THE JOURNAL OF SYMBOLIC LOGIC
Volume 35, Number 1 March, 1970

ON AN ALGEBRA OF SETS OF FINITE SEQUENCES[1]

J. DONALD MONK

The algebras studied in this paper were suggested to the author by William Craig as a possible substitute for cylindric algebras. Both kinds of algebras may be considered as algebraic versions of first-order logic. Cylindric algebras can be introduced as follows. Let $\mathscr{L}$ be a first-order language, and let $\mathfrak{A}$ be an $\mathscr{L}$-structure. We assume that $\mathscr{L}$ has a simple infinite sequence $v_0, v_1, \cdots$ of individual variables, and we take as known what it means for a sequence $x = \langle x_0, x_1, \cdots \rangle$ of elements of $\mathfrak{A}$ to satisfy a formula ϕ of $\mathscr{L}$ in $\mathfrak{A}$. Let $\widetilde{\phi}^{\mathfrak{A}}$ be the collection of all sequences x which satisfy ϕ in $\mathfrak{A}$. We can perform certain natural operations on the sets $\widetilde{\phi}^{\mathfrak{A}}$, of basic model-theoretic significance: Boolean operations $\sim\widetilde{\phi}^{\mathfrak{A}} = \widetilde{\neg\phi}^{\mathfrak{A}}$, $\widetilde{\phi}^{\mathfrak{A}} \cup \widetilde{\psi}^{\mathfrak{A}} = \widetilde{\phi \vee \psi}^{\mathfrak{A}}$; cylindrifications $c_i\widetilde{\phi}^{\mathfrak{A}} = \widetilde{\exists v_i \phi}^{\mathfrak{A}}$; diagonal elements (0-ary operations) $d_{ij} = \widetilde{v_i \equiv v_j}^{\mathfrak{A}}$. In this way we make the class of all sets $\widetilde{\phi}^{\mathfrak{A}}$ into an algebra; a natural abstraction gives the class $\mathscr{C}$ of all cylindric set algebras (of dimension ω). Thus this method of constructing an algebraic counterpart of first-order logic is based upon the notion of satisfaction of a formula by an infinite sequence of elements. Since, however, a formula has only finitely many variables occurring in it, it may seem more natural to consider satisfaction by a finite sequence of elements; then $\widetilde{\phi}^{\mathfrak{A}}$ becomes a collection of finite sequences of varying ranks (cf. Tarski [10]). In forming an algebra of sets of finite sequences it turns out to be possible to get by with only finitely many operations instead of the infinitely many c_i's and d_{ij}'s of cylindric algebras. Let $\mathscr{S}$ be the class of all algebras of sets of finite sequences (an exact definition is given in §1).

The purpose of this paper is to investigate some fundamental properties of the class $\mathscr{S}$. Like the class $\mathscr{C}$, the class of isomorphs of members of $\mathscr{S}$ is not elementary (Theorem 1.2). Corresponding to the class RCA_ω of representable cylindric algebras we can introduce in a natural way the class K of representable algebras over $\mathscr{S}$. But, while RCA_ω is an equational class, K is not even an elementary class (Theorem 1.3). These two facts do not lie very deep. A harder question concerns the equations which hold in all members of $\mathscr{S}$. The main portion of the paper is devoted to showing that a natural set of equations valid over $\mathscr{S}$ has no finite basis; for this purpose we have to use a construction related to one used for a similar purpose in the theory of cylindric algebras (see Monk [9]). We do not have explicit equations for the variety $\mathscr{S}'$ over $\mathscr{S}$, but it appears likely that methods developed by Craig will lead to a simple primitive recursive set of equations for $\mathscr{S}'$. Indeed, many of the ideas in this paper should be useful in developing such a set of equations. Thus the situation is probably different from the case of cylindric algebras; RCA_ω is not

Received August 28, 1968, revised May 2, 1969.

[1] Research supported in part by NSF grants GP7387 and GP6232-x.

finitely axiomatizable even by a schema of a certain sort, and the known equations characterizing RCA_ω are rather complicated (see Monk [9]).

The results obtained in this paper are, however, quite analogous to those obtained in [9] for cylindric algebras. They contribute to the conjecture that no equational form of first-order logic is finitely axiomatizable—more precisely, with respect to any conception $\mathscr{S}$ of a set algebra (corresponding to the notion of satisfaction), and any choice of basic operations, the corresponding class $\mathscr{S}'$ is not finitely axiomatizable. It appears difficult to give this conjecture a very precise form, since there is a wide latitude of choice with regard to the fundamental operations as well as the kinds of sequences considered in the satisfaction relation. The conjecture has been verified for most brands of algebraic logic known to the author. In addition to cylindric algebras and the modified conjecture in this paper it has been verified for polyadic algebras with or without equality (Johnson [7]), diagonal-free cylindric algebras (Johnson [7]), relation algebras (Monk [8]), some algebras considered by C. Howard (cf. Craig [3]), and Copeland algebras (Copeland [2]). In the last two cases (unpublished) a nonfinitizability result has been shown by D. Demaree, using for Howard's algebras a modification of the argument of the present paper.

I am indebted to William Craig for the suggestion to consider the algebras discussed in this paper, and also for many conversations which have clarified the relationship of these algebras with cylindric algebras. As will be seen, these relationships, which were first seen by Craig, are essential to the proof of the main result (cf. also Bernays [1]). I am also grateful to J. S. Johnson for his help in preparing this paper for publication.

We will use standard set-theoretical and logical notation. The set of all natural numbers is ω; letters $i, j, k, l, \cdots$ always denote natural numbers; each natural number is identical with the set of its predecessors, $i = \{j : j < i\}$. The set of all functions mapping A into B is denoted ${}^{A}B$. "$\mathrm{Dmn} f$" is the *domain* of the function f. If u is a finite sequence, $u \neq 0$, then we let lu be $\mathrm{Dmn}\, u - 1$; thus $u = \langle u_0, \cdots, u_{lu}\rangle$. ${}^{\omega}U$ is the set of all finite sequences of members of U. If f and g are two finite sequences, then $f \frown g$ is their concatentation: $f \frown g = \langle f_0, \cdots, f_{\mathrm{Dmn} f - 1}, g_0, \cdots, g_{\mathrm{Dmn}\, g - 1}\rangle$. $\langle s\rangle$ is the one-termed sequence with term s. If f is a function mapping A into A, then f^m is the mth iterate of f (f^0 is the identity function on A). $S(A)$ is the set of all subsets of A. If f is a function, $a_0, \cdots, a_{k-1} \in \mathrm{Dmn} f$, all distinct, and $b_0, \cdots, b_{k-1}$ are arbitrary, then $f^{a_0 \cdots a_{k-1}}_{b_0 \cdots b_{k-1}}$ is the function g such that $\mathrm{Dmn} f = \mathrm{Dmn}\, g$, $gx = fx$ for all $x \in \mathrm{Dmn} f \sim \{a_0, \cdots, a_{k-1}\}$, and $ga_i = b_i$ for each $i < k$. If $\tau(i)$ is a set-theoretical term, $\langle \tau(i) : i \in I\rangle$ is the function with domain I whose value at $i \in I$ is $\tau(i)$. Algebras are denoted by capital German letters and their universes by the corresponding Roman letter. In a Boolean algebra, $\oplus$ is the operation of symmetric difference.

