ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 3, Number 1, Winter 1973

CONSTRUCTION OF THE REALS VIA ULTRAPOWERS

G. H. MEISTERS AND J. D. MONK[1]

1. **Introduction.** In courses on real analysis it is convenient and not unusual to define the real number system R axiomatically as a "complete ordered field", that is, as an ordered field which satisfies the least upper bound axiom. Then all other properties of the reals (e.g., uniqueness up to isomorphism, the Heine-Borel theorem, etc.) can be proved directly from these axioms and it is not necessary to argue from some particular "construction" such as Dedekind cuts or equivalence classes of Cauchy sequences of rationals. (This latter method of construction we shall refer to as "Cantor's method".) In particular, it is easily proved that an ordered field satisfies the least upper bound axiom iff it is archimedean ordered and Cauchy sequentially complete.

Of course this axiomatic definition of the reals leaves open the question of the *existence* of a complete ordered field; and for this purpose a construction of some sort can hardly be avoided. Furthermore, Cantor's method can hardly be improved upon for its simplicity, directness and freedom from transfinite existence principles (such as the axiom of choice or Zorn's lemma). Cantor's method will certainly remain an important method of "completion".

Nevertheless, we sketch here another method for constructing the reals from the rationals Q which uses the notion of an "ultrapower" (introduced in [8] and developed in [3]; but see also [5]) and which we feel is not without interest even though it is neither shorter nor simpler than Cantor's method. It has the additional disadvantage that it depends on the following transfinite existence principle, which we shall refer to as "the ultrafilter hypothesis".

(U) *Every filter is contained in some ultrafilter.*

On the positive side we make three points which we consider to be pedagogical advantages of our construction.

1. Transfinite principles of some sort seem to be here to stay, and

Received by the editors January 29, 1971 and, in revised form, January 20, 1972.
AMS (MOS) subject classifications (1970). Primary 12L10, 12J15, 12F20, 12D99, 12D15, 10M15, 10N10; Secondary 10N15, 12L15, 98A05, 98A10, 98A20, 98D05, 98E05.
[1]This research was supported in part by NSF Grants GP 5965 and GP 7387 respectively.

so it is perhaps good to illustrate their use in a variety of ways so that the student can master them.

2. The above principle (U) is easily derived from Zorn's lemma (exercise for the student). On the other hand the ultrafilter hypothesis (U) is known to be a strictly weaker hypothesis for set theory than Zorn's lemma (cf. [**4**]).

3. The transfinite principle (U), and each separate stage of our construction of the reals to be presented here, is of interest in itself. That is, each of the separate parts, which once in hand fit together easily to prove the existence of a complete ordered field, has many other uses in mathematics. Consequently, the entire construction, while longer in detail, has much less of an *ad hoc* appearance than do the usual constructions. As a matter of fact, our method relates to "nonstandard analysis" (cf. [**9**], and [**10**]; see also the remarks in §6 of the present paper), and serves to provide some useful preliminaries for understanding that subject. It came to the authors' attention after this paper was written that this new method for constructing the reals was already known to Abraham Robinson and its possibility was mentioned by him in [**11**] on page 842, line 9.

Furthermore, any reader who is already familiar with the constructions and notions which make up the parts of our construction of the reals will be able to understand immediately the meaning (if not the proofs) of the following statement of our construction.

THEOREM I. *If F is an ordered field then the residue class ring F_1/F_0, of the ring F_1 of finite elements of F modulo its maximum ideal F_0 of infinitesimals, is an archimedean ordered field.*

THEOREM II. *If $\mathcal{U}$ is a countably free ultrafilter then the $\mathcal{U}$-ultrapower of any ordered field F (denoted $F^{\mathcal{U}}$) is an ordered field and the archimedean ordered field $(F^{\mathcal{U}})_1/(F^{\mathcal{U}})_0$ is Cauchy sequentially complete.*

COROLLARY. *For any countably free ultrafilter $\mathcal{U}$ and any ordered field F,*

$$(F^{\mathcal{U}})_1/(F^{\mathcal{U}})_0 \cong R.$$

In particular, when F is the field of rationals Q,

$$(Q^{\mathcal{U}})_1/(Q^{\mathcal{U}})_0 \cong R.$$

Here the symbol "$\cong$" means "is isomorphic to".

Thus the question of the existence of a complete ordered field R is reduced (in this setting) to the question of the existence of a count-

ably free ultrafilter. The existence of countably free ultrafilters on any given infinite set X is an easy consequence of the ultrafilter hypothesis (U). This will be proved in §3 where we have outlined, for the reader's convenience, the definitions and facts concerning filters which are needed for the discussion of ultrapowers given in §4. Theorem I and the first part of Theorem II (that $F^{\mathcal{U}}$ is an ordered field) are well known. However, for the sake of completeness, we have included their proofs here (in §§2 and 4 respectively) along with the necessary definitions concerning ordered fields (in §2). At the end of §4 (Theorem 4.1) we prove that $(F^{\mathcal{U}})_1/(F^{\mathcal{U}})_0$ is Cauchy sequentially complete, and this then completes our construction of the reals via ultrapowers.

In §5 we give some additional results for ultrapowers of ordered fields, and we close in §6 with some miscellaneous remarks.

2. **Ordered fields and the proof of Theorem I.** An *ordered field* is a pair (F, F^+) where F is a field and F^+ is a subset of F such that F^+ is closed under addition and multiplication, F^+ does not contain zero, and for every nonzero element a of F either $a \in F^+$ or $-a \in F^+$. The elements of F^+ are called the *positive elements* of F, and an *order relation*, $a < b$, is defined on F by the condition $(b - a) \in F^+$. This order relation *totally orders* F and satisfies the following properties:

(O1) For each c in F, $a < b$ implies $a + c < b + c$.

(O2) For each $c > 0$, $a < b$ implies $ac < bc$.

(O3) $a < b$ implies $-b < -a$.

(O4) $0 < a < b$ implies $0 < b^{-1} < a^{-1}$.

