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ON THE FOUNDATIONS OF SET THEORY

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I want to discuss here the relevance to mathematicians, as teachers and researchers, of some of the recent discoveries about axiomatic set theory. Most readers have heard of these advances, which began just a few years ago with Cohen's work. The results are certainly intellectually amazing to all of us. I think they may even give rise to certain changes in our teaching and research, and the purpose of this paper is to describe some possibilities along these lines. To set the stage and fix the ideas I shall first describe a few of these discoveries in a fairly precise way. Then, in the nonexact portion of the paper, I shall discuss some possible changes in teaching and research, and also some philosophical views which are affected by these discoveries.

1. A survey of results. A much more comprehensive (and more technical) survey can be found in Mathias [7]. Here I state just a very few results, but I wish to emphasize that the nonmathematical arguments of the next section apply in some form to virtually all of the results described in [7]. I assume that the reader has a modest acquaintance with the idea of a language and a metalanguage, and with the precise notions of a (first-order) sentence, a (formal) proof, and a theorem. In this section I work in a metalanguage and talk about the language of mathematics. I leave the metalanguage unspecified in detail; to begin with I assume that it is rather weak, with just enough machinery to

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formulate the above precise notions. If T is a list of sentences (thought of as axioms for a certain theory), and ϕ is a single sentence, I write $T\vdash\phi$ to indicate that there is a formal proof of ϕ from T, i.e., that there is a list of sentences, each of which is either a logical axiom, appears in the list T, or is obtained from earlier sentences of the list by applying a rule of inference. I write $T\nvdash\phi$ if there is no proof of ϕ from T. Before embarking on a description of set-theoretical matters I want to mention two famous theorems of Gödel which form a background upon which to view the latest results.

THEOREM 1. (Incompleteness theorem.) *If T is a consistent, sufficiently strong, effective list of sentences, then there is a sentence ϕ such that $T\nvdash\phi$ and $T\nvdash\neg\phi$.*

Here "$\neg$" is an abbreviation for "not." T is *consistent* if $T\nvdash\psi$ for some ψ. To say that T is *sufficiently strong* means, roughly speaking, that T embodies enough mathematics to develop elementary number theory; technically speaking, Peano arithmetic, P, is relatively interpretable in T. To say that T is *effective* means that the list T is presented in a reasonable manner—reasonable enough for oe tno be able to recognize by some algorithm when a sentence is in the list T. Certainly T is effective if T is finite: to check if a sentence ϕ is in T just look through the whole list, a process which in principle terminates with the last member of T. But Theorem 1 applies to some infinite lists also. Precisely speaking, one assigns numbers, called *Gödel numbers*, to all sentences, and T is called effective if the set of Gödel numbers of members of T is a recursive set. Clearly, to express this notion of effectiveness my original weak metalanguage must be strengthened enough to work in an elementary way with recursive functions, integers, and sets of integers. The other theorems of this section are also formulated in this stronger metalanguage, which is still much weaker than the ordinary language of mathematics.

Theorem 1 itself has a profound philosophical significance. According to it, one cannot hope to base all of conceivable mathematics on a single axiomatic basis; it points out the necessity of a continuous search for additional axioms. The importance of the theorem can be more appreciated in connection with the opposed philosophical views of formalism and platonism which will be discussed in section 2. I want to point out now, though, that the incompleteness theorem has not had much effect on the attitude of the working mathematician. In contrast, the latest results concerning the independence of the Axiom of Choice and the Continuum Hypothesis are already having an effect on teaching and research. I think that one of the main reasons for the lesser practical significance of the incompleteness theorem lies in the nature of the known proofs of the theorem. The sentence ϕ whose existence is asserted in the theorem *is* effectively constructed, but its intuitive meaning is not to be found in ordinary mathematics. It can be interpreted as asserting a relationship between formal proofs and its own Gödel number, and its construction is another instance of the well-known Cantor diagonal method. Theorem 1, essentially due to Gödel, can be found in Tarski, Mostowski, Robinson [10] (it follows immediately from Theorems I1, I7, I10, and II9 there).

If T is an effective list of sentences, one can effectively construct a number-theoretic sentence $ConT$ which expresses the statement that T is consistent (under a natural interpretation, via Gödel numbers).

