7. Morley’s theorem

This chapter is devoted to the proof of Morley’s theorem, which says that in a countable language, if $\Gamma$ is a theory with only infinite models and $\Gamma$ is $\kappa$-categorical for some uncountable cardinal $\kappa$, then it is $\kappa$-categorical for every uncountable cardinal $\kappa$. In the course of developing the proof we will introduce several new model-theoretic concepts. We follow Marker, Model theory, an introduction.

Unless otherwise mentioned, $T$ is a complete theory in a countable language having only infinite models.

Some useful notation is as follows. If $\mathcal{M}$ is a structure and $\varphi(\overline{v})$ is a formula with parameters in $M$, with $\overline{v}$ of length $m$, then by $\varphi(M)$ we mean the set $\{\overline{v} \in M : M \models \varphi(\overline{v})\}$. Here we make a slight abuse of notation, in that we write $\mathcal{M} \models \varphi(\overline{v})$ when we should write something like $(\mathcal{M}, \overline{b}) \models \varphi(\overline{v})$, where $\overline{b}$ is the sequence of parameters in $\varphi$. Similar abuses will take place later without comment.

Let $\mathcal{M}$ be an $\mathcal{L}$-structure, and suppose that $A \cup \{b\} \subseteq M$. We say that $b$ is algebraic over $A$ iff there is a formula $\varphi(x, \overline{a})$ with $\overline{a} \in A$ such that $\varphi(M, \overline{a})$ is finite and $\mathcal{M} \models \varphi(b, \overline{a})$. Now if $A \subseteq D \subseteq M$ we define

$$\text{acl}_D(A) = \{b \in D : b \text{ is algebraic over } A\}.$$ 

Lemma 7.1. (i) $A \subseteq \text{acl}_D(A)$.

(ii) $\text{acl}_D(\text{acl}_D(A)) = \text{acl}_D(A)$.

(iii) If $A \subseteq B$ then $\text{acl}_D(A) \subseteq \text{acl}_D(B)$.

(iv) If $a \in \text{acl}_D(A)$, then $a \in \text{acl}_D(A_0)$ for some finite $A_0 \subseteq A$.

Proof. (i): For any $a \in A$, let $\varphi(x, a)$ be the formula $x = a$.

(ii): Suppose that $b \in \text{acl}_D(\text{acl}_D(A))$. Accordingly, choose $\varphi(v, \overline{w})$ and $\overline{a} \in \text{acl}_D(A)$ such that

$$\mathcal{M} \models \varphi(b, \overline{a}) \text{ and } \{y \in M : \mathcal{M} \models \varphi(y, \overline{a})\} \text{ is finite.}$$

Say $\overline{w}$ has length $n$. Then for all $i < n$ we get $\psi_i(v, \overline{w}^i)$ and $\overline{a}^i \in A$ such that

$$\mathcal{M} \models \psi_i(a_i, \overline{c}^i) \text{ and } \{y \in M : \mathcal{M} \models \psi_i(y, \overline{c}^i)\} \text{ is finite.}$$

Let $k = |\{y \in M : \mathcal{M} \models \varphi(y, \overline{a})\}|$. Now let $\chi(v, \overline{w}^0, \ldots, \overline{w}^{n-1})$ be the formula

$$\exists v_0 \ldots v_{n-1} \left[ \bigwedge_{i<n} \psi_i(v_i, \overline{w}^i) \land \varphi(v, v_0, \ldots, v_{n-1}) \land (\exists! k)v_j \varphi(v_j, v_0, \ldots, v_{n-1}) \right].$$

Here $(\exists! k)v_j \ldots$ abbreviates “there are exactly $k$ $v_j$ such that ...”, which is easy to express in our language.

Now we want to show that $\mathcal{M} \models \chi(b, \overline{w}^0, \ldots, \overline{w}^{n-1})$. For any $i < n$ we have $\mathcal{M} \models \psi_i(a_i, \overline{c}^i)$, and so

$$\mathcal{M} \models \bigwedge_{i<n} \psi_i(a_i, \overline{c}^i) \land \varphi(b, \overline{a}) \land (\exists! k)v_j \varphi(v_j, \overline{a}).$$

Hence $\mathcal{M} \models \chi(b, \overline{w}^0, \ldots, \overline{w}^{n-1})$. 

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Next we want to show that \( \{ y \in M : \overline{M} \models \chi(y, \overline{a}^0, \ldots, \overline{a}^{n-1}) \} \) is finite. Let
\[
K = \prod_{i < m} \{ y \in M : \overline{M} \models \psi_i(y, \overline{a}^i) \}.
\]

Thus \( K \) is finite. Suppose that \( \overline{M} \models \chi(y, \overline{a}^0, \ldots, \overline{a}^{n-1}) \). Choose \( \overline{c} \) such that
\[
\overline{M} \models \bigwedge_{i < n} \psi_i(e_i, \overline{c}) \land \varphi(y, \overline{c}) \land (\exists ! k) v_j \varphi(v_j, \overline{c}).
\]

Then \( \overline{c} \in K \). Hence there are at most \( |K| \cdot k \) elements \( y \) such that \( \overline{M} \models \chi(y, \overline{a}^0, \ldots, \overline{a}^{n-1}) \).

(iii) and (iv) are clear.

If \( \overline{M} \) is a structure, \( m \) is a positive integer, and \( D \subseteq {}^m M \), then we say that \( D \) is definable with parameters iff there is a formula \( \varphi(\overline{a}) \) in \( L_M \) with \( \overline{a} \) of length \( m \) such that \( D = \{ a \in {}^m M : \overline{M} \models \varphi(a) \} \).

A subset \( D \) of \( M^n \) is minimal in \( \overline{M} \) iff \( D \) is infinite, and for any set \( Y \subseteq D \) definable with parameters, either \( Y \) is finite or \( D \setminus Y \) is finite. In case \( \varphi(\overline{a}, \overline{a}) \) defines \( D \), we also say that \( \varphi \) is minimal.

**Lemma 7.2.** Suppose that \( D \subseteq M \) is definable and minimal in \( \overline{M} \) and \( A \cup \{ a, b \} \subseteq D \). Suppose that \( a \in \text{acl}_D(A \cup \{ b \}) \setminus \text{acl}_D(A) \). Then \( b \in \text{acl}_D(A \cup \{ a \}) \).

**Proof.** Assume the hypotheses. Thus there is a formula \( \varphi(a, b) \) with additional parameters from \( A \), and a positive integer \( n \), such that \( \overline{M} \models \varphi(a, b) \) and \( |\{ x \in D : \overline{M} \models \varphi(x, b) \}| \leq n \). Let \( \psi(w) \) be the formula with parameters from \( A \) asserting that \( |\{ x \in D : \varphi(x, w) \}| = n \). If \( \psi(w) \) defines a finite subset of \( D \), then \( b \in \text{acl}_D(A) \). Hence \( A \cup \{ b \} \subseteq \text{acl}_D(A) \), hence by Lemma 7.1 \( a \in \text{acl}_D(A \cup \{ b \}) \subseteq \text{acl}_D(\text{acl}_D(A)) = \text{acl}_D(A) \), contradiction. It follows that \( \psi(w) \) defines a cofinite subset of \( D \).

If \( \{ y \in D : \overline{M} \models \varphi(a, y) \land \psi(y) \} \) is finite then since \( b \) is in this set we get \( b \in \text{acl}_D(A \cup \{ a \}) \), as desired. Thus we may assume that \( \{ y \in D : \overline{M} \models \varphi(a, y) \land \psi(y) \} \) is cofinite in \( D \); say that its complement has size \( l \). Let \( \chi(x) \) be the formula expressing that
\[
|D \setminus \{ y \in D : \varphi(x, y) \land \psi(y) \}| = l.
\]

since \( \overline{M} \models \chi(a) \), our assumption that \( a \notin \text{acl}_D(A) \) implies that \( \chi(\overline{M}) \) is cofinite. Let \( a_0, \ldots, a_n \) be distinct members of \( \chi(\overline{M}) \). Then for each \( i \leq n \) the set \( B_i \equiv \{ y \in D : \overline{M} \models \varphi(a_i, y) \land \psi(y) \} \) is cofinite. Let \( c \in \bigcap_{i \leq n} B_i \). Thus \( \varphi(a_i, c) \) for each \( i \leq n \), so \( |\{ x \in D : \overline{M} \models \varphi(x, c) \}| \geq n + 1 \), contradicting the choice of \( \psi(c) \).

Suppose that \( D \subseteq M^n \). We say that \( D \) is strongly minimal in \( \overline{M} \) iff \( D \) is minimal in any elementary extension of \( \overline{M} \). Similarly for a formula \( \varphi \).

Given \( A \subseteq D \), we call \( A \) independent iff \( \forall a \in A[a \notin \text{acl}_D(A \setminus \{ a \})] \). For \( C \subseteq D \) we say that \( A \) is independent over \( C \) iff \( \forall a \in A[a \notin \text{acl}_D(C \cup (A \setminus \{ a \}))] \). Note then that \( A \cap C = \emptyset \).

For a sequence of elements of \( M \) and \( A \subseteq M \) we define
\[
\text{tp}_{\overline{M}/A}(a) = \{ \varphi(\overline{a}) : \varphi \text{ is a formula with parameters from } A \text{ and } \overline{M} \models \varphi(\overline{a}) \}.
\]

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Note that if $\overline{\pi}$ is the empty sequence, then $tp(\overline{\pi}/A)$ is simply the set of all sentences with parameters from $A$ that hold in $\overline{M}$. If $A$ is empty, we just omit it.

**Lemma 7.3.** Suppose that $\overline{M}, \overline{N} \models T$ and one of the following conditions holds:

(i) $A = \emptyset$.

(ii) $A \subseteq \overline{M}_0 < \overline{M}, \overline{N}$.

Assume that $\varphi(v)$ is strongly minimal over $\overline{M}$ and has parameters from $A$, $n \in \omega$, $a \in n\varphi(\overline{M})$, $rng(\overline{a})$ is independent over $A$, and $b \in n\varphi(\overline{N})$, $rng(\overline{b})$ is independent over $A$.

Then $tp(\overline{M}/A) = tp(\overline{N}/A)$.

**Proof.** Induction on $n$. For $n = 0$ the conclusion is clear if (i) holds, since $\overline{M} \equiv \overline{N}$. The conditions in (ii) also clearly give the conclusion.

Now assume the result for $n$, and suppose that $a \in n+1\varphi(\overline{M})$, $rng(\overline{a})$ is independent over $A$, $b \in n+1\varphi(\overline{N})$, and $rng(\overline{b})$ is independent over $A$. So by the inductive hypothesis,

\[(1)\quad tp(\overline{M}(\overline{\pi} \upharpoonright n)/A) = tp(\overline{N}(\overline{\xi} \upharpoonright n)/A).\]

Let $\psi(\overline{v})$ be a formula with parameters from $A$ such that $\overline{M} \models \psi(\overline{\pi})$. Now $a_n \in \varphi(\overline{M}) \cap \psi(a_0, \ldots, a_{n-1}, \overline{M})$ and $a_n \notin acl_D(A \cup \{a_0, \ldots, a_{n-1}\})$, so $\varphi(\overline{M}) \cap \psi(a_0, \ldots, a_{n-1}, \overline{M})$ is infinite. Since $\varphi$ is strongly minimal, this set is actually cofinite in $\varphi(\overline{M})$. So there is an integer $m$ such that

\[
\overline{M} \models |\{v : \varphi(v) \land \neg \psi(a_0, \ldots, a_{n-1}, v)\}| = m.
\]

Thus the formula $\chi(\overline{v})$ expressing that

\[
|\{v : \varphi(v) \land \neg \psi(w_0, \ldots, w_{n-1}, v)\}| = m
\]

is in $tp(\overline{M}(\overline{\pi} \upharpoonright n)/A)$, and hence by (1) we get

\[
\overline{N} \models |\{v : \varphi(v) \land \neg \psi(b_0, \ldots, b_{n-1}, v)\}| = m.
\]

Since $b_n \notin acl_D(A \cup \{b_0, \ldots, b_{n-1}\})$, it follows that $\overline{N} \models \psi(\overline{b})$, as desired. $\square$

If $X$ is an infinite subset of $M$, then $X$ is an indiscernible set over $\overline{M}$ iff for any formula $\varphi(\overline{v})$ and any two sequences $\overline{x}, \overline{y}$ of distinct elements of $X$ we have $\overline{M} \models \varphi(\overline{x}) \leftrightarrow \varphi(\overline{y})$.

**Corollary 7.4.** Suppose that $\overline{M}, \overline{N} \models T$ and one of the following conditions holds:

(i) $A = \emptyset$.

(ii) $A \subseteq \overline{M}_0 < \overline{M}, \overline{N}$.

Assume that $\varphi(v)$ is strongly minimal over $\overline{M}$ and has parameters from $A$, and $B$ and $C$ are infinite subsets of $\varphi(\overline{M})$ each independent over $A$. Then $B$ and $C$ are sets of indiscernibles over $\overline{M}$, and for any $n \in \omega$ and one-one sequences $\overline{b} \in nB$ and $\overline{c} \in nC$ we have $tp(b/A) = tp(\overline{c}/A)$. $\square$

If $Y \subseteq D$, we say that $A \subseteq Y$ is a basis for $Y$ iff $A$ is independent and $acl_D(A) = acl_D(Y)$.

**Lemma 7.5.** Assume that $D \subseteq M$ is minimal in $\overline{M}$.
(i) A union of a chain of independent sets over a set \( C \subseteq D \) is again independent over \( C \). (Hence we can apply Zorn’s lemma in this context.)

(ii) For any \( Y \subseteq D \), any maximal independent subset subset of \( Y \) is a basis for \( Y \).

