Countable models

A structure $\overline{M}$ is atomic iff $\text{tp}(\overline{M}(\bar{a}))$ is isolated, for all positive integers $m$ and all $\bar{a} \in {}^m M$.

**Theorem 10.1.** Let $\mathcal{L}$ be a countable language, and $T$ a complete theory in $\mathcal{L}$ with infinite models. Let $\overline{M}$ be a model of $T$. Then $\overline{M}$ is prime iff it is countable and atomic.

**Proof.** $\Rightarrow$: Assume that $\overline{M}$ is prime. If $t$ is a type of $T$ which is not isolated, then by the omitting types theorem 6.26, $T$ has a model $\overline{N}$ which omits $t$. Since $\overline{M}$ can be elementarily embedded in $\overline{N}$, it follows that $\overline{M}$ also omits $t$. Thus for any type $t$ of $T$, $t$ isolated $\Rightarrow T$ omits $t$. Hence any type which $\overline{M}$ realizes is isolated; this means that $\overline{M}$ is atomic. Since $T$ has countable models by the downward Löwenheim-Skolem theorem, and $\overline{M}$ can be elementarily embedded in any model of $T$, $\overline{M}$ is countable.

$\Leftarrow$: Suppose that $\overline{M}$ is countable and atomic, and $\overline{N}$ is any model of $T$; we want to construct an elementary embedding of $\overline{M}$ into $\overline{N}$. Let $(a_i : i \in \omega)$ enumerate the elements of $M$, and for each $i \in \omega$ let $\theta_i(\bar{v})$ isolate $\text{tp}(a_0, \ldots, a_i)$. We will now construct elementary maps $f_0 \subseteq f_1 \subseteq \ldots$ from subsets of $M$ into $N$, where the domain of $f_i$ is $\{a_0, \ldots, a_i\}$. Let $f_0 = \emptyset$. It is elementary since $\overline{M} \equiv \overline{N}$ (because $T$ is complete). Suppose that $f_i$ has been constructed, an elementary map. Then $\overline{M} \models \theta_i(a_0, \ldots, a_i)$, hence $\overline{M} \models \exists v \theta_i(a_0, \ldots, a_i, v)$. Since $f_i$ is an elementary map, it follows that $\overline{N} \models \exists v \theta_i(f_i(a_0), \ldots, f_i(a_i), v)$. Choose $b \in N$ such that $\overline{N} \models \theta_i(f_i(a_0), \ldots, f_i(a_i), b)$, and let $f_{i+1} = f_i \cup \{(i, b)\}$. To see that $f_{i+1}$ is elementary, suppose that $\overline{M} \models \psi(a_0, \ldots, a_i)$. Thus $\psi(\bar{v}) \in \text{tp}(a_0, \ldots, a_i)$, so $T \models \theta_i \rightarrow \psi$. Since $\overline{N} \models \theta_i(f_i(a_0), \ldots, f_i(a_i), b)$, it follows that $\overline{N} \models \psi(f_i(a_0), \ldots, f_i(a_i), b)$. Thus $f_{i+1}$ is elementary.

Now $\bigcup_{i \in \omega} f_i$ is an elementary, as desired. $\Box$

**Corollary 10.2.** If $\mathcal{L}$ is a countable language and $T$ is a complete theory with infinite models, then $T$ has a prime model iff $T$ has an atomic model.

**Proof.** $\Rightarrow$: by Theorem 10.1.

$\Leftarrow$: Suppose that $\overline{M}$ is an atomic model of $T$. Let $\overline{N}$ be a countable elementary substructure of $\overline{M}$. Then for any $n \in \omega$ and any $\bar{a} \in {}^n N$ we have $\text{tp}(\overline{N}(\bar{a})) = \text{tp}(\overline{M}(\bar{a}))$, and hence $\text{tp}(\overline{N}(\bar{a}))$ is isolated. So $\overline{N}$ is prime by Theorem 10.1. $\Box$

**Proposition 10.3.** If $\overline{M}$ is an atomic model of $T$, then it is $\omega$-homogeneous.

**Proof.** See just before Lemma 7.28 for the definition of $\omega$-homogeneous. Suppose that $\mathcal{A}$ is a finite subset of $M$ and $f : \mathcal{A} \rightarrow \overline{M}$ is a partial elementary map. Let $\bar{a}$ enumerate $\mathcal{A}$. Let $b \in M$. Let $\varphi(\bar{v}, w)$ isolate the type $\text{tp}(\overline{M}(\bar{a}, b))$. Then $\overline{M} \models \exists w \varphi(\bar{a}, w)$. So, since $f$ is partial elementary, we get $\overline{M} \models \exists w \varphi(f \circ \bar{a}, w)$. Choose $d \in M$ such that $\overline{M} \models \varphi(f \circ \bar{a}, d)$. Let $g = f \cup \{(b, d)\}$. To see that $g$ is partial elementary, suppose that $\overline{M} \models \psi(\bar{a}, b)$. Then $T \models \varphi \rightarrow \psi$ and $\overline{M} \models \varphi(f \circ \bar{a}, d)$, so $\overline{M} \models \psi(f \circ \bar{a}, d)$, as desired. $\Box$

