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MODEL-THEORETIC METHODS AND RESULTS IN THE THEORY OF CYLINDRIC ALGEBRAS *

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We shall deal in this paper with the cylindric algebras of Tarski, in particular with the class of representable cylindric algebras. The principal model-theoretic method exploited here is that of ultraproducts. The model-theoretic results are connected with the equational character of the class of representable cylindric algebras and with the problem of finite axiomatizability of this class.

The intent of Section 1 is mainly expository, and consists of a simplified development of some general theorems about representable algebras. Theorems 1.3, 1.4, 1.5, 1.8 and 1.13 seem to be new results. Section 2 contains the main new result of the paper. There we show that the class of representable three-dimensional cylindric algebras is not finitely axiomatizable. The methods—apart from the use of an ultraproduct construction—are similar to methods introduced by Jónsson and Lyndon in the study of relation algebras. The fairly "deep" theorem of Bruck and Ryser concerning orders of finite projective planes is used in an essential way.

1. Representable cylindric algebras. We assume as known the basic definitions in the theory of cylindric algebras. We use the notation of Henkin and Tarski [61]. Proofs of some elementary facts about cylindric algebras may be found in Henkin [56]; in one essential theorem below we use results of Halmos [62] and Galler [57]. Most of the results of this section are due to Henkin and Tarski. We recall that $\mathscr{RCA}_\alpha$ and $\mathscr{CS}_\alpha$ are the classes of all representable cylindric algebras and of all cylindric set algebras respectively, of dimension α.

Theorem 1.1. *Every $\beta\gamma$-reduct of an $\mathscr{RCA}_\alpha$ is an $\mathscr{RCA}_\alpha$.*

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Proof. It suffices to prove that, given a $\mathscr{CS}_\alpha$ $\mathfrak{A}$ with base U and a non-zero element x of A, there is a homomorphism f of the $\beta\gamma$-reduct $\mathfrak{B}$ of $\mathfrak{A}$ into a $\mathscr{CS}_\beta$ such that $f(x) \neq 0$. Choose $t \in x$. If $s \in U^\beta$, define $s^+ \in U^\alpha$ as follows:

$$s^+(\varkappa) = \begin{cases} t(\varkappa) \text{ if } \varkappa \notin \text{range of } \gamma, \\ s(\gamma^{-1}(\varkappa)) \text{ if } \varkappa \in \text{range of } \gamma. \end{cases}$$

The required function f is defined as follows: $f(y) = \{s \in U^\beta : s^+ \in y\}$, for all $y \in A$.

Theorem 1.2. *If $\alpha \leq \beta$, then every $\mathscr{RCA}_\alpha$ can be neatly embedded in an $\mathscr{RCA}_\beta$.*

Proof. It suffices to prove the theorem for a $\mathscr{CS}_\alpha$ $\mathfrak{A}$ with base U. For each $x \in A$ let $f(x) = \{t \in U^\beta : t \restriction \alpha \in x\}$. Then f neatly embeds $\mathfrak{A}$ in a $\mathscr{CS}_\beta$.

These two simple theorems are quite important for the present development. The following theorem is a simple generalization of Corollary 1.16 of Frayne–Morel–Scott [62]. For this theorem we suppose that L is a first-order language with relation symbols, operation symbols and individual constants, and we suppose that N is the set of non-logical constants of L.

Theorem 1.3. *Suppose $\mathscr{K}$ is a class of L-structures and $\mathfrak{A}$ is an L-structure. Let $\mathscr{F} = \{\langle F, K\rangle : F$ is a finite subset of A and K is a finite subset of $N\}$. Suppose $\mathscr{M}$ is a subset of $\mathscr{F}$ such that for every $\langle F, K\rangle \in \mathscr{F}$ there exists a pair $\langle G, H\rangle \in \mathscr{M}$ such that $F \subseteq G$ and $K \subseteq H$. Suppose that for every $\langle F, K\rangle \in \mathscr{M}$ the F-generated substructure of the K-reduct of $\mathfrak{A}$ is a substructure of the K-reduct of some member of $\mathscr{K}$. Then $\mathfrak{A}$ can be isomorphically embedded in an ultraproduct of members of $\mathscr{K}$.*

Proof. For each $\langle F, K\rangle \in \mathscr{M}$ let $M_{FK} = \{\langle G, H\rangle \in \mathscr{M} : F \subseteq G$ and $K \subseteq H\}$. Then $\{M_{FK} : \langle F, K\rangle \in \mathscr{M}\}$ has the finite intersection property and so is included in an ultrafilter D on $\mathscr{M}$. For every $\langle F, K\rangle \in \mathscr{M}$ choose $\mathfrak{B}_{FK} \in \mathscr{K}$ such that the F-generated substructure of the K-reduct of $\mathfrak{A}$ is a substructure of the K-reduct of $\mathfrak{B}_{FK}$. Then there is a function f on A into $\prod_{\langle F,K\rangle \in \mathscr{M}} B_{FK}/D$ such that $f(a) = g/D$ with $g(F, K) = a$ whenever $a \in F$ and $\langle F, K\rangle \in \mathscr{M}$, and f is the required isomorphism.

From Theorems 1.1 through 1.3 we obtain the following two results at once. Recall that $\mathscr{LCA}_\alpha$ is the class of all locally finite dimensional cylindric algebras of dimension α.

Lemma 1.4. *If $\mathfrak{A}$ is a $\mathscr{CA}_\alpha$ such that every finitely generated subalgebra of every finite reduct of $\mathfrak{A}$ is representable, then $\mathfrak{A}$ is isomorphically embeddable in an ultraproduct of representable $\mathscr{LCA}_\alpha$'s.*

Theorem 1.5. *Every $\mathscr{RCA}_\alpha$ is isomorphically embeddable in an ultraproduct of representable $\mathscr{LCA}_\alpha$'s.*

Problem 1. Is every $\mathscr{RCA}_\alpha$ elementarily embeddable in an ultraproduct of representable $\mathscr{LCA}_\alpha$'s?

Using a modified form of Theorem 1.3 it can be seen that appropriate generalizations of Theorems 1.1 and 1.2 would serve to answer this question positively.

