SUBSTITUTIONLESS PREDICATE LOGIC WITH IDENTITY*

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Introduction. The main object of this paper is to simplify the usual framework for predicate logic (with identity). The motivation is analogous to that of the preceding papers of Tarski [14] and of Kalish and Montague [11]. In those papers, working with the ordinary rules of formula formation, the authors give very simple axiom systems for the universally valid formulas and sentences of predicate logic. Thus in describing the axioms of system $\mathfrak{S}_2$ of [14] only two notions of any degree of complexity at all appear: the notion of the set of variables occurring in a formula, and the notion of substituting one variable for another in an atomic formula. By modifying the formation rules of predicate logic we will be able to eliminate this last notion.

The modification consists in having after each non-logical predicate a fixed string of variables depending upon the particular predicate but not changing for different occurrences of the same predicate. Identity is considered as a logical notion, and so full substitution is allowed in atomic identity formulas. The possibility of still obtaining an adequate system of predicate logic using this formation rule may be illustrated by the following example. In ordinary logic, let π be a binary predicate. Then the formula

$$\pi\, v_1\, v_2 \leftrightarrow \bigwedge v_0\, [v_0 \equiv v_1 \rightarrow \bigwedge v_1\, (v_1 \equiv v_2 \rightarrow \pi\, v_0\, v_1)]$$

is universally valid. If $\langle v_0\, v_1 \rangle$ is considered to be the fixed string of variables associated with π, then this formula may be considered as a definition of the formula $\pi\, v_1\, v_2$.

The possibility of formalizing predicate logic in this way was recognized by Henkin and Tarski in [9]; there they state (without proof) that the power of expression and proof does not change upon going over to the new formation rule.

We shall now describe the contents of the paper in detail. In section 1 we define precisely the new formation rule mentioned above. Then we prove the theorem of Henkin and Tarski about the preservation of the power of expression and proof. Our first axiom system, $\mathfrak{L}$, for the substitutionless predicate logic is then described. Using the theory of cylindric algebras and the

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known completeness of the system $\mathfrak{S}_2$ of [14], $\mathfrak{L}$ is proven complete (in Theorem 6). A more direct proof, in the style of Henkin's completeness proof, is then outlined. Our second system, $\mathfrak{L}_1$, is then defined. This system seems to us especially simple; in its definition the only non-elementary notion required is that of the set of occurrences of a variable in a formula. Some systems related to $\mathfrak{L}$ and $\mathfrak{L}_1$ are described. All of these systems are much simpler than the systems one would get in a natural way from the theorem of Henkin and Tarski. Next, primarily with the view of indicating the utility of $\mathfrak{L}$ or $\mathfrak{L}_1$ as a basis for higher logical investigations, we indicate the respect in which operations and individuals can be discussed within these systems; this is the known and trivial method of "pushing" the operation symbols and individual constants into the metalanguage.

None of these observations can be applied to predicate logic without identity; we close section 1 with more specific remarks to this effect. In section 2 we discuss independence questions for the systems $\mathfrak{L}$, $\mathfrak{L}_1$, etc. introduced in the first section. The methods used in § 2 are applied in § 3 to show the independence of the system $\mathfrak{S}_1$ and variants of [14].

We use the notation of [14], with the following additions. If f and g are functions, fg is the composition of f and g, i. e., the function h such that $h\,(i) = f\,(g\,(i))$ for all i such that i is in the domain of g and $g\,(i)$ is in the domain of f; i is not in the domain of h except under these conditions. If f is a function and a is in the domain of f, by $f\,(b/a)$ we understand the function h such that $h\,(x) = f\,(x)$ for x in the domain of f, $x \neq a$, and $h\,(a) = b$. For any function f, *rng* f is the *range* of f. If A is any set, $\overline{\overline{A}}$ is the number of elements of A. The function v is biunique and has domain ω and range VR. The set VR is well-ordered by the relation $<$ such that $\alpha < \beta$ if and only if $v^{-1}\,(\alpha) < v^{-1}\,(\beta)$, for all $\alpha, \beta \in VR$. If $\varphi, \psi \in FM$, we let $\varphi \wedge \psi = \neg\,(\varphi \rightarrow \neg\,\psi)$, $\varphi \vee \psi = \neg\,\varphi \rightarrow \psi$, $\varphi \leftrightarrow \psi = (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$, and $\bigvee \alpha\,\varphi = \neg \bigwedge \alpha \neg\,\varphi$. If $\varphi \in FM$ and $\alpha, \beta \in VR$ with $\alpha \neq \beta$, we let $S(\beta/\alpha)\,\varphi = \bigwedge \alpha\,(\alpha \equiv \beta \rightarrow \varphi)$. If $\alpha = \beta$, let $S(\beta/\alpha)\,\varphi = \varphi$. As mentioned in [14], the formula $S(\beta/\alpha)\,\varphi$ is logically equivalent to the formula obtained from φ by replacing each free occurrence of α in φ by a free occurrence of β, after renaming bound variables, if necessary[1].

§ 1. Substitutionless Predicate Logic

The new rule of formation mentioned above is embodied in the following definitions. We now assume that with each $\pi \in PR$ there is associated a biunique function $u^{\pi} \in {}^{n}VR$, where $r\,(\pi) = n$. If $\varphi \in {}^{n}VR$, we let $D\,(\pi, \varphi) = \{i : i < n$

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and $\varphi_i \neq u^\pi(i)\}$. The *standard atomic formulas* are the expressions $\alpha \equiv \beta$ and $\pi \circ u^\pi$ for arbitrary $\alpha, \beta \in VR$ and $\pi \in PR$; the set of all standard atomic formulas is denoted by ATS. A *standard formula* is an expression belonging to every set Γ such that (i) $ATS \subseteq \Gamma$; (ii) $\varphi \to \psi \in \Gamma$ whenever $\varphi, \psi \in \Gamma$; (iii) $\neg\, \varphi \in \Gamma$ whenever $\varphi \in \Gamma$; (iv) $\bigwedge \alpha\, \varphi \in \Gamma$ whenever $\varphi \in \Gamma$ and $\alpha \in VR$. The set of all standard formulas is denoted by FMS, and the set of all standard sentences by STS. Note that $ATS \subseteq AT$, $FMS \subseteq FM$ and $STS \subseteq ST$. A subset Γ of FMS is *complete* if $\overline{\overline{\Gamma}} = UF \cap FMS$, and a subset of Γ of STS is *complete* if $\overline{\Gamma} = US \cap STS$. Thus *complete* is used in four different senses, applying to subsets of FM, ST, FMS or STS; but it will always be clear from the context which sense is intended.

These rules of formation assume a very simple form if we assume that for each $\pi \in PR$ we have $u^\pi(i) = v_i$ for all $i < r(\pi)$. We shall discuss this *normalized* case later.

Now we want to explicitly prove that, working in ordinary logic, every formula is equivalent to a standard formula. The main burden of the proof is just to construct the equivalent standard formula.

There is a unique function $g \in {}^{FM}FM$ satisfying the following conditions:

(1) $g(\alpha \equiv \beta) = \alpha \equiv \beta$ if $\alpha, \beta \in VR$;

(2) if $\pi \in PR$, $r(\pi) = n$, $\varphi \in {}^nVR$, $\overline{\overline{D(\pi, \varphi)}} = m$, $k \in {}^mD(\pi, \varphi)$ with $k_i < k_{i+1}$ whenever $i < m-1$, $\psi \in {}^mVR$ with $\psi_i < \psi_{i+1}$ whenever $i < m-1$, and *rng* ψ consists of the first m members of $VR - (\mathit{rng}\,\varphi \cup \mathit{rng} u^\pi)$ in the well-ordering $<$, then

$g(\pi \circ \varphi) = S(\varphi(k_0)/\psi_0) \ldots S(\varphi(k_{m-1})/\psi_{m-1})\; S(\psi_{m-1}/u^\pi(k_{m-1})) \ldots$
$S(\psi_0/u^\pi(k_0))\,(\pi \circ u^\pi)$;

(3) if $\varphi, \psi \in FM$ and $\alpha \in VR$, then $g(\neg\, \varphi) = \neg\, g(\varphi)$, $g(\varphi \to \psi) = g(\varphi) \to g(\psi)$ and $g(\bigwedge \alpha\, \varphi) = \bigwedge \alpha\, g(\varphi)$.

In condition (2), if $m = 0$ we mean of course to let $g(\pi \circ \varphi) = \pi \circ u^\pi$. The following theorem is now easily established:

THEOREM 1. (i) $g \in {}^{FM}FMS$;

(ii) $g(\varphi) = \varphi$ *whenever* $\varphi \in FMS$;

(iii) $FV(\varphi) = FV(g(\varphi))$ *for all* $\varphi \in FM$;

(iv) $\varphi \leftrightarrow g(\varphi)$ *is universally valid.*

Condition (iv) can be established, e. g., by making use of Lemmas 16 and 17 of [14], or by a direct semantical argument. The essential import of Theorem 1 is that *the substitutionless predicate calculus has the same power of expression as ordinary predicate calculus* (Henkin-Tarski).

Turning to proof-theoretical matters we have

THEOREM 2. *Let* Σ_2 *be the set of axioms of the system* S_2 *of* [14]. *Let* $\Delta = \{g(\varphi) : \varphi \in \Sigma_2\}$: *Then* Δ *is complete, i. e.,* $\overline{\overline{\Delta}} = UF \cap FMS$.

PROOF. By Theorem 1 and Theorem 5 of [14], $\Delta \subseteq UF$. If $\varphi \in UF \cap FMS$, then $\Sigma_2 \vdash \varphi$; it is easily seen that, therefore, $\Delta \vdash g(\varphi)$. Since $g(\varphi) = \varphi$, we get $\Delta \vdash \varphi$. The proof is complete.

Thus *the new version of predicate logic does not differ from the old version in power of proof* (this statement is also due to Henkin and Tarski).