§1. Representable $\mathscr{F}$-algebras. An *$\mathscr{F}$-algebra* is an algebra $\mathfrak{A} = \langle A, +, \cdot, -, P, P', Q, Q', D\rangle$ such that $\langle A, +, \cdot, -\rangle$ is a Boolean algebra, $P, P', Q, Q' \in {}^{A}A$, and $D \in A$. $\mathscr{F}$-algebras which play a central role in this paper are defined as follows. Let U be a nonempty set. We set $\mathfrak{A}(U) = \langle S({}^{\omega}U), \cup, \cap, \sim, P, P', Q, Q', D\rangle$, where

$$D = \{u \in {}^{\omega}U : u \neq 0 \text{ and } u_0 = u_{lu}\}$$

and for any $X \subseteq {}^{\omega}U$,

$$\begin{aligned} PX &= \{u \in {}^{\omega}U : \exists s \in U(\langle s \rangle \cap u \in X)\}, \\ P'X &= \{u \in {}^{\omega}U : \exists s \in U(u \cap \langle s \rangle \in X)\}, \\ QX &= \{u \in {}^{\omega}U : u \neq 0 \text{ and } \langle u_1, \cdots, u_{lu} \rangle \in X\}, \\ Q'X &= \{u \in {}^{\omega}U : u \neq 0 \text{ and } \langle u_0, \cdots, u_{lu-1} \rangle \in X\}. \end{aligned}$$

Subalgebras of the algebras $\mathfrak{A}(U)$ are called *set algebras*. We let $\mathscr{S}$ be the collection of all algebras $\mathfrak{A}(U)$.

In any $\mathscr{F}$-algebra $\mathfrak{A}$ as above, we introduce certain defined operations. Let $x \in A$; then

$$\begin{aligned} R_l x &= P'(Qx \cdot D), \\ R_r x &= P(Q'x \cdot D), \\ C_m x &= R_l^m QPR_r^m x \cdot Q^{m+1}1, \\ D_{m,m+1} &= R_l^{m+1} D \cdot Q^{m+2}1, \\ D_{mn} &= C_0 \cdots C_{m-1}C_{m+1} \cdots C_{n-1}(D_{01} \cdot D_{12} \cdot \cdots \cdot D_{n-1,n}) \quad \text{for } m+1 < n. \end{aligned}$$

The following theorem expresses some elementary properties of the algebras $\mathfrak{A}(U)$, and a proof can be easily supplied.

THEOREM 1.1. *In an algebra* $\mathfrak{A}(U)$, *with* $x \subseteq {}^{\omega}U$, *we have*

(i) $Q^m({}^{\omega}U) = \{u \in {}^{\omega}U : \mathrm{Dmn}\, u \geq m\}$.

(ii) $Q^m({}^{\omega}U) \sim Q^{m+1}({}^{\omega}U) = \{u \in {}^{\omega}U : \mathrm{Dmn}\, u = m\}$.

(iii) $R_l x = \{u \in {}^{\omega}U : (u = 0 \text{ and } 0 \in x) \text{ or } (u \neq 0 \text{ and } \langle u_1 \cdots u_{lu}u_0 \rangle \in x)\}$.

(iv) $R_r x = \{u \in {}^{\omega}U : (u = 0 \text{ and } 0 \in x) \text{ or } (u \neq 0 \text{ and } \langle u_{lu}u_0 \cdots u_{lu-1} \rangle \in x)\}$.

(v) $C_m x = \{u \in {}^{\omega}U : \mathrm{Dmn}\, u \geq m+1 \text{ and } \exists s \in U(u_s^m \in x)\}$.

(vi) *If* $k \geq 0$ *and* $m_0 < m_1 < \cdots < m_k$, *then*

$$C_{m_0}C_{m_1} \cdots C_{m_k}x = \{u \in {}^{\omega}U : \mathrm{Dmn}\, u \geq m_k + 1 \text{ and } \exists s_0, \cdots, s_k \in U, u_{s_0 \cdots s_k}^{m_0 \cdots m_k} \in x\}.$$

(vii) $D_{m,m+1} = \{u \in {}^{\omega}U : \mathrm{Dmn}\, u \geq m+2 \text{ and } u_m = u_{m+1}\}$.

(viii) *If* $m+1 < n$, *then* $D_{mn} = \{u \in {}^{\omega}U : \mathrm{Dmn}\, u \geq n+1 \text{ and } u_m = u_n\}$.

In analogy with the theory of cylindric algebras, an $\mathscr{F}$-algebra is called *strongly representable* if it is isomorphic to a set algebra.

THEOREM 1.2. *The class of strongly representable* $\mathscr{F}$*-algebras is not closed under ultrapowers and hence is not elementary or even* $\mathbf{PC}_\Delta$.

PROOF. By Frayne, Morel and Scott [4] choose I, F such that $|{}^I\mathfrak{A}(2)/F| > 2^{\aleph_0}$. Now it is easily verified that

$$Q^3 1 \cdot -D_{01} \cdot -D_{12} \cdot -D_{02} = 0 \tag{1}$$

holds in $\mathfrak{A}(2)$ and hence also in ${}^I\mathfrak{A}(2)/F$. Now suppose that ${}^I\mathfrak{A}(2)/F$ is strongly representable; let f be an isomorphism from ${}^I\mathfrak{A}(2)/F$ into $\mathfrak{A}(U)$. Then $|U| \leq 2$; for, suppose on the contrary that $|U| > 2$. Let $u \in {}^3U$ be one-one. Then $u \in Q^3({}^{\omega}U) \cap \sim D_{01} \cap \sim D_{12} \cap \sim D_{02}$, contradicting the fact that (1) holds in ${}^I\mathfrak{A}(2)/F$ and hence in $\mathfrak{A}(U)$. Thus $|U| \leq 2$. But then $|{}^I\mathfrak{A}(2)/F| \leq 2^{\aleph_0}$, contradiction.