Lemma 1.6. *An ultraproduct of $\mathscr{RCA}_\alpha$'s is an $\mathscr{RCA}_\alpha$.*

Proof. Suppose that for each $t \in T$ $\mathfrak{A}_t$ is a $\mathscr{CS}_\alpha$ with base U_t and x is in the universe B of $\prod_{t\in T} \mathfrak{A}_t$, D is an ultrafilter over T, and $x/D \neq 0$. It suffices to find a homomorphism f of $\prod_{t\in T} \mathfrak{A}_t/D$ into a $\mathscr{CS}_\alpha$ such that $f(x/D) \neq 0$. Now there is an $s \in \prod_{t\in T} U_t^\alpha$ such that $s_t \in x_t$ whenever $t \in T$ and $x_t \neq 0$. Let $s'(\varkappa)_t = s_t(\varkappa)$ for all $\varkappa < \alpha$ and $t \in T$, and let $s''(\varkappa) = s'(\varkappa)/D$ for all $\varkappa < \alpha$. Let $\mathscr{C}$ be a function on $\prod_{t\in T} U_t/D$ into $\prod_{t\in T} U_t$ such that $\mathscr{C}(y/D)/D = y/D$ for all $y \in \prod_{t\in T} U_t$. Now for each $w \in (\prod_{t\in T} U_t/D)^\alpha$ define π_w as follows:

$$\pi_w(\varkappa) = \begin{cases} s'(\varkappa) \text{ if } s''(\varkappa) = w(\varkappa), \\ \mathscr{C}(w(\varkappa)) \text{ if } s''(\varkappa) \neq w(\varkappa), \end{cases}$$

for all $\varkappa < \alpha$. The desired function f is defined as follows:

$$f(y/D) = \{w \in (\textstyle\prod_{t\in T} U_t/D)^\alpha : \{t : pr_t \circ \pi_w \in y_t\} \in D\}$$

for all $y \in B$, where for each $t \in T$ pr_t is the function mapping B into A_t such that $pr_t(z) = z_t$ for all $z \in B$.

The problem arises whether an ultraproduct of $\mathscr{CS}_\alpha$'s is isomorphic to a $\mathscr{CS}_\alpha$. For α finite the preceding proof can easily be modified to give a positive answer. The answer is negative for α infinite, e.g., for $\alpha = \omega$. For, let T be a set of cardinality 2^{2^ω} and for each $t \in T$ let $\mathfrak{A}_t$ be the $\mathscr{CS}_\omega$ of all subsets of 2^ω; thus $U_t = 2$ for all $t \in T$. By Theorem 1.25 of Frayne–Morel–Scott [62] choose D such that $\prod_{t\in T} \mathfrak{A}_t/D$ has cardinality $2^{2^{2^\omega}}$. Now $\prod_{t\in T} U_t/D$ has just two elements, and a $\mathscr{C}\mathscr{S}_\omega$ with base 2 has $\leqslant 2^{2^\omega}$ elements. The equation $c_0c_1c_2(-d_{01}\cdot -d_{02}\cdot -d_{12})=0$ holds in $\mathfrak{A}_t$ for each $t\in T$ and hence also in $\prod_{t\in T} A_t/D$. It follows that $\prod_{t\in T}\mathfrak{A}_t/D$ is not isomorphic to a $\mathscr{C}\mathscr{S}_\omega$.

Theorem 1.7. *Suppose $\mathfrak{A}\in\mathscr{C}\mathscr{A}_\alpha$. Then the following conditions are equivalent:*

(i) $\mathfrak{A}\in\mathscr{R}\mathscr{C}\mathscr{A}_\alpha$;

(ii) *every finite reduct of $\mathfrak{A}$ is representable;*

(iii) *every finitely generated subalgebra of $\mathfrak{A}$ is representable;*

(iv) *every finitely generated subalgebra of every finite reduct of $\mathfrak{A}$ is representable.*

This theorem can be easily derived from the preceding results.

For the next theorem we suppose that we are given first-order languages $L_0, L_1, \ldots, L_\omega$ with sets of non-logical constants $N_0, N_1, \ldots, N_\omega$ respectively, such that $N_0\subseteq N_1\subseteq\ldots\subseteq N_\omega=\bigcup_{\varkappa<\omega} N_\varkappa$. Also suppose that for each $\varkappa\leqslant\omega$ $\Delta_\varkappa$ (resp., $\Gamma_\varkappa$) is a set of $L_\varkappa$-sentences (resp., $L_\varkappa$-formulas with sole free variable v_0) such that $\Delta_0\subseteq\Delta_1\subseteq\ldots\subseteq\Delta_\omega=\bigcup_{\varkappa<\omega}\Delta_\varkappa$ (resp., $\Gamma_0\subseteq\Gamma_1\subseteq\ldots\subseteq\Gamma_\omega=\bigcup_{\varkappa<\omega}\Gamma_\varkappa$).

Theorem 1.8. *Suppose that $\mathfrak{A}$ is an L_0-structure and that for every $\varkappa<\omega$ $\mathfrak{A}$ is a substructure of the N_0-reduct of an $L_\varkappa$-structure $\mathfrak{B}_\varkappa$ such that $\mathfrak{B}_\varkappa$ is a model of $\Delta_\varkappa$ and x satisfies φ in $\mathfrak{B}_\varkappa$ for each $x\in A$ and each φ in $\Gamma_\varkappa$.*

Then $\mathfrak{A}$ is a substructure of the N_0-reduct of an L_ω-structure $\mathfrak{C}$ such that $\mathfrak{C}$ is a model of Δ_ω and x satisfies φ in $\mathfrak{C}$ for each $x\in A$ and each φ in Γ_ω.

Proof. For each $\varkappa<\omega$, let $M_\varkappa=\{\lambda\in\omega : \varkappa\leqslant\lambda\}$. Then $\{M_\varkappa : \varkappa<\omega\}$ is included in an ultrafilter D on ω. The set $\prod_{\varkappa<\omega} B_\varkappa/D$ can be given a natural L_ω-structure, and one can easily define an isomorphic embedding of $\mathfrak{A}$ into the N_0-reduct of this structure so that the conclusion of the theorem can be satisfied.

Theorems 1.1–1.8 are all of a rather elementary nature. The more profound results of the representation theory stem from the following theorem, which will not be proved here. A proof can be found by consulting Halmos [1] and Galler [1] [1].

[1] Several simpler but unpublished proofs, due to Henkin and Tarski, are known. Perhaps the simplest was used in a modified form in Daigneault–Monk [63].