Conversely, if a relation $a < b$ totally orders F and satisfies (O1) and (O2) then $F^+ = \{a \in F \mid a > 0\}$ satisfies the above conditions for a set of "positive" elements for F, and $a < b$ iff $b - a \in F^+$. The *absolute value* $|a|$ of an element a of F is defined to be a if $a \in F^+$ and $-a$ otherwise. In an ordered field the multiplicative identity element 1 is necessarily positive (i.e., a member of F^+) as are all squares of nonzero elements. No positive integral multiple of 1, $n \cdot 1 = 1 + \cdots + 1$ (n terms), can be zero since F^+ is closed under addition and does not contain zero. Consequently, an ordered field has zero characteristic and contains (an isomorphic copy of) the rationals Q. An ordered field is said to be archimedean iff every element a of F^+ is exceeded in the order by some positive integer. If F is not archimedean it is called *nonarchimedean*. A sequence a_n is said to *converge* to an element b of F iff for every $\epsilon \in F^+$ there is an integer k such that $|a_n - b| < \epsilon$ whenever $n > k$. A sequence a_n in F is called *Cauchy* iff for every $\epsilon \in F^+$, there is an integer k such that $|a_n - a_m| < \epsilon$ whenever $n > k$ and $m > k$. An ordered field F is

called *Cauchy sequentially complete* iff every Cauchy sequence in F converges to an element of F.

Elements of an ordered field are classified as follows. If $|a| > q$ for all rationals q, then a is called *infinite*. If $|a| < q$ for some rational q, then a is called *finite*. If $|a| < q$ for all positive rational q, then a is called *infinitesimal*. We denote by F_∞ the set of all infinite elements, by F_1 the set of all finite elements, and by F_0 the set of all infinitesimals. Obviously, $\{0\} \subset F_0 \subset F_1$, $F = F_1 \cup F_\infty$, $F_1 \cap F_\infty = \emptyset$, $Q \subset F_1$ and $Q \cap F_0 = \{0\}$. Furthermore, if F is archimedean then $F_0 = \{0\}$ and $F_\infty = \emptyset$ while if F is nonarchimedean, then $F_0 \neq \{0\}$ and F_∞ consists of the inverses of the nonzero elements of F_0 (and vice versa).

LEMMA 2.1. *The set of finite elements F_1 is a commutative ring with identity and has no proper divisors of zero.*

PROOF. If $a, b \in F_1$ then there exist rationals p and q such that $|a| < p$ and $|b| < q$. Then $|ab| = |a| \cdot |b|$ and $|a - b| \leqq |a| + |b|$ imply that ab and $a - b$ are both finite also. The rest is obvious.

LEMMA 2.2. *F_0 consists of all nonregular elements of F_1.*

PROOF. If $a \in F_0$, then $a = 0$ or $a^{-1} \in F_\infty$ and so $a^{-1} \notin F_1$. That is, a is a nonregular element of F_1. Conversely, if a is a nonregular element of F_1, then $a = 0$ or else $a^{-1} \in F_\infty$. In either case $a \in F_0$.

LEMMA 2.3. *F_0 is a maximum ideal in the ring F_1.*

PROOF. Every proper ideal consists only of nonregular elements and hence is contained in F_0 by Lemma 2.2. Thus it only remains to show that F_0 is an ideal in F_1. If a and b belong to F_0, then the inequality $|a - b| \leqq |a| + |b|$ implies that $a - b$ belongs to F_0. If $a \in F_0$ and $b \in F_1$, then there exists a rational q such that $|b| < q$, and for every positive rational p we have $|a| < p/q$ so that $|ab| < p$; thus $ab \in F_0$. Finally $F_0 \neq F_1$ since 1 is finite but not infinitesimal.

LEMMA 2.4. *F_0 is an interval. That is, if a and b belong to F_0 and $a < c < b$, then $c \in F_0$.*

PROOF. $a < c < b$ implies $|c| < \max\{|a|, |b|\}$. But then a and b in F_0 implies $c \in F_0$.

We will say that an element a in F is *infinitely close* to an element b, denoted $a \simeq b$, iff $a - b \in F_0$. This is obviously an equivalence relation on F and also on F_1. We denote by (a) the equivalence class containing a. The set of these equivalence classes in F_1 is denoted by F_1/F_0 and is made into a ring (the *residue class ring of F_1 modulo*

F_0) by defining addition and multiplication as follows.

Multiplication in F_1/F_0 : $(a)(b) = (ab)$.

Addition in F_1/F_0 : $(a) + (b) = (a + b)$.

These definitions are independent of the representatives a, b of (a), (b) because F_0 is an ideal in F_1. Thus if $a' \simeq a$ and $b' \simeq b$, then

$$a'b' - ab = a'(b' - b) + (a' - a)b \in F_0$$

so that $a'b' \simeq ab$, and

$$(a' + b') - (a + b) = (a' - a) - (b - b') \in F_0$$

so that $a' + b' \simeq a + b$.

LEMMA 2.5. *If* $a \not\simeq b$ *then, for every pair* ϵ, $\delta \in F_0$, $a < b$ *iff* $a + \epsilon < b + \delta$.

PROOF. $b - a > 0$ and $b - a \notin F_0$ implies by Lemma 2.4 that $b - a > h$ for all $h \in F_0$. In particular for $h = \epsilon - \delta$, this gives

$$a + \epsilon < b + \delta.$$

For the converse, $a \not\simeq b$ implies $a + \epsilon \not\simeq b + \delta$ so that we may add $-\epsilon$ to the left and $-\delta$ to the right of $a + \epsilon < b + \delta$ to obtain $a < b$.

LEMMA 2.6. *Each equivalence class* (a) *is an interval in* F.

PROOF. $a < b < c$ and $a \simeq c$ implies that $0 < c - b < c - a \in F_0$ and so, by Lemma 2.4, $c - b$ also belongs to F_0. That is, $b \simeq c$.

It follows easily from the above lemmas that the relation

$$(a) < (b) \quad \text{iff } a \not\simeq b \text{ and } a < b$$

defines a *total order* for F_1/F_0 which satisfies the conditions (O1) and (O2).

THEOREM 2.1. F_1/F_0 *with the definitions of multiplication, addition and order given above is an archimedean ordered field with identity element* (1) *and zero element* $(0) = F_0$. *The rationals* Q *are embedded in* F_1/F_0 *by the mapping* $q \to (q)$, *and we will frequently identify* (q) *with* q.

PROOF. Actually if F_0 is a maximal ideal in a commutative ring F_1 with identity, then the residue class ring F_1/F_0 is always a field. But the proof here is a little simpler because of the special properties of our case. F_1/F_0 is obviously an ordered commutative ring with identity element (1) and zero element $(0) = F_0$. If $(a) \neq (0)$, then $a \in F_1$ and $a \notin F_0$ so that, by Lemma 2.2, $a^{-1} \in F_1$ also. Hence

$(a)(a^{-1}) = (1)$ which means that $(a)^{-1}$ exists and is equal to (a^{-1}). Thus F_1/F_0 is an ordered field.