Another natural representability notion is as follows. An $\mathscr{F}$-algebra is *representable* if it is isomorphic to a subdirect product of strongly representable $\mathscr{F}$-algebras.

THEOREM 1.3. *The class of representable* $\mathscr{F}$*-algebras is not closed under ultrapowers and hence is not elementary.*

PROOF. First note

(1) $$\text{If } \mathfrak{A} \text{ is representable, then } \prod_{m\in\omega} Q^m 1 = 0.$$

Indeed, suppose $0 \neq a \leq Q^m 1$ for each $m \in \omega$. Let f be a homomorphism of $\mathfrak{A}$ into $\mathfrak{A}(U)$ such that $Fa \neq 0$. But $\bigcap_{m\in\omega} Q^m({}^{\omega}U) = 0$, so this is a contradiction.

Now let F be a nonprincipal ultrafilter over ω. For each $i \in \omega$ let u_i be the unique element of ${}^i 1$, and let $a_i = \{u_i\}$. Then for any $m \in \omega$, $\{i: a_i \in Q^m({}^{\omega}2)\} = \{m, m+1, \cdots\} \in F$, so $0 \neq a/F \leq Q^m 1$ for each $m \in \omega$. Hence by (1) ${}^{\omega}\mathfrak{A}(2)/F$ is nonrepresentable.

§2. **The main result.** Let Γ be the set of all equations of the form $\sigma\cdot \mathbf{Q}^m\mathbf{1}\cdot -\mathbf{Q}^{m+1}\mathbf{1} \equiv \tau\cdot \mathbf{Q}^m\mathbf{1}\cdot -\mathbf{Q}^{m+1}\mathbf{1}$ which hold in all members of $\mathscr{S}$. Our main result is that Γ is not finitely based. It is an open question whether the equational closure $\mathscr{S}'$ of $\mathscr{S}$ is finitely based.

The proof requires the consideration of a new kind of $\mathscr{F}$-algebra. For any natural number m, let $\mathscr{U}_m$ consist of all triples $\langle R, f, n\rangle$ satisfying the following conditions:

(1) R is an equivalence relation on n;

(2) f maps $n \times n$ into m;

(3) $\forall i, j < n(fij = fji)$;

(4) $\forall i, j, k, l < n(iRk \text{ and } jRl \Rightarrow fij = fkl)$;

(5) $\forall i, j, k < n(i\bar{R}j\bar{R}k\bar{R}i \Rightarrow |\{fij, fjk, fki\}| \neq 1)$.

Further, we let $\mathfrak{B}_m = \langle S(\mathscr{U}_m), \cup, \cap, \sim, P, P', Q, Q', D\rangle$, where $D = \{\langle R, f, n\rangle \in \mathscr{U}_m : n \neq 0 \text{ and } OR(n-1)\}$, while for any $\langle R, f, n\rangle \in \mathscr{U}_m$ and $x \subseteq \mathscr{U}_m$,

$$\begin{aligned} P\{\langle R, f, n\rangle\} &= 0 \quad \text{if } n = 0,\\ &= \{\langle S, g, n-1\rangle \in \mathscr{U}_m : \forall i, j < n-1([iSj \text{ iff } (i+1)R(j+1)] \text{ and } \\ &\qquad gij = f(i+1, j+1))\} \quad \text{if } n \neq 0; \end{aligned}$$

$Px = \bigcup_{\langle R,f,n\rangle \in x} P\{\langle R, f, n\rangle\}$;

$$\begin{aligned} P'\{\langle R, f, n\rangle\} &= 0 \quad \text{if } n = 0,\\ &= \{\langle S, g, n-1\rangle \in \mathscr{U}_m : \forall i, j < n-1([iSj \text{ iff } iRj] \text{ and } \\ &\qquad gij = fij)\} \quad \text{if } n \neq 0; \end{aligned}$$

$P'x = \bigcup_{\langle R,f,n\rangle \in x} P'\{\langle R, f, n\rangle\}$;

$$\begin{aligned} Q\{\langle R, f, n\rangle\} &= \{\langle S, g, n+1\rangle \in \mathscr{U}_m : \forall i, j < n([iRj \text{ iff } (i+1)S(j+1)] \text{ and } \\ &\qquad fij = g(i+1, j+1))\}; \end{aligned}$$

$Qx = \bigcup_{\langle R,f,n\rangle \in x} Q\{\langle R, f, n\rangle\}$;

$Q'\langle R, f, n\rangle\} = \{\langle S, g, n+1\rangle \in \mathscr{U}_m : \forall i, j < n([iRj \text{ iff } iSj] \text{ and } fij = gij)\}$;

$Q'x = \bigcup_{\langle R,f,n\rangle \in x} Q'\{\langle R, f, n\rangle\}$.

Thus $\mathfrak{B}_m$ is an $\mathscr{F}$-algebra. Note that if $m < n$ then $\mathscr{U}_m \subseteq \mathscr{U}_n$, $B_m \subseteq B_n$, but $\mathfrak{B}_m$ is not a subalgebra of $\mathfrak{B}_n$. In order to formulate some elementary properties of the algebras $\mathfrak{B}_m$ we introduce the following notation. For $n \in \omega \sim 1$ define $\rho_n : n \to n$ by $\rho_n 0 = 1$, $\rho_n 1 = 2, \cdots, \rho_n(n-2) = n-1$, $\rho_n(n-1) = 0$; and let $\lambda_n = \rho_n^{-1}$. If $\langle R, f, n\rangle \in \mathscr{U}_m$ and $M \subseteq n$, let $\langle R, f, n\rangle \upharpoonright M = \langle S, g, M\rangle$, where $S = R \cap {}^2M$ and for $i, j \in M$, $gij = fij$.

THEOREM 2.1. *In an algebra $\mathfrak{B}_m$, with $x \subseteq \mathscr{U}_m$, we have*

(i) $Q^p(\mathscr{U}_m) = \{\langle R, f, n\rangle \in \mathscr{U}_m : n \geq p\}$.

