8. Recursiveness of syntactic notions

August 18, 2014

In this chapter we finish the proof of Gödel's incompleteness theorem by proving the recursiveness of syntactic notions, proving Theorems B–D.

We start with terms. We repeat the general definition in Chapter 2 in the special case of our number-theoretic language. The non-logical symbols are $+$, the integer 7, $\bullet$, the integer 9, $\mathbf{S}$, the integer 6, and $\mathbf{0}$, the integer 8. We also have variables, with v_i the integer $5(i+1)$. A term construction sequence is a finite sequence $\langle \tau_0, \ldots, \tau_{m-1}\rangle$ such that each τ_i is a sequence of some of these integers, and for each $i < m$ one of the following conditions holds:

τ_i is $\langle v_j\rangle$ for some $j \in \omega$.

There are $j, k < i$ such that τ_i is $\langle +\rangle^\frown \tau_j^\frown \tau_k$.

There are $j, k < i$ such that τ_i is $\langle \bullet\rangle^\frown \tau_j^\frown \tau_k$.

There is a $j < i$ such that τ_i is $\langle \mathbf{S}\rangle^\frown \tau_j$.

τ_i is $\langle \mathbf{0}\rangle$.

Let TRMCON be the set of all Gödel numbers of term construction sequences:

$$\mathrm{TRMCON} = \{gn_1(\Phi) : \Phi \text{ is a term construction sequence}\}.$$

Now for any $m, n \in \omega$ we define

$$\mathrm{CAT}(m, n) = m \cdot \prod_{i<\mathrm{len}(n)} \mathbf{p}_{\mathrm{len}(m)+i}^{(n)_i}.$$

Thus if $m = gn(\varphi)$ and $n = gn(v)$, then $\mathrm{CAT}(m, n) = gn(\varphi^\frown\psi)$.

Lemma 8.1. CAT *is recursive.*

Proof. We define

$$\begin{aligned}
f(m,n,i) &= (n)_i; \text{ recursive, since } f = \mathbf{C}^2_3((\ ), \mathbf{I}^3_1, \mathbf{I}^3_2);\\
g(m,n,i) &= \mathrm{len}(m); \text{ recursive, since } g = \mathbf{C}^1_3(\mathrm{len}, \mathbf{I}^3_0);\\
h(m,n,i) &= \mathrm{len}(m) + i; \text{ recursive, since } h = \mathbf{C}^2_3(+, g, \mathbf{I}^3_2);\\
k(m,n,i) &= \mathbf{p}_{\mathrm{len}(m)+i}; \text{ recursive, since } k = \mathbf{C}^1_3(\mathbf{p}, h);\\
l(m,n,i) &= \mathbf{p}_{\mathrm{len}(m)+i}^{(n)_i}; \text{ recursive, since } l = \mathbf{C}^2_3(\exp, k, f);\\
t(m,n,z) &= \prod_{i<z} \mathbf{p}_{\mathrm{len}(m)+i}^{(n)_i}; \text{ recursive, by Proposition 7.13}\\
u(m,n) &= \mathrm{len}(n); \text{ recursive, since } u = \mathbf{C}^1_2(\mathrm{len}, \mathbf{I}^2_1);\\
w(m,n) &= \prod_{i<\mathrm{len}(n)} \mathbf{p}_{\mathrm{len}(m)+i}^{(n)_i}; \text{ recursive, since } w = \mathbf{C}^3_2(t, \mathbf{I}^2_0, \mathbf{I}^2_1, w);
\end{aligned}$$

hence CAT is recursive, since it is $\mathbf{C}^2_2(\cdot, \mathbf{I}^2_0, w)$. □

Now we define

$$\begin{aligned} \text{PLUS}(m,n) &= \text{CAT}(\text{CAT}(2^7, m), n); \\ \text{TIMES}(m,n) &= \text{CAT}(\text{CAT}(2^9, m), n); \\ \text{SU}(m) &= \text{CAT}(2^6, m); \\ \text{ZERO} &= 2^8. \end{aligned}$$

Thus if σ and τ are terms, then $\text{PLUS}(gn(\sigma), gn(\tau)) = gn(+\sigma\tau)$, $\text{TIMES}(gn(\sigma), gn(\tau)) = gn(\bullet\sigma\tau)$, $\mathbf{SU}(gn(\sigma)) = gn(\mathbf{S}\sigma)$, and $\mathbf{ZERO} = GN(\mathbf{0})$.

Lemma 8.2. PLUS*,* TIMES*, and* SU *are recursive.*

Proof.

Let $f_0(m,n) = \text{CAT}(2^7, m)$; recursive since $f_0 = \mathbf{C}^2_2(\text{CAT}, \mathbf{k}^2_{2^7}, \mathbf{I}^2_0)$;
PLUS is recursive, since $\text{PLUS} = \mathbf{C}^2_2(\text{CAT}, f_0, \mathbf{I}^2_1)$;
Let $f_1(m,n) = \text{CAT}(2^9, m)$; recursive since $f_1 = \mathbf{C}^2_2(\text{CAT}, \mathbf{k}^2_{2^9}, \mathbf{I}^2_0)$;
TIMES is recursive, since $\text{TIMES} = \mathbf{C}^2_2(\text{CAT}, f_1, \mathbf{I}^2_1)$;
SU is recursive, since $\text{SU} = \mathbf{C}^2_1(\text{CAT}, \mathbf{k}^1_{2^6}, \mathbf{I}^1_0)$. □

Lemma 8.3. TRMCON *is recursive.*

Proof. An outline of the proof is as follows. $m \in \text{TRMCON}$ iff for every $i < \text{len}(m)$ one of the following holds:

(1) $(m)_i = gn(\langle \mathbf{0} \rangle)$;

(2) There is a j such that $(m)_i = gn(\langle v_j \rangle)$, where to insure recursiveness we need to bound j, and m itself is a suitable bound;

(3) There is a $j < i$ such that $(m)_i = gn(\langle \mathbf{S} \rangle ^\frown \sigma)$, with $gn(\sigma) = (m)_j$;

(4) There are $j, k < i$ such that $(m)_i$ is $gn(\langle + \rangle ^\frown \sigma ^\frown \tau)$, with $(m)_j = gn(\sigma)$ and $(m)_k = gn(\tau)$;

(5) There are $j, k < i$ such that $(m)_i$ is $gn(\langle \bullet \rangle ^\frown \sigma ^\frown \tau)$, with $(m)_j = gn(\sigma)$ and $(m)_k = gn(\tau)$.

The details:

Let $R_0 = \{(m,i) : (m)_i = 2^8\}$; recursive since $\chi_{R_0} = \mathbf{C}^2_2(\chi_=, (\ ), \mathbf{k}^2_{2^8})$;
(R_0 corresponds to (1))
Let $f_0(m,i,j) = 5(j+1)$; recursive since $f_0 = \mathbf{C}^2_3(\cdot, \mathbf{k}^3_5, \mathbf{C}^1_3(\mathbf{s}, \mathbf{I}^3_2))$;
Let $f_1(m,i,j) = 2^{5(j+1)}$; recursive since $f_1 = \mathbf{C}^2_3(\exp, f_0)$;