The mapping $q \to (q)$ of the rationals Q into F_1/F_0 is obviously an ordered-field homomorphism and it only remains to show that it is one-to-one. But if $(q_1) = (q_2)$ then $q_1 - q_2 \in Q \cap F_0 = \{0\}$ so that $q_1 = q_2$.

Finally F_1/F_0 is archimedean because every element a of F_1 is bounded above by some (rational) integer n, and so $(a) \leqq (n)$. This completes the proof of Theorem 2.1.

Theorem I is now proved.

The following simple facts will be useful in the proof of Theorem 4.1.

LEMMA 2.7. $|(a)| = (|a|)$.

PROOF. If $a > 0$ and $a \neq 0$, then $(a) > 0$ and $|(a)| = (a) = (|a|)$. If $a < 0$ and $a \neq 0$, then $(a) < 0$ and $|(a)| = -(a) = (-a) = (|a|)$. If $a = 0$, then $|a| = 0$, and $|(a)| = |0| = 0 = (|a|)$. Thus in all cases $|(a)| = (|a|)$.

LEMMA 2.8. *For each finite element a of an ordered field F (in particular for every element a of an archimedean ordered field F) and for every positive rational $p \in Q \subset F$, there is a rational q such that $|a - q| < p$.*

PROOF. We may assume that $a > 0$. Let m be a positive integer such that $(1/m) < p$. Since a is finite so is ma, and consequently there exists an integer $n > ma$. Let n be the smallest such integer. Then

$$q = (n - 1)/m \leqq a$$

and

$$|a - q| = a - (n - 1)/m = (a - n/m) + 1/m < 1/m < p.$$

3. **Filters.** For the readers' convenience we give here all definitions and facts needed in this paper concerning filters. For a more detailed discussion see [1].

A filter $\mathcal{F}$ on a nonempty set X is a nonempty family $\mathcal{F}$ of subsets of X satisfying the following:

(F1) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$.

(F2) $A \in \mathcal{F}$ and $A \subset S \subset X$ implies $S \in \mathcal{F}$.

(F3) $\emptyset \notin \mathcal{F}$.

Some examples of filters are

(a) the set of all subsets S of X containing a fixed nonempty subset X_0 of X. When $X_0 = \{x_0\}$, a singleton, this filter is said to be *fixed at the point* x_0.

(b) If X is infinite, the set of all subsets of X whose complements are finite. This filter is called the *Fréchet filter* on X.

A filter $\mathcal{F}$ is said to be *free* if $\bigcap \mathcal{F} = \emptyset$; and *countably free* if $\bigcap_n U_n = \emptyset$ for some denumerable sequence U_n of members of $\mathcal{F}$. An *ultrafilter* is a maximal filter; that is, a filter not properly contained in any other filter. Two important properties of ultrafilters which we will use many times are the following:

(U1) A filter $\mathcal{U}$ on X is an ultrafilter iff, for every subset A of X, either $A \in \mathcal{U}$ or its complement $A' \in \mathcal{U}$.

(U2) If a finite union $A_1 \cup \cdots \cup A_n$ belongs to an ultrafilter $\mathcal{U}$, then at least one of the sets A_i belongs to $\mathcal{U}$.

If the A_i are pairwise disjoint in (U2), then *exactly* one belongs to $\mathcal{U}$. It follows easily from these facts that every ultrafilter on a finite set is fixed at some point while every ultrafilter on an infinite set is either fixed at some point or else is free. Also an ultrafilter on an infinite set is free iff it contains (is *finer* than) the Fréchet filter. Zorn's lemma easily implies that every filter is contained in an ultrafilter, but the latter condition is known to be a strictly weaker hypothesis for set theory than is Zorn's lemma [4]. Free ultrafilters exist on any infinite index set since any ultrafilter containing the Fréchet filter is necessarily free. A family S of subsets of X belongs to some filter on X iff no finite intersection of members of S is empty. Such a family S is said to have the *finite intersection property*.

LEMMA 3.1. *Every free ultrafilter $\mathcal{U}$ on the set of positive integers N (or on any denumerable set X) is countably free.*

PROOF. For each $n \in N$, the set

$$U_n = \{m \in N \mid m > n\}$$

belongs to $\mathcal{U}$ since its complement U_n' is finite and $\mathcal{U}$ contains the Fréchet filter. But $\bigcap_n U_n = \emptyset$.

LEMMA 3.2. *If X is a nondenumerable set, then there exist countably free ultrafilters on X.*

PROOF. The set X can be expressed in many ways as a denumerable union of mutually disjoint nonempty sets. Select one way, say $X = \bigcup_n U_n'$. Then the family $S = \{U_n \mid n \in N\}$ of their complements U_n has the property that no finite intersection of its members is empty. Therefore, S is contained in some filter on X which in turn is contained in some ultrafilter $\mathcal{U}$ on X. But $X = \bigcup_n U_n'$ implies that $\bigcap_n U_n = \emptyset$ so that $\mathcal{U}$ is countably free.

LEMMA 3.3. *An ultrafilter $\mathcal{U}$ on a set X is countably free iff X can be written as a denumerable union of subsets whose complements belong to $\mathcal{U}$.*

PROOF. If $\mathcal{U}$ is countably free then there exists a denumerable sequence $U_n \in \mathcal{U}$ such that $\bigcap_n U_n = \emptyset$. Therefore $\bigcup_n U_n' = X$. Conversely, suppose $X = \bigcup_n U_n'$, with complements $U_n \in \mathcal{U}$. Then $\emptyset = \bigcap_n U_n$ so that $\mathcal{U}$ is countably free.

LEMMA 3.4. *If $\mathcal{U}$ is a countably free ultrafilter on a set X, then X can be expressed as a denumerable union of pairwise disjoint subsets whose complements belong to $\mathcal{U}$.*

PROOF. Suppose $X = \bigcup_n U_n'$ with $U_n \in \mathcal{U}$. Define $V_1' = U_1'$ and, for $n \geqq 2$, define

$$\begin{aligned} V_n' &= U_n' \sim (U_1' \cup \cdots \cup U_{n-1}') \\ &= U_n' \cap U_1 \cap \cdots \cap U_{n-1}. \end{aligned}$$

Then $V_n = U_n \cup U_1' \cup \cdots \cup U_{n-1}' \supset U_n \in \mathcal{U}$. Therefore $V_n \in \mathcal{U}$, $V_m' \cap V_n' = \emptyset$ for $m \neq n$, and $X = \bigcup_n V_n'$.

