13. Well-founded relations

Here we introduce the usual hierarchy of sets, give a final generalized recursion theorem, and prove Mostowski’s collapsing theorem.

The hierarchy of sets

The hierarchy of sets is defined recursively as follows:

\[ V_0 = \emptyset; \]
\[ V_{\alpha+1} = \mathcal{P}(V_\alpha); \]
\[ V_\gamma = \bigcup_{\alpha < \gamma} V_\alpha \text{ for } \gamma \text{ limit.} \]

**Theorem 13.1.** For every ordinal \( \alpha \) the following hold:

(i) \( V_\alpha \) is transitive.

(ii) \( V_\beta \subseteq V_\alpha \) for all \( \beta < \alpha \).

**Proof.** We prove these statements simultaneously by induction on \( \alpha \). They are clear for \( \alpha = 0 \). Assume that both statements hold for \( \alpha \); we prove them for \( \alpha + 1 \). First we prove

(1) \( V_\alpha \subseteq V_{\alpha+1} \).

In fact, suppose that \( x \in V_\alpha \). By (i) for \( \alpha \), the set \( V_\alpha \) is transitive. Hence \( x \subseteq V_\alpha \), so \( x \in \mathcal{P}(V_\alpha) = V_{\alpha+1} \). So (1) holds.

Now (ii) follows. For, suppose that \( \beta < \alpha + 1 \). Then \( \beta \leq \alpha \), so \( V_\beta \subseteq V_\alpha \) by (ii) for \( \alpha \) (or trivially if \( \beta = \alpha \)). Hence by (1), \( V_\beta \subseteq V_{\alpha+1} \).

To prove (i) for \( \alpha + 1 \), suppose that \( x \in y \in V_{\alpha+1} \). Then \( y \in \mathcal{P}(V_\alpha) \), so \( y \subseteq V_\alpha \), hence \( x \in V_\alpha \). By (1), \( x \in V_{\alpha+1} \), as desired.

For the final inductive step, suppose that \( \gamma \) is a limit ordinal and (i) and (ii) hold for all \( \alpha < \gamma \). To prove (i) for \( \gamma \), suppose that \( x \in y \in V_\gamma \). Then by definition of \( V_\gamma \), there is an \( \alpha < \gamma \) such that \( y \in V_\alpha \). By (i) for \( \alpha \) we get \( x \in V_\alpha \). So \( x \in V_\gamma \) by the definition of \( V_\gamma \).

Condition (ii) for \( \gamma \) is obvious. \( \square \)

A very important fact about this hierarchy is that every set is a member of some \( V_\alpha \). To prove this, we need to introduce the notion of transitive closure, which is itself important.

**Theorem 13.2.** For any set \( a \) there is a transitive set \( b \) with the following properties:

(i) \( a \subseteq b \).

(ii) For every transitive set \( c \) such that \( a \subseteq c \) we have \( b \subseteq c \).

**Proof.** By recursion we define \( d_0 = a \) and \( d_{m+1} = d_m \cup \bigcup d_m \) for every \( m \in \omega \). Let \( b = \bigcup_{m \in \omega} d_m \). Then \( a = d_0 \subseteq b \). Suppose that \( x \in y \in b \). Choose \( m \in \omega \) such that \( y \in d_m \). Then \( x \in d_m \subseteq d_{m+1} \subseteq b \). Thus \( b \) is transitive. Now suppose that \( c \) is a transitive set such that \( a \subseteq c \). We show by induction that \( d_m \subseteq c \) for every \( m \in \omega \).

First, \( d_0 = a \subseteq c \), so this is true for \( m = 0 \). Now assume that it is true for \( m \). Then \( d_{m+1} = d_m \cup d_m \subseteq c \cup c = c \), completing the inductive proof.
Hence \( b = \bigcup_{m \in \omega} d_m \subseteq c \). \(\square\)

The set shown to exist in Theorem 13.2 is called the transitive closure of \( a \), and is denoted by \( \text{trcl}(a) \).