Let $S_0 = \{(m,i,j) : (m)_i = 2^{5(j+1)}\}$; recursive since $\chi_{S_0} = \mathbf{C}_3^2(\chi_=, (\ ), f_1)$;
Let $R_1 = \{(m,i) : \exists j < m[(m)_i = 2^{5(j+1)}]\}$; recursive by Proposition 7.20;
(R_1 corresponds to (2))
Let $f_2(i,m,j) = (m)_j$; recursive since $f_2 = \mathbf{C}_3^2((\ ), \mathbf{I}_1^3, \mathbf{I}_2^3)$;
Let $f_3(i,m,j) = \mathrm{SU}((m)_j)$; recursive since $f_3 = \mathbf{C}_1^3(\mathrm{SU}, f_3)$;
Let $f_4(i,m,j) = (m)_i$; recursive since $f_4 = \mathbf{C}_3^2((\ ), \mathbf{I}_1^3, \mathbf{I}_0^3)$;
Let $S_1 = \{(i,m,j) : (m)_i = \mathrm{SU}((m)_j)\}$; recursive since $\chi_{S_1} = \mathbf{C}_3^2(\chi_=, f_4, f_3)$;
Let $S_2 = \{(i,m) : \exists j < i[(m)_i = \mathrm{SU}((m)_j)]\}$; recursive by Proposition 7.20
Let $R_2 = \{(m,i) : \exists j < i[(m)_i = \mathrm{SU}((m)_j)]\}$; recursive since $\chi_{R_2} = \mathbf{C}_2^2(\chi_{S_2}, \mathbf{I}_1^2, \mathbf{I}_0^2)$;
(R_2 corresponds to (3))
Let $f_5(i,m,j,k) = (m)_k$; recursive since $f_5 = \mathbf{C}_4^2((\ ), \mathbf{I}_1^4.\mathbf{I}_3^4)$;
Let $f_6(i,m,j,k) = (m)_j$; recursive since $f_6 = \mathbf{C}_4^2((\ ), \mathbf{I}_1^4.\mathbf{I}_2^4)$;
Let $f_7(i,m,j,k) = \mathbf{PLUS}((m)_j, (m)_k)$; recursive since $f_7 = \mathbf{C}_4^2(\mathbf{PLUS}, f_6, v_5)$;
Let $f_8(i,m,j,k) = (m)_i$; recursive since $f_8 = \mathbf{C}_4^2((\ ), \mathbf{I}_1^4.\mathbf{I}_0^4)$;
Let $S_3 = \{(i,m,j,k) : (m)_i = \mathbf{PLUS}((m)_j, (m)_k)\}$;
recursive since $\chi_{S_3} = \mathbf{C}_4^2(\chi_=, f_8, f_7)$;
Let $S_4 = \{(i,m,j) : \exists k < i[(m)_i = \mathbf{PLUS}((m)_j, (m)_k)]\}$;
recursive by Proposition 7.20;
Let $S_5 = \{(i,m) : \exists j < i \exists k < i[(m)_i = \mathbf{PLUS}((m)_j, (m)_k)]\}$;
recursive by Proposition 7.20;
Let $R_3 = \{(m,i) : \exists j < i \exists k < i[(m)_i = \mathbf{PLUS}((m)_j, (m)_k)]\}$;
recursive since $\chi_{R_3} = \mathbf{C}_2^2(\chi_{S_5}, \mathbf{I}_1^2, \mathbf{I}_0^2)$;
(R_3 corresponds to (4))
Let $f_9(i,m,j,k) = \mathbf{TIMES}((m)_j, (m)_k)$; recursive since $f_9 = \mathbf{C}_4^2(\mathbf{TIMES}, f_7, v_6)$;
Let $S_6 = \{(i,m,j,k) : (m)_i = \mathbf{TIMES}((m)_j, (m)_k)\}$;
recursive since $\chi_{S_6} = \mathbf{C}_4^2(\chi_=, f_8, f_9)$;
Let $S_7 = \{(i,m,j) : \exists k < i[(m)_i = \mathbf{TIMES}((m)_j, (m)_k)]\}$;
recursive by Proposition 7.20;
Let $S_8 = \{(i,m) : \exists j < i \exists k < i[(m)_i = \mathbf{TIMES}((m)_j, (m)_k)]\}$;
recursive by Proposition 7.20;
Let $R_4 = \{(m,i) : \exists j < i \exists k < i[(m)_i = \mathbf{TIMES}((m)_j, (m)_k)]\}$;
recursive since $\chi_{R_4} = \mathbf{C}_2^2(\chi_{S_8}, \mathbf{I}_1^2, \mathbf{I}_0^2)$;
(R_4 corresponds to (5))

Now let $T = R_0 \cup \ldots \cup R_4$; so T is recursive by Proposition 7.16. By Corollary 7.22 the set $U \overset{\text{def}}{=} \{(m,n) : \text{for all } i < n\ (m,i) \in T\}$ is recursive. Now TRMCON $= \{m : m > 1$

and $(m, \mathrm{len}(m)) \in U\}$; it is recursive by the following steps:

$$\begin{aligned}
&\text{Let } S_9 = \{m : m > 1\};\ \text{recursive by Corollary 7.17;}\\
&\text{Let } S_{10} = \{m : (m, \mathrm{len}(m)) \in U\};\ \text{recursive since}\\
&\qquad \chi_{S_{10}} = \mathbf{C}^2_1(\chi_U, \mathbf{I}^1_0, \mathbf{C}^1_1(\mathrm{len}, \mathbf{I}^1_0));\\
&\mathrm{TRMCON} = S_9 \cap S_{10};\ \text{recursive by Proposition 7.16}
\end{aligned}$$

□

Lemma 8.4. *For any term σ there is a term construction sequence $\Phi = \langle \tau_0, \ldots, \tau_{m-1} \rangle$ with the following properties:*

(i) $\tau_{m-1} = \sigma$.

(ii) Each τ_i is a subterm of σ.

(iii) m is the length of σ.

Proof. We prove this by induction on σ, thus using Proposition 2.1. If σ is $\mathbf{0}$ or a variable, we can take $\Phi = \langle \sigma \rangle$, which clearly satisfies the conditions (i)–(iii). If Φ is a term construction sequence with properties (i)–(iii) for σ, then $\Phi^\frown\langle \mathbf{S}\sigma \rangle$ has properties (i)–(iii) for $\mathbf{S}\sigma$. If Φ is a term construction sequence with properties (i)–(iii) for σ and Ψ is a term construction sequence with properties (i)–(iii) for τ, then $\Phi^\frown\Psi^\frown\langle +\sigma\tau \rangle$ is a term construction sequence with properties (i)–(iii) for $+\sigma\tau$, and $\Phi^\frown\Psi^\frown\langle \bullet\sigma\tau \rangle$ is a term construction sequence with properties (i)–(iii) for $\bullet\sigma\tau$. □

Note that an upper bound on $gn_1(\Phi)$ for Φ a term construction sequence satisfying the conditions of the lemma is

$$\prod_{i < \mathrm{len}(gn(\sigma))} \mathbf{p}_i^{gn(\sigma)}.$$

This explains some steps in the proof of the following lemma.

Let TRM be the set of all Gödel numbers of terms.

Lemma 8.5. TRM *is recursive.*

Proof.

$$\begin{aligned}
&\text{Let } f_0(m,n) = \mathbf{p}_n^m;\ \text{recursive since } f_0 = \mathbf{C}^2_2(\exp, \mathbf{C}^1_2(\mathbf{p}, \mathbf{I}^2_1), \mathbf{I}^2_0);\\
&\text{let } f_1(m,n) = \prod_{i<n} \mathbf{p}_i^m;\ \text{recursive by Proposition 7.13}\\
&\text{let } f_2(m) = \prod_{i<\mathrm{len}(m)} \mathbf{p}_i^m;\ \text{recursive since } f_2 = \mathbf{C}^2_1(f_1, \mathbf{I}^1_0, \mathbf{C}^1_1(\mathrm{len}, \mathbf{I}^1_0));\\
&\text{let } f_3(m,n) = \mathrm{len}(n);\ \text{recursive since } f_3 = \mathbf{C}^1_2(\mathrm{len}, \mathbf{I}^2_1);\\
&\text{let } f_4(m,n) = \mathbb{P}(\mathrm{len}(n));\ \text{recursive since } f_4 = \mathbf{C}^1_2(\mathbb{P}, f_3);\\
&\text{let } f_5(m,n) = (n)_{\mathbb{P}(\mathrm{len}(n))};\ \text{recursive since } f_5 = \mathbf{C}^2_2((\ ), \mathbf{I}^2_1, f_4);\\
&\text{let } S_0 = \{(m,n) : (n)_{\mathbb{P}(\mathrm{len}(n))} = m\};\ \text{recursive since } \chi_{S_0} = \mathbf{C}^2_2(\chi_=, f_5, \mathbf{I}^2_0);\\
&\text{let } S_1 = \{(m,n) : n \in \mathrm{TRMCON}\};\ \text{recursive since } \chi_{S_1} = \mathbf{C}^1_2(\chi_{\mathrm{TRMCON}}, \mathbf{I}^2_1);\\
&\text{let } S_2 = S_0 \cap S_1;\ \text{recursive by Proposition 7.16}\\
&\text{let } S_3 = \{(m,n) : \text{ there is an } i \le n \text{ such that } (m,i) \in S_2\}.\ \text{recursive by Prop. 7.21}
\end{aligned}$$

Now TRM is recursive, since $\chi_{\mathrm{TRM}} = \mathbf{C}_1^2(\chi_{S_3}, \mathbf{I}_0^1, f_2)$; in fact,

$$\begin{aligned}
\mathbf{C}_1^2(\chi_{S_3}, \mathbf{I}_0^1, f_2)(m) &= \chi_{S_3}(m, f_2(m)) \\
&= \chi_{S_3}\left(m, \prod_{i<\mathrm{len}(m)} \mathbf{p}_i^m\right) \\
&= \begin{cases} 1 & \text{if } \left(m, \prod_{i<\mathrm{len}(m)} \mathbf{p}_i^m\right) \in S_3, \\ 0 & \text{otherwise,} \end{cases} \\
&= \begin{cases} 1 & \text{if there is a } j \le \prod_{i<\mathrm{len}(m)} \mathbf{p}_i^m \\ & \text{such that } (m, j) \in S_2, \\ 0 & \text{otherwise,} \end{cases} \\
&= \begin{cases} 1 & \text{if there is a } j \le \prod_{i<\mathrm{len}(m)} \mathbf{p}_i^m \\ & \text{such that } (j)_{\mathrm{len}(j)-1} = m \text{ and } j \in \mathrm{TRMCON} \\ 0 & \text{otherwise,} \end{cases} \\
&= \chi_{\mathrm{TRM}}(m).
\end{aligned}$$

□

Now we define $\mathrm{EQ}(m,n) = \mathrm{CAT}(2^3, \mathrm{CAT}(m,n))$. Thus if σ and τ are terms, then $\mathrm{EQ}(gn(\sigma), gn(\tau)) = gn(= \sigma\tau)$. AT is the set of all Gödel numbers of atomic formulas. Recall that $=$ is the integer 3.