What we have here called a "countably free" ultrafilter is equivalent to what some authors call a "countably incomplete" ultrafilter. We are using the former terminology in order to avoid confusion later with "countably complete" which is used by many authors for what we have called "Cauchy sequentially complete".

4. **Ultrapowers of ordered fields and the proof of Theorem II.** Let X denote an infinite set and let F^X denote the set of all functions, A, B, etc., from X to an ordered field F. Equipped with the pointwise operations of addition and multiplication,

$$(A + B)(x) = A(x) + B(x) \quad \text{and}$$

$$(AB)(x) = A(x) \cdot B(x) \quad \text{for } x \in X,$$

F^X is a commutative ring with zero element $0(x) \equiv 0$ and identity element $1(x) \equiv 1$; these are denoted 0 and 1, respectively.

Now let $\mathcal{U}$ be an ultrafilter on X and define $A \overset{\mathcal{U}}{=} B$ to mean that

$$\{x \in X \mid A(x) = B(x)\} \in \mathcal{U}.$$

It follows easily from the fact that $\mathcal{U}$ is a filter that $A \overset{\mathcal{U}}{=} B$ is an equivalence relation in F^X. The set of equivalence classes thus determined in F^X is denoted $F^X/\mathcal{U}$, or simply $F^{\mathcal{U}}$ when the set X is understood. The equivalence class with representative element A will be denoted $a = [A]$. Addition, multiplication and order are defined for $F^{\mathcal{U}}$ as follows.

Addition in $F^{\mathcal{U}}$: $[A] + [B] = [A + B]$.
Multiplication in $F^{\mathcal{U}}$: $[A]\,[B] = [AB]$.
Order in $F^{\mathcal{U}}$: $[A] < [B]$ iff $\{x \in X \mid A(x) < B(x)\} \in \mathcal{U}$.
It is easily verified that these definitions are independent of the representatives A and B of $[A]$ and $[B]$, respectively.

LEMMA 4.1. *With the above definitions of addition, multiplication and order,* $F^{\mathcal{U}}$ *forms a totally ordered field with zero element* $[0]$ *and identity element* $[1]$.

PROOF. We prove that every nonzero element of $F^{\mathcal{U}}$ has a multiplicative inverse and that the order relation is total (i.e., linear), and we leave the rest to the reader.

If $[A] \neq [0]$, then

$$U' = \{x \in X \mid A(x) = 0\} \notin \mathcal{U}.$$

Since $\mathcal{U}$ is an ultrafilter, it follows that

$$U = \{x \in X \mid A(x) \neq 0\} \in \mathcal{U}.$$

Define

$$\begin{aligned} B(x) &= 1/A(x) \quad \text{if } x \in U, \\ &= 1 \qquad\quad\ \ \text{if } x \in U'. \end{aligned}$$

Then $[A] \cdot [B] = [1]$, since $AB = 1$ on $U \in \mathcal{U}$. Therefore $[A]^{-1} = [B] \in F^{\mathcal{U}}$.

If $[A] \neq [B]$, then

$$V' = \{x \in X \mid A(x) = B(x)\} \notin \mathcal{U}.$$

Since X is the disjoint union of V' and the two sets

$$V_1 = \{x \in X \mid A(x) < B(x)\}, \qquad V_2 = \{x \in X \mid A(x) > B(x)\},$$

and since $\mathcal{U}$ is an ultrafilter it follows from (U2) of §3 that exactly one of the sets V_1 or V_2 belongs to $\mathcal{U}$. But this means that exactly one of the relations $[A] < [B]$ or $[B] < [A]$ holds. Thus the order relation for $F^{\mathcal{U}}$ is total.

This ordered field $F^{\mathcal{U}}$ is called the *$\mathcal{U}$-ultrapower* of F. (Cf. [3].)

Actually part of Lemma 4.1 follows directly from a more general algebraic fact. Namely, if we let $I_{\mathcal{U}}$ denote the set of all functions $A \in F^X$ such that $\{x \mid A(x) = 0\} \in \mathcal{U}$, then $I_{\mathcal{U}}$ is easily shown to be a maximal ideal in the ring F^X. Consequently, as is well known, the residue class ring $F^X/I_{\mathcal{U}}$ (denoted $F^{\mathcal{U}}$ above) is a field. Another interesting fact which we will not use is that there is a one-to-one

correspondence between ultrafilters on X and maximal ideals in F^X. See [5].

For each $q \in F$ define $\hat{q} \in F^X$ by $\hat{q}(x) = q$ for all $x \in X$. That is, $\hat{q}$ is the constant function with value q. Define the mapping $\iota : F \to F^{\mathcal{U}}$ by $\iota(q) = [\hat{q}]$, and let $\hat{F} = \iota(F) \subset F^{\mathcal{U}}$.

LEMMA 4.2. *The mapping $\iota : F \to \hat{F} \subset F^{\mathcal{U}}$ is an ordered-field isomorphism of F onto $\hat{F}$.*

PROOF. We prove that the mapping is one-to-one and leave the rest to the reader. Now $\iota(q_1) = \iota(q_2)$ means $[\hat{q}_1] = [\hat{q}_2]$ and the latter means that

$$V = \{x \in X \mid \hat{q}_1(x) = \hat{q}_2(x)\} \in \mathcal{U}.$$

Since $\emptyset \notin \mathcal{U}$, V is not empty. Let x_0 be an element of V. Then

$$q_1 = \hat{q}_1(x_0) = \hat{q}_2(x_0) = q_2.$$

Therefore the mapping $\iota : F \to \hat{F}$ is one-to-one.

We are now ready for the main result.

THEOREM 4.1. *If $\mathcal{U}$ is a countably free ultrafilter and if F is any ordered field, then $(F^{\mathcal{U}})_1/(F^{\mathcal{U}})_0$ is an archimedean ordered field which is Cauchy sequentially complete.*

PROOF. It follows from Lemma 4.1 and Theorem 2.1 that $(F^{\mathcal{U}})_1/(F^{\mathcal{U}})_0$ is an archimedean ordered field, and it only remains to be shown that it is Cauchy sequentially complete.

