Solutions to exercises in Chapter 4

$\boxed{\text{E4.1}}$ *Suppose that $\Gamma \vdash \varphi \to \psi$, $\Gamma \vdash \varphi \to \neg\psi$, and $\Gamma \vdash \neg\varphi \to \varphi$. Prove that Γ is inconsistent.*

The formula $(\neg\varphi \to \varphi) \to \varphi$ is a tautology. Hence by Lemma 3.3, $\Gamma \vdash (\neg\varphi \to \varphi) \to \varphi$. Since also $\Gamma \vdash \neg\varphi \to \varphi$, it follows that $\Gamma \vdash \varphi$. Hence $\Gamma \vdash \psi$ and $\Gamma \vdash \neg\psi$. Hence by Lemma 4.1, Γ is inconsistent.

$\boxed{\text{E4.2}}$ *Let $\mathscr{L}$ be a language with just one non-logical constant, a binary relation symbol $\mathbf{R}$. Let Γ consist of all sentences of the form $\exists v_1\forall v_0[\mathbf{R}v_0v_1 \leftrightarrow \varphi]$ with φ a formula with only v_0 free. Show that Γ is inconsistent. Hint: take φ to be $\neg\mathbf{R}v_0v_0$.*

By Theorem 3.27 we have

$$\Gamma \vdash \forall v_0[\mathbf{R}v_0v_1 \leftrightarrow \neg\mathbf{R}v_0v_0] \to [\mathbf{R}v_1v_1 \leftrightarrow \neg\mathbf{R}v_1v_1]. \tag{1}$$

Now $[\mathbf{R}v_1v_1 \leftrightarrow \neg\mathbf{R}v_1v_1] \to \neg(v_0 = v_0)$ is a tautology, so from (1) we obtain

$$\Gamma \vdash \forall v_0[\mathbf{R}v_0v_1 \leftrightarrow \neg\mathbf{R}v_0v_0] \to \neg(v_0 = v_0);$$

then generalization gives

$$\Gamma \vdash \forall v_1[\forall v_0[\mathbf{R}v_0v_1 \leftrightarrow \neg\mathbf{R}v_0v_0] \to \neg(v_0 = v_0)].$$

Then by Proposition 3.39 we get

$$\Gamma \vdash \exists v_1\forall v_0[\mathbf{R}v_0v_1 \leftrightarrow \neg\mathbf{R}v_0v_0] \to \neg(v_0 = v_0).$$

But the hypothesis here is a member of Γ, so we get $\Gamma \vdash \neg(v_0 = v_0)$. Hence by Lemma 4.1, Γ is inconsistent.

Alternate proof (due to a couple of students). Suppose that Γ is consistent. By the completeness theorem let $\overline{A}$ be a model of Γ. Taking φ to be $\neg\mathbf{R}v_0v_0$, we get $\overline{A} \models \exists v_1\forall v_0[\mathbf{R}v_0v_1 \leftrightarrow \neg\mathbf{R}v_0v_0]$. Let $a : \omega \to A$ be any assignment. Then by Proposition 2.8(iv) there is a $b \in A$ such that $\overline{A} \models \forall v_0[\mathbf{R}v_0v_1 \leftrightarrow \neg\mathbf{R}v_0v_0][a^1_b]$. By the definition of satisfaction of $\forall$, it follows that for any $c \in A$ we have $\overline{A} \models [\mathbf{R}v_0v_1 \leftrightarrow \neg\mathbf{R}v_0v_0][a^{0\ 1}_{c\ b}]$. Hence $(c, b) \in \mathbf{R}^{\overline{A}}$ iff $(c, b) \notin \mathbf{R}^{\overline{A}}$, contradiction.

