12. Non-standard analysis

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In this section we give a brief introduction to non-standard analysis. This is a fairly well-developed field of mathematics based on model theory. It deals not just with the reals, functions on them, continuity, etc., but also with a proper elementary extension of the reals, infinitesimals, infinite integers, etc. The deeper theory of non-standard analysis requires some subtle set-theoretical considerations, but for elementary considerations such as those in this section, the apparatus of the preceding section is sufficient. For a thorough development of this elementary portion of non-standard analysis, two books of H. J. Keisler are recommended: **Foundations of infinitesimal calculus** (211 pp.) and **Elementary calculus** (880pp.). The latter is a complete calculus course based on non-standard analysis.

Infinitesimals.

The basic theory of infinitesimals can be formulated with respect to any ordered field $\mathbb{R}^*$ which is a proper extension of the field of real numbers. So for a while that is all we assume about $\mathbb{R}^*$. We introduce some of the basic notions of non-standard analysis:

- An element ε of $\mathbb{R}^*$ is an *infinitesimal* provided that $|\varepsilon| < r$ for every positive real number r.

- An element $x \in \mathbb{R}^*$ is *finite* if $|x| < r$ for some real number r. $\text{Fin}(\mathbb{R}^*)$ is the collection of all finite elements of $\mathbb{R}^*$. Of course, if an element $x \in \mathbb{R}^*$ is not finite, we say that it is *infinite*.

- Elements $x, y \in \mathbb{R}^*$ are *infinitely close*, in symbols $x \approx y$, if $|x - y|$ is infinitesimal.

Proposition 12.1. *Fin*$(\mathbb{R}^*)$ *is a subring of* $\mathbb{R}^*$. $\square$

Proposition 12.2. *The set I of all infinitesimals is an ideal in* $\text{Fin}(\mathbb{R}^*)$.

Proof. It is easy enough to check that I is a subset, even a subring, of $\text{Fin}(\mathbb{R}^*)$. To check the ideal property, suppose that $x \in \text{Fin}(\mathbb{R}^*)$ and ε is an infinitesimal. Then there is a real number r such that $|x| < r$. Now if s is any positive real number, then $\frac{s}{r}$ is also a positive real number, hence $|\varepsilon| < \frac{s}{r}$ and so $|\varepsilon x| = |\varepsilon|\,|x| < \frac{s}{r}r = s$. $\square$

Proposition 12.3. $\approx$ *is an equivalence relation on* $\mathbb{R}^*$. $\square$

Theorem 12.4. *Every finite $x \in \mathbb{R}^*$ is infinitely close to a unique real number.*

Proof. Let $X = \{s \in \mathbb{R} : s < x\}$. Now X is nonempty, since we can choose a real number t such that $|x| < t$, so $-t < x < t$, hence $-t \in X$. With the same choice of t we see that X is bounded above. So, let r be the least upper bound of X. We claim that r is as desired. Let u be any positive real number. Then $r - u$ is no longer an upper bound for X, so there is an $s \in X$ such that $r - u < s$, hence $r - u < x$ and $r - x < u$. Also, $r + \frac{u}{2}$ is not in X, so $x \le r + \frac{u}{2} < r + u$, hence $-u < r - x$. So we have shown that $-u < r - x < u$; hence $|r - x| < u$. This shows that r is infinitely close to x.

For uniqueness, suppose that also $r' \in \mathbb{R}$ is infinitely close to x. By 12.3, $r \approx r'$. Clearly then $r = r'$. $\square$

A notation connected with 12.4 is useful. If x is a finite element of $\mathbb{R}^*$, the *standard part* of x, denoted by $\mathrm{st}(x)$, is the unique real number infinitely close to x.

Theorem 12.5. *There are positive and negative infinitesimals.*

Proof. It suffices to show that there are positive infinitesimals. Let $a \in \mathbb{R}^*\backslash\mathbb{R}$.

Case 1. $r < |a|$ for every real number r. Then if s is any positive real number we have $\frac{1}{s} < |a|$, hence $\frac{1}{|a|} < s$, showing that $\frac{1}{|a|}$ is a positive infinitesimal.