(ii) $Q^p(\mathscr{U}_m) \sim Q^{p+1}(\mathscr{U}_m) = \{\langle R, f, n\rangle \in \mathscr{U}_m : n = p\}$.

(iii) $R_l x = \{\langle R, f, n\rangle \in \mathscr{U}_m : (n = 0 \textit{ and } \langle 0, 0, 0\rangle \in x) \textit{ or } (n \neq 0 \textit{ and } \exists\langle S, g, n\rangle \in x \forall i, j < n([iSj \textit{ iff } (\rho_n i)R(\rho_n j)] \textit{ and } gij = f(\rho_n i, \rho_n j)))\}$.

(iv) $R_r x = \{\langle R, f, n\rangle \in \mathscr{U}_m : (n = 0 \textit{ and } \langle 0, 0, 0\rangle \in x) \textit{ or } (n \neq 0 \textit{ and } \exists\langle S, g, n\rangle \in x \forall i, j < n([iSj \textit{ iff } (\lambda_n i)R(\lambda_n j)] \textit{ and } gij = f(\lambda_n i, \lambda_n j)))\}$.

(v) $C_i x = \{\langle R, f, n\rangle \in \mathscr{U}_m : n \geq i + 1 \textit{ and } \exists\langle S, g, n\rangle \in x(\langle R, f, n\rangle \restriction (n \sim \{i\}) = \langle S, g, n\rangle \restriction (n \sim \{i\}))\}$.

(vi) $D_{i,i+1} = \{\langle R, f, n\rangle \in \mathscr{U}_m : n \geq i + 2 \textit{ and } iR(i + 1)\}$.

A proof is routine.

Let $\mathfrak{A}$ be an $\mathscr{F}$-algebra, $m < \omega$. $\boldsymbol{CA}_m\mathfrak{A}$ is then the algebra $\langle A', +', \cdot', -', c_i', d_{ij}'\rangle_{i,j<m}$ such that $A' = \{x \in A : x \leq Q^m 1 \cdot - Q^{m+1}1\}$, $+'$ and $\cdot'$ are the restrictions of the corresponding operations of $\mathfrak{A}$, and for any $x \in A'$,

$$\begin{aligned} c_i'x &= c_i x \cdot Q^m 1 \cdot - Q^{m+1}1 && \text{if } i < m \\ -'x &= -x \cdot Q^m 1 \cdot - Q^{m+1}1 && \\ d_{ij}' &= D_{ij} \cdot Q^m 1 \cdot - Q^{m+1}1 && \text{if } i < j < m \\ d_{ij}' &= D_{ji} \cdot Q^m 1 \cdot - Q^{m+1}1 && \text{if } j < i < m \\ d_{ii}' &= Q^m 1 \cdot - Q^{m+1}1 && \text{if } i < m. \end{aligned}$$

Thus $\boldsymbol{CA}_m\mathfrak{A}$ is an algebra in the similarity class of $\boldsymbol{CA}_m$'s.

We now assume the definition and basic properties of the algebras $\mathfrak{A}_m^n$ of Monk [9].

THEOREM 2.2. *$\mathfrak{A}_m^n$ can be isomorphically imbedded in $\boldsymbol{CA}_n\mathfrak{B}_m$.*

PROOF. For $\langle R, f\rangle \in m_n'$, let

$$F\{\langle R, f\rangle\} = \{\langle R, g, n\rangle : \forall i, j < n(i\bar{R}j \Rightarrow fij = gij)\};$$

for $x \subseteq m_n'$, let

$$Fx = \bigcup_{\langle R, f\rangle \in x} F\{\langle R, f\rangle\}.$$

Clearly F is a Boolean isomorphism into, and $Fd_{ij} = d_{ij}'$ for all $i, j < n$. Next, suppose $\langle R, g, n\rangle \in Fc_i x$; say $\langle R, g, n\rangle \in F\{\langle R, f\rangle\}$ with $\langle R, f\rangle \in c_i\{\langle S, h\rangle\}$ and $\langle S, h\rangle \in x$. Define $k : n \times n \to m$:

$$\begin{aligned} kij &= hij && \text{if } i\bar{S}j, \\ &= gij && \text{if } iSj. \end{aligned}$$

Then $\langle S, k, n\rangle \in \mathscr{U}_m$, $\langle R, g, n\rangle \in C_i\{\langle S, k, n\rangle\}$, and $\langle S, k, n\rangle \in F\{\langle S, h\rangle\}$, so $\langle R, g, n\rangle \in C_i Fx$. The converse is similarly shown.

If $\mathfrak{A}$ is an algebra similar to $\boldsymbol{CA}_m$'s, $m \geq 3$, by $\mathrm{Rd}_3\,\mathfrak{A}$ we mean the 3-reduct of $\mathfrak{A}$. An $\mathscr{F}$-algebra $\mathfrak{A}$ is *weakly representable* provided that for every $m \geq 3$, $\mathrm{Rd}_3\, \boldsymbol{CA}_m\mathfrak{A}$ is representable. Now by Corollary 1.9 of Monk [9] choose $3 \leq m_0 < m_1 < m_2 < \cdots$ such that for each i, $\mathrm{Rd}_3\, \mathfrak{A}_{m_i+i}^{m_i}$ is nonrepresentable. From Theorem 2.2 we infer that $\mathfrak{B}_{m_i+i}$ is not weakly representable. Let $\mathscr{F}$ be any nonprincipal ultrafilter over ω; and let $\mathfrak{B}' = \mathsf{P}_{i<\omega}\mathfrak{B}_{m_i+i}/\mathscr{F}$. We now prove a sequence of lemmas aimed at showing that a certain algebra related to $\mathfrak{B}'$ is in $\mathscr{S}$. From this the fact that Γ is not finitely based will follow easily.

Let $\equiv\, = \{(x, y) \in {}^2\mathsf{P}_{i<\omega}B_{m_i+i}$: for every $n \in \omega$, $\{i : x_i \oplus y_i \leq Q^n 1\} \in \mathscr{F}\}$. Clearly $\equiv$ is a congruence relation on the Boolean part of $\mathsf{P}_{i<\omega}\mathfrak{B}_{m_i+i}$. The proof that it is a congruence on all of $\mathsf{P}_{i<\omega}\mathfrak{B}_{m_i+i}$ is illustrated by checking the congruence property with respect to P. Assume that $x \equiv y$. Let $n \in \omega$. We know that $I = \{i : x_i \cdot -y_i \leq Q^{n+1}1\} \in \mathscr{F}$. Let i be an arbitrary member of I, and assume that $\langle R, f, p\rangle \in Px_i \cdot -Py_i$.