Theorem 1.9 (Tarski). *For* $\alpha \geqslant \omega$, $\mathscr{L}\mathscr{C}\mathscr{A}_\alpha \subseteq \mathscr{R}\mathscr{C}\mathscr{A}_\alpha$.

Theorem 1.10 [2]. *Suppose* $\mathfrak{A} \in \mathscr{C}\mathscr{A}_\alpha$. *Then the following conditions are equivalent:*

(i) $\mathfrak{A} \in \mathscr{R}\mathscr{C}\mathscr{A}_\alpha$;
(ii) *for every* $\varkappa < \omega$, $\mathfrak{A}$ *can be neatly embedded in a* $\mathscr{C}\mathscr{A}_{\alpha+\varkappa}$;
(iii) $\mathfrak{A}$ *can be neatly embedded in a* $\mathscr{C}\mathscr{A}_{\alpha+\omega}$.

Proof. That (i) implies (ii) follows from Theorem 1.2; that (ii) implies (iii) is seen by Theorem 1.8. Now assume (iii) holds. Then each finite reduct of $\mathfrak{A}$ is a subalgebra of a reduct of an $\mathscr{L}\mathscr{C}\mathscr{A}_\omega$, and so by Theorem 1.9 is representable. Hence by Theorem 1.7 $\mathfrak{A}$ is representable.

We are now in a position to describe the class $\mathscr{R}\mathscr{C}\mathscr{A}_\alpha$ in model-theoretic terms.

Theorem 1.11. $\mathscr{R}\mathscr{C}\mathscr{A}_\alpha$ *is an equational class (in the wider sense); if* $\alpha \geqslant \omega$, *then* $\mathscr{R}\mathscr{C}\mathscr{A}_\alpha$ *is the least* $\mathbf{UC}_\delta$ *including* $\mathscr{L}\mathscr{C}\mathscr{A}_\alpha$.

Proof. The equivalence – 1.10 (i) equivalent to 1.10 (iii) – can be used to show that $\mathscr{R}\mathscr{C}\mathscr{A}_\alpha$ is closed under the operation of taking homomorphic images, so $\mathscr{R}\mathscr{C}\mathscr{A}_\alpha$ is equational [3]. If $\mathscr{K}$ is a $\mathbf{UC}_\delta$ including $\mathscr{L}\mathscr{C}\mathscr{A}_\alpha$ and if $\mathfrak{A} \in \mathscr{R}\mathscr{C}\mathscr{A}_\alpha$, then by Theorem 1.5 $\mathfrak{A}$ is a subalgebra of a member of $\mathscr{K}$, so $\mathfrak{A} \in \mathscr{K}$.

Problem 3. For $\alpha \geqslant \omega$: is there an $\mathbf{EC}_\delta$ $\mathscr{K}$ such that $\mathscr{L}\mathscr{C}\mathscr{A}_\alpha \subseteq \subseteq \mathscr{K} \subset \mathscr{R}\mathscr{C}\mathscr{A}_\alpha$?

Actually we can give a more detailed description of the equational character of the class $\mathscr{R}\mathscr{C}\mathscr{A}_\alpha$, a description which shows the universal character of $\mathscr{L}\mathscr{C}\mathscr{A}_\omega$ for equations. To this end we introduce the following notation: if ε is an equation, Γ is the set of $\varkappa < \alpha$ such that a symbol $c_\varkappa$ or $d_{\varkappa\lambda}$ occurs in ε, and δ is a one-to-one function mapping Γ into ω, then by $\delta^*(\varepsilon)$ we mean the equation obtained from ε by replacing each index $\varkappa$ of symbols occurring in ε by $\delta(\varkappa)$; δ is called an *ω-mapping for ε*.

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[2] From this theorem one can derive many generalizations of Theorem 1.9 in a rather rapid fashion; see Monk [61]. Using the notation of that paper, the following new result follows from the present Theorem 1.5: $\mathscr{G}\mathscr{C}\mathscr{A}_\alpha$, $\mathscr{W}\mathscr{D}\mathscr{C}\mathscr{A}_\alpha$, and the class of semi-simple $\mathscr{C}\mathscr{A}_\alpha$'s ($\alpha \geqslant \omega$) are not $\mathbf{EC}_\delta$'s; and in the same way one can prove that $\mathscr{L}\mathscr{C}\mathscr{A}_\alpha$ ($\alpha \geqslant \omega$) is not an $\mathbf{EC}_\delta$ (this was known previously because of Feferman–Vaught [59], Corollary 6.7.2).

[3] For further details, see Monk [61].

Theorem 1.12. *For each equation ε in the elementary theory of $\mathscr{CA}_\alpha$'s let δ_ε be an ω-mapping for ε. Then $\mathscr{RCA}_\alpha$ is the class of all models of all equations ε such that $\delta_\varepsilon^*(\varepsilon)$ holds in every $\mathscr{LCA}_\omega$.*

Proof. Let $\Sigma = \{\varepsilon : \varepsilon$ is an equation identically satisfied in every $\mathscr{RCA}_\alpha\}$ and let $\Gamma = \{\varepsilon : \delta_\varepsilon^*(\varepsilon)$ holds in every $\mathscr{LCA}_\omega\}$. It suffices to show that $\Sigma = \Gamma$. First we take the case $\alpha \geqslant \omega$. Note first that if $\delta_\varepsilon \subseteq \gamma$ and γ is a permutation of α, then ε holds in a $\mathscr{CA}_\alpha$ $\mathfrak{A}$ iff $\delta_\varepsilon^*(\varepsilon)$ holds in the $\alpha\gamma$-reduct $\mathfrak{A}'$ of $\mathfrak{A}$; moreover $\mathfrak{A}$ is representable iff $\mathfrak{A}'$ is, and $\mathfrak{A}$ is locally finite iff $\mathfrak{A}'$ is. Hence $\varepsilon \in \Sigma$ iff $\delta_\varepsilon^*(\varepsilon) \in \Sigma$. Now suppose $\varepsilon \in \Sigma$. Then $\delta_\varepsilon^*(\varepsilon) \in \Sigma$, so in particular $\delta_\varepsilon^*(\varepsilon)$ holds in every $\mathscr{LCA}_\alpha$ and hence by 1.9 and 1.2 in every $\mathscr{LCA}_\omega$; thus $\varepsilon \in \Gamma$. Suppose conversely that $\varepsilon \in \Gamma$. Then $\delta_\varepsilon^*(\varepsilon)$ holds in every $\mathscr{LCA}_\omega$ and hence in every $\mathscr{LCA}_\alpha$, since the ω-reduct of an $\mathscr{LCA}_\alpha$ is an $\mathscr{LCA}_\omega$. By Theorem 1.5 $\delta_\varepsilon^*(\varepsilon) \in \Sigma$, so $\varepsilon \in \Sigma$.