Clearly Theorem 2 could be modified by taking various other axiom systems in place of S_2. With all the usual axiom systems, however, the resulting set Δ of axioms would have a complicated nature, stemming primarily from the substitution law for identity (schema (B8) of S_2). The problem rises, then, to give a simple and natural axiomatization for $UF \cap FMS$.

To this end we describe the system $\mathfrak{L}$ of predicate logic. Let Λ consist of all formulas of the following kinds, where $\varphi, \psi, \chi, \in FMS$ and $\alpha, \beta, \gamma \in VR$ are arbitrary, unless otherwise stated:

(C1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$;

(C2) $(\neg \varphi \to \varphi) \to \varphi$;

(C3) $\varphi \to (\neg \varphi \to \psi)$;

(C4) $\bigwedge \alpha (\varphi \to \psi) \to (\bigwedge \alpha \varphi \to \bigwedge \alpha \psi)$;

(C5) $\varphi \to \bigwedge \alpha \varphi$, *if* $\alpha \notin FV(\varphi)$;

(C6) $\neg \bigwedge \alpha \neg \alpha \equiv \beta$, *if* $\alpha \neq \beta$;

(C7) $\alpha \equiv \beta \to (\alpha \equiv \gamma \to \beta \equiv \gamma)$;

(C8) $\alpha \equiv \beta \to (\varphi \to \bigwedge \alpha (\alpha \equiv \beta \to \varphi))$, *if* $\alpha \neq \beta$.

Note that (C1)–(C6) correspond to schemata of S_1 or S_2, (C7) consists of certain instances of (B8) while (C8) is a version of (B8) translated by means of the substitution notation (and applying now to arbitrary formulas and not just to atomic formulas).

The system $\mathfrak{L}$ has Λ as its set of axioms and detachment and generalization as its rules of inference. Our proof of the completeness of $\mathfrak{L}$ can be carried out purely syntactically, but to emphasize the algebraic nature of our arguments and to shorten the proof we shall apply the elementary theory of cylindric algebras.

A (locally finite) *cylindric algebra* (of dimension ω) is a system $\mathfrak{A} = \langle A, +, \cdot, -, c_i, d_{ij} \rangle_{i, j \in \omega}$ such that $\langle A, +, \cdot, - \rangle$ is a Boolean algebra, $c_i \in {}^A A$ for each $i \in \omega$, $d_{ij} \in A$ for $i, j \in \omega$, and the following conditions hold, for all $x, y \in A$ and $i, j, k \in \omega$:

P1 $x \leq c_i x$;

P2 $c_i (x \cdot c_i y) = c_i x \cdot c_i y$;

P3 $c_i c_j x = c_j c_i x$;

P4 $d_{ii} = 1$;

P5 $d_{ij} = c_k (d_{ik} \cdot d_{kj})$ if $k \neq i, j$;

P6 $c_i (d_{ij} \cdot x) = -c_i (d_{ij} \cdot -x)$ if $i \neq j$;

P7 *there is a finite subset* F *of* ω *such that* $c_i x = x$ *whenever* $i \in \omega - F$.

We sometimes write $c(i)$ and $d(i, j)$ instead of c_i and d_{ij}. The elementary properties of cylindric algebras are easily derived; cf. [3] and [8]. We list here a few of these properties.

THEOREM 3. *Let* $\mathfrak{A} = \langle A, +, \cdot, -, c_i, d_{ij}\rangle_{i, j \in \omega}$ *be a cylindric algebra Suppose* $x, y \in A$ *and* $i, j \in \omega$. *Then*

(*i*) $d_{ij} = d_{ji}$;

(*ii*) $c_i\, d_{ij} = 1$;

(*iii*) $c_i\, 0 = 0$;

(*iv*) $c_i\, c_i\, x = c_i\, x$;

(*v*) *if* $x \leq y$, *then* $c_i\, x \leq c_i\, y$;

(*vi*) $c_i\, (-\, c_i\, x) = -c_i\, x$;

(*vii*) $c_i\, (x + y) = c_i\, x + c_i\, y$;

(*viii*) $x \cdot c_i\, y = 0$ *if and only if* $c_i\, x\, .\, y = 0$;

(*ix*) *if* $i \neq j, k$, *then* $c_i\, d_{jk} = d_{jk}$.

On any cylindric algebra $\mathfrak{A}$ as above we define operators $S(i/j)$ by the stipulation that $S(i/j)x = x$ if $i = j \in \omega$, while if $i, j \in \omega$ and $i \neq j$, $S(i/j)x = c_j\, (d_{ji} \cdot x)$, for all $x \in A$. The following theorem is then easily proved; cf [2] and [4].

THEOREM 4. *Let* $\mathfrak{A} = \langle A, +, \cdot, -, c_i, d_{ij}\rangle_{i, j \in \omega}$ *be a cylindric algebra. Then for all* $x \in A$ *and* $i, j, k, l \in \omega$,

(*i*) $S(i/j)$ *is an endormophism of the Boolean algebra* $\langle A, +, \cdot, -\rangle$;

(*ii*) *if* $k \neq i, j$, *then* $S(i/j)c_k\, x = c_k S(i/j)x$;

(*iii*) *if* $i \neq j$, *then* $c_j S(i/j)x = S(i/jx)$;

(*iv*) $c_i S(i/j)x = c_j S(j/i)x$;

(*v*) $d_{ij} \cdot S(i/j)x = d_{ij} \cdot x$;

(*vi*) $S(i/j)S(j/i)x = S(i/j)x$;

(*vii*) *if* $k \neq j$, *then* $S(i/j)S(k/j)x = S(k/j)x$;

(*viii*) $S(j/i)S(i/k)x = S(j/i)S(j/k)x$;

(*ix*) $S(i/j)S(i/k)x = S(i/k)S(i/j)x$;

(*x*) *if* i, j, k, l *are distinct, except possibly* $i = k$, *then* $S(i/j)S(k/l)x = S(k/l)S(i/j)x$;

(*xi*) $S(i/j)S(j/k)c_j x = S(i/k)c_j x$;

(*xii*) $S(i/jS(j/k)c_j c_l x = S(i/l)S(l/k)c_j c_l x$;

(*xiii*) *if* $k \neq i, j$, *then* $d_{ij} \cdot S(i/k)x = d_{ij} \cdot S(j/k)x$.

We now turn to a lemma needed for our completeness proof.

LEMMA 1[2]. *Let* $\mathfrak{A} = \langle A, +, \cdot, -, c_i, d_{ij}\rangle_{i,j\in\omega}$ *be a cylindric algebra. Suppose* $m \in \omega$, *and* $a, b, e, f \in {}^m\omega$. *Assume that* e *and* f *are biunique and* $(rnga \cup rngb) \cap (rnge \cup rngf) = 0$. *Also assume that* $x \in A$, *and that* $c(e_i)\, x = c(f_i)\, x = x$ for all $i < m$. Then

$$(*)\ S(a_0/e_0)\dots S(a_{m-1}/e_{m-1})S(e_{m-1}/b_{m-1})\dots S(e_0/b_0)x =$$
$$S(a_0/f_0)\dots S(a_{m-1}/f_{m-1})S(f_{m-1}/b_{m-1})\dots S(f_0/b_0)x.$$

PROOF. Assume first that

(1) $rnge \cap rngf = 0$.

Then we prove (*) by induction on m. Since (*) is trivial for $m = 0$, assume inductively that $m > 0$. Then

$$S(a_0/e_0)\dots S(a_{m-1}/e_{m-1})S(e_{m-1}/b_{m-1})\dots S(e_0/b_0)x =$$
$$S(a_0/e_0)\dots S(a_{m-1}/e_{m-1})S(e_{m-1}/b_{m-1})c(e_{m-1})c(f_{m-1})S(e_{m-2}/b_{m-2})\dots S(e_0/b_0)x$$
by Theorem 4 (ii);
$$= S(a_0/e_0)\dots S(a_{m-1}/f_{m-1})S(f_{m-1}/b_{m-1})c(f_{m-1})c(e_{m-1})S(e_{m-2}/b_{m-2})\dots S(e_0/b_0)x$$
by Theorem 4 (*xii*);
$$= S(a_{m-1}/f_{m-1})S(a_0/e_0)\dots S(a_{m-2}/e_{m-2})S(e_{m-2}/b_{m-2})\dots S(e_0/b_0)S(f_{m-1}/b_{m-1})x$$
by Theorem 4 (*ii*), (*x*);
$$= S(a_{m-1}/f_{m-1})S(a_0/f_0)\dots S(a_{m-2}/f_{m-2})S(f_{m-2}/b_{m-2})\dots S(f_0/b_0)S(f_{m-1}/b_{m-1})x$$
by induction hypothesis; f
$$= S(a_0/f_0)\dots S(a_{m-1}/f_{m-1})S(f_{m-1}/b_{m-1})\dots S(f_0/b_0)x \text{ by Theorem 4 (x)}$$
This completes the induction.

To treat the general case, where (1) does not necessarily hold, using the postulate P7 of cylindric algebras choose $g \in {}^m\omega$ such that g is biunique, $(rnga \cup rngb \cup rnge \cup rngf) \cap rng\, g = 0$, and $c(g_i)x = x$ for all $i < m$. Then, using the first part of this proof twice,

$$S(a_0/e_0)\dots S(a_{m-1}/e_{m-1})S(e_{m-1}/b_{m-1})\dots S(e_0/b_0)x =$$
$$S(a_0/g_0)\dots S(a_{m-1}/g_{m-1})S(g_{m-1}/b_{m-1})\dots S(g_0/b_0)x =$$
$$S(a_0/f_0)\dots S(a_{m-1}/f_{m-1})S(f_{m-1}/b_{m-1})\dots S(f_0/b_0)x.$$

This completes the proof.

Now we shall establish a connection between our axiom system and the theory of cylindric algebras[3]. The idea is to form the cylindric algebra associated with our axioms; to see that a cylindric algebra is obtained we need a few simple lemmas.