Choose $\langle S, g, p+1\rangle \in x_i$ such that $\langle R, f, p\rangle \in P\{\langle S, g, p+1\rangle\}$. Thus $\langle S, g, p+1\rangle \in x_i \cdot -y_i$, so $\langle S, g, p+1\rangle \in Q^{n+1}1$ and hence $p+1 \geq n+1$, i.e., $p \geq n$. Hence $I \subseteq \{i: Px_i \cdot -Py_i \leq Q^n 1\}$, so the latter set is in $\mathscr{F}$. By symmetry we easily obtain $Px \equiv Py$. Thus $\equiv$ *is* a congruence relation on $\mathbf{P}_{i<\omega}\mathfrak{B}_{m_i+i}$. Note that $x/\mathscr{F} = y/\mathscr{F}$ implies that $x \equiv y$. Let $\mathfrak{C} = \mathbf{P}_{i<\omega}\mathfrak{B}_{m_i+i}/\equiv$.

LEMMA 2.3. *If an equation* $\sigma \cdot Q^m 1 \cdot - Q^{m+1}1 = \tau \cdot Q^m 1 \cdot - Q^{m+1}1$ *holds in* $\mathfrak{C}$ *then the same equation holds in* $\mathfrak{B}'$.

PROOF. For any term σ, any structure $\mathfrak{M}$, and any $x \in {}^\omega M$, let $\bar{\sigma}^{\mathfrak{M}}x$ denote the value of σ in $\mathfrak{M}$ under the assignment x. Now let φ be the equation indicated in the statement of the lemma, and let $x \in {}^\omega D$, where $D = \mathbf{P}_{i<\omega}B_{m_i+i}$. For each $i < \omega$ let pr_i be the canonical homomorphism of $\mathfrak{D}$ onto $\mathfrak{B}_{m_i+i}$. Since φ holds in $\mathfrak{C}$ we have

$$\bar{\sigma}^{\mathfrak{D}}x \cdot Q^m 1 \cdot - Q^{m+1}1 \equiv \bar{\tau}^{\mathfrak{D}}x \cdot Q^m 1 \cdot - Q^{m+1}1;$$

in particular,

$$\{i: [\bar{\sigma}(pr_i \circ x) \cdot Q^m 1 \cdot - Q^{m+1}1] \oplus [\bar{\tau}(pr_i \circ x) \cdot Q^m 1 \cdot - Q^{m+1}1] \leq Q^{m+1}1\} \in \mathscr{F}.$$

But the set on the left is just

$$\{i: \bar{\sigma}(pr_i \circ x) \cdot Q^m 1 \cdot - Q^{m+1}1 = \bar{\tau}(pr_i \circ x) \cdot Q^m 1 \cdot - Q^{m+1}1\}.$$

By the basic theorem on ultraproducts it follows that φ holds in $\mathfrak{B}'$, as desired.

Lemma 2.3 will essentially yield later that $\mathfrak{B}'$ is a model of Γ. This will follow from the fact that $\mathfrak{C}$ is strongly representable. The purpose of the next few lemmas is to establish this last fact.

LEMMA 2.4. *For each n,* $CA_n\mathfrak{B}'$ *is a simple* CA_n.

PROOF. It is easily seen that $CA_n\mathfrak{B}'$ is a CA_n; for example, to check that $c_i c_j(x/\mathscr{F}) = c_j c_i(x/\mathscr{F})$ it is enough to show that $c_i c_j x_k \subseteq c_j c_i x_k$ for any $k \geq n$, and this is easily carried out along the lines of the proof of 1.1 of Monk [9]. To show that $CA_n\mathfrak{B}'$ is simple, it suffices to prove the following:

(1) If $p \geq n$, $x \subseteq B_{m_p+p}$, $x \subseteq Q^n 1 \cdot - Q^{n+1}1$, and $q_0 < q_1 < \cdots < q_h < n$, then $C_{q_0}C_{q_1} \cdots C_{q_h}x = \{\langle R, g, n\rangle \in \mathscr{U}_{m_p+p}$:

$$\exists\langle S, g, n\rangle \in x(\langle R, f, n\rangle \upharpoonright (n \sim \{q_0, \cdots, q_h\}) = \langle S, g, n\rangle \upharpoonright (n \sim \{q_0, \cdots, q_h\}))\},$$

where C_{q_i} is the operation in $\mathfrak{B}_{m_p+p}$.

We prove (1) by induction on h; it is clear for $h = 0$. Now assume (1) for h, and suppose $q_0 < q_1 < \cdots < q_{h+1}$. The inclusion $\subseteq$ follows directly from the induction hypothesis. Assume $\langle S, g, n\rangle \in x$ and $\langle R, f, n\rangle \upharpoonright (n \sim \{q_0, \cdots, q_{h+1}\}) = \langle S, g, n\rangle \upharpoonright (n \sim \{q_0, \cdots, q_{h+1}\})$. Let

$$T = S \cap {}^2(n \sim \{q_{h+1}\}) \cup \{(q_{h+1}, s), (s, q_{h+1}): q_{h+1} = s \text{ or } \exists t \in n \sim \{q_0, \cdots, q_{h+1}\}(q_{h+1}RtSs)\}.$$

It is easily checked that T is an equivalence relation on n, and also that $T \cap {}^2(n \sim \{q_0, \cdots, q_h\}) = R \cap {}^2(n \sim \{q_0, \cdots, q_h\})$. Now for $s, t \in n \sim \{q_{h+1}\}$, let $kst = gst$. If $q_{h+1}Rt$ for some $t \in n \sim \{q_0, \cdots, q_{h+1}\}$, let $kq_{h+1}s = ksq_{h+1} = kst$ for all $s \in n \sim \{q_{h+1}\}$, and let $kq_{h+1}q_{h+1} = ktt$. In this case it is clear that $\langle T, k, n\rangle \in C_{q_{h+1}}\{\langle S, g, n\rangle\}$ and $\langle R, f, n\rangle \upharpoonright (n \sim \{q_0, \cdots, q_h\}) = \langle T, k, n\rangle \upharpoonright (n \sim \{q_0, \cdots, q_h\})$, so the induction hypothesis gives the desired result. Now suppose $\neg\exists t \in n \sim \{q_0, \cdots, q_{h+1}\}(q_{h+1}Rt)$. Then for any $s \in n \sim \{q_0, \cdots, q_h\}$ let $kq_{h+1}s = ksq_{h+1} = fsq_{h+1}$. Let $\{kq_{h+1}q_i: i \leq h\}$ be distinct elements of $m_p + p$ different from all of

$kq_{h+1}s$, $s \in n \sim \{q_0, \cdots, q_h\}$; and let $kq_iq_{h+1} = kq_{h+1}q_i$ for $i \leq h$. The desired conclusion again follows.