Let (a_n) be a Cauchy sequence in $(F^{\mathcal{U}})_1/(F^{\mathcal{U}})_0$. Recall from §2 that (a_n) denotes the equivalence class of all elements in $(F^{\mathcal{U}})_1$ which are infinitely close to a_n. By Lemma 2.8, for each n we may choose a rational q_n such that

$$|(a_n) - (q_n)| < (1/n).$$

(Note that since the rationals can be effectively well ordered, this choice can be made without applying the axiom of choice.) But then, by Lemma 2.7, also $|a_n - q_n| < 1/n$, which means that, for $A_n \in a_n$,

$$V_n = \{x \in X \mid |A_n(x) - q_n| < 1/n\} \in \mathcal{U}. \tag{1}$$

Now (q_n) is a Cauchy sequence in $(F^{\mathcal{U}})_1/(F^{\mathcal{U}})_0$. Indeed, given $(\epsilon) > 0$ choose a positive integer k such that $|(a_m) - (a_n)| < (\epsilon/3)$ for all $m, n > k$ and such that $(1/k) < (\epsilon/3)$, as we may since (a_n) is Cauchy and $(F^{\mathcal{U}})_1/(F^{\mathcal{U}})_0$ is archimedean. Then, for $m, n > k$, we have

$$|(q_m) - (q_n)| \leqq |(q_m) - (a_m)| + |(a_m) - (a_n)| + |(a_n) - (q_n)|$$
$$< (1/m) + (\epsilon/3) + (1/n) < (\epsilon).$$

Now since $\mathcal{U}$ is countably free there exists a denumerable sequence of nonempty pairwise disjoint sets U_n' such that

$$X = \bigcup_n U_n' \quad \text{and} \quad U_n \in \mathcal{U}.$$

We define an element $a = [A]$ of $F^{\mathcal{U}}$ by

$$A(x) = q_n \quad \text{for } x \in U_n' \text{ and } n \geqq 1.$$

We claim that $a = [A]$ is finite so that $(a) \in (F^{\mathcal{U}})_1/(F^{\mathcal{U}})_0$. To prove this choose l so large that $|(q_n) - (q_m)| < (1)$ whenever $m, n > l$. But then also $|q_n - q_m| < 1$ for $m, n > l$. Choose $m > l$ and consider the inequality

$$|A(x)| = |q_n| \leqq |q_n - q_m| + |q_m|$$

which holds when $x \in U_n'$. Now if $n > l$ it follows that

$$|A(x)| \leqq 1 + |q_m|.$$

Therefore,

$$\{x \in X \mid |A(x)| > 1 + |q_m|\} \subset U_1' \cup \cdots \cup U_l'$$

so that

$$\{x \in X \mid |A(x)| \leqq 1 + |q_m|\} \supset U_1 \cap \cdots \cap U_l.$$

But since $U_1 \cap \cdots \cap U_l \in \mathcal{U}$, $|a| \leqq 1 + |q_m|$; that is, the element $a = [A]$ is finite as claimed.

Finally we show that (a_n) converges to (a). Let $(\epsilon) > 0$ be given. Let r be a rational such that $(\epsilon) > (r) > 0$. Let k be a positive integer such that $1/k < r/2$ and such that, for all $m, n > k$,

$$|(q_m) - (q_n)| < (r/2).$$

This implies in particular that $|q_m - q_n| < r/2$. Thus if $m > k$, $x \in U_n'$ with $n > k$, and $x \in V_m$ (defined in equation (1)), then

$$|A_m(x) - A(x)| = |A_m(x) - q_n|$$
$$\leqq |A_m(x) - q_m| + |q_m - q_n|$$
$$\leqq 1/m + r/2 < r.$$

Therefore, for $m > k$,

$$\{x \in X \mid |A_m(x) - A(x)| \geqq r\} \subset V_m' \cup U_1' \cup \cdots \cup U_k'$$

so that

$$\{x \in X \mid |A_m(x) - A(x)| < r\} \supset V_m \cap U_1 \cap \cdots \cap U_k.$$

But since $V_m \cap U_1 \cap \cdots \cap U_k \in \mathcal{U}$, $|a_m - a| < r$; and therefore

$$|(a_m) - (a)| \leqq (r) < (\epsilon) \quad \text{for } m > k$$

which shows that (a_m) converges to (a) as claimed.

Theorem II is now proved and we may now conclude that if $\mathcal{U}$ is any countably free ultrafilter (e.g., any free ultrafilter on the set N of positive integers) then

$$(Q^{\mathcal{U}})_1/(Q^{\mathcal{U}})_0 \cong R.$$

This completes our construction of the reals via an ultrapower of the rationals.

5. **Some additional remarks on ultrapowers of ordered fields.** We give here some results which are related to the results obtained in §4 and may serve to further clarify the nature of ultrapowers of ordered fields.

LEMMA 5.1. *If $\mathcal{U}$ is countably free then there is no (denumerable) sequence a_n of positive elements of $F^{\mathcal{U}}$ which converges to zero (in the order topology of $F^{\mathcal{U}}$).*

PROOF. Suppose a_n is a sequence of positive elements of $F^{\mathcal{U}}$ which converges to zero. Then there exists an integer $k_1 > 1$ such that $a_{k_1} < a_1$, an integer $k_2 > k_1$ such that $a_{k_2} < a_{k_1}$, etc. Consequently there exists a subsequence a_{k_n} of a_n such that a_{k_n} is strictly decreasing and also converges to zero. Hence we may assume without loss of generality that the sequence a_n is strictly decreasing. This means that, for any sequence of representatives $A_n \in a_n$,

$$(1) \qquad \text{for every } n \leqq 1, \qquad \{x \in X \mid A_{n+1}(x) < A_n(x)\} \in \mathcal{U}.$$

However, we may assume more; namely, that

$$(1') \qquad \text{for every } n \geqq 1, \qquad A_{n+1}(x) < A_n(x) \textit{ for all } x \in X.$$