From axioms (C1)–(C3) and the sentential completeness theorem we obtain

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[2] This lemma occurs in a more general form in the unpublished work [12] where it forms a part of a general theory of substitution operators on cylindric algebras.
[3] We are applying here to our axiom system the procedure described in [9].

Lemma 2. *If φ is a standard tautologous formula, then $\Lambda \vdash \varphi$.*

As a consequence we have

Lemma 3. *If $\varphi, \psi, \varphi', \psi' \in FMS$, $\Lambda \vdash \varphi \leftrightarrow \psi$ and $\Lambda \vdash \varphi' \leftrightarrow \psi'$, then*

(*i*) $\Lambda \vdash \neg\varphi \leftrightarrow \neg\psi$;

(*ii*) $\Lambda \vdash (\varphi \vee \varphi') \leftrightarrow (\psi \vee \psi')$;

(*iii*) $\Lambda \vdash (\varphi \wedge \varphi') \leftrightarrow (\psi \wedge \psi')$.

Making use of (C4), we can prove

Lemma 4. *If $\varphi, \psi \in FMS$ and $\Lambda \vdash \varphi \leftrightarrow \psi$, then $\Lambda \vdash \bigwedge \alpha\varphi \leftrightarrow \bigwedge \alpha\psi$.*

Now let $\asymp = \{\langle\varphi\,\psi\rangle : \varphi, \psi \in FMS \text{ and } \Lambda \vdash \varphi \leftrightarrow \psi\}$. Then using Lemmas 3 and 4 it is easy to see that $\asymp$ is a congruence relation (cf. [1]) on the algebra $\mathfrak{F} = \langle FMS, \vee, \wedge, \neg, \bigwedge v_i, v_i \equiv v_j\rangle_{i,j\in\omega}$. We denote by $\mathfrak{F}/\asymp$ the quotient algebra of $\mathfrak{F}$ under $\asymp$; the congruence class of $\varphi \in FMS$ under $\asymp$ is denoted by $[\varphi]$.

The following three lemmas are established analogously to Lemmas 5, 6 and 7 of [14].

Lemma 5. *If $\varphi \in FMS$ and $\alpha \notin FV(\varphi)$, then $\Lambda \vdash \bigwedge\alpha(\varphi \to \psi) \to (\varphi \to \bigwedge\alpha\psi)$.*

Lemma 6. $\Lambda \vdash \alpha \equiv \alpha$.

Lemma 7. $\Lambda \vdash \alpha \equiv \beta \to \beta \equiv \alpha$.

Hence by (C7) we obtain

Lemma 8. $\Lambda \vdash \alpha \equiv \beta \to (\gamma \equiv \alpha \to \gamma \equiv \beta)$.

Lemma 9. *If $\varphi \in FMS$, then $\Lambda \vdash \bigwedge\alpha\varphi \to \varphi$.*

Proof. Let β be a variable not occurring in $\bigwedge\alpha\varphi$. Then

$\Lambda \vdash \beta \equiv \alpha \to (\neg\varphi \to \bigwedge\alpha(\alpha \equiv \beta \to \neg\varphi))$ by (C8) and Lemma 7;

$\Lambda \vdash \beta \equiv \alpha \to (\neg\varphi \to (\bigwedge\alpha\varphi \to \bigwedge\alpha\neg\alpha \equiv \beta))$ by (C4);

$\Lambda \vdash \beta \equiv \alpha \to (\neg\varphi \to \neg\bigwedge\alpha\varphi)$ by (C6);

$\Lambda \vdash \bigwedge\beta(\neg(\bigwedge\alpha\varphi \to \varphi) \to \neg\beta \equiv \alpha)$ by Lemma 2;

$\Lambda \vdash \neg(\bigwedge\alpha\varphi \to \varphi) \to \bigwedge\beta\neg\beta \equiv \alpha$ by Lemma 5;

$\Lambda \vdash \bigwedge\alpha\varphi \to \varphi$ by (C6).

From this lemma we infer

Lemma 10. *If $\varphi \in FMS$, then $\Lambda \vdash \varphi \to \bigvee\alpha\varphi$.*

Lemma 11. *If $\varphi, \psi \in FMS$, then $\Lambda \vdash \bigvee\alpha(\varphi \wedge \bigvee\alpha\psi) \leftrightarrow \bigvee\alpha\varphi \wedge \bigwedge\alpha\psi$.*

Proof. We have $\bigvee\alpha(\varphi \wedge \bigvee\alpha\psi) = \neg\bigwedge\alpha\neg\neg(\varphi \to \neg\neg\bigwedge\alpha\neg\psi)$ and $\bigvee\alpha\varphi \wedge \bigvee\alpha\psi = \neg(\neg\bigwedge\alpha\neg\varphi \to \neg\neg\bigwedge\alpha\neg\psi)$. Hence

$\Lambda \vdash \bigwedge\alpha\neg\neg(\varphi \to \neg\neg\bigwedge\alpha\neg\psi) \to \bigwedge\alpha(\neg\bigwedge\alpha\neg\psi \to \neg\varphi)$
by Lemma 2 and (C4);

$\Lambda \vdash \bigwedge \alpha \neg \neg (\varphi \rightarrow \neg \neg \bigwedge \alpha \neg \psi) \rightarrow (\neg \bigwedge \alpha \neg \psi \rightarrow \bigwedge \alpha \neg \varphi)$ by Lemma 5;

$\Lambda \vdash \bigvee \alpha \varphi \wedge \bigvee \alpha \psi \rightarrow \bigvee \alpha (\varphi \wedge \bigvee \alpha \psi)$ by Lemma 2.

For the other implication, we have

$\Lambda \vdash (\bigvee \alpha \varphi \rightarrow \neg \bigvee \alpha \psi) \rightarrow \neg \neg (\varphi \rightarrow \neg \bigvee \alpha \psi)$ by Lemma 10;

$\Lambda \vdash (\bigvee \alpha \varphi \rightarrow \neg \bigvee \alpha \psi) \rightarrow \bigwedge \alpha \neg \neg (\varphi \rightarrow \neg \bigvee \alpha \psi)$ by Lemma 5;

$\Lambda \vdash \bigvee \alpha (\varphi \wedge \bigvee \alpha \psi) \rightarrow \bigvee \alpha \varphi \wedge \bigvee \alpha \psi$ by Lemma 2.

This completes the proof.

The following lemma can be proved exactly like Lemma 32 of [14].

LEMMA 12. *If* $\varphi \in FMS$, *then* $\Lambda \vdash \bigwedge \alpha \bigwedge \beta \varphi \rightarrow \bigwedge \beta \bigwedge \alpha \varphi$.

From this lemma we obtain immediately

LEMMA 13. *If* $\varphi \in FMS$, *then* $\Lambda \vdash \bigvee \alpha \bigvee \beta \varphi \leftrightarrow \bigvee \beta \bigvee \alpha \varphi$.

Now it is possible to prove the following lemma in a way analogous to the proof of Lemma 14 of [14].

LEMMA 14. *If* $\gamma \neq \alpha, \beta$, *then* $\Lambda \vdash \bigwedge \gamma \neg \neg (\alpha \equiv \gamma \rightarrow \neg \gamma \equiv \beta) \rightarrow \neg \neg (\alpha \equiv \alpha \rightarrow \neg \alpha \equiv \beta)$.

LEMMA 15. *If* $\gamma \neq \alpha, \beta$, *then* $\Lambda \vdash \alpha \equiv \beta \leftrightarrow \bigvee \gamma (\alpha \equiv \gamma \wedge \gamma \equiv \beta)$.

PROOF. We have $\bigvee \gamma (\alpha \equiv \gamma \wedge \gamma \equiv \beta) = \neg \bigwedge \gamma \neg \neg (\alpha \equiv \gamma \rightarrow \neg \gamma \equiv \beta)$. Hence

$\Lambda \vdash \bigwedge \gamma \neg \neg (\alpha \equiv \gamma \rightarrow \neg \gamma \equiv \beta) \rightarrow \neg \neg (\alpha \equiv \alpha \rightarrow \neg \alpha \equiv \beta)$ by Lemma 14;

$\Lambda \vdash \alpha \equiv \beta \rightarrow \bigvee \gamma (\alpha \equiv \gamma \wedge \gamma \equiv \beta)$ by Lemma 6.

The other implication is obtained as follows:

$\Lambda \vdash \alpha \equiv \gamma \rightarrow (\gamma \equiv \beta \rightarrow \alpha \equiv \beta)$ by (C7) and Lemma 7;

$\Lambda \vdash \neg \alpha \equiv \beta \rightarrow \neg \neg (\alpha \equiv \gamma \rightarrow \neg \gamma \equiv \beta)$ by Lemma 2;

$\Lambda \vdash \neg \alpha \equiv \beta \rightarrow \bigwedge \gamma \neg \neg (\alpha \equiv \gamma \rightarrow \neg \gamma \equiv \beta)$ by Lemma 5;

$\Lambda \vdash \bigvee \gamma (\alpha \equiv \gamma \wedge \gamma \equiv \beta) \rightarrow \alpha \equiv \beta$ by Lemma 2.

The proof is complete.

LEMMA 16. *If* $\alpha \neq \beta$ *and* $\varphi \in FMS$, *then* $\Lambda \vdash \bigvee \alpha (\alpha \equiv \beta \wedge \varphi) \leftrightarrow \neg \bigvee \alpha (\alpha \equiv \beta \wedge \neg \varphi)$.