**Proof.** (i): Let \( \mathscr{A} \) be a chain of independent sets over \( C \). Suppose that \( a \in \operatorname{acl}_D(C \cup (\bigcup \mathscr{A} \setminus \{a\})) \). Thus \( a \) is algebraic over \( C \cup (\bigcup \mathscr{A} \setminus \{a\}) \), and so there is a formula \( \varphi(x, \overline{a}, \overline{b}) \) with \( \overline{a} \in C \) and \( \overline{b} \in \bigcup \mathscr{A} \setminus \{a\} \) such that \( \varphi(M, \overline{a}, \overline{b}) \) is finite and \( M \models \varphi(a, \overline{a}, \overline{b}) \). Then there is an \( X \in \mathscr{A} \) such that \( \overline{b} \in X \), so that \( a \in \operatorname{acl}_D(C \cup (X \setminus \{a\})) \), contradiction.

(ii): Suppose that \( A \) is a maximal independent subset of \( Y \). Obviously \( \operatorname{acl}_D(A) \subseteq \operatorname{acl}_D(Y) \). Suppose that \( a \in \operatorname{acl}_D(Y) \setminus \operatorname{acl}_D(A) \). Let \( \varphi(x, \overline{y}) \) be a formula with \( \overline{y} \in Y \) such that \( \varphi(M, \overline{a}, \overline{y}) \) is finite and \( M \models \varphi(a, \overline{a}, \overline{y}) \). If each \( y_i \in \operatorname{acl}_D(A) \), then \( a \notin \operatorname{acl}_D(\operatorname{ring}(\overline{y})) \subseteq \operatorname{acl}_D(\operatorname{acl}_D(A)) = \operatorname{acl}_D(A) \), contradiction. So there is an \( i \) such that \( y_i \notin \operatorname{acl}_D(A) \). If \( b \in A \) and \( b \in \operatorname{acl}_D(\{y_i \cup (A \setminus \{b\})) \), then \( b \notin \operatorname{acl}_D(A \setminus \{b\}) \) by independence, and so \( y_i \in \operatorname{acl}_D(A) \) by Lemma 7.2, contradiction. Hence \( A \cup \{y_i\} \) is independent, contradiction.

**Lemma 7.6.** Let \( D \) be strongly minimal over \( \overline{M} \). Then:

(i) Let \( A, B \subseteq D \) be independent with \( A \subseteq \operatorname{acl}(B) \). Then:

(a) Suppose that \( A_0 \subseteq A, B_0 \subseteq B, A_0 \cup B_0 \) is a basis for \( \operatorname{acl}(B) \), and \( a \in A \setminus A_0 \). Then there is a \( b \in B_0 \) such that \( A_0 \cup \{a\} \cup (B_0 \setminus \{b\}) \) is a basis for \( \operatorname{acl}(B) \).

(b) \( |A| \leq |B| \).

(ii) If \( A \) and \( B \) are bases for \( Y \subseteq D \), then \( |A| = |B| \).

**Proof.** (i): Assume the hypotheses. (a): Assume the hypotheses. Then \( a \in A \subseteq \operatorname{acl}_D(B) = \operatorname{acl}_D(A_0 \cup B_0) \), so by Lemma 7.1(iv) there is a finite \( X \subseteq A_0 \cup B_0 \) such that \( a \in \operatorname{acl}_D(X) \). Let \( C \subseteq B_0 \) be of smallest size such that \( a \in \operatorname{acl}_D(A_0 \cup C) \). Thus \( C \) is finite, and \( A_0 \cap C = \emptyset \) by the minimality of \( C \). Since \( A \) is independent and \( a \notin A_0 \), we have \( C \neq \emptyset \). Fix \( b \in C \). Now \( a \in \operatorname{acl}_D(A_0 \cup (C \setminus \{b\}) \cup \{b\}) \setminus \operatorname{acl}_D(A_0 \cup (C \setminus \{b\})) \), so by Lemma 7.2, \( b \in \operatorname{acl}_D(A_0 \cup (C \setminus \{b\}) \cup \{a\}) \). Hence \( A_0 \cup B_0 \subseteq \operatorname{acl}_D(A_0 \cup \{a\} \cup (B_0 \setminus \{b\}) \), and hence

\[
\operatorname{acl}_D(B) = \operatorname{acl}_D(A_0 \cup B_0) \\
\subseteq \operatorname{acl}_D(\operatorname{acl}_D(A_0 \cup \{a\} \cup (B_0 \setminus \{b\}))) \\
= \operatorname{acl}_D(A_0 \cup \{a\} \cup (B_0 \setminus \{b\})) \\
\subseteq \operatorname{acl}_D(B).
\]

Thus \( \operatorname{acl}_D(A_0 \cup \{a\} \cup (B_0 \setminus \{b\})) = \operatorname{acl}_D(B) \). We claim that \( X \overset{\text{def}}{=} A_0 \cup \{a\} \cup (B_0 \setminus \{b\}) \) is independent. For, suppose that \( x \in X \) and \( x \in \operatorname{acl}(X \setminus \{x\}) \).

Case 1. \( x = a \). Thus \( a \in \operatorname{acl}_D(A_0 \cup (B_0 \setminus \{b\})) \), hence \( A_0 \cup \{a\} \cup (B_0 \setminus \{b\}) \subseteq \operatorname{acl}_D(A_0 \cup (B_0 \setminus \{b\})) \), hence \( b \in \operatorname{acl}_D(B) = \operatorname{acl}_D(A_0 \cup \{a\} \cup (B_0 \setminus \{b\}) \subseteq \operatorname{acl}_D(A_0 \cup (B_0 \setminus \{b\}) \), contradicting the fact that \( A_0 \cup B_0 \) is independent. (Recall that \( A_0 \cap C = \emptyset \), hence \( b \notin A_0 \).)

Case 2. \( x \neq a \). Now \( X \setminus \{x\} = \{a\} \cup (A_0 \cup (B_0 \setminus \{b\})) \setminus \{x\} \) and \( x \notin \operatorname{acl}_D((A_0 \cup (B_0 \setminus \{b\})) \setminus \{x\}) \) by the independence of \( A_0 \cup B_0 \). So by Lemma 2 we get \( a \in \operatorname{acl}_D(A_0 \cup (B_0 \setminus \{b\})) \), i.e., Case 1, contradiction.

(b): Case 1. \( B \) is finite; say \( |B| = n \). Suppose that \( a_0, \ldots, a_n \) are distinct elements of \( A \). We now define distinct elements \( b_i \) of \( B \) for \( i < n \) by recursion. Suppose they have been defined for all \( j < i \), where \( 0 \leq i < n - 1 \), so that \( \{a_j : j < i\} \cup (B \setminus \{b_j : j < i\}) \) is
a basis for $\text{acl}_D(B)$. Since $a_i \in A \setminus \{a_j : j < i\}$, we can apply (a) to obtain $b_i$ such that
\{a_j : j \leq i\} \cup (B \setminus \{b_j : j \leq i\}$ is a basis for $\text{acl}_D(B)$.

It follows that $\{a_j : j < n\}$ is a basis for $\text{acl}_D(B)$. Hence $a_n \in \text{acl}_D(\{a_j : j < n\})$, contradicting $A$ independent.

Thus we must have $|A| \leq |B|$.

Case 2. $B$ is infinite. By Case 1, $|A \cap \text{acl}(B_0)| \leq |B_0|$ for each finite subset $B_0$ of $B$. Now

$$A \subseteq \bigcup_{B_0 \subseteq B, \ B_0 \text{ finite}} (A \cap \text{acl}(B_0),]$$

so clearly $|A| \leq |B|$.

(ii) follows from (i)(b).

For $D$ strongly minimal, the dimension of $D$, $\text{dim}(D)$, is the cardinality of a basis for $D$.

**Lemma 7.7.** If $D$ is strongly minimal and uncountable, then $\text{dim}(D) = |D|$.

**Proof.** Since the language is countable, also $\text{acl}(B)$ is countable for every finite subset $B$ of $D$. If $X \subseteq D$ and $|X| < |D|$, then

$$|\text{acl}(X)| \leq \bigcup_{B \subseteq X, \ B \text{ finite}} \text{acl}(B) \leq |X| \cdot \omega < |D|.$$

Let $\pi$ be a sequence of elements of $M$ and $A \subseteq M$. We say that $\text{tp}^M(\pi/A)$ is isolated if there is a formula $\varphi(\pi) \in \text{tp}^M(\pi/A)$ such that for every formula $\chi(\pi) \in \text{tp}^M(\pi/A)$ we have $M \models \forall \pi[\varphi(\pi) \rightarrow \chi(\pi)]$.

**Lemma 7.8.** If $A \cup \{b\} \subseteq M$ and $b$ is algebraic over $A$, then $\text{tp}^M(b/A)$ is isolated.

**Proof.** Let $\pi \in A$ and $\varphi(v, \pi)$ be such that $M \models \varphi(b, \pi)$ and $\{y \in M : M \models \varphi(y, \pi)\}$ is finite. Let

$$B = \{d \in M : M \models \varphi(d, \pi) \text{ and there exist a formula } \psi(v, \pi) \text{ with } \pi \in A \text{ such that } M \models \psi(b, \pi) \text{ and } M \models \neg \psi(d, \pi)\}.$$  

Note that $B$ is finite. For each $d \in B$, choose $\psi_d$ and $\pi_d$ as indicated. Let $\varphi'(v, \pi)$ be the formula

$$\varphi(v, \pi) \land \bigwedge_{d \in B} \psi_d(v, \pi_d).$$

Thus $M \models \varphi'(b, \pi)$, and so $\varphi'(v, \pi) \in \text{tp}^M(b/A)$. Now suppose that $\chi(v, \pi) \in \text{tp}^M(b/A)$, but there is a $d \in M$ such that $M \models \varphi'(d, \pi) \land \neg \chi(d, \pi)$; we want to get a contradiction. We have $M \models \varphi(d, \pi)$, so it follows that $d \in B$; hence $M \models \neg \psi_d(d, \pi)$; but this contradicts $M \models \varphi'(d, \pi)$.  

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Lemma 7.9. Suppose that $\mathcal{M}, \mathcal{N} \models T$, $\varphi(v)$ is strongly minimal, and $\dim(\varphi(\mathcal{M})) = \dim(\varphi(\mathcal{N}))$. Then there is a bijection $f : \varphi(\mathcal{M}) \to \varphi(\mathcal{N})$ such that for every formula $\psi(w)$ and every $\overline{a} \in \varphi(\mathcal{M})$, $\mathcal{M} \models \psi(\overline{a})$ iff $\mathcal{N} \models \psi(f \circ \overline{a})$.

Proof. Assume the hypotheses. Let $B$ be a base for $\varphi(\mathcal{M})$, and let $C$ be a base for $\varphi(\mathcal{N})$. Thus $|B| = |C|$, and we let $h : B \to C$ be a bijection. Let

$$I = \{ g : g : B' \to C' \text{ is a surjection, } B \subseteq B' \subseteq \varphi(\mathcal{M}), C \subseteq C' \subseteq \varphi(\mathcal{N}) \text{ and }$$

$$\forall \chi \forall \overline{a} \in B'[\mathcal{M} \models \chi(\overline{a}) \leftrightarrow \mathcal{N} \models \chi(g \circ \overline{a})].$$

Note that every $g \in I$ is injective; consider the formula $x \neq y$. Now $h \in I$, since for any $\chi(\overline{a})$ and any $\overline{a} \in B$,

$$\mathcal{M} \models \chi(\overline{a}) \iff \chi(\overline{w}) \in \text{tp}^\mathcal{M}(\overline{a})$$

$$\iff \chi(\overline{w}) \in \text{tp}^\mathcal{N}(h \circ \overline{a}) \text{ by Corollary 7.4}$$

Clearly we can apply Zorn’s lemma to $I$ and obtain a maximal member $g$ of it, with associated sets $B', C'$. We claim that $\text{dnn}(g) = \varphi(\mathcal{M})$ and $\text{rng}(g) = \varphi(\mathcal{N})$. By symmetry we prove only that $\text{dnn}(g) = \varphi(\mathcal{M})$. In fact, suppose that this is not true. Let $b \in \varphi(\mathcal{M}) \setminus B'$. Since $\text{acl}(B) = \varphi(\mathcal{M})$, we also have $\text{acl}(B') = \varphi(\mathcal{M})$, and so $b \in \text{acl}(B')$. Hence by Lemma 7.8 let $\psi(v, \overline{a}) \in \text{tp}^\mathcal{M}(b/B')$ isolate $\text{tp}^\mathcal{M}(b/B')$, where $\overline{a} \in B'$. Now $\mathcal{M} \models \exists x \psi(x, \overline{a})$, so from $g \in I$ we get $\mathcal{N} \models \exists x \psi(x, g \circ \overline{a})$. Say $\mathcal{N} \models \psi(d, g \circ \overline{a})$. Extend $g$ to $g' : B' \cup \{b\} \to C' \cup \{d\}$ by setting $g'(b) = d$. So $g'$ is a surjection from $B' \cup \{b\}$ to $C' \cup \{d\}$. Now take any formula $\chi(v, \overline{w})$ and any $\overline{c} \in B'$. Then

$$\mathcal{M} \models \chi(b, \overline{c}) \Rightarrow \chi(v, \overline{c}) \in \text{tp}^\mathcal{M}(b)$$

$$\Rightarrow \mathcal{M} \models \forall v[\psi(v, \overline{c}) \to \chi(v, \overline{c})]$$

$$\Rightarrow \mathcal{N} \models \forall v[\psi(v, g \circ \overline{c}) \to \chi(v, g \circ \overline{c})]$$

$$\Rightarrow \mathcal{N} \models \chi(d, g \circ \overline{c});$$

this shows that $g' \in I$, contradiction. □

A theory $T$ is strongly minimal iff the formula $v = v$ is strongly minimal for each model $\mathcal{M}$ of $T$.

For each infinite cardinal $\kappa$, $I(T, \kappa)$ is the number of nonisomorphic models of $T$ of size $\kappa$.