Case 2. a is finite. Then a is infinitely close to some real number r, and so $|a - r|$ is a positive infinitesimal. □

Theorem 12.6. *If a is a nonzero element of $\mathbb{R}^*$, then a is infinitesimal iff $\frac{1}{a}$ is infinite. Hence there exist positive and negative infinite elements.*

Proof. Suppose that a is infinitesimal. Let r be any positive real number. Then $|a| < \frac{1}{r}$, and so $r < \frac{1}{|a|}$. This shows that $\frac{1}{|a|}$ is infinite. The converse is proved similarly, and the last statement of the theorem is clear too. □

Non-standard arithmetic.

As a first result in non-standard mathematics we prove a little fact about non-standard arithmetic. Let

$$\overline{N} = \langle \omega, +, \cdot, 0, \mathbb{S}, < \rangle.$$

This is the usual arithmetic structure, with associated language $\mathscr{L}$ having the obvious relation and operation symbols. For each $m \in \omega$ we define a term $\overline{m}$ of $\mathscr{L}$ as follows, by recursion: $\overline{0} = 0$, $\overline{m+1} = \mathbb{S}\overline{m}$. Here, as in much of our exposition of logic, we are using 0 and $\mathbb{S}$ in two different ways: first as in the set-theory part (empty set and successor operation on sets), and second as symbols of $\mathscr{L}$. The context should make clear which one we really mean—for example, in the definition of $\overline{N}$ we mean the set-theoretic notions, and in the definition of $\overline{m}$ we mean, on the right side of the defining equalities, the symbols of $\mathscr{L}$. Note that $\overline{m}^{\overline{N}} = m$ for every $m \in \omega$. Recall that $\overline{m}^{\overline{N}}$ is the value of the term $\overline{m}$ in the structure $\overline{N}$; the general notation is $\overline{m}^{\overline{N}}(a)$ for $a \in {}^{\omega}\omega$, but the "a" can be omitted since $\overline{m}$ does not have any variables in it.

The following formula $\varphi(v_0)$ says "v_0 is a prime":

$$\overline{1} < v_0 \wedge \forall v_1 \forall v_2 (v_0 = v_1 \cdot v_2 \rightarrow v_0 = v_1 \vee v_0 = v_2).$$

The *twin prime conjecture* (still unresolved, as far as I know) says that for every natural number m there is a prime $p > m$ such that $p + 2$ is also a prime. (Consider $3, 5$; $11, 13$; $17, 19$; etc.)

Proposition 12.7. *Suppose that $\overline{A} = \langle A, +, \cdot, 0, \mathbb{S}, < \rangle$ is a proper elementary extension of $\overline{N}$ ("proper" just means that $A \neq \omega$). Then the following conditions are equivalent:*

(i) The twin prime conjecture.

(ii) There is an $a \in A\backslash\omega$ such that $\overline{A} \models (\varphi(v_0) \wedge \varphi(v_0 + \overline{2}))[a]$.

Note that (ii) can be loosely stated like this: there is an infinite prime a such that also $a + 2$ is a prime. Only one is needed, rather than infinitely many things as in (i).

Proof. We prove some simple facts first.

(1) $\overline{m}^{\overline{A}} = m$ for all $m \in \omega$.

This is easily proved by induction on m.

(2) If $m \in \omega$, $a \in A$, and $a < m$, then $a \in \omega$.

In fact,

$$\overline{N} \models \forall v_0 \left(v_0 < \overline{m} \leftrightarrow \bigvee_{i<m} v_0 = \overline{i} \right),$$

so this holds in $\overline{A}$ too. It follows that $\overline{N} \models \bigvee_{i<m} v_0 = \overline{i}[a]$, hence there is an $i < m$ such that $\overline{N} \models v_0 = \overline{i}[a]$, so $a = i$ by (1).

(3) If $a \in A \backslash \omega$, then $m < a$ for all $m \in \omega$.

This is true by (2), since the sentence $\forall v_0 \forall v_1 (v_0 < v_1 \vee v_0 = v_1 \vee v_1 < v_0)$ holds in $\overline{N}$ and hence in $\overline{A}$.

