s| \leqslant \mathfrak{m}$ and $|\mathrm{Aut}\,(\mathfrak{A} \upharpoonright a)| > 1$, contradicting the maximility of M. Thus $\mathfrak{A} \upharpoonright c$ is rigid.

Now suppose I has no largest element. Let $B = I \cup \{x\colon -x \in I\}$. It is easily verified that B is closed under $+$ and $-$, so $\mathfrak{B} \subseteq \mathfrak{A}$. Obviously $|B| = |A|$. Suppose $\mathfrak{B}$ is not rigid. By 1.6 (iii) choose disjoint non-zero elements x, y of B with $\mathfrak{B} \upharpoonright x \cong \mathfrak{B} \upharpoonright y$; say $f\colon \mathfrak{B} \upharpoonright x \rightarrowtail\twoheadrightarrow \mathfrak{B} \upharpoonright y$. If $x + y \in I$, then $\mathfrak{A} \upharpoonright x + y = \mathfrak{B} \upharpoonright x + y$ is non-rigid and again we easily obtain a contradiction to the maximility of M. Assume that $x + y \notin I$. Then $-x \cdot -y \in I$. Choose $c \in I$ with $-x \cdot -y < c$. Thus $c \cdot (x + y) \neq 0$. Say $c \cdot x \neq 0$. Thus $f \upharpoonright (B \upharpoonright c \cdot x)\colon \mathfrak{B} \upharpoonright c \cdot x \rightarrowtail\twoheadrightarrow \mathfrak{B} \upharpoonright f(c \cdot x)$. If $f(c \cdot x) \in I$, we again obtain a contradiction. If $f(c \cdot x) \notin I$, then $-f(c \cdot x) \in I$ and we may choose $d \in I$ with $-f(c \cdot x) < d$. Thus $d \cdot f(c \cdot x) \neq 0$, and $\mathfrak{B} \upharpoonright f^{-1}(d \cdot f(c \cdot x)) \cong \cong \mathfrak{B} \upharpoonright d \cdot f(c \cdot x)$ with $d \cdot f(c \cdot x)$, $f^{-1}(d \cdot f(c \cdot x)) \in I$, which again yields a contradiction.

Corollary 3.12. *For every* $\mathfrak{m} \geqslant \exp \aleph_0$ *the following conditions are equivalent:*

(i) *there is a rigid* BA *of power* $\mathfrak{m}$*;*

(ii) *there is a* BA *of power* $\mathfrak{m}$ *with denumerable automorphism group.*

Proof. (i) $\to$ (ii). Let $\mathfrak{A}$ be rigid, $|A| = \mathfrak{m}$. By 3.1 let $\mathfrak{B}$ be an atomic BA of power $\exp \aleph_0$ with $|\mathrm{Aut}\,\mathfrak{B}| = \aleph_0$. We may assume that $\mathfrak{A}$ is atomless. $\mathfrak{A} \times \mathfrak{B}$ satisfies (ii).

(ii) $\to$ (i). By Katetov [8] we may assume that $\mathfrak{m} > \exp \aleph_0$. Then (i) follows by 3.11.

We conclude the paper with a result concerning a possible improvement of 3.4.

Lemma 3.13. *Assume that* $\mathfrak{n}$ *and* $\mathfrak{p}$ *are infinite cardinals with* $\mathfrak{n} < \exp \mathfrak{p} < \exp \mathfrak{n} < \exp \exp \mathfrak{p}$. *Then there is a* BA $\mathfrak{A}$ *of power* $\exp \mathfrak{p}$ *with* Aut $\mathfrak{A}$ *of power* $\exp \mathfrak{n}$.

Proof. Let $\mathfrak{C}$ be the BA of finite and cofinite subsets of $\mathfrak{n}$, and let $\mathfrak{B}$ be an atomless rigid BA of power $\exp \mathfrak{p}$. Then $\mathfrak{C} \times \mathfrak{B} \times \mathfrak{B}$ satisfies the desired conditions.

Theorem 3.14. Con (ZFC) $\to$ Con (ZFC + $\forall \mathfrak{m}$ ($\mathfrak{m}$ *regular* $\to \exists$ BA $\mathfrak{A}$ *of power* $\mathfrak{m}^{++}$ *with* $\mathfrak{m}^{++} < |\mathrm{Aut}\,\mathfrak{A}| < \exp \mathfrak{m}^{++}$)).

Proof. We shall apply Easton [4]. Let $F\mathfrak{m} = \mathfrak{m}^{++}$ for every regular cardinal $\mathfrak{m}$. Then [4] gives the desired result.

From the problems left open in this section, we may mention the following.

Problem 10. Does $|\mathrm{Aut}\,\mathfrak{A}| = \aleph_0$ imply $|A| \geqslant \exp \aleph_0$ without MA?

Problem 11. Con (ZFC) $\to$ Con (ZFC + $\forall \mathfrak{m} > \aleph_0\ \exists \mathfrak{A}$ of power $\mathfrak{m}$ with $\mathfrak{m} < |\mathrm{Aut}\,\mathfrak{A}| < \exp \mathfrak{m}$)?

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THE NUMBER OF SPERNER FAMILIES OF SUBSETS OF AN n ELEMENT SET

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1. INTRODUCTION

Let S be an n-element set and let 2^S represent the collection of subset of S. A subset of 2^S no member of which contains another will be called a Sperner family. To each Sperner family we can correspond a monotone $0-1$ function (or Boolean function) defined on 2^S, by assigning the value 1 to those members of 2^S that are contained in no member of the family. One may also correspond a member of the free distributive lattice on n generators to each Sperner family.

Thus the number of Sperner families represents the number of monotone Boolean functions definable on 2^S and the size of the free distributive lattice on n generators as well.

A number of authors, especially Korobkov, Hansel and the present author have obtained upper bounds on this number. The general

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