Theorem 7.10. Suppose that $T$ is strongly minimal.

(i) If $\mathcal{M}, \mathcal{N} \models T$, then $\mathcal{M} \cong \mathcal{N}$ iff $\dim(\mathcal{M}) = \dim(\mathcal{N})$.

(ii) $T$ is $\kappa$-categorical for each uncountable cardinal $\kappa$.

(iii) $I(T, \omega) \leq \omega$.

Proof. (i) is immediate from Lemma 7.9. (ii) follows from (i) by Lemma 7.7. (iii) follows from (i) since $\dim(\mathcal{M}) \leq \omega$ for any countable model $\mathcal{M}$ of $T$. □
A set $\Gamma$ of formulas is \textit{finitely satisfiable} in $\mathcal{M}$ iff for every finite subset $\Delta$ of $\Gamma$ there is an $a \in \omega \mathcal{M}$ such that $\mathcal{M} \models \varphi[a]$ for all $\varphi \in \Delta$. For any model $\mathcal{M}$ of $T$, any subset $A$ of $\mathcal{M}$, and any positive integer $n$, an $n$-type over $\mathcal{M}$ is a set of formulas with free variables among $v_0, \ldots, v_{n-1}$ and with parameters from $A$ which is finitely satisfiable over $\mathcal{M}$. It is a \textit{complete} $n$-type iff for any formula $\varphi$ with free variables among $v_0, \ldots, v_{n-1}$ and parameters from $A$, either $\varphi$ or $\neg \varphi$ is a member of it. We let $S_n^\mathcal{M}(A)$ be the set of all complete $n$-types over $A$ with respect to $\mathcal{M}$. Note that $|S_n^\mathcal{M}(A)| \leq 2^{\max(\omega, \kappa)}$. $T$ is $\kappa$-stable iff for every $\mathcal{M} \models T$, every $A \subseteq M$ of size $\kappa$, and every positive integer $n$ we have $|S_n^\mathcal{M}(A)| = \kappa$.

**Lemma 7.11.** If $T$ is $\omega$-stable and $\mathcal{M} \models T$, then there is a minimal formula for $\mathcal{M}$.

**Proof.** Suppose not. We define formulas $\varphi_f$ for each $f \in \omega 2$ by induction on $\text{dmm}(f)$. Let $\varphi_\emptyset$ be the formula $v = v$. Now suppose that $\varphi_f$ has been defined so that $\varphi_v(\mathcal{M})$ is infinite. Since $\varphi_f$ is not minimal, there is a formula $\psi$ with parameters such that $\varphi_f(\mathcal{M}) \cap \psi(\mathcal{M})$ and $\varphi_f(\mathcal{M}) \land \neg \psi(\mathcal{M})$ are infinite. We let $\varphi_{f \land \neg \psi}$ be $\varphi_f \land \psi$ and $\varphi_{f \land \neg(\psi)}$ be $\varphi_f \land \neg \psi$.

Let $A$ be the set of all parameters appearing in any formula $\varphi_f$ for $f \in \omega 2$. So $A$ is countable. For each $f \in \omega 2$ the set

$$\{ \varphi_f|_n : n \in \omega \}$$

is finitely satisfiable in $\mathcal{M}$ and hence is contained in a complete type $t_f$ over $\mathcal{M}$. This gives $2^\omega$ complete types over $A$, contradicting $\omega$-stability. \hfill $\square$

**Lemma 7.12.** If $\mathcal{M}$ is $\omega$-saturated and $\varphi(\overline{v}, \overline{a})$ is a minimal formula in $\mathcal{M}$, then $\varphi(\overline{v}, \overline{a})$ is a strongly minimal.

**Proof.** Suppose not. Let $\mathcal{M} \prec \mathcal{N}$ with $\psi(\overline{nN}, \overline{b})$ an infinite and cofinite subset of $\varphi(\overline{nN}, \overline{a})$, where $\overline{b} \in \mathcal{N}$. Then $\text{tp}^\mathcal{N}(\overline{b}/\overline{a})$ is a complete type in $\mathcal{N}$, hence it is finitely satisfiable in $\mathcal{N}$, so it is finitely satisfiable in $\mathcal{M}$. Thus it is a complete type in $\mathcal{M}$ over $\overline{a}$. So by the $\omega$-saturation of $\mathcal{M}$, it is satisfiable in $\mathcal{M}$, say by $\overline{b}'$. Thus $\text{tp}^\mathcal{N}(\overline{b}/\overline{a}) = \text{tp}^\mathcal{M}(\overline{b}'/\overline{a})$.

Now for any positive integer $p$,

$$\mathcal{N} \models \exists_{\geq p} \overline{v}[\varphi(\overline{v}, \overline{a}) \land \psi(\overline{v}, \overline{b})],$$

hence

$$\exists_{\geq p} \overline{v}[\varphi(\overline{v}, \overline{a}) \land \psi(\overline{v}, \overline{w})] \in \text{tp}^\mathcal{N}(\overline{b}/\overline{a}),$$

hence

$$\exists_{\geq p} \overline{v}[\varphi(\overline{v}, \overline{a}) \land \psi(\overline{v}, \overline{w})] \in \text{tp}^\mathcal{M}(\overline{b}/\overline{a}),$$

$$\mathcal{M} \models \exists_{\geq p} \overline{v}[\varphi(\overline{v}, \overline{a}) \land \psi(\overline{v}, \overline{b}')].$$

It follows that $\varphi(\overline{nM}) \cap \psi(\overline{nM})$ is infinite. Similarly, $\varphi(\overline{nM}) \cap \neg \psi(\overline{nM})$ is infinite, contradiction. \hfill $\square$

A \textit{Vaughtian pair} for $T$ is a pair $(\mathcal{M}, \mathcal{N})$ of models of $T$ such that there is a formula $\varphi(\overline{v})$ such that $\mathcal{M} \prec \mathcal{N}$, $\mathcal{M} \neq \mathcal{N}$, $\varphi(\mathcal{M})$ is infinite, and $\varphi(\mathcal{M}) = \varphi(\mathcal{N})$.  

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Lemma 7.13. Suppose that $T$ does not have any Vaughtian pairs, $\mathcal{M} \models T$, and $\varphi(\bar{v}, \bar{w})$ is a formula with parameters from $M$, with $\bar{v}$ of length $m$ and $\bar{w}$ of length $k$. Then there is a natural number $n$ such that for all $\bar{a} \in M$, if $|\varphi(\mathcal{M}, \bar{a})| > n$, then $\varphi(\mathcal{M}, \bar{a})$ is infinite.

Proof. Suppose not. For each $n \in \omega$ let $\bar{a}_n \in M$ be such that $\varphi(\mathcal{M}, \bar{a}_n)$ is finite, but of size $> n$.

Adjoin to the language a new one-place relation symbol $U$. Let $\Gamma$ be the set of formulas of the following four types:

1. $\forall x \left[ \bigwedge_{i<p} Ux_i \rightarrow [\psi \leftrightarrow \psi^U] \right]$, for each formula $\psi$ with free variables among $\bar{x}$, where $\bar{x} = \langle x_i : i < p \rangle$, and $\psi^U$ indicates relativization of quantifiers to $U$.
2. $\exists x \neg Ux$.
3. $\exists \geq s \forall v \varphi(v, w)$ for each $s \in \omega$.
4. $\varphi(v, w) \rightarrow \bigwedge_{i<k} Uw_i$.

Now let $\mathcal{N}$ be a proper elementary extension of $\mathcal{M}$. For each $n \in \omega$ we have $\varphi(\mathcal{M}, \bar{a}_n) = \varphi(\mathcal{N}, \bar{a}_n)$, since $\varphi(\mathcal{M}, \bar{a}_n)$ is finite. Each finite subset of $\Gamma$ is satisfiable in the structure $(\mathcal{N}, M)$. Hence by the compactness theorem we get an elementary extension $(\mathcal{N}', M')$ of $(\mathcal{N}, M)$ such that $\Gamma$ is realizable in $(\mathcal{N}', M')$, say by $\bar{a}$. Let $\mathcal{M}'$ be the structure with universe $M'$. Then by (1), $\mathcal{N}'$ is an elementary extension of $\mathcal{M}'$, and it is a proper extension by (2). By (3), $\varphi(\mathcal{N}', \bar{a})$ is infinite, and by (4) we have $\varphi(\mathcal{N}', \bar{a}) \subseteq M'$, hence $\varphi(\mathcal{N}', \bar{a}) = \varphi(\mathcal{M}', \bar{a})$ by elementarity. Thus $(\mathcal{M}', \mathcal{N}')$ is a Vaughtian pair, contradiction.

Lemma 7.14. If $T$ has no Vaughtian pairs, then for every $\mathcal{M} \models T$ and every formula $\varphi$ with parameters from $\mathcal{M}$, if $\varphi$ is minimal for $\mathcal{M}$ then it is strongly minimal for $\mathcal{M}$.

Proof. Suppose not. Let $\varphi = \varphi(\bar{v})$, with parameters from $M$. Then there is an elementary extension $\mathcal{N}$ of $\mathcal{M}$ and a formula $\psi(\bar{v}, \bar{b})$ with $\bar{b} \in N$ such that $\varphi(\mathcal{N}) \cap \psi(\mathcal{N}, \bar{b})$ and $\varphi(\mathcal{N}) \cap \neg \psi(\mathcal{N}, \bar{b})$ are infinite. By Lemma 7.13 applied twice, let $n \in \omega$ be such that for all $\bar{a} \in M$,

$$|\varphi(\mathcal{M}) \cap \psi(\mathcal{M}, \bar{a})| > n \rightarrow \varphi(\mathcal{M}) \cap \psi(\mathcal{M}, \bar{a})$$

is infinite, and

$$|\varphi(\mathcal{M}) \cap \neg \psi(\mathcal{M}, \bar{a})| > n \rightarrow \varphi(\mathcal{M}) \cap \neg \psi(\mathcal{M}, \bar{a})$$

is infinite.

Thus by the minimality of $\varphi$,

$$\mathcal{M} \models \forall \bar{w}(|\varphi(\mathcal{M}) \cap \psi(\mathcal{M}, \bar{w})| \leq n \lor |\varphi(\mathcal{M}) \cap \neg \psi(\mathcal{M}, \bar{w})| \leq n].$$

So this also holds in $\mathcal{N}$, and it follows that $\varphi(\mathcal{N}) \cap \psi(\mathcal{N}, \bar{b})$ is finite or $\varphi(\mathcal{N}) \cap \neg \psi(\mathcal{N}, \bar{b})$ is finite, contradiction.

Corollary 7.15. If $T$ is $\omega$-stable and has no Vaughtian pairs, then for every $\mathcal{M} \models T$ there is a strongly minimal formula over $\mathcal{M}$.
Corollary 7.16. If \( T \) has no Vaughtian pairs, \( \overline{M} \models T \), and \( \varphi(\overline{a}) \) is a formula with parameters from \( M \), and if \( \varphi(\overline{M}) \) is infinite, then no proper elementary submodel of \( \overline{M} \) contains both \( \varphi(\overline{M}) \) and the parameters of \( \varphi(\overline{a}) \).

**Proof.** Suppose that \( \overline{N} \) is a proper elementary submodel of \( \overline{M} \) which contains both \( \varphi(\overline{M}) \) and the parameters of \( \varphi(\overline{a}) \). Then for any \( \overline{a} \in N \), \( \overline{N} \models \varphi(\overline{a}) \) implies that \( \overline{M} \models \varphi(\overline{a}) \) by elementarity. Conversely, if \( \overline{M} \models \varphi(\overline{a}) \) with \( \overline{a} \in M \), then \( \overline{a} \in N \) by assumption, so \( \overline{N} \models \varphi(\overline{a}) \) by elementarity. Thus \( \varphi(\overline{M}) = \varphi(\overline{N}) \). So \( (\overline{M}, \overline{N}) \) is a Vaughtian pair, contradiction. \( \square \)

**Lemma 7.17.** Suppose that \( T \) is \( \omega \)-stable, \( \overline{M} \models T \), \( A \subseteq M \), \( \varphi(\overline{a}) \) is a formula with parameters from \( A \), and \( \overline{M} \models \exists \overline{v} \varphi(\overline{v}) \). Then there is an \( \overline{a} \in M \) such that \( \varphi(\overline{a}) \in \text{tp}^\overline{M}(\overline{a}/A) \) and \( \text{tp}^\overline{M}(\overline{a}/A) \) is isolated.

**Proof.** Suppose that this does not hold. We construct formulas \( \psi_f \) for each \( f \in \omega^\omega \). Let \( \psi_0 = \varphi \). Suppose that we have constructed \( \psi_f(\overline{a}) \), a formula with parameters from \( A \), so that

(*) \( \overline{M} \models \exists \overline{v} \psi_f(\overline{v}) \), and for all \( \overline{a} \in M \), if \( \varphi_f(\overline{a}) \in \text{tp}^\overline{M}(\overline{a}/A) \), then \( \text{tp}^\overline{M}(\overline{a}/A) \) is not isolated.

This is true for \( f = \emptyset \) by assumption. We claim

(**) There is a formula \( \chi(\overline{v}) \) with parameters from \( A \) such that \( \overline{M} \models \exists \overline{v} \left[ \psi_f(\overline{v}) \land \chi(\overline{v}) \right] \) and \( \overline{M} \models \exists \overline{v} \left[ \psi_f(\overline{v}) \land \neg \chi(\overline{v}) \right] \).

Suppose not. Take any \( \overline{v} \) such that \( \overline{M} \models \psi_f(\overline{v}) \). Suppose that \( \chi(\overline{v}) \in \text{tp}^\overline{M}(\overline{a}/A) \). Now by (** failing) we have

\( \overline{M} \models \forall \overline{v} \left[ \psi_f(\overline{v}) \rightarrow \chi(\overline{v}) \right] \) or \( \overline{M} \models \forall \overline{v} \left[ \psi_f(\overline{v}) \rightarrow \neg \chi(\overline{v}) \right] \).

