11. Martin's axiom

Martin’s axiom is not an axiom of ZFC, but it can be added to those axioms. It has many important consequences. Actually, the continuum hypothesis implies Martin’s axiom, so it is of most interest when combined with the negation of the continuum hypothesis. The consistency of $\text{MA} + \neg \text{CH}$ involves iterated forcing, and is proved much later in these notes.

The formulation of the axiom involves some new notions about partially ordered sets. These notions are basic for the method of forcing also. Let $P = (P, \leq)$ be a partial order.

- A subset $D$ of $P$ is dense in $P$ iff for every $p \in P$ there is a $d \in D$ such that $d \leq p$.
- A subset $G$ of $P$ is a filter on $P$ iff the following two conditions hold:
  (i) For all $p, q \in G$ there is an $r \in G$ such that $r \leq p, q$.
  (ii) For all $p \in G$ and $q \in P$, if $p \leq q$ then $q \in G$.
- Elements $p, q \in P$ are compatible iff there is an $r \in P$ such that $r \leq p, q$. Otherwise they are incompatible, and we write $p \perp q$.
- A subset $A$ of $P$ is an antichain in $P$ iff any two distinct elements of $A$ are incompatible.

To become familiar with these notions we start with some simple examples and facts. If $P$ is a linear order, then a dense subset of $P$ is just a set which is cofinal in $(P, \geq)$. Actually, linear orders are not very interesting as partial orders to which to apply MA, as we will see. At the other extreme, if $P = (P, \emptyset)$, then the only dense subset of $P$ is $P$ itself. This is also not an interesting thing to apply MA to. A more interesting partial order is the following:

$$P = \{ f : f \text{ is a function, } \text{dom}(f) \subseteq \omega_1, \text{rng}(f) \subseteq 2, |f| < \omega \};$$

$$f \leq g \text{ iff } f, g \in P \text{ and } f \supseteq g.$$  

Here a subset $D$ is dense iff every $f \in P$ can be extended to some $g \in D$. A filter on $P$ is a collection of pairwise compatible functions closed under taking subsets. Two elements $f, g \in P$ are compatible iff $f \cup g$ is a function. An antichain consists of pairwise incompatible functions. Somewhat deeper is the fact that $P$ does have ccc. In fact, suppose that $\mathcal{F}$ is an uncountable subset of $P$. Then $\langle \text{dom}(f) : f \in \mathcal{F} \rangle$ is a system of finite sets.
By the $\Delta$-system lemma, there is an uncountable $\mathcal{F}' \subseteq \mathcal{F}$ such that $\langle \text{dom}(f) : f \in \mathcal{F}' \rangle$ is a $\Delta$-system, say with root $\Gamma$. Then

$$\mathcal{F}' = \bigcup_{g \in \Gamma^2} \{ f \in \mathcal{F}' : g \subseteq f \},$$

and the index set is finite, so there is a $g \in \Gamma^2$ such that $\mathcal{F}'' \overset{\text{def}}{=} \{ f \in \mathcal{F}' : g \subseteq f \}$ is uncountable. Clearly any two members of $\mathcal{F}''$ are compatible. Thus the original set $\mathcal{F}$ could not be an antichain. So, as claimed, $\mathbb{P}$ has ccc. This argument is rather typical of arguments showing ccc.

Clearly if $\kappa < \lambda$ and $\text{MA}(\lambda)$, then also $\text{MA}(\kappa)$.

**Theorem 11.1.** $\text{MA}(\omega)$ holds.

**Proof.** Let $\mathbb{P}$ be a ccc partial order and $\mathcal{D}$ a countable collection of dense sets in $\mathbb{P}$. If $\mathcal{D}$ is empty, we can fix any $p \in \mathbb{P}$ and let $G = \{ q \in \mathbb{P} : p \leq q \}$. Then $G$ is a filter on $\mathbb{P}$, which is all that is required in this case.

Now suppose that $\mathcal{D}$ is nonempty, and let $\langle D_n : n \in \omega \rangle$ enumerate all the members of $\mathcal{D}$; repetitions are needed if $\mathcal{D}$ is finite. We now define a sequence $\langle p_n : n \in \omega \rangle$ of elements of $P$ by recursion. Let $p_0$ be any element of $P$. If $p_n$ has been defined, by the denseness of $D_n$ let $p_{n+1}$ be such that $p_{n+1} \leq p_n$ and $p_{n+1} \in D_n$. This finishes the construction. Let $G = \{ q \in \mathbb{P} : p_n \leq q \text{ for some } n \in \omega \}$. Clearly $G$ is as desired. \qed

Note that ccc was not used in this proof.

**Corollary 11.2.** CH implies MA.

**Theorem 11.3.** $\text{MA}(2^\omega)$ does not hold.

**Proof.** Suppose that it does hold. Let

$$P = \{ f : f \text{ is a finite function with } \text{dom}(f) \subseteq \omega \text{ and } \text{rng}(f) \subseteq 2 \};$$

$$f \leq g \text{ iff } f, g \in P \text{ and } f \supseteq g;$$

$$\mathbb{P} = (P, \leq).$$

Then $\mathbb{P}$ has ccc, since $P$ itself is countable. Now for each $n \in \omega$ let

$$D_n = \{ f \in P : n \in \text{dom}(f) \}.$$

Each such set is dense in $\mathbb{P}$. For, if $g \in P$, either $g$ is already in $D_n$, or $n \notin \text{dom}(g)$, and then $g \cup \{(n, 0)\}$ is in $D_n$ and it is $\leq g$.

For each $h \in \omega^2$ let

$$E_h = \{ f \in P : \text{there is an } n \in \text{dom}(f) \text{ such that } f(n) \neq h(n) \}.$$

Again, each such set $E_h$ is dense in $\mathbb{P}$. For, let $f \in P$. If $f \not\subseteq h$, then already $f \in E_h$, so suppose that $f \subseteq h$. Take any $n \in \omega \setminus \text{dom}(f)$, and let $g = f \cup \{(n, 1 - h(n))\}$. Then $g \in E_h$ and $g \leq f$, as desired.