Lemma 8.6. EQ *is recursive.*

Proof. $\mathrm{EQ} = \mathbf{C}_2^2(\mathrm{CAT}, \mathbf{k}_{2^3}^2, \mathrm{CAT})$. □

Lemma 8.7. AT *is recursive.*

Proof.

Let $S_0 = \{(z,x,y) : z = \mathrm{EQ}(x,y)\}$; recursive since $\chi_{S_0} = \mathbf{C}_3^2(\chi_=, \mathbf{I}_0^3, \mathbf{C}_3^2(\mathrm{EQ}, \mathbf{I}_1^3, \mathbf{I}_2^3))$;
let $S_1 = \{(z,x,y) : y \in \mathrm{TRM}\}$; recursive since $\chi_{S_1} = \mathbf{C}_3^1(\chi_{\mathrm{TRM}}, \mathbf{I}_2^3)$;
let $S_2 = \{(z,x,y) : x \in \mathrm{TRM}\}$; recursive since $\chi_{S_2} = \mathbf{C}_3^1(\chi_{\mathrm{TRM}}, \mathbf{I}_1^3)$;
let $S_3 = S_0 \cap S_1 \cap S_2$; recursive by Prop. 7.16
let $S_4 = \{\langle z, x\rangle :$ there is a $y < z$ such that $\langle z,x,y\rangle \in S_3\}$; recursive by Prop. 7.20.

Now AT is recursive, since $\mathrm{AT} = \{z :$ there exist $x < z$ such that $(z,x) \in S_3\}$, and Prop. 7.20 applies. □

Now we define

$$\begin{aligned}
\mathrm{NEG}(m) &= \mathrm{CAT}(2^1, m); \\
\mathrm{IMP}(m,n) &= \mathrm{CAT}(2^2, \mathrm{CAT}(m,n)); \\
\mathrm{ALL}(m,n) &= \mathrm{CAT}(2^4, \mathrm{CAT}(2^{5(m+1)}, n)).
\end{aligned}$$

Thus for any formulas φ, ψ and any $m \in \omega$, $\text{NEG}(gn(\varphi)) = gn(\neg\varphi)$, $\text{IMP}(gn(\varphi), gn(\psi)) = gn(\varphi \to \psi)$, and $\text{ALL}(m, gn(\varphi)) = gn(\forall v_m \varphi)$.

Lemma 8.8. NEG, IMP *and* ALL *are recursive.*

Proof.

$$
\begin{aligned}
&NEG = \mathbf{C}^2_1(\text{CAT}, \mathbf{k}^1_1, \mathbf{I}^1_0));\\
&IMP = \mathbf{C}^2_2(\text{CAT}, \mathbf{k}^2_{2^2}, \mathbf{C}^2_2(\mathbf{I}^2_0, \mathbf{I}^2_1));\\
&\text{Let } f_0(m,n) = m+1; \text{ recursive since } f_0 = \mathbf{C}^1_2(\mathbf{s}, \mathbf{I}^2_0);\\
&\text{Let } f_1(m,n) = 5(m+1); \text{ recursive since } f_1 = \mathbf{C}^2_2(\cdot, \mathbf{k}^2_5, f_0);\\
&\text{Let } f_2(m,n) = 2^{5(m+1)}; \text{ recursive since } f_2 = \mathbf{C}^2_2(exp, \mathbf{k}^2_2, f_2);\\
&\text{Let } f_3(m,n) = \text{CAT}(2^{5(m+1)}, n); \text{ recursive since } f_3 = \mathbf{C}^2_2(\text{CAT}, f_2, \mathbf{I}^2_1;\\
&\text{ALL is recursive, since ALL} = \mathbf{C}^2_2(\mathbf{k}^2_{2^4}, f_3).
\end{aligned}
$$

□

We recall now the definition of a formula construction sequence; a sequence $\langle \varphi_0, \ldots, \varphi_{m-1} \rangle$ of sequences of natural numbers such that for each $i < m$ one of the following conditions holds:

(1) φ_i is an atomic formula.

(2) There is a $j < i$ such that φ_i is $\langle 1 \rangle ^\frown \varphi_j$.

(3) There exist $j, k < i$ such that φ_i is $\langle 2 \rangle ^\frown \varphi_j {}^\frown \varphi_k$.

(4) There exist $j < i$ and k such that φ_i is $\langle 4, 5(k+1) \rangle ^\frown \varphi_j$.

Recall here that $\neg$ corresponds to 1, $\to$ to 2 and $\forall$ to 4. Let FMLACON be the set of all Gödel numbers of formula construction sequences.

Lemma 8.9. FMLACON *is recursive.*

Proof.

Let $R_0 = \{(m,i) : (m)_i \in \text{AT}\}$; recursive, since $\chi_{R_0} = \mathbf{C}^1_2(\chi_{\text{AT}}, \mathbf{C}^2_2((\ ), \mathbf{I}^2_0, \mathbf{I}^2_1));$
(R_0 corresponds to (1))
let $f_0(i,m,j) = (m)_j$; recursive since $f_0 = \mathbf{C}^2_3((\ ), \mathbf{I}^3_1, \mathbf{I}^3_2);$
let $f_1(i,m,j) = (m)_i$; recursive since $f_0 = \mathbf{C}^2_3((\ ), \mathbf{I}^3_1, \mathbf{I}^3_0);$
let $f_2(i,m,j) = \text{NEG}((m)_j)$; recursive since $f_2 = \mathbf{C}^1_3(\mathbf{C}^1_3(\text{NEG}, f_0);$
let $S_0 = \{(i,m,j) : (m)_i = \text{NEG}((m)_j)\}$; recursive since $\chi_{S_0} = \mathbf{C}^2_3(\chi_=, f_1, f_2);$
let $S_1 = \{(i,m) :$ there is a $j < i$ such that $(m)_i = \text{NEG}((m)_j)\}$;
recursive by Prop. 7.20
let $R_1 = \{(m,i) :$ there is a $j < i$ such that $(m)_i = \text{NEG}((m)_j)\}$;
recursive since $\chi_{R_1} = \mathbf{C}^2_2(\chi_{S_1}, \mathbf{I}^2_1, \mathbf{I}^2_0);$
(R_1 corresponds to (2))

let $f_3(i,m,j,k) = (m)_k$; recursive since $f_4 = \mathbf{C}^2_4((\ ), \mathbf{I}^4_1, \mathbf{I}^4_3)$;
let $f_4(i,m,j,k) = (m)_j$; recursive since $f_5 = \mathbf{C}^2_4((\ ), \mathbf{I}^4_1, \mathbf{I}^4_2)$;
let $f_5(i,m,j,k) = \mathrm{IMP}((m)_j, (m)_k)$; recursive since $f_5 = \mathbf{C}^2_4(\mathrm{IMP}, f_4, f_3)$;
let $f_6(i,m,j,k) = (m)_i$; recursive since$f_6 = \mathbf{C}^2_4((\ ), \mathbf{I}^4_1, \mathbf{I}^4_0)$;
let $S_2 = \{\langle i,m,j,k\rangle : (m)_i = \mathrm{IMP}((m)_j), (m)_k)\}$; recursive since
$\chi_{S_2} = \mathbf{C}^2_4(\xi_=, f_6, f_5)$;
let $S_3 = \{\langle i,m,j\rangle :$ there is a $k < i$ such that
$(m)_i = \mathrm{IMP}((m)_j), (m)_k)\}$; recursive by Prop. 7.20;
let $S_4 = \{\langle i,m\rangle :$ there exist $j,k < i$ such that
$(m)_i = \mathrm{IMP}((m)_j), (m)_k)\}$; recursive by Prop. 7.20;
let $R_2 = \{(m,i) :$ there exist $j,k < i$ such that $(m)_i = \mathrm{IMP}((m)_j), (m)_k)\}$;
recursive since $\chi_{R_2} = \mathbf{C}^2_2(\chi_{S_4}, \mathbf{I}^2_1, \mathbf{I}^2_0)$:
(R_2 corresponds to (3))
let $f_7(i,m,j,k) = \mathrm{ALL}(j,(m)_k)$; recursive since $f_7 = \mathbf{C}^3_4(\mathrm{ALL}, \mathbf{I}^4_2, f_3)$;
let $S_5 = \{(i,m,j,k) : (m)_i = \mathrm{ALL}(j,(m)_k)\}$; recursive since
$\chi_{S_5} = \mathbf{C}^2_4(\chi_=, f_6, f_7)$;
let $S_6 = \{(i,m,j) :$ there is a $k < i$ such that $(m)_i = \mathrm{ALL}(j,(m)_k)\}$;
recursive by Prop. 7.20
let $S_7 = \{(m,i,j) :$ there is a $k < i$ such that $(m)_i = \mathrm{ALL}(j,(m)_k)\}$;
recursive since $\chi_{S_7} = \mathbf{C}^3_3(\chi_{S_6}, \mathbf{I}^3_1, \mathbf{I}^3_0, \mathbf{I}^3_2)$;
let $R_3 = \{(m,i) :$ there exist $j < m$ and $k < i$ such that
$(m)_i = \mathrm{ALL}(j,(m)_k)\}$; recursive by Prop. 7.20;
(R_3 corresponds to (4))

Now let $T = R_0 \cup \ldots \cup R_3$; by Proposition 7.16, T is recursive. Hence by Corollary 8.22 the set $U \stackrel{\text{def}}{=} \{(m,n) :$ for all $i < n$ $(m,i) \in T\}$ is recursive. Now FMLACON $= \{m : m > 1$ and $(m, \mathrm{len}(m)) \in U\}$; it is recursive by the following steps:

let $S_8 = \{m : m > 1\}$; recursive by Cor. 7.17
let $S_9 = \{m : (m, \mathrm{len}(m)) \in U\}$; recursive since
$\chi_{S_9} = \mathbf{C}^2_1(\chi_U, \mathbf{I}^1_0, \mathbf{C}^1_1(\mathrm{len}, \mathbf{I}^1_0))$;
FMLACON $= S_8 \cap S_9$; recursive by Prop. 7.16 $\square$

The following lemma is similar to Lemma 8.4.