But \( \overline{M} \models \chi(\overline{a}) \) and \( \overline{M} \models \psi_f(\overline{a}) \), so it follows that \( \overline{M} \models \forall \overline{v} \left[ \psi_f(\overline{v}) \rightarrow \chi(\overline{v}) \right] \). This proves that \( \psi_f(\overline{v}) \) isolates \( \text{tp}^\overline{M}(\overline{a}/A) \), contradiction. Hence (***) holds. We take such a formula \( \chi(\overline{v}) \) and define \( \psi_{f-0} \) to be \( \psi_f(\overline{v}) \land \chi(\overline{v}) \) and \( \psi_{f-1} \) to be \( \psi_f(\overline{v}) \land \neg \chi(\overline{v}) \). This finishes the construction.

But this clearly gives \( 2^\omega \) types over \( A \), contradicting \( \omega \)-stability. \( \square \)

If \( \overline{M}, \overline{N} \) are structures, \( A \subseteq M \), and \( f : A \rightarrow N \), we say that \( f \) is partial elementary iff for every formula \( \varphi(\overline{v}) \) without parameters and every \( \overline{a} \in A \), \( \overline{M} \models \varphi(\overline{a}) \) iff \( \overline{N} \models \varphi(f \circ \overline{a}) \).

\( \overline{M} \) is a prime model of \( T \) iff \( \overline{M} \) can be elementarily embedded in every model of \( T \).

If \( \overline{M} \models T \) and \( A \subseteq M \), we say that \( \overline{M} \) is prime over \( A \) for \( T \) iff for every model \( \overline{N} \) of \( T \), every partial elementary \( f : A \rightarrow N \) can be extended to an elementary \( f^+ : \overline{M} \rightarrow \overline{N} \).

**Lemma 7.18.** If \( \overline{a} \in mM, \overline{b} \in nM, A \subseteq M \), and \( \text{tp}^\overline{M}(\overline{a}/\overline{b}) \) is isolated, then \( \text{tp}^\overline{M}(\overline{a}/\overline{b}) \) is isolated.

**Proof.** Let \( \varphi(\overline{v}, \overline{w}) \), a formula with parameters in \( A \), isolate \( \text{tp}^\overline{M}(\overline{a}/\overline{b}) \). We claim that \( \exists \overline{v} \varphi(\overline{a}, \overline{w}) \) isolates \( \text{tp}^\overline{M}(\overline{a}/\overline{b}) \). First, \( \overline{M} \models \varphi(\overline{a}, \overline{w}) \), so \( \overline{M} \models \exists \overline{v} \varphi(\overline{a}, \overline{w}) \). Second, suppose that \( \overline{M} \models \chi(\overline{a}) \), where \( \chi \) has parameters in \( A \). Then \( \chi(\overline{v}) \in \text{tp}^\overline{M}(\overline{a}/\overline{b}) \), so \( \overline{M} \models \forall \overline{v} \varphi(\overline{a}, \overline{w}) \). Hence \( \overline{M} \models \forall \overline{v} \left[ \exists \overline{w} \varphi(\overline{v}, \overline{w}) \rightarrow \chi(\overline{v}) \right] \) by elementary logic. \( \square \)
Lemma 7.19. Suppose that $A \subseteq B \subseteq M$, and $\overline{M} \models T$. Suppose that every $b \in B$ realizes an isolated type over $A$, and suppose that $\text{tp}^\overline{M}(\overline{a}/B)$ is isolated. Then $\text{tp}^\overline{M}(\overline{a}/A)$ is isolated.

Proof. Suppose that $\varphi(\overline{v}, \overline{b})$ isolates $\text{tp}^\overline{M}(\overline{a}/B)$, where $\overline{b} \in B$ are the parameters of $\varphi$. By hypothesis, let $\theta(\overline{w})$ isolate $\text{tp}^\overline{M}(b/A)$. We claim that $\varphi(\overline{v}, \overline{w}) \land \theta(\overline{w})$ isolates $\text{tp}^\overline{M}(\overline{a}/\overline{b}/A)$. For, $\overline{M} \models \varphi(\overline{v}, \overline{b})$ and $\overline{M} \models \theta(\overline{b})$, so $\overline{M} \models \varphi(\overline{v}, \overline{b}) \land \theta(\overline{b})$. Now suppose that $\overline{M} \models \chi(\overline{a}, \overline{b})$. Hence $\overline{M} \models \forall\overline{v}[\varphi(\overline{v}, \overline{b}) \rightarrow \chi(\overline{v}, \overline{b})]$. Hence the formula

$$\forall\overline{v}[\varphi(\overline{v}, \overline{b}) \rightarrow \chi(\overline{v}, \overline{b})]$$

is in $\text{tp}^\overline{M}(\overline{b}/A)$, and it follows that

$$\overline{M} \models \forall\overline{w}[\theta(\overline{w}) \rightarrow \forall\overline{v}[\varphi(\overline{v}, \overline{b}) \rightarrow \chi(\overline{v}, \overline{b})]].$$

Hence by elementary logic,

$$\overline{M} \models \forall\overline{w}\forall\overline{v}[\theta(\overline{w}) \land \varphi(\overline{v}, \overline{b}) \rightarrow \chi(\overline{v}, \overline{b})].$$

So we have shown that $\varphi(\overline{v}, \overline{w}) \land \theta(\overline{w})$ isolates $\text{tp}^\overline{M}(\overline{a}/\overline{b}/A)$. Now by Lemma 7.18 it follows that $\text{tp}^\overline{M}(\overline{a}/A)$ is isolated.

Theorem 7.20. Let $T$ be $\omega$-stable. Suppose that $\overline{M} \models T$ and $A \subseteq M$. Then there is an $\overline{M}_0 \preceq \overline{M}$ which is prime over $A$ for $T$, and is such that every element of $M_0$ realizes an isolated type over $A$ with respect to $\overline{M}_0$.

Proof. We define a sequence $\langle A_\alpha : \alpha \leq \delta \rangle$ by recursion, where $\delta$ is also defined in the construction. Let $A_0 = A$. If $\alpha$ is a limit ordinal and $A_\beta$ has been defined for all $\beta < \alpha$, then we let $A_\alpha = \bigcup_{\beta < \alpha} A_\beta$. Now suppose that $A_\alpha$ has been defined. If no element of $M \setminus A_\alpha$ realizes an isolated type over $A_\alpha$ (in particular, if $M = A_\alpha$), we stop and let $\delta = \alpha$. Otherwise we pick an element $a_\alpha \in M \setminus A_\alpha$ realizing an isolated type over $A_\alpha$ and let $A_{\alpha+1} = A_\alpha \cup \{a_\alpha\}$.

\begin{enumerate}
\item $A_\delta$ is closed under the fundamental functions of $\overline{M}$.
\item In fact, suppose that $F$ is an $m$-ary function symbol and $\overline{a} \in m A_\delta$. Now $\text{tp}^\overline{M}(F^\overline{M}(\overline{a})/A_\delta)$ is isolated over $A_\delta$. For, suppose that $\varphi(\overline{v}) \in \text{tp}^\overline{M}(F^\overline{M}(\overline{a})/A_\delta)$. Thus $\overline{M} \models \varphi(F^\overline{M}(\overline{a}))$, and so $\overline{M} \models \forall\overline{v}[F^\overline{M}(\overline{a}) = v \rightarrow \psi(\overline{v})]$, so that $F^\overline{M}(\overline{a}) = v$ isolates $F^\overline{M}(\overline{a})/A_\delta$. It follows that $F^\overline{M}(\overline{a}) \in A_\delta$.
\end{enumerate}

Let $\overline{M}_0$ be the substructure of $\overline{M}$ with universe $A_\delta$.

We apply Tarski’s lemma. Suppose that $\varphi(v, \overline{a})$ is a formula with parameters $\overline{a} \in A_\delta$, and $\overline{M} \models \exists v \varphi(v, \overline{a})$. By Lemma 7.17, choose $b \in M$ such that $\varphi(v, \overline{a}) \in \text{tp}^\overline{M}(b/\overline{a})$ and $\text{tp}^\overline{M}(b/\overline{a})$ is isolated. By construction we have $b \in A_\delta$, as desired.

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Now suppose that $\overline{N} \models T$ and $f : A \to \overline{N}$ is partial elementary. We now define $f_0 \subseteq \cdots \subseteq f_\delta$ by recursion so that $f_\alpha : A_\alpha \to \overline{N}$ is partial elementary. Let $f_0 = f$. If $\alpha \leq \delta$ is a limit ordinal and $f_\beta$ has been defined for all $\beta < \alpha$, let $f_\alpha = \bigcup_{\beta < \alpha} f_\beta$. Clearly $f_\alpha$ is partial elementary. Now suppose that $f_\alpha$ has been defined, where $\alpha < \delta$, with $f_\alpha : A_\alpha \to \overline{N}$ partial elementary. Then by construction, $A_{\alpha+1} = A_\alpha \cup \{a_\alpha\}$, where $a_\alpha \in M \setminus A_\alpha$ and $\text{tp}^M(a_\alpha/A_\alpha)$ is isolated. Let $\varphi(v, \overline{b})$ be a formula with parameters $\overline{b} \in A_\alpha$ which isolates $\text{tp}^M(a_\alpha/A_\alpha)$. Thus the following conditions hold:

(3) $\overline{M} \models \varphi(a_\alpha, \overline{b})$.

(4) For every formula $\chi(v, \overline{c})$ with parameters $\overline{c} \in A_\alpha$, if $\overline{M} \models \chi(a_\alpha, \overline{c})$ then $\overline{M} \models \forall v[\varphi(v, \overline{b}) \rightarrow \chi(v, \overline{c})]$.

Now by (3) we have $\overline{M} \models \exists v \varphi(v, \overline{b})$, so by the assumption that $f_\alpha$ is partial elementary we have $\overline{N} \models \varphi(d, f_\alpha \circ \overline{b})$. Choose $d \in N$ so that $\overline{N} \models \varphi(d, f_\alpha \circ \overline{b})$. Let $f_{\alpha+1} = f_\alpha \cup \{(a_\alpha, d)\}$. To show that $f_{\alpha+1}$ is partial elementary, suppose that $\chi(v, \overline{c})$ is a formula with parameters $\overline{c} \in A_\alpha$, and $\overline{M} \models \chi(a_\alpha, \overline{c})$. So by (4) we have $\overline{M} \models \forall v[\varphi(v, \overline{b}) \rightarrow \chi(v, \overline{c})]$, hence $\overline{N} \models \forall v[\varphi(v, f_\alpha \circ \overline{b}) \rightarrow \chi(v, f_\alpha \circ \overline{c})]$. Now $\overline{N} \models \varphi(d, f_\alpha \circ \overline{b})$, so $\overline{N} \models \chi(d, f_\alpha \circ \overline{c})$. Hence $f_{\alpha+1}$ is partial elementary.

This finishes the construction of the $f_\alpha$‘s. In particular, $f_\delta$ is an elementary mapping of $\overline{M}_0$ into $\overline{N}$, as desired.

It remains to show that every element of $M_0$ realizes an isolated type over $A$ with respect to $\overline{M}_0$. We prove by induction on $\alpha$ that every element of $A_\alpha$ realizes an isolated type over $A$ with respect to $\overline{M}$, for each $\alpha \leq \delta$. This is true for $\alpha = 0$, since any element $a \in A$ is isolated over $A$ by the formula $v = a$. The inductive step to a limit ordinal $\alpha$ is obvious. Now suppose that $b \in A_{\alpha+1}$. Then $b$ is isolated over $A_\alpha$ by construction, so $b$ is isolated over $A$ by the inductive hypothesis and Lemma 7.19.

Clearly being isolated over $A$ with respect to $\overline{M}$ implies isolated over $A$ with respect to $\overline{M}_0$.

**Corollary 7.21.** If $T$ is $\omega$-stable, then it has a prime model.

**Proof.** Take $A = \emptyset$ in Theorem 7.20.

**Corollary 7.22.** If $T$ is $\omega$-stable and has no Vaughtian pairs, and if $\varphi(\overline{v})$ is a formula with parameters in $M$ such that $\varphi(\overline{M})$ is infinite, then $\overline{M}$ is prime over $\varphi(\overline{M})$.

**Proof.** By Theorem 7.20 there is an $\overline{N} \preceq \overline{M}$ which is prime over $\varphi(\overline{M})$. Since $\varphi(\overline{M}) \subseteq N$, we have $\varphi(\overline{N}) = \varphi(\overline{M})$. Since $T$ has no Vaughtian pairs, it follows that $\overline{N} = \overline{M}$.

**Theorem 7.23.** If $T$ is $\omega$-stable and has no Vaughtian pairs, then $T$ is $\kappa$-categorical for every uncountable cardinal $\kappa$.

**Proof.** Assume the hypotheses, with $\kappa$ uncountable. Suppose that $\overline{M}, \overline{N} \models T$ with $|M| = |N| = \kappa$. Let $\overline{M}_0$ be a prime model of $T$ by Corollary 7.21. Wlog $\overline{M}_0 \preceq \overline{M}, \overline{N}$. By Corollary 7.15 let $\varphi(\overline{v})$ be strongly minimal over $\overline{M}_0$.

(1) $|\varphi(\overline{M})| = |\varphi(\overline{N})| = \kappa$. 

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For, suppose that $|\varphi(M)| < \kappa$. By the downward L"owenheim-Skolem theorem, let $\overline{P}$ be an elementary substructure of $\overline{M}$ containing both $\varphi(M)$ and the parameters of $\varphi$, with $|P| < \kappa$. This contradicts Corollary 7.16. Hence $|\varphi(M)| = \kappa$. By symmetry, $|\varphi(N)| = \kappa$.