**Theorem 13.3.** Every set is a member of some \( V_\alpha \).

**Proof.** Suppose that this is not true, and let \( a \) be a set which is not a member of any \( V_\alpha \). Let \( A = \{ x \in \text{trcl}(a \cup \{ a \}) : x \text{ is not in any of the sets } V_\alpha \} \). Then \( a \in A \), so \( A \) is nonempty. By the foundation axiom, choose \( x \in A \) such that \( x \cap A = 0 \). Suppose that \( y \in x \). Then \( y \in \text{trcl}(a \cup \{ a \}) \), so \( y \) is a member of some \( V_\alpha \). Let \( \alpha_y \) be the least such \( \alpha \). Let \( \gamma = \bigcup_{y \in x} \alpha_y \). Then by 13.1(ii), \( x \subseteq V_\beta \). So \( x \in V_{\beta+1} \), contradiction. \(\square\)

Thus by Theorem 13.3 we have \( V = \bigcup_{\alpha \in \text{On}} V_\alpha \). An important technical consequence of Theorem 13.3 is the following definition, known as Scott’s trick:

- Let \( R \) be a class equivalence relation on a class \( A \). For each \( a \in A \), let \( \alpha \) be the smallest ordinal such that there is a \( b \in V_\alpha \) with \( (a,b) \in R \), and define
  \[
  \text{type}_R(a) = \{ b \in V_\alpha : (a,b) \in R \}.
  \]

This is the “reduced” equivalence class of \( a \). It could be that the collection of \( b \) such that \( (a,b) \in R \) is a proper class, but \( \text{type}_R(a) \) is always a set. An important use of this trick is in defining order types, where we modify this procedure slightly. Define the class \( R \) to be the collection of all ordered pairs \( (L,M) \) such that \( L \) and \( M \) are order-isomorphic linear orders. For any linear order \( L \), we define its order type to be

\[
\text{o.t.}(L) = \begin{cases} 
\alpha & \text{if } L \text{ is a well-order which is order-isomorphic to the ordinal } \alpha, \\
\text{type}_R(L) & \text{if } L \text{ is not well-ordered}.
\end{cases}
\]

On the basis of this definition one can introduce the usual terminology for order types, e.g., \( \eta \) for the order type of the rationals, \( \omega^* \) for the order type of \( (\omega,\rhd) \), etc.

On the basis of our hierarchy we can define the important notion of rank of sets:

- For any set \( x \), the rank of \( x \), denoted by \( \text{rank}(x) \), is the smallest ordinal \( \alpha \) such that \( x \in V_{\alpha+1} \).

We take \( \alpha + 1 \) here instead of \( \alpha \) just for technical reasons. Some of the most important properties of ranks are given in the following theorem.

**Theorem 13.4.** Let \( x \) be a set and \( \alpha \) an ordinal. Then
  
  (i) \( V_\alpha = \{ y : \text{rank}(y) < \alpha \} \).
  (ii) For all \( y \in x \) we have \( \text{rank}(y) < \text{rank}(x) \).
  (iii) \( \text{rank}(y) \leq \text{rank}(x) \) for every \( y \subseteq x \).
  (iv) \( \text{rank}(x) = \sup_{y \in x} (\text{rank}(y) + 1) \).
  (v) \( \text{rank}(\alpha) = \alpha \).
  (vi) \( V_\alpha \cap \text{On} = \alpha \).
Proof. (i): Suppose that \( y \in V_\alpha \). Then \( \alpha \neq 0 \). If \( \alpha \) is a successor ordinal \( \beta + 1 \), then \( \text{rank}(y) \leq \beta < \alpha \). If \( \alpha \) is a limit ordinal, then \( y \in V_\beta \) for some \( \beta < \alpha \), hence \( y \in V_{\beta+1} \) also, so \( \text{rank}(y) \leq \beta < \alpha \). This proves \( \subseteq \).

For \( \supseteq \), suppose that \( \beta \overset{\text{def}}{=} \text{rank}(y) < \alpha \). Then \( y \in V_{\beta+1} \subseteq V_\alpha \), as desired.