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So, by MA($2^\omega$) let $G$ be a filter on $\mathbb{P}$ which intersects each of the sets $D_n$ and $E_h$. Let $k = \bigcup G$.

(*) $k : \omega \to \omega$.

In fact, $k$ is obviously a relation. Suppose that $(m, \varepsilon), (m, \delta) \in k$. Choose $f, g \in G$ such that $(m, \varepsilon) \in f$ and $(m, \delta) \in g$. Then choose $s \in G$ such that $s \leq f, g$. So $f, g \subseteq s$, and $s$ is a function. It follows that $\varepsilon = \delta$. Thus $k$ is a function.

If $n \in \omega$, choose $f \in G \cap D_n$. So $n \in \text{dmn}(f)$, and so $n \in \text{dmn}(k)$. So we have proved (*).

Now take any $f \in G \cap E_k$. Choose $n \in \text{dmn}(f)$ such that $f(n) \neq k(n)$. But $f \subseteq k$, contradiction. \qed

There is one more fact concerning the definition of MA which should be mentioned. Namely, for $\kappa > \omega$ the assumption of ccc is essential in the statement of MA($\kappa$). (Recall our comment above that ccc is not needed in order to prove that MA(\omega) holds.) To see this, define

$$P = \{f : f \text{ is a finite function, } \text{dmn}(f) \subseteq \omega, \text{ and } \text{rng}(f) \subseteq \omega_1\};$$
$$f \leq g \iff f, g \in P \text{ and } f \supseteq g;$$
$$\mathbb{P} = (P, \leq).$$

This example is similar to two of the partial orders above. Note that $\mathbb{P}$ does not have ccc, since for example $\{(0, \alpha) : \alpha < \omega_1\}$ is an uncountable antichain. Defining $D_n$ as in the proof of Theorem 11.3, we clearly get dense subsets of $\mathbb{P}$. Also, for each $\alpha < \omega_1$ let

$$F_\alpha = \{f \in P : \alpha \in \text{rng}(f)\}.$$

Then $F_\alpha$ is dense in $\mathbb{P}$. For, suppose that $g \in P$. If $\alpha \in \text{rng}(g)$, then $g$ itself is in $F_\alpha$, so suppose that $\alpha \notin \text{rng}(g)$. Choose $n \in \omega \setminus \text{dmn}(g)$. Let $f = g \cup \{(n, \alpha)\}$. Then $f \in F_\alpha$ and $f \leq g$, as desired. Now if MA($\omega_1$) holds without the assumption of ccc, then we can apply it to our present partial order. Suppose that $G$ is a filter on $\mathbb{P}$ which intersects each of these sets $D_n$ and $F_\alpha$. As in the proof of Theorem 11.3, $k \overset{\text{def}}{=} \bigcup G$ is a function mapping $\omega$ into $\omega_1$. For any $\alpha < \omega_1$ choose $f \in G \cap F_\alpha$. Thus $\alpha \in \text{rng}(f)$, and so $\alpha \in \text{rng}(k)$. Thus $k$ has range $\omega_1$. This is impossible.

Now we proceed beyond the discussion of the definition of MA in order to give several typical applications of it. First we consider again almost disjoint sets of natural numbers. Our result here will be used to derive some important implications of MA for cardinal arithmetic. We proved in Theorem 8.1 that there is a family of size $2^\omega$ of almost disjoint sets of natural numbers. Considering this further, we may ask what the size of maximal almost disjoint families can be; and we may consider the least such size. This is one of many min-max questions concerning the natural numbers which have been considered recently. There are many consistency results saying that numbers of this sort can be less than $2^\omega$; in particular, it is consistent that there is a maximal family of almost disjoint subsets of $\omega$ which has size less than $2^\omega$. MA, however, implies that this size, and most of the similarly defined min-max functions, is $2^\omega$. 127
Let \( \mathcal{A} \subseteq \mathcal{P}(\omega) \). The almost disjoint partial order for \( \mathcal{A} \) is defined as follows:

\[
P_{\mathcal{A}} = \{(s, F) : s \in [\omega]^\omega \text{ and } F \in [\mathcal{A}]^\omega \};
\]

\((s', F') \leq (s, F)\) iff \( s \subseteq s' \), \( F \subseteq F' \), and \( x \cap s' \subseteq s \) for all \( x \in F \);

\( \mathbb{P}_{\mathcal{A}} = (P_{\mathcal{A}}, \leq) \).

We give some useful properties of this construction.

**Lemma 11.4.** Let \( \mathcal{A} \subseteq \mathcal{P}(\omega) \).

(i) \( \mathbb{P}_{\mathcal{A}} \) is a partial order.

(ii) Let \((s, F), (s', F') \in P_{\mathcal{A}}\). Then the following conditions are equivalent:

(a) \((s, F)\) and \((s', F')\) are compatible.

(b) \( \forall x \in F (x \cap s' \subseteq s) \) and \( \forall x \in F' (x \cap s \subseteq s') \).

(c) \( (s \cup s', F \cup F') \leq (s, F), (s', F') \).

(iii) Suppose that \( x \in \mathcal{A} \), and let \( D_x = \{(s, F) \in P_{\mathcal{A}} : x \in F\} \). Then \( D_x \) is dense in \( \mathbb{P}_{\mathcal{A}} \).

(iv) \( \mathbb{P}_{\mathcal{A}} \) has ccc.

**Proof.**

(i): Clearly \( \leq \) is reflexive on \( P_{\mathcal{A}} \) and it is antisymmetric, i.e. \((s, F) \leq (s', F') \leq (s, F)\) implies that \((s, F) = (s', F')\). Now suppose that \((s'', F'') \leq (s', F') \leq (s, F)\). Thus \( s \subseteq s' \subseteq s'' \), so \( s \subseteq s'' \). Similarly, \( F \subseteq F'' \). Now take any \( x \in F \). Then \( x \in F'' \), so \( x \cap s'' \subseteq s' \) because \((s'', F'') \leq (s', F')\). Hence \( x \cap s'' \subseteq x \cap s' \). And \( x \cap s' \subseteq s \) because \((s', F') \leq (s, F)\). So \( x \cap s'' \subseteq s \), as desired.