By Lemma 7.7, $\dim(\varphi(M)) = \dim(\varphi(N))$, and hence there is a bijection $f : \varphi(M) \to \varphi(N)$ which is a partial elementary embedding of $\varphi(M)$ into $\varphi(N)$, by Lemma 7.9. By Corollary 7.22, $\overline{M}$ is prime over $\varphi(M)$, and hence $f$ can be extended to an elementary embedding of $\overline{M}$ into $\overline{N}$. By Corollary 7.16, $f$ maps onto $\overline{N}$. \qed

We now give some results of a general nature before turning to the converse of Theorem 7.23. We will use Ramsey’s theorem from set theory, and we begin with a proof of it.

**Ramsey’s Theorem.** Suppose that $M$ is an infinite set, $n$ and $r$ are positive integers, and $f : [M]^n \to r$. (is considered as equal to $\{0, \ldots, r-1\}$.) Then there exist an $i < r$ and an infinite $N \subseteq M$ such that $f(a) = i$ for all $a \in [N]^n$.

**Proof.** We may assume that $M = \omega$. We proceed by induction on $n$. First suppose that $n = 1$. Thus $f : [\omega]^1 \to r$, so $\omega = \bigcup_{i \in r} \{j : j \in \omega : f(\{j\}) = i\}$. It follows that there is an $i \in r$ such that $N \triangleq \{j : j \in \omega : f(\{j\}) = i\}$ is infinite, as desired.

Now assume that the theorem holds for $n \geq 1$, and suppose that $f : [\omega]^{n+1} \to r$. For each $m \in \omega$ define $g_m : [\omega \setminus \{m\}]^n \to r$ by:

$$g_m(X) = f(X \cup \{m\}).$$

Then by the inductive hypothesis, for each $m \in \omega$ and each infinite $S \subseteq \omega$ there is an infinite $H^S_m \subseteq S \setminus \{m\}$ such that $g_m$ is constant on $[H^S_m]^n$. We now construct by recursion two sequences $\langle S_i : i \in \omega \rangle$ and $\langle m_i : i \in \omega \rangle$. Each $m_i$ will be in $\omega$, and we will have $S_0 \supseteq S_1 \supseteq \cdots$. Let $S_0 = \omega$ and $m_0 = 0$. Suppose that $S_i$ and $m_i$ have been defined, with $S_i$ an infinite subset of $\omega$. We define

$$S_{i+1} = H^S_{m_i} \quad \text{and} \quad m_{i+1} = \text{the least element of } S_{i+1} \text{ greater than } m_i.$$

Clearly $S_0 \supseteq S_1 \supseteq \cdots$ and $m_0 < m_1 < \cdots$. Moreover, $m_i \in S_i$ for all $i \in \omega$.

(1) For each $i \in \omega$, the function $g_{m_i}$ is constant on $[\{m_j : j > i\}]^n$.

In fact, $\{m_j : j > i\} \subseteq S_{i+1}$ by the above, and so (1) is clear by the definition.

Let $p_i < r$ be the constant value of $g_{m_i} \upharpoonright [\{m_j : j > i\}]^n$, for each $i \in \omega$. Hence

$$\omega = \bigcup_{j < r} \{i : i \in \omega : p_i = j\};$$

so there is a $j < r$ such that $K \triangleq \{i : i \in \omega : p_i = j\}$ is infinite. Let $L = \{m_i : i \in K\}$. We claim that $f([L]^{n+1}) \subseteq \{j\}$, completing the inductive proof. For, take any $X \in [L]^{n+1}$; say $X = \{m_{i_0}, \ldots, m_{i_n}\}$ with $i_0 < \cdots < i_n$. Then

$$f(X) = g_{m_{i_0}}(\{m_{i_1}, \ldots, m_{i_n}\}) = p_{i_0} = j. \qed$$

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Now we return to model theory. Let \((I, <)\) be a linear order, \(\overline{M}\) a structure, and \(\langle a_i : i \in I \rangle\) a system of distinct elements of \(M\). We say that \(\langle a_i : i \in I \rangle\) is a system of order indiscernibles for \(\overline{M}\) iff for every formula \(\varphi(w_1, \ldots, w_m)\) with free variables among the distinct variables \(w_1, \ldots, w_m\) and all sequences \(i_1 < \cdots < i_m\) and \(j_1 < \cdots < j_m\) of elements of \(I\) we have

\[
\overline{M} \models \varphi(a_{i_1}, \ldots, a_{i_m}) \iff \varphi(a_{j_1}, \ldots, a_{j_m}).
\]

**Theorem 7.24.** Let \(T\) be a theory with infinite models, and let \((I, <)\) be an infinite linear order. Then \(T\) has a model with a system \(\langle a_i : i \in I \rangle\) of order indiscernibles.

**Proof.** We will work with the standard sequence \(v_1, v_2, \ldots\) of variables; all variables are assumed to be among these. Adjoin to the language a system \(\langle c_i : i \in I \rangle\) of distinct new individual constants. Let \(\Gamma\) be the union of the following set of sentences:

1. \(T\);
2. \(c_i \neq c_j\) for \(i \neq j\).
3. \(\varphi(c_{i_1}, \ldots, c_{i_p}) \iff \varphi(c_{j_1}, \ldots, c_{j_p})\) for every formula \(\varphi(v_1, \ldots, v_p)\) with free variables exactly the variables \(v_1, \ldots, v_p\) and all sequences \(i_1 < I \cdots < I\) and \(j_1 < I \cdots < I\) of elements of \(I\).

We claim that every finite subset of \(\Gamma\) has a model. So, suppose that \(\Delta \subseteq \Gamma\) is finite. Let \(I_0\) be the set of all \(i \in I\) such that \(c_i\) occurs in one of the formulas in \(\Delta\). Let \(\varphi_1, \ldots, \varphi_m\) be all of the formulas occurring in the third part of \(\Delta\) as above, and for each \(k \in \{1, m\}\) let \(p_k\) be the “p” involved. Let \(n = \max\{p_k : 1 \leq k \leq n\}\). Let \(\overline{M}\) be an infinite model of \(T\), and fix any linear order \(\prec_M\) of \(M\). We now define \(F : [M]^n \to \mathcal{P}(m)\) as follows. Given \(A \in [M]^n\) with \(A = \{a_1, \ldots, a_n\}, a_1 < M \cdots < M a_n\), let

\[
F(A) = \{k : \overline{M} \models \varphi_k[a_1, \ldots, a_n]\}.
\]

By Ramsey’s theorem let \(X \in [M]^\omega\) and \(\eta \in \mathcal{P}(m)\) be such that \(F(A) = \eta\) for all \(A \in [X]^n\). Let \(I_0 = \{s_0, \ldots, s_{m-1}\}\) with \(s_0 < I \cdots < I s_{m-1}\). Let \(x_0 < M \cdots < M x_{m+n-1}\) be elements of \(X\). Define \(a_{s_k} = x_k\) for all \(k < m\). Thus for any \(i, j \in I_0\) we have \(i < I j\) iff \(a_i < M a_j\). Now \((\overline{M}, a_i)_{i \in I_0}\) is a model of \(\Delta\). In fact, this is clear for the first two kinds of sentences above. Now take one of the third sort:

\[
\varphi_k(c_{i_1}, \ldots, c_{i_{p_k}}) \iff \varphi_k(c_{j_1}, \ldots, c_{j_{p_k}})
\]

where \(\varphi_k(v_1, \ldots, v_{p_k})\) is a formula with free variables exactly the variables \(v_1, \ldots, v_{p_k}\) and with sequences \(i_1 < I \cdots < I i_{p_k}\) and \(j_1 < I \cdots < I j_{p_k}\) of elements of \(I_0\). Using the additional \(n\) elements of \(X\) mentioned above, extend \(a_{i_1}, \ldots, a_{i_{p_k}}\) to a sequence \(\vec{b} \in \omega X\) strictly increasing in the sense of \(\prec_M\), and extend \(a_{j_1}, \ldots, a_{j_{p_k}}\) to a sequence \(\vec{c} \in \omega X\) strictly increasing in the sense of \(\prec_M\). Then

\[
(\overline{M}, a_i)_{i \in I_0} \models \varphi_k(c_{i_1}, \ldots, c_{i_{p_k}})
\]

iff

\[
\overline{M} \models \varphi_k[\vec{b}]
\]

iff \(k \in F(\text{rng}(\vec{b}))\)

iff \(k \in \eta\)

iff \(k \in F(\text{rng}(\vec{c}))\)

iff \(\overline{M} \models \varphi_k[\vec{c}]\)

iff \((\overline{M}, a_i)_{i \in I_0} \models \varphi_k(c_{j_1}, \ldots, c_{j_{p_k}})\).

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This finishes the proof that \( (\overline{M}, a_i)_{i \in I_0} \) is a model of \( \Delta \).

Hence by the compactness theorem, let \( (\overline{N}, d_i)_{i \in I} \) be a model of \( \Gamma \). We claim that \( \overline{N} \) is as desired. For, suppose that \( \varphi(\overline{w}) \) is a formula with every free variable occurring in the sequence \( \overline{w} = (w_1, \ldots, w_q) \), and \( i_1 < i_2 \cdots < I_i q, j_1 < i_2 \cdots < I_i j_q \). Let the variables actually occurring free in \( \varphi \) be \( w_{s(1)}, \ldots, w_{s(r)} \), with \( 1 \leq s(1) < \cdots < s(r) \leq q \). Let \( \varphi' \) be obtained from \( \varphi \) by replacing \( w_{s(1)}, \ldots, w_{s(r)} \) by \( v_1, \ldots, v_r \) respectively, after changing bound variables to avoid clashes. Then \( \varphi' \) is a formula with exactly the free variables \( v_1, \ldots, v_r \). Moreover, \( i_{s(1)} < i_1 \cdots < I_i j_{s(r)} \) and \( j_{s(1)} < i_1 \cdots < I_i j_{s(r)} \). Hence

\[
(\overline{N}, d_i)_{i \in I} \models \varphi'(c_{i_{s(1)}}, \ldots, c_{i_{s(r)}}) \iff \varphi'(c_{j_{s(1)}}, \ldots, c_{j_{s(r)}}).
\]

It follows that

\[
\overline{N} \models \varphi'(d_{i_{s(1)}}, \ldots, d_{i_{s(r)}}) \iff \varphi'(d_{j_{s(1)}}, \ldots, d_{j_{s(r)}});
\]

\[
\overline{N} \models \varphi(d_{i_{s(1)}}, \ldots, d_{i_{s(r)}}) \iff \varphi(d_{j_{s(1)}}, \ldots, d_{j_{s(r)}});
\]

\[
\overline{N} \models \varphi(d_{i_1}, \ldots, d_{i_q}) \iff \varphi(d_{j_1}, \ldots, d_{j_q}). \quad \square
\]

A theory \( T \) in a language \( \mathcal{L} \) has built-in Skolem functions iff for every positive integer \( n \), every system \( v, w_1, \ldots, w_n \) of distinct variables, and every formula \( \varphi(v, w_1, \ldots, w_n) \) without parameters whose free variables are among \( v, w_1, \ldots, w_n \), there is an \( m \)-ary function symbol \( f \) such that

\[
T \models \forall \overline{w} \exists v \varphi(v, \overline{w}) \rightarrow \varphi(f(\overline{w}), \overline{w})].
\]

**Theorem 7.25.** Let \( T \) be a theory in a language \( \mathcal{L} \). Then there exist a language \( \mathcal{L}^* \supseteq \mathcal{L} \) and a theory \( T^* \supseteq T \) in \( \mathcal{L}^* \) such that:

(i) \( T^* \) has built-in Skolem functions.

(ii) Each model of \( T \) can be expanded to a model of \( T^* \).

(iii) \( |\mathcal{L}^*| = |\mathcal{L}| + \omega \).

**Proof.** Fix \( c \in M \) We define \( \mathcal{L}_0, \mathcal{L}_1, \ldots \) and \( T_0, T_1, \ldots \) by recursion. Let \( \mathcal{L}_0 = \mathcal{L} \) and \( T_0 = T \). Having defined \( \mathcal{L}_m \) and \( T_m \), for each formula \( \varphi(v, w_1, \ldots, w_n) \) as in the above definition, introduce an \( n \)-ary function symbol \( f_\varphi \), and add the following sentence to \( T_m \):

\[
\forall \overline{w} \exists v \varphi(v, \overline{w}) \rightarrow \varphi(f_\varphi(\overline{w}), \overline{w})].
\]

This finishes the construction. Let \( \mathcal{L}^* = \bigcup_{m \in \omega} \mathcal{L}_m \) and \( T^* = \bigcup_{m \in \omega} T_m \). The desired conditions are easy to check. \( \square \)

**Theorem 7.26.** Let \( \mathcal{L} \) be countable and let \( T \) be an \( \mathcal{L} \)-theory with an infinite model. Suppose that \( \kappa \) is an infinite cardinal. Then there is a model \( \overline{M} \) of \( T \) of size \( \kappa \) such that for every \( A \subseteq M \) and every positive integer \( n \), \( \overline{M} \) realizes at most \( |A| + \omega \) \( n \)-types over \( A \).