(ii): Suppose that \( x \in y \). Let \( \text{rank}(y) = \alpha \). Thus \( y \in V_{\alpha+1} = \mathcal{P}(V_\alpha) \), so \( y \subseteq V_\alpha \) and hence \( x \in V_\alpha \). Then by (i), \( \text{rank}(x) < \alpha \).

(iii): Let \( \text{rank}(x) = \alpha \). Then \( x \in V_{\alpha+1} \), so \( x \subseteq V_\alpha \). Let \( y \subseteq x \). Then \( y \subseteq V_\alpha \), and so \( y \in V_{\alpha+1} \). Thus \( \text{rank}(y) \leq \beta < \alpha \).

(iv): Let \( \alpha \) be the indicated sup. Then \( \geq \) holds by (ii). Now if \( y \in x \), then \( \text{rank}(y) < \alpha \), and hence \( y \in V_{\text{rank}(y)+1} \subseteq V_\alpha \). This shows that \( x \subseteq V_\alpha \), hence \( x \in V_{\alpha+1} \), hence \( \text{rank}(x) \leq \alpha \), finishing the proof of (iv).

(v): We prove this by transfinite induction. Suppose that it is true for all \( \beta < \alpha \). Then by (iv),
\[
\text{rank}(\alpha) = \sup_{\beta < \alpha} (\text{rank}(\beta) + 1) = \sup_{\beta < \alpha} (\beta + 1) = \alpha.
\]

Finally, for (vi), using (i) and (v),
\[
V_\alpha \cap \text{On} = \{ \beta \in \text{On} : \beta \in V_\alpha \} = \{ \beta \in \text{On} : \text{rank}(\beta) < \alpha \} = \{ \beta \in \text{On} : \beta < \alpha \} = \alpha.
\]

We now define a sequence of cardinals by recursion:
\[
\beth_0 = \omega;
\beth_{\alpha+1} = 2^{\beth_\alpha};
\beth_{\gamma} = \bigcup_{\alpha < \gamma} \beth_\alpha \quad \text{for } \gamma \text{ limit.}
\]

Thus under GCH, \( \kappa_\alpha = \beth_\alpha \) for every ordinal \( \alpha \); in fact, this is just a reformulation of GCH.

**Theorem 13.5.** (i) \( n \leq |V_n| \in \omega \) for any \( n \in \omega \).

(ii) For any ordinal \( \alpha \), \( |V_{\omega+\alpha}| = \beth_\alpha \).

**Proof.** (i) is clear by ordinary induction on \( n \). We prove (ii) by the three-step transfinite induction (where \( \gamma \) is a limit ordinal below):

\[
|V_\omega| = \left| \bigcup_{n \in \omega} V_n \right| = \omega = \beth_0 \quad \text{by (i)};
\]

\[
|V_{\omega+1}| = |\mathcal{P}(V_{\omega+1})| = 2^{\omega} = 2^{\beth_\omega} = \beth_{\alpha+1};
\]

\[
|V_{\omega+\gamma}| = \left| \bigcup_{\beta < \gamma} V_{\omega+\beta} \right|.
\]

155
\[ \leq \sum_{\beta < \gamma} |V_{\omega+\beta}| \]
\[ = \sum_{\beta < \gamma} \exists_\beta \quad \text{(inductive hypothesis)} \]
\[ \leq \sum_{\beta < \gamma} \exists_\gamma \]
\[ = |\gamma| \cdot \exists_\gamma \]
\[ = \exists_\gamma \quad \text{by a normal function theorem;} \]

to finish this last inductive step, note that for each \( \beta < \gamma \) we have \( \exists_\beta = |V_{\omega+\beta}| \leq |V_{\omega+\gamma}| \), and hence \( \exists_\gamma \leq |V_{\omega+\gamma}| \).

\[ \square \]

**Well-founded class relations**

We now introduce a kind of generalization of set membership.

- If \( A \) is a class, a class relation \( R \) is **well-founded on** \( A \) iff for every nonempty subset \( X \) of \( A \) there is an \( x \in X \) such that for all \( y \in X \) it is not the case that \((y, x) \in R\). Such a set \( x \) is called **\( R \)-minimal**.