Now we get down to the proof of the proposition. (i)$\Rightarrow$(ii): Under the assumption (i), the sentence

$$\forall v_0 \exists v_1 (v_0 < v_1 \wedge \varphi(v_1) \wedge \varphi(v_1 + \overline{2}))$$

holds in $\overline{N}$, hence in $\overline{A}$. Now take any $b \in A \backslash \omega$. Then because this sentence holds in $\overline{A}$, we can choose $a \in A$ such that $b < a$ and $\overline{A} \models (\varphi(v_0) \wedge \varphi(v_0 + \overline{2}))[a]$. By (2) and $b < a$, $b \notin \omega$, we get $a \notin \omega$. This proves (ii).

(ii)$\Rightarrow$(i): Assume (ii). Now take any $m \in \omega$; we want to find a prime $p > m$ such that $p + 2$ is also a prime. By (3) and (ii),

$$\begin{aligned} \overline{A} &\models (v_0 < v_1 \wedge \varphi(v_1) \wedge \varphi(v_1 + \overline{2}))[m, a], \\ \text{hence } \overline{A} &\models \exists v_1 (v_0 < v_1 \wedge \varphi(v_1) \wedge \varphi(v_1 + \overline{2}))[m], \\ \text{hence } \overline{N} &\models \exists v_1 (v_0 < v_1 \wedge \varphi(v_1) \wedge \varphi(v_1 + \overline{2}))[m], \end{aligned}$$

and the desired conclusion follows. □

The full elementary extension.

Now we introduce the full apparatus that is going to be used in this section. Let $\mathscr{F}$ be the collection of all finitary operations on $\mathbb{R}$, and $\mathscr{R}$ the set of all finitary relations on $\mathbb{R}$. Recall that this means that

$$\mathscr{F} = \bigcup_{n \in \omega} {}^{^n\mathbb{R}}\mathbb{R} \quad \text{and} \quad \mathscr{R} = \bigcup_{n \in \omega \backslash \{0\}} {}^n\mathbb{R}.$$

Now we take a language $\mathscr{L}$ which has an operation symbol $\mathbb{G}_f$ and a relation symbol $\mathbb{S}_R$ corresponding to each member of $f \in \mathscr{F}$ and $R \in \mathscr{R}$ respectively. Note that $\mathscr{L}$ has 2^{2^ω} symbols; this is interesting, but is not very relevant to the work of this section. *Now we introduce some notation and assumptions used for the rest of this section.* Let

$\overline{R} = (\mathbb{R}, \mathbb{f}, \mathbb{r})$, where for each $f \in \mathscr{F}$ we have $\mathbb{f}_{\mathbb{G}_f} = f$, and for each $R \in \mathscr{R}$ we have $\mathbb{r}_{\mathbb{S}_R} = R$. Let $\overline{R}^* = (\mathbb{R}^*, \mathbb{g}, \mathbb{s})$ be a proper elementary extension of $\overline{A}$. Thus in particular, $\mathbb{R}^*$ is an ordered field which is a proper extension of $\mathbb{R}$, so the initial terminology and results about infinitesimals and such are applicable. This is the official notation, but in practice we will just use f for $\mathbb{f}_{\mathbb{G}_f}$, R for $\mathbb{r}_{\mathbb{S}_R}$, f^* for $\mathbb{g}_{\mathbb{G}_f}$, and R^* for $\mathbb{s}_{\mathbb{S}_R}$; the context should make clear what is meant. If f is a member of $\mathscr{F}$, then we let f^* be the corresponding member of $\overline{R}^*$; so f is the restriction of f^* to tuples from $\mathbb{R}$. Similarly for relations. Sometimes we use "$\in$" when we really should be using a relation symbol notation. Also, instead of using the [...] notation in talking about a formula being modeled in a structure, we simply substitute the relevant elements into the formula.