Now we take the case $\alpha < \omega$. If $\varepsilon \in \Sigma$, then ε holds in every $\mathscr{RCA}_\omega$ by 1.1, so that $\delta_\varepsilon^*(\varepsilon)$ holds in every $\mathscr{RCA}_\omega$ and in particular in every $\mathscr{LCA}_\omega$, and $\varepsilon \in \Gamma$. Assume $\varepsilon \in \Gamma$. Then by 1.5 $\delta_\varepsilon^*(\varepsilon)$ holds in every $\mathscr{RCA}_\omega$ and so ε holds in every $\mathscr{RCA}_\omega$. Hence by 1.2 $\varepsilon \in \Sigma$.

The well known argument of Craig [53] can be modified to prove the following theorem.

Theorem 1.13. *For $\alpha < \omega$ there is a primitive recursive set Σ of equations such that $\mathscr{RCA}_\alpha$ is the set of all models of Σ.*

Proof [4]. Let $\Gamma = \{\varepsilon : \varepsilon$ holds in every $\mathscr{RCA}_\alpha\}$. Then by Theorem 2.11 of Henkin–Tarski [61], Γ is recursively enumerable. Let f be a primitive recursive function enumerating Γ, and let $\Delta = \{\varepsilon :$ there is a $p \leqslant \varepsilon$ such that ε has the form $(\varphi - \psi) + \ldots + (\varphi - \psi) = 0$, where $f(p) = \varphi \doteq \psi$ and $-$ is symmetric difference$\}$. Clearly Δ is primitive recursive and $\mathscr{RCA}_\alpha$ is the set of all models of Δ.

The problem still remains whether or not $\mathscr{RCA}_\alpha$ is finitely axiomatizable for $\alpha < \omega$ [5]. We treat this problem in the next section.

2. Non-finite axiomatizability of $\mathscr{RCA}_3$. Since $\mathscr{RCA}_0 = \mathscr{CA}_0$ and $\mathscr{RCA}_1 = \mathscr{CA}_1$, these two classes are finitely axiomatizable. Also, $\mathscr{RCA}_2$ is

[4] For the purposes of this proof we shall identify formulas with natural numbers.

[5] If is also of interest to consider the case $\alpha \geqslant \omega$, where of course finite axiomatizability fails but where the problem of a kind of uniform infinite axiomatizability is still open.

finitely axiomatizable, since $\mathscr{RCA}_2$ is the class of all models of the equations defining $\mathscr{CA}_2$'s together with two further equations (see Henkin and Tarski [61], Theorem 2.18). The main purpose of this section, on the other hand, is to show that $\mathscr{RCA}_3$ is not finitely axiomatizable. The method used involves two steps. First, non-representable $\mathscr{CA}_3$'s are associated with numbers m for which there is no projective plane of order m, following the pattern of Lyndon [61]. Second, an ultraproduct of these non-representable algebras is shown to be representable.

If R is an equivalence relation on 3, let $\binom{R}{2} = \{\{\alpha\beta\} : \alpha$ and β are distinct R-equivalence classes$\}$. Let G be any set with at least three elements. Let G' be the set of all pairs $\langle R, f\rangle$ such that R is an equivalence relation on 3, f maps $\binom{R}{2}$ into G, and the range of f does not consist of exactly two elements. Let $\langle A_G, \cup, \cap, \sim\rangle$ be the Boolean algebra formed from all subsets of G'. For $\varkappa, \lambda < 3$ define

$$\begin{aligned} d_{\varkappa\lambda} &= \{\langle R, f\rangle : \langle \varkappa\lambda\rangle \in R\};\\ c_\varkappa\{\langle R, f\rangle\} &= \{\langle S, g\rangle : S \cap (3 \sim \{\varkappa\})^2 = R \cap (3 \sim \{\varkappa\})^2\\ &\qquad \text{and } f\{\alpha\beta\} = g\{\gamma\delta\} \text{ whenever}\\ &\qquad 0 \neq \alpha \cap \gamma \neq \{\varkappa\} \text{ and } 0 \neq \beta \cap \delta \neq \{\varkappa\}\};\\ c_\varkappa a &= \textstyle\bigcup_{t\in a} c_\varkappa\{t\} \text{ if } a \in A_G. \end{aligned}$$

Finally, let $\mathfrak{A}_G = \langle A_G, \cup, \cap, \sim, c_\varkappa, d_{\varkappa\lambda}\rangle_{\varkappa,\lambda<3}$.

Lemma 2.1. *$\mathfrak{A}_G$ is a simple $\mathscr{CA}_3$, in which the condition*

$$(*) \qquad c_\varkappa c_\lambda a = 1$$

holds for $a \neq 0$ and $\varkappa \neq \lambda$.

Proof. We shall verify postulates P1 – P8 of Henkin–Tarski [61], simplicity, and (*). Of these, P1, P2, P3 and P6 obviously hold for $\mathfrak{A}_G$. Since $c_\varkappa$ is clearly completely additive, it suffices for P4 to assume that x and y are atoms (singletons). Note also that P5 and simplicity are implied by (*), and that (*) only needs to be proved for a a singleton, in virtue of P3.

