ON THE NUMBER OF COMPLETE BOOLEAN ALGEBRAS

J. D. MONK[1]) and R. M. SOLOVAY[2])

It is known that for any infinite cardinal $\mathfrak{m}$ there are exactly $2^{\mathfrak{m}}$ isomorphism types of Boolean algebras of power $\mathfrak{m}$. This result and generalizations to the counting of more restricted kinds of Boolean algebras were established independently by Efimov and Kuznetzov [4], Shelah [9], and Carpintero [1], [2], [3] (Shelah's result is much more general). Still open in these papers is the counting problem for complete, or $\mathfrak{m}$-complete, Boolean algebras. In the present note we shall give a partial solution to the counting problem for complete Boolean algebras. Namely, we shall prove that for any infinite cardinal $\mathfrak{m}$, there are exactly $2^{2^{\mathfrak{m}}}$ isomorphism types of complete Boolean algebras of power $2^{\mathfrak{m}}$. Now Pierce [8] has shown that a complete Boolean algebra of infinite power $\mathfrak{m}$ exists iff $\mathfrak{m}^{\aleph_0}=\mathfrak{m}$. Hence the following problem remains open.

PROBLEM. If $\mathfrak{m}$ is infinite, $\mathfrak{m}^{\aleph_0}=\mathfrak{m}$, but $\mathfrak{m}$ does not have the form $2^{\mathfrak{n}}$, are there $2^{\mathfrak{m}}$ isomorphism types of complete Boolean algebras of power $\mathfrak{m}$?

The simplest cases of this problem are $\mathfrak{m}=\beth_{\omega_1}$ (where $\beth_0=\aleph_0$, $\beth_{\alpha+1}=2^{\beth_\alpha}$, $\beth_\lambda=\bigcup_{\alpha<\lambda}\beth_\alpha$ for λ a limit ordinal), $\mathfrak{m}=\aleph_{\omega_1}$ assuming GCH, or $\mathfrak{m}=\aleph_2$ assuming $2^{\aleph_0}=\aleph_1$ and $2^{\aleph_1}>\aleph_2$.

Throughout this note $\mathfrak{m}$ will be a fixed but arbitrary infinite cardinal. 'CBA' is an abbreviation for 'complete Boolean algebra'. SA is the set of all subsets of A. A Boolean algebra $\mathfrak{A}$ satisfies the $\mathfrak{m}$-chain condition if every disjointed subset of A has power $<\mathfrak{m}$.

By a well-known theorem of Hausdorff [6] let $M\subseteq S\mathfrak{m}$ be a family of independent sets with $|M|=2^{\mathfrak{m}}$. Thus if F and G are disjoint finite subsets of M then

$$\bigcap_{X\in F} X \cap \bigcap_{X\in G} (\mathfrak{m}\sim X)\neq 0. \tag{1}$$

Note that there are infinitely many elements in each of these intersections. Let t be a one-one map from $S\mathfrak{m}$ onto M. For each $R\subseteq S\mathfrak{m}$ such that $|S\mathfrak{m}\sim R|=2^{\mathfrak{m}}$ we now define a CBA $\mathfrak{C}_R$. Let $A_R=\{t_a: a\in S\mathfrak{m}\sim R\}$. Let $\mathscr{P}_R$ consist of all pairs (k, K) such that k is a finite subset of $\mathfrak{m}$ and K is a finite subset of A_R. We partially order $\mathscr{P}_R$ by setting $(k_1, K_1)\leq(k_2, K_2)$ iff $k_1\subseteq k_2$, $K_1\subseteq K_2$, and $k_2\cap\bigcup K_1\subseteq k_1$. For each $(k, K)\in\mathscr{P}_R$ let $\mathcal{O}_{(k,K)}=\{(k_1, K_1)\in\mathscr{P}_R: (k, K)\leq(k_1, K_1)\}$. Then the collection of all sets

[1]) Research supported in part by NSF grant GP-28070.
[2]) Research supported in part by NSF grant GP-33951.

Presented by R. S. Pierce. Received July 10, 1972. Accepted for publication in final form October 2, 1972.