**Proof.** By Theorem 7.24, let \( \overline{N} \) be a model of \( T \) with a system \( \langle a_\alpha : \alpha < \kappa \rangle \) of order indiscernibles with respect to \( (\kappa, <) \). Let \( I = \{ a_\alpha : \alpha < \kappa \} \). Let \( \mathcal{L}^* \) and \( T^* \) be as in Theorem 7.25. Let \( M \) be the closure under all of the functions of \( \overline{N}^* \) of \( I \). Then \( M \) is the universe of some substructure \( \overline{M}^* \) of \( \overline{N}^* \). Let \( \overline{M} \) be the reduct of \( \overline{M}^* \) to the language \( \mathcal{L} \).
So $\mathcal{M} \models T$, and $|M| = \kappa$. Suppose that $A \subseteq M$. For each $b \in M$ we can write $b = t_b(x_b)$, where $t_b$ is a term and $x_b$ is a strictly increasing sequence $\langle x_b(0), \ldots, x_b(mb-1) \rangle$ of elements of $I$. Let $X = \{ y \in I : y = x_b(u) \text{ for some } b \in A \text{ and } u < m_b \}$. Now for any $c \in {}^nM$ we define (with $c = \langle c(i) : i < n \rangle$)

$$L_c = \langle \tau_{c(i)} : i < n \rangle;$$

$$N_c = \{ (i, j, u, v) : i < j, u < m_{c(i)}, v < m_{c(j)} \};$$

for $(i, j, u, v) \in N_c$, $F_c(i, j, u, v) = \begin{cases} 0 & \text{if } x_{c(i)}(u) < x_{c(j)}(v), \\ 1 & \text{if } x_{c(i)}(u) = x_{c(j)}(v), \\ 2 & \text{if } x_{c(i)}(u) > x_{c(j)}(v); \end{cases}$

$$P_c = \{ (i, u, y) : i < n, u < m_i, y \in X \};$$

for $(i, u, y) \in P_c$, $G_c(i, u, y) = \begin{cases} 0 & \text{if } x_{c(i)}(u) = y, \\ 1 & \text{if } x_{c(i)}(u) < y, \\ 2 & \text{if } x_{c(i)}(u) > y; \end{cases}$

$$T(c) = \langle L_c, F_c, G_c \rangle.$$

Now we claim that if $c, d \in {}^nM$ and $T(c) = T(d)$, then $\text{tp}^{\mathcal{M}}(c/A) = \text{tp}^{\mathcal{M}}(d/A)$. For, assume that $T(c) = T(d)$, and let $\varphi(\bar{v}, a)$ be given, with $a \in {}^lA$. Let

$$Y_c = \{ x_{c(i)}(u) : i < n, u < m_i \} \cup \{ x_{a(i)}(u) : i < l, u < m_i \};$$

$$Y_d = \{ x_{d(i)}(u) : i < n, u < m_i \} \cup \{ x_{a(i)}(u) : i < l, u < m_i \}.$$

Clearly $|Y_c| = |Y_d|$. Let $\langle z^c_i : i < e \rangle$ and $\langle z^d_i : i < e \rangle$ enumerate $Y_c$ and $Y_d$ respectively, in the order $<_I$. Let $\langle w_i : i < e \rangle$ be a sequence of new variables. Say $x_{c(i)}(u) = z^c_{k(i,u)}$ and $x_{a(i)}(u) = z^c_{l(i,u)}$. Then by $T(c) = T(d)$ we have $x_{d(i)}(u) = z^d_{k(i,u)}$ and $x_{a(i)}(u) = z^d_{l(i,u)}$. Let $\varphi'$ be the formula

$$\varphi(\langle t_{c(i)}(w_{k(i,0)}, \ldots, w_{k(i,m_i-1)}) : i < n \rangle, \langle t_{a(i)}(w_{l(i,0)}, \ldots, w_{l(i,m_i-1)}) : i < l \rangle).$$

Then

$$\mathcal{M} \models \varphi(c, a) \iff \mathcal{M} \models \varphi'(z^c)$$

iff

$$\mathcal{M} \models \varphi'(z^d)$$

iff

$$\mathcal{M} \models \varphi(d, a).$$

This proves our claim. Now clearly there are at most $|A| + \omega$ choices for $T(c)$, so the conclusion of the theorem follows. \hfill \Box

Now we again make the standing assumption that $T$ is a complete theory in a countable language with only infinite models.

**Theorem 7.27.** If $T$ is $\kappa$-categorical for some uncountable $\kappa$, then $T$ is $\omega$-stable.

**Proof.** Suppose that $T$ is not $\omega$-stable. Then there is a model $\mathcal{M}$ of $T$, a countable subset $A$ of $M$, and a positive integer $n$, such that $|S^n_M(A)| > \omega$. Let $\overline{\mathcal{M}}$ be a countable
elementary submodel of $\bar{M}$ containing $A$. Then $\bar{M}' \models T$ and $|S^n_\alpha^M(A)| > \omega$. Hence $\bar{M}'$ has an elementary extension $\bar{N}_0$ of size $\kappa$ which realises uncountably many $n$-types over $A$. By Theorem 7.26 there is a model $\bar{N}_1$ of $T$ such that for every countable $B \subseteq N_1$, $\bar{N}_1$ realizes only countably many $n$ types over $B$. Hence $\bar{N}_0$ and $\bar{N}_1$ are not isomorphic. □

If $\bar{M}$ is an infinite structure and $\kappa$ is an infinite cardinal, we say that $\bar{M}$ is $\kappa$-homogeneous iff for every $A \in [M]^{<\kappa}$, every partial elementary map $f : A \to \bar{M}$, and every $a \in M$, there is a partial elementary map $f^+ : A \cup \{a\} \to \bar{M}$ which extends $f$. We say that $\bar{M}$ is homogeneous iff it is $|M|$-homogeneous.

**Lemma 7.28.** Suppose that $\bar{M}$ and $\bar{N}$ are $\mathcal{L}$ structures, $n$ is a positive integer, $a \in {}^nM$, and $b \in {}^nN$. Then the following conditions are equivalent:

(i) $tp^\bar{M}(a) = tp^\bar{N}(b)$.

(ii) There is a partial elementary map $f : rng(a) \to N$ such that $b = f \circ a$.

**Proof.** (i)⇒(ii): Assume (i). Define $f(a_i) = b_i$ for all $i < n$. $f$ is well defined, since $a_i = a_j$ implies that $v_i = v_j \in tp^\bar{M}(a) = tp^\bar{N}(b)$, hence $b_i = b_j$. Clearly $f$ is partial elementary.

(ii)⇒(i): clear. □

**Lemma 7.29.** Suppose that $\kappa$ is an infinite cardinal, $\bar{M}$ is $\kappa$-homogeneous, $n$ is a positive integer, $\bar{a}, b \in {}^nM$, $tp^\bar{M}(\bar{a}) = tp^\bar{M}(b)$, and $c \in M$. Then there is a $d \in M$ such that $tp^\bar{M}((\bar{a})^\kappa\langle c \rangle) = tp^\bar{M}(b^\kappa\langle d \rangle)$.

**Proof.** This is immediate from Lemma 7.28. □

**Lemma 7.30.** The following are equivalent:

(i) $\bar{M}$ is $\omega$-homogeneous.

(ii) For every positive integer $n$, all $\bar{a}, b \in {}^nM$, and all $c \in M$, if $tp^\bar{M}(\bar{a}) = tp^\bar{M}(b)$, then there is a $d \in M$ such that $tp^\bar{M}((\bar{a})^\omega\langle c \rangle) = tp^\bar{M}(b^\omega\langle d \rangle)$.

**Proof.** (i)⇒(ii): Assume (i) and the hypothesis of (ii). So by Lemma 7.28 there is an elementary map $f : rng(\bar{a}) \to M$ such that $b = f \circ \bar{a}$. By (i), extend $f$ to an elementary map $f^+ : rng(\bar{a}) \cup \{c\} \to M$. Let $d = f(c)$. Then by Lemma 7.28 again, $tp^\bar{M}((\bar{a})^\omega\langle c \rangle) = tp^\bar{M}(b^\omega\langle d \rangle)$.

(ii)⇒(i): Assume (ii) and suppose that $f : A \to M$ is partial elementary, where $A$ is a finite subset of $M$, and suppose that $c \in M$. Say $rng(\bar{a}) = A$. By Lemma 7.28 we have $tp^\bar{M}(\bar{a}) = tp^\bar{M}(f \circ \bar{a})$. Hence by (ii) choose $d \in M$ such that $tp^\bar{M}((\bar{a})^\omega\langle c \rangle) = tp^\bar{M}((f \circ \bar{a})^\omega\langle d \rangle)$. By Lemma 28 we get a partial elementary map $g$ such that $(f \circ \bar{a})^\omega\langle d \rangle = g \circ (\bar{a}^\omega\langle c \rangle)$. Thus $g$ extends $f$ and $g(c) = d$, as desired. □

**Theorem 7.31.** If $\bar{M}$ and $\bar{N}$ are countable homogeneous models of $T$ and for each positive integer $n$ they realize the same $n$-types, then they are isomorphic.

**Proof.** Let $a_0, a_1, \ldots$ enumerate $M$ and $b_0, b_1, \ldots$ enumerate $N$. We now define by recursion partial elementary maps $f_0, f_1, \ldots$ from subsets of $M$ into $\bar{N}$. Let $f = \emptyset$; so it is partial elementary into $\bar{N}$ because $T$ is complete. Now suppose that a partial elementary
map $f_s$ has been defined from a finite subset of $M$ into $\bar{N}$. Let $\tau$ be a sequence enumerating the domain of $f$.

**Case 1.** $s$ is even, say $s = 2i$. By hypothesis, let $d, e \in N$ such that $\text{tp}^M(\bar{\tau} \langle a_i \rangle) = \text{tp}^{\bar{N}}(d \langle e \rangle)$. Hence $\text{tp}^M(\tau) = \text{tp}^{\bar{N}}(d)$. Also, by Lemma 7.28, $\text{tp}^M(\tau) = \text{tp}^{\bar{N}}(f_s \circ \tau)$. So $\text{tp}^{\bar{N}}(d) = \text{tp}^{\bar{N}}(f_s \circ \tau)$. Since $\bar{N}$ is homogeneous, by Lemma 7.29 there is a $u \in N$ such that $\text{tp}^{\bar{N}}(d \langle e \rangle) = \text{tp}^{\bar{N}}((f_s \circ \tau) \circ \langle u \rangle)$. Let $f_{s+1} = f_s \cup \{(a_i, u)\}$. Then

$$\text{tp}^M(\bar{\tau} \langle a_i \rangle) = \text{tp}^{\bar{N}}(d \langle e \rangle) = \text{tp}^{\bar{N}}((f_s \circ \tau) \circ \langle u \rangle) = \text{tp}^{\bar{N}}(f_{s+1} \circ (\bar{\tau} \langle a_i \rangle)),$$

so by Lemma 7.28 $f_{s+1}$ is partial elementary.

**Case 2.** $s$ is odd, say $s = 2i + 1$. This is treated similarly. Choose $d, e \in M$ such that $\text{tp}^M(d \langle e \rangle) = \text{tp}^{M}(f \circ \tau) \langle b_i \rangle$. Hence $\text{tp}^M(d) = \text{tp}^{\bar{N}}(f \circ \tau)$. Also, by Lemma 7.28 $\text{tp}^M(\tau) = \text{tp}^{\bar{N}}(d \langle e \rangle)$. Since $\bar{M}$ is homogeneous, by Lemma 7.27 there is a $u \in M$ such that $\text{tp}^M(\bar{\tau} \langle u \rangle) = \text{tp}^M(d \langle e \rangle)$. Now if there is an $i$ such that $c_i = u$, then $d_i = e$, hence $f(c_i) = b_i$. Hence $f_{\sigma + 1} \overset{\text{def}}{=} f_s \cup \{(u, b_i)\}$ is a function. Also,

$$\text{tp}(\bar{\tau} \langle u \rangle) = \text{tp}(d \langle e \rangle) = \text{tp}^{\bar{N}}((f \circ \tau) \langle b_i \rangle) = \text{tp}^{\bar{N}}(f_{s+1} \circ (\bar{\tau} \langle u \rangle)),$$

so by Lemma 7.28 $f_{s+1}$ is partial elementary.

Clearly $\bigcup_{s \in \omega} f_s$ is as desired. \hfill \Box

We consider an expansion $\bar{L}_U$ of our language $\bar{L}$ obtained by adjoining a one-place relation symbol $U$. For each formula $\varphi(v_0, \ldots, v_{n-1})$ of $\bar{L}$ we associate a formula $\varphi_U(v_0, \ldots, v_{n-1})$ of $\bar{L}_U$, as follows:

If $\varphi$ is atomic, then $\varphi_U$ is $Uv_0 \land \ldots \land Uv_{n-1} \land \varphi$.

$(\neg \psi)_U = \neg \psi_U$.

$(\psi \land \chi)_U = \psi_U \land \chi_U$.

$(\exists w \psi)_U = \exists w[Uw \land \psi_U]$.

**Proposition 7.32.** If $\bar{M}$ is a substructure of $\bar{N}$, $\varphi(v_0, \ldots, v_{n-1})$ is a formula of $\bar{L}$, and $a \in ^n M$, then $\bar{M} \models \varphi(\bar{a})$ iff $(\bar{N}, U) \models \varphi_U(\bar{a})$.

**Proof.** An easy induction on $\varphi$. \hfill \Box

**Theorem 7.33.** If there is a Vaughtian pair $(\bar{M}, \bar{N})$, then there is one in which $N$ is countable.

**Proof.** Let $\varphi$ be a formula such that $\varphi(\bar{M})$ is infinite and $\varphi(\bar{M}) = \varphi(\bar{N})$. Let $\bar{a}$ be the parameters from $M$ occurring in $\varphi$. We consider the structure $(\bar{N}, M)$ in the language $\bar{L}_U$. Let $(\bar{N}_0, M_0)$ be a countable elementary substructure of $(\bar{N}, M)$ such that $\bar{a} \in M_0$. Among the sentences holding in $(\bar{N}, M)$ are those asserting that $M$ is closed under the fundamental function of $\bar{N}$. Hence $M_0$ is closed under the fundamental functions of $\bar{N}_0$, and hence $M_0$ is the universe of a substructure $\bar{M}_0$ of $\bar{N}_0$. For any formula $\psi(b)$ with
Thus $\overline{M}_0 \preceq \overline{N}_0$. Moreover, the sentence $\exists x - Ux$ holds in $(\overline{N}, M)$, hence also in $(\overline{N}_0, M_0)$, so that $\overline{M}_0 \neq \overline{N}_0$.

