7. Recursive functions

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In this chapter we show that many functions are recursive. This chapter does not require any knowledge of the formal languages and logic developed in earlier in the notes; it depends only on the definition of recursive functions given in chapter 5.

Proposition 7.1. $+$ *is recursive.*

Proof. Let $g(a_0, a_1, a_2) = a_2 + 1$ for all $a_0, a_1, a_2 \in \omega$. Then g is recursive, since $g = \mathbf{C}^1_3(\mathbf{s}, \mathbf{I}^3_2)$. We claim that $+$ is $\mathbf{Q}_1(\mathbf{I}^1_0, g)$, and this will establish the proposition We show that $a + b = \mathbf{Q}_1(\mathbf{I}^1_0, g)(a, b)$ for all $a, b \in \omega$ by induction on b, for fixed a. We have

$$\begin{aligned}
\mathbf{Q}_1(\mathbf{I}^1_0, g)(a, 0) &= \mathbf{I}^1_0(a) = a = a + 0;\\
\mathbf{Q}_1(\mathbf{I}^1_0, g)(a, b + 1) &= g(a, b, \mathbf{Q}_1(\mathbf{I}^1_0, g)(a, b))\\
&= g(a, b, a + b) = \mathbf{s}(a + b) = a + b + 1,
\end{aligned}$$

as desired. □

For each $a \in \omega$, let $\mathbf{k}_a : \omega \to \omega$ be the constant function with value a, i.e., $\mathbf{k}_a(b) = a$ for all $b \in \omega$.

Proposition 7.2. $\mathbf{k}_a$ *is recursive for every* $a \in \omega$.

Proof. We claim that $\mathbf{k}_a = \mathbf{Q}_0(a, \mathbf{I}^2_1)$, and we prove $\mathbf{Q}_0(a, \mathbf{I}^2_1)(m) = a$ for all m by induction on m. $\mathbf{Q}_0(a, \mathbf{I}^2_1)(0) = a$ by definition. Assume that $\mathbf{Q}_0(a, \mathbf{I}^2_1)(m) = a$. Then $\mathbf{Q}_0(a, \mathbf{I}^2_1)(m + 1) = \mathbf{I}^2_1(m, \mathbf{Q}_0(a, \mathbf{I}^2_1)(m)) = \mathbf{I}^2_1(m, a) = a$. □

For each positive integer n and each $a \in \omega$ let $\mathbf{k}^n_a$ be defined by $\mathbf{k}^n_a(x_0, \ldots, x_{n-1}) = a$. Thus $\mathbf{k}_a = \mathbf{k}^1_a$.

Proposition 7.3. *For each positive integer* n *and each* $a \in \omega$, *the function* $\mathbf{k}^n_a$ *is recursive.*

Proof. $\mathbf{k}^n_a = \mathbf{C}^1_n(\mathbf{k}_a, \mathbf{I}^n_0)$. □

Proposition 7.4. $\cdot$ *is recursive.*

Proof. Let $g(a, b, c) = a + c$ for any $a, b, c \in \omega$. Then g is recursive, since $g = \mathbf{C}^2_3(+, \mathbf{I}^3_0, \mathbf{I}^3_2)$. We claim that $\cdot$ is $\mathbf{Q}_1(\mathbf{k}_0, g)$, and we prove that $\mathbf{Q}_1(\mathbf{k}_0, g)(m, n) = m \cdot n$ by induction on n, with m fixed. $\mathbf{Q}_1(\mathbf{k}_0, g)(m, 0) = \mathbf{k}_0(m) = 0 = m \cdot 0$. Assume that $\mathbf{Q}_1(\mathbf{k}_0, g)(m, n) = m \cdot n$. Then $\mathbf{Q}_1(\mathbf{k}_0, g)(m, n+1) = g(m, n, \mathbf{Q}_1(\mathbf{k}_0, g)(m, n)) = g(m, n, m \cdot n) = m + m \cdot n = m \cdot (n + 1)$. □

The exponent is defined like this:

$$\begin{aligned}
a^0 &= 1\\
a^{b+1} &= a^b \cdot a.
\end{aligned}$$

Note that then $0^0 = 1$. This differs from elementary calculus, where 0^0 is undefined. We also write exp for this two-place exponential function.

