Solutions for exercises in chapter 1

$\boxed{\text{E1.1}}$ *Verify that*

$$S_0 \to \neg S_1 = \langle 2,3,1,4\rangle$$

and

$$(S_0 \to S_1) \to (\neg S_1 \to \neg S_0) = \langle 2,2,3,4,2,1,4,1,3\rangle.$$

$$\begin{aligned} S_0 \to \neg S_1 &= \langle 2\rangle^\frown S_0^\frown \neg S_1 \\ &= \langle 2\rangle^\frown \langle 3\rangle^\frown \langle 1\rangle^\frown S_1 \\ &= \langle 2,3,1,4\rangle; \end{aligned}$$

$$\begin{aligned} (S_0 \to S_1) \to (\neg S_1 \to \neg S_0) &= \langle 2\rangle^\frown (S_0 \to S_1)^\frown (\neg S_1 \to \neg S_0) \\ &= \langle 2\rangle^\frown \langle 2\rangle^\frown S_0^\frown S_1^\frown \langle 2\rangle^\frown \neg S_1^\frown \neg S_0 \\ &= \langle 2,2,3,4,2\rangle^\frown \langle 1\rangle^\frown S_1^\frown \langle 1\rangle^\frown S_0 \\ &= \langle 2,2,3,4,2,1,4,1,3\rangle. \end{aligned}$$

$\boxed{\text{E1.2}}$ *Show that the function h defined after the definition of sentential formula construction satisfies the conditions for a sentential formula construction.*

$h(0) = S_4$; $h(1) = \neg h(0)$; $h(2) = \neg h(1)$; and $h(3) = \neg h(2)$.

$\boxed{\text{E1.3}}$ *Prove that there is a sentential formula of each positive integer length.*

If m is a positive integer, then

$$\langle \overbrace{1,1,\ldots,1}^{m-1 \text{ times}}, S_0\rangle$$

is a formula of length m, it is

$$\overbrace{\neg\neg\cdots\neg}^{m-1 \text{ times}} S_0.$$

$\boxed{\text{E1.4}}$ *Prove that m is the length of a sentential formula not involving $\neg$ iff m is odd.*

Proof. $\Rightarrow$: We prove by induction on φ that if φ is a sentential formula not involving $\neg$, then the length of φ is odd. This is true of sentential variables, which have length 1. Suppose that it is true of φ and ψ, which have length $2m+1$ and $2n+1$ respectively. Then $\varphi \to \psi$, which is $\langle 1\rangle^\frown \varphi^\frown \psi$, has length $1 + 2m + 1 + 2n + 1 = 2(m + n + 1) + 1$, which is again odd. This finishes the inductive proof.

$\Leftarrow$. We construct formulas without $\neg$ with length any odd integer by induction. $\langle S_0\rangle$ is a formula of length 1. If φ has been constructed of length $2m + 1$, then $S_0 \to \varphi$, which is $\langle 1, S_0\rangle^\frown \varphi$, has length $2m + 3$. This finishes the inductive construction.

$\boxed{\text{E1.5}}$ *Prove Proposition 1.3 as follows. Let f be a sentential assignment. For each positive integer m, let A_m be the set of all sentential formulas of length at most m. An*

m-approximation is a function G assigning to each member of A_m a value 0 or 1 so that the following conditions hold:

(1) *If $S_i \in A_m$, then $G(S_i) = f(i)$.*
(2) *If $\neg\varphi \in A_m$, then $G(\neg\varphi) = 1 - G(\varphi)$.*
(3) *If $\varphi \to \psi$ is in A_m, then $G(\varphi \to \psi) = 0$ iff $G(\varphi) = 1$ and $G(\psi) = 0$.*

Prove:

(4) *If G and G' are m-approximations, then $G = G'$.*

(5) *For each positive integer m there is an m-approximation.*

Then one can define the desired function F by setting $F(\varphi) = G(\varphi)$ where G is an m-approximation with φ of length m.

Following the outline, to prove (4), suppose that G and G' are m-approximations. We prove by induction on $i \leq m$ that if φ is a formula of length i, then $G(\varphi) = G'(\varphi)$. Suppose that we know the result for formulas φ of length less than i, and ψ has length i, where $1 \leq i \leq m$. By Proposition 1.2(ii) we have three cases.