(ii): For (a)\(\Rightarrow\)(b), assume (a). Choose \((s'', F'') \leq (s, F), (s', F')\). Now take any \( x \in F \). Then \( x \cap s' \subseteq x \cap s'' \) since \( s' \subseteq s'' \), and \( x \cap s'' \subseteq s \) since \((s'', F'') \leq (s, F)\); so \( x \cap s' \subseteq s'' \).

The other part of (b) follows by symmetry.

(b)\(\Rightarrow\)(c): By symmetry it suffices to show that \((s \cup s', F \cup F') \leq (s, F)\), and for this we only need to check the last condition in the definition of \( \leq \). So, suppose that \( x \in F \). Then \( x \cap (s \cup s') = (x \cap s) \cup (x \cap s') \subseteq s \) by (b).

(c)\(\Rightarrow\)(a): Obvious.

(iii): For any \((s, F) \in P_{\mathcal{A}}\), clearly \((s, F \cup \{x\}) \leq (s, F)\).

(iv) Suppose that \( \langle (s_\xi, F_\xi) : \xi < \omega_1 \rangle \) is a pairwise incompatible system of elements of \( P_{\mathcal{A}} \). Clearly then \( s_\xi \neq s_\eta \) for distinct \( \xi, \eta < \omega_1 \), contradiction. \( \square \)

**Theorem 11.5.** Let \( \kappa \) be an infinite cardinal, and assume \( \text{MA}(\kappa) \). Suppose that \( \mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(\omega) \), and \( |\mathcal{A}|, |\mathcal{B}| \leq \kappa \). Also assume that

(i) For all \( y \in \mathcal{B} \) and all \( F \in [\mathcal{A}]^{<\omega} \) we have \( |y \setminus \bigcup F| = \omega \).

Then there is a \( d \subseteq \omega \) such that \( |d \cap x| < \omega \) for all \( x \in \mathcal{A} \) and \( |d \cap y| = \omega \) for all \( y \in \mathcal{B} \).

**Proof.** For each \( y \in \mathcal{B} \) and each \( n \in \omega \) let

\[
E_n^y = \{(s, F) \in \mathbb{P}_{\mathcal{A}} : s \cap y \not\subseteq n\}.
\]

We claim that each such set is dense. For, suppose that \((s, F) \in \mathbb{P}_{\mathcal{A}}\). Then by assumption, \(|y \setminus F| = \omega\), so we can pick \( m \in y \setminus \bigcup F \) such that \( m > n \). Then \((s \cup \{m\}, F) \leq (s, F)\),
since for each \( z \in F \) we have \( z \cap (s \cup \{m\}) \subseteq s \) because \( m \notin z \). Also, \( m \in y \setminus n \), so \((s \cup \{m\}) \in E_n^v\). This proves our claim.

There are clearly at most \( \kappa \) sets \( E_n^v \); and also there are at most \( \kappa \) sets \( D_x \) with \( x \in \mathcal{A} \), with \( D_x \) as in Lemma 11.4(iii). Hence by MA(\( \kappa \)) we can let \( G \) be a filter on \( \mathcal{P}_\mathcal{A} \) intersecting all of these dense sets. Let \( d = \bigcup_{(s,F) \in G} s \).

(1) For all \( x \in \mathcal{A} \), the set \( d \cap x \) is finite.

For, by the denseness of \( D_x \), choose \( (s,F) \in G \cap D_x \). Thus \( x \in F \). We claim that \( d \cap x \subseteq s \).

To prove this, suppose that \( n \in d \cap x \). Choose \( (s,F) \in G \) such that \( n \in s \). Now \((s,F) \) and \((s',F')\) are compatible. By Lemma 11.4(ii), \( \forall y \in F(y \cap s \subseteq s) \); in particular, \( x \cap s' \subseteq s \).

Since \( n \in x \cap s' \), we get \( n \in s \). This proves our claim, and so (1) holds.

The proof will be finished by proving

(2) For all \( y \in \mathcal{B} \), the set \( d \cap y \) is infinite.

To prove (2), given \( n \in \omega \) choose \( (s,F) \in E_n^y \cap G \). Thus \( s \cap y \nsubseteq n \), so we can choose \( m \in s \cap y \setminus n \). Hence \( m \in d \cap y \setminus n \), proving (2). \( \square \)

**Corollary 11.6.** Let \( \kappa \) be an infinite cardinal and assume MA(\( \kappa \)). Suppose that \( \mathcal{A} \subseteq \mathcal{P}(\omega) \) is an almost disjoint set of infinite subsets of \( \omega \) of size \( \kappa \). Then \( \mathcal{A} \) is not maximal.

**Proof.** If \( F \) is a finite subset of \( \mathcal{A} \), then we can choose \( a \in \mathcal{A} \setminus F \); then \( a \cap \bigcup F = \bigcap_{b \in F} (a \cap b) \) is finite. Thus \( \omega \setminus \bigcup F \) is infinite. Hence we can apply Theorem 11.5 to \( \mathcal{A} \) and \( \mathcal{B} = \{ \omega \} \) to obtain the desired result. \( \square \)

**Corollary 11.7.** Assuming MA, every maximal almost disjoint set of infinite sets of natural numbers has size \( 2^\omega \). \( \square \)

**Lemma 11.8.** Suppose that \( \mathcal{B} \subseteq \mathcal{P}(\omega) \) is an almost disjoint family of infinite sets, and \( |\mathcal{B}| = \kappa \), where \( \omega \leq \kappa < 2^\omega \). Also suppose that \( \mathcal{A} \subseteq \mathcal{B} \). Assume MA(\( \kappa \)).