Lemma 8.10. *For any formula φ there is a formula construction sequence $\Phi = \langle \psi_0, \ldots, \psi_{m-1}\rangle$ with the following properties:*
(i) $\psi_{m-1} = \varphi$.
(ii) Each ψ_i is a subformula of φ.

(iii) m is the less or equal the length of φ.

Proof. We prove this by induction on φ, thus using Proposition 2.5. If φ is an atomic formula we can take $\Phi = \langle\varphi\rangle$, and clearly (i)–(iii) hold. If Φ is a formula construction sequence with the properties (i)–(iii) for φ, then $\Phi^\frown\langle\neg\varphi\rangle$ has properties (i)–(iii) for $\neg\varphi$. If Φ is a formula construction sequence with properties (i)–(iii) for φ and Ψ is a formula construction sequence with properties (i)–(iii) for ψ, then $\Phi^\frown\Psi^\frown\langle\varphi \to \psi\rangle$ is a formuls construction sequence with properties (i)–(iii) for $\varphi \to \psi$. If Φ is a formula construction sequence with properties (i)–(iii) for φ and $i \in \omega$, then $\Phi^\frown\langle\forall v_i\varphi\rangle$ is a formula construction sequence with properties (i)–(iii) for $\forall v_i\varphi$. □

Now let FMLA be the set of all Gödel numbers of formulas.

Lemma 8.11. FMLA *is recursive.*

Proof. See the proof of Lemma 8.5; replace TRMCON by FMLACON and TRM by FMLA. □

Now let NUM be defined by $\mathrm{NUM}(m) = gn(\overline{m})$.

Lemma 8.12. NUM *is recursive.*

Proof. Define $f(m, n) = \mathrm{SU}(n)$ for all $m, n \in \omega$. Then f is recursive, since $f = \mathbf{C}^1_2(\mathrm{SU}, \mathbf{I}^2_1)$. It follows that $\mathbf{Q}_0(2^8, f)$ is recursive, and we claim that $\mathbf{Q}_0(2^8, f) = \mathrm{NUM}$. We prove that $\mathbf{Q}_0(2^8, f)(m) = \mathrm{NUM}(m)$ by induction on m. First, $\mathbf{Q}_0(2^8, f)(0) = 2^8 = gn(\langle\mathbf{0}\rangle) = \mathrm{NUM}(0)$. Now suppose that $\mathbf{Q}_0(2^8, f)(m) = \mathrm{NUM}(m)$. Then

$$\begin{aligned}
\mathbf{Q}_0(2^8, f)(m+1) &= f(m, \mathbf{Q}_0(2^8, f)(m)) \\
&= f(m, \mathrm{NUM}(m)) \\
&= \mathrm{SU}(\mathrm{NUM}(m)) \\
&= \mathrm{NUM}(m+1)
\end{aligned}$$

□

Theorem 8.13. (Theorem C) *G is recursive, hence representable.*

Proof. Recall that $G : \omega \to \omega$ is defined as follows:

$$G(m) = \begin{cases} gn(\mathrm{Subf}^{v_0}_{\overline{m}}\varphi) & \text{if } m = gn(\varphi) \text{ for some formula } \varphi, \\ 0 & \text{otherwise.} \end{cases}$$

By the definition of Subf, this means that if $m = gn(\varphi)$ then $G(m)$ is the Gödel number of the formula

$$\forall v_{gn(\varphi)}[v_{gn(\varphi)} = \overline{m} \to \forall v_0[v_0 = v_{gn(\varphi)} \to \varphi]].$$

Thus the definition of G can be given as follows:

$$G(m) = \begin{cases} \mathrm{ALL}(m, \mathrm{IMP}(\mathrm{EQ}(2^{5(m+1)}, \mathrm{NUM}(m)), \\ \mathrm{ALL}(0, \mathrm{IMP}(\mathrm{EQ}(2^5, 2^{5(m+1)}), m))) & \text{if } m \in \mathrm{FMLA}, \\ 0 & \text{otherwise.} \end{cases}$$

We see that G is recursive in the following steps.

Let $f_0(m) = 5(m+1)$; recursive since $f_0 = \mathbf{C}^2_1(\cdot, \mathbf{k}^1_5, \mathbf{s})$;
Let $f_1(m) = 2^{5(m+1)}$; recursive since $f_1 = \mathbf{C}^2_1(\mathbf{k}^1_2, f_0)$;
Let $f_2(m) = \mathrm{EQ}(2^5, 2^{5(m+1)})$; recursive since $f_2 = \mathbf{C}^2_1(\mathrm{EQ}, \mathbf{k}^1_{2^5}, f_1)$;
Let $f_3(m) = \mathrm{IMP}(\mathrm{EQ}(2^5, 2^{5(m+1)}), m)$; recursive since $f_3 = \mathbf{C}^2_1(\mathrm{IMP}, f_2, \mathbf{I}^1_0)$;
Let $f_4(m) = \mathrm{ALL}(0, \mathrm{IMP}(\mathrm{EQ}(2^5, 2^{5(m+1)}), m))$; recursive since $f_4 = \mathbf{C}^2_1(\mathrm{ALL}, \mathbf{k}^1_0, f_3)$;
Let $f_5(m) = \mathrm{EQ}(2^{5(m+1)}, \mathrm{NUM}(m))$; recursive since $f_5 = \mathbf{C}^2_1(\mathrm{EQ}, f_1, \mathrm{NUM})$;
Let $f_6(m) = \mathrm{IMP}(\mathrm{EQ}(2^{5(m+1)}, \mathrm{NUM}(m)), \mathrm{ALL}(0, \mathrm{IMP}(\mathrm{EQ}(2^5, 2^{5(m+1)}), m)))$;
recursive since $f_6 = \mathbf{C}^2_1(\mathrm{IMP}, f_5, f_4)$;
Let $f_7(m) = \mathrm{ALL}(m, \mathrm{IMP}(\mathrm{EQ}(2^{5(m+1)}, \mathrm{NUM}(m)),$
$\mathrm{ALL}(0, \mathrm{IMP}(\mathrm{EQ}(2^5, 2^{5(m+1)}), m)))$; recursive since $f_7 = \mathbf{C}^2_1(\mathrm{ALL}, \mathbf{I}^1_0, f_6)$.

Now we have

$$G(m) = \begin{cases} f_7(m) & \text{if } m \in \mathrm{FMLA}, \\ \mathbf{k}^1_0(m) & \text{otherwise.} \end{cases}$$

Hence G is recursive, by Proposition 7.23. □

Theorem D involves the notion of proof. Basic to the notion of proof are the logical axioms. Let LOGAX be the set of all Gödel numbers of logical axioms.

Lemma 8.14. LOGAX *is recursive.*

Proof. Axiom (L1a) has the form $\varphi \to (\psi \to \varphi)$.

Let $f_0(m,n,p) = \mathrm{IMP}(n,p)$; recursive since $f_0 = \mathbf{C}^2_3(\mathrm{IMP}, \mathbf{I}^3_1, \mathbf{I}^3_2)$;
Let $f_1(m,n,p) = \mathrm{Imp}(p, \mathrm{IMP}(n,p))$; recursive since $f_1 = \mathbf{C}^2_3(\mathrm{IMP}, \mathbf{I}^3_2, f_0)$;
Let $S_0 = \{(m,n,p) : m = \mathrm{Imp}(p, \mathrm{IMP}(n,p))\}$; recursive since $\chi_{S_0} = \mathbf{C}^2_3(\chi_=, \mathbf{I}^3_0, f_1)$;
Let $S_1 = \{(m,n,p) : p \in \mathrm{FMLA}\}$; recursive since $\chi_{S_1} = \mathbf{C}^1_3(\chi_{\mathrm{FMLA}}, \mathbf{I}^3_2)$;
Let $S_2 = \{(m,n,p) : n \in \mathrm{FMLA}\}$; recursive since $\chi_{S_2} = \mathbf{C}^1_3(\chi_{\mathrm{FMLA}}, \mathbf{I}^3_1)$;
Let $S_3 = S_0 \cap S_1 \cap S_2$; recursive by Proposition 7.16;
Let $S_4 = \{(m,n) : \exists p < m[p, n \in \mathrm{FMLA} \text{ and } m = \mathrm{Imp}(p, \mathrm{IMP}(n,p))]\}$;
recursive by Proposition 7.20
Let $R_0 = \{m : \exists n, p < m[n, p \in \mathrm{FMLA} \text{ and } m = \mathrm{Imp}(p, \mathrm{IMP}(n,p))]\}$;
recursive by Proposition 7.20
(R_0 corresponds to axiom (L1a)).