Clearly $\varphi(\overline{M}_0)$ is infinite and $\varphi(\overline{M}_0) = \varphi(\overline{N}_0)$. \hfill \square

**Lemma 7.34.** Suppose that $\overline{M} \preceq \overline{N}$ and in the language $\mathcal{L}_U$ we have $(\overline{N}, M) \preceq (\overline{N}', M')$. Then $M'$ is the universe of a structure $\overline{M}'$, and $\overline{M} \preceq \overline{M}' \preceq \overline{N}'$.

**Proof.** Clearly $M \subseteq M'$, and $M'$ is closed under the fundamental functions of $\overline{N}'$, and hence is the universe of a structure $\overline{M}'$. If $\varphi$ is a formula and $\overline{a} \in M$, then by Proposition 7.32,

$$\overline{M} \models \varphi(\overline{a}) \iff (\overline{N}, M) \models \varphi^U(\overline{a}) \iff (\overline{N}', M') \models \varphi^U(\overline{a}) \iff \overline{M}' \models \varphi(\overline{a}).$$

Thus $\overline{M} \preceq \overline{M}'$.

Next we claim that

$$(1) \quad (\overline{N}, M) \models \forall \overline{v}[Uv_0 \land \ldots \land Uv_{n-1} \rightarrow (\varphi(\overline{v}) \leftrightarrow \varphi^U(\overline{v}))].$$

In fact, suppose that $\overline{a} \in M$ is given. Then

$$(\overline{N}, M) \models \varphi(\overline{a}) \iff \overline{N} \models \varphi(\overline{a}) \iff \overline{M} \models \varphi(\overline{a}) \iff (\overline{N}, M) \models \varphi^U(\overline{a}).$$

This proves (1). Hence we also get

$$(2) \quad (\overline{N}', M') \models \forall \overline{v}[Uv_0 \land \ldots \land Uv_{n-1} \rightarrow (\varphi(\overline{v}) \leftrightarrow \varphi^U(\overline{v}))].$$

Now let $b \in M'$. Then using (2),

$$\overline{M}' \models \varphi(b) \iff (\overline{N}', M') \models \varphi^U(b) \iff (\overline{N}', M') \models \varphi(b) \iff \overline{N}' \models \varphi(b). \hfill \square$$

**Lemma 7.35.** Suppose that $\overline{M} \preceq \overline{N}$, $\overline{N}$ countable, $\overline{a} \in M$, $\overline{b} \in \overline{N}$.

Then there exist countable $\overline{M}'$, $\overline{N}'$ and $\overline{c} \in M$ such that $(\overline{N}, M) \prec (\overline{N}', M')$ and $\text{tp}^{\overline{N}}(\overline{b}/\overline{a}) = \text{tp}^{\overline{M}'}(\overline{c}/\overline{a})$.

**Proof.** Say $\overline{b}$ is of length $n$. In $\mathcal{L}_U$ let $\Gamma(\overline{v})$ be the following set of formulas:

\[\text{Eldiag}(\overline{N}, M) \{ \wedge_{i<n} Uv_i \land \varphi^U(\overline{v}, \overline{a}) : \overline{N} \models \varphi(\overline{b}, \overline{a}) \}.\]
If $\varphi_0, \ldots, \varphi_{m-1}$ are such that $\overline{N} \models \varphi_i(b, c)$ for all $i < m$, then $\overline{N} \models \exists v \wedge \bigwedge_{i < m} \varphi_i(v, \overline{a})$, hence by Proposition 7.32,

$$(\overline{N}, M) \models \exists v \left( \bigwedge_{i < n} U v_i \wedge \bigwedge_{i < m} \varphi_i^U(v, \overline{a}) \right).$$

This shows that every finite subset of $\Gamma(\overline{v})$ is satisfiable. Hence there exist a countable $(\overline{N}, M')$ and $\overline{v} \in M'$ such that $(\overline{N}, M) \preceq (\overline{N}', M')$ and $(\overline{N}', M') \models \varphi^U(\overline{v}, \overline{a})$ whenever $\overline{N} \models \varphi(b, c)$. If $\overline{N} \models \varphi(b, c)$, then $\overline{M}' \models \varphi(\overline{v}, \overline{a})$.  

\textbf{Corollary 7.36.} Suppose that $\overline{M} \preceq \overline{N}$ and $\overline{N}$ is countable. Then there exist countable $\overline{M}'$, $\overline{N}'$ such that $(\overline{N}, M) \preceq (\overline{N}', M')$, and for every $\overline{a} \in M$ and every $\overline{b} \in N$ there is an $\overline{c} \in M^*$ such that $tp^\overline{N}(\overline{b}/\overline{a}) = tp^\overline{M'}(\overline{c}/\overline{a})$.

\textbf{Proof.} Iterate Lemma 7.35. \hfill $\square$

\textbf{Lemma 7.37.} Suppose that $\overline{M} \preceq \overline{N}$, $\overline{N}$ is countable, $b, c \in N$, $tp^\overline{N}(a) = tp^\overline{N}(b)$.

Then there exist countable $\overline{M}'$, $\overline{N}'$ and $d$ such that $(\overline{N}, M) \preceq (\overline{N}', M^*)$, $d \in N^*$ and $tp^\overline{N'}(\overline{a} \cup c) = tp^\overline{N'}(\overline{b} \cup d)$.

\textbf{Proof.} Apply the compactness theorem to the set

$$\text{Eldiag}(\overline{N}, M) \{ \varphi(b, u) : \overline{N} \models \varphi(\overline{a}, c) \} \ (u \ \text{a new constant}) \ \square$$

\textbf{Corollary 7.38.} Suppose that $\overline{M} \preceq \overline{N}$ and $\overline{N}$ is countable. Then there exist countable $\overline{M}'$, $\overline{N}'$ and $d$ such that $(\overline{N}, M) \preceq (\overline{N}', M^*)$, and for all $\overline{a}, b, c \in N$, if $tp^\overline{N}(\overline{a}) = tp^\overline{N}(b)$, then there is a $d \in N^*$ such that $tp^\overline{N'}(\overline{a} \cup c) = tp^\overline{N'}(\overline{b} \cup d)$.

\textbf{Proof.} Iterate Lemma 7.37. \hfill $\square$

\textbf{Lemma 7.37a.} Suppose that $\overline{M} \preceq \overline{N}$, $\overline{N}$ is countable, $b, c \in M$, $tp^\overline{M}(a) = tp^\overline{M}(b)$.

Then there exist countable $\overline{M}'$, $\overline{N}'$ and $d$ such that $(\overline{N}, M) \preceq (\overline{N}', M^*)$, $d \in M^*$ and $tp^\overline{M'}(\overline{a} \cup c) = tp^\overline{M'}(\overline{b} \cup d)$.

\textbf{Proof.} Apply the compactness theorem to the set

$$\text{Eldiag}(\overline{N}, M) \{ \varphi^U(b, u) : \overline{M} \models \varphi(\overline{a}, c) \} \ (u \ \text{a new constant}) \ \square$$

\textbf{Corollary 7.38a.} Suppose that $\overline{M} \preceq \overline{N}$, $\overline{N}$ is countable. Then there exist countable $\overline{M}'$, $\overline{N}'$ and $d$ such that $(\overline{N}, M) \preceq (\overline{N}', M^*)$, and for all $\overline{a}, b, c \in M$, if $tp^\overline{M}(\overline{a}) = tp^\overline{N}(b)$, then there is a $d \in M^*$ such that $tp^\overline{M'}(\overline{a} \cup c) = tp^\overline{M'}(\overline{b} \cup d)$.

\textbf{Proof.} Iterate Lemma 7.37a. \hfill $\square$

\textbf{Lemma 7.39.} Suppose that $\overline{M} \preceq \overline{N}$ (so $M \neq N$), and $\overline{N}$ is countable. Then there exist countable $\overline{M}'$, $\overline{N}'$ such that $(\overline{N}, M) \preceq (\overline{N}', M')$, $\overline{N}'$ and $\overline{M}'$ are homogeneous and they realize the same $n$-types for all positive integers $n$. Moreover, they are isomorphic.
Proof. We define an elementary chain \( \langle P_i, Q_i \rangle : i \in \omega \) by recursion. Let \( P_0 = N \) and \( Q_0 = M \). Suppose that \( \langle P_{3i}, Q_{3i} \rangle \) has been defined. Apply Corollary 7.36 to get an elementary extension \( P_{3i+1}, Q_{3i+1} \) of \( \langle P_{3i}, Q_{3i} \rangle \) such that every type realized in \( P_{3i} \) is realized in \( Q_{3i+1} \). Note that these types are realized in \( P_{3i+1} \). Next, apply Corollary 7.38a to obtain an elementary extension \( P_{3i+2}, Q_{3i+2} \) of \( \langle P_{3i+1}, Q_{3i+1} \rangle \) such that for all \( a,b,c \in Q_{3i+1} \), if \( \text{tp}_{P_{3i+1}}(a) = \text{tp}_{Q_{3i+1}}(b) \), then there is a \( d \in Q_{3i+2} \) such that \( \text{tp}_{P_{3i+2}}(a \in c) = \text{tp}_{Q_{3i+2}}(b \in c) \). Finally, apply Corollary 7.38 to obtain an elementary extension \( P_{3i+3}, Q_{3i+3} \) of \( \langle P_{3i+2}, Q_{3i+2} \rangle \) such that for all \( a,b,c \in P_{3i+2} \), \( \text{tp}_{P_{3i+2}}(a) = \text{tp}_{P_{3i+3}}(a \in c) \).}

This finishes the construction. Let \( N' = \bigcup_{i \in \omega} P_i \) and \( M' = \bigcup_{i \in \omega} Q_i \). The desired conclusion is clear, using Theorem 7.31 for the last statement. \( \square \)

Suppose that \( \omega \leq \lambda < \kappa \). We say that \( T \) has a \( (\kappa, \lambda) \)-model iff there exist an \( \bar{M} \models T \) and a formula \( \varphi(\bar{a}) \) such that \( |M| = \kappa \) and \( |\varphi(\bar{M})| = \lambda \).

Lemma 7.40. If \( \omega \leq \lambda < \kappa \) and \( T \) has a \( (\kappa, \lambda) \)-model, then \( T \) has a Vaughtian pair.

Proof. Let \( \bar{N} \) be a \( (\kappa, \lambda) \)-model, with associated formula \( \varphi(\bar{a}) \). By the downward Löwenheim-Skolem theorem, let \( \bar{M} \) be an elementary substructure of \( \bar{N} \) of size \( \lambda \) such that \( \varphi(\bar{N}) \subseteq M \). Clearly then \( \varphi(\bar{N}) = \varphi(\bar{M}) \), so that \( (\bar{M}, \bar{N}) \) is a Vaughtian pair. \( \square \)

Theorem 7.41. If \( T \) has a Vaughtian pair, then \( T \) has an \( (\aleph_1, \aleph_0) \)-model.

Proof. Assume that \( T \) has a Vaughtian pair. By Lemma 7.33 we may assume that \( (\bar{M}, \bar{N}) \) is a Vaughtian pair with \( N \) countable. Say \( \varphi(\bar{M}) = \varphi(\bar{N}) \) is infinite. Also, \( M \neq N \). By Lemma 7.39 there are countable \( \bar{M}', \bar{N}' \) such that \( (\bar{N}, M) \preceq (\bar{N}', M') \), \( \bar{M}' \) and \( \bar{M} \) are homogeneous, the realize the same \( n \)-types for every positive integer \( n \), and they are isomorphic. Still \( M' \neq N' \). Now \( (\bar{N}, M) \models \forall \bar{v}[\varphi(\bar{v})] \iff \bigwedge_{i < \lambda} Uv_i \land \varphi(U\bar{v}) \), so also \( (\bar{N}', M') \models \forall \bar{v}[\varphi(\bar{v})] \iff \bigwedge_{i < \lambda} Uv_i \land \varphi(U\bar{v}) \), and this implies that \( \varphi(\bar{M'}) = \varphi(\bar{N'}) \).

We now define by recursion a sequence \( \langle P_\alpha : \alpha < \omega_1 \rangle \) of models. Let \( P_0 = \bar{N}' \). Now suppose that \( P_\alpha \) has been defined so that \( P_\alpha \cong \bar{N}' \). Then also \( P_\alpha \cong \bar{M}' \), so \( P_\alpha \) has an elementary extension \( P_{\alpha+1} \) such that \( \langle \bar{N}', M' \rangle \cong \langle P_{\alpha+1}, P_\alpha \rangle \). To see this, let \( g \) be an isomorphism from \( P_\alpha \) onto \( \bar{M}' \), and let \( Q \) be a set such that \( Q \cap (N' \setminus M') = Q \cap P_\alpha = \emptyset \) and \( |Q| = |N' \setminus M'| \). Let \( P_{\alpha+1} = P_\alpha \cup Q \), and let \( f : P_{\alpha+1} \to N' \) be a bijection such that \( f \upharpoonright P_\alpha = g \) while \( f \upharpoonright Q \) is a bijection from \( Q \) onto \( N' \setminus M' \). We can make \( P_{\alpha+1} \) into a structure so that \( f \) is an isomorphism from \( P_{\alpha+1} \) onto \( \bar{N}' \). Then \( P_\alpha \) is an elementary substructure of \( P_{\alpha+1} \), since for \( a \in P_\alpha \) we have

\[
P_\alpha \models \varphi[a] \iff \bar{M}' \models \varphi[g \circ a] \iff \bar{N}' \models \varphi[g \circ a] \iff P_{\alpha+1} \models \varphi[a].
\]

For \( \alpha \) limit, let \( P_\alpha = \bigcup_{\beta < \alpha} P_\beta \). Since then \( P_\alpha \) is the union of models isomorphic to \( \bar{N}' \), it is clearly homogeneous and realizes the same types as \( \bar{N}' \). Hence it is isomorphic to \( \bar{N}' \). This finishes the construction.