Then there is a \( d \subseteq \omega \) such that \( |d \cap x| < \omega \) for all \( x \in \mathcal{A} \) and \( |d \cap x| = \omega \) for all \( x \in \mathcal{B} \setminus \mathcal{A} \).

**Proof.** We apply 11.5 with \( \mathcal{B} \setminus \mathcal{A} \) in place of \( \mathcal{B} \). If \( y \in \mathcal{B} \setminus \mathcal{A} \) and \( F \in [\mathcal{A}]^{<\omega} \), then \( y \cup F \subseteq \mathcal{B} \), and hence \( y \cap z \) is finite for all \( y \in F \). Hence also \( y \cap \bigcup F \) is finite. Since \( y \) itself is infinite, it follows that \( y \setminus \bigcup F \) is infinite.

Thus the hypotheses of 11.5 hold, and it then gives the desired result. \( \square \)

We now come to two of the most striking consequences of Martin’s axiom.

**Theorem 11.9.** If \( \kappa \) is an infinite cardinal and MA(\( \kappa \)) holds, then \( 2^\kappa = 2^\omega \).

**Proof.** By Theorem 10.1 let \( \mathcal{B} \) be an almost disjoint family of infinite subsets of \( \omega \) such that \( |\mathcal{B}| = \kappa \). For each \( d \subseteq \omega \) let \( F(d) = \{ b \in \mathcal{B} : |b \cap d| < \omega \} \). We claim that \( F \) maps \( \mathcal{P}(\omega) \) onto \( \mathcal{P}(\mathcal{B}) \); from this it follows that \( 2^\kappa \leq 2^\omega \), hence \( 2^\kappa = 2^\omega \). To prove the claim, suppose that \( \mathcal{A} \subseteq \mathcal{B} \). A suitable \( d \) with \( F(d) = \mathcal{A} \) is then given by Lemma 11.8. \( \square \)

**Corollary 11.10.** MA implies that \( 2^\omega \) is regular.
Proof. Assume MA, and suppose that $\omega \leq \kappa < 2^\omega$. Then $2^\kappa = 2^\omega$ by Theorem 11.9, and so $\text{cf}(2^\omega) = \text{cf}(2^\kappa) > \kappa$ by Corollary 4.411.

Another important application of Martin’s axiom is to the existence of Suslin trees; in fact, Martin’s axiom arose out of the proof of this theorem:

**Theorem 11.11.** MA($\omega_1$) implies that there are no Suslin trees.

**Proof.** Suppose that $(T, \leq)$ is a Suslin tree. By 8.7 and the remarks before it, we may assume that $T$ is well-pruned. We are going to apply MA($\omega_1$) to the partial order $(T, \geq)$, i.e., to $T$ turned upside down. Because $T$ has no uncountable antichains in the tree sense, $(T, \geq)$ has no uncountable antichains in the incompatibility sense. Now for each $\alpha < \omega_1$ let

$$D_\alpha = \{ t \in T : \text{ht}(t, T) > \alpha \}.$$  

Then each $D_\alpha$ is dense in $(T, \geq)$. For, suppose that $s \in T$. By well-prunedness, choose $t \in T$ such that $s < t$ and $\text{ht}(t, T) > \alpha$. Thus $t \in D_\alpha$ and $t > s$, as desired.

Now we let $G$ be a filter on $(T, \geq)$ which intersects each $D_\alpha$. Any two elements of $G$ are compatible in $(T, \geq)$, so they are comparable in $(T, \leq)$. Since $G \cap D_\alpha \neq \emptyset$ for all $\alpha < \omega_1$, $G$ has a member of $T$ of height greater than $\alpha$, for each $\alpha < \omega_1$. Hence $G$ is an uncountable chain, contradiction.

Our last application of Martin’s axiom involves Lebesgue measure. In order not to assume too much about measures, we give some results of measure theory that will be used in our application but may have been omitted in your standard study of measure theory.

**Lemma 11.12.** Suppose that $\mu$ is a measure and $E, F, G$ are $\mu$-measurable. Then

$$\mu(E \triangle F) \leq \mu(E \triangle G) + \mu(G \triangle F).$$

**Proof.**

$$\mu(E \triangle F) = \mu(E \setminus F) + \mu(F \setminus E)$$

$$= \mu((E \setminus F) \cap G) + \mu((E \setminus F) \setminus G) + \mu(F \setminus E) \cap G) + \mu((F \setminus E) \setminus G)$$

$$\leq \mu(G \setminus F) + \mu(E \setminus G) + \mu(G \setminus E) + \mu(F \setminus G)$$

$$= \mu(E \triangle G) + \mu(G \triangle F).$$

**Lemma 11.13.** If $E$ is Lebesgue measurable with finite measure, then for any $\varepsilon > 0$ there is an open set $U \supseteq E$ such that $\mu(E) \leq \mu(U) \leq \mu(E) + \varepsilon$. Moreover, there is a system $\langle K_j : j < \omega \rangle$ of open intervals such that $U = \bigcup_{j<\omega} K_j$ and $\mu(U) \leq \sum_{j<\omega} \mu(K_j) \leq \mu(E) + \varepsilon$.

**Proof.** By the basic definition of Lebesgue measure,

$$\mu(E) = \inf \left\{ \sum_{j \in \omega} \mu(I_j) : \langle I_j : j \in \omega \rangle \text{ is a sequence of half-open intervals such that } E \subseteq \bigcup_{j \in \omega} I_j \right\}.$$
Hence we can choose a sequence $\langle I_j : j \in \omega \rangle$ of half-open intervals such that $E \subseteq \bigcup_{j \in \omega} I_j$ and
\[
\mu \left( \bigcup_{j \in \omega} I_j \right) \leq \sum_{j \in \omega} \mu(I_j) \leq \mu(E) + \frac{\varepsilon}{2}.
\]
Write $I_j = [a_j, b_j)$ with $a_j < b_j$. Define
\[
K_j = \left( a_j - \frac{\varepsilon}{2j+2}, b_j \right); \quad \text{then}
\]
\[
E \subseteq \bigcup_{j \in \omega} K_j \quad \text{and}
\]
\[
\mu \left( \bigcup_{j \in \omega} K_j \right) \leq \sum_{j \in \omega} \mu(K_j) \leq \sum_{j \in \omega} \left( \frac{\varepsilon}{2j+2} + \mu(I_j) \right) \leq \frac{\varepsilon}{2} + \mu(E) + \frac{\varepsilon}{2} = \mu(E) + \varepsilon.
\]

The following is an elementary lemma concerning the topology of the reals.