Let R_1 and R_2 be the sets of all Gödel numbers of formulas in axioms (L1b) and (L1c) respectively. We leave to an exercise the fact that these sets are recursive.

We turn to axioms (L2).They have the form $\forall v_j(\varphi \to \psi) \to (\forall v_j\varphi \to \forall v_j\psi)$.

Let $f_2(m,n,p,j) = \mathrm{ALL}(j,p)$; recursive since $f_2 = \mathbf{C}^2_4(\mathrm{ALL}, \mathbf{I}^4_3, \mathbf{I}^4_2)$;

Let $f_3(m,n,p,j) = \mathrm{ALL}(j,n)$; recursive since $f_2 = \mathbf{C}^2_4(\mathrm{ALL}, \mathbf{I}^4_3, \mathbf{I}^4_1)$;

Let $f_4(m,n,p,j) = \mathrm{IMP}(\mathrm{ALL}(j,n), \mathrm{ALL}(j,p))$; recursive since $f_4 = \mathbf{C}^2_4(\mathrm{IMP}, f_3, f_2)$;

Let $f_5(m,n,p,j) = \mathrm{IMP}(n,p)$; recursive since $f_5 = \mathbf{C}^2_4(\mathrm{IMP}, \mathbf{I}^5_1, \mathbf{I}^5_2)$;

Let $f_6(m,n,p,j) = \mathrm{ALL}(j, \mathrm{IMP}(n,p))$; recursive since $f_6 = \mathbf{C}^2_4(\mathrm{ALL}, \mathbf{I}^4_3, f_5)$;

Let $f_7(m,n,p,j) = \mathrm{IMP}(\mathrm{ALL}(j,\mathrm{IMP}(n,p)), \mathrm{IMP}(\mathrm{ALL}(j,n), \mathrm{ALL}(j,p))$;
recursive since $f_7 = \mathbf{C}^2_4(\mathrm{IMP}, f_6, f_4)$;

Let $S_5 = \{(m,n,p,j) : m = \mathrm{IMP}(\mathrm{ALL}(j,\mathrm{IMP}(n,p)), \mathrm{IMP}(\mathrm{ALL}(j,n), \mathrm{ALL}(j,p))\}$;
recursive since $\chi_{S_5} = \mathbf{C}^2_4(\chi_=, \mathbf{I}^4_0, f_7)$;

Let $S_6 = \{(m,n,p) : \exists j < m[m = \mathrm{IMP}(\mathrm{ALL}(j,\mathrm{IMP}(n,p)), \mathrm{IMP}(\mathrm{ALL}(j,n), \mathrm{ALL}(j,p))]\}$;
recursive by Proposition 7.20;

Let $S_7 = \{(m,n,p) : n \in \mathrm{FMLA}\}$; recursive since $\chi_{S_7} = \mathbf{C}^1_3(\chi_{\mathrm{FMLA}}, \mathbf{I}^3_1$;

Let $S_8 = \{(m,n,p) : p \in \mathrm{FMLA}\}$; recursive since $\chi_{S_8} = \mathbf{C}^1_3(\chi_{\mathrm{FMLA}}, \mathbf{I}^3_2$;

Let $S_9 = S_6 \cap S_1 \cap S_2 \cap S_7 \cap S_8$; recursive by Prop. 7.16;

Let $S_{10} = \{(m,n) : \exists p,j < m[n,p \in \mathrm{FMLA}\mathrm{and}$
$m = \mathrm{IMP}(\mathrm{ALL}(j,\mathrm{IMP}(n,p)), \mathrm{IMP}(\mathrm{ALL}(j,n), \mathrm{ALL}(j,p))]\}$;
recursive by Proposition 7.20;

Let $R_3 = \{m : \exists n,p,j < m[n,p \in \mathrm{FMLA}$ and
$m = \mathrm{IMP}(\mathrm{ALL}(j,\mathrm{IMP}(n,p)), \mathrm{IMP}(\mathrm{ALL}(j,n), \mathrm{ALL}(j,p))]\}$;
recursive by Proposition 7.20;

(R_3 corresponds to axiom (L2)).

Instances of (L3) have the form $\varphi \to \forall v_j\varphi$, where v_j does not occur in φ.

Let $f_8(m,n,j,i) = (n)_i$; recursive since $f_8 = \mathbf{C}^2_4((\ ), \mathbf{I}^4_1, \mathbf{I}^4_3)$;

Let $f_9(m,n,j,i) = 5(j+1)$; recursive since $f_9 = \mathbf{C}^2_4(\cdot, \mathbf{k}^4_5, \mathbf{C}^1_4(\mathrm{SU}, \mathbf{I}^4_2))$;

Let $S_9 = \{(m,n,j,i) : (n)_i \neq 5(j+1)\}$; recursive since $\chi_{S_9} = \mathbf{C}^2_4(\chi_{\neq}, f_8, f_9)$;

Let $S_{10} = \{(m,n,j,s) : \forall i < s[(n)_i \neq 5(j+1)]\}$; recursive by Prop. 7.22;

Let $S_{11} = \{(m,n,j) : \forall i < \mathrm{len}(n)[(n)_i \neq 5(j+1)]\}$; recursive since
$\chi_{S_{11}} = \mathbf{C}^4_3(\chi_{S_{10}}, \mathbf{I}^3_0, \mathbf{I}^3_1, \mathbf{I}^3_2, \mathbf{C}^1_3(\mathrm{len}, \mathbf{I}^3_1))$;

Let $f_{10}(m,n,j) = \mathrm{ALL}(j,n)$; recursive since $f_{10} = \mathbf{C}^2_3(\mathrm{ALL}, \mathbf{I}^3_2, \mathbf{I}^3_1)$;

Let $f_{11}(m,n,j) = \mathrm{IMP}(n, \mathrm{ALL}(j,n))$; recursive since $f_{11} = \mathbf{C}^2_3(\mathrm{IMP}, \mathbf{I}^3_1, f_{10})$;

Let $S_{12} = \{(m,n,j) : m = \mathrm{IMP}(n, \mathrm{ALL}(j,n))\}$; recursive since $\chi_{S_{12}} = \mathbf{C}^2_3(\chi_=, \mathbf{I}^3_0, f_{11})$;

Let $S_{13} = \{(m,n,j) : n \in \mathrm{FMLA}\}$; recursive since $\chi_{S_{13}} = \mathbf{C}^1_3(\chi_{\mathrm{FMLA}}, \mathbf{I}^3_1$;

Let $S_{14} = S_{11} \cap S_{12} \cap S_{13}$; recursive by Prop. 7.16;

Let $S_{15} = \{(m,n) : \exists j < m[n \in \mathrm{FMLA}$ and $\forall i < \mathrm{len}(n)[(n)_i \neq 5(j+1)]$

and $m = \mathrm{IMP}(n, \mathrm{ALL}(j, n))]\}$; recursive by Prop. 7.20

Let $R_4 = \{m : \exists n, j < m[n \in \mathrm{FMLA}$ and $\forall i < \mathrm{len}(n)[(n)_i \neq 5(j+1)]$

and $m = \mathrm{IMP}(n, \mathrm{ALL}(j, n))]\}$; recursive by Prop. 7.20

(R_4 corresponds to axiom (L3)).

Axioms in (L4) have the form $\exists v_i(v_i = \sigma)$ with v_i not in the term σ; using the definition of $\exists$, this is $\neg\forall v_i \neg(v_i = \sigma)$.