$\boxed{\text{E4.3}}$ *Show that the first-order deduction theorem fails if the condition that φ is a sentence is omitted. Hint: take $\Gamma = \emptyset$, let φ be the formula $v_0 = v_1$, and let ψ be the formula $v_0 = v_2$.*

$$\begin{aligned}
&\{v_0 = v_1\} \vdash v_0 = v_1\\
&\{v_0 = v_1\} \vdash \forall v_1(v_0 = v_1)\\
&\{v_0 = v_1\} \vdash \forall v_1(v_0 = v_1) \to v_0 = v_2 \quad \text{by Theorem 3.27}\\
&\{v_0 = v_1\} \vdash v_0 = v_2.
\end{aligned}$$

On the other hand, let $\overline{A}$ be the structure with universe ω and define $a = \langle 0, 0, 1, 1, \ldots \rangle$. Clearly $\overline{A} \not\models [v_0 = v_1 \to v_0 = v_2][a]$. Hence $\not\models v_0 = v_1 \to v_0 = v_2$ by Theorem 3.2.

$\boxed{\text{E4.4}}$ *In the language for $\overline{A} \stackrel{\text{def}}{=} (\omega, S, 0, +, \cdot)$, let τ be the term $v_0 + v_1 \cdot v_2$ and ν the term $v_0 + v_2$. Let a be the sequence $\langle 0, 1, 2, \ldots \rangle$. Let ρ be obtained from τ by replacing the occurrence of v_1 by ν.*

(a) Describe ρ as a sequence of integers.

(b) What is $\rho^{\overline{A}}(a)$?

(c) What is $\nu^{\overline{A}}(a)$?

(d) Describe the sequence $a^1_{\nu^{\overline{A}}(a)}$ as a sequence of integers.

(e) Verify that $\rho^{\overline{A}}(a) = \tau^{\overline{A}}(a^1_{\nu^{\overline{A}}(a)})$ (cf. Lemma 4.4.)

(a) ρ is $v_0 + (v_0 + v_2) \cdot v_2$; as a sequence of integers it is $\langle 7, 5, 9, 7, 5, 15, 15 \rangle$.

(b) $\rho^{\overline{A}}(a) = 0 + (0 + 2) \cdot 2 = 4$.

(c) $\nu^{\overline{A}}(a) = 0 + 2 = 2$.

(d) $a^1_{\nu^{\overline{A}}(a)} = \langle 0, 2, 2, 3, \ldots \rangle$.

(e) $\rho^{\overline{A}}(a) = 4$, as above; $\tau^{\overline{A}}(a^1_{\nu^{\overline{A}}(a)}) = 0 + 2 \cdot 2 = 4$.

$\boxed{\text{E4.5}}$ *In the language for $\overline{A} \stackrel{\text{def}}{=} (\omega, S, 0, +, \cdot)$, let φ be the formula $\forall v_0 (v_0 \cdot v_1 = v_1)$, let ν be the formula $v_1 + v_1$, and let $a = \langle 1, 0, 1, 0, \ldots \rangle$.*

(a) Describe $\mathrm{Subf}^{v_1}_{\nu} \varphi$ as a sequence of integers

(b) What is $\nu^{\overline{A}}(a)$?

(c) Describe $a^1_{\nu^{\overline{A}}(a)}$ as a sequence of integers.

(d) Determine whether $\overline{A} \models \mathrm{Subf}^{v_1}_{\nu} \varphi[a]$ or not.

(e) Determine whether $\overline{A} \models \varphi[a^1_{\nu^{\overline{A}}(a)}]$ or not.

(a) $\mathrm{Subf}^{v_1}_{\nu} \varphi$ is $\forall v_0 (v_0 \cdot (v_1 + v_1) = v_1 + v_1$; as a sequence of integers it is

$$\langle 4, 5, 3, 9, 5, 7, 10, 10, 7, 10, 10 \rangle.$$

(b) $\nu^{\overline{A}}(a) = (v_1 + v_1)^{\overline{A}}(\langle 1, 0, 1, 0, \ldots \rangle) = 0 + 0 = 0$.

(c) $a^1_{\nu^{\overline{A}}(a)} = \langle 1, 0, 1, 0, \ldots \rangle$.