Many of the concepts of analysis are concerned not with functions defined on all of $\mathbb{R}$, but with functions f defined only on a subset D. Since the functions in first-order logic are assumed to be everywhere defined, we have to use a circumlocution to take this kind of thing into account. We can imagine any such function f extended to all of $\mathbb{R}$ in any way, say by setting its values equal to 0 outside of D, and work with D and the extension, both of which are expressed in our language. The extension of f defined this way will be called the *natural extension of f to $\mathbb{R}$.*

Our first result in non-standard analysis proper gives a nonstandard version of the definition of limit.

Proposition 12.8. *Suppose that $f : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}$, $a, u \in \mathbb{R}$, and some deleted open interval about a is a subset of D. Let g be the natural extension of f to $\mathbb{R}$. Then the following conditions are equivalent:*

(i) $\lim_{x\to a} f(x) = u$.

(ii) For every $y \in \mathbb{R}^*$, if $y \in D^*$, $y \neq a$, and y is infinitely close to a, then $g^*(y)$ is infinitely close to u.

Proof. We take g as indicated in the remark. (i)⇒(ii): Assume (i), let y be as in the hypothesis of (ii), and suppose that ε is a positive real number. We want to show that $|g^*(y) - u| < \varepsilon$. Choose $\delta > 0$ in $\mathbb{R}$ so that for all $x \in D$, if $|x - a| < \delta$ and $x \neq a$, then $|f(x) - u| < \varepsilon$. Thus

$$(*) \qquad \overline{R} \models \forall x(x \in D \land x \neq a \land |x - a| < \delta \to |g(x) - u| < \varepsilon).$$

Since $a, \delta, u, \varepsilon$ are all in $\mathbb{R}$, the elementary extension definition implies, from (*),

$$(**) \qquad \overline{R}^* \models \forall x(x \in D^* \land x \neq a \land |x - a| < \delta \to |g^*(x) - u| < \varepsilon).$$

Now $y \in D^*$, $y \neq a$, and since y is infinitely close to a, by definition of that notion $|y - a| < \delta$. Hence by (**), $|g^*(y) - u| < \varepsilon$. Since ε was an arbitrary positive element of $\mathbb{R}$, this shows that $g^*(y)$ is infinitely close to u.

(ii)⇒(i). Suppose that (i) fails. Accordingly, choose a positive ε in $\mathbb{R}$ so that

$$\overline{R} \models \forall \delta[\delta > 0 \to \exists x(x \in D \land x \neq a \land |x - a| < \delta \land |g(x) - u| \geq \varepsilon)].$$

Again, this implies

$$(*) \qquad \overline{R}^* \models \forall \delta[\delta > 0 \to \exists x(x \in D^* \land x \neq a \land |x - a| < \delta \land |g^*(x) - u| \geq \varepsilon)].$$

Let $\delta > 0$ be a positive infinitesimal (by 12.5). By (*), choose $y \in D^*$ such that $y \neq a$, $|y-a| < \delta$, and $|g^*(y)-u| \geq \varepsilon$. Since δ is a positive infinitesimal, also $|y-a|$ is infinitesimal. So y is infinitely close to a, but $g^*(y)$ is not infinitely close to u, so that (ii) fails. $\square$

A version of 12.8 for one-sided limits is just as easy to prove; we omit the formulation and proof of this, and when we use 12.8 we might tacitly apply the one-sided version.

Lemma 12.9. *(i) If $u \approx v$ and u is finite, then v is finite.*
(ii) If $u \approx v$ and u is finite, then $\mathrm{st}(u) = \mathrm{st}(v)$.
(iii) If $a, b \in \mathbb{R}$ and $a < b$, and if $u \in [a, b]^*$, then $\mathrm{st}(u) \in [a, b]$.
(iv) If a and b are finite, then $\mathrm{st}(a + b) = \mathrm{st}(a) + \mathrm{st}(b)$.

Proof. (i): say $|u| < r \in \mathbb{R}$. Thus $-r < u < r$. We claim that $-r - 1 < v < r + 1$. For if, for example, $v \leq -r - 1$, then $|u - v| = u - v > 1$, contradiction. So $|v| < r + 1$ and so v is finite.

(ii): The assumption, and (i), imply that $\mathrm{st}(u) \approx u \approx v \approx \mathrm{st}(v)$, so $\mathrm{st}(u) = \mathrm{st}(v)$.