$\mathcal{O}_{(k,K)}$ for $(k, K)\in\mathscr{P}_R$ forms a base for topology on $\mathscr{P}_R$, as is easily checked. We let $\mathfrak{C}_R$ be the complete Boolean algebra of regular open sets in this topology (see Halmos [5]). The remainder of this note is devoted to showing that each CBA $\mathfrak{C}_R$ has power $2^{\mathfrak{m}}$, and that there are $2^{2^{\mathfrak{m}}}$ isomorphism types among them. The construction of $\mathfrak{C}_R$ is taken from Martin, Solovay [7], and many parts of the proofs below are adapted from that paper to the present simpler situation.[3]) Some further notation: if $z\in\mathscr{P}_R$ we let $b_R z$ be the interior of the closure of $\mathcal{O}_z$; thus $b_R z\in\mathfrak{C}_R$. For $\alpha<\mathfrak{m}$, let $a_\alpha^R=b_R(\{\alpha\}, 0)$. For some of the proofs below the following two facts are useful:

$$b_R z=\{w\in\mathscr{P}_R\colon \forall w'\geq w\ \exists z'\geq z\,(z'\geq w')\};$$
$$-\,b_R z=\{w\in\mathscr{P}_R\colon \forall z'\geq z\,(z'\ngeq w)\}.$$

These facts are easily established, using the observation that $\mathcal{O}_z$ is the smallest neighborhood of z.

LEMMA 1. *$\mathfrak{C}_R$ satisfies the $\mathfrak{m}^+$-chain condition.*

Proof. $\mathcal{O}_{(k,K)}\cap\mathcal{O}_{(l,L)}=0$ implies that $k\neq l$; the $\mathfrak{m}^+$-chain condition follows.

LEMMA 2. $b_R(k, K)=\{(l, L)\in\mathscr{P}_R\colon k\subseteq l\cup(\mathfrak{m}\sim\bigcup A_R),\ K\subseteq L,\ l\cap\bigcup K\subseteq k\}$.

Proof. First suppose that $(l, L)\in b_R(k, K)$. If $\alpha\in k\cap\bigcup A_R$, say $\alpha\in x\in A_R$. Then $(l, L)\leq(l, L\cup\{x\})$, so there is an (m, M) with $(l, L\cup\{x\})\leq(m, M)$ and $(k, K)\leq(m, M)$. It follows easily that $\alpha\in l$. Thus $k\subseteq l\cup(\mathfrak{m}\sim\bigcup A_R)$. Next, suppose that $y\in K\sim L$. By independence and the fact that each intersection (1) is infinite, choose $\alpha\in y\sim(\bigcup L\cup l)$. Then $(l, L)\leq(l\cup\{\alpha\}, L)$, so there is an (m, M) with $(l\cup\{\alpha\}, L)\leq(m, M)$ and $(k, K)\leq(m, M)$. Thus $\alpha\in k$, and hence by what has already been established, $\alpha\in l$, contradiction. Thus $K\subseteq L$. Finally, suppose that $\alpha\in l\cap\bigcup K$. Choosing (m, M) so that $(l, L)\leq(m, M)$ and $(k, K)\leq(m, M)$, we easily infer that $\alpha\in k$. This finishes the proof of $\subseteq$ in the equality of the lemma. The converse inclusion $\supseteq$ is easily established.

LEMMA 3. $|\mathfrak{C}_R|\geq 2^{\mathfrak{m}}$.

Proof. By Lemma 2, $b_R(0, \{t\})=\{(l, L)\in\mathscr{P}_R\colon t\in L,\ l\subseteq\mathfrak{m}\sim t\}$ for each $t\in A_R$. Thus $b_R(0, \{s\})\neq b_R(0, \{t\})$ for $s\neq t$, and Lemma 3 follows.

LEMMA 4. *$\mathfrak{C}_R$ is completely generated by a set with $\leq\mathfrak{m}$ elements.*

Proof. First note, using Lemma 2:

$$a_\alpha^R=\{(l, L)\colon \alpha\in l\}\quad\text{if}\quad \alpha\in\bigcup A_R \tag{2}$$

[3]) Thanks are due to R. S. Pierce for comments on an earlier draft of this note, which led to making the proofs independent of [7].

$$a_\alpha^R = \mathscr{P}_R \quad \text{if} \quad \alpha \in \mathfrak{m} \sim \bigcup A_R \tag{3}$$

$$-a_\alpha^R = \{(l, L) \colon \alpha \in \bigcup L \sim l\} \quad \text{if} \quad \alpha \in \bigcup A_R \tag{4}$$

From (2)–(4) and Lemma 2 we easily obtain

$$\begin{aligned} b_R(k, K) &= \bigcap_{\alpha \in k} a_\alpha^R \cap \bigcap_{\alpha \in \bigcup K \sim k} - a_\alpha^R \cdot \\ &= \prod_{\alpha \in k} a_\alpha^R \cdot \prod_{\alpha \in \bigcup K \sim k} - a_\alpha^R \end{aligned} \tag{5}$$

Thus $\mathfrak{C}_R$ is completely generated by all elements a_α^R, as desired.