PROOF. By definition $\bigvee \alpha (\alpha \equiv \beta \wedge \varphi) = \neg \bigwedge \alpha \neg \neg (\alpha \equiv \beta \rightarrow \neg \varphi)$ and $\neg \bigvee \alpha (\alpha \equiv \beta \wedge \neg \varphi) = \neg \neg \bigwedge \alpha \neg \neg (\alpha \equiv \beta \rightarrow \neg \neg \varphi)$. Hence, first,

$\Lambda \vdash \alpha \equiv \beta \rightarrow (\neg \neg \varphi \rightarrow \bigwedge \alpha (\alpha \equiv \beta \rightarrow \neg \neg \varphi))$ by (C8);

$\Lambda \vdash \alpha \equiv \beta \rightarrow (\neg \neg \varphi \rightarrow \bigwedge \alpha \neg \neg (\alpha \equiv \beta \rightarrow \neg \neg \varphi))$ by (C4);

$\Lambda \vdash \neg \bigwedge \alpha \neg \neg (\alpha \equiv \beta \rightarrow \neg \neg \varphi) \rightarrow \neg \neg (\alpha \equiv \beta \rightarrow \neg \varphi)$ by Lemma 2;

$\Lambda \vdash \neg \bigwedge \alpha \neg \neg (\alpha \equiv \beta \rightarrow \neg \neg \varphi) \rightarrow \bigwedge \alpha \neg \neg (\alpha \equiv \beta \rightarrow \neg \varphi)$ by Lemma 5;

$\Lambda \vdash \bigvee \alpha (\alpha \equiv \beta \wedge \varphi) \rightarrow \neg \bigvee \alpha (\alpha \equiv \beta \wedge \neg \varphi)$ by Lemma 2.

Second,

$\Lambda \vdash \bigwedge \alpha \neg \neg (\alpha \equiv \beta \rightarrow \neg \neg \varphi) \rightarrow \neg \neg (\alpha \equiv \beta \rightarrow \neg \neg \varphi)$ by Lemma 9;

$\Lambda \vdash \bigwedge \alpha \neg \neg (\alpha \equiv \beta \rightarrow \neg \varphi) \rightarrow \neg \neg (\alpha \equiv \beta \rightarrow \neg \varphi)$ by Lemma 9;

$\Lambda \vdash \neg \bigvee \alpha (\alpha \equiv \beta \wedge \neg \varphi) \rightarrow (\bigwedge \alpha \neg \neg (\alpha \equiv \beta \rightarrow \neg \varphi) \rightarrow \neg \alpha \equiv \beta)$ by Lemma 2;

$\Lambda \vdash \neg \bigvee \alpha (\alpha \equiv \beta \wedge \neg \varphi) \rightarrow (\bigwedge \alpha \neg \neg (\alpha \equiv \beta \rightarrow \neg \varphi) \rightarrow \bigwedge \alpha \neg \alpha \equiv \beta)$ by Lemma 5;

$\Lambda \vdash \neg \bigvee \alpha (\alpha \equiv \beta \wedge \neg \varphi) \rightarrow \bigvee \alpha (\alpha \equiv \beta) \wedge \varphi)$ by (C6).

Q. E. D.

The following lemmas are easily established.

LEMMA 17. *If* $\varphi \in FMS$, *then* $\Lambda \vdash \bigwedge \alpha (\alpha \equiv \beta \rightarrow \varphi) \leftrightarrow \bigvee a (\alpha \equiv \beta \wedge \varphi)$.

LEMMA 18. *If* $\varphi \in FMS$ *and* $\alpha \in VR - FV(\varphi)$, *then* $\Lambda \vdash \varphi \leftrightarrow \bigvee \alpha \varphi$.

It is clear from the ordinary theory of Boolean algebras that $\langle FMS, \wedge, \wedge, \neg \rangle / \asymp$ is a Boolean algebra; hence there is a unit element 1 and a natural ordering $\leq$. With regard to them we have:

LEMMA 19. *If* $\varphi, \psi \in FMS$, *then*

(*i*) $\Lambda \vdash \varphi$ *if and only if* $[\varphi] = 1$;

(*ii*) $\Lambda \vdash \varphi \rightarrow \psi$ *if and only if* $[\varphi] \leq [\psi]$;

(*iii*) *if* $\alpha \neq \beta$, *then* $[S(\alpha/\beta)\varphi] = S(v^{-1}\alpha/v^{-1}\beta)[\varphi]$.

From the lemmas now demonstrated the following theorem, which shows a connection between the system $\mathfrak{L}$ and the theory of cylindric algebras, follows.

THEOREM 5. $\mathfrak{F}/\asymp$ is a cylindric algebra.

Making use of this theorem we can now prove the basic lemma for the completeness proof.

LEMMA 20. *If* $\varphi_0 \varphi_1 \in FM$ and $P(\varphi_0, \varphi_1, \alpha, \beta)$, *then* $\Lambda \vdash g(\alpha \equiv \beta \rightarrow (\varphi_0 \rightarrow \varphi_1))$.

PROOF We have $g(\alpha \equiv \beta \rightarrow (\varphi_0 \rightarrow \varphi_1)) = \alpha \equiv \beta \rightarrow (g(\varphi_0) \not\rightarrow g(\varphi_1))$. Since $P(\varphi_0, \varphi_1, \alpha, \beta)$ holds, there exist $\pi \in PR$, expressions φ_0', φ_1', and an $n \in \omega$ such that, with $r(\pi) = m$, (*i*) $\varphi_0', \varphi_1' \in {}^m VR$; (*ii*) $\varphi_0 = \pi \circ \varphi_0'$ and $\varphi_1 = \pi \circ \varphi_1'$;

(*iii*) $n < m$; and (*iv*) $\varphi_0'(n) = \alpha$ and $\varphi_1'(n) = \beta$, while $\varphi_0'(k) = \varphi_1(k)$ for $k < m$, $k \neq n$. Let $E = D(\pi, \varphi_0') \cup \{n\}$. Thus also $E = D(\pi, \varphi_1') \cup \{n\}$. Let $\overline{\overline{E}} = p$, and choose $k \in {}^pE$ such that $k_i < k_{i+1}$ whenever $i < p-1$; say $k_j = n$.

Let $D(\pi, \varphi_0') = q$. Choose $\psi \in {}^qVR$ such that $\psi_i < \psi_{i+1}$ whenever $i < q-1$ and $rng\,\psi$ consists of the first q members of VR–($rng\,\varphi_0' \cup rng\,u^\pi$) in the well-ordering $<$. If $q = p$, then

$$g(\varphi_0) = g(\pi \circ \varphi_0') = S(\varphi_0'(k_0)/\psi_0) \ldots S(\varphi_0'(k_{p-1})/\psi_{p-1}) \\ S(\psi_{p-1}/u^\pi(k_{p-1})) \ldots S(\psi_0/u^\pi(k_0))\,(\pi \circ u^\pi);$$

moreover, $(rng\,\varphi_0' \cup rng\,u^\pi) \cap rng\,\psi = 0$. If on the other hand $q \neq p$, then $q = p$–1. In this case, let γ be any member of VR–($rng\,\varphi_0' \cup rng\,u^\pi \cup rng\,\psi$). Then

$$[g(\varphi_0)] = [g(\pi \circ \pi_0')]$$
$$= [S(\varphi_0'(k_0)/\psi_0) \ldots S(\varphi_0'(k_{j-1})/\psi_{j-1})\, S(\varphi_0'(k_{j+1})/\psi_j) \ldots$$
$$S(\varphi_0'(k_{p-1})/\psi_{q-1})\, S(\psi_{q-1}/u^\pi(k_{p-1})) \ldots S(\psi_j/u^\pi(k_{j+1}))\, S(\psi_{j-1}/u^\pi(k_{j-1})) \ldots$$
$$S(\psi_0/u^\pi(k_0))\,(\pi \circ u^\pi)]$$
$$= S(v^{-1}\varphi_0'(k_0)/v^{-1}\psi_0) \ldots S(v^{-1}\varphi_0'(k_{j-1})/v^{-1}\psi_{j-1})\, S(v^{-1}\varphi_0'(k_{j+1})/v^{-1}\psi_j) \ldots$$
$$S(v^{-1}\varphi_0'(k_{p-1})/v^{-1}\psi_{q-1})\, S(v^{-1}\psi_{q-1}/v^{-1}u^\pi(k_{p-1})) \ldots$$
$$S(v^{-1}\psi_j/v^{-1}u^\pi(k_{j+1}))\, S(v^{-1}\psi_{j-1}/v^{-1}u^\pi(k_{j-1})) \ldots$$
$$S(v^{-1}\psi_0/v^{-1}u^\pi(k_0))\, S(v^{-1}\varphi_0'(k_j)/v^{-1}\gamma)\, S(v^{-1}\gamma/v^{-1}u^\pi(k_j))\, c(v^{-1}\gamma)\,[\pi \circ u^\pi]$$

by Lemma 19 (*iii*) and Theorem 4 (*xi*);

$$= S(v^{-1}\varphi_0'(k_0)/v^{-1}\psi_0) \ldots$$
$$S(v^{-1}\varphi_0'(k_{j-1}/v^{-1}\psi_{j-1})\, S(v^{-1}\varphi_0'(k_j)/v^{-1}\gamma)\, S(v^{-1}\varphi_0'(k_{j+1})/v^{-1}\psi_j) \ldots$$
$$S(v^{-1}\varphi_0'(k_{p-1})/v^{-1}\psi_{q-1})\, S(v^{-1}\psi_{q-1}/v^{-1}u^\pi(k_{p-1})) \ldots$$
$$S(v^{-1}\psi_j/v^{-1}u^\pi(k_{j+1}))\, S(v^{-1}\gamma/v^{-1}u^\pi(k_j))\, S(v^{-1}\psi_{j-1}/v^{-1}u^\pi(k_{j-1})) \ldots$$
$$S(v^{-1}\psi_0/v^{-1}u^\pi(k_0))\,[\pi \circ u^\pi]$$ by Theorem 4 (*x*).