(iii): We have $\mathbb{R} \models \forall x \in [a, b](a \leq x \leq b)$, so also $\mathbb{R}^* \models \forall x \in [a, b]^*(a \leq x \leq b)$. Hence $a \leq u \leq b$. Suppose that $\mathrm{st}(u) \notin [a, b]$. By symmetry, say $\mathrm{st}(u) < a$. Thus $0 < a - \mathrm{st}(u) \leq u - \mathrm{st}(u) = |u - \mathrm{st}(u)|$, contradiction since u and $\mathrm{st}(u)$ are infinitely close.

(iv): Let $u = \mathrm{st}(a)$ and $v = \mathrm{st}(b)$. Then for any positive real ε,

$$\begin{aligned} |(a + b) - (u + v)| &= |(a - u) + (b - v)| \\ &\leq |a - u| + |b - v| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

It follows that $a + b \approx u + v$. Hence $\mathrm{st}(a + b) = u + v = \mathrm{st}(a) + \mathrm{st}(b)$. $\square$

Theorem 12.10. *Suppose that f is defined on a closed interval $[a, b]$, with $a < b$. Let g be the natural extension of f to $\mathbb{R}$. Then the following conditions are equivalent:*
(i) f is continuous on $[a, b]$;
(ii) For any $u \in [a, b]^*$, $f(st(u)) = st(g^*(u))$.
(iii) For all $u, v \in [a, b]^*$, if $u \approx v$ then $g^*(u) \approx g^*(v)$.

Proof. (i)⇒(ii): Assume (i) and suppose that $u \in [a, b]^*$. We may assume that $u \neq \mathrm{st}(u)$. Now $\lim_{x \to \mathrm{st}(u)} f(x) = f(\mathrm{st}(u))$, $u \neq \mathrm{st}(u)$, and u is infinitely close to $\mathrm{st}(u)$, so by 12.8, $g^*(u)$ is infinitely close to $f(\mathrm{st}(u))$. Hence $f(\mathrm{st}(u)) = \mathrm{st}(g^*(u))$. (ii)⇒(iii): Assume (ii) and suppose that $u, v \in [a, b]^*$ with $u \approx v$. Then $\mathrm{st}(g^*(u)) = f(\mathrm{st}(u)) = f(\mathrm{st}(v)) = \mathrm{st}(g^*(v))$. (iii)⇒(i): Assume (iii) and suppose that $c \in [a, b]$. Take any $y \in \mathbb{R}^*$ such that $y \neq c$, $y \in [a, b]^*$, and y is infinitely close to c. Then $g^*(y)$ is infinitely close to $f(c)$ by (iii), as desired. $\square$

Next we want to give a genuine piece of non-standard reasoning, by proving the intermediate value theorem for continuous functions using non-standard analysis. To do this the following simple lemma is useful.

Lemma 12.11. *There is an infinite integer, i.e., there is an infinite element of ω^*.*

Proof. By 12.6 let a be an positive infinite element of $\mathbb{R}^*$. Now $\overline{R} \models \forall x \exists y(y \in \omega \wedge x < y)$, so $\overline{R}^* \models \forall x \exists y(y \in \omega^* \wedge x < y)$. Hence choose $n \in \omega^*$ so that $a < n$. This n is as desired. □

Theorem 12.12. (Intermediate value theorem) *Suppose that $f : D \to \mathbb{R}$, $[a, b] \subseteq D$, f is continuous on $[a, b]$, and $f(a) < d < f(b)$. Then there is an element $c \in (a, b)$ such that $f(c) = d$.*

Remark: note that this is purely about $\mathbb{R}$; it is a usual formulation of the intermediate value theorem.

Proof. Let g be the natural extension of f to $\mathbb{R}$. Note that

$$\overline{R} \models \forall \text{ nonzero } n \in \omega \exists i \in \omega \left(i < n \wedge \left[f\left(a + i\frac{b-a}{n} \right) \leq d < f\left(a + (i+1)\frac{b-a}{n} \right) \right] \right).$$

In fact, given a nonzero $n \in \omega$, note first that $d < f(b) = f(a + n \cdot \frac{b-a}{n})$, and hence there is a $j \leq n$ such that $d < f(a + j \cdot \frac{b-a}{n})$. Take the least such j. Now $f(a + 0 \cdot \frac{b-a}{n}) = f(a) < d$, so $j \neq 0$. Write $j = i + 1$. Clearly i is as desired.