This notion is important even if \( A \) and \( R \) are mere sets. The prototypical example of a well-founded relation is \( \in \) itself; it is well-founded on \( V \) as one sees by the foundation axiom.

Recall that our intuitive notion of class is made rigorous by using formulas instead. Thus we would talk about a formula \( \varphi(x,y) \) being well-founded on another formula \( \psi(x) \). In the case of \( \in \), we are really looking at the formula \( x \in y \) being well-founded on the formula \( x = y \). So, rigorously we are associating with two formulas \( \varphi(x,y) \) and \( \psi(x) \) another formula “\( \varphi(x,y) \) is well-founded on \( \psi(x) \)”, namely the following formula:

\[ \forall X[\forall x \in X \psi(x) \land X \neq \emptyset \rightarrow \exists x \in X \forall y \in X \neg \varphi(y, x)]. \]

We are going to formulate and prove a recursion principle for well-founded relations. But first we need some technical preparation.

- A class relation \( R \) is **set-like on a class** \( A \) iff for every \( a \in A \) the class \( \{b \in A : (b, a) \in R\} \) is a set.

Again we give a rigorous version. Given two formulas \( \varphi(x,y) \) and \( \psi(x) \), the formula “\( \varphi(x,y) \) is set-like on \( \psi(x) \)” is the following formula:

\[ \forall x[\psi(x) \rightarrow \exists z \forall y(y \in z \leftrightarrow \psi(y) \land \varphi(y, x))]. \]

- Let the class relation \( R \) be set-like on the class \( A \), and suppose that \( x \in A \). We define \( \text{pred}_n(A, x, R) \) for every natural number \( n \) by recursion.

\[ \text{pred}_0(A, x, R) = \{y \in A : (y, x) \in R\}; \]
\[ \text{pred}_{n+1}(A, x, R) = \bigcup \{\text{pred}_0(A, y, R) : y \in \text{pred}_n(A, x, R)\}. \]

156
Then we define
\[ \text{cl}(A, x, R) = \bigcup_{n \in \omega} \text{pred}_n(A, x, R). \]

**Proposition 13.6.** If R is well-founded and set-like on A, x \in A, and y is an element of cl(A, x, R), then \( \text{pred}_0(A, y, R) \subseteq \text{cl}(A, x, R) \).

**Proof.** Choose \( n \in \omega \) such that \( y \in \text{pred}_n(A, x, R) \). Clearly then \( \text{pred}_0(A, y, R) \subseteq \text{pred}_{n+1}(A, x, R) \subseteq \text{cl}(A, x, R) \).

**Proposition 13.7.** If R is well-founded and set-like on A, then every nonempty subclass of A has an R-minimal element.

**Proof.** Suppose that \( X \) is a nonempty subclass of A and \( x \in X \), but \( x \) is not an R-minimal element of \( X \). So there is a \( y \in X \) such that \( (y, x) \in R \), and so \( y \in \text{pred}_0(A, x, R) \cap X \subseteq \text{cl}(A, x, R) \cap X \). Since thus \( \text{cl}(A, x, R) \cap X \) is a nonempty subset of A, we can take an R-minimal element \( y \) of it. Suppose that \( (z, y) \in R \). Then by Proposition 13.6, \( z \in \text{pred}_0(A, y, R) \subseteq \text{cl}(A, x, R) \), and so \( z \notin X \). Hence \( y \) is an R-minimal element of \( X \). \( \square \)

Now we are ready for the general recursion theorem.

**Theorem 13.8.** Suppose that R is a class relation which is well-founded and set-like on A. Also suppose that \( F : A \times V \to V \). Then there is a unique \( G : A \to V \) such that for every \( x \in A \),
\[ G(x) = F(x, G \upharpoonright \text{pred}(A, x, R)). \]

Before beginning the proof of this theorem, we make some comments about it. The fact that R is set-like on A implies that \( \text{pred}(A, x, R) \) is a set, and hence so is \( F \upharpoonright \text{pred}(A, x, R) \), using the replacement axiom. Recalling that classes are really a shorthand for formulas, we can formulate the theorem more rigorously, first introducing the following definitions.