Verification of P4. We wish to show that $c_\varkappa(\{\langle R, f\rangle\} \cap c_\varkappa\{\langle S, g\rangle\}) = c_\varkappa\{\langle R, f\rangle\} \cap c_\varkappa\{\langle S, g\rangle\}$. First suppose that $\langle T, h\rangle$ is a member of the left-hand side of this equation. Then $\langle R, f\rangle \in c_\varkappa\{\langle S, g\rangle\}$ and $\langle T, h\rangle \in c_\varkappa\{\langle R, f\rangle\}$. Thus $T \cap (3 \sim \{\varkappa\})^2 = R \cap (3 \sim \{\varkappa\})^2 = S \cap (3 \sim \{\varkappa\})^2$. If

$0 \neq \alpha \cap \gamma \neq \{\varkappa\}$ and $0 \neq \beta \cap \delta \neq \{\varkappa\}$ with α and β T-equivalence classes while γ and δ are S-equivalence classes, let ε and ξ be R-equivalence classes such that $0 \neq \alpha \cap \gamma \cap \varepsilon \neq \{\varkappa\}$ and $0 \neq \beta \cap \delta \cap \xi \neq \{\varkappa\}$; then $h\{\alpha\beta\} = f\{\varepsilon\xi\} = g\{\gamma\delta\}$. Thus $\langle T, h\rangle \in c_\varkappa\{\langle R, f\rangle\} \cap c_\varkappa\{\langle S, g\rangle\}$. Assume conversely that this membership relation holds. Then $R \cap (3 \sim \{\varkappa\})^2 = T \cap (3 \sim \{\varkappa\})^2 = S \cap (3 \sim \{\varkappa\})^2$. Suppose α and β are R-equivalence classes, γ and δ are S-equivalence classes, and $0 \neq \alpha \cap \gamma \neq \{\varkappa\}$, $0 \neq \beta \cap \delta \neq \{\varkappa\}$. Then there are T-equivalence classes ε and ξ such that $0 \neq \alpha \cap \gamma \cap \varepsilon \neq \{\varkappa\}$ and $0 \neq \beta \cap \delta \cap \xi \neq \{\varkappa\}$. Hence $f\{\alpha\beta\} = h\{\varepsilon\xi\} = g\{\gamma\delta\}$. Thus $\langle R, f\rangle \in c_\varkappa\{\langle S, g\rangle\}$, and $\langle T, h\rangle \in c_\varkappa(\{\langle R, f\rangle\} \cap c_\varkappa\{\langle S, g\rangle\})$.

Verification of (*). We wish to show that $\langle S, g\rangle \in c_\varkappa c_\lambda\{\langle R, f\rangle\}$, with $\varkappa \neq \lambda$. Let $3 = \{\varkappa, \lambda, \mu\}$. Let $T = [R \cap (3 \sim \{\lambda\})^2] \cup [S \cap (3 \sim \{\varkappa\})^2] \cup \{\langle\varkappa\lambda\rangle, \langle\lambda\varkappa\rangle : \langle\varkappa\mu\rangle \in R \text{ and } \langle\lambda\mu\rangle \in S\}$. Clearly T is an equivalence relation on 3. Let $\mathscr{H}$ be the set of functions on $\binom{T}{2}$ into G whose range has just one, or exactly three, elements. If $\langle\mu\lambda\rangle \in T$, choose $h \in \mathscr{H}$ so that $h\{[\mu]_T, [\varkappa]_T\} = f\{[\mu]_R, [\varkappa]_R\}$ if $\langle\mu\varkappa\rangle \notin R$ ($[\mu]_T$ is the T-equivalence class of which μ is a member). If $\langle\mu\varkappa\rangle \in T$, choose $h \in \mathscr{H}$ so that $h\{[\mu]_T, [\lambda]_T\} = g\{[\mu]_S, [\lambda]_S\}$ if $\langle\mu\lambda\rangle \notin S$. Finally, if $\langle\mu\lambda\rangle, \langle\mu\varkappa\rangle \notin T$, choose $h \in \mathscr{H}$ such that $h\{[\mu]_T, [\varkappa]_T\} = f\{[\mu]_R, [\varkappa]_R\}$ if $\langle\mu\varkappa\rangle \notin R$ and $h\{[\mu]_T, [\lambda]_T\} = g\{[\mu]_S, [\lambda]_S\}$ if $\langle\mu\lambda\rangle \notin S$. Then in any case $\langle T, h\rangle \in c_\lambda\{\langle R, f\rangle\}$ and $\langle S, g\rangle \in c_\varkappa\{\langle T, h\rangle\}$, so $\langle S, g\rangle \in c_\varkappa c_\lambda\{\langle R, f\rangle\}$.

Verification of P7. We wish to show that $d_{\varkappa\lambda} = c_\mu(d_{\varkappa\mu} \cdot d_{\mu\lambda})$, where $\varkappa, \lambda \neq \mu$. First suppose $\varkappa \neq \lambda$. Clearly $d_{\varkappa\mu} \cdot d_{\mu\lambda} = \{\langle 3^2, 0\rangle\}$ and hence the desired result follows. Second, suppose $\varkappa = \lambda$. Let $\langle R, F\rangle \in G'$. Let $3 = \{\varkappa\mu\nu\}$, and let

$$S = \begin{cases} 3^2 \text{ if } \langle\varkappa\nu\rangle \in R \\ \{\langle\varkappa\mu\rangle, \langle\varkappa\varkappa\rangle, \langle\mu\varkappa\rangle, \langle\mu\mu\rangle, \langle\nu\nu\rangle\} \text{ if } \langle\varkappa\nu\rangle \notin R. \end{cases}$$

Clearly S is an equivalence relation on 3.

Let $g = 0$ if $\langle\varkappa\nu\rangle \in R$, and let g be such that $g([\varkappa]_S, [\nu]_S) = f([\varkappa]_R, [\nu]_R)$ if $\langle\varkappa\nu\rangle \notin R$. Then $\langle S, g\rangle \in d_{\varkappa\mu}$ and $\langle R, f\rangle \in c_\mu\{\langle S, g\rangle\}$, so $\langle R, f\rangle \in c_\mu d_{\varkappa\mu}$.

Verification of P8. By the complete additivity of $c_\varkappa$ it suffices to show that $c_\varkappa\{\langle R, f\rangle\} \cap c_\varkappa\{\langle S, g\rangle\} = 0$, where $\langle\varkappa\lambda\rangle \in R \cap S$, $\langle R, f\rangle \neq \langle S, g\rangle$, and $\varkappa \neq \lambda$. We cannot have $R = S = 3^2$, since then $f = g = 0$ and $\langle R, f\rangle = \langle S, g\rangle$. Say $\{\varkappa\lambda\mu\} = 3$. Suppose $\langle T, h\rangle \in c_\varkappa\{\langle R, f\rangle\} \cap c_\varkappa\{\langle S, g\rangle\}$. If $R = S$, then since $R \neq 3^2$ and $\langle\varkappa\lambda\rangle \in R$, we have $\langle\lambda\mu\rangle \notin R$, and so $\langle\lambda\mu\rangle \notin S$; hence $\langle\lambda\mu\rangle \notin T$ and $f([\lambda]_R, [\mu]_R) = h([\lambda]_T, [\mu]_T) = g([\lambda]_S, [\mu]_S)$, so that $f = g$ and

$\langle R, f\rangle = \langle S, g\rangle$, which contradicts an earlier assumption. Thus $R \neq S$. By symmetry we may assume that $R = 3^2$, while $S \neq 3^2$. In particular, $\langle \lambda\mu\rangle \in R \sim S$, which contradicts $\langle T, h\rangle \in c_\varkappa\{\langle R, f\rangle\} \cap c_\varkappa\{\langle S, g\rangle\}$.