It follows that

$$(*) \quad \begin{aligned} \overline{R}^* \models \forall \text{ nonzero } n \in \omega^* \exists i \in \omega^* \Bigg(i < n \wedge \Bigg[g^*\left(a + i\frac{b-a}{n} \right) \leq d \\ < g^*\left(a + (i+1)\frac{b-a}{n} \right) \Bigg] \Bigg). \end{aligned}$$

By the lemma, let n be an infinite integer. Then by (*) choose $i < n$, $i \in \omega^*$, so that $g^*(a + i\frac{b-a}{n}) \leq d < g^*(a + (i+1)\frac{b-a}{n})$. Note that i might be infinite (in fact, it is easy to see that it really is, but we do not need that). The difference between $a + i\frac{b-a}{n}$ and $a + (i+1)\frac{b-a}{n}$ is $\frac{b-a}{n}$, which is infinitesimal; hence by the continuity of f and 12.10, also $g^*(a + i\frac{b-a}{n})$ is infinitely close to $g^*(a + (i+1)\frac{b-a}{n})$; so $g^*(a + i\frac{b-a}{n})$ is infinitely close to d. By the continuity again, $f(\text{st}(a + i\frac{b-a}{n})) = \text{st}(g^*(a + i\frac{b-a}{n})) = d$, as desired. □

We devote the rest of this section to the nonstandard definition of the Riemann integral and description of its simplest properties. Suppose that $f : D \to \mathbb{R}$, $a, b \in \mathbb{R}$, $a < b$, and $[a, b] \subseteq D$. We define the *Riemann sum function* $s_{fab} : (0, \infty) \to \mathbb{R}$ as follows. Let Δx be a given positive real number. Let n be the largest integer such that $a + n\Delta x \leq b$; maybe $n = 0$. Then we define

$$s_{fab}(\Delta x) = \sum_{i<n} f(a + i\Delta x)\Delta x + f(a + n\Delta x)(b - a - n\Delta x).$$

Theorem 12.13. *Suppose that $f : D \to \mathbb{R}$, $a, b \in \mathbb{R}$, $a < b$, $[a, b] \subseteq D$, and f is continuous on $[a, b]$. Suppose that dx is a positive infinitesimal. Then $s^*_{fab}(dx)$ is finite.*

Proof. Let M be greater than the maximum value of f on $[a, b]$, and let m be smaller than the minimum value of f on $[a, b]$. Then for any positive real number Δx, with n as above,

$$s_{fab}(\Delta x) \leq \sum_{i<n} M\Delta x + M(b - a - n\Delta x) = M(b - a),$$

and similarly $m(b-a) \leq s_{fab}(\Delta s)$. So $\overline{R} \models \forall v_0(v_0 > 0 \rightarrow m(b-a) \leq s_{fab}(v_0) \leq M(b-a))$, and so $\overline{R}^* \models \forall v_0(v_0 > 0 \rightarrow m(b-a) \leq s^*_{fab}(v_0) \leq M(b-a))$. In particular, $s^*_{fab}(dx)$ is finite. $\square$

This justifies the following definition of the definite integral: for any positive infinitesimal dx,

$$\int_a^b f(x)dx = \mathrm{st}(s^*_{fab}(dx)).$$

Note that "dx" here is not just a vacuous expression as in the standard definition of integral; it is a definite positive infinitesimal, and on the face of it we might get different integrals by taking different infinitesimals. We prove two elementary properties of integrals which will enable us to show that the definition does not really depend on the particular positive infinitesimal dx chosen.

