ON THE NUMBER OF COMPLETE BOOLEAN ALGEBRAS

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It is known that for any infinite cardinal $m$ there are exactly $2^m$ isomorphism types of Boolean algebras of power $m$. This result and generalizations to the counting of more restricted kinds of Boolean algebras were established independently by Efimov and Kuznetzov [4], Shelah [9], and Carpintero [1], [2], [3] (Shelah’s result is much more general). Still open in these papers is the counting problem for complete, or $m$-complete, Boolean algebras. In the present note we shall give a partial solution to the counting problem for complete Boolean algebras. Namely, we shall prove that for any infinite cardinal $m$, there are exactly $2^{2m}$ isomorphism types of complete Boolean algebras of power $2^m$. Now Pierce [8] has shown that a complete Boolean algebra of infinite power $m$ exists iff $m^{<m} = m$. Hence the following problem remains open.

PROBLEM. If $m$ is infinite, $m^{<m} = m$, but $m$ does not have the form $2^n$, are there $2^m$ isomorphism types of complete Boolean algebras of power $m$?

The simplest cases of this problem are $m = \exists_{\omega_1}$ (where $\exists_0 = \aleph_0$, $\exists_{n+1} = 2^{\exists_n}$, $\exists_\lambda = \bigcup_{\lambda < \gamma} \exists_{\gamma}$ for $\lambda$ a limit ordinal), $m = \aleph_\omega$, assuming GCH, or $m = \aleph_2$ assuming $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} > \aleph_2$.

Throughout this note $m$ will be a fixed but arbitrary infinite cardinal. ‘CBA’ is an abbreviation for ‘complete Boolean algebra’. $SA$ is the set of all subsets of $A$. A Boolean algebra $\mathcal{A}$ satisfies the $m$-chain condition if every disjoint subset of $A$ has power $< m$.

By a well-known theorem of Hausdorff [6] let $M \subseteq \text{Sm}$ be a family of independent sets with $|M| = 2^m$. Thus if $F$ and $G$ are disjoint finite subsets of $M$ then

$$
\bigcap_{X \in F} X \cap \bigcap_{X \in G} (m \sim X) \neq 0.
$$

Note that there are infinitely many elements in each of these intersections. Let $t$ be a one-one map from $\text{Sm}$ onto $M$. For each $R \subseteq \text{Sm}$ such that $|\text{Sm} \sim R| = 2^m$ we now define a CBA $\mathcal{C}_R$. Let $A_R = \{t_\alpha : \alpha \in \text{Sm} \sim R\}$. Let $\mathcal{P}_R$ consist of all pairs $(k, K)$ such that $k$ is a finite subset of $m$ and $K$ is a finite subset of $A_R$. We partially order $\mathcal{P}_R$ by setting $(k_1, K_1) \leq (k_2, K_2)$ iff $k_1 \subseteq k_2$, $K_1 \subseteq K_2$, and $k_2 \cap K_1 \subseteq k_1$. For each $(k, K) \in \mathcal{P}_R$ let $\vartheta(k, K) = \{(k_1, K_1) \in \mathcal{P}_R : (k, K) \leq (k_1, K_1)\}$. Then the collection of all sets

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\( \mathcal{C}_K \) for \((k, K) \in \mathcal{P}_R \) forms a base for topology on \( \mathcal{P}_R \), as is easily checked. We let \( \mathcal{C}_R \) be the complete Boolean algebra of regular open sets in this topology (see Halmos [5]). The remainder of this note is devoted to showing that each CBA \( \mathcal{C}_R \) has power \( 2^m \), and that there are \( 2^m \) isomorphism types among them. The construction of \( \mathcal{C}_R \) is taken from Martin, Solovay [7], and many parts of the proofs below are adapted from that paper to the present simpler situation.\(^9\) Some further notation: if \( z \in \mathcal{P}_R \) we let \( b_R z \) be the interior of the closure of \( \mathcal{C}_z \); thus \( b_R z \in \mathcal{C}_R \). For \( \alpha < m \), let \( a^\alpha = b_R(\{x\}, 0) \). For some of the proofs below the following two facts are useful:

\[
\begin{align*}
& b_R z = \{ w \in \mathcal{P}_R : \forall w' \geq w \exists z' \geq z (z' \geq w') \} ; \\
& -b_R z = \{ w \in \mathcal{P}_R : \forall z' \geq z (z' \geq w') \} .
\end{align*}
\]

These facts are easily established, using the observation that \( \mathcal{C}_z \) is the smallest neighborhood of \( z \).

**Lemma 1.** \( \mathcal{C}_R \) satisfies the \( m^+ \)-chain condition.

**Proof.** \( \mathcal{C}_{(k, k)} \cap \mathcal{C}_{(l, l)} = 0 \) implies that \( k \neq l \); the \( m^+ \)-chain condition follows.

**Lemma 2.** \( b_R(k, K) = \{(l, L) \in \mathcal{P}_R : k \leq l \cup (m \sim \bigcup A_R), K \leq L, l \cap \bigcup K \leq k \} \).