THEOREM 2. (Gödel's second underivability theorem.) *If T is a consistent, sufficiently strong, effective list of sentences, then $T \nvdash ConT$.*

See Feferman [2] for a proof of this theorem; it is also pointed out in [2] that one must take some care in the construction of the sentence $ConT$. This second theorem of Gödel again puts a limitation on what one can do in the foundations of mathematics. If T is a list of sentences which provides axioms for a large portion of mathematics, then by Theorem 1 not all mathematics is encompassed by T. One is thus forced to be somewhat modest. Theorem 2 forces a more severe degree of modesty: it is impossible to prove the consistency of T (without using devices not available in T itself). One would like to know that working with T has some significance, which it does not if T is inconsistent; so no method is available for logically proving that mathematics is significant.

I now turn to set theory. Students are taught these days that all mathematics can be based on set theory, indeed, that ordinary mathematics logically speaking is just a branch of set theory. It is, in fact, well established that almost all mathematics can be reduced to set theory. The only doubts that may arise concerning such a reduction have to do with the recently developed theory of categories. Anyway, in talking about set theory I think one is talking about essentially all of mathematics. I shall fix upon a particular list ZF of axioms for set theory, the *Zermelo-Fraenkel axioms*. These are essentially the axioms used in Bourbaki [1] and Halmos [5], but I assume that the axiom of regularity is also included (see Monk [8]), while the Axiom of Choice is excluded. By the *Axiom of Choice*, for short AC, I mean the statement that if A is a family of nonempty sets, then there is a function f (called a *choice function* for A) such that $f(x) \in x$ for each $x \in A$. If A is a finite family (*finite* defined conventionally), then such a function can be proved to exist within ZF, using only the simplest set-theoretical axioms (this was overlooked in Hall, Spencer [4], p. 282). In general, however, the existence of f requires infinitely many choices, and there is no principle within ZF which makes this legal. The axiom of choice can be proved equivalent, within ZF, to Zorn's lemma, to the Well-ordering Principle, to Tukey's lemma, etc. By the *Continuum Hypothesis*, for short CH, I mean the statement that any infinite set of real numbers can be placed in one-to-one correspondence either with the integers or with the set of all real numbers. Finally, the *Generalized Continuum Hypothesis*, GCH for short, asserts that for any infinite set X and any family Y of subsets of X, Y can be placed in one-to-one correspondence either with a subset of X or with the set of all subsets of X. Since in $ZF+AC$ one can show that the reals can be placed in one-to-one correspondence with the set of all subsets of Z (the set of integers), it follows that

$ZF+AC\vdash GCH\rightarrow CH$. Now I shall list a few of the important recent results in the foundations of set theory.

THEOREM 3. (Gödel, 1938.) $Con(ZF)\Rightarrow Con(ZF+AC+GCH)$.

I could just as well formulate Theorem 3 as follows:

If ZF is consistent, then so is $ZF+AC+GCH$.

Theorem 3 is written as a theorem of number theory as so to make more precise the assumptions in the metalanguage and to establish a connection with Theorem 2. The question of consistency of $ZF+AC+GCH$ reduces to the same question for ZF. By Theorem 2 this latter question cannot be given a rigorous affirmative answer. Since set theory, in particular ZF, has been used so much in the last century, mathematicians have grown confident that it is, in fact, consistent. Thus one can assert with the same degree of confidence that $ZF+AC+GCH$ is consistent.

THEOREM 4. (Cohen, 1963.) $Con(ZF)\Rightarrow Con(ZF+AC+\neg CH)$.

In Theorem 4, $\neg CH$ can be taken in various very specific forms, such as $2^{\aleph_0}=\aleph_2$, or $2^{\aleph_a}=\aleph_3$, or $2^{\aleph_0}=\aleph_{\omega+1}$. The most general results of this sort have been established by Easton and Solovay; see, e.g. [9].

THEOREM 5. (Fraenkel, Mostowski, Cohen, 1929–1963.) $Con(ZF)\Rightarrow Con(ZF+\neg AC)$.

Again, $\neg AC$ can be taken in many more definite forms, for example: there is a countable collection of unordered pairs without a choice function.

THEOREM 6. (Vitali, 1905.) $ZF+AC\vdash \exists x$ (*x is a set of real numbers, but x is not Lebesgue measurable*).