**Lemma 11.14.** Suppose that $U$ is a bounded open set.

(i) There is a collection $\mathcal{A}$ of pairwise disjoint open intervals such that $U = \bigcup \mathcal{A}$.

(ii) There exist a countable subset $C$ of $\mathbb{R}$ and a collection $\mathcal{B}$ of pairwise disjoint open intervals with rational endpoints such that $U = C \cup \bigcup \mathcal{B}$ and $C \cap \bigcup \mathcal{B} = \emptyset$.

**Proof.** (i): For $x, y \in \mathbb{R}$, define $x \equiv y$ iff one of the following conditions holds: (1) $x = y$; (2) $x < y$ and $[x, y] \subseteq U$; (3) $y < x$ and $[y, x] \subseteq U$. Clearly $\equiv$ is an equivalence relation on $\mathbb{R}$. If $x < z \leq y$ and $x \equiv y$, then obviously $x \equiv z$. Thus each equivalence class is convex. If $C$ is an equivalence class with more than one element, then it must be an open interval $(a, b)$, since if for example the left endpoint $a$ is in $C$ then some real to the left of $a$ must be in $C$, contradiction. It follows now that the collection $\mathcal{A}$ of all equivalence classes with more than one element is as desired in (i).

(ii): First note that the set $\mathcal{A}$ of (i) must be countable. Now take any $(a, b) \in \mathcal{A}$, $a < b$. Let $c_0 < c_1 < \cdots < c_m < \cdots$ be rational numbers in $(a, b)$ which converge to $b$, and $d_0 = d_1 > \cdots > d_m > \cdots$ rational numbers which converge to $a$. Then let $L_{2i}^{ab} = (c_i, c_{i+1})$ and $L_{2i+1}^{ab} = (d_{i+1}, d_i)$ for all $i \in \omega$. Let $D^{ab} = \{c_i : i < \omega\} \cup \{d_i : i < \omega\}$. Define $\mathcal{B} = \{L_i^{ab} : (a, b) \in \mathcal{A}, i < \omega\}$ and $C = \bigcup_{(a, b) \in \mathcal{A}} D^{ab}$. Clearly this works for (ii).

**Lemma 11.15.** If $E$ is Lebesgue measurable and $\varepsilon > 0$, then there is an $m \in \omega$ and a sequence $\langle I_i : i < m \rangle$ of open intervals with rational endpoints such that $\mu \left( E \triangle \bigcup_{i < m} I_i \right) \leq \varepsilon$. 131
Proof. By Lemma 11.13 let $U \supseteq E$ be open such that $\mu(E) \leq \mu(U) \leq \mu(E) + \frac{\varepsilon}{2}$. Then choose $C$ and $\mathcal{B}$ as in Lemma 11.14(ii). Let $W = \bigcup \mathcal{B}$. So $\mu(W) = \sum_{I \in \mathcal{B}} \mu(I)$. Then choose $m \in \omega$ and $\langle I_i : i < m \rangle$ elements of $\mathcal{B}$ such that $\sum_{I \in \mathcal{B}} \mu(I) - \sum_{i < m} \mu(I_i) \leq \frac{\varepsilon}{2}$. Now $\mu(W) = \sum_{I \in \mathcal{B}} \mu(I)$ and $\mu(\bigcup_{i < m} I_i) = \sum_{i < m} \mu(I_i)$. Let $V = \bigcup_{i < m} I_i$. Thus $\mu(W) - \mu(V) \leq \frac{\varepsilon}{2}$. Hence $V \subseteq W \subseteq U$, and

$$\mu(E \triangle V) \leq \mu(E \triangle U) + \mu(U \triangle W) + \mu(W \triangle V)$$

$$= \mu(U \setminus E) + \mu(C) + \mu(W \setminus V)$$

$$= \mu(U) - \mu(E) + \mu(W) - \mu(V)$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Now we are ready for an application of Martin’s axiom to Lebesgue measure.

Theorem 11.16. Suppose that $\kappa$ is an infinite cardinal and $\text{MA}(\kappa)$ holds. If $\langle M_\alpha : \alpha < \kappa \rangle$ is a system of subsets of $\mathbb{R}$ each of Lebesgue measure $0$, then also $\bigcup_{\alpha < \kappa} M_\alpha$ has Lebesgue measure $0$.

Proof. Let $\varepsilon > 0$. We are going to find an open set $U$ such that $\bigcup_{\alpha < \kappa} M_\alpha \subseteq U$ and $\mu(U) \leq \varepsilon$; this will prove our result. Let

$$\mathbb{P} = \{ p \subseteq \mathbb{R} : p \text{ is open and } \mu(p) < \varepsilon \}.$$ 

The ordering, as usual, is $\supseteq$.

(1) Elements $p, q \in \mathbb{P}$ are compatible iff $\mu(p \cup q) < \varepsilon$.

In fact, the direction $\Leftarrow$ is clear, while if $p$ and $q$ are compatible, then there is an $r \in \mathbb{P}$ with $r \supseteq p, q$, hence $p \cup q \subseteq r$ and $\mu(r) < \varepsilon$, hence $\mu(p \cup q) < \varepsilon$.