Let $f_{12}(m, n, j) = 5(j+1)$; recursive since $f_{12} = \mathbf{C}^2_3(\cdot, \mathbf{k}^3_5, \mathbf{C}^1_3(\mathrm{SU}, \mathbf{I}^3_2))$;

Let $f_{13}(m, n, j) = 2^{5(j+1)}$; recursive since $f_{13} = \mathbf{C}^2_3(\exp, \mathbf{k}^3_2, f_{12})$;

Let $f_{14}(m, n, j) = \mathrm{EQ}(2^{5(j+1)}, n)$; recursive since $f_{12} = \mathbf{C}^2_3(\mathrm{EQ}, f_{13}, \mathbf{I}^3_1)$;

Let $f_{15}(m, n, j) = \mathrm{NEG}(\mathrm{EQ}(2^{5(j+1)}, n))$; recursive since $f_{15} = \mathbf{C}^1_3(\mathrm{NEG}, f_{14})$;

Let $f_{16}(m, n, j) = \mathrm{ALL}(j, \mathrm{NEG}(\mathrm{EQ}(2^{5(j+1)}, n)))$; recursive since

$f_{16} = \mathbf{C}^2_3(\mathrm{ALL}, \mathbf{I}^3_2, f_{15})$;

Let $f_{17}(m, n, j) = \mathrm{NEG}(\mathrm{ALL}(j, \mathrm{NEG}(\mathrm{EQ}(2^{5(j+1)}, n))))$;

recursive since $f_{17} = \mathbf{C}^1_3(\mathrm{NEG}, f_{16})$;

Let $S_{16} = \{(m, n, j) : m = \mathrm{NEG}(\mathrm{ALL}(j, \mathrm{NEG}(\mathrm{EQ}(2^{5(j+1)}, n))))\}$;

recursive since $\chi_{S_{16}} = \mathbf{C}^2_3(\chi_=, \mathbf{I}^3_0, f_{17})$;

Let $S_{17} = \{(m, n, j) : n \in \mathrm{TRM}\}$; recursive since $\chi_{S_{17}} = \mathbf{C}^1_3(\chi_{\mathrm{TRM}}, \mathbf{I}^3_1)$;

Let $S_{18} = S_{11} \cap S_{16} \cap S_{17}$; recursive by Prop. 7.16;

Let $S_{19} = \{(m, n) : \exists j[\forall i < \mathrm{len}(n)[(n)_i \neq 5(j+1)]$ and

$n \in \mathrm{TRM}$ and $m = \mathrm{NEG}(\mathrm{ALL}(j, \mathrm{NEG}(\mathrm{EQ}(2^{5(j+1)}, n))))]\}$;

recursive by Prop. 7.20;

Let $R_5 = \{m : \exists n, j < m[\forall i < \mathrm{len}(n)[(n)_i \neq 5(j+1)]$ and

$n \in \mathrm{TRM}$ and $m = \mathrm{NEG}(\mathrm{ALL}(j, \mathrm{NEG}(\mathrm{EQ}(2^{5(j+1)}, n))))]\}$;

recursive by Prop. 7.20;

R_5 corresponds to axiom (L4).

An instance of (L5) has the form $\sigma = \tau \to (\sigma = \rho \to \tau = \rho)$ for some terms σ, τ, ρ.

Let $f_{18}(m, n, p, q) = \mathrm{EQ}(p, q)$; recursive since $f_{18} = \mathbf{C}^2_4(\mathrm{EQ}, \mathbf{I}^4_2, \mathbf{I}^4_3)$;

Let $f_{19}(m, n, p, q) = \mathrm{EQ}(n, q)$; recursive since $f_{19} = \mathbf{C}^2_4(\mathrm{EQ}, \mathbf{I}^4_1, \mathbf{I}^4_3)$;

Let $f_{20}(m, n, p, q) = \mathrm{EQ}(n, p)$; recursive since $f_{18} = \mathbf{C}^2_4(\mathrm{EQ}, \mathbf{I}^4_1, \mathbf{I}^4_2)$;

Let $f_{21}(m, n, p, q) = \mathrm{IMP}(\mathrm{EQ}(n, q), \mathrm{EQ}(p, q))$; recursive since

$f_{21} = \mathbf{C}^2_4(\mathrm{IMP}, f_{19}, f_{18})$;

Let $f_{22}(m, n, p, q) = \mathrm{IMP}(\mathrm{EQ}(n, p), \mathrm{IMP}(\mathrm{EQ}(n, q), \mathrm{EQ}(p, q))$;

recursive since $f_{22} = \mathbf{C}^2_4(\mathrm{IMP}, f_{20}, f_{21})$;

Let $S_{20} = \{(m, n, p, q) : m = \mathrm{IMP}(\mathrm{EQ}(n, p), \mathrm{IMP}(\mathrm{EQ}(n, q), \mathrm{EQ}(p, q))\}$;

recursive since $\chi_{S_{20}} = \mathbf{C}_4^2(\chi_=, \mathbf{I}_0^4, f_{22})$;

Let $S_{21} = \{(m, n, p, q) : n \in \mathrm{TRM}\}$; recursive since $\chi_{S_{21}} = \mathbf{C}_4^1(\chi_{\mathrm{TRM}}, \mathbf{I}_1^4)$;

Let $S_{22} = \{(m, n, p, q) : p \in \mathrm{TRM}\}$; recursive since $\chi_{S_{22}} = \mathbf{C}_4^1(\chi_{\mathrm{TRM}}, \mathbf{I}_2^4)$;

Let $S_{23} = \{(m, n, p, q) : q \in \mathrm{TRM}\}$; recursive since $\chi_{S_{23}} = \mathbf{C}_4^1(\chi_{\mathrm{TRM}}, \mathbf{I}_3^4)$;

Let $S_{24} = S_{20} \cap S_{21} \cap S_{22} \cap S_{23}$; recursive by Prop. 7.16;

Let $S_{25} = \{(m, n, p) : \exists q < m[n, p, q \in \mathrm{TRM}$ and
$m = \mathrm{IMP}(\mathrm{EQ}(n, p), \mathrm{IMP}(\mathrm{EQ}(n, q), \mathrm{EQ}(p, q))]\}$;
recursive by Prop. 7.20

Let $S_{26} = \{(m, n) : \exists p, q < m[n, p, q \in \mathrm{TRM}$ and
$m = \mathrm{IMP}(\mathrm{EQ}(n, p), \mathrm{IMP}(\mathrm{EQ}(n, q), \mathrm{EQ}(p, q))]\}$;
recursive by Prop. 7.20

Let $R_6 = \{m : \exists n, p, q < m[n, p, q \in \mathrm{TRM}$ and
$m = \mathrm{IMP}(\mathrm{EQ}(n, p), \mathrm{IMP}(\mathrm{EQ}(n, q), \mathrm{EQ}(p, q))]\}$;

(R_6 corresponds to (L5)).

We leave (L6) to an exercise; R_7 is the set of all Gödel numbers of instances of (L6).

(L7) consists of formulas of the following forms:

$$
\begin{array}{ll}
(L7a) & \sigma = \tau \rightarrow \mathbf{S}\sigma = \mathbf{S}\tau; \\
(L7b) & \sigma = \tau \rightarrow (\sigma + \xi = \tau + \xi); \\
(L7c) & \sigma = \tau \rightarrow (\xi + \sigma = \xi + \tau); \\
(L7d) & \sigma = \tau \rightarrow (\sigma \bullet \xi = \tau \bullet \xi); \\
(L7e) & \sigma = \tau \rightarrow (\xi \bullet \sigma = \xi \bullet \tau).
\end{array}
$$

Here σ, τ, ξ are terms. We treat only (L7a) and (L7b), and leave the others for an exercise.

Let $f_{23}(m, n, p) = \mathrm{SU}(n)$; recursive since $f_{23} = \mathbf{C}_3^1(\mathrm{SU}, \mathbf{I}_1^3)$;

Let $f_{24}(m, n, p) = \mathrm{SU}(p)$; recursive since $f_{24} = \mathbf{C}_3^1(\mathrm{SU}, \mathbf{I}_2^3)$;

Let $f_{25}(m, n, p) = \mathrm{EQ}(\mathrm{SU}(n), \mathrm{SU}(p))$; recursive since $f_{25} = \mathbf{C}_3^2(\mathrm{EQ}, f_{24}, f_{23})$;

Let $f_{26}(m, n, p) = \mathrm{EQ}(n, p)$; recursive since $f_{23} = \mathbf{C}_3^2(\mathrm{EQ}, \mathbf{I}_1^3, \mathbf{I}_2^3)$;

Let $f_{27}(m, n, p) = \mathrm{IMP}(\mathrm{EQ}(n, p), \mathrm{EQ}(\mathrm{SU}(n), \mathrm{SU}(p)))$;
recursive since $f_{27} = \mathbf{C}_3^2(\mathrm{IMP}, f_{26}, f_{25})$;

Let $S_{27} = \{(m, n, p) : m = \mathrm{IMP}(\mathrm{EQ}(n, p), \mathrm{EQ}(\mathrm{SU}(n), \mathrm{SU}(p)))\}$;
recursive since $\chi_{S_{27}} = \mathbf{C}_3^2(\chi_=, \mathbf{I}_0^3, f_{27})$;

Let $S_{28} = \{(m, n, p) : n \in \mathrm{TRM}\}$; recursive since $\chi_{S_{28}} = \mathbf{C}_3^1(\chi_{\mathrm{TRM}}, \mathbf{I}_1^3)$;

Let $S_{29} = \{(m, n, p) : p \in \mathrm{TRM}\}$; recursive since $\chi_{S_{29}} = \mathbf{C}_3^1(\chi_{\mathrm{TRM}}, \mathbf{I}_2^3)$;

Let $S_{30} = S_{27} \cap S_{28} \cap S_{29}$; recursive by Prop. 7.16;

Let $S_{31} = \{(m, n) : \exists p < m[n, p \in \mathrm{TRM}$ and

$m = \mathrm{IMP}(\mathrm{EQ}(n,p), \mathrm{EQ}(\mathrm{SU}(n), \mathrm{SU}(p)))]\}$; recursive by Prop. 7.20;

Let $R_8 = \{m : \exists n, p < m[n, p \in \mathrm{TRM}$ and

$m = \mathrm{IMP}(\mathrm{EQ}(n,p), \mathrm{EQ}(\mathrm{SU}(n), \mathrm{SU}(p)))]\}$;

recursive by Prop. 7.20;