**Corollary 10.4.** If $T$ is a complete theory in a countable language, then any two prime models of $T$ are isomorphic.

**Proof.** Let $\overline{M}$ and $\overline{N}$ be prime models of $T$. Then by Theorem 10.1 they are both countable and atomic. By Proposition 10.3 they are also both $\omega$-homogeneous. Now every
type realized in $\mathcal{M}$ is isolated. Also, if $t$ is an isolated type, say isolated by $\varphi(\overline{v})$, then $T \models \exists \overline{v} \varphi(\overline{v})$, hence $\mathcal{M}$ realizes $t$. So a type is isolated iff it is realized in $\mathcal{M}$. The same is true of $\mathcal{N}$, so $\mathcal{M}$ and $\mathcal{N}$ realize the same types. Hence they are isomorphic by Theorem 7.31. \qed

**Theorem 10.5.** If $\mathcal{M}$ is $\kappa$-saturated, then $\mathcal{M}$ is $\kappa$-homogeneous.

**Proof.** Suppose that $A \subseteq [M]^{<\kappa}$, $f : A \rightarrow M$ is partial elementary, and $b \in M \setminus A$. Let

$$\Gamma = \{ \varphi(v, f \circ \overline{a}) : \exists m \in \omega[\overline{a} \in {}^m A \text{ and } \mathcal{M} \models \varphi(b, \overline{a})] \}.$$ 

Let $\Delta$ be a finite subset of $\Gamma$. For each member $\chi$ of $\Delta$, choose $m_\chi \in \omega$, $\varphi_\chi$, and $\overline{a}_\chi \in {}^m A$ such that $\mathcal{M} \models \varphi_\chi(b, \overline{a}_\chi)$ and $\chi = \varphi_\chi(v, f \circ \overline{a}_\chi)$. Then there is an $n \in \omega$, $\overline{a}' \in {}^n A$, and a formula $\psi$ such that the following conditions hold:

1. $\mathcal{M} \models \psi(b, \overline{a}')$.
2. $\models \bigwedge \Delta \leftrightarrow \psi(v, f \circ \overline{a}')$.

Now from (1) we get $\mathcal{M} \models \exists v \psi(v, \overline{a}')$. Hence, since $f$ is elementary, $\mathcal{M} \models \exists v \psi(v, f \circ \overline{a}')$. Hence by (2), $\mathcal{M} \models \exists v \bigwedge \Delta$. Thus $\Gamma$ is finitely satisfiable. So since $\mathcal{M}$ is $\omega$-saturated we get $c \in M$ such that $\mathcal{M} \models \varphi(c, f \circ \overline{a}_\chi)$ for each $\varphi(v, f \circ \overline{a}_\chi) \in \Gamma$. So $f \cup \{(b, c)\}$ is elementary. \qed

**Theorem 10.6.** Suppose that $\mathcal{M}$ is a model of $T$. Then $\mathcal{M}$ is $\omega$-saturated iff $\mathcal{M}$ is $\omega$-homogeneous and for every $m \in \omega$, $\mathcal{M}$ realizes all types in $S_m(T)$.

**Proof.** $\Rightarrow$: by Theorem 10.5.

$\Leftarrow$: Let $m, n \in \omega$, $\overline{a} \in {}^m M$, and $p \in S_n^M(\overline{a})$. Let $q \in S_{m+n}(T)$ be the type $\{ \varphi(\overline{v}, \overline{w}) : \varphi(\overline{v}, \overline{w}) \in p \}$. Since $\mathcal{M}$ realizes all types in $S_{m+n}(T)$, choose $(\overline{b}, \overline{c})$ realizing $q$. Now

(*) $\text{tp}^\mathcal{M}(\overline{a}) = \text{tp}^\mathcal{M}(\overline{c})$.