Next we check that $\mathbb{P}$ has ccc. Suppose that $\langle p_\alpha : \alpha < \omega_1 \rangle$ is a system of pairwise incompatible elements of $\mathbb{P}$. Now

$$\omega_1 = \bigcup_{n \in \omega} \left\{ \alpha < \omega_1 : \mu(p_\alpha) \leq \varepsilon - \frac{1}{n+1} \right\},$$

so there exist an uncountable $\Gamma \subseteq \omega_1$ and a positive integer $m$ such that $\mu(p_\alpha) \leq \varepsilon - \frac{1}{m}$ for all $\alpha \in \Gamma$. Let $\mathcal{C}$ be the collection of all finite unions of open intervals with rational coefficients. Note that $\mathcal{C}$ is countable. By Lemma 11.15, for each $\alpha \in \Gamma$ let $C_\alpha$ be a member of $\mathcal{C}$ such that $\mu(p_\alpha \triangle C_\alpha) \leq \frac{1}{3m}$. Now take any two distinct members $\alpha, \beta \in \Gamma$. Then

$$\varepsilon \leq \mu(p_\alpha \cup p_\beta) = \mu(p_\alpha \cap p_\beta) + \mu(p_\alpha \Delta p_\beta) \leq \varepsilon - \frac{1}{m} + \mu(p_\alpha \Delta p_\beta),$$

and hence $\mu(p_\alpha \Delta p_\beta) \geq \frac{1}{m}$. Thus, using Lemma 11.12,

$$\frac{1}{m} \leq \mu(p_\alpha \Delta p_\beta) \leq \mu(p_\alpha \Delta C_\alpha) + \mu(C_\alpha \Delta C_\beta) + \mu(C_\beta \Delta p_\beta) \leq \frac{1}{3m} + \mu(C_\alpha \Delta C_\beta) + \frac{1}{3m};$$

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Hence \(\mu(C_\alpha \triangle C_\beta) \geq \frac{1}{3m}\). It follows that \(C_\alpha \neq C_\beta\). But this means that \(\langle C_\alpha : \alpha \in \Gamma \rangle\) is a one-one system of members of \(\mathcal{C}\), contradiction. So \(\mathbb{P}\) has ccc.

Now for each \(\alpha < \kappa\) let

\[ D_\alpha = \{ p \in \mathbb{P} : M_\alpha \subseteq p \}. \]

To show that \(D_\alpha\) is dense, take any \(p \in \mathbb{P}\). Thus \(\mu(p) < \varepsilon\). By Lemma 11.13, let \(U\) be an open set such that \(M_\alpha \subseteq U\) and \(\mu(U) < \varepsilon - \mu(p)\). Then \(\mu(p \cup U) \leq \mu(p) + \mu(U) < \varepsilon\); so \(p \cup U \in D_\alpha\) and \(p \cup U \supseteq p\), as desired.

Now let \(G\) be a filter on \(\mathbb{P}\) which intersects each \(D_\alpha\). Set \(V = \bigcup G\). So \(V\) is an open set. For each \(\alpha < \kappa\), choose \(p_\alpha \in G \cap D_\alpha\). Then \(M_\alpha \subseteq p_\alpha \subseteq V\). It remains only to show that \(\mu(V) \leq \varepsilon\). Let \(\mathcal{B}\) be the set of all open intervals with rational endpoints. We claim that \(V = \bigcup (G \cap \mathcal{B})\). In fact, \(\supseteq\) is clear, so suppose that \(x \in V\). Then \(x \in p\) for some \(p \in G\), hence there is a \(U \in \mathcal{B}\) such that \(x \in U \subseteq p\), since \(p\) is open. Then \(U \in G\) since \(G\) is a filter and the partial order is \(\supseteq\). So we found a \(U \in G \cap \mathcal{B}\) such that \(x \in U\); hence \(x \in \bigcup (G \cap \mathcal{B})\). This proves our claim. Now if \(F\) is a finite subset of \(G\), then \(\bigcup F \in G\) since \(G\) is a filter. In particular, \(\bigcup F \in \mathbb{P}\), so its measure is less than \(\varepsilon\). Now \(G \cap \mathcal{B}\) is countable; let \(\langle p_i : i \in \omega \rangle\) enumerate it. Define \(q_i = p_i \setminus \bigcup_{j < i} p_j\) for all \(i \in \omega\). Then by induction one sees that \(\bigcup_{i < m} p_i = \bigcup_{i < m} q_i\), and hence \(\bigcup (G \cap \mathcal{B}) = \bigcup_{i \in \omega} q_i\). So

\[
\mu(V) = \mu \left( \bigcup (G \cap \mathcal{B}) \right) = \mu \left( \bigcup_{i < \omega} q_i \right)
= \sum_{i < \omega} \mu(q_i) = \lim_{m \to \infty} \sum_{i < m} \mu(q_i) = \lim_{m \to \infty} \mu \left( \bigcup_{i < m} q_i \right) = \lim_{m \to \infty} \mu \left( \bigcup_{i < m} p_i \right) \leq \varepsilon. \quad \square
\]

**EXERCISES**

11.1. Assume MA(\(\kappa\)). Suppose that \(X\) is a compact Hausdorff space, and any pairwise disjoint collection of open sets in \(X\) is countable. Suppose that \(U_\alpha\) is dense open in \(X\) for each \(\alpha < \kappa\). Show that \(\bigcap_{\alpha < \kappa} U_\alpha \neq \emptyset\).

11.2. A partial order \(P\) is said to have \(\omega_1\) as a precaliber iff for every system \(\langle p_\alpha : \alpha < \omega_1 \rangle\) of elements of \(P\) there is an \(X \in [\omega_1]^{\omega_1}\) such that for every finite subset \(F\) of \(X\) there is a \(q \in P\) such that \(q \leq p_\alpha\) for all \(\alpha \in F\).

Show that MA(\(\omega_1\)) implies that every ccc partial order \(P\) has \(\omega_1\) as a precaliber.