(d) $\overline{A} \models \mathrm{Subf}^{v_1}_{\nu} \varphi[a]$ iff $\overline{A} \models [\forall v_0 (v_0 \cdot (v_1 + v_1) = v_1 + v_1][\langle 1, 0, 1, 0, \ldots \rangle]$ iff for all $a \in \omega$, $a \cdot (0 + 0) = 0 + 0$; this is true.

(e) $\overline{A} \models \varphi[a^1_{\nu^{\overline{A}}(a)}]$ iff $\overline{A} \models [\forall v_0 (v_0 \cdot v_1 = v_1][\langle 1, 0, 1, 0, \ldots \rangle]$ iff for all $a \in \omega$, $a \cdot 0 = 0$; this is true.

$\boxed{\text{E4.6}}$ *Show that the condition in Lemma 4.6 that*

no free occurrence of v_i in φ is within a subformula of the form $\forall v_k \mu$ with v_k a variable occurring in ν

is necessary for the conclusion of the lemma.

In the language for $\overline{A} = (\omega, S, 0, +, \cdot)$, let φ be the formula $\exists v_1[Sv_1 = v_0]$, $\nu = v_1$, and $a = \langle 1, 1, \ldots\rangle$. Note that the condition on v_0 fails. Now $\mathrm{Subf}^{v_0}_{v_1}\varphi$ is the formula $\exists v_1[Sv_1 = v_1]$, and there is no $a \in \omega$ such that $Sa = a$, and hence $\overline{A} \not\models \mathrm{Subf}^{v_0}_{v_1}\varphi[a]$. Also, $\nu^{\overline{A}}(a) = v_1^{\overline{A}}(a) = a_1 = 1$, and hence $a^0_{\nu^{\overline{A}}(a)} = \langle 1, 1, \ldots\rangle$. Since $S0 = 1$, it follows that $\overline{A} \models \varphi[a^0_{\nu^{\overline{A}}(a)}]$.

$\boxed{\text{E4.7}}$ *Let $\overline{A}$ be an $\mathscr{L}$-structure, with $\mathscr{L}$ arbitrary. Define $\Gamma = \{\varphi : \varphi$ is a sentence and $\overline{A} \models \varphi[a]$ for some $a : \omega \to A\}$. Prove that Γ is complete and consistent.*

Note by Lemma 4.4 that $\overline{A} \models \varphi[a]$ for some $a : \omega \to A$ iff $\overline{A} \models \varphi[a]$ for every $a : \omega \to A$. Let φ be any sentence. Take any $a : \omega \to A$. If $\overline{A} \models \varphi[a]$, then $\varphi \in \Gamma$ and hence $\Gamma \vdash \varphi$. Suppose that $\overline{A} \not\models \varphi[a]$. Then $\overline{A} \models \neg\varphi[a]$, hence $\neg\varphi \in \Gamma$, hence $\Gamma \vdash \neg\varphi$.

This shows that Γ is complete. Suppose that Γ is not consistent. Then $\Gamma \vdash \neg(v_0 = v_0)$ by Lemma 4.1. Then $\Gamma \models \neg(v_0 = v_0)$ by Theorem 3.2. Since $\overline{A}$ is a model of Γ, it is also a model of $\neg(v_0 = v_0)$, contradiction.

$\boxed{\text{E4.8}}$ *Call a set Γ strongly complete iff for every formula φ, $\Gamma \vdash \varphi$ or $\Gamma \vdash \neg\varphi$. Prove that if Γ is strongly complete, then $\Gamma \vdash \forall v_0\forall v_1(v_0 = v_1)$.*

Assume that Γ is strongly complete. Then $\Gamma \vdash v_0 = v_1$ or $\Gamma \vdash \neg(v_0 = v_1)$. If $\Gamma \vdash v_0 = v_1$, then by generalization, $\Gamma \vdash \forall v_0\forall v_1(v_0 = v_1)$. Suppose that $\Gamma \vdash \neg(v_0 = v_1)$. Then by generalization, $\Gamma \vdash \forall v_0\neg(v_0 = v_1)$. By Theorem 3.27, $\Gamma \vdash \forall v_0\neg(v_0 = v_1) \to \neg(v_1 = v_1)$. Hence $\Gamma \vdash \neg(v_1 = v_1)$. But also $\Gamma \vdash v_1 = v_1$ by Proposition 3.4, so Γ is inconsistent by Lemma 4.1, and hence again $\Gamma \vdash \forall v_0\forall v_1(v_0 = v_1)$.