**Proof.** First suppose that \( (l, L) \in b_R(k, K) \). If \( \alpha \neq k \cap \bigcup A_R \), say \( \alpha \in x \in A_R \). Then \( (l, L) \leq (l, L \cup \{x\}) \), so there is an \( (m, M) \) with \( (l, L \cup \{x\}) \leq (m, M) \) and \( (k, K) \leq (m, M) \). It follows easily that \( \alpha \notin l \). Thus \( k \leq l \cup (m \sim \bigcup A_R) \). Next, suppose that \( y \in K \sim L \). By independence and the fact that each intersection \((1)\) is infinite, choose \( \alpha \in y \sim (\bigcup l \cup l) \). Then \( (l, L) \leq (l \cup \{x\}, L) \), so there is an \( (m, M) \) with \( (l \cup \{x\}, L) \leq (m, M) \) and \( (k, K) \leq (m, M) \). Thus \( \alpha \neq k \), and hence by what has already been established, \( \alpha \notin l \), contradiction. Thus \( K \leq L \). Finally, suppose that \( \alpha \in l \cap \bigcup K \). Choosing \( (m, M) \) so that \( (l, L) \leq (m, M) \) and \( (k, K) \leq (m, M) \), we easily infer that \( \alpha \neq k \). This finishes the proof of \( \leq \) in the equality of the lemma. The converse inclusion \( \supseteq \) is easily established.

**Lemma 3.** \( |\mathcal{C}_R| \geq 2^m \).

**Proof.** By Lemma 2, \( b_R(0, \{t\}) = \{(l, L) \in \mathcal{P}_R : t \in L, l \leq m \sim t \} \) for each \( t \in A_R \). Thus \( b_R(0, \{s\}) \neq b_R(0, \{t\}) \) for \( s \neq t \), and Lemma 3 follows.

**Lemma 4.** \( \mathcal{C}_R \) is completely generated by a set with \( \leq m \) elements.

**Proof.** First note, using Lemma 2:

\[
\alpha^R = \{(l, L) : \alpha \in l \} \quad \text{if} \quad \alpha \in \bigcup A_R
\]

\(^9\) Thanks are due to R. S. Pierce for comments on an earlier draft of this note, which led to making the proofs independent of [7].
\[ a^R_\alpha = \mathcal{P}_R \quad \text{if} \quad \alpha \in m \sim \bigcup A_R \quad (3) \]
\[ -a^R_\alpha = \{(l, L) : \alpha \in \bigcup L \sim l\} \quad \text{if} \quad \alpha \in \bigcup A_R \quad (4) \]

From (2)–(4) and Lemma 2 we easily obtain
\[ b_R(k, K) = \bigcap_{\alpha \in k} a^R_\alpha \cap \bigcap_{\alpha \in \bigcup K \sim k} -a^R_\alpha. \]
\[ = \prod_{\alpha \in k} a^R_\alpha \cdot \prod_{\alpha \in \bigcup K \sim k} -a^R_\alpha \quad (5) \]

Thus \( \mathcal{C}_R \) is completely generated by all elements \( a^R_\alpha \), as desired.

By Lemmas 1, 3, 4 it follows easily that

**LEMMA 5.** \( |\mathcal{C}_R| = 2^m \).

Now we turn to the proof that many of the algebras \( \mathcal{C}_R \) are non-isomorphic.

To this end, we say that a set \( R \subseteq Sm \) is represented in a complete Boolean algebra \( D \) by \( x \in m \) \( D \) provided that
\[ R = \{ c \subseteq m : \sum \{ x \alpha : \alpha \in t_c \} = 1 \}. \quad (6) \]

Obviously we have

**LEMMA 6.** If \( \mathcal{D} \) is a CBA of power \( 2^m \), then there are at most \( 2^m \) sets \( R \subseteq Sm \) representable in \( \mathcal{D} \) by some \( x \in m \mathcal{D} \).

**LEMMA 7.** For any \( R \subseteq Sm \) such that \( |Sm \sim R| = 2^m \), the function \( a^R \) represents \( R \) in \( \mathcal{C}_R \).

**Proof.** If \( c \in Sm \sim R \), then by (5) above,
\[ 0 \neq b_R(0, \{ t_c \}) = \prod \{ -a^R_\alpha : \alpha \in t_c \} \]
and hence \( c \) is not in the right hand side of (6). Now assume that \( c \in R \). Using (2) and (3) it is clear that \( \bigcup \{ a^R_\alpha : \alpha \in t_c \} \) is dense; in fact, if \( (k, K) \in \mathcal{P}_R \) is arbitrary, we may choose \( \alpha \in t_c \sim \bigcup K \) by independence; then \( (k \cup \{ \alpha \}, K) \in \mathcal{C}_R \cap a^R_\alpha \). Hence \( \sum \{ a^R_\alpha : \alpha \in t_c \} = 1 \), i.e., \( c \) is in the right hand side of (6). This completes the proof.

Immediately from Lemmas 5–7 we have the main result of this note:

**THEOREM.** For any infinite cardinal \( m \) there are exactly \( 2^{2^m} \) isomorphism types of complete Boolean algebras of power \( 2^m \).

**REFERENCES**


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