Theorem 12.14. *Suppose that $f(x) = c$ for all $x \in [a, b]$, where $a < b$. Then for any positive infinitesimal dx,*

$$\int_a^b f(x)dx = c(b-a).$$

Proof. For any positive real number Δx we have, with n chosen as above,

$$s_{fab}(\Delta x) = \sum_{i<n} c\Delta x + c(b-a-n\Delta x) = c(b-a).$$

Thus for all $\Delta x > 0$ in $\mathbb{R}$ we have $s_{fab}(\Delta x) = c(b-a)$. That is, $\mathbb{R} \models \forall \Delta > 0(s_{fab}(\Delta) = c(b-a))$. Hence by the definition of elementary extension it follows that for every positive $d \in \mathbb{R}^*$ we have $s^*_{fab}(d) = c(b-a)$, and the desired conclusion follows. $\square$

Theorem 12.15. *Suppose that f and g are both defined and continuous on $[a, b]$, with $a < b$, and dx is a positive infinitesimal. Then*

$$\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

Proof. Take any positive real number Δx and choose n as usual. Then

$$\begin{aligned} s_{f+g,a,b}(\Delta x) &= \sum_{i<n}(f(a+i\Delta x) + g(a+i\Delta x))\Delta x + (f(a+n\Delta x) \\ &\qquad + g(a+n\Delta x))(b-a-n\Delta x) \\ &= \sum_{i<n} f(a+i\Delta x)\Delta x + f(a+n\Delta x)(b-a-n\Delta x) \\ &\qquad + \sum_{i<n} g(a+i\Delta x)\Delta x + g(a+n\Delta x)(b-a-n\Delta x) \\ &= s_{fab}(\Delta x) + s_{gab}(\Delta x). \end{aligned}$$

Hence $\mathbb{R} \models \forall\Delta > 0(s_{f+g,a,b}(\Delta) = s_{fab}(\Delta)+s_{gab}(\Delta)$. It follows that $s^*_{f+g,a,b}(d) = s^*_{fab}(d)+ s^*_{gab}(d)$ for every positive $d \in \mathbb{R}^*$, and the desired conclusion follows, using 12.9(iv). $\square$

So, now we can prove the independence of the definition on the particular infinitesimal chosen:

Theorem 12.16. *Suppose that $f : D \to \mathbb{R}$, $a, b \in \mathbb{R}$, $a < b$, $[a, b] \subseteq D$, and f is continuous on $[a, b]$. Suppose that dx and du are positive infinitesimals. Then*

$$\int_a^b f(x)dx = \int_a^b f(u)du.$$

Proof. Let g be the natural extension of f to $\mathbb{R}$.

This will be a funny proof. We show that for every positive real number r we have

$$\int_a^b f(x)dx \leq \int_a^b f(u)du + r;$$

since r is arbitrary, this shows one inequality of the desired equality in the theorem, and the other one is true by symmetry.

Let $c = \frac{r}{b-a}$, and let $h(x) = f(x) + c$ for all $x \in [a, b]$. We claim

(*) If Δx and Δu are two positive real numbers and $s_{fab}(\Delta x) > s_{hab}(\Delta u)$, then there are $x, u \in [a, b]$ such that $x - \Delta u \leq u \leq x + \Delta x$ and $f(x) > f(u) + c$.

To see this, according to the definition of s_{fab} and s_{gab} choose integers $n, m \in \omega$ such that

$$\begin{aligned} & s_{fab}(\Delta x) = \sum_{i<n} f(a + i\Delta x)\Delta x + f(a + n\Delta x)(b - a - n\Delta x) \\ \text{and} \qquad & s_{hab}(\Delta u) = \sum_{i<m} g(a + i\Delta u)\Delta u + g(a + m\Delta x)(b - a - m\Delta u). \end{aligned}$$

Now let

$$\{c_0, \ldots, c_p\} = \{a + i\Delta x : i \leq n\} \cup \{a + j\Delta u : j \leq m\} \cup \{b\},$$

where $c_0 < \cdots < c_p$. Then for each $k < p$ there are unique $i_k \leq n$ and $j_k \leq m$ such that

$$[c_k, c_{k+1}] \subseteq [a + i_k\Delta x, a + (i_k + 1)\Delta x] \cap [a + j_k\Delta u, a + (j_k + 1)\Delta u],$$

where if $i_k = n$ then we replace $a + (i_k + 1)\Delta x$ by b, and similarly if $j_k = m$. Hence

$$\begin{aligned} & s_{fab}(\Delta x) = \sum_{k<p} f(a + i_k\Delta x)|c_{k+1} - c_k| \\ \text{and} \qquad & s_{hab}(\Delta u) = \sum_{k<p} h(a + j_k\Delta u)|c_{k+1} - c_k|. \end{aligned}$$