$\boxed{\text{E4.9}}$ *Prove that if Γ is rich, then for every term σ with no variables occurring in σ there is an individual constant $\mathbf{c}$ such that $\Gamma \vdash \sigma = \mathbf{c}$.*

By richness we have $\Gamma \vdash \exists v_0(v_0 = \sigma) \to \mathbf{c} = \sigma$ for some individual constant $\mathbf{c}$. Then using (L4) it follows that $\Gamma \vdash \mathbf{c} = \sigma$.

$\boxed{\text{E4.10}}$ *Prove that if Γ is rich, then for every sentence φ there is a sentence ψ with no quantifiers in it such that $\Gamma \vdash \varphi \leftrightarrow \psi$.*

We proceed by induction on the number m of symbols $\neg$, $\to$, $\forall$ in φ. (More exactly, by the number of the integers 1,2,4 that occur in the sequence φ.) If $m = 0$, then φ is atomic and we can take $\psi = \varphi$. Assume the result for m and suppose that φ has $m + 1$ integers 1,2,4 in it. Then there are three possibilities. First, $\varphi = \neg\varphi'$. Let ψ' be a quantifier-free sentence such that $\Gamma \vdash \varphi' \leftrightarrow \psi'$. Then $\Gamma \vdash \varphi \leftrightarrow \neg\psi'$. Second, $\varphi = (\varphi' \to \varphi'')$. Choose quantifier-free sentences ψ' and ψ'' such that $\Gamma \vdash \varphi' \leftrightarrow \psi'$ and $\Gamma \vdash \varphi'' \leftrightarrow \psi''$. Then $\Gamma \vdash \varphi \leftrightarrow (\psi' \to \psi'')$. Third, $\varphi = \forall v_i\varphi'$. By richness, let c be an individual constant such that $\Gamma \vdash \exists v_i\neg\varphi' \to \mathrm{Subf}^{v_i}_c\neg\varphi'$. Then by Theorem 3.33 we get

(1) $\Gamma \vdash \exists v_i\neg\varphi' \leftrightarrow \mathrm{Subf}^{v_i}_c\neg\varphi'$.

Now $\mathrm{Subf}^{v_i}_c\varphi'$ has only m integers 1,2,4 in it, so by the inductive hypothesis there is a sentence ψ with no quantifiers in it such that $\Gamma \vdash \mathrm{Subf}^{v_i}_c\varphi' \leftrightarrow \psi$ and hence

(2) $\Gamma \vdash \mathrm{Subf}^{v_i}_c\neg\varphi' \leftrightarrow \neg\psi$.

From (1) and (2) and a tautology we get $\Gamma \vdash \neg\exists v_i \neg\varphi' \leftrightarrow \psi$. Then by Proposition 3.31, $\Gamma \vdash \forall v_i \varphi' \leftrightarrow \psi$, finishing the inductive proof.

$\boxed{\text{E4.11}}$ *Describe sentences in a language for ordering which say that $<$ is a linear ordering and there are infinitely many elements. Prove that the resulting set Γ of sentences is not complete.*

Let Γ consist of the following sentences:

$$\begin{aligned}
&\neg\exists v_0(v_0 < v_0);\\
&\forall v_0\forall v_1\forall v_2[v_0 < v_1 \wedge v_1 < v_2 \rightarrow v_0 < v_2];\\
&\forall v_0\forall v_1[v_0 < v_1 \vee v_0 = v_1 \vee v_1 < v_0];\\
&\bigwedge_{i<j<n} \neg(v_i = v_j) \quad \text{for every positive integer } n.
\end{aligned}$$