THEOREM 7. (Solovay, 1965.) $Con(ZF'+AC)\Rightarrow Con(ZF+\neg AC+\forall x$ (*if x is a set of real numbers, then x is Lebesgue measurable*)).

Here ZF' is obtained from ZF by replacing the usual axiom of infinity by a stronger one which asserts the existence of an uncountable strongly inaccessible cardinal. This stronger axiom is coming more and more to be an accepted part of set theory. For example, category theory appears to require this axiom, or even stronger axioms, to justify its methods. Intuitively, $Con(ZF'+AC)$ seems as plausible as $Con(ZF+AC)$; like $ZF+AC$ itself, the consequences of $ZF'+AC$ have been rather well worked-out and some confidence can be placed in its consistency. Since, however, $ZF'+AC\vdash Con(ZF+AC)$, the new axiom of infinity is much stronger than the old axiom (see Theorem 2).

THEOREM 8. (Solovay.) $Con(ZF)\Rightarrow Con(ZF+\neg AC+\exists x$ (*x is a set of real numbers and x is not Lebesgue measurable*)).

By Theorem 8, the Axiom of Choice is not equivalent to the existence of non-Lebesgue-measurable sets of real numbers.

These theorems I have listed represent a small sampling of results known in this area; they are sufficient to form a basis for the non-mathematical arguments in the next section. The interested reader can think of many familiar theorems whose proofs involve the Axiom of Choice, for example, and ask whether results similar to the above for Lebesgue measure hold. Thus the existence of Hamel bases, the Hahn-Banach extension theorem, the Boolean prime ideal theorem, the Banach-Tarski paradox, and the extendability of a partial order to a linear order are theorems which give rise to independence questions of this sort. Even assuming the Axiom of Choice, many statements of a set-theoretical nature have until recently been open. Examples are: the existence of a nontrivial measure on the set of all subsets of a set, and Souslin's hypothesis. Some of these many natural hypotheses have now been settled (in the sense of being shown independent), while others are still under attack. The intuitive remarks in the next section apply to all of these questions.

2. Meaning of the results. What do these results "do" for the ordinary mathematician? Before indicating some specific possibilities along these lines, it is worthwhile briefly to take a deeper view of the significance of the results. The results throw a great deal of light on a certain dichotomy in the philosophy of mathematics which now has a long history. Without trying to connect theories in the philosophy of mathematics with broader philosophical trends, I will distinguish two extreme views, *platonism* and *formalism*. These are not the only possible philosophies of mathematics. For example, intuitionism has a great appeal and is close to the beliefs of many practicing mathematicians. But the philosophies of most mathematicians can be construed as somewhere in the range between extreme platonism and extreme formalism. Practicing mathematicians, consciously or not, subscribe to some philosophy of mathematics (if unstudied, it is usually inconsistent). If you make a simple reference to the real numbers, you express a tendency toward platonism. And if you refer to a theorem as correct because it follows from the axioms of set theory, you tend toward formalism.

According to extreme platonism, mathematical objects are real, as real as the world we live in. Thus infinite sets exist, not just as a mental construct but in a real sense, perhaps in a "hyperworld." Similarly, nondenumerable sets, real numbers, choice functions, Lebesgue measure, etc., have a real existence. From the point of view of platonism, the purpose of a mathematician is to discover some of the facts of nature. His job is thus quite similar to that of a physicist, chemist, or biologist. The various possible axioms of set theory are then either true or false, and one of the main aims in the foundations of mathematics is to develop correct intuitions so as to determine which are the true axioms; these may then be taken as a rigorous basis for set theory. Actually, for a platonist, axiomatic development of set theory is not essential, but is perhaps useful to keep from making mistakes. (Even a platonist, however, will admit the importance of axiomatic treatments outside set theory, as in topology or group theory, because of the usefulness of axiomatizations for abstractions and classifications.)