“\( \varphi(x, y, z) \) is a function from \( \psi(x) \times V \) into \( V \)” abbreviates
\[ \forall x, y, z, w[\varphi(x, y, z) \land \varphi(x, y, w) \to z = w] \land \forall x, y[\exists z \varphi(x, y, z) \iff \psi(x)]. \]

“\( \theta(x, y) \) is a function from \( \psi(x) \) into \( V \)” abbreviates
\[ \forall x, y, z[\theta(x, y) \land \theta(x, z) \to y = z] \land \forall x[\exists y \theta(x, y) \iff \psi(x)]. \]

“\( a \) is the set of predecessors of \( x \) under \( \chi(x, y) \)” abbreviates
\[ \forall y[y \in a \iff \chi(y, x)]. \]

“\( f \) is the restriction of \( \theta(x, y) \) to the set of predecessors of \( x \) under \( \chi(x, y) \)” abbreviates
\[ \forall z[z \in f \iff \exists x, y, a[\text{“}a \text{ is the set of predecessors of } x \text{ under } \chi(x, y)\text{”} \land z = (x, y) \land x \in a \land \theta(x, y)]]]. \]

157
Now the rigorous version of the theorem is as follows:

*Suppose that $\varphi(x, y), \psi(x), \chi(x, y, z), \theta'(x, y)$ are formulas. Then there is a formula $\theta(x, y)$ (given explicitly in the proof below) such that*

$$ZFC \vdash \[ \chi(x, y) \text{ is well-founded on } \psi(x) \] \land \chi(x, y) \text{ is set-like on } \psi(x)$$

$$\land \varphi(x, y, z) \text{ is a function from } \psi(x) \times V \text{ into } V$$

$$\land \theta(x, y) \text{ is a function from } \psi(x) \text{ into } V \land \forall x \exists f, z[\text{"}f \text{ is the restriction of } \theta(x, y) \text{ to the set of predecessors of } x \text{ under } \chi(x, y)\text{"} ]$$

$$\land (\forall x, z) \land \varphi(x, f, z)$$

$$\land \{ [\text{"}\theta'(x, y) \text{ is a function from } \psi(x) \text{ into } V\text{"} \land \forall x \exists f, z[\text{"}f \text{ is the restriction of } \theta'(x, y) \text{ to the set of predecessors of } x \text{ under } \chi(x, y)\text{"} ]$$

$$\land \theta'(x, z) \land \varphi(x, f, z) \}]$$

$$\rightarrow \forall x, z[\theta(x, z) \leftrightarrow \theta'(x, z)]\}. $$

**Proof of Theorem 13.8.** This proof is very similar to that of Theorem 2.12. Consider the following condition:

(*) $f$ is a function with domain $d \subseteq A$, $\forall x \in d(\text{pred}(A, x, R) \subseteq d)$, and for every $x \in d$ we have $f(x) = F(x, f \upharpoonright \text{pred}(A, x, R))$.

First we show

(1) If $f, d$ and $g, e$ satisfy (*), then $f \upharpoonright (d \cap e) = g \upharpoonright (d \cap e)$.

To prove (1) suppose that $x \in d \cap e$ and $f(x) \neq g(x)$. Thus the set $X = \{ y \in (d \cap e) : f(y) \neq g(y) \}$ is a nonempty subset of $A$, so by well-foundedness we can take an $R$-minimal element $z$ of $X$. Then $f(w) = g(w)$ for all $w \in d \cap e$ such that $wRz$. Now clearly $\text{pred}(A, z, R) \subseteq (d \cap e)$, so $f \upharpoonright \text{pred}(A, z, R) = g \upharpoonright \text{pred}(A, z, R)$. Hence

$$f(z) = F(z, f \upharpoonright \text{pred}(A, z, R)) = F(z, g \upharpoonright \text{pred}(A, z, R)) = g(z),$$

contradiction.