Case 1. ψ is S_j for some j. Then $G(\psi) = G(S_j) = f(j) = G'(S_j) = G'(\psi)$.

Case 2. ψ is $\langle 0\rangle^\frown\chi$ for some formula χ. Thus the length of χ is $i - 1 < i$, so $G(\chi) = G'(\chi)$ by the inductive assumption. Hence $G(\psi) = 1 - G(\chi) = 1 - G'(\chi) = G'(\psi)$.

Case 3. ψ is $\langle 1\rangle^\frown\chi^\frown\theta$ for some formulas χ, θ. Then the lengths of χ and θ are less than i, and so $G(\chi) = G'(\chi)$ and $G(\theta) = G'(\theta)$ by the inductive assumption. Hence

$$\begin{aligned} G(\psi) = 0 \quad &\text{iff} \quad G(\chi \to \theta) = 0 \\ &\text{iff} \quad G(\chi) = 1 \text{ and } G(\theta) = 0 \\ &\text{iff} \quad G'(\chi) = 1 \text{ and } G'(\theta) = 0 \\ &\text{iff} \quad G'(\chi \to \theta) = 0 \\ &\text{iff} \quad G'(\psi) = 0. \end{aligned}$$

and it follows that $G(\psi) = G'(\psi)$.

This finishes the inductive proof.

We prove (5) by induction. For $m = 1$, define $G(S_i) = f(i)$ for all $i \in \omega$. Clearly G is a 1-approximation. Now assume that we know that there is an n-approximation for every $n < m$, where $m > 1$. For each $n < m$ let G_n be an n-approximation. Let $\varphi \in A_m$. If φ has length $n < m$, let $H(\varphi) = G_n(\varphi)$. Now suppose that φ has length m. By Proposition 1.2(ii) we have the following cases:

Case 1. $\varphi = S_i$ for some i. But S_i has length 1 and $m > 1$, contradiction.

Case 2. $\varphi = \neg\psi$ for some formula ψ. Then ψ has length $m - 1$. We define $H(\varphi) = 1 - G_{m-1}(\psi)$.

Case 3. $\varphi = \psi \to \chi$ for some formulas ψ, χ. Say ψ has length $n < m$ and χ has length $p < m$. We define $H(\varphi) = 0$ iff $G_n(\psi) = 1$ and $G_p(\chi) = 0$.

Clearly H is an m-approximation.

$\boxed{\text{E1.6}}$ *Prove that a truth table for a sentential formula involving n basic formulas has 2^n rows.*

We prove this by induction on n. For $n = 1$, there are two rows. Assume that for n basic formulas there are 2^n rows. Given $n + 1$ basic formulas, let φ be one of them. For the others, by the inductive hypothesis there are 2^n rows. For each such row there are two possibilities, 0 or 1, for φ. So for the $n + 1$ basic formulas there are $2^n \cdot 2 = 2^{n+1}$ rows.

$\boxed{\text{E1.7}}$ *Use the truth table method to show that the formula*

$$(\varphi \to \psi) \leftrightarrow (\neg\varphi \vee \psi)$$

is a tautology.

φ	ψ	$\varphi \to \psi$	$\neg\varphi$	$\neg\varphi \vee \psi$	$(\varphi \to \psi) \leftrightarrow (\neg\varphi \vee \psi)$
1	1	1	0	1	1
1	0	0	0	0	1
0	1	1	1	1	1
0	0	1	1	1	1

$\boxed{\text{E1.8}}$ *Use the truth table method to show that the formula*

$$[\varphi \vee (\psi \wedge \chi)] \leftrightarrow [(\varphi \vee \psi) \wedge (\varphi \vee \chi)]$$

is a tautology.

Let θ be the indicated formula.

φ	ψ	χ	$\varphi \vee \psi$	$\varphi \vee \chi$	$(\varphi \vee \psi) \wedge (\varphi \vee \chi)$	$\psi \wedge \chi$	$\varphi \vee (\psi \wedge \chi)$	θ
1	1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1	1
1	0	1	1	1	1	0	1	1
1	0	0	1	1	1	0	1	1
0	1	1	1	1	1	1	1	1
0	1	0	1	0	0	0	0	1
0	0	1	0	1	0	0	0	1
0	0	0	0	0	0	0	0	1

$\boxed{\text{E1.9}}$ *Use the truth table method to show that the formula*

$$(\varphi \to \psi) \to (\varphi \to \neg\psi)$$

is not a tautology. It is not necessary to work out the full truth table.