A strict formalist, on the other hand, does not believe that any mathematical objects have a real existence. For him, mathematics is just the business of deriving sentences from axioms. It is a game, in that some definite rules must be followed in such derivations. Unsolved problems give rise to goals for the game; the winner is the one who solves the problem. The analogy with games like chess and go is very close. A formalist chooses which game to play, that is, which axioms to take and which problem to work on, using practical and artistic criteria. One set of axioms may be best suited to be a base for a physical theory like relativity, for example, and hence because of the predictive ability of the physical theory this set of axioms has a practical value. And, of course, some problems are more practical in nature than others. The criteria for choosing axioms and problems, when not practical in nature, are extremely varied. A certain axiom may enable one to resolve many questions that are difficult without its aid; assuming that GCH, for example, infinite cardinal arithmetic is very much simplified. Many investigations are made in order to try to relate two seemingly distant areas in mathematics; one may cite the duality theory for Boolean algebras, relating algebraic to topological structures, or the investigation of closure algebras—doing topology within algebra. These two criteria for choosing the right game are just examples, and are undoubtedly not the most important of those criteria which are not based on practical considerations.

The results of section 1 have different meanings for platonists and formalists. The incompleteness theorem (Theorem 1) shows the platonist that he cannot hope to capture all of mathematics in a completely rigorous form for once and for all. There will remain beyond any fixed rigorous framework a statement whose truth must be determined by intuition. On the other hand, a strict formalist may even doubt that Theorem 1 says something relevant to his activity, since, as I indicated in section 1, the formulation and proof of Theorem 1 require a metalanguage stronger than the minimal one needed to understand the notions of proof and theorem which are the basic "rules of the game" for the formalist. But if the formalist does admit the usual intuitive meaning of Theorem 1, the theorem will just be taken as evidence that one cannot be content with just one axiom system if one wants to develop a comprehensive part of mathematics. To a platonist, Theorem 2 shows again the weakness of axiom systems; to him a system such as ZF, for example, is obviously consistent, since all the axioms of ZF are intuitively true. A formalist would also view Theorem 2 as showing a weakness of formal systems, again, if he admitted the usual intuitive meaning of the theorem. The other results in Section 1, independence results in set theory, give several examples of important statements which cannot be decided on the basis of the usual axioms of set theory. Here the platonist will try to investigate the situation further, in hopes of finding an impelling intuitive principle with the aid of which these statements can be resolved one way or the other. The formalist will simply view the results as giving rise to several alternative paths of development of set theory.

These two philosophical viewpoints do not make much difference in the communication of mathematical results to other mathematicians. A correct mathematical argument from given premises is recognized as such by a platonist and by a formalist. The philosophical questions inherent in the dichotomy can be, and are, ignored in the writing of most mathematicians. In the main this indifference to the philosophy of mathematics has had a good effect on the progress of mathematics; mathematicians have insisted on proving theorems rather than spending a bulk of their time with difficult and ultimately inclusive philosophical speculations. Nonetheless, the two views have an effect on the direction of mathematical research. For example, a platonist may convince himself that CH is false (cf. Gödel [**3**]). He is then less likely to try to derive consequences from CH. But a formalist may very well like CH and even GCH because he can prove many nice theorems with their aid.

My remarks so far in this section concern mathematics itself, or at least mathematical research. I now turn to some specific possibilities for change in teaching and research which might come about because of the recent independence results. Mainly I will discuss the definition of the real number system. In beginning analysis courses it is customary to give the main properties of the real numbers and perhaps to carry out one of the constructions of the reals from the rationals. Many constructions of the reals are known; the ones using Dedekind cuts and Cauchy sequences are the most popular. All of these constructions turn out to be equivalent. This fact is rightly used, I think, as evidence for the naturalness of the notion of real number. Another basic, and satisfying, result here is uniqueness: any two Dedekind-complete ordered fields are isomorphic. But what are the real numbers? Under a platonistic point of view, the real numbers exist in nature; in teaching beginning mathematicians it would be nice to be able to point and say: here they are. A natural definition would be as an isomorphism equivalence class of Dedekind complete ordered fields. However, such classes are too big, and are not admitted as existing in ZF. There is a sophisticated way of chopping such classes down to a manageable size, but the method is not suitable for elementary classes (cf. Monk [**8**], p. 114). The only natural way out seems to be to fix upon a definite construction of the reals in order to give a specific definition for them. Then different mathematicians will have different definitions, but at least they can be shown equivalent.