For $n \leq p$ we define $F'_{np}: \mathbf{P}_{i<\omega}B_{m_i+i} \to \mathbf{P}_{i<\omega}B_{m_i+i}$ by

$$(F'_{np}x)_i = \{\langle R, f, p\rangle : \langle R, f, p\rangle \upharpoonright n \in x_i\},$$

for $x \in P_{i<\omega}B_{m_i+i}$ and $i < \omega$. If $x \equiv y$, then $F'_{np}x \equiv F'_{np}y$. Thus there is a function $F_{np}: CA_nC \to CA_pC$ such that $F_{np}(x/\equiv) = F'_{np}x \equiv$ for any $x/\equiv \in CA_nC$.

LEMMA 2.5. *F_{np} is a neat embedding of $CA_n\mathfrak{C}$ into $CA_p\mathfrak{C}$.*

PROOF. The nontrivial parts of the proof are to show that F_{np} is one-one and that F_{np} preserves C_i for $i < n$. Suppose $0 \neq x/\equiv \in CA_nC$. Thus $x\cdot(-Q^n1 + Q^{n+1}1) \equiv 0$ and $\{i: x_i \subseteq Q^s1\} \notin \mathscr{F}$ for a certain $s \in \omega$. Thus $I = \{i: x_i \cap (\sim Q^n1 \cup Q^{n+1}1) \subseteq Q^s1$ and $x_i \nsubseteq Q^s1\} \in \mathscr{F}$. Let i be an arbitrary member of I, $i \geq n$. Choose $\langle R, f, t\rangle \in x_i \sim Q^s1$. Thus $\langle R, f, t\rangle \in \sim(\sim Q^n1 \cup Q^{n+1}1)$, so $t = n$. Choose $\langle S, g, p\rangle \in \mathscr{U}_{m_i+i}$ such that $\langle S, g, p\rangle \upharpoonright n = \langle R, f, n\rangle$. Thus $\langle S, g, p\rangle \in (F'_{np}x)_i \sim Q^{s+p-n}1$. We have thus shown that $I \cap \{i: i \geq n\} \subseteq \{i: (F'_{np}x)_i \nsubseteq Q^{s+p-n}1\}$, so it follows that $F_{np}(x/\equiv) \neq 0$.

That F_{np} preserves C_i for $i < n$ is seen as in the proof of 1.2 of Monk [9].

Let $D = \{x \in \mathbf{P}_{n<\omega}CA_nC: \exists i < \omega \forall j \geq i(x_j = F_{ij}x_i)\}$. Let $\approx = \{(x, y) \in {}^2D: \exists i < \omega \forall j \geq i(x_j = y_j)\}$. Clearly $\approx$ is an equivalence relation on D. For $x, y \in D$ let

$$x/\approx + y/\approx = \langle x_n + y_n : n < \omega\rangle/\approx,$$
$$x/\approx \cdot y/\approx = \langle x_n \cdot y_n : n < \omega\rangle/\approx,$$
$$-(x/\approx) = \langle -x_n : n < \omega\rangle/\approx.$$

For $i < \omega$ let $c_i(x/\approx) = y/\approx$, where for $j < \omega$

$$y_j = 0 \quad \text{if } j \leq i,$$
$$= c_ix_j \quad \text{if } j > i.$$

Finally, for $i, j < \omega$, let $d_{ij} = x/\approx$, where

$$x_h = 0 \quad \text{if } h \leq i \text{ or } h \leq j,$$
$$= dij \quad \text{if } i, j < h.$$

Let $\mathfrak{C}$ be $D/\approx$ with all of these operations. The following lemma is easily checked.

LEMMA 2.6. *$\mathfrak{C}$ is a simple locally finite CA_ω.*

For $n \in \omega$ and $x \in CA_nC$, let x' be the member of D such that for any $i < \omega$,

$$x'_i = 0 \quad \text{if } i < n,$$
$$= F_{ni}x \quad \text{if } n \leq i.$$

Furthermore, let $G_nx = x'/\approx$. Then, as is easily checked,

LEMMA 2.7. *G_n is a neat embedding of $CA_n\mathfrak{C}$ into $\mathfrak{C}$.*

Our final lemma is as follows:

LEMMA 2.8. *$\mathfrak{C}$ is strongly representable.*

PROOF. Let H be an isomorphism of $\mathfrak{C}$ onto a set algebra with base U. We may assume that H has the following additional property:

(1) If $e \in E$, $u \in He$, $v \in {}^\omega U$, and $u \upharpoonright \Delta e = v \upharpoonright \Delta e$, then $v \in He$. (See Henkin and Tarski [5].) Now for any $a \in C$ we let

$$Ka = \{u \in {}^\omega U: \text{if } u \in {}^nU \text{ then } \exists v \in HG_n(a\cdot Q^n1 \cdot -Q^{n+1}1)(u \subseteq v)\}.$$

It is easily checked that K preserves $+$, $K0 = 0$, and $K1 = 1$. To check that K

preserves $-$, suppose that $Ka \cap Kb \neq 0$, say $u \in Ka \cap Kb$; we will show that $a \cdot b \neq 0$. Say $u \subseteq v \in HG_n(a \cdot Q^n 1 \cdot - Q^{n+1}1)$ and $u \subseteq w \in HG_n(b \cdot Q^n 1 \cdot - Q^{n+1}1)$. Thus $\Delta G_n(a \cdot Q^n 1 \cdot - Q^{n+1}1) \subseteq n$ and $w \upharpoonright n = v \upharpoonright n$, so $w \in HG_n(a \cdot Q^n 1 \cdot - Q^{n+1}1)$ by (1). Hence $a \cdot b \neq 0$, as desired.