Thus no matter whether $q = p$ or $q = p$–1 we have, for $i = 0$,

(1) There exists $e_i \in {}^p\omega$ such that e_i is biunique. $rng\,e_i \cap (rng\,v^{-1}u^\pi \cup rng\,v^{-1}\varphi_i') = 0$, $c(e_i(j))\,[\pi \circ u^\pi] = [\pi \circ u^\pi]$ for each $j < p$, and $[g(\varphi_i)] = S(v^{-1}\varphi_i'(k_0)/e_i(0)) \ldots S(v^{-1}\varphi_i'(k_{p-1})/e_i(p-1))\, S(e_i(p-1)/v^{-1}u^\pi(k_{p-1})) \ldots S(e_i(0)/v^{-1}u^\pi(k_0))\,[\pi \circ u^\pi]$.

In exactly the same way, (1) can be established for $i = 1$. Hence

$$d(v^{-1}(\alpha), v^{-1}(\beta)) \cdot [g(\varphi_0)] = d(v^{-1}\varphi_0'(k_j), v^{-1}\varphi_1'(k_j)) \cdot S(v^{-1}\varphi_0'(k_j)/e_0(j))$$
$$S(v^{-1}\varphi_0'(k_0)/e_0(0)) \ldots S(v^{-1}\varphi_0'(k_{j-1})/e_0(j-1))\, S(v^{-1}\varphi_0'(k_{j+1})/e_0(j+1)) \ldots$$
$$S(v^{-1}\varphi_0'(k_{p-1})/e_0(p-1))\, S(e_0(p-1)/v^{-1}u^\pi(k_{p-1})) \ldots S(e_0(0)/v^{-1}u^\pi(k_0))$$
$[\pi \circ u^\pi]$ by Theorem 4 (*x*)

$$= d(v^{-1}\varphi_0'(k_j), v^{-1}\varphi_1'(k_j)) \cdot [g(\varphi_1)]$$ by Theorem 4 (*xiii*), (*x*) and Lemma 1.

It follows that $[g(\alpha \equiv \beta \rightarrow (\varphi_0 \rightarrow \varphi_1))] = 1$. Hence by Lemma 19 (*i*), $\Lambda \vdash g(\alpha \equiv \beta \rightarrow (\varphi_0 \rightarrow \varphi_1))$, Q. E. D.

The main result of this section is as follows.

THEOREM 6. *Λ is complete, i. e., $\overline{\overline{\Lambda}} = UF \cap FMS$*[4].

PROOF. Clearly $\overline{\overline{\Lambda}} \subseteq UF \cap FMS$. Now suppose $\varphi \in UF \cap FMS$. Then by Theorem 5 of [14], $\Sigma_2 \vdash \varphi$. Now if ψ is an instance of one of the schemas (B1) to (B8), then $\Lambda \vdash g(\psi)$ in accordance with (C1)–(C4), Lemma 9, (C5)–(C6), and Lemma 20, respectively. Hence we infer that $\Lambda \vdash g(\psi)$, i.e., $\Lambda \vdash \varphi$. This completes the proof.

The proof of Theorem 6 uses in an essential way Theorem 5 of [14]. Thus the proof of the completeness of the system which we have given would be quite long if written out in all detail: first there would be the reduction to the system $\mathfrak{S}_2$ we have made, then the reduction to a familiar system of logic made in [14], and finally a completeness proof for this last system (see, e. g., [6]). We want to sketch here a more direct proof of the completeness of $\mathfrak{L}$, patterned after Henkin's completeness proof, as modified by Hasenjäger (see [5])[5].

First we adjoin a new set W of variables, with $\overline{\overline{W}} = \overline{\overline{FMS}}$. The fundamental notions such as FM, ST, etc., then change; we denote the notions associated with $VR \cup W$, by VR^+, FM^+, ST^+, etc. Thus in particular $VR^+ = VR \cup W$. For $\Gamma \subseteq FMS^+$ and $\varphi \in FMS^+$ we write $\Gamma \overset{+}{\vdash} \varphi$ if there is an $m \in \omega$ and a $\psi \in {}^m\Gamma$ such that $\Lambda^+ \overset{+}{\vdash} \psi_0 \wedge \ldots \wedge \psi_{m-1} \rightarrow \varphi$. The set Γ is *consistent* if it is not the case that $\Gamma \overset{+}{\vdash} \varphi \wedge \neg \varphi$ for some $\varphi \in FMS^+$. As is usual one can prove that $\Lambda \vdash \varphi$ if $\Lambda^+ \overset{+}{\vdash} \varphi$, whenever $\varphi \in FMS$. Consequently to prove $\mathfrak{L}$ complete it suffices to show that any consistent subset Γ of FMS is satisfiable.

Let $\mathfrak{F} = \{\varphi \in FMS^+$: the bound variables of φ are in $VR\}$. Thus $\Gamma \subseteq \mathfrak{F}$. Now using a transfinite recursion, one can obtain a subset Δ of $\mathfrak{F}$ satisfying the following two conditions:

(1) Δ is maximal consistent in $\mathfrak{F}$, and $\Gamma \subseteq \Delta$;

(2) for any $\psi \in \mathfrak{F}$ and $\alpha \in VR$ there is a $\beta \in W$ such that $S(\beta/\alpha)\psi \rightarrow \bigwedge \alpha\, \psi \in \Delta$.

Now let $\equiv = \{\langle \beta, \gamma \rangle : \beta, \gamma \in W$ and $\Delta \overset{+}{\vdash} \beta \equiv \gamma\}$. Then $\equiv$ is an equivalence relation with field W. We let $A = W/\equiv$, the set of all $\equiv$-equivalence classes. (A is to be the underlying set of the model of Γ). For the expanded formalism

[4] Using this theorem one may establish Theorem 1. 12 of [9] much more easily than with ordinary axiom systems. Thus the proof just outlined (in Lemmas 1–20) may be used in an integrated introduction to logic and cylindric algebra.

[5] The problem of supplying such a shorter proof was posed to the author by Professor Richard Montague. In this sketch we use various familiar notions which have not been formally defined here or in [14].

based on VR^+ we can obtain the analogue of Theorem 5 (referring now to an α-dimensional locally finite cylindric algebra, where α is an ordinal with $\overline{\overline{\alpha}} = \overline{\overline{VR^+}}$). Hence using the analogue of Theorem 4 (x), ($xiii$) the following lemma can be easily established:

LEMMA A. *If* $r(\pi) = m$, $\beta \in {}^mW$, $i < m$, *and* $\gamma \in W$, *then* $\Lambda^+ \vdash^{\pm} \beta_i \equiv \gamma \rightarrow (S(\beta_0/u^\pi(0)) \ldots S(\beta_{m-1}/u^\pi(m-1))\,(\pi \circ u^\pi) \rightarrow S(\beta_0/u^\pi(0)) \ldots S(\beta_{i-1}/u^\pi(i-1))\, S(\gamma/u^\pi(i))\, S(\beta_{i+1}/u^\pi(i+1)) \ldots S(\beta_{m-1}/u^\pi(m-1)\,(\pi \circ u^\pi)$.

Let p be the natural mapping of W onto A: $p = \{\langle w, w/\equiv\rangle : w \in W\}$. From Lemma A we obtain at once

LEMMA B. If $r(\pi) = m$; $\beta, \gamma \in {}^mW$, and $p\,\beta = p\,\gamma$, then $\Delta \vdash^{\pm} S(\beta_0/u^\pi(0)) \ldots S(\beta_{m-1}/u^\pi(m-1))\,(\pi \circ u^\pi) \leftrightarrow S(\gamma_0/u^\pi(0)) \ldots S(\gamma_{m-1}/u^\pi(m-1))\,(\pi \circ u^\pi)$. From Lemma B we see that if $r(\pi) = m$, then there is a relation $R_\pi \subseteq {}^mA$ such that for all $\beta \in {}^mW$, $p\,\beta \in R^\pi$ if and only if $\Delta \vdash^{\pm} S(\beta_0/u^\pi(0)) \ldots S(\beta_{m-1}/u^\pi(m-1))\,(\pi \circ u^\pi)$. The system $\mathfrak{A} = \langle A, R_\pi\rangle_{\pi \varepsilon PR}$ is, we claim, the desired model of Γ.

To prove this, a little auxiliary notation is needed. With each $f \in {}^{VR}W$ we associate a function $f^* \in {}^{FMS^+}FMS^+$, as follows. If $\varphi \in FMS^+$, let $M = VR \cap FV(\varphi)$. Choose $m \in \omega$ and $\alpha \in {}^mM$ such that $rng\ \alpha = M$ and $\alpha_i < \alpha_{i+1}$ for $i < m-1$. Let

$$f^*(\varphi) = S(f(\alpha_0)/\alpha_0) \ldots S(f(\alpha_{m-1})/\alpha_{m-1})\,\varphi.$$

If $x \in {}^{VR^+}A$, we assume ask nown what is meant in saying that x *satisfies a formula* φ *in* $\mathfrak{A}$, a phrase abbreviated by writing x *sat* φ $\mathfrak{A}$. The following lemma leads directly to the conclusion that $\mathfrak{A}$ is a model of Γ.

LEMMA C. *Suppose* $x \in {}^{VR^+}A$ *and* $x(w) = w/\equiv$ *for all* $w \in W$. *Then for all* $f \in {}^{VR}W$ *and for all* $\varphi \in \mathfrak{F}$ *we have that* x *sat* $f^*(\varphi)$ $\mathfrak{A}$ *if and only if* $\Delta \vdash^{\pm} f^*(\varphi)$.

The proof, by induction on φ, may be omitted.

Now suppose $\varphi \in \Gamma$, and let ψ be a closure of φ (using variables in VR). Then $\psi \in \mathfrak{F}$, and $f^*(\psi) = \psi$. Clearly $\Delta \vdash^{\pm} \psi$, so by Lemma C, x *sat* ψ $\mathfrak{A}$. Thus $\mathfrak{A}$ is a model of Γ, and this completes the sketch of the second proof of Theorem 6. This proof, carried out in all detail, does not have any steps of the complexity of Lemmas 1 and 20.