In fact, let $\psi(\overline{w}) \in \text{tp}^\mathcal{M}(\overline{a})$. Let $\psi'$ be $\overline{v} = \overline{v} \land \psi$. Then $\psi'(\overline{v}, \overline{a}) \in p$, since otherwise $\neg \psi'(\overline{v}, \overline{a}) \in p$ and hence, since every finite subset of $p$ is satisfiable in $\mathcal{M}$, we get $\overline{v}$ such that $\mathcal{M} \models \neg \psi'(\overline{v}, \overline{a})$, i.e., $\mathcal{M} \models \neg \psi(\overline{a})$, contradiction. So $\psi'(\overline{v}, \overline{a}) \in p$, hence $\psi'(\overline{v}, \overline{w}) \in q$, hence $\mathcal{M} \models \psi'(\overline{c}, \overline{a})$, so $\mathcal{M} \models \psi(\overline{v})$. This proves (*).

Now by $\omega$-homogeneity we get $\overline{d}$ such that $\text{tp}^\mathcal{M}(\overline{b}, \overline{c}) = \text{tp}^\mathcal{M}(\overline{d}, \overline{c})$. Thus for any $\varphi(\overline{v}, \overline{a}) \in p$ we have $\varphi(\overline{v}, \overline{w}) \in q$, hence $\mathcal{M} \models \psi(\overline{c}, \overline{a})$, so $\varphi(\overline{v}, \overline{w}) \in \text{tp}^\mathcal{M}(\overline{b}, \overline{c})$, hence $\varphi(\overline{v}, \overline{w}) \in \text{tp}^\mathcal{M}(\overline{d}, \overline{c})$, so $\mathcal{M} \models \varphi(\overline{v}, \overline{w})$. This shows that $\mathcal{M}$ realizes $p$. \qed

**Corollary 10.7.** If $\mathcal{M}$ and $\mathcal{N}$ are countable saturated models of $T$, then $\mathcal{M} \cong \mathcal{N}$.

**Proof.** By Theorem 10.6, both $\mathcal{M}$ and $\mathcal{N}$ are $\omega$-homogeneous and realize all types in any $S_m(T)$. By Theorem 7.31 they are isomorphic. \qed

Now we need some variants of 7.38–7.39.

**Lemma 10.8.** Suppose that $\mathcal{M}$ is a model of $T$ and $\overline{b}, c \in M$, $\text{tp}^\mathcal{M}(\overline{a}) = \text{tp}^\mathcal{M}(\overline{b})$. Then there exist an elementary extension $\mathcal{N}$ of $\mathcal{M}$ such that $|M| = |N|$ and an element $d \in N$ such that $\text{tp}^\mathcal{N}(\overline{c}, c) = \text{tp}^\mathcal{N}(\overline{b}, d)$.  

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Proof. Apply the compactness theorem to the set
\[ \text{Eldiag}(\mathcal{M}) \cup \{ \varphi(\overline{b}, u) : \mathcal{M} \models \varphi(\overline{a}, c) \} \quad (u \text{ a new constant}) \]

**Theorem 10.9.** Suppose that \( \mathcal{M} \) is a model of \( T \). Then there exists an elementary extension \( \mathcal{N} \) of \( \mathcal{M} \) such that \( |\mathcal{M}| = |\mathcal{N}| \) and for all \( \overline{a}, \overline{b}, c \in M \) there is a \( d \in N \) such that \( \text{tp}^\mathcal{N}(\overline{a}, c) = \text{tp}^\mathcal{N}(\overline{b}, d) \).

**Proof.** Iterate Lemma 10.8.

**Theorem 10.11.** \( T \) has a countable saturated model iff \( |S_n(T)| \leq \aleph_0 \) for all \( n \).

**Proof.** \( \Rightarrow \): If \( \mathcal{M} \) is a countable saturated model, then it realizes only countably many types; but it realizes all types, so \( |S_n(T)| \leq \aleph_0 \) for all \( n \).

\( \Leftarrow \): Let \( t_0, t_1, \ldots \) list all members of \( \bigcup_{n \in \omega} S_n(T) \). By the compactness theorem, for any countable model \( \mathcal{N} \) of \( T \) and any \( i \in \omega \) there is a countable elementary extension \( \mathcal{P} \) of \( \mathcal{N} \) with an element which realizes \( t_i \). So if we start with \( \mathcal{M} \) and iterate this process \( \omega \) times we obtain an elementary chain \( \mathcal{M} = \mathcal{N}_0 \leq \mathcal{N}_1 \leq \cdots \) such that each \( \mathcal{N}_i \) is countable and \( \mathcal{N}_{i+1} \) has an element realizing \( t_i \). Let \( \mathcal{P} = \bigcup_{i \in \omega} \mathcal{N}_i \). Then \( \mathcal{P} \) is an elementary extension of \( \mathcal{M} \) which realizes every type over \( T \). By Theorem 10.10 let \( \mathcal{Q} \) be a countable elementary extension of \( \mathcal{P} \) which is \( \omega \)-homogeneous. By Theorem 10.6, \( \mathcal{Q} \) is \( \omega \)-saturated.