Let \( P_{\omega_1} = \bigcup_{\alpha < \omega_1} P_\alpha \). Then \( |P_{\omega_1}| = \omega_1 \). Now by induction we have \( \varphi(P_\alpha) = \varphi(\bar{M}') \) for all \( \alpha \leq \omega_1 \). Hence \( |\varphi(P_{\omega_1})| = \omega \).

\( \square \)
Lemma 7.42. Suppose that $T$ is $\omega$-stable, $\mathbb{M} \models T$, and $|M| \geq \aleph_1$. Then $\mathbb{M}$ has a proper elementary extension $\mathbb{N}$ such that for every finite sequence $\overline{w}$ of variables and every $\Gamma(\overline{w})$ of formulas with free variables among $\overline{w}$ and with parameters from $M$ and with $\Gamma(\overline{w})$ countable, if $\Gamma(\overline{w})$ is realized in $\mathbb{N}$, then it is also realized in $\mathbb{M}$.

Proof. First we claim

1. There is a formula $\varphi(v)$ with parameters from $M$ such that $|\varphi(\mathbb{M})| \geq \aleph_1$, and for every formula $\psi(v)$ with parameters from $M$, either $|\varphi(\mathbb{M}) \cap \psi(\mathbb{M})| < \aleph_1$ or $|\varphi(\mathbb{M}) \cap \neg \psi(\mathbb{M})| < \aleph_1$.

Suppose not. Then it is easy to define formulas $\varphi_f$ for $f \in \mathcal{P}\mathcal{P}_2$ such that the following conditions hold for each $f$:

2. $\varphi_\emptyset$ is the formula $v = v$.

3. $|\varphi_f(\mathbb{M})| \geq \aleph_1$.

4. $\varphi_{f \neg 0}(\mathbb{M}) \cap \varphi_{f \neg 1}(\mathbb{M}) = \emptyset$.

This gives $2^\omega$ types over $M$, contradicting the $\omega$-stability of $T$. So (1) holds.

Choose $\varphi(v)$ as in (1), and let $p = \{\psi(v) : \psi(v)$ is a formula with parameters from $M$, and $|\varphi(\mathbb{M}) \cap \psi(\mathbb{M})| \geq \aleph_1\}$.

Note that $\varphi(\mathbb{M}) \cap \psi(\mathbb{M})$ is a co-countable subset of $\varphi(\mathbb{M})$, and an intersection of countably many co-countable subset of a set is still co-countable. Hence

5. $p$ is finitely satisfiable.

From (1) it also follows that $p$ is a complete type.

Let $\mathbb{M}'$ be a proper elementary extension of $\mathbb{M}$ containing an element $c$ which realizes $p$, and choose $d \in M' \setminus M$. Now we apply Theorem 20 to get an elementary substructure $\mathbb{N}$ of $\mathbb{M}'$ which is prime over $M \cup \{c, d\}$ for $T$ and is such that every finite sequence of elements of $N$ realizes an isolated type over $M \cup \{c, d\}$. Thus $M \cup \{c, d\} \subseteq N$, so clearly $\mathbb{M} \prec \mathbb{N}$. Now suppose that $\Gamma(\overline{w})$ is a set of formulas with free variables among $\overline{w}$, with parameters from $M$ and $\Gamma(\overline{w})$ is countable, and such that it is realized in $\mathbb{N}$, say by $b$. Let $\theta(\overline{w}, v)$ be a formula which isolates $\text{tp}^\mathbb{N}(b/M \cup \{c\})$.

6. $\exists\overline{w}\theta(\overline{w}, v) \in p$.

In fact, otherwise $\neg\exists\overline{w}\theta(\overline{w}, v) \in p$, hence $\mathbb{M}' \models \neg\exists\overline{w}\theta(\overline{w}, c)$, hence $\mathbb{N} \models \neg\exists\overline{w}\theta(\overline{w}, c)$. This contradicts $\mathbb{N} \models \theta(b, c)$.

7. $\forall\overline{w}[\theta(\overline{w}, v) \rightarrow \gamma(\overline{w})] \in p$ for every $\gamma(\overline{w}) \in \Gamma(\overline{w})$.

For, otherwise $\exists\overline{w}[\theta(\overline{w}, v) \land \neg \gamma(\overline{w})] \in p$, hence $\mathbb{M}' \models \exists\overline{w}[\theta(\overline{w}, c) \land \neg \gamma(\overline{w})]$, hence $\mathbb{N} \models \exists\overline{w}[\theta(\overline{w}, c) \land \neg \gamma(\overline{w})]$, contradicting $\mathbb{N} \models \varphi(b, c) \land \gamma(b)$.
Now let
\[ \Delta = \{ \exists w \theta(w, v) \} \cup \{ \forall w [\theta(w, v) \rightarrow \gamma(w)] : \gamma(w) \in \Gamma(w) \}. \]

If \( \delta(v) \in \Delta \), then \( \delta(v) \in p \), and so \( |\varphi(\overline{M}) \delta(\overline{M})| < \aleph_1 \). It follows that \( \bigcap_{\delta(v) \in \Delta} \delta(\overline{M}) \neq \emptyset \), i.e. there is a \( c' \in M \) such that \( \overline{M} \models \delta(c) \) for every \( \delta(v) \in \Gamma(v) \). In particular, \( \overline{M} \models \exists w \theta(w, c') \), so we can choose \( b' \in M \) such that \( \overline{M} \models \theta(b', c) \). Now for each \( \gamma(w) \in \Gamma(w) \) the formula \( \forall w [\theta(w, v) \rightarrow \gamma(w)] \) is in \( \Delta \), so it follows that \( \overline{M} \models \gamma(b') \).

**Theorem 7.43.** Suppose that \( T \) is \( \omega \)-stable and has an \((\aleph_1, \aleph_0)\)-model. Then for any \( \kappa > \aleph_1 \) it has a \((\kappa, \aleph_0)\)-model.

**Proof.** Let \( \overline{M} \models T \) with \( |M| = \aleph_1 \), and let \( \varphi(\overline{v}) \) be a formula with \( |\varphi(\overline{M})| = A_0 \). We now construct an elementary chain \( \langle N_\alpha : \alpha < \kappa \rangle \) by recursion. Let \( N_0 = \overline{M} \). Now suppose that \( N_\alpha \) has been defined so that \( \varphi(\overline{M}) = \varphi(N_\alpha) \). We apply Lemma 7.42 to obtain a proper elementary extension \( N_{\alpha+1} \) of \( N_\alpha \) such that if \( G(\overline{w}) \) is a countable type over \( M \) realized in \( N_{\alpha+1} \), then it is realized in \( N_\alpha \). Let
\[ \Gamma_\alpha(\overline{v}) = \{ \varphi(\overline{v}) \} \cup \{ \overline{v} \neq \overline{u} : \overline{v} \in M \text{ and } \overline{M} \models \varphi(\overline{u}) \} \]

Thus \( \Gamma_\alpha \) is a countable type over \( \overline{M} \), but it is not realized in \( N_\alpha \). Hence it is not realized in \( N_{\alpha+1} \). It follows that \( \varphi(N_{\alpha+1}) = \varphi(\overline{M}) \).

For \( \alpha \) limit we let \( N_\alpha = \bigcup_{\beta < \alpha} N_\beta \). Clearly still \( \varphi(N_\alpha) = \varphi(\overline{M}) \).

Finally, \( \bigcup_{\alpha < \kappa} N_\alpha \) is as desired. \( \square \)

**Theorem 7.44.** If \( \overline{M} \) is an infinite structure and \( \kappa \) is a cardinal \( \geq |M| \), then \( \overline{M} \) has an elementary extension \( \overline{N} \) of cardinality \( \kappa \) such that for every formula \( \varphi(\overline{v}) \) with parameters from \( N \), if \( \varphi(\overline{N}) \) is infinite then \( |\varphi(\overline{N})| = \kappa \).

**Proof.** For each formula \( \varphi(\overline{v}) \) adjoin \( \kappa \) many tuples of new constants of the length of \( \overline{v} \), and apply the compactness theorem to the set consisting of \( \text{Eldia}(\overline{M}) \) together with sentences saying, for each \( \varphi(\overline{v}) \) such that \( \varphi(\overline{M}) \) is infinite, that the \( \kappa \) many tuples for this formula are all distinct and satisfy \( \varphi \). \( \square \)

**Theorem 7.45.** Suppose that \( \kappa \) is uncountable and \( T \) is \( \kappa \)-categorical. Then \( T \) has no Vaughtian pairs.

**Proof.** Assume the hypothesis. By Theorem 7.27, \( T \) is \( \omega \)-stable. Suppose that there is a Vaughtian pair. Then by Theorems 7.41 and 7.43, \( T \) has a \((\kappa, \aleph_0)\)-model, and then by Theorem 7.43 it has a \((\kappa, \aleph_0)\)-model \( \overline{M} \). So \( |M| = \kappa \) and \( |\varphi(\overline{M})| = \aleph_0 \) for some formula \( \varphi(\overline{v}) \). By Theorem 7.44, there is a model \( \overline{N} \) of \( T \) in which \( |\varphi(\overline{N})| = |N| = \kappa \). This contradicts \( \kappa \)-categoricity.

**Theorem 7.46.** (Baldwin, Lachlan) Let \( \kappa \) be uncountable. Then the following conditions are equivalent:

(i) \( T \) is \( \kappa \)-categorical

(ii) \( T \) is \( \omega \)-stable and has no Vaughtian pairs.

**Proof.** (i) \( \Rightarrow \) (ii): Theorems 7.27 and 7.45.
Theorem 7.47. (Morley) T is \(\kappa\) categorical for some uncountable \(\kappa\) iff it is \(\kappa\)-categorical for every uncountable \(\kappa\).

EXERCISES

Exc. 7.1. Let \(\overline{M}\) be a field, \(A\) a subfield, and \(a \in M\). Suppose that \(a\) is algebraic over \(A\) in the usual sense of field theory. Show that \(a\) is algebraic over \(A\) in the model-theoretic sense.

Exc. 7.2. Let \(\overline{M} = (\omega, <)\). Show that every element of \(\omega\) is algebraic over \(\emptyset\).

Exc. 7.3. Let \(\overline{A} = ([\omega]^2, R)\), where

\[
R = \{(a, b) : a, b \in [\omega]^2, a \neq b \text{ and } a \cap b \neq \emptyset\}.
\]

(i) Show that \(\{a \in [\omega]^2 : (a, \{0, 1\}) \in R\}\) is neither finite nor cofinite.

(ii) Infer from (i) that \([\omega]^2\) is not minimal.

(iii) If \(f\) is a permutation of \(\omega\), define \(f^+ : [\omega]^2 \to [\omega]^2\) by setting \(f^+(a) = f[a]\) for any \(a \in [\omega]^2\). Show that \(f^+\) is an automorphism of \(\overline{A}\).

(iv) Let \(X = \{a \in [\omega]^2 : 0 \in a \text{ and } a \cap \{1, 2\} = \emptyset\}\). Show that \(X\) is definable in \(\overline{A}\) with parameters.

(v) Show that \(X\) is minimal.

Exc. 7.4. Let \(V\) be an infinite vector space over a finite field \(F\). We consider \(V\) as a structure \((V, +, f_\alpha)_{\alpha \in F}\), where \(f_\alpha(v) = av\) for any \(v \in V\) and \(a \in F\). Show that \(V\) is minimal.

Exc. 7.5. (continuing exc. 7.4) Prove that for any subset \(A\) of \(V\), \(\text{acl}(A) = \text{span}(A)\).

Exc. 7.6. (continuing excs. 7.4, 7.5) By exercise 7.4 and Lemma 7.2, the following holds in \(V\): if \(a \in \text{span}(A \cup \{b\}) \setminus \text{span}(A)\), then \(b \in \text{span}(A \cup \{a\})\). Prove this statement using ordinary linear algebra.

Exc. 7.7. Give an example of a set \(\Gamma\) of sentences and two sentences \(\varphi\) and \(\psi\), such that \(\Gamma \models \varphi\) iff \(\Gamma \models \psi\), but \(\Gamma \not\models (\varphi \leftrightarrow \psi)\).

Exc. 7.8. Show that for \(\Gamma\) a set of sentences and for sentences \(\varphi, \psi\), if \(\Gamma \models \varphi \leftrightarrow \psi\) then \(\Gamma \models \varphi\) iff \(\Gamma \models \psi\).

Exc. 7.9. Prove that the following two conditions are equivalent:

(i) \(\overline{M} \models \varphi[a]\) iff \(\overline{M} \models \psi[a]\).

(ii) \(\overline{M} \models (\varphi \leftrightarrow \psi)[a]\).

Exc. 7.10. Prove that the following two conditions are equivalent, for any sentences \(\varphi, \psi\):

(i) \(\overline{M} \models \varphi\) iff \(\overline{M} \models \psi\).

(ii) \(\overline{M} \models (\varphi \leftrightarrow \psi)\).

Exc. 7.11. In the language with no non-logical symbols, show that \(\omega\) is an indiscernible set in \(\omega\).
Exc. 7.12. (Continuing exercises 7.4, 7.5, 7.6) Let $A = \{w_1, w_2\}$, two members of $V$, and let $b = w_1$. Thus $b \in \text{span}(A)$. According to Lemma 7.8, $\text{tp}^V(b/A)$ is isolated. Give a formula $\varphi(v_0, \overline{a})$ with $\overline{a} \in A$ which isolates $\text{tp}^V(b/A)$.

Exc. 7.13. Suppose that $\overline{M}$ is an infinite structure, $\varphi(v_0)$ is a formula with at most $v_0$ free, and $\varphi(\overline{M})$ is infinite. Show that $\overline{M}$ has a proper elementary extension $\overline{N}$ such that $(\overline{M}, \overline{N})$ is not a Vaughtian pair.