The following sentence φ holds in $(\mathbb{Q}, <)$ but not in $(\omega, <)$:

$$\forall v_0\forall v_1[v_0 < v_1 \rightarrow \exists v_2(v_0 < v_2 \wedge v_2 < v_1)].$$

Since φ does not hold in $(\omega, <)$, we have $\Gamma \not\vdash \varphi$, by Theorem 4.2. But since φ holds in $(\mathbb{Q}, <)$, we also have $\Gamma \not\vdash \neg\varphi$ by Theorem 4.2. So Γ is not complete.

$\boxed{\text{E4.12}}$ *Prove that if a sentence φ holds in every infinite model of a set Γ of sentences, then there is an $m \in \omega$ such that it holds in every model of Γ with at least m elements.*

Suppose that φ holds in every infinite model of a set Γ of sentences, but for every $m \in \omega$ there is a model $\overline{M}$ of Γ with at least m elements such that φ does not hold in $\overline{M}$. Let Δ be the following set:

$$\Gamma \cup \left\{ \bigwedge_{i<j<n} \neg(v_i = v_j) : n \text{ a positive integer} \right\} \cup \{\neg\varphi\}.$$

Our hypothesis implies that every finite subset Δ' of Δ has a model; for if m is the maximum of all n such that the above big conjunction is in Δ', then the hypothesis yields a model of Δ'. By the compactness theorem we get a model $\overline{N}$ of Δ. Thus $\overline{N}$ is an infinite model of Γ in which φ does not hold, contradiction.

$\boxed{\text{E4.13}}$ *Let $\mathscr{L}$ be the language of ordering. Prove that there is no set Γ of sentences whose models are exactly the well-ordering structures.*

Suppose there is such a set. Let us expand the language $\mathscr{L}$ to a new one $\mathscr{L}'$ by adding an infinite sequence $\mathbf{c}_m$, $m \in \omega$, of individual constants. Then consider the following set Θ of sentences: all members of Γ, plus all sentences $\mathbf{c}_{m+1} < \mathbf{c}_m$ for $m \in \omega$. Clearly every finite subset of Θ has a model, so let $\overline{A} = (A, <, a_i)_{i<\omega}$ be a model of Θ itself. (Here a_i is the 0-ary function, i.e., element of A, corresponding to $\mathbf{c}_i$.) Then $a_0 > a_1 > \cdots$; so $\{a_i : i \in \omega\}$ is a nonempty subset of A with no least element, contradiction. $\square$

$\boxed{\text{E4.14}}$ *Suppose that Γ is a set of sentences, and φ is a sentence. Prove that if $\Gamma \models \varphi$, then $\Delta \models \varphi$ for some finite $\Delta \subseteq \Gamma$.*

We prove the contrapositive: Suppose that for every finite subset Δ of Γ, $\Delta \not\models \varphi$. Thus every finite subset of $\Gamma \cup \{\neg\varphi\}$ has a model, so $\Gamma \cup \{\neg\varphi\}$ has a model, proving that $\Gamma \not\models \varphi$.

$\boxed{\text{E4.15}}$ *Suppose that f is a function mapping a set M into a set N. Let $R = \{(a,b) : a, b \in M$ and $f(a) = f(b)\}$. Prove that R is an equivalence relation on M.*

If $a \in M$, then $f(a) = f(a)$, so $(a,a) \in R$. Thus R is reflexive on M. Suppose that $(a,b) \in R$. Then $f(a) = f(b)$, so $f(b) = f(a)$ and hence $(b,a) \in R$. Thus R is symmetric. Suppose that $(a,b) \in R$ and $(b,c) \in R$. Then $f(a) = f(b)$ and $f(b) = f(c)$, so $f(a) = f(c)$ and hence $(a,c) \in R$.