φ	ψ	$\varphi \to \psi$	$\neg\psi$	$\varphi \to \neg\psi$	$(\varphi \to \psi) \to (\varphi \to \neg\psi)$
1	1	1	0	0	0

$\boxed{\text{E1.10}}$ *Use the informal method described in the notes to determine whether or not the following is a tautology:*

$$S_0 \to (S_1 \to (S_2 \to (S_3 \to S_1))).$$

2	1	3	2	4	3	5	4	5
S_0	$\to$	$(S_1$	$\to$	$(S_2$	$\to$	$(S_3$	$\to$	$S_0)))$
1	0	1	0	1	0	1	0	0

Values 0 and 1 have been tentatively assigned to S_0, a contradiction, so the formula is a tautology.

$\boxed{\text{E1.11}}$ *Use the informal method described in the notes to determine whether or not the following is a tautology:*

$$(\{[(\varphi \to \psi) \to (\neg\chi \to \neg\theta)] \to \chi\} \to \tau) \to [(\tau \to \varphi) \to (\theta \to \varphi)].$$

5	6		11	13	14	12	6	5	9	10	2	8	1	7	3	5	2	4	3	4
$(\{[(\varphi$	$\to$	$\psi)$	$\to$	$(\neg$	χ	$\to$	$\neg$	$\theta)]$	$\to$	$\chi\}$	$\to$	$\tau)$	$\to$	$[(\tau$	$\to$	$\varphi)$	$\to$	$(\theta$	$\to$	$\varphi)]$
0	1		1	0	1	1	0	1	0	0	1	0	0	0	1	0	0	1	0	0

Both 0 and 1 have tentatively been assigned to χ, so the formula is a tautology.

$\boxed{\text{E1.12}}$ *Determine whether the following statements are logically consistent. If the contract is valid, then Horatio is liable. If Horation is liable, he will go bankrupt. Either Horatio will go bankrupt or the bank will lend him money. However, the bank will definitely not lend him money.*

Let S_0 correspond to "the contract is valid", S_1 to "Horatio is liable", S_2 to "Horatio will go bankrupt", and S_3 to "the bank will lend him money". Then we want to see if there is an assignment of values which makes the following sentence true:

$$(S_0 \to S_1) \wedge (S_1 \to S_2) \wedge (S_2 \vee S_3) \wedge \neg S_3.$$

We can let $f(0) = f(1) = f(2) = 1$ and $f(3) = 0$, and this gives the sentence the value 1.

$\boxed{\text{E1.13}}$ *Prove Proposition 1.5. Hint: For m a positive integer, let G_m be the set of all functions f with the following properties.*

(1) The domain of f is m'.

(2) For each $i < m$, f_i is itself a function whose domain is the set of all $(i+1)$-tuples of sentential formulas.

(3) $f_0(\varphi) = \varphi$ for every sentential formula φ.

(4) If $0 < i < m$ and $\langle \psi_0, \ldots, \psi_i \rangle$ is a sequence of sentential formulas, then

$$f_i(\psi_0, \ldots, \psi_i) = f_{i-1}(\psi_0, \ldots, \psi_{i-1}) \vee \psi_i.$$

Prove:

(5) If $0 < n < m$ and $f \in G_m$, then f restricted to n' is in G_n.

(6) If m is a positive integer and $f, g \in G_m$, then $f = g$.

(7) For each positive integer m the set G_m is nonempty

Then one can define, for each positive integer m, $F_m = f_m$ for f the unique member of G_m.

Condition (5) is clear.

We prove (6) by induction. For $m = 1$, suppose that $f, g \in G_1$. Thus both f and g have domain $1' = \{0\}$, and both f_0 and g_0 are functions with domain the set of all sentential formulas. Moreover, for any sentential formula φ we have $f_0(\varphi) = \varphi = g_0(\varphi)$. So $f = g$.