By Lemmas 1, 3, 4 it follows easily that

LEMMA 5. $|\mathfrak{C}_R| = 2^{\mathfrak{m}}$.

Now we turn to the proof that many of the algebras $\mathfrak{C}_R$ are non-isomorphic. To this end, we say that a set $R \subseteq S\mathfrak{m}$ is *represented in a complete Boolean algebra* D *by* $x \in^{\mathfrak{m}} D$ provided that

$$R = \{c \subseteq \mathfrak{m} \colon \sum \{x\alpha \colon \alpha \in t_c\} = 1\}. \tag{6}$$

Obviously we have

LEMMA 6. *If* $\mathfrak{D}$ *is a* CBA *of power* $2^{\mathfrak{m}}$, *then there are at most* $2^{\mathfrak{m}}$ *sets* $R \subseteq S\mathfrak{m}$ *representable in* $\mathfrak{D}$ *by some* $x \in^{\mathfrak{m}} \mathfrak{D}$.

LEMMA 7. *For any* $R \subseteq S\mathfrak{m}$ *such that* $|S\mathfrak{m} \sim R| = 2^{\mathfrak{m}}$, *the function* a^R *represents* R *in* $\mathfrak{C}_R$.

Proof. If $c \in S\mathfrak{m} \sim R$, then by (5) above,

$$0 \neq b_R(0, \{t_c\}) = \prod \{-a_\alpha^R \colon \alpha \in t_c\}$$

and hence c is not in the right hand side of (6). Now assume that $c \in R$. Using (2) and (3) it is clear that $\bigcup \{a_\alpha^R \colon \alpha \in t_c\}$ is dense; in fact, if $(k, K) \in \mathscr{P}_R$ is arbitrary, we may choose $\alpha \in t_c \sim \bigcup K$ by independence; then $(k \cup \{\alpha\}, K) \in \mathscr{O}_{(k, K)} \cap a_\alpha^R$. Hence $\sum \{a_\alpha^R \colon \alpha \in t_c\} = 1$, i.e., c is in the right hand side of (6). This completes the proof.

Immediately from Lemmas 5–7 we have the main result of this note:

THEOREM. *For any infinite cardinal* $\mathfrak{m}$ *there are exactly* $2^{2^{\mathfrak{m}}}$ *isomorphism types of complete Boolean algebras of power* $2^{\mathfrak{m}}$.

REFERENCES

[1] P. Carpintero, *Número de tipos diferentes de álgebras de Boole de cardinal* $\mathfrak{m}$ *infinito*, Acta Salmanticensia, Universidad de Salamanca *38* (1971), 59 pp.

[2] P. Carpintero, *Cuatro trabajos sobre topología, álgebras de Boole, hipotesis general del continuo y espacios funcionales*, Acta Salmanticensia, Universidad de Salamanca *40* (1971), 120 pp.
[3] P. Carpintero, *Número de tipos diferentes de álgebras de Boole de cardinal* $\mathfrak{m}$ *infinito que poseen* $2^{\mathfrak{m}}$ *ideals primos*, Revista Matemática Hispano-Americana *31* (1971), 93–97.
[4] B. Efimov, and V. Kuznetsov, *On the topological types of dyadic spaces*, (in Russian) Dokl. AN SSSR, vol. *195* (1970), 20–23.
[5] P. R. Halmos, *Lectures on Boolean algebras*, Van Nostrand 1963, 147 pp.
[6] F. Hausdorff, *Über zwei Sätze von G. Fichtenholz und L. Kantorovitch*, Studia Math. *6* (1936), 18–19.
[7] D. Martin, and R. M. Solovay, *Internal Cohen extensions*, Annals Math. Logic *2* (1970), 143–178.
[8] R. Pierce, *A note on complete Boolean algebras*, Proc. Amer. Math. Soc. *9* (1958), 892–896.
[9] S. Shelah, *The number of non-isomorphic models of an unstable first-order theory*, Israel J. Math. *9* (1971), 473–487.

University of California
Berkeley, California
University of Colorado
Boulder, Colorado