Lemma 2.2. *If G has at least four elements and if G is a line in a projective plane P, then $\mathfrak{A}_G \in \mathscr{RCA}_3$* [6].

Proof. Let $U = P \sim G$. Define

$$F\{\langle R, f\rangle\} = \{x \in U^3 : x_\varkappa = x_\lambda \text{ iff } \langle \varkappa\lambda\rangle \in R,\ \text{and } f\{\alpha\beta\} \in \overline{x_\varkappa x_\lambda} \text{ whenever } \varkappa \in \alpha,\ \lambda \in \beta, \text{ and } \alpha \neq \beta\};$$

$$F(a) = \bigcup_{t \in a} F\{t\}$$

for $a \in A_G$. Clearly F preserves $\bigcup$. If $x \in F\{\langle R, f\rangle\} \cap F\{\langle S, g\rangle\}$, then $\langle \varkappa\lambda\rangle \in R$ iff $x_\varkappa = x_\lambda$ iff $\langle \varkappa\lambda\rangle \in S$, and so $R = S$; since furthermore two distinct lines intersect in exactly one point, we have $f = g$. It follows that $F(a) \cap F(-a) = 0$ for all $a \in A_G$. If $x \in U^3$, let $R = \{\langle \varkappa\lambda\rangle : x_\varkappa = x_\lambda\}$; thus R is an equivalence relation on 3. Let $f\{\alpha\beta\} = \overline{x_\varkappa x_\lambda} \cdot G$ whenever $\varkappa \in \alpha$, $\lambda \in \beta$, and $\alpha \neq \beta$. Then f maps $\binom{R}{2}$ into G, and the range of f does not consist of exactly two elements. Clearly $x \in F\{\langle R, f\rangle\}$. Hence $F(a) \cup F(-a) = 1$ for all $a \in A_G$, and F preserves $-$. Let $\langle R, f\rangle \in G'$ be given. If $R = 3^2$, let $x \in U^3$ be such that $x_0 = x_1 = x_2$. If R has exactly two equivalence classes $\alpha = \{\varkappa\lambda\}$ and $\beta = \{\mu\}$, let x_μ be a point not on G and let $x_\varkappa = x_\lambda$ be a further point on the line $\overline{x_\mu f\{\alpha\beta\}}$. If R has three equivalence classes but the range of f is a singleton $\{u\}$, let x_0 be a point not on G and let x_1 and x_2 be two further points on the line $\overline{x_0 u}$. Finally, if R has three equivalence classes and the range of f consists of three distinct elements, let x_0 be a point not on G, let x_1 be a further point on the line $\overline{x_0 f\{\{0\}\{1\}\}}$, and let $x_2 = \overline{x_0 f\{\{0\}\{2\}\}} \cdot \overline{x_1 f\{\{1\}\{2\}\}}$. In any of these cases, $x \in F\{\langle R, f\rangle\}$. Hence $F(a) \neq 0$ whenever $a \neq 0$, and so F is one-to-one. Clearly $F(d_{\varkappa\lambda}) = D_{\varkappa\lambda}$ whenever $\varkappa, \lambda < 3$. Finally, suppose $x \in F(c_\varkappa\{\langle R, f\rangle\})$. Say $x \in F\{\langle S, g\rangle\}$, where $S \cap (3 \sim \{\varkappa\})^2 = R \cap (3 \sim \{\varkappa\})^2$ and $f\{\alpha\beta\} = g\{\gamma\delta\}$ whenever $0 \neq \alpha \cap \gamma \neq \{\varkappa\}$ and $0 \neq \beta \cap \delta \neq \{\varkappa\}$. Let $y_\lambda = x_\lambda$ for every $\lambda \in 3 \sim \{\varkappa\}$. If $\{\varkappa\}$ is not an R-equivalence class, say

[6] Throughout the remainder of the paper we follow the notation of Pickert [55] for projective planes, except that: a line is considered to be the set of all points lying on it, $\overline{PQ}$ is the line joining P and Q, and $l \cdot m$ is the intersection point of l and m.

$\langle\varkappa\lambda\rangle \in R$ and let $y_\varkappa = x_\lambda$. If the R-equivalence classes are $\{\varkappa\}$ and $\{\lambda\mu\}$, let $y_\varkappa$ be a third point on the line $\overline{x_\lambda f\{\{\varkappa\}\{\lambda\mu\}\}}$. If the R-equivalence classes are $\{\varkappa\}$, $\{\lambda\}$ and $\{\mu\}$ and if the range of f is a singleton, let $y_\varkappa$ be a fourth point on the line through $f\{\{\lambda\}\{\mu\}\}$, x_λ and x_μ. Lastly if the R-equivalence classes are $\{\varkappa\}$, $\{\lambda\}$ and $\{\mu\}$ and if the range of f has exactly three elements, let $y_\varkappa = \overline{x_\lambda f\{\{\varkappa\}\{\lambda\}\}} \cdot \overline{x_\mu f\{\{\varkappa\}\{\mu\}\}}$. In any of these cases it is easily seen that $y \in F\{\langle R, f\rangle\}$, so that $x \in C_\varkappa F\{\langle R, f\rangle\}$. Thus $F(c_\varkappa\{\langle R, f\rangle\}) \subseteq C_\varkappa F\{\langle R, f\rangle\}$. The converse is verified in a similar manner.