Hint: for each \(\alpha < \omega_1\) let

\[ W_\alpha = \{ q \in P : \exists \beta > \alpha(q \text{ and } p_\alpha \text{ are compatible}) \}. \]

Show that there is an \(\alpha < \omega_1\) such that \(W_\alpha = W_\beta\) for all \(\beta > \alpha\), and apply MA(\(\omega_1\)) to \(W_\alpha\).

11.3. Call a topological space \(X\) ccc iff every collection of pairwise disjoint open sets in \(X\) is countable. Show that \(\prod_{i \in I} X_i\) is ccc iff \(\forall F \in [I]^{<\omega} [\prod_{i \in F} X_i\text{ is ccc}]\). Hint: use the \(\Delta\)-system theorem.
11.4 Assuming MA(\(\omega_1\)), show that any product of ccc spaces is ccc.

11.5. Assume MA(\(\omega_1\)). Suppose that \(P\) and \(Q\) are ccc partially ordered sets. Define \(\leq\) on \(P \times Q\) by setting \((a, b) \leq (c, d)\) iff \(a \leq c\) and \(b \leq d\). Show that \(<\) is a ccc partial order on \(P \times Q\). Hint: use exercise 11.1.

11.6. We define \(<^*\) on \(\omega\) by setting \(f \ <^*\) \(g\) iff \(f, g \in \omega\) and \(\exists n \forall m > n(f(m) < g(m))\). Suppose that MA(\(\kappa\)) holds and \(\mathcal{F} \in [\omega]^\kappa\). Show that there is a \(q \in \omega\) such that \(f <^*\) \(g\) for all \(f \in \mathcal{F}\). Hint: let \(P\) be the set of all pairs \((p, F)\) such that \(p\) is a finite function mapping a subset of \(\omega\) into \(\omega\) and \(F\) is a finite subset of \(\mathcal{F}\). Define \((p, F) \leq (q, G)\) iff \(q \subseteq p, G \subseteq F,\) and
\[
\forall f \in G \forall n \in \text{dmn}(p) \setminus \text{dmn}(q)[p(n) > f(n)].
\]

11.7. Let \(\mathcal{B} \subseteq [\omega]^\omega\) be almost disjoint of size \(\kappa\), with \(\omega \leq \kappa < 2^\omega\). Let \(\mathcal{A} \subseteq \mathcal{B}\) with \(\mathcal{A}\) countable. Assume MA(\(\kappa\)). Show that there is a \(d \subseteq \omega\) such that \(|d \cap x| < \omega\) for all \(x \in \mathcal{A}\), and \(|x \setminus d| < \omega\) for all \(x \in \mathcal{B}\setminus \mathcal{A}\). Hint: Let \(|\mathcal{D}| = \omega\) enumerate \(\mathcal{A}\). Let \(\mathbb{P} = \{(s, F, m) : s \in [\omega]^{<\omega}, F \in [\mathcal{B}\setminus \mathcal{A}]^{<\omega},\) and \(m \in \omega\}\);
\[
(s', F', m') \leq (s, F, m)\] if \(s \subseteq s', F \subseteq F', m \leq m',\) and
\[
\forall x \in F \left[\left(x \setminus \bigcup_{i \in m} a_i\right) \cap s' \subseteq s\right].
\]

Show that \(\mathbb{P}\) satisfies ccc. To apply MA(\(\kappa\)), one needs various dense sets. The most complicated is defined as follows. Let \(\mathcal{D} = \{(s, F, m, i, n) : (s, F, m) \in \mathbb{P}, i < m,\) and \(n \in a_i \setminus s\}\). Clearly \(|\mathcal{D}| = \kappa\). For each \((s, F, m, i, n) \in \mathcal{D}\) let
\[
E_{(s, F, m, i, n)} = \{(s', F', m') \in \mathbb{P} : (s, F, m)\) and \((s', F', m')\) are incompatible or \((s', F', m') \leq (s, F, m)\) and \(n \in s'\}\}.
\]

11.8. [The condition that \(\mathcal{A}\) is countable is needed in exercise 7.] Show that there exist \(\mathcal{A}, \mathcal{B}\) such that \(\mathcal{B}\) is an almost disjoint family of infinite subsets of \(\omega\), \(\mathcal{A} \subseteq \mathcal{B}\), \(|\mathcal{A}| = |\mathcal{B}\setminus \mathcal{A}| = \omega_1\), and there does not exist a \(d \subseteq \omega\) such that \(|d \cap x| < \omega\) for all \(x \in \mathcal{A}\), and \(|x \setminus d| < \omega\) for all \(x \in \mathcal{B}\setminus \mathcal{A}\). Hint: construct \(\mathcal{A} = \{a_\alpha : \alpha < \omega_1\}\) and \(\mathcal{B}\setminus \mathcal{A} = \{b_\alpha : \alpha < \omega_1\}\) by constructing \(a_\alpha, b_\alpha\) inductively, making sure that the elements are infinite and pairwise almost disjoint, and also \(a_\alpha \cap b_\beta \neq \emptyset\), while for \(\alpha \neq \beta\) we have \(a_\alpha \cap b_\beta \neq \emptyset\).

11.9. Suppose that \(\mathcal{A}\) is a family of infinite subsets of \(\omega\) such that \(\bigcap F\) is infinite for every finite subset \(F\) of \(\mathcal{A}\). Suppose that \(|\mathcal{A}| \leq \kappa\). Assuming MA(\(\kappa\)), show that there is an infinite \(X \subseteq \omega\) such that \(X \setminus A\) is finite for every \(A \in \mathcal{A}\). Hint: use Theorem 11.5.