$\boxed{\text{E4.16}}$ *Suppose that R is an equivalence relation on a set M. Prove that there is a function f mapping M into some set N such that $R = \{(a,b) : a, b \in M$ and $f(a) = f(b)\}$.*

Let N be the collection of all equivalence classes under R. For each $a \in M$ let $f(a) = [a]_R$. Then $(a,b) \in R$ iff $a, b \in M$ and $[a]_R = [b]_R$ iff $a, b \in M$ and $f(a) = f(b)$.

$\boxed{\text{E4.17}}$ *Let Γ be a set of sentences in a first-order language, and let Δ be the collection of all sentences holding in every model of Γ. Prove that $\Delta = \{\varphi : \varphi$ is a sentence and $\Gamma \vdash \varphi\}$.*

For $\subseteq$, suppose that $\varphi \in \Delta$. To prove that $\Gamma \vdash \varphi$ we use the compactness theorem, proving that $\Gamma \models \varphi$. Let $\overline{A}$ be any model of Γ. Since $\varphi \in \Delta$, it follows that $\overline{A}$ is a model of Γ, as desired.

For $\supseteq$, suppose that φ is a sentence and $\Gamma \vdash \varphi$. Then by the easy direction of the completeness theorem, $\Gamma \models \varphi$. That is, every model of Γ is a model of φ. Hence $\varphi \in \Delta$.

$\boxed{\text{E4.18}}$ *Prove (2) in the proof of Theorem 4.24.*

By the competeness theorem it suffices to show that

$$\models \varphi \leftrightarrow \exists v_n \ldots \exists v_{n+m-1} \left[\bigwedge_{j<m} (\sigma_j = v_{n+j}) \wedge \mathbf{R} v_n \ldots v_{n+m-1} \right].$$

So, let $\overline{A}$ be any structure, and suppose that $a : \omega \to A$. First suppose that $\overline{A} \models \varphi[a]$. Then $\langle \sigma_0^{\overline{A}}(a), \ldots, \sigma_{m-1}^{\overline{A}}(a) \rangle \in \mathbf{R}^{\overline{A}}$. Let

$$b = (\cdots (a^n_{\sigma_0^{\overline{A}}(a)})^{n+1}_{\sigma_1^{\overline{A}}(a)}) \cdots)^{n+m-1}_{\sigma_{m-1}^{\overline{A}}(a)}.$$

Let $j < m$. Since n is greater than eack k such that v_k occurs in σ_j, we have $\sigma_j^{\overline{A}}(a) = \sigma^{\overline{A}}(b) = b_{n+j}$. Hence $\overline{A} \models (\sigma_j = v_{n+j})[b]$, and $\overline{A} \models \mathbf{R} v_n \ldots v_{n+m-1}[b]$. It follows that

$$(*) \qquad \overline{A} \models \exists v_n \ldots \exists v_{n+m-1} \left[\bigwedge_{j<m} (\sigma_j = v_{n+j}) \wedge \mathbf{R} v_n \ldots v_{n+m-1} \right] [a].$$

Thus we have shown that $\overline{A} \models \varphi[a]$ implies $(*)$. Conversely, assume $(*)$. Then there exist $x(0), \ldots, x(m-1) \in A$ such that $\left[\bigwedge_{j<m}(\sigma_j = v_{n+j}) \wedge \mathbf{R}v_n \ldots v_{n+m-1}\right][b]$, where $b = (\cdots(a^n_{x(0)})^{n+1}_{x(1)}) \cdots)^{n-m+1}_{x(m-1)}$. Let $j < m$. Then $\sigma_j^{\overline{A}}(a) = \sigma_j^{\overline{A}}(b) = b_{n+j}$. Also, we have $\langle b_n, \ldots; b_{n+m-1}\rangle \in \mathbf{R}^{\overline{A}}$. So $\langle \sigma_0^{\overline{A}}(a), \ldots, \sigma_{m-1}^{\overline{A}}(a)\rangle \in \mathbf{R}^{\overline{A}}$. Hence $\overline{A} \models \varphi[a]$.