Of the schemata (C1)–(C8), all but (C5) have a quite elementary character. It is possible to replace (C5) by the following three more elementary schemata.[6]

(C5[1]) $\varphi \rightarrow \bigwedge \alpha\, \varphi$, if $\alpha \notin OC(\varphi)$;

(C5[2]) $\neg \bigwedge \alpha\, \varphi \rightarrow \bigwedge \alpha\, \neg \bigwedge \alpha\, \varphi$;

(C5[3]) $\bigwedge \alpha \bigwedge \beta\, \varphi \rightarrow \bigwedge \beta \bigwedge \alpha\, \varphi$.

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[6] I am indebted to Richard Montague for the simplified form of (C5[2]) which follows. The schema originally used was $(\bigwedge \alpha\, \varphi \rightarrow \bigwedge \alpha\, \psi) \rightarrow \bigwedge \alpha\, (\bigwedge \alpha\, \varphi \rightarrow \bigwedge \alpha\, \psi)$.

Indeed, let Λ_1, consist of all instances of (C1)–(C4), ($C5^1$)–($C5^3$), and (C6) to (C8). By the proof of Lemma 9 we have

Lemma 21, *If $\varphi \in FMS$, then* $\Lambda_1 \vdash \bigwedge \alpha \varphi \to \varphi$.

Lemma 22. *If $\varphi \in FMS$, then* $\Lambda_1 \vdash \bigwedge \alpha \varphi \to \bigwedge \alpha \bigwedge \alpha \varphi$.

Proof. We have

$$\Lambda_1 \vdash \bigwedge \alpha \neg \bigwedge \alpha \varphi \to \neg \bigwedge \alpha \varphi \text{ by Lemma 21;}$$
$$\Lambda_1 \vdash \bigwedge \alpha \varphi \to \neg \bigwedge \alpha \neg \bigwedge \alpha \varphi;$$
$$\Lambda_1 \vdash \bigwedge \alpha \varphi \to \bigwedge \alpha \neg \bigwedge \alpha \neg \bigwedge \alpha \varphi.$$

Also,

$$\Lambda_1 \vdash \neg \bigwedge \alpha \varphi \to \bigwedge \alpha \neg \bigwedge \alpha \varphi \text{ by } (C5^1);$$
$$\Lambda_1 \vdash \neg \bigwedge \alpha \neg \bigwedge \alpha \varphi \to \bigwedge \alpha \varphi;$$
$$\Lambda_1 \vdash \bigwedge \alpha \neg \bigwedge \alpha \neg \bigwedge \alpha \varphi \to \bigwedge \alpha \bigwedge \alpha \varphi.$$

The desired result now easily follows.

Lemma 23. *If $\varphi, \psi \in FMS$, then* $\Lambda_1 \vdash (\bigwedge \alpha \varphi \to \bigwedge \alpha \psi) \to \bigwedge \alpha (\bigwedge \alpha \varphi \to \bigwedge \alpha \psi)$.

Proof. We have

$$\Lambda_1 \vdash \neg \bigwedge \alpha \varphi \to (\bigwedge \alpha \varphi \to \bigwedge \alpha \psi);$$
$$\Lambda_1 \vdash \bigwedge \alpha \neg \bigwedge \alpha \varphi \to \bigwedge \alpha (\bigwedge \alpha \varphi \to \bigwedge \alpha \psi);$$
$$\Lambda_1 \vdash \neg \bigwedge \alpha \varphi \to \bigwedge \alpha (\bigwedge \alpha \varphi \to \bigwedge \alpha \psi) \text{ by } (C5^2);$$
$$\Lambda_1 \vdash \bigwedge \alpha \psi \to (\bigwedge \alpha \varphi \to \bigwedge \alpha \psi);$$
$$\Lambda_1 \vdash \bigwedge \alpha \bigwedge \alpha \psi \to \bigwedge \alpha (\bigwedge \alpha \varphi \to \bigwedge \alpha \psi);$$
$$\Lambda_1 \vdash \bigwedge \alpha \psi \to \bigwedge \alpha (\bigwedge \alpha \varphi \to \bigwedge \alpha \psi);$$
$$\Lambda_1 \vdash (\bigwedge \alpha \varphi \to \bigwedge \alpha \psi) \to \bigwedge \alpha (\bigwedge \alpha \varphi \to \bigwedge \alpha \psi).$$

Lemma 24 $\Lambda_1 \vdash \bigwedge \beta \bigwedge \alpha \varphi \to \bigwedge \alpha \bigwedge \beta \bigwedge \alpha \varphi$.

Proof. $\Lambda_1 \vdash \bigwedge \beta \bigwedge \alpha \varphi \to \bigwedge \alpha \bigwedge \beta \varphi$ by ($C5^3$)

$$\Lambda_1 \vdash \bigwedge \beta \bigwedge \alpha \varphi \to \bigwedge \alpha \bigwedge \alpha \bigwedge \beta \varphi \text{ by Lemma 22;}$$
$$\Lambda_1 \vdash \bigwedge \alpha \bigwedge \alpha \bigwedge \beta \varphi \to \bigwedge \alpha \bigwedge \beta \bigwedge \alpha \varphi \text{ by } (C5^3) \text{ and } (C4);$$
$$\Lambda_1 \vdash \bigwedge \beta \bigwedge \alpha \varphi \to \bigwedge \alpha \bigwedge \beta \bigwedge \alpha \varphi.$$

Hence by an easy induction we have

Lemma 25. *If $\varphi \in FMS$ and $\alpha \notin FV(\varphi)$, then* $\Lambda_1 \vdash \varphi \to \bigwedge \alpha \varphi$.

From Theorem 6 we obtain at once

Theorem 7. *Λ_1 is complete.*

The system of predicate logic L_1 based upon Λ_1 is, to the author's knowledge, the simplest known formulation of ordinary logic.

Other axiomatizations of logic can be obtained easily from L and L_1. Thus let $\Lambda' = \{\bigwedge \alpha_0 \ldots \bigwedge \alpha_{m-1}\, \varphi \colon \varphi \in \Lambda\}$, and similarly form Λ_1', from Λ_1. The corresponding systems of predicate logic L' and L_1' have detachment as their only rule of inference. We have:

THEOREM 8. *The sets Λ_1' and Λ' are complete.*

Again, let Λ'' (resp. Λ_1'') be the set of all closures of sentences of Λ (resp. Λ_1). Then clearly the subsets $\Lambda'' \cup \{\bigwedge \alpha\, \varphi \to \varphi \colon \varphi \in STS\}$ and $\Lambda_1'' \cup \{\bigwedge \alpha\, \varphi \to \varphi \colon \varphi \in STS\}$ of STS are complete. Moreover, Montague has observed that Λ'' has the following properties, and that this can be shown by essentially the same methods which were used in [11] to show that Σ_6 has analogous properties: (1) if $\neg \bigwedge \alpha\, \varphi \in US \cap FMS$, then $\Lambda'' \cup \{\neg \bigwedge \alpha\, \varphi\}$ is complete, i.e., $\overline{\Lambda'' \cup \{\neg \bigwedge \alpha\, \varphi\}} = US \cap FMS$; (2) $\overline{\Lambda''} = APS \cap FMS$.

A particular logic is obtained by fixing upon a set PR of predicates and the associated rank function r. Clearly the choice of the standard variable sequence u^π for each $\pi \in PR$ is irrelevant. The simplest choice is to assume that $u^\pi(i) = v_i$ for each $i < r(\pi)$ and each $\pi \in PR$; this gives the *normalized* substitutionless predicate logic. It is then not at all necessary to write down the variables after a predicate; one could explicitly modify the formation rules in this way. The definition of $OC(\varphi)$ would then be rather artificial, however. For example, the following sentence φ of ordinary logic:

$$\bigwedge v_2 \bigwedge v_3\, (\pi\, v_2\, v_3 \to \pi\, v_3\, v_2)$$

would have the following translation φ':

$$\bigwedge v_2 \bigwedge v_3\, (\bigwedge v_0 \bigwedge v_1\, (v_0 \equiv v_2 \wedge v_1 = v_3 \to \pi) \to \\ \bigwedge v_0 \bigwedge v_1\, (v_0 \equiv v_3 \wedge v_1 = v_2 \to \pi)).$$

The variables v_0, v_1 must be considered to occur in π.

We would like to make some remarks about the use of the system $\mathfrak{L}$ or $\mathfrak{L}'$ in conjunction with individual constants and operation symbols. We assume given a set C of constants and a set OP of operation symbols in addition to PR. There is a function r' which associates with each $o \in OP$ a positive integer $r'(o)$. The set TM of *terms* is the smallest set Γ of expressions such that $C \cup VR \subseteq \Gamma$ and $\langle o \rangle^\frown f \in \Gamma$ whenever $o \in OP$ and $f \in {}^n\Gamma$, with $n = r'(o)$.