(R_8 corresponds to axiom (L7a));

Let $f_{28}(m,n,p,q) = \mathrm{PLUS}(p,q)$; recursive since $f_{28} = \mathbf{C}^2_4(\mathrm{PLUS}, \mathbf{I}^4_2, \mathbf{I}^4_3)$;

Let $f_{29}(m,n,p,q) = \mathrm{PLUS}(n,q)$; recursive since

$f_{29} = \mathbf{C}^2_4(\mathrm{PLUS}, \mathbf{I}^4_1, \mathbf{I}^4_3)$;

Let $f_{30}(m,n,p,q) = \mathrm{EQ}(\mathrm{PLUS}(n,q), \mathrm{PLUS}(p,q))$;

recursive since $f_{30} = \mathbf{C}^2_4(\mathrm{EQ}, f_{29}, v_{28})$;

Let $f_{31}(m,n,p,q) = \mathrm{EQ}(n,p)$; recursive since $f_{31} = \mathbf{C}^2_4(\mathrm{EQ}, \mathbf{I}^4_1, \mathbf{I}^4_2)$;

Let $f_{32}(m,n,p,q) = \mathrm{IMP}(\mathrm{EQ}(n,p), \mathrm{EQ}(\mathrm{PLUS}(n,q), \mathrm{PLUS}(p,q)))$;

recursive since $f_{32} = \mathbf{C}^2_4(\mathrm{IMP}, f_{31}, f_{30})$;

Let $S_{32} = \{(m,n,p,q) : m = \mathrm{IMP}(\mathrm{EQ}(n,p), \mathrm{EQ}(\mathrm{PLUS}(n,q),$

$\mathrm{PLUS}(p,q)))\}$; recursive since $\chi_{S_{32}} = \mathbf{C}^2_4(\chi_=, \mathbf{I}^4_0, f_{32})$;

Let $S_{33} = \{(m,n,p,q) : n \in \mathrm{TRM}\}$; recursive since $\chi_{S_{33}} = \mathbf{C}^1_4(\chi_{\mathrm{TRM}}, \mathbf{I}^4_1)$;

Let $S_{34} = \{(m,n,p,q) : p \in \mathrm{TRM}\}$; recursive since $\chi_{S_{34}} = \mathbf{C}^1_4(\chi_{\mathrm{TRM}}, \mathbf{I}^4_2)$;

Let $S_{35} = \{(m,n,p,q) : q \in \mathrm{TRM}\}$; recursive since $\chi_{S_{35}} = \mathbf{C}^1_4(\chi_{\mathrm{TRM}}, \mathbf{I}^4_3)$;

Let $S_{36} = S_{32} \cap S_{33} \cap S_{34} \cap S_{35}$; recursive by Prop. 7.16;

Let $S_{37} = \{(m,n,p) : \exists q < m[n,p,q \in \mathrm{TRM}$ and

$m = \mathrm{IMP}(\mathrm{EQ}(n,p), \mathrm{EQ}(\mathrm{PLUS}(n,q)))]\}$; recursive by Prop. 7.20

Let $S_{38} = \{(m,n) : \exists p, q < m[n,p,q \in \mathrm{TRM}$ and

$m = \mathrm{IMP}(\mathrm{EQ}(n,p), \mathrm{EQ}(\mathrm{PLUS}(n,q)))]\}$; recursive by Prop. 7.20

Let $R_9 = \{m : \exists n, p, q < m[n,p,q \in \mathrm{TRM}$ and

$m = \mathrm{IMP}(\mathrm{EQ}(n,p), \mathrm{EQ}(\mathrm{PLUS}(n,q)))]\}$; recursive by Prop. 7.20

(R_9 corresponds to axiom (L7b)).

Let R_{10}, R_{11}, R_{12} be the sets of Gödel numbers of formulas in axioms (L7c), (L7d), (L7e) respectively. Then $\mathrm{LOGAX} = R_0 \cup \ldots \cup R_{12}$, and hence LOGAX is recursive. $\square$

Theorem 8.15. (Theorem D) *If Γ is a set of formulas and $gn[\Gamma]$ is recursive, then the following binary relation is also recursive:*

$$\mathrm{Prf}_\Gamma \stackrel{\text{def}}{=} \{(n,m) : \textit{there is a } \Gamma\textit{-proof } \Phi \textit{ with last entry } \varphi \\ \textit{such that } m = gn_1(\Phi) \textit{ and } n = gn(\varphi)\}$$

Proof.

Let $R_0 = \{(m,i) : (m)_i \in gn[\Gamma]\};$ recursive since $\chi_{R_0} = \mathbf{C}^1_2(\chi_{gn[\Gamma]}, (\ ));$

Let $R_1 = \{(m,i) : (m)_i \in \text{LOGAX}\};$ recursive since $\chi_{R_0} = \mathbf{C}^1_2(\chi_{\text{LOGAX}}, (\ ));$

Let $f_0(i,m,j,k) = (m)_j;$ recursive since $f_0 = \mathbf{C}^2_4((\ ), \mathbf{I}^4_0, \mathbf{I}^4_2);$

Let $f_1(i,m,j,k) = (m)_k;$ recursive since $f_1 = \mathbf{C}^2_4((\ ), \mathbf{I}^4_0, \mathbf{I}^4_3);$

Let $f_2(i,m,j,k) = (m)_i;$ recursive since $f_2 = \mathbf{C}^2_4((\ ), \mathbf{I}^4_0, \mathbf{I}^4_0;$

Let $f_3(i,m,j,k) = \text{IMP}((m)_j, (m)_i);$ recursive since $f_3 = \mathbf{C}^2_4(\text{IMP}, f_0, f_2);$

Let $S_0 = \{(i,m,j,k) : (m)_k = \text{IMP}((m)_j, (m)_i)\};$
recursive since $\chi_{S_0} = \mathbf{C}^2_4(\chi_=, f_1, f_3);$

Let $S_1 = \{(i,m,j,k) : (m)_i \in \text{FMLA}\};$
recursive since $\chi_{S_1} = \mathbf{C}^1_4(\chi_{\text{FMLA}}, f_2);$

Let $S_2 = \{(i,m,j,k) : (m)_j \in \text{FMLA}\};$
recursive since $\chi_{S_2} = \mathbf{C}^1_4(\chi_{\text{FMLA}}, f_0);$

Let $S_3 = \{(i,m,j,k) : (m)_k \in \text{FMLA}\};$
recursive since $\chi_{S_3} = \mathbf{C}^1_4(\chi_{\text{FMLA}}, f_1);$

Let $S_5 = S_0 \cap S_1 \cap S_2 \cap S_3;$ recursive by Prop. 7.16;

Let $S_6 = \{(i,m,j) : \exists k < i[(m)_i, (m)_j, (m)_k \in \text{FMLA}$ and
$(m)_k = \text{IMP}((m)_j, (m)_i)]\};$ recursive by Prop. 7.20;

Let $S_7 = \{(i,m) : \exists j,k < i[(m)_i, (m)_j, (m)_k \in \text{FMLA}$ and
$(m)_k = \text{IMP}((m)_j, (m)_i)]\};$ recursive by Prop. 7.20;

Let $R_2 = \{(m,i) : \exists j,k < i[(m)_i, (m)_j, (m)_k \in \text{FMLA}$ and
$(m)_k = \text{IMP}((m)_j, (m)_i)]\};$ recursive since $\chi_{R_2} = \mathbf{C}^2_2(\chi_{S_7}, \mathbf{I}^2_1, \mathbf{I}^2_0);$

(R_2 corresponds to modus ponens);

Let $f_4(i,m,k,j) = (m)_j;$ recursive since $f_4 = \mathbf{C}^2_4((\ ), \mathbf{I}^4_1, \mathbf{I}^4_3);$

Let $f_5(i,m,k,j) = (m)_i;$ recursive since $f_4 = \mathbf{C}^2_4((\ ), \mathbf{I}^4_1, \mathbf{I}^4_0);$

Let $f_6(i,m,k,j) = \text{ALL}(k, (m)_j);$ recursive since $f_6 = \mathbf{C}^2_4(\mathbf{ALL}, \mathbf{I}^4_2, f_4);$

Let $S_8 = \{(i,m,k,j) : (m)_i = \text{ALL}(k, (m)_j)\};$ recursive since
$\chi_{S_8} = \mathbf{C}^2_4(\chi_=, f_5, f_6);$

Let $S_9 = \{(i,m,k,j) : (m)_j \in \text{FMLA}\};$ recursive since
$\chi_{S_9} = \mathbf{C}^1_4(\chi_{\text{FMLA}}, f_4);$

Let $S_{10} = S_8 \cap S_9;$ recursive by Prop. 7.16;

Let $S_{11} = \{(i,m,k) : \exists j < i[(m)_j \in \text{FMLA}$ and $(m)_i = \text{ALL}(k, (m)_j)]\};$
recursive by Prop. 7.20;

Let $S_{12} = \{(m,i,k) : \exists j < i[(m)_j \in \text{FMLA}$ and $(m)_i = \text{ALL}(k, (m)_j)]\};$
recursive since $\chi_{S_{12}} = \mathbf{C}^3_3(\chi_{S_{11}}, \mathbf{I}^3_1, \mathbf{I}^3_0, \mathbf{I}^3_2);$