Now suppose that (6) holds for all positive integers $\leq m$ and $f, g \in G_{m+1}$. Thus both f and g have domain $(m+1)'$. Let f' and g' be the restrictions of f and g respectively to m'. Then $f', g' \in G_m$ by (5), and so $f' = g'$ by the induction hypothesis.

For any formula φ we have $f_0(\varphi) = \varphi = g_0(\varphi)$. Now suppose that $0 < i < m+1$ and $\langle \psi_0, \ldots, \psi_i \rangle$ is a sequence of sentential formulas. If $i < m$, then

$$\begin{aligned} f_i(\psi_0, \ldots, \psi_i) &= f'_i(\psi_0, \ldots, \psi_i) \\ &= g'_i(\psi_0, \ldots, \psi_i) \\ &= g_i(\psi_0, \ldots, \psi_i). \end{aligned}$$

If $i = m$, then

$$\begin{aligned} f_m(\psi_0, \ldots, \psi_m) &= f_{m-1}(\psi_0, \ldots, \psi_{m-1}) \vee \psi_m \\ &= f'_{m-1}(\psi_0, \ldots, \psi_{m-1}) \vee \psi_m \\ &= g'_{m-1}(\psi_0, \ldots, \psi_{m-1}) \vee \psi_m \\ &= g(\psi_0, \ldots, \psi_m). \end{aligned}$$

Thus $f = g$, finishing the inductive proof of (6).

We also prove (7) by induction on m. For $m = 1$, let f be the function with domain $\{0\}$ such that $f_0(\varphi) = \varphi$ for every formula φ. Clearly $f \in G_1$.

Suppose we have shown that G_i is nonempty for all positive $i < m$, where $m \geq 2$. By (6), for each $i < m$ there is a unique member g_i of G_i. We now define f with domain m' by setting $f_i = g_{i+1}(i)$ for all i with $i + 1 < m$, and

$$f_{m-1}(\varphi_0, \ldots, \varphi_m) = (g_{m-1}(m-2))(\varphi_0, \ldots, \varphi_{m-1}) \vee \varphi_m$$

for any formulas $\varphi_0, \ldots, \varphi_m$.

To show that $f \in G_m$, first note that $f_0(\varphi) = (g_1(0))(\varphi) = \varphi$. Now suppose that $0 < i < m$ and $\langle \psi_0, \ldots, \psi_i \rangle$ is a sequence of sentential formulas. Then if $i+1 < m$ we have

$$\begin{aligned} f_i(\psi_0, \ldots, \psi_i) &= (g_{i+1}(i))(\psi_0, \ldots, \psi_i) \\ &= (g_{i+1}(i-1))(\psi_0, \ldots, \psi_{i-1}) \vee \psi_i \\ &= (g_i(i-1))((\psi_0, \ldots, \psi_{i-1}) \vee \psi_i \\ &= f_{i-1}(\psi_0, \ldots, \psi_{i-1}) \vee \psi_i \end{aligned}$$

For $i + 1 = m$,

$$\begin{aligned} f_{m-1}(\varphi_0, \ldots, \varphi_m) &= (g_{m-1}(m-2))(\varphi_0, \ldots, \varphi_{m-1}) \vee \varphi_m \\ &= f_{m-2}(\varphi_0, \ldots, \varphi_{m-1}) \vee \varphi_m. \end{aligned}$$

$\boxed{\text{E1.14}}$ *Let $\varphi \mid \psi$ be defined by the following truth table:*

φ	ψ	$\varphi \mid \psi$
1	*1*	*0*
1	*0*	*0*
0	*1*	*0*
0	*0*	*1*

Prove that for any k, any function mapping k-tuples of members of $\{0, 1\}$ into $\{0, 1\}$ can be obtained from $\mid$.