11.10. Show that MA(\(\kappa\)) is equivalent to MA(\(\kappa\)) restricted to ccc partial orders of cardinality \(\leq \kappa\). Hint: Assume the indicated special form of MA(\(\kappa\)), and assume given a ccc partially ordered set \(P\) and a family \(\mathcal{D}\) of at most \(\kappa\) dense sets in \(P\); we want to find a filter on \(P\) intersecting each member of \(\mathcal{D}\). We introduce some operations on \(P\). For each \(D \in \mathcal{D}\) define \(f_D : P \rightarrow P\) by setting, for each \(p \in P\), \(f_D(p)\) to be some element of \(D\) which is \(\leq p\). Also we define \(g : P \times P \rightarrow P\) by setting, for all \(p, q \in P\),
\[
g(p, q) = \begin{cases} p & \text{if } p \text{ and } q \text{ are incompatible}, \\ r & \text{with } r \leq p, q \text{ if there is such an } r. \end{cases}
\]

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Here, as in the definition of $f_D$, we are implicitly using the axiom of choice; for $g$, we choose any $r$ of the indicated form.

We may assume that $\mathcal{P} \neq \emptyset$. Choose $D \in \mathcal{P}$, and choose $s \in D$. Now let $Q$ be the intersection of all subsets of $P$ which have $s$ as a member and are closed under all of the operations $f_D$ and $g$. We take the order on $Q$ to be the order induced from $P$. Apply the special form to $Q$.

11.11. Define $x \subset^* y$ iff $x, y \subseteq \omega$, $x \backslash y$ is finite, and $y \backslash x$ is infinite. Assume $\text{MA}(\kappa)$, and suppose that $L, < \subseteq Q$ is a linear ordering of size at most $\kappa$. Show that there is a system $(a_x : x \in L)$ of infinite subsets of $\omega$ such that for all $x, y \in L$, $x < y$ iff $a_x \subset^* a_y$. Hint: let $P$ consist of all pairs $(p, n)$ such that $n \in \omega$, $p$ is a function whose domain is a finite subset of $L$, and $\forall x \in \text{dmn}(p)[p(x) \subseteq n]$. Define $(p, n) \leq (q, m)$ iff $m \leq n$, $\text{dmn}(q) \subseteq \text{dmn}(p)$, $\forall x \in \text{dmn}(q)[p(x) \cap m = q(x)]$, and $\forall x, y \in \text{dmn}(q)[x < y \rightarrow p(x) \setminus p(y) \subseteq m]$.

For the remaining exercises we use the following definitions.

$$a \subseteq^* b \iff a \backslash b \text{ is finite;}$$

$$a \subset^* b \iff a \subseteq^* b \text{ and } b \backslash a \text{ is infinite.}$$

11.12. If $\mathcal{A}, \mathcal{B}$ are nonempty countable subsets of $[\omega]^\omega$ and $a \subseteq^* b$ whenever $a \in \mathcal{A}$ and $b \in \mathcal{B}$, then there is a $c \in [\omega]^\omega$ such that $a \subseteq^* c \subseteq^* b$ whenever $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

11.13. Suppose that $\mathcal{A}$ is a nonempty countable family of members of $[\omega]^\omega$, and $\forall a, b \in \mathcal{A}[a \subseteq^* b \text{ or } b \subseteq^* a]$. Also suppose that $\forall a \in \mathcal{A}[a \subset^* d]$, where $d \in [\omega]^\omega$. Then there is a $c \in [\omega]^\omega$ such that $\forall a \in \mathcal{A}[a \subseteq^* c \subset^* d]$.

11.14. If $a, b \in [\omega]^\omega$ and $a \subset^* b$, then there is a $c \in [\omega]^\omega$ such that $a \subset^* c \subset^* b$.

11.15. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are nonempty countable subsets of $[\omega]^\omega$, $\forall x, y \in \mathcal{A}[x \subseteq^* y \text{ or } y \subseteq^* x]$, $\forall x, y \in \mathcal{B}[x \subseteq^* y \text{ or } y \subseteq^* x]$, and $\forall x \in \mathcal{A}\forall y \in \mathcal{B}[a \subset^* b]$. Then there is a $c \in [\omega]^\omega$ such that $a \subset^* c \subset^* b$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

Now we need some more terminology. Let $\mathcal{A} \subseteq [\omega]^\omega$, $b \in [\omega]^\omega$, and $\forall a \in \mathcal{A}[a \subset^* b]$. We say that $b$ is near to $\mathcal{A}$ iff for all $m \in \omega$ the set $\{a \in \mathcal{A} : a \backslash b \subseteq m\}$ is finite.

11.16. Suppose that $a_m \in [\omega]^\omega$ for all $m \in \omega$, $a_m \subset^* a_n$ whenever $m < n \in \omega$, $b \in [\omega]^\omega$, and $a_m \subset^* b$ for all $m \in \omega$. Then there is a $c \in [\omega]^\omega$ such that $\forall m \in \omega[a_m \subset^* c \subset^* b]$ and $c$ is near to $\{a_n : n \in \omega\}$.

11.17. Suppose that $\mathcal{A} \subseteq [\omega]^\omega$, $\forall x, y \in \mathcal{A}[x \subset^* y \text{ or } y \subset^* x]$, $b \in [\omega]^\omega$, $\forall x \in \mathcal{A}[x \subset^* b]$, and $\forall a \in \mathcal{A}[b$ is near to $\{d \in \mathcal{A} : d \subset^* a\}]$.

Then there is a $c \in [\omega]^\omega$ such that $\forall a \in \mathcal{A}[a \subset^* c \subset^* b]$ and $c$ is near to $\mathcal{A}$.

11.18. (The Hausdorff gap) There exist sequences $(a_\alpha : \alpha < \omega_1)$ and $(b_\alpha : \alpha < \omega_1)$ of members of $[\omega]^\omega$ such that $\forall \alpha, \beta < \omega_1[\alpha < \beta \rightarrow a_\alpha \subset^* a_\beta \text{ and } b_\beta \subset^* b_\alpha]$, $\forall \alpha, \beta < \omega_1[a_\alpha \subset^* b_\beta]$, and there does not exist a $c \subseteq \omega$ such that $\forall \alpha < \omega_1[a_\alpha \subset^* c \text{ and } c \subset^* b_\alpha]$.

**Reference**

Fremlin, D. *Consequences of Martin’s axiom*. Cambridge Univ. Press, 325pp.