For if $(1')$ does not hold we can define a new sequence, $B_n \in a_n$ such that $(1')$ holds for B_n as follows. Define $B_1(x) = A_1(x)$ for all $x \in X$, and for $n \geqq 1$ define $B_{n+1}(x) = A_{n+1}(x)$ if $A_{n+1}(x) < B_n(x)$ and $B_{n+1}(x) = \frac{1}{2} B_n(x)$ otherwise. Obviously $(1')$ holds for this new sequence, and it only remains to show that, for each $n \geqq 1$, $B_n \in a_n$;

that is $B_n \overset{\mathcal{U}}{=} A_n$. But this is easily established by mathematical induction on n as follows. Clearly $B_1 \overset{\mathcal{U}}{=} A_1$ (since in fact $B_1 = A_1$), and if $B_n \overset{\mathcal{U}}{=} A_n$ for some $n \geqq 1$, then

$$\{x \in X \mid A_n(x) = B_n(x)\} \in \mathcal{U}. \tag{2}$$

But (2) and (1) together imply $[B_{n+1}] = [A_{n+1}]$ for

$$\begin{aligned}\{x \mid B_{n+1}(x) = A_{n+1}(x)\} &\supset \{x \mid A_{n+1}(x) < B_n(x)\} \\ &\supset \{x \mid A_{n+1}(x) < A_n(x)\} \cap \{x \mid A_n(x) = B_n(x)\}.\end{aligned}$$

This completes the proof that we may assume (1′). Now since $\mathcal{U}$ is countably free, Lemma 3.4 allows us to write, for some nonempty sets U_n,

$$X = \bigcup_n U_n', \qquad U_n' \cap U_m' = \emptyset \quad \text{for } n \neq m, U_n \in \mathcal{U}. \tag{3}$$

We define $D(x) = A_n(x)$ for all $x \in U_n'$ and all $n \geqq 1$. Then $d = [D]$ is a positive element of $F^{\mathcal{U}}$ and (1′) implies that, for each $m \geqq 1$,

$$V_m' = \{x \mid D(x) \geqq A_m(x)\} = U_1' \cup \cdots \cup U_m'.$$

Therefore, for every $m \geqq 1$,

$$V_m = \{x \mid D(x) < A_m(x)\} = U_1 \cap \cdots \cap U_m,$$

so that $V_m \in \mathcal{U}$. This means that $0 < d < a_m$ for all $m \geqq 1$, which contradicts the original hypothesis that a_m converges to zero. This completes the proof of Lemma 5.1.

LEMMA 5.2. *If an ordered field F satisfies the conclusion of Lemma 5.1, then F is trivially Cauchy sequentially complete. That is, if a_n is a Cauchy sequence in such an ordered field, then $a_n = a_m$ for all sufficiently large n and m.*

PROOF. Suppose a_n is a Cauchy sequence in F. Define $d_{nm} = |a_n - a_m|$. Then either $d_{nm} = 0$ for all n, m larger than some integer l (and the lemma is proved) or else for every integer l there exist integers m_l and n_l larger than l such that $d_{n_l m_l} > 0$. Let $b_l = d_{n_l m_l}$. Then b_l is a sequence of positive elements of F and b_l is easily seen to converge to zero as follows. For every positive ϵ in F there is an integer k such that $d_{nm} < \epsilon$ for n, $m > k$. Therefore if $l > k$, then $m_l > l > k$ and $n_l > l > k$ so that $b_l = d_{n_l m_l} < \epsilon$. But the existence of such a sequence b_l contradicts the conclusion of Lemma 4.3. Therefore $d_{nm} = 0$ for all sufficiently large m and n, and this proves the lemma.

LEMMA 5.3. *If $\mathcal{U}$ is countably free then $F^{\mathcal{U}}$ is nonarchimedean.*

PROOF. By Lemma 3.4 we can write

$$(1) \qquad X = \bigcup_n U_n', \qquad U_n' \cap U_m' = \emptyset \quad \text{for } m \neq n, \text{ and } U_n \in \mathcal{U}.$$

Define $A(x) = n$ for $x \in U_n'$ and all integers $n \geqq 1$. Then $A \in F^X$ and so $a = [A] \in F^{\mathcal{U}}$. For each $n \geqq 1$ consider the set

$$V_n = \{x \in X \mid A(x) > n\}.$$

$$V_n' = \{x \in X \mid A(x) \leqq n\} = U_1' \cup \cdots \cup U_n'.$$

Therefore $V_n = U_1 \cap \cdots \cap U_n$ and so V_n belongs to $\mathcal{U}$. That is, by definition of order in $F^{\mathcal{U}}$, $a > n$ for every integer $n \geqq 1$. But this means that $F^{\mathcal{U}}$ is not archimedean.

A partial converse is given by

LEMMA 5.4. *If F is archimedean and $F^{\mathcal{U}}$ is nonarchimedean, then $\mathcal{U}$ is countably free.*

PROOF. $F^{\mathcal{U}}$ nonarchimedean implies the existence of an element $[A] \in F^{\mathcal{U}}$ such that, for every positive integer m,

$$U_m = \{x \in X \mid |A(x)| > m\} \in \mathcal{U}.$$

Then since F is archimedean, for each x in X there exists an integer m such that $|A(x)| < m$. Therefore every element x of X belongs to U_m' for some m; that is, $X = \bigcup_m U_m'$. But then $\bigcap_m U_m = \emptyset$ and $U_m \in \mathcal{U}$ so that $\mathcal{U}$ is countably free.

LEMMA 5.5. *If $\mathcal{U}$ is countably free then $F^{\mathcal{U}} \neq \hat{F}$.*

(See the paragraph preceding Lemma 4.2.)

PROOF. Again by Lemma 3.4, we may define, as in the proof of Lemma 5.3, $A(x) = n$ for $x \in U_n'$ and $n \geqq 1$. Then $A \in F^X$ and $[A] \in F^{\mathcal{U}}$. Suppose that $[A] \in \hat{F}$. Then, for some $q \in F$,

$$(4) \qquad V_q = \{x \in X \mid A(x) = q\} \in \mathcal{U}.$$

But for each $q \in F$, V_q is either empty (if q is not an integer) or equal to U_m'(if q is the integer m). In either case $V_q \notin \mathcal{U}$ contrary to (4).