Frequently the uniqueness theorem is used as a basis for asserting that all questions about the real numbers can be resolved, at least theoretically: anything true of one Dedekind-complete ordered field is true of another. But then Theorems 3 and 4 pose a puzzle. CH is a property of the real numbers (in a broad sense), and it is consistent to assume CH, but also consistent to assume $\neg CH$ (assuming ZF consistent). A closer analysis reveals the true state of affairs. Call a Dedekind-complete ordered field *a system of real numbers*. It is then provable within ZF that CH holds with respect to one system of real numbers if and only if it holds with respect to any other system of real numbers.

But neither side of this biconditional is *actually* provable in ZF. One may say that the uniqueness theorem (theoretically) reduces all questions about the reals to purely foundational, set-theoretical questions. It seems to me to be appropriate to bring discussions such as this down to the teaching level. Students should be aware of the possibility of getting different conceptions of the real numbers by choosing one or another of various hypotheses such as CH. It also seems to me that it would be useful to make students aware of the alternative philosophies of platonism and formalism.

Similarly, in the important applications of the Axiom of Choice I think it is appropriate to point out various alternatives that exist. Above all, I think a retreat from dogmatism is called for. In proving the existence of non-Lebesgue-measurable sets, it should be pointed out that, at a price, one can as well assume that every set of real numbers is Lebesgue-measurable. Similar remarks are appropriate whenever the mathematical or foundational results are of significant import to most mathematicians; certainly when one discusses such topics as the Hahn-Banach theorem, Souslin's hypothesis, the existence of maximal ideals, Tychonoff's theorem, etc.

The possibilities for remarks in classroom teaching thus appear to be very great, even though one cannot expect of teachers more than a mention of the independence results in question, since the proofs seem as yet inaccessible to an audience not able to devote some months to a study of these matters.

Possible uses of these results in research are rather obvious, but limited. Thus most mathematicians do not work in areas where a choice of CH or $\neg CH$ would make a big difference. And most modern mathematics depends fundamentally on the Axiom of Choice, so that the independence results such as I have mentioned are not of great practical import. But there are some instances in research where a new foundational hypothesis might prove useful. An analyst might like to assume that every set of real numbers is Lebesgue measurable. This can be done even while retaining a weak form of the Axiom of Choice. Again, the assumption that $2^{\aleph_0}=\aleph_2$ might facilitate the construction of counterexamples in certain contexts. Research based on Souslin's hypothesis has not been done very much. Many more possibilities could be stated, and research using these unusual hypotheses is needed. From a platonistic point of view such research might lead to a better insight into the nature of our underlying set theory.

I hope that the discussion I have given will convince more mathematicians to become familiar with the results obtained in the foundations of set theory and keep these results in mind in their teaching and research.

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THINKING GEOMETRICALLY*

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To many people the word "geometry" inevitably suggests a figure, a drawing. We are aware of the fact, apparently overlooked by Euclid, that we have to be very careful in arguing from a figure, that we may unwittingly assume properties which are not deducible from the given hypotheses, and may therefore arrive at incorrect logical conclusions. This may be a partial explanation of the fact that the whole subject of geometry, especially elementary geometry, is under attack these days. The leader of the attack, and a very formidable person he is, seems to be my old friend Prof. Jean Dieudonné. He is, of course, a very fine geometer, and a well-known member of the Bourbaki school.

Dieudonné has made his views known on a number of occasions, and most explicitly perhaps in a long preface to a book *Linear algebra and geometry*, pub-

* This paper is based on lectures given at the Universities of South Carolina and Toronto in the spring of 1968, and at Makerere College, Uganda and the University of the Witwatersrand in the summer of 1969. The invited address to the North Central Section of the MAA in April, 1969 contained similar material.

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Dan Pedoe is well known for his elegant geometrical expositions in many articles, films, and books. The latter include the 3 vol. *Methods of Algebraic Geometry* (with W. Hodge, Cambridge U. Press, 1947–1953), *Circles* (Pergamon Press, 1957), *Gentle Art of Mathematics* (English U. Press, 1958, Penguin, 1963), *Geometric Introduction to Linear Algebra* (Wiley, 1963), *Introduction to Projective Geometry* (Pergamon, 1964), and *Course of Geometry for Colleges and Universities* (Cambridge U. Press, forthcoming). He received an MAA Lester Ford Award in 1968. *Editor.*