Let $R_3 = \{(m,i)\exists j,k < i[(m)_j \in \text{FMLA and } (m)_i = \text{ALL}(k,(m)_j)]\}$;
recursive by Prop. 7.20;
Let $S_{13} = R_0 \cup \ldots \cup R_3$; recursive by Prop. 7.16;
Let $S_{14} = \{(m,s) : \forall i < s[(m,i) \in S_{13}]\}$; recursive by Prop. 7.22;
Let $S_{15} = \{m : (m, \text{len}(m)) \in S_{14}\}$; recursive since
$\chi_{S_{15}} = \mathbf{C}_1^2(\chi_{S_{14}}, \mathbf{I}_0^1, \mathbf{C}_1^1(\text{len}, \mathbf{I}_0^1))$;
Let $S_{16} = \{m : m > 1\}$; recursive by Cor. 7.17;
Let $S_{17} = S_{15} \cap S_{16}$; recursive by Prop. 7.16;
Let $S_{18} = \{(n,m) : m \in S_{17}\}$; recursive since $\chi_{S_{18}} = \mathbf{C}_2^1(\chi_{S_{17}}, \mathbf{I}_1^2)$;
Let $S_{19} = \{(n,m) : n = \text{len}(m)\}$; recursive since $\chi_{S_{19}} = \mathbf{C}_2^2(\chi_=, \mathbf{I}_0^2, \mathbf{C}_2^1(\text{len}, \mathbf{I}_1^2))$.

Clearly $S_{19} = \text{Prf}_\Gamma$. □

To complete the proofs connected with Gödel's incompleteness theorem, it remains only to treat Theorem B:

Theorem 8.16. (Theorem B) *The set $gn[\mathbf{P}']$ is recursive. Moreover, if Δ is a finite set of formulas, then $gn[\mathbf{P}' \cup \Delta]$ is recursive.*

Proof. The assertion concerning Δ follows from the first statement using Proposition 7.16 and Corollary 7.17. Moreover, for $\mathbf{P}'$ itself, each of axioms (P1)–(P6) are single formulas, so by the same reasons, we only need to treat (P7′). We recall the form of (P7′):

$$\text{Subff}^{v_0}_{\mathbf{0}}\varphi \wedge \forall v_0(\varphi \to \text{Subff}^{v_0}_{\mathbf{S}v_0}\varphi) \to \forall v_0\varphi$$

for any formula φ. By the definition of Subff, this is

$$\begin{aligned}
&\forall v_{gn(\varphi)}[v_{gn(\varphi)} = \mathbf{0} \to \forall v_0(v_0 = v_{gn(\varphi)} \to \varphi)] \wedge \\
&\forall v_0[\varphi \to \forall v_{gn(\varphi)}[v_{gn(\varphi)} = \mathbf{S}v_0 \to \forall v_0(v_0 = v_{gn(\varphi)} \to \varphi)]] \\
&\ \to \forall v_0\varphi.
\end{aligned}$$

Using the definition of $\wedge$, this is

$$\begin{aligned}
&\neg[\forall v_{gn(\varphi)}[v_{gn(\varphi)} = \mathbf{0} \to \forall v_0(v_0 = v_{gn(\varphi)} \to \varphi)] \to \\
&\neg\forall v_0[\varphi \to \forall v_{gn(\varphi)}[v_{gn(\varphi)} = \mathbf{S}v_0 \to \forall v_0(v_0 = v_{gn(\varphi)} \to \varphi)]]] \\
&\ \to \forall v_0\varphi.
\end{aligned}$$

Thus we want to show that the set T of all Gödel numbers of formulas of this form is recursive.

Let $f_0(m,n) = \text{ALL}(0,n)$; recursive since $f_0 = \mathbf{C}_2^2(\text{ALL}, \mathbf{k}_0^2, \mathbf{I}_1^2)$;
Let $f_1(m,n) = 5(n+1)$; recursive since $f_1 = \mathbf{C}_2^2(\cdot, \mathbf{k}_5^2, \mathbf{C}_2^1(\mathbf{s}, \mathbf{I}_1^2))$;
Let $f_2(m,n) = 2^{5(n+1)}$; recursive since $f_2 = \mathbf{C}_2^2(\exp, \mathbf{k}_2^2, f_1)$;

Let $f_3(m,n) = \mathrm{EQ}(2^5, 2^{5(n+1)})$; recursive since $f_3 = \mathbf{C}^2_2(\mathrm{EQ}, \mathbf{k}^2_{2^5}, f_2)$;

Let $f_4(m,n) = \mathrm{IMP}(\mathrm{EQ}(2^5, 2^{5(n+1)}), n)$;

recursive since $f_4 = \mathbf{C}^2_2(\mathrm{IMP}, f_3, \mathbf{I}^2_1)$;

Let $f_5(m,n) = \mathrm{ALL}(\mathrm{IMP}(0, \mathrm{EQ}(2^5, 2^{5(n+1)}), n))$;

recursive since $f_5 = \mathbf{C}^2_2(\mathrm{ALL}, f_4, \mathbf{I}^2_1)$;

Let $f_6(m,n) = \mathbf{k}^2_{2^6 3^5}$; clearly recursive;

Let $f_7(m,n) = \mathrm{EQ}(2^{5(n+1)}, 2^6 3^5)$; recursive since $f_7 = \mathbf{C}^2 2_2(\mathrm{EQ}, f_1, f_6)$;

Let $f_8(m,n) = \mathrm{IMP}(\mathrm{EQ}(2^{5(n+1)}, 2^6 3^5), \mathrm{ALL}(\mathrm{IMP}(0, \mathrm{EQ}(2^5, 2^{5(n+1)}), n)))$;

recursive since $f_8 = \mathbf{C}^2_2(\mathrm{IMP}, f_7, f_5)$;

Let $f_9(m,n) = \mathrm{ALL}(n, \mathrm{IMP}(\mathrm{EQ}(2^{5(n+1)}, 2^6 3^5), \mathrm{ALL}(\mathrm{IMP}(0, \mathrm{EQ}(2^5, 2^{5(n+1)}), n)))$;

recursive since $f_9 = \mathbf{C}^2_2(\mathrm{ALL}, \mathbf{I}^2_1, f_8)$;

Let $f_{10}(m,n) = \mathrm{IMP}(n, f_9(m,n))$; recursive since $f_{10} = \mathbf{C}^2_2(\mathrm{IMP}, \mathbf{I}^2_1, f_9)$;

Let $f_{11}(m,n) = \mathrm{ALL}(0, f_{10}(m,n))$; recursive since $f_{11} = \mathbf{C}^2_2(\mathrm{ALL}, \mathbf{k}^2_0, f_{10})$;

Let $f_{12}(m,n) = \mathrm{NEG}(f_{11}(m,n))$; recursive since $f_{12} = \mathbf{C}^1_2(\mathrm{NEG}, f_{11})$;

Let $f_{13}(m,n) = \mathrm{EQ}(2^{5(n+1)}, \mathrm{ZERO})$; recursive since $f_{13} = \mathbf{C}^2_2(\mathrm{EQ}, f_2, \mathbf{k}^2_{2^8})$;

Let $f_{14}(m,n) = \mathrm{IMP}(f_{13}(m,n), f_5(m,n))$; recursive since $f_{14} = \mathbf{C}^2_2(\mathbf{IMP}, f_{13}, f_5)$;

Let $f_{15}(m,n) = \mathrm{ALL}(n, f_{14}(m,n))$; recursive since $f_{15} = \mathbf{C}^2_2(\mathrm{ALL}, \mathbf{I}^2_1, f_{14})$;

Let $f_{16}(m,n) = \mathrm{IMP}(f_{15}(m,n), f_{11}(m,n))$; recursive since $f_{16} = \mathbf{C}^2_2(\mathrm{IMP}, f_{15}, f_{11})$;

Let $f_{17}(m,n) = \mathrm{NEG}(f_{16}(m,n))$; recursive since $f_{17} = \mathbf{C}^1_2(\mathrm{NEG}, f_{16})$;

Let $f_{18}(m,n) = \mathrm{IMP}(f_{17}(m,n), f_0(m,n))$; recursive since $f_{18} = \mathbf{C}^2_2(\mathrm{IMP}, f_{17}, f_0)$;

Let $S = \{(m,n) : m = f_{18}(m,n)\}$; recursive since $\chi_S = \mathbf{C}^2_2(\chi_=, \mathbf{I}^2_0, f_{18})$;

$T = \{m : \exists n < m[(m,n) \in S]\}$; recursive by Prop. 7.20. $\square$

EXERCISES

E8.1. Show that the set of all Gödel numbers of formulas in logical axiom (L1b) is recursive.

E8.2. Show that the set of all Gödel numbers of formulas in logical axiom (L1c) is recursive.

E8.3. Show that the set of all Gödel numbers of formulas in logical axiom (L7c) is recursive.

E8.4. Show that the set of all Gödel numbers of formulas in logical axiom (L7d) is recursive.

E8.5. Show that the set of all Gödel numbers of formulas in logical axiom (L7e) is recursive.