(2) For every $x \in A$, let $d(x) = \{ x \} \cup \text{cl}(A, x, R)$. Then for every $x \in A$ there is a $f$ such that $f$ and $d(x)$ satisfy (*).

Suppose that (2) is not true, and let $x \in A$ be $R$-minimal such that there is no such $f$. For each $y$ such that $(y, x) \in R$ there is a $g$ so that $g$ and $d(y)$ satisfy (*). By (1) this $g$ is unique, and so by the replacement axiom we can associate with each such $y$ the corresponding function $g_y$. Let

$$h = \bigcup_{(y, x) \in R} g_y \quad \text{and} \quad f = h \cup \{(x, F(x, h))\}.$$ 

It is straightforward to check that $f$ and $d(x)$ satisfy (*), contradiction. So (2) holds.
Now for any \( x \in A \), let \( G(x) = f(x) \), where \( f \) and \( d(x) \) satisfy (*). This definition is unambiguous by (1) and (2). It is easy to see that \( G \) is what is needed in the theorem.

The uniqueness of \( G \) follows by an easy argument by contradiction. \( \square \)

As a first application of this theorem we can define rank for well-founded relations.

- If \( R \) is well-founded and set-like on \( A \), then for any \( x \in A \),
  \[
  \text{rank}(x, A, R) = \sup \{ \text{rank}(y, A, R) + 1 : y \in A \text{ and } (y, x) \in R \}.
  \]

By induction, this is always an ordinal.

**Proposition 13.9.** If \( A \) is a transitive class, then for any \( x \in A \), \( \text{rank}(x, A, \in) = \text{rank}(x) \). In particular, \( \text{rank}(x, V, \in) = \text{rank}(x) \) for any set \( x \).

**Proof.** Otherwise, let \( x \) be \( R \)-minimal such that \( x \in A \) and \( \text{rank}(x, A, \in) \neq \text{rank}(x) \).

Then
\[
\text{rank}(x, A, \in) = \sup \{ \text{rank}(y, A, \in) + 1 : y \in A \text{ and } y \in x \} = \sup \{ \text{rank}(y) + 1 : y \in A \text{ and } y \in x \} \quad (\text{R-minimality of } x)
\]
\[
= \sup \{ \text{rank}(y) + 1 : y \in x \} \quad (\text{transitivity of } A)
\]
\[
= \text{rank}(x) \quad \text{by Theorem 13.4(iii)).}
\]

This is a contradiction. \( \square \)

**The Mostowski collapse**

We give here an important technical result about well-founded relations.

- Suppose that \( R \) is well-founded and set-like on \( A \). The Mostowski collapsing function is a class function \( G : A \to V \) defined by recursion as follows: for any \( x \in A \),
  \[
  G(x) = \{ G(y) : y \in A \text{ and } (y, x) \in R \}.
  \]

The Mostowski collapse of \( A, R \) is defined as the range of this function \( G \).

**Proposition 13.10.** Suppose that \( R \) is well-founded and set-like on \( A \), \( G \) is the Mostowski collapsing function for \( A, R \), and \( M \) is the Mostowski collapse. Then

(i) For all \( x, y \in A \), if \( (x, y) \in R \) then \( G(x) \in G(y) \).

(ii) \( M \) is transitive.

(iii) For any \( x \in A \) we have \( \text{rank}(x, A, R) = \text{rank}(G(x)) \).

**Proof.** (i) is obvious from the definition. If \( a \in b \in M \), choose \( y \in A \) such that \( b = G(y) \). Since \( a \in b \), by the definition we have \( a \in \text{rng}(G) = M \). So (ii) holds. For (iii), suppose it is not true, and let \( x \) be \( R \)-minimal such that \( x \in A \) and \( \text{rank}(x, A, R) \neq \text{rank}(G(x)) \).