Lemma 2.3. *If G has at least four elements but is finite, and if $\mathfrak{A}_G$ is representable, then G is a line in some projective plane.*

Proof. Since $\mathfrak{A}_G$ is simple, the hypothesis implies the existence of an isomorphism F of $\mathfrak{A}_G$ onto a $\mathscr{C}\mathscr{S}_3$ with base U. We may assume that $U \cap G = 0$. Let $P = G \cup U$; members of P are the points of our projective plane. For the lines of P we take G together with all sets

$$L(x, u) = \{x, u\} \cup \{v : \langle vuu\rangle \in F\{\langle R, f\rangle\}\},$$

where $x \in G$, $u \in U$, R has equivalence classes $\{0\}$ and $\{12\}$, and $f\{\{0\}\{12\}\} = x$. To show that we have a projective plane, we shall verify the conditions (1.1), (1.2), (1.16), (1.17) and (1.18) of Pickert [55]. To do this we first need two auxiliary results:

(a) If $u, v \in U$, $x \in G$, and $v \in L(x, u)$, then $u \in L(x, v)$.

To prove (a), we may assume that $u \neq v$. Let S have equivalence classes $\{01\}$, $\{2\}$, and let g be such that $g\{\{01\}\{2\}\} = x$. Then with R and f as above we have $d_{01} \cdot c_1\{\langle R, f\rangle\} = \{\langle S, g\rangle\}$. Hence $D_{01} \cap C_1 F\{\langle R, f\rangle\} = F\{\langle S, g\rangle\}$. By assumption $\langle vuu\rangle \in F\{\langle R, f\rangle\}$, so $\langle vvu\rangle \in F\{\langle S, g\rangle\}$. Now let T have equivalence classes $\{02\}$, $\{1\}$, and let h be such that $h\{\{02\}\{1\}\} = x$. Then $d_{02} \cdot c_0\{\langle S, g\rangle\} = \{\langle T, h\rangle\}$, so $\langle uvu\rangle \in F\{\langle T, h\rangle\}$. Also, $d_{12} \cdot c_2\{\langle T, h\rangle\} = \{\langle R, f\rangle\}$, so $\langle uvv\rangle \in F\{\langle R, f\rangle\}$ and $u \in L(x, v)$.

(b) If $u, v \in U$, $x \in G$, and $v \in L(x, u)$, then $L(x, u) = L(x, v)$.

To prove (b), by (a) and symmetry it suffices to show that $L(x, u) \subseteq L(x, v)$. Suppose $w \in L(x, u)$ and $u \neq v \neq w \neq u$. We are given that $\langle wuu\rangle \in F\{\langle R, f\rangle\}$ and $\langle vuu\rangle \in F\{\langle R, f\rangle\}$. As in the proof of (a) we see that $\langle vvu\rangle \in F\{\langle S, g\rangle\}$. Let V have equivalence classes $\{0\}$, $\{1\}$, $\{2\}$, with k mapping $\binom{V}{2}$ onto $\{x\}$. Then $c_1\{\langle R, f\rangle\} \cap c_0\{\langle S, g\rangle\} = \{\langle S, g\rangle, \langle V, k\rangle\}$.

Hence $\langle wvu\rangle \in F\{\langle S, g\rangle\} \cup F\{\langle V, k\rangle\}$; since clearly $\langle wvu\rangle \notin F\{\langle S, g\rangle\}$, we have $\langle wvu\rangle \in F\{\langle V, k\rangle\}$. Now $d_{12}\cdot c_2\{\langle V, k\rangle\}=\{\langle R, f\rangle\}$, so $\langle wvv\rangle \in F\{\langle R, f\rangle\}$ and hence $w \in L(x, v)$.

Now to verify the axioms for a projective plane, we note that (1.1) and (1.16) are obvious. To verify (1.2), suppose that P_i lies on l_k, $i, k=1, 2$, while $\mathrm{l}_1 \neq \mathrm{l}_2$. If one of l_1, l_2 is G, then obviously $P_1=P_2$. Suppose $\mathrm{l}_1=L(x, u)$ and $\mathrm{l}_2=L(y, v)$. If $P_1 \in G$, then $P_1=x=y$. If $P_2 \in G$, then $P_1=P_2$. If $P_2 \notin G$, then $L(x, u)=L(x, P_2)=L(y, v)$, which contradicts the assumption $\mathrm{l}_1 \neq \mathrm{l}_2$. One argues similarly if $P_2 \in G$. Suppose $P_1, P_2 \notin G$. Then $L(x, u)=L(x, P_1)$ and $L(y, v)=L(y, P_1)$; since $P_2 \in L(x, P_1) \cap L(y, P_1)$ we have $\langle P_2P_1P_1\rangle \in F\{\langle R, f\rangle\} \cap F\{\langle R, g\rangle\}$, where the range of f is $\{x\}$ and the range of g is $\{y\}$. Hence $x=y$ since F preserves $-$, and $L(x, u)=L(y, v)$, again a contradiction. Thus (1.2) holds.

To verify (1.17), note that G intersects each line $L(x, u)$. Now suppose two distinct lines $L(x, u)$ and $L(y, v)$ are given. We may assume that $x \neq y$ and $u \neq v$. Let R have equivalence classes $\{0\}$ and $\{12\}$ and let f and g map $\binom{R}{2}$ onto $\{x\}$ and $\{y\}$ respectively. Let S have equivalence classes $\{01\}$ and $\{2\}$. Then it is easily seen that

(c) $d_{01}\cdot c_0(c_1\{\langle R, f\rangle\} \cap c_2\{\langle R, g\rangle\})=\{\langle S, h\rangle : h$ maps $\binom{S}{2}$ onto $\{z\}$, with $z \neq x, y\}$.

Now since $\mathfrak{A}_G$ is finite, there is an h such that $\langle vvu\rangle \in F\{\langle S, h\rangle\}$ [7]. Say h maps $\binom{S}{2}$ onto $\{z\}$. If $z=x$, then $\langle vuu\rangle \in F\{\langle R, f\rangle\}$ so that $v \in L(x, u) \cap L(y, v)$. If $z=y$, then $\langle vuu\rangle \in F\{\langle R, g\rangle\}$ so that $u \in L(y,v) \cap L(x, u)$. Assume $z \neq x, y$. Then by (c) choose w such that $\langle wvu\rangle \in F(c_1\{\langle R,f\rangle\}\cap c_2\{\langle R,g\rangle\})$. Then $\langle wuu\rangle \in F\{\langle R,f\rangle\}$ and $\langle wvv\rangle \in F\{\langle R,g\rangle\}$, so $w \in L(x, u) \cap L(y, v)$. Thus (1.17) holds.