It follows that there is a $k < p$ such that $f(a + i_k\Delta x) > h(a + j_k\Delta u)$, i.e., $f(a + i_k\Delta x) > f(a + j_k\Delta u) + c$. Let $x = a + i_k\Delta x$ and $u = a + j_k\Delta u$. So $f(x) > f(u) + c$. Now

$$\begin{aligned} & a + i_k\Delta x \le c_k < c_{k+1} \le a + (i_k + 1)\Delta x \\ \text{and} \qquad & a + j_k\Delta u \le c_k < c_{k+1} \le a + (j_k + 1)\Delta u. \end{aligned}$$

It follows that $u \le c_{k+1} \le x + \Delta x$ and $x < c_{k+1} \le u + \Delta u$, as desired in (*).

It follows that the extended version of (*) holds in $\mathbb{R}^*$:

(**) If dx and du are two positive members of $\mathbb{R}^*$ and $s^*_{fab}(dx) > s^*_{hab}(du)$, then there are $x, u \in [a, b]$ such that $x - du \le u \le x + dx$ and $g^*(x) > g^*(u) + c$.

Now we claim:

(***) If dx and du are positive infinitesimals, then $s^*_{fab}(dx) \le s^*_{hab}(du)$.

In fact, if not we can apply (**) to get $x, u \in [a, b]^*$ such that $x - du \le u \le x + dx$ and $g^*(x) > g^*(u) + c$. But then $x \approx u$ and $g^*(x) \not\approx g^*(u)$, contradicting the continuity of f. This proves (***).

Using (***), the proof is easily finished:

$$\begin{aligned} \int_a^b f(x)dx &= \mathrm{st}(s^*_{fab}(dx)) \\ &\le \mathrm{st}(s^*_{hab}(du) \\ &= \int_a^b (f(u) + c)du \\ &= \int_a^b f(u)du + \int_a^b c du \\ &= \int_a^b f(u)du + c(b - a) \\ &= \int_a^b f(u)du + r. \end{aligned}$$

□

Exercises, section 12

For the first three exercises, suppose that ε and δ are positive infinitesimals, and M is a positive infinite number, in a proper extension of $\mathbb{R}$. Determine whether the given expression is 0, a positive infinitesimal, a positive finite number not infinitesimal, or a positive infinite number; if more than one of these is possible, indicate that, and give reasons.

E12.1. $(\varepsilon + \delta)/\sqrt{\varepsilon^2 + \delta^2}$.

E12.2.

$$\frac{2M + 1}{3M + 2}.$$

E12.3. ε/δ.

E12.4. Let M be positive and infinite. Show that the following is finite, and compute its standard part:

$$\frac{2M^3 + 5M^2 + 3M}{17M^3 + M + 1}.$$

E12.5. Suppose that $\mathrm{st}(b) = 5$ and $b \neq 5$. Compute the standard part of

$$\frac{b^2 - 25}{b - 5}.$$

E12.6. Let $\overline{A}$ be as in Proposition 12.7. Prove that for every $m \in A\backslash\omega$ and all positive (ordinary) integers we have $x^m + y^m \neq z^m$.

E12.7. Let $f : [0,1] \to \mathbb{R}$, and let g be its natural extension to $\mathbb{R}$. Prove that the following conditions are equivalent:

(i) f is uniformly continuous on $[a,b]$.

(ii) For all $x, y \in [0,1]^*$, if x is infinitely close to y, the $g^*(x)$ is infinitely close to $g^*(y)$.

(iii) There is a positive δ in $\mathbb{R}^*$ such that for all $x, y \in [0,1]^*$, if $|x-y| < \delta$ then $g^*(x)$ is infinitely close to $g^*(y)$.