Then
\[
\text{rank}(x, A, R) = \sup \{ \text{rank}(y, A, R) + 1 : y \in A \text{ and } (y, x) \in R \}
\]
\[
= \sup \{ \text{rank}(G(y)) + 1 : y \in A \text{ and } (y, x) \in R \} \quad (\text{minimality of } x)
\]
\[
= \sup \{ \text{rank}(u) + 1 : u \in G(x) \} \quad (\text{definition of } G)
\]
\[
= \text{rank}(G(x)) \quad (\text{by 13.4(iii)).}
\]
The Mostowski collapse is especially important for extensional relations, defined as follows.

- Let $R$ be a class relation and $A$ a class. We say that $R$ is extensional on $A$ iff the following generalization of the extensionality axiom holds:

$$\forall x, y \in A [\forall z \in A [(z, x) \in R \iff (z, y) \in R] \rightarrow x = y].$$

**Proposition 13.11.** Suppose that $R$ is well-founded and set-like on $A$. Let $G$ and $M$ be the Mostowski collapsing function and Mostowski collapse, respectively. Then the following conditions are equivalent:

(i) $R$ is extensional on $A$.

(ii) $G$ is one-one, and for all $x, y \in A$ we have $(x, y) \in R$ iff $G(x) \in G(y)$.

**Proof.** (i)⇒(ii): Assume (i). Suppose that $G$ is not one-one. Then the set

$$(*) \quad \{x \in A : \text{there is a } y \in A \text{ such that } x \neq y \text{ and } G(x) = G(y)\}$$

is nonempty, and we take an $R$-minimal element of this set. Also, let $y \in A$ with $x \neq y$ and $G(x) = G(y)$. Since both $x$ and $y$ are in $A$, and $x \neq y$, the extensionality condition gives two cases.

Case 1. There is a $z \in A$ such that $(z, x) \in R$ and $(z, y) \notin R$. Since $(z, x) \in R$, it follows that $z$ is not in the set $(*)$. Now $G(z) \in G(x)$ by Proposition 13.10(i), so the fact that $G(x) = G(y)$ implies that $G(z) \in G(y)$. Hence by definition of $G$ we can choose $w \in A$ such that $(w, y) \in R$ and $G(z) = G(w)$. Then from $z$ not in $(*)$ we infer that $z = w$, hence $(z, y) \in R$, contradiction.

Case 2. There is a $z \in A$ such that $(z, y) \in R$ and $(z, x) \notin R$. Since $(z, y) \in R$, by Proposition 13.10(i) we get $G(z) \in G(y) = G(x)$, and so there is a $v \in A$ such that $(v, x) \in R$ and $G(z) = G(v)$. Now $v$ is not in $(*)$ by the minimality of $x$, so $z = v$ and $(z, x) \in R$, contradiction.

Therefore, $G$ is one-one. Now the implication $\Rightarrow$ in the second part of (ii) holds by Proposition 13.10(i). Suppose now that $G(x) \in G(y)$. Choose $w \in A$ such that $(w, y) \in R$ and $G(x) = G(w)$. Then $x = w$ since $G$ is one-one, so $(x, y) \in R$, as desired.

(ii)⇒(i): Assume (ii), and suppose that $x, y \in A$, and $\forall z \in A [(z, x) \in R \iff (z, y) \in R]$. Take any $u \in G(x)$. By the definition of $G$, choose $z \in A$ such that $(z, x) \in R$ and $u = G(z)$. Then also $(z, y) \in R$, so $u = G(z) \in G(y)$. This shows that $G(x) \subseteq G(y)$. Similarly $G(y) \subseteq G(x)$, so $G(x) = G(y)$. Since $G$ is one-one, it follows that $x = y$. \qed

**Theorem 13.12.** Suppose that $R$ is a well-founded class relation that is set-like and extensional on a class $A$. Then there are unique $G, M$ such that $M$ is a transitive class and $G$ is an isomorphism from $(A, R)$ onto $(M, \in)$.