A partial converse is given by

LEMMA 5.6. *If $F^{\mathcal{U}} \neq \hat{F}$, then $\mathcal{U}$ is free. If $F^{\mathcal{U}} \neq \hat{F}$ and F is denumerable, then $\mathcal{U}$ is countably free.*

PROOF. Let $[A] \in F^{\mathcal{U}}$ with $[A] \notin \hat{F}$. For each $q \in F$ define

$$V_q' = \{x \in X \mid A(x) = q\}.$$

Since $[A] \notin \hat{F}$, for every $q \in F$, $V_q' \notin \mathcal{U}$. Since $\mathcal{U}$ is an ultrafilter, $V_q \in \mathcal{U}$. Now, for each x, $A(x) \in F$ so that

$$X = \bigcup_{q \in F} V_q' \quad \text{or} \quad \emptyset = \bigcap_{q \in F} V_q.$$

Thus $\mathcal{U}$ is free, and if F is denumerable $\mathcal{U}$ is countably free.

A sequence of positive elements in an ordered field F is said to be *co-initial* with F^+ if it converges to zero (in the order topology of F). It is the content of Lemma 5.1 that *if $\mathcal{U}$ is a countably free ultrafilter then $F^{\mathcal{U}}$ has no sequence co-initial with its positive elements;* and of Lemma 5.2 that *if there is no sequence co-initial with F^+ then every Cauchy sequence in F converges trivially to an element of F.* The following equivalences are proved in [2] but are perhaps "well known".

LEMMA 5.7. *If F is an ordered field, the following statements are pairwise equivalent.*

(i) *F^+ contains a (denumerable) co-initial sequence.*

(ii) *F is metrizable in its order topology.*

(iii) *(The order topology of) F is first countable.*

(iv) *Sequences are adequate (to describe closure in the order topology) in F.*

PROOF (An outline only).

If F^+ contains a (denumerable) sequence co-initial with F^+, then a metric can be constructed for F in a manner quite analogous to the usual proof of the metrization theorem for uniform spaces (see any modern book on topology). In fact, the metric can be constructed so as to be "invariant": $\rho(a + c, b + c) \equiv \rho(a, b)$. Thus (i) implies (ii).

If F is metrizable then it is certainly first countable; and if F is first countable, then sequences are adequate in F. Again see any modern book on topology. Thus (ii) implies (iii) and (iii) implies (iv).

Finally, if sequences are adequate in F, then in particular, since zero is a limit point (in the order topology of F) of the set F^+, there must be a sequence $a_n \in F^+$ such that a_n converges to 0. But then this sequence is co-initial with F^+. Thus (iv) implies (i), and the proof of Lemma 5.5 is complete.

For any denumerable subfield of the reals, and notably for the rationals Q, the above lemmas can be summarized as follows.

THEOREM 5.1. *The following six statements are pairwise equivalent:*

(i) *$\mathcal{U}$ is countably free.*

(ii) *$Q^{\mathcal{U}}$ is nonarchimedean.*

(iii) *$Q^{\mathcal{U}}$ is not isomorphic to Q.*

(iv) $Q^{\mathcal{U}} \neq \hat{Q}$.
(v) $Q^{\mathcal{U}}$ *is not metrizable in its order topology.*
(vi) $Q^{\mathcal{U}}$ *is Cauchy sequentially complete.*

PROOF. (i) and (ii) are equivalent by Lemmas 5.3 and 5.4. (i) and (iv) are equivalent by Lemmas 5.5 and 5.6. Lemmas 5.1 and 5.7 show that (i) implies (v), and Lemmas 5.1 and 5.2 show that (i) implies (vi). Now (v) implies (i), for if $\mathcal{U}$ is not countably free then (since (i) and (iv) are already proved equivalent) $Q^{\mathcal{U}} = \hat{Q}$, which is metrizable. Next (vi) implies (i), for if $\mathcal{U}$ is not countably free then again $Q^{\mathcal{U}} = \hat{Q}$, which is not Cauchy sequentially complete. This leaves only the equivalence of (i) and (iii). First if $\mathcal{U}$ is countably free, then $Q^{\mathcal{U}}$ is nonarchimedean and so $Q^{\mathcal{U}}$ is certainly not isomorphic to Q. Finally if $Q^{\mathcal{U}}$ is not isomorphic to Q, then in particular $Q^{\mathcal{U}} \neq \hat{Q}$. But (iv) is already known to imply (i). This completes the proof of Theorem 5.1.

Of course it follows from Lemmas 5.1 and 5.2 that $R^{\mathcal{U}}$ is Cauchy sequentially complete, and a theorem holds for $R^{\mathcal{U}}$ analogous to Theorem 5.1.

THEOREM 5.2. *The following five statements are pairwise equivalent*:
(i) $\mathcal{U}$ *is countably free.*
(ii) $R^{\mathcal{U}}$ *is nonarchimedean.*
(iii) $R^{\mathcal{U}}$ *is not isomorphic to* R.
(iv) $R^{\mathcal{U}} \neq \hat{R}$.
(v) $R^{\mathcal{U}}$ *is not metrizable in its order topology.*

PROOF. All parts of the proof of this theorem are exactly analogous to the corresponding part of the proof of Theorem 5.1 with *one exception*: Namely, the proof that (iv) implies (i). This requires a different argument for $R^{\mathcal{U}}$ as follows. First (iv) implies (ii). For let $a \in R^{\mathcal{U}}$ with $a \notin \hat{R}$. If $R^{\mathcal{U}}$ is archimedean then there exists some member of $\hat{R}$ larger than $|a| \neq 0$ so that the set $S = \{r \in \hat{R} \mid r \leqq |a|\}$ is a nonempty subset of $\hat{R}$ bounded above by a member of $\hat{R}$. Hence it has a least upper bound b in $\hat{R}$. But then $|a| - b \neq 0$ (since $b \in \hat{R}$ and $|a| \notin \hat{R}$) and $|a| - b$ is infinitesimal, so that $R^{\mathcal{U}}$ is nonarchimedean. To see that $|a| - b$ is infinitesimal we consider two cases, $|a| < b$ and $b < |a|$ (both possible). If $|a| < b$ and $b - |a|$ is not infinitesimal then there is some $\epsilon > 0$ in $\hat{R}$ such that $0 < \epsilon < b - |a|$. But then $|a| < b - \epsilon$ and $b - \epsilon \in \hat{R}$ so that $b - \epsilon$ is an upper bound (in $\hat{R}$) for S and hence cannot be less than the least upper bound b. But $b \leqq b - \epsilon$ implies $\epsilon \leqq 0$ contrary to hypothesis. On the other hand, if $b < |a|$ and $|a| - b$ is not infinitesi-

mal, then there is some $\epsilon > 0$ in $\hat{R}$ such that $0 < \epsilon < |a| - b$. But then $b + \epsilon < |a|$ which implies that $b + \epsilon \in S$, so that $b + \epsilon \leqq b$, or $\epsilon \leqq 0$, again a contradiction.