**Corollary 10.13.** If \( \mathcal{L} \) is countable and \( T \) is a theory in \( \mathcal{L} \) which has an \( \omega \)-saturated model, then \( T \) has a countable atomic model.

**Proof.** By Theorems 10.11 and 10.12.

**Theorem 10.14.** For \( T \) a theory in a countable language the following are equivalent:

(i) \( T \) is \( \aleph_0 \)-categorical.

(ii) For every \( n < \omega \), every type in \( S_n(T) \) is isolated.

(iii) \( |S_n(T)| < \aleph_0 \) for every \( n \in \omega \).

(iv) For every \( n \in \omega \) there is a finite set \( \Gamma \) of formulas with free variables among \( v_0, \ldots, v_{n-1} \) such that for every formula \( \varphi \) with free variables among \( v_0, \ldots, v_{n-1} \) there is a \( \psi \in \Gamma \) such that \( T \models \varphi \iff \psi \).

**Proof.** (i) \( \Rightarrow \) (ii): Suppose that (ii) fails: there exist \( n < \omega \) and a type \( p \in S_n(T) \) which is not isolated. By the omitting types theorem 6.26, there is a countable model \( \mathcal{M} \) of \( T \) which omits \( p \). But clearly there is also a countable model \( \mathcal{N} \) which admits \( p \). Thus \( \mathcal{M} \not\equiv \mathcal{N} \), so (i) fails.
(ii)⇒(iii): Assume (ii), but suppose that (iii) fails: there is an $n \in \omega$ such that $S_n(T)$ is infinite. For each $p \in S_n(T)$ let $\varphi_p$ isolate $p$. Let $\bar{c}$ be a sequence of new constants of length $n$, and consider the set

$$T' \overset{\text{def}}{=} T \cup \{\neg \varphi_p(\bar{c}) : p \in S_n(T)\}.$$  

We claim that $T'$ has a model. For, take any finite subset $T''$ of $T$. Let $P$ be the set of all types $p$ such that $\neg \varphi_p(\bar{c})$ is in $T''$. Let $q \in S_n(T)$ be different from each member of $P$. Now $T \models \varphi_q \rightarrow \neg \varphi_p$ for each $p \in P$; otherwise $T \models \varphi_q \rightarrow \varphi_p$ for some $p \in P$ and then $T \models \varphi_q \rightarrow \psi$ for each $\psi \in p$, so that $p \subseteq q$, hence $p = q$, contradiction. Now take a model $\bar{M}$ of $T$ which realizes $q$, say $\bar{M} \models \varphi_q(\bar{a})$. Interpreting $\bar{c}$ by $\bar{a}$ in this model gives a model of $T''$, as desired.

Thus $T'$ has a model, which gives a model $\bar{N}$ of $T$ with a sequence $\bar{b}$ satisfying in $\bar{N}$ each formula $\neg \varphi_p$. Thus $\text{tp}^\bar{N}(\bar{b})$ cannot be in $S_n(T)$, contradiction.

(iii)⇒(iv): Assume (iii), and suppose that $n \in \omega$. For distinct $p, q \in S_n(T)$ choose $\varphi_{pq} \in p \setminus q$. For each $p \in S_n(T)$ let $\psi_p = \bigwedge \{\varphi_{pq} : q \neq p\}$. Thus $\psi_p \in p$ while $\psi_p \notin q$ for all $q \neq p$. Now given $\chi$ with variables among $v_0, \ldots, v_{n-1}$ we have $T \models \chi \iff \bigvee \{\psi_p : \chi \in p\}$.

(iv)⇒(i): Assume (iv). Let $\bar{M}$ be a countable model of $T$; we show that $\bar{M}$ is atomic; so $T$ is $\aleph_0$-categorical by Theorems 10.1 and 10.4. If $\bar{\pi} \in \text{tn} \bar{M}$, then $\text{tp}^\bar{M}(\bar{\pi})$ is isolated by

$$\bigwedge \{\varphi_i(\bar{\pi}) : \bar{M} \models \varphi_i(\bar{\pi})\} \land \bigwedge \{\neg \varphi_i(\bar{\pi}) : \bar{M} \models \neg \varphi_i(\bar{\pi})\} \quad \Box$$

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