Note that we have formulated this in the usual fashion for isomorphism of structures, but of course we cannot form the ordered pairs $(A, R)$ and $(M, \in)$ if $A, R, M$ are proper classes. So we understand the above as an abbreviation for a longer statement, that $G$ is a bijection from $A$ onto $M$, etc.
Proof of 13.12. The existence of $G$ and $M$ is immediate from Propositions 13.10 and 13.13. Now suppose that $G'$ and $M'$ also work. Since $M$ is the range of $G$ and $M'$ is the range of $G'$, it suffices to show that $G = G'$. Suppose not. Let $a$ be $R$-minimal such that $G(a) \neq G'(a)$. Take any $x \in G(a)$. Then since $M$ is transitive and $G(a) \in M$, it follows that $x \in M$. And since $G$ maps onto $M$, it then follows that there is a $b \in A$ such that $G(b) = x$. So $G(b) \in G(a)$ so, by the isomorphism property, $(b,a) \in R$. Then the minimality of $a$ yields $G'(b) = G(b) \in G(a)$. But also $(b,a) \in R$ implies that $G'(b) \in G'(a)$ by the isomorphism property, so $x = G(b) \in G'(a)$. Thus we have proved that $G(a) \subseteq G'(a)$. By symmetry $G'(a) \subseteq G(a)$, so $G(a) = G'(a)$, contradiction. □

EXERCISES

E13.1. Write out all the elements of $V_\alpha$ for $\alpha = 0, 1, 2, 3, 4$.

E13.2. Define by recursion

$$S(\alpha) = \bigcup_{\beta < \alpha} \mathcal{P}(S(\beta))$$

for every ordinal $\alpha$. Prove that $V_\alpha = S(\alpha)$ for every ordinal $\alpha$.

E13.3. Determine exactly the ranks of the following sets in terms of the ranks of the sets entering into their definitions. In some cases the rank is not completely determined by the ranks of the constituents; in such cases, describe all possibilities.

(i) $\{x\}$
(ii) $\{x, y\}$
(iii) $\langle x, y \rangle$
(iv) $x \cup y$
(v) $x \cap y$
(vi) $x \setminus y$
(vii) $\bigcup x$
(viii) $\text{dmn}(R)$
(ix) $\mathcal{P}(x)$

E13.4. Let $R$ be a relation $\subseteq A \times A$ for some set $A$. Show that $R$ is well-founded on $A$ if there does not exist a sequence $\langle a_i : i \in \omega \rangle$ of elements of $A$ such that $a_{i+1}Ra_i$ for all $i \in \omega$.

E13.5. Define $xRy$ iff $(x, 1) \in y$. Show that $R$ is well-founded and set-like on $V$.

E13.6. (Continuing E13.5) By recursion let $\hat{y} = \{(\hat{x}, 1) : x \in y\}$ for any set $y$. Let $G$ be the Mostowski collapsing function for $R, V$ in exercise E13.5. Prove that $G(\hat{y}) = y$ for every set $y$.

E13.7. Prove that in the replacement axioms (page 8) the part $\exists y$ can be replaced by $\exists y$. Hint: apply the replacement axiom to a formula that says there is a member $y$ of $V_\alpha$ such that $\varphi$ holds, and $\alpha$ is minimum with this property.

E13.8. Prove that if $a$ is transitive, then $\{\text{rank}(b) : b \in a\}$ is an ordinal.

E13.9. Show that for any set $a$ we have $\text{rank}(\text{trcl}(a)) = \text{rank}(a)$.

E13.10. Define $xRy$ iff $x \in \text{trcl}(y)$. Show that $R$ is well-founded and set like on $V$.

E13.11. (Continuing exercise E13.10) Let $G$ be the Mostowski collapsing function for $R, V$. Show that $G(x) = \text{rank}(x)$ for every set $x$.

E13.12. Define $A$ and $R$ such that $R$ is well-founded on $A$ but not set-like on $A$. 161
E13.13. Define $A$ and $R$ such that $R$ is neither well-founded nor set-like on $A$.

E13.14. Let $R = \{(m, n) \in \omega \times \omega : n < m\}$. Note that $R$ is not well-founded on $\omega$, but it is set-like on $\omega$. Define a function $F : \omega \times V \to V$ such that there is no function $G$ as in the recursion theorem, using $\omega$ and $R$.

References


Kunen, K. Set Theory, 313pp.