Finally, (ii) implies (i) by Lemma 5.4. This completes the proof of Theorem 5.2.

6. **Some final remarks.** The simplest form of our construction of the reals is as follows. By the ultrafilter hypothesis (U), select an ultrafilter $\mathcal{U}$ containing the Fréchet filter on the integers N, and form the ultrapower $Q^{\mathcal{U}}$ of the rationals Q. Let $Q_1^{\mathcal{U}}$ denote the ring of finite elements of $Q^{\mathcal{U}}$ and let $Q_0^{\mathcal{U}}$ denote the maximum ideal in $Q_1^{\mathcal{U}}$ of infinitesimals of $Q^{\mathcal{U}}$. Then the residue class ring $Q_1^{\mathcal{U}}/Q_0^{\mathcal{U}}$ is (isomorphic to) the reals R.

More generally, let $\mathcal{U}$ be any countably free ultrafilter. Then

1. Both $Q^{\mathcal{U}}$ and $R^{\mathcal{U}}$ are (trivially) Cauchy sequentially complete nonarchimedean ordered fields and

$$Q_1^{\mathcal{U}}/Q_0^{\mathcal{U}} = R_1^{\mathcal{U}}/R_0^{\mathcal{U}} = R.$$

2. $Q^{\mathcal{U}}$ contains (an isomorphic copy of) Q as an ordered subfield, but not R.

3. $R^{\mathcal{U}}$ contains R (and Q) as an ordered subfield.

4. $R_1^{\mathcal{U}} = \hat{R} \oplus R_0^{\mathcal{U}}$, but $Q_1^{\mathcal{U}} \neq R \oplus Q_0$ and $Q_1^{\mathcal{U}} \neq Q \oplus Q_0^{\mathcal{U}}$.

5. If $\mathcal{U}$ is a countably free ultrafilter then $F^{\mathcal{U}}$ is a "nonstandard model" for the ordered field F (see [**9**], [**10**] or [**11**]). That is, $F^{\mathcal{U}}$ is a proper extension of F, which satisfies the same "first-order" sentences as does F, and every "higher-order" true sentence for F has a true canonical interpretation in $F^{\mathcal{U}}$. In particular, $R^{\mathcal{U}}$ is a nonstandard model of the reals, and $Q^{\mathcal{U}}$ is a nonstandard model for the rationals. Furthermore, $Q^{\mathcal{U}}$ is isomorphic to a subfield of $R^{\mathcal{U}}$, which in turn is isomorphic to a subfield of $C^{\mathcal{U}}$. Finally $R^{\mathcal{U}}$ is a real closed field.

6. It is known (cf. [**3**]) that

$$\operatorname{card} F \leqq \operatorname{card} F^{\mathcal{U}} \leqq (\operatorname{card} F)^{\operatorname{card} X}.$$

Since any algebraically closed field of characteristic zero which has the same cardinality as the complexes C is isomorphic to C, it follows from the above inequality that $C^{\mathcal{U}}$ is isomorphic to C whenever $\mathcal{U}$ is an ultrafilter on a denumerable set X. Nevertheless, if $\mathcal{U}$ is countably free then $C^{\mathcal{U}} \neq \hat{C}$, $R^{\mathcal{U}}$ is isomorphic to a subfield of $C^{\mathcal{U}}$ and $\hat{R}$ is isomorphic to a subfield of $\hat{C}$. Of course the isomorphism from C onto $C^{\mathcal{U}}$ does not map R onto (nor even into) $\hat{R}$.

Concerning the existence of countably free ultrafilters, we have

already observed that (1) every free ultrafilter on a denumerable set is countably free and (2) on every infinite set there is a countably free ultrafilter. It is further known that it is consistent with the usual axioms of set theory to assume that *every free ultrafilter is countably free*. Also the continuum hypothesis implies that *every free ultrafilter on $X = R$ is countably free*. For these facts see [7]. Finally we remark that the fields Q^{μ} and R^{μ} may have very large cardinality: For any X there is a countably free ultrafilter on X such that Q^{μ} has the same cardinality as Q^X (cf. [6]).

References

1. N. Bourbaki, *Elements of general topology*. Part I, Addison-Wesley, Reading, Mass., 1966. MR **34** #5044b.

2. Hugh S. Carroll, *Topological aspects of ordered division rings*, Masters Thesis, University of Nebraska, Lincoln, Neb., 1963.

3. T. Frayne, A. C. Morel and D. S. Scott, *Reduced direct products*, Fund. Math. **51** (1962/63), 195–228. MR **26** #28.

4. J. D. Halpern, *The independence of the axiom of choice from the Boolean prime ideal theorem*, Fund. Math. **55** (1964), 57–66. MR **29** #2182.

5. E. Hewitt, *Rings of real-valued continuous functions*. I, Trans. Amer. Math. Soc. **64** (1948), 45–99. MR **10**, 126.

6. H. J. Keisler, *On cardinalities of ultraproducts*, Bull. Amer. Math. Soc. **70** (1964), 644–647. MR **29** #3384.

7. H. J. Keisler and A. Tarski, *From accessible to inaccessible cardinals. Results holding for all accessible cardinal numbers and the problem of their extension to inaccessible ones*, Fund. Math. **53** (1963/64), 225–308. MR **29** #3385.

8. J. Łoś, *Quelques remarques, théorèmes, et problèmes sur les classes définissables d'algébres*, Mathematical Interpretation of Formal Systems (Sympos., Amsterdam, 1954), Studies in Logic and the Foundations of Mathematics, North-Holland, Amsterdam, 1955, pp. 98–113. MR **17**, 700.

9. W. A. J. Luxemburg, *Non-standard analysis*, Lectures on A. Robinson's Theory of Infinitesimals and Infinitely Large Numbers, California Institute of Technology, Pasadena, Calif., 1964.

10. A. Robinson, *Non-standard analysis*, Studies in Logic and the Foundations of Mathematics, North-Holland, Amsterdam, 1966. MR **34** #5680.

11. ——, *Nonstandard arithmetic*, Bull. Amer. Math. Soc. **73** (1967), 818–843. MR **36** #1319.

University of Colorado, Boulder, Colorado 80302