To express these notions in our system, expand PR to PR' by including new predicates π_c for each $c \in C$ and π_o for each $o \in OP$; let $r(\pi_c) = 1$ for each $c \in C$, and $r(\pi_o) = r'(o) + 1$ for each $o \in OP$. With each term σ we associate a function $R_\sigma \in {}^{VR}FMS$ as follows (assuming that we are working in the normalized case, for simplicity); for each $\beta \in VR$,

$$R_\alpha(\beta) = \beta \equiv \alpha \text{ if } \alpha \in VR;$$

$$R_c(\beta) = S(\beta/v_0)\, \pi_c\, v_0;$$

$$R_{\langle o \rangle^\frown f}(\beta) = \bigwedge v_0 \ldots \bigwedge v_{n-1}\, (R_{f_0}(v_0) \wedge \ldots \wedge R_{f_{n-1}}(v_{n-1}) \\ \to S(\beta/v_n)\, \pi_o\, v_0 \ldots v_n), \text{ if } r'(o) = n \text{ and } f \in {}^nTM.$$

We define $eq \in {}^{(^2TM)}FMS$ as follows: for $\sigma, \tau \in TM$,

$$\sigma\, eq\, \tau = \bigwedge v_0 \bigwedge v_1 (R_\sigma(v_0) \wedge R_\tau(v_1) \rightarrow v_0 \equiv v_1).$$

For each $\pi \in PR$ with $r(\pi) = n$ we define $S_\pi \in {}^{(^nTM)}FMS$ as follows: for $\sigma \in {}^nTM$,

$$S_\pi(\sigma_0, \ldots, \sigma_{n-1}) = \bigwedge v_0 \ldots \bigwedge v_{n-1} (R\sigma_0(v_0) \wedge \ldots \wedge R\sigma_{n-1}(v_{n-1}) \rightarrow \pi\, v_0 \ldots v_{n-1}).$$

The following axioms should be added to Λ or Λ':

(D_1) $\bigvee v_0\, \pi_c\, v_0$, for each $c \in C$;

(D_2) $\bigwedge v_0 \bigwedge v_1 (\pi_c\, v_0 \wedge \bigwedge v_0 (v_0 \equiv v_1 \rightarrow \pi_c\, v_0) \rightarrow v_0 \equiv v_1)$, for each $c \in C$;

(O_1) $\bigwedge v_0 \ldots \bigwedge v_{n-1} \bigwedge v_n\, \pi_o\, v_0 \ldots v_n$, for each $o \in OP$;

(O_2) $\bigwedge v_0 \ldots \bigwedge v_{n+1} (\pi_o\, v_0 \ldots v_n \wedge \bigwedge v_n (v_n \equiv v_{n+1} \rightarrow \pi_o\, v_0 \ldots v_n) \rightarrow v_n \equiv v_{n+1})$, for each $o \in OP$.

These definitions make the systems $\mathfrak{L}$ and $\mathfrak{L}'$ as managable as the ordinary systems for most purposes. These remarks also apply to ordinary predicate logic without individual constants or operations symbols.

In all of the preceding discussion identity plays an essential role. For predicate logic without identity Theorem 1 fails: there are sentences of ordinary logic which are not equivalent to standard formulas. Thus the power of expression is not as great in substitutionless logic without identity as in ordinary logic. An interesting open problem is to give a simple set of axioms for the universally valid formulas or sentences of this weaker logic. A related but different problem has been treated in the literature; see [7], [10] and [15].

§ 2. Independence of the Schemata and Rules of the System $\mathfrak{L}$[7].

In this section we establish the independence of the schemas and rules of the system $\mathfrak{L}$ of § 1, and discuss independence questions for variants of $\mathfrak{L}$.

THEOREM 9. *In general, the schemata and rules of $\mathfrak{L}$ are independent. In more detail:*

(*i*) *none of the schemata* (C1)–(C7), *or detachment or generalization can be omitted*;

(*ii*) *if* $r(\pi) = 0$ *for each* $\pi \in PR$ (*in particular if* $PR = 0$) *then* (C8) *can be omitted*;

(*iii*) *if* $r(\pi) > 0$ *for some* $\pi \in PR$, *then* (C8) *cannot be omitted.*

PROOF. The proof naturally breaks into ten parts. In most parts a function V with domain FMS is defined. Unless otherwise stated, $V(\varphi) = 1$ for formulas φ derivable without the schema of the given part. Also unless otherwise stated $V(\neg\varphi)$ and $V(\varphi \rightarrow \psi)$ will be given by the *usual table*:

[7] In constructing the independence proofs of this and the next section the author has used in some instances methods found in the unpublished work [13].

$\to$	0	1	$\neg$
0	1	1	1
1	0	1	0

Part 1 (C1). For $\varphi \in ATS$ let $V(\varphi) = 2$. The values $V(\neg\varphi)$ and $V(\varphi \to \psi)$ for $\varphi, \psi \in FMS$ are given by the following table:

$\to$	0	1	2	3	$\neg$
0	1	1	1	1	1
1	0	1	0	0	0
2	0	1	1	0	3
3	1	1	2	1	1

Finally, for $\alpha \in VR$ and $\varphi \in FMS$ we let $V(\bigwedge \alpha\, \varphi) = V(\varphi)$ Then $V(\varphi) = 0$ for the following instance φ of (C1):

$$(\neg\, \alpha \equiv \alpha \to \neg\,\neg\,\neg\, \alpha \equiv \alpha) \to ((\neg\,\neg\,\neg\, \alpha \equiv \alpha \to \alpha \equiv \alpha) \to (\neg\, \alpha \equiv \alpha \to \alpha \equiv \alpha))$$

Part 2 (C2). For $\varphi \in ATS$ let $V(\varphi) = 1$. If $\varphi \in FMS$ and the implication symbol does not occur in φ, let $V(\neg\varphi) = 0$ if $V(\varphi) = 1$, and $V(\neg\varphi) = 1$ if $V(\varphi) = 0$. If the implication symbol occurs in φ, let $V(\varphi) = 0$. Let $V(\bigwedge \alpha\, \varphi) = V(\varphi)$ for all $\varphi \in FMS$ and $\alpha \in VR$. Then for $\psi = (\alpha \equiv \alpha \to \neg\, \alpha \equiv \alpha)$ we have $V((\neg\,\psi \to \psi) \to \psi) = 0$.

Part 3 (C3). Let $V(\varphi) = 0$ for $\varphi \in ATS$. For all $\varphi \in FMS$ and $\alpha \in VR$ let $V(\neg\varphi) = 1$ and $V(\bigwedge \alpha\, \varphi) = V(\varphi)$.
Then $V(\neg\, \alpha \equiv \alpha \to (\neg\,\neg\, \alpha \equiv \alpha \to \alpha \equiv \alpha)) = 0$.

Part 4 (C4). Let $V(\varphi) = 1$ for $\varphi \in ATS$. Let $V(\wedge\, \alpha\, (\varphi \to \psi)) = 1$, and $V(\bigwedge \alpha\, \chi) = V(\chi)$ if χ is not of the form $\varphi \to \psi$. Then

$$V(\bigwedge \alpha\, (\alpha \equiv \alpha \to \neg\, \alpha \equiv \alpha) \to (\bigwedge \alpha\, \alpha \equiv \alpha \to \bigwedge \alpha\, \neg\, \alpha \equiv \alpha)) = 0.$$

Part 5 (C5). Define $g \in {}^{FMS}FMS$ as follows. Set $g(\varphi) = \varphi$ for $\varphi \in ATS$. For $\varphi, \psi \in FMS$, $m \in \omega$ and $m \neq 0$, and $\alpha \in {}^{m}VR$, let $g(\neg\varphi) = \neg\, g(\varphi)$, $g(\varphi \to \psi) = g(\varphi) \to g(\psi)$, $g(\bigwedge \alpha_0 \dots \bigwedge \alpha_{m-1}\, \varphi) = \bigwedge v_0\, \varphi$ for $\varphi \in ATS$, $g(\bigwedge \alpha_0 \dots \bigwedge \alpha_{m-1}\, \neg\, \varphi) = \neg\, g(\bigwedge \alpha_0 \dots \bigwedge a_{m-1}\, \varphi)$ and $g(\bigwedge \alpha_0 \dots \bigwedge \alpha_{m-1}\, (\varphi \to \psi)) = g(\bigwedge \alpha_0 \dots \bigwedge \alpha_{m-1}\, \varphi) \to g(\bigwedge \alpha_0 \dots \bigwedge \alpha_{m-1}\, \psi)$. Define $k(\varphi) = 0$ for $\varphi \in ATS$, $k(\neg\varphi)$ and $k(\varphi \to \psi)$ by the usual table, and $k(\bigwedge \alpha\, \varphi) = 1$ for all α. Finally, let $V(\varphi) = 1$ if and only if $kg(\varphi) = 1$ and $kg(\bigwedge v_0\, \varphi) = 1$. Then

$$V(\neg\, v_1 \equiv v_2 \to \bigwedge v_0\, \neg\, v_1 \equiv v_2) = 0.$$

Part 6 (C6). Let $V(\varphi) = 0$ for $\varphi \in ATS$, and $V(\bigwedge \alpha\, \varphi) = 1$ for $\varphi \in FMS$ and $\alpha \in VR$. Then $V(\neg \bigwedge \alpha\, \neg\, \alpha \equiv \beta) = 0$ for $\alpha, \beta \in VR$, $\alpha \neq \beta$.

Part 7 (C7). Let $V(\alpha \equiv \beta) = 1$ for $\alpha \neq \beta$, and $V(\varphi) = 0$ for $\varphi \in ATS$ otherwise. Let $V(\bigwedge \alpha\, \varphi) = V(\varphi)$. Then $V(v_0 \equiv v_1 \to (v_0 \equiv v_1 \to v_1 \equiv v_1)) = 0$.

Part 8 (C8). Under the hypothesis of (*ii*) (C8) can be omitted by Theorem of [11]. Now assume $\pi \in PR$ and $r(\pi) > 0$. Let $\equiv$ and π be interpreted by E and P respectively, such that E is an equivalence relation and P does not have the substitution property relative to E. Then clearly (C8) fails.

Part 9 (Detachment). Without detachment, formulas shorter than the axioms cannot be derived. In particular, $v_0 \equiv v_0$ cannot be derived.

Part 10 (Generalization). Let $V(\varphi) = 0$ for $\varphi \in AT$, $V(\bigwedge \alpha\, \varphi) = 0$ if $\alpha \in FV(\varphi)$, and $V(\bigwedge \alpha\, \varphi) = 1$ otherwise. Then $V(\bigwedge v_0 \neg \bigwedge v_1 \neg v_1 \equiv v_0) = 0$.

Q. E. D.

By the same proof we have:

THEOREM 10. *In general, the schemata and rules of* $\mathfrak{L}'$ *are independent.*

The more detailed statement of Theorem 10 reads exactly like Theorem 9.

We have not determined whether or not the schemata and rules of $\mathfrak{L}_1$ are independent. The preceding proofs do, however, give the following theorem.