To show that K is one-one, suppose $0 \neq a \in C$. Say $a = x/\equiv$, and choose $n \in \omega$ such that $\{i : x_i \subseteq Q^n 1\} \notin \mathscr{F}$. Thus $\{i : x_i \nsubseteq Q^n 1\} \in \mathscr{F}$. Now

$$\{i : x_i \nsubseteq Q^n 1\} \subseteq \bigcap_{j<n} \{i : x_i \cdot Q^j 1 \cdot - Q^{j+1}1 \neq 0\};$$

hence we may choose $j < n$ such that $\{i : x_i \cdot Q^j 1 \cdot - Q^{j+1}1 \neq 0\} \in \mathscr{F}$. Thus $a \cdot Q^j 1 \cdot - Q^{j+1}1 \neq 0$, so we may choose $v \in HG_j(a \cdot Q^j 1 \cdot - Q^{j+1}1)$. Clearly then $v \upharpoonright j \in Ka$.

To show that K preserves P, we first show that

$$G_n(Pa \cdot Q^n 1 \cdot - Q^{n+1}1) = S^n_{n-1} S^{n-1}_{n-2} \cdots S^1_0 c_0 G_{n+1}(a \cdot Q^{n+1}1 \cdot - Q^{n+2}1), \tag{2}$$

where $S^i_j x = c_i(d_{ij} \cdot x)$ in any cylindric algebra. To prove (2), let $a = x/\equiv$, and let $i \geq n+1$. Then

$$\begin{aligned}(Pa \cdot Q^n 1 \cdot - Q^{n+1}1)^\prime_i &= F_{ni}(Pa \cdot Q^n 1 \cdot - Q^{n+1}1) \\ &= \langle \{\langle R, f, i \rangle : \langle R, f, i \rangle \upharpoonright n \in Px_j \cdot Q^n 1 \cdot - Q^{n+1}1\} : j < \omega \rangle / \equiv.\end{aligned}$$

On the other hand,

$$\begin{aligned}&S^n_{n-1} S^{n-1}_{n-2} \cdots S^1_0 c_0 G_{n+1}(a \cdot Q^{n+1}1 \cdot - Q^{n+2}1) \\ &\qquad = G_{n+1}(S^n_{n-1} S^{n-1}_{n-2} \cdots S^1_0 c_0(a \cdot Q^{n+1}1 \cdot - Q^{n+2}1)).\end{aligned}$$

Now, with x and i as above,

$$\begin{aligned}&[S^n_{n-1} S^{n-1}_{n-2} \cdots S^1_0 c_0(a \cdot Q^{n+1}1 \cdot - Q^{n+2}1)]^\prime_i \\ &\qquad = \langle \{\langle R, f, i \rangle : \langle R, f, i \rangle \upharpoonright (n+1) \\ &\qquad\quad \in S^n_{n-1} S^{n-1}_{n-2} \cdots S^1_0 c_0(x_j \cdot Q^{n+1}1 \cdot - Q^{n+2}1)\} : j < \omega \rangle / \equiv.\end{aligned}$$

Hence it suffices to take any $j < \omega$ with $j \geq n+1$ and show that

$$\begin{aligned}&\{\langle R, f, i \rangle : \langle R, f, i \rangle \upharpoonright n \in Px_j \cdot Q^n 1 \cdot - Q^{n+1}1\} \\ &\quad = \{\langle R, f, i \rangle : \langle R, f, i \rangle \upharpoonright (n+1) \in S^n_{n-1} S^{n-1}_{n-2} \cdots S^1_0 c_0(x_j \cdot Q^{n+1}1 \cdot - Q^{n+2}1)\},\end{aligned} \tag{3}$$

where the operations take place in $\mathfrak{B}_{m_j+j}$. If $\langle R, f, i \rangle$ is a member of the set on the right-hand side, then there exist $\langle T_n g_n, n+1 \rangle, \langle T_{n-1}, g_{n-1}, n+1 \rangle, \cdots, \langle T_0, g_0, n+1 \rangle$ such that $\langle R, f, i \rangle \upharpoonright n+1 \in C_n\{\langle T_n, g_n, n+1 \rangle\}$ and $\langle T_n, g_n, n+1 \rangle \in D_{n-1,n}$, $\langle T_n, g_n, n+1 \rangle \in C_{n-1}\{\langle T_{n-1}, g_{n-1}, n+1 \rangle\}$ and $\langle T_{n-1}, g_{n-1}, n+1 \rangle \in D_{n-2,n-1}, \cdots, \langle T_1, g_1, n+1 \rangle \in c_0\{\langle T_0, g_0, n+1 \rangle\}$ and $\langle T_0, g_0, n+1 \rangle \in x_j$. Recalling the definition of P in the algebra $\mathfrak{B}_{m_j+j}$ we easily infer that $\langle R, f, i \rangle$ is a member of the left-hand side of (3). The opposite inclusion in (3) follows similarly; here the sequence $\langle T_0, g_0, n+1 \rangle, \cdots, \langle T_n, g_n, n+1 \rangle$ must be constructed. Thus (3) and (2) both hold. From (2) it easily follows that $KPa = PKa$.

That K preserves P', Q, and Q' is similarly shown; the counterparts of (2) are then, respectively,

$$G_n(P'a \cdot Q^n 1 \cdot - Q^{n+1}1) = G_{n+1}(a \cdot Q^{n+1}1 \cdot - Q^{n+2}1), \tag{4}$$

$$G_{n+1}(Qa \cdot Q^{n+1}1 \cdot - Q^{n+2}1) = S^0_1 S^1_2 \cdots S^{n-1}_n G_n(a \cdot Q^n 1 \cdot - Q^{n+1}1), \tag{5}$$

$$G_{n+1}(Q'a \cdot Q^{n+1}1 \cdot - Q^{n+2}1) = G_n(a \cdot Q^n 1 \cdot - Q^{n+1}1). \tag{6}$$

This completes the proof.

With Lemma 2.8 we have completed the main technical results of the paper. We now want to indicate the metamathematical significance of these results, and to this end we describe connections between the language of cylindric algebras and the language for $\mathscr{F}$-algebras.