By theorem 1.7, it suffices to show that $\neg$ and $\rightarrow$ can be obtained from $\mid$:

φ	$\varphi \mid \varphi$
1	0
0	1

φ	ψ	$\neg\varphi$	$\neg\varphi \mid \psi$	$\neg(\neg\varphi \mid \psi)$
1	1	0	0	1
1	0	0	1	0
0	1	1	0	1
0	0	1	0	1

$\boxed{\text{E1.15}}$ *Give a formula in disjunctive normal form equivalent to the following formula:*

$$(S_0 \to (S_1 \to S_2)) \to (S_1 \to S_0).$$

In order to follow the proof of Theorem 1.8, we first write out a truth table for this formula:

S_0	S_1	S_2	$S_1 \to S_2$	$S_0 \to (S_1 \to S_2)$	$S_1 \to S_0$	$(S_0 \to (S_1 \to S_2)) \to (S_1 \to S_0)$
1	1	1	1	1	1	1
1	1	0	0	0	1	1
1	0	1	1	1	1	1
1	0	0	1	1	1	1
0	1	1	1	1	0	0
0	1	0	0	1	0	0
0	0	1	1	1	1	1
0	0	0	1	1	1	1

So by the proof of Theorem 1.8 the following is a formula in disjunctive normal form equivalent to the given formula:

$$\begin{aligned}&(S_0 \wedge S_1 \wedge S_2) \vee (S_0 \wedge S_1 \wedge \neg S_2) \vee (S_0 \wedge \neg S_1 \wedge S_2)\\ &\vee (S_0 \wedge \neg S_1 \wedge \neg S_2) \vee (\neg S_0 \wedge \neg S_1 \wedge S_2) \vee (\neg S_0 \wedge \neg S_1 \wedge \neg S_2).\end{aligned}$$

$\boxed{\text{E1.16}}$ *Write out an actual proof for $\{\psi\} \vdash \neg\psi \to \varphi$. This can be done by following the proof of Lemma 1.13, expanding it using the proof of the deduction theorem.*

Following the proof of Lemma 1.13, the following is a $\{\psi, \neg\psi\}$-proof:

(a)	$\neg\psi$	
(b)	$\neg\psi \to (\neg\varphi \to \neg\psi)$	(1)
(c)	$\neg\varphi \to \neg\psi$	(a), (b), MP
(d)	$(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$	(3)
(e)	$\psi \to \varphi$	(c), (d), MP
(f)	ψ	
(g)	φ	(e), (f), MP

Now applying the proof of the deduction theorem, the following is a $\{\psi\}$-proof:

(a)	$[\neg\psi \to [(\neg\psi \to \neg\psi) \to \neg\psi]] \to [[\neg\psi \to (\neg\psi \to \neg\psi)]$ $\to (\neg\psi \to \neg\psi)]$	(2)
(b)	$\neg\psi \to [(\neg\psi \to \neg\psi) \to \neg\psi]$	(1)
(c)	$[\neg\psi \to (\neg\psi \to \neg\psi)] \to (\neg\psi \to \neg\psi)$	(a), (b), MP

(d)	$\neg\psi \to (\neg\psi \to \neg\psi)$	(1)
(e)	$\neg\psi \to \neg\psi$	(c), (d), MP
(f)	$[\neg\psi \to (\neg\varphi \to \neg\psi)] \to [\neg\psi \to [\neg\psi \to (\neg\varphi \to \neg\psi)]]$	(1)
(g)	$\neg\psi \to (\neg\varphi \to \neg\psi)$	(1)
(h)	$\neg\psi \to [\neg\psi \to (\neg\varphi \to \neg\psi)]]$	(f), (g), MP
(i)	$[(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)] \to [\neg\psi \to [(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)]]$	(1)
(j)	$(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$	(3)
(k)	$\neg\psi \to [(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)]$	(i), (j), MP
(l)	$[\neg\psi \to [(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)]] \to [[\neg\psi \to (\neg\varphi \to \neg\psi)]$ $\to [\neg\psi \to (\psi \to \varphi)]]$	(2)
(m)	$[\neg\psi \to (\neg\varphi \to \neg\psi)] \to [\neg\psi \to (\psi \to \varphi)]$	(k), (l), MP
(n)	$\neg\psi \to (\psi \to \varphi)$	(g), (m), MP
(o)	$\psi \to (\neg\psi \to \psi)$	(1)
(p)	ψ	
(q)	$\neg\psi \to \psi$	(o), (p), MP
(r)	$[\neg\psi \to (\psi \to \varphi)] \to [(\neg\psi \to \psi) \to (\neg\psi \to \varphi)]$	(2)
(s)	$(\neg\psi \to \psi) \to (\neg\psi \to \varphi)$	(n), (r), MP
(t)	$\neg\psi \to \varphi$	(q), (s), MP