To prove (1.18), let x, y, z be distinct elements of G, and let $u \in U$. If R has equivalence classes $\{0\}$ and $\{12\}$ and f maps $\binom{R}{2}$ onto $\{z\}$, then $c_0\{\langle R, f\rangle\}=d_{12}$. Hence $\langle uuu\rangle \in C_0F\{\langle R, f\rangle\}$, so choose v such that $\langle vuu\rangle \in F\{\langle R, f\rangle\}$. It is easily seen that x, y, u, v are four points no three of which are collinear.

[7] It is only here that the assumption that G is finite is used. The necessity for some assumption to assure the existence of h was first noticed in another context by Tarski; see Lyndon [56].

The above three lemmas were suggested by the work of Lyndon [61] on relation algebras. Now we can prove the main result of this section.

Theorem 2.4. *$\mathscr{RCA}_3$ is not finitely axiomatizable* [8].

Proof. Let $M = \{m \in \omega : 3 \leqslant m$ and there is no projective plane of order $m\}$. By the Bruck–Ryser theorem [49], M is infinite. For each $m \in M$ let $K_m = \{n \in M : m \leqslant n\}$. Then $\{K_m : m \in M\}$ has the finite intersection property and so there is an ultrafilter D on the set of all subsets of M such that $\{K_m : m \in M\} \subseteq D$. For each $m \in M$ let G_m be a set with $m+1$ elements. Then by Lemma 2.3, $\mathfrak{A}_{G_m}$ is non-representable for each $m \in M$. The proof will be complete when we show that $\mathfrak{B} = \prod_{m \in M} \mathfrak{A}_{G_m}/D$ is representable.

Let $H = \prod_{m \in M} G_m/D$. By Theorem 6.5 of Kochen [61], H has $2^{\aleph_0}$ elements. Hence by Lemma 2.2, $\mathfrak{A}_H$ is representable. We shall show that $\mathfrak{B}$ is isomorphic to a subalgebra of $\mathfrak{A}_H$. Using Theorem 2.2 of Frayne–Morel–Scott [62], we easily see that $\mathfrak{B}$ is atomic and that for each atom b of $\mathfrak{B}$ there is an equivalence relation R on 3 and a function

$$g \in \prod_{m \in M} G_m{}^{\binom{R}{2}}$$

such that $b = h/D$, where $h_m = \{\langle R, g_m \rangle\}$ for each $m \in M$. We let $h = \mathscr{A}(R, g)$, and we define g' mapping $\binom{R}{2}$ into $\prod_{m \in M} G_m$ as follows:

$$(g'\{\alpha\beta\})_m = g_m\{\alpha\beta\}$$

for all $\{\alpha\beta\} \in \binom{R}{2}$ and all $m \in M$. If k maps $\binom{R}{2}$ into $\prod_{m \in M} G_m$, define k^* as follows:

$$(k_m^*)\ \{\alpha\beta\} = (k\{\alpha\beta\})_m;$$

thus $k^* \in \prod_{m \in M} G_m{}^{\binom{R}{2}}$.

Let $\mathscr{C}$ be a choice function for H and let π be the natural mapping of $\prod_{m \in M} G_m$ onto H. The isomorphism F embedding $\mathfrak{B}$ in $\mathfrak{A}_H$ is defined as follows:

$$F(b) = \{\langle R, g \rangle : \mathscr{A}(R, (\mathscr{C} \circ g)^*)/D \leqslant b\}$$

for all $b \in B$. Clearly $F(\sum X) = \bigcup_{x \in X} F(x)$ whenever $\sum X$ exists; F preserves $-$; and F is one-to-one. If $\varkappa, \lambda < 3$, then

[8] Building upon Lyndon [61] the corresponding result for relation algebras can also be proved in a fashion analogous to the proof below. Theorem 2.4 is not surprising then, in view of Monk [61a].

$$\begin{aligned}\langle R, g\rangle \in F(d_{\varkappa\lambda}) \ &\text{iff } \{m : \langle R, (\mathscr{C} \circ g)^*_m\rangle \in d_{\varkappa\lambda}\} \in D \\ &\text{iff } \{m : \langle \varkappa\lambda\rangle \in R\} \in D \\ &\text{iff } \langle \varkappa\lambda\rangle \in R \\ &\text{iff } \langle R, g\rangle \in d_{\varkappa\lambda}.\end{aligned}$$

To show that $F(c_\varkappa b)=c_\varkappa F(b)$ for all $b \in B$ it suffices to take the case in which b is an atom $\mathscr{A}(R, f)/D$. Assume $\langle S, g\rangle \in F(c_\varkappa\mathscr{A}(R, f)/D)$. Then $\mathscr{A}(S, (\mathscr{C}\circ g)^*)/D \leq c_\varkappa\mathscr{A}(R, f)/D$ and hence $\{m : \langle S, (\mathscr{C}\circ g)^*_m\rangle \in c_\varkappa\{\langle R, f_m\rangle\}\} \in D$. Thus $S \cap (3 \sim \{\varkappa\})^2 = R \cap (3 \sim \{\varkappa\})^2$ and $\{m : (\mathscr{C} \circ g)^*_m\{\alpha\beta\} = f_m\{\gamma\delta\}$ whenever $0 \neq \alpha \cap \gamma \neq \{\varkappa\}$ and $0 \neq \beta \cap \delta \neq \{\varkappa\}\} \in D$. It follows that $(\mathscr{C} \circ g)\{\alpha\beta\}/D = f'\{\gamma\delta\}/D$ whenever $0 \neq \alpha \cap \gamma \neq \{\varkappa\}$ and $0 \neq \beta \cap \delta \neq \{\varkappa\}$; hence $g\{\alpha\beta\} = (\pi \circ f')\{\gamma\delta\}$ under these circumstances, and thus $\langle S, g\rangle \in$ $\in c_\varkappa\{\langle R, \pi \circ f'\rangle\}$. Clearly $\{\langle R, \pi \circ f'\rangle\} = F(\mathscr{A}(R, f)/D)$, so $\langle S, g\rangle \in$ $\in c_\varkappa F(\mathscr{A}(R, f)/D)$. Hence $F(c_\varkappa\mathscr{A}(R, f)/D) \subseteq c_\varkappa F(\mathscr{A}(R, f)/D)$, and the converse is proved in a similar fashion. This completes the proof.

The above method does not immediately extend to higher dimensions, so the following problem is open.

Problem 4. For $3<\alpha<\omega$, is $\mathscr{RCA}_\alpha$ finitely axiomatizable?