Let $\mathscr{L}$ be a first-order language for $\mathscr{F}$-algebras, and let $\mathscr{L}_m$ be a first-order language for CA_m's. We define a mapping ξ_m: Terms$_{\mathscr{L}_m} \to$ Terms$_{\mathscr{L}}$:

$$\begin{aligned}
\xi_m v_i &= v_i \cdot Q^m\mathbf{1} \cdot -Q^{m+1}\mathbf{1}\\
\xi_m(\sigma + \tau) &= \xi_m\sigma + \xi_m\tau\\
\xi_m(\sigma \cdot \tau) &= \xi_m\sigma \cdot \xi_m\tau\\
\xi_m(-\sigma) &= -\xi_m\sigma \cdot Q^m\mathbf{1} \cdot -Q^{m+1}\mathbf{1}\\
\xi_m(\mathbf{c}_i\sigma) &= C_i\xi_m\sigma
\end{aligned}$$

$$\xi_m(d_{ij}) = \begin{cases} v_0 + -v_0 & \text{if } i = j\\ D_{ij}\cdot Q^m\mathbf{1} \cdot -Q^{m+1}\mathbf{1} & \text{if } i < j\\ D_{ji}\cdot Q^m\mathbf{1} \cdot -Q^{m+1}\mathbf{1} & \text{if } j < i.\end{cases}$$

If σ is a term in a language $\mathscr{M}$ and $x \in {}^{\omega}A$, where $\mathfrak{A}$ is an $\mathscr{M}$-structure, $\bar{\sigma}^{\mathfrak{A}}x$ denotes the value of σ in $\mathfrak{A}$ under the assignment x of values to the variables.

LEMMA 2.10. *Let σ and τ be terms in $\mathscr{L}_m$; then the following conditions are equivalent*:

(i) *$\sigma = \tau$ holds in every representable CA_m.*
(ii) *$\xi_m\sigma = \xi_m\tau$ holds in every member of $\mathscr{S}$.*

PROOF. (i) $\Rightarrow$ (ii). Let $\mathfrak{A} = \mathfrak{A}(U) \in \mathscr{S}$, and let $\mathfrak{B}$ be the CA_m of all subsets of ${}^mU$. For any $x \in {}^{\omega}S({}^{\omega}U)$ define $y = \langle x_i \cap {}^mU : i < \omega\rangle$. Then for any term ρ of $\mathscr{L}_m$, $\bar{\rho}^{\mathfrak{B}}y = \widetilde{\xi_m\rho}^{\mathfrak{A}}x$. Hence (ii) easily follows from (i).

(ii) $\Rightarrow$ (i). For any $x \in {}^{\omega}S({}^mU)$, $\bar{\rho}^{\mathfrak{B}}\widetilde{x =} \xi_m\rho^{\mathfrak{A}}x$ for any term ρ.

Hence (i) follows from (ii).

The same proof yields the following result.

LEMMA 2.10. *Let σ and τ be terms in $\mathscr{L}_m$; then the following conditions are equivalent*:

(i) *$\sigma = \tau$ holds in every representable CA_m.*
(ii) *$\xi_m\sigma\cdot Q^m\mathbf{1}\cdot - Q^{m+1}\mathbf{1} = \xi_m\tau\cdot Q^m\mathbf{1}\cdot - Q^{m+1}\mathbf{1}$ holds in every member of $\mathscr{S}$.*

The following lemma, easily proved, is also needed.

LEMMA 2.11. *Let $i < \omega$ and let σ and τ be terms in $\mathscr{L}_3$. Let $n = m_i$, and assume that $\xi_n\sigma\cdot Q^n\mathbf{1}\cdot - Q^{n+1}\mathbf{1} = \xi_n\tau\cdot Q^n\mathbf{1}\cdot - Q^{n+1}\mathbf{1}$ holds in $\mathfrak{B}_{n+i}$. Then $\sigma = \tau$ holds in $\mathfrak{Rd}_3 CA_n \mathfrak{B}_{n+i}$.*

THEOREM 2.12. *Γ does not have a finite basis.*

PROOF. By 2.3 and 2.8, $\mathfrak{B}'$ is a model of Γ. We shall show that each algebra $\mathfrak{B}_{m_i+i}$ fails to be a model of Γ, so that the desired result follows from the basic result on ultraproducts. From 2.2 and the discussion following it we see that $\mathfrak{Rd}_3 CA_n \mathfrak{B}_{n+i}$ is a nonrepresentable CA_3, where $n = m_i$. Hence there is an equation $\sigma = \tau$ of $\mathscr{L}_3$ which holds in every representable CA_3 but fails in $\mathfrak{Rd}_3 CA_n \mathfrak{B}_{n+i}$. By 2.10 the equation $\xi_n\sigma\cdot Q^n\mathbf{1}\cdot - Q^{n+1}\mathbf{1} = \xi_n\tau\cdot Q^n\mathbf{1}\cdot - Q^{n+1}\mathbf{1}$ is a member of Γ, while by 2.11 it fails to hold in $\mathfrak{B}_{n+i}$. This completes the proof.

REFERENCES

[1] P. BERNAYS, *Über eine natürliche Erweiterung des Relationenkalkuls*, ***Constructivity in mathematics***, North-Holland, Amsterdam, 1957, pp. 1–14.

[2] A. H. COPELAND, SR., *A note on cylindric and polyadic algebras*, ***The Michigan mathematical journal***, vol. 3 (1955), pp. 155–157.

[3] W. CRAIG, *Boolean notions extended to higher dimensions*, ***Theory of models***, North-Holland, Amsterdam, 1965, pp. 55–69.

[4] T. FRAYNE, A. MOREL and D. SCOTT, *Reduced direct products*, ***Fundamenta mathematicae***, vol. 51 (1962), pp. 195–228.

[5] L. HENKIN and A. TARSKI, *Cylindric algebras*, ***Lattice theory***, Proceedings of the Symposium in Pure Mathematics, vol. 2, Amer. Math. Soc., 1961, 83–113.

[6] L. HENKIN, J. D. MONK and A. TARSKI, ***Cylindric algebras***. Part I, North-Holland, Amsterdam (to appear).

[7] J. JOHNSON, *Nonfinitizability of classes of representable polyadic algebras*, this JOURNAL, vol. 34 (1969), pp. 344–352.

[8] J. D. MONK, *On representable relation algebras*, ***The Michigan mathematical journal***, vol. 11 (1964), pp. 207–210.

[9] J. D. MONK, *Nonfinitizability of classes of representable cylindric algebras*, this JOURNAL, vol. 34 (1969), pp. 331–343.

[10] A. TARSKI, *Der Wahrheitsbegriff in den formalisierten Sprachen*, ***Studia phil.***, vol. 1 (1935), pp. 261–405.

UNIVERSITY OF COLORADO