THEOREM 11. *In the system* $\mathfrak{L}_1$, or $\mathfrak{L}_1'$:

(*i*) *none of the schemata* (C1)–(C4), (C5′), (C6), (C7), *or detachment or* (*for* $\mathfrak{L}_1$) *generalization, can be omitted*;
(*ii*) *if* $r(\pi) = 0$ *for each* $\pi \in PR$ (*in particular if* $PR = 0$), *then* ($C5^2$), ($C5^3$) *and* (C8) *can be omitted*;
(*iii*) *if* $r(\pi) > 0$ *for some* $\pi \in PR$, *then* (C8) *cannot be omitted.*

§ 3. Independence of an axiom system of Tarski.

In this section we prove that the system $\mathfrak{S}_1$, and certain variants, of [14] are independent.

THEOREM 12. *The schemata and rules of the system* $\mathfrak{S}_1$ *are independent.*

PROOF. Schemata (A1), (A2) and (A3), and the rule of detachment can be proved independent as in Parts 1, 2, 3 and 9 respectively of the proof of Theorem 9. The remainder of the proof splits into six parts. We use the same conventions about the function V as in the proof of Theorem 9.

Part 1 (A4). For $\varphi \in AT$ we let $V(\varphi) = 1$. For $\alpha \in VR$ and $\varphi \in FM$ let

$$V(\bigwedge \alpha\, \varphi) = \begin{cases} V(\varphi), \text{ if } \alpha \notin FV(\varphi); \\ 1 \text{ if } V(\varphi) = 1 \text{ and } \alpha \text{ is the last} \\ \text{free variable of } \varphi \text{ in the ordering} <; \\ 0 \text{ otherwise.} \end{cases}$$

Then $V(\bigwedge v_0 \bigwedge v_1\, v_0 \equiv v_1 \to \bigwedge v_1 \bigwedge v_0\, v_0 \equiv v_1) = 0$.

Part 2 (A5). Let $g(\varphi) = \varphi$ for $\varphi \in AT$, $g(\neg\, \varphi) = \neg\, g(\varphi)$, and $g(\varphi \to \psi) = g(\varphi) \to g(\psi)$ for $\varphi, \psi \in FM$. Let
$g(\bigwedge \alpha\, \varphi) = g(\varphi)$ for $\alpha \neq v_0$, and $g(\bigwedge v_0\, \varphi) = \bigwedge v_0\, g(\varphi)$.
Let $k(\varphi) = 1$ for $\varphi \in AT$, let $k(\neg\, \varphi)$ and $k(\varphi \to \psi)$ be given by the usual table, let $k(\bigwedge \alpha\; \alpha \equiv \alpha) = 0$ and let $k(\bigwedge \alpha\, \varphi) = k(\varphi)$ otherwise. Finally, let $V(\varphi) = 1$ if and only if $kg(\varphi) = 1$. Then $V(\bigwedge v_1\, (\bigwedge v_0 \equiv v_0 \to v_0 \equiv v_0) \to (\bigwedge v_0\, v_0 \equiv v_1 \to \bigwedge v_0\, v_0 \equiv v_0))) = 0$.

Part 3 (A6). For $\varphi \in AT$ let $V(\varphi) = 1$. If $\alpha \in VR$ and $\varphi \in FM$ let $V(\bigwedge \alpha\, \varphi) = 1$. Then $V(\bigwedge \alpha\, \neg \bigwedge \alpha\; \alpha \equiv \alpha \to \neg \bigwedge \alpha\; \alpha \equiv \alpha\,) = 0$.

Part 4 (A7). Let $V(\varphi) = 1$ if $\varphi \in AT$. For $\alpha \in VR - \{v_0\}$ and $\varphi \in FM$, let $V(\bigwedge \alpha\, \varphi) = V(\varphi)$. Finally, for $\varphi \in FM$ let

$$V(\bigwedge v_0\, \varphi) = \begin{cases} V\,(\varphi) \text{ if } \varphi \text{ is universally valid,} \\ 0 \text{ otherwise.} \end{cases}$$

Then $V([v_1 \equiv v_2 \to \bigwedge v_0\, v_1 \equiv v_2]) = 0$.

Part 5 (A8). Let $V(\varphi) = 0$ for $\varphi \in AT$, and for $\alpha \in VR$ and $\varphi \in FM$ let $V(\bigwedge \alpha\, \varphi) = V(\varphi)$. Then $V[\neg \bigwedge \alpha\, \neg \equiv \beta] = 0$.

Part 6 (A9). Let $\mathfrak{A} = \langle\{0, 1\}, \{\langle 0,1\rangle, \langle 0,0\rangle\}\rangle = \langle A, R\rangle$. Interpreting the equality symbol by the relation R we see that (A1)–(A8) and detachment hold but (A9) fails.

With regard to variants of $\mathfrak{S}_1$, we note first that upon replacing (A6) by a special case of (A6′) redundances may develop. The following theorem illustrates this possibility.

THEOREM 13. *In the system* $\mathfrak{S}_1$ *replace* (A6) *by the following schema*:
(*i*) $\neg \bigwedge \alpha\, \neg\, [\bigwedge \beta \bigwedge \gamma\, \varphi \to \bigwedge \gamma \bigwedge \beta\, \varphi]$.
The resulting system is complete, but (A4) *is then redundant.*

The easy proof, using (A1)–(A3) and (A7), may be omitted. Because of this theorem it is appropriate when discussing independence questions for (A6′) to consider only special cases of that condition. We consider the following three cases, which are mentioned in [14]:

(A6′$_1$) $\neg \bigwedge \alpha\, \neg\, \alpha \equiv \alpha$;

(A6′$_2$) $\neg \bigwedge \alpha \bigwedge \beta\, \neg\, \alpha \equiv \beta$, for $\alpha \neq \beta$;

(A6′$_3$) $\neg \bigwedge \alpha\, \neg\, (\varphi \to \varphi)$, where $\varphi \in ST$.

We have:

THEOREM 14. *The schemata and rules of the system* $\mathfrak{S}_1$ *remain independent after applying independently any number of the following three modifications*:

(*i*) *replace* (A6) *by* (A6*), (A6′$_1$), (A6′$_2$) *or* (A6′$_3$);
(*ii*) *in the description of* S_1 *let* $[\varphi]$ *be the Berry closure of* φ *for each* $\varphi \in FM$;
(*iii*) *replace* (A1)–(A3) *by the stipulation that* $[\varphi]$ *be an axiom whenever* φ *is a tautologus formula.*

Proof. The preceding proofs apply except in the following two cases.

Case 1. (A6) is replaced by (A6$'_1$) and we wish to show that (A8) is not redundant. Let $V(\alpha \equiv \beta) = 0$ for $\alpha, \beta \in VR$, $\alpha \neq \beta$, and $V(\varphi) = 1$ for $\varphi \in AT$ otherwise. For $\alpha \in VR$ and $\varphi \in FM$ let $V(\bigwedge \alpha \varphi) = V(\varphi)$. Then

$V[\neg \bigwedge \alpha \neg \alpha \equiv \beta] = 0$ for $\alpha \neq \beta$.

Case 2 (A6) is replaced by (A6$'_2$) and we wish to show that (A8) is not redundant. Let $\mathfrak{A} = \langle\{0,1\}, \{\langle 00\rangle\}\rangle = \langle A, R\rangle$. Interpreting the equality symbol by R gives the desired result.

It is natural to consider the possibility of replacing the schema A7 by the following weaker schema:

(A7′) $[\varphi \rightarrow \wedge\ \alpha\ \varphi]$ for $\alpha \notin OC\ (\varphi)$.

In the original system $\mathfrak{S}_1$ it is open whether or not, upon replacing (A7) by (A7′), completeness is preserved. With regard to the systems weakened by the condition (A6′), however, the following result holds.

Theorem 15. *The system obtained from $\mathfrak{S}_1$ by repacing* (A7) *by* (A7′), *and* (A6) *by* (A6$'_1$), (A6$'_2$) *or* (A6$'_3$) *is incomplete*; *the same applies if moreover* (A1) to (A3) *are replaced by the stipulation that* $[\varphi]$ *be an axiom for each tautological formula* φ, *or if Berry closure is used instead of Quine closure.*

Proof. Let $g(\varphi) = \varphi$ for $\varphi \in AT$; let $g(\neg \varphi) = \neg g(\varphi)$ and $g(\varphi \rightarrow \psi) = g(\varphi) \rightarrow g(\psi)$. Let $g(\bigwedge \alpha \varphi) = g(\varphi)$ if $\alpha \neq v_0$. Let $g(\bigwedge v_0 \varphi) = \bigwedge v_0 \varphi$ for $\varphi \in AT$, $g(\bigwedge v_0 \neg \varphi) = \neg g(\bigwedge v_0 \varphi)$ and $g(\bigwedge v_0 (\varphi \rightarrow \psi)) = g(\bigwedge v_0 \varphi) \rightarrow g(\bigwedge v_0 \psi)$. Finally, let $g(\bigwedge v_0 \bigwedge \alpha \varphi) = g(\bigwedge v_0 \varphi)$ for $\alpha \neq v_0$, and $g(\bigwedge v_0 \bigwedge v_0 \varphi) = \bigwedge v_0\, g(\bigwedge v_0 \varphi)$. Let $k(\varphi) = 1$ for $\varphi \in AT$, let $k(\neg \varphi)$ and $k(\varphi \rightarrow \psi)$ be given by the usual table, and let $k(\bigwedge v_0 \bigwedge v_0\, v_0 \equiv v_0) = 0$, $k(\bigwedge v_0 \varphi) = k(\varphi)$ otherwise. Let $V(\varphi) = kg(\varphi)$. Then $V(\bigwedge v_0\, v_0 \equiv v_0 \rightarrow \bigwedge v_0 \bigwedge v_0\, v_0 \equiv v_0) = 0$.

The following theorem has the same proof as Theorem 12.

Theorem 16. *The schemata and rules of the system $\mathfrak{S}'_1$ are independent.*

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