Proposition 7.5. exp *is recursive.*

Proof. Let $f(x, y, z) = x \cdot z$. Then f is recursive, since $f = \mathbf{C}^2_3(\cdot, \mathbf{I}^3_0, \mathbf{I}^3_2)$. Now the exponent is $\mathbf{C}_1(\mathbf{k}_1.f)$; we prove that $\mathbf{C}_1(\mathbf{k}_1.f)(a, b) = a^b$ by induction on b, with a fixed. $\mathbf{C}_1(\mathbf{k}_1.f)(a, 0) = \mathbf{k}_1(a) = 1$. Assume that $\mathbf{C}_1(\mathbf{k}_1.f)(a, b) = a^b$. Then $\mathbf{C}_1(\mathbf{k}_1.f)(a, b+1) = f(a, b, \mathbf{C}_1(\mathbf{k}_1.f)(a, b)) = f(a, b, a^b) = a \cdot a^b = a^{b+1}$. □

Let $\mathbb{P}$ be the *predecessor function*:

$$\mathbb{P}(m) = \begin{cases} m-1 & \text{if } m > 0, \\ 0 & \text{if } m = 0, \end{cases}$$

for any $m \in \omega$.

Proposition 7.6. $\mathbb{P}$ *is recursive.*

Proof. $\mathbb{P} = \mathbf{Q}_0(0, \mathbf{I}^2_0)$. We prove by induction on a that $\mathbb{P}(a) = \mathbf{Q}_0(0, \mathbf{I}^2_0)(a)$ for all $a \in \omega$. We have $\mathbf{Q}_0(0, \mathbf{I}^2_0)(0) = 0 = \mathbb{P}(0)$. Assume that $\mathbf{Q}_0(0, \mathbf{I}^2_0)(a) = \mathbb{P}(a)$. Then $\mathbf{Q}_0(0, \mathbf{I}^2_0)(a+1) = \mathbf{I}^2_0(a, \mathbf{Q}_0(0, \mathbf{I}^2_0)(a)) = a = \mathbb{P}(a+1)$. □

Next we define

$$m \ominus n = \begin{cases} m-n & \text{if } m \geq n, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 7.7. $\ominus$ *is recursive.*

Proof. First let $f(x, y, z) = \mathbb{P}(z)$ for all $x, y, z \in \omega$. Then f is recursive, since $f = \mathbf{C}^1_3(\mathbb{P}, \mathbf{I}^3_2)$. And $\ominus$ is $\mathbf{Q}_1(\mathbf{I}^1_0, f)$. In fact, we prove that $\mathbf{Q}_1(\mathbf{I}^1_0, f)(m, n) = m \ominus n$ by induction on n, with m fixed. $\mathbf{Q}_1(\mathbf{I}^1_0, f)(m, 0) = \mathbf{I}^1_0(m) = m = m \ominus 0$. Now assume that $\mathbf{Q}_1(\mathbf{I}^1_0, f)(m, n) = m \ominus n$. Then $\mathbf{Q}_1(\mathbf{I}^1_0, f)(m, n+1) = f(m, n, \mathbf{Q}_1(\mathbf{I}^1_0, f)(m, n)) = f(m, n, m \ominus n) = \mathbb{P}(m \ominus n) = m \ominus (n+1)$. □

$|\ -\ |$ is the two-place function such that for any $m, n \in \omega$, $|m - n|$ is the absolute value of the difference of m and n.

Proposition 7.8. $|\ -\ |$ *is recursive.*

Proof. $|x - y| = (x \ominus y) + (y \ominus x)$ for all $x, y \in \omega$. Thus

$$|\ -\ | = \mathbf{C}^2_2(+, \mathbf{C}^2_2(\ominus, I^2_0, I^2_1), \mathbf{C}^2_2(\ominus, I^2_1, I^2_0)).$$ □

sg and $\overline{\text{sg}}$ are the one-place functions defined as follows, for any $m \in \omega$:

$$\text{sg}(m) = \begin{cases} 0 & \text{if } m = 0, \\ 1 & \text{if } m \neq 0; \end{cases} \quad \overline{\text{sg}}(m) = \begin{cases} 1 & \text{if } m = 0, \\ 0 & \text{if } m \neq 0. \end{cases}$$

Proposition 7.9. sg *and* $\overline{\text{sg}}$ *are recursive.*

Proof. We claim that $\overline{\mathrm{sg}} = \mathbf{Q}_0(0, \mathbf{k}_0^2)$; and we prove that $\mathbf{Q}_0(1, \mathbf{k}_0^2)(a) = \overline{\mathrm{sg}}(a)$ for all $a \in \omega$ by induction. $\mathbf{Q}_0(1, \mathbf{k}_0^2)(0) = 1 = \overline{\mathrm{sg}}(0)$. Assume that $\mathbf{Q}_0(1, \mathbf{k}_0^2)(a) = \overline{\mathrm{sg}}(a)$. Then $\mathbf{Q}_0(1, \mathbf{k}_0^2)(a+1) = \mathbf{k}_0^2(a, \mathbf{Q}_0(1, \mathbf{k}_0^2)(a)) = 0 = \overline{\mathrm{sg}}(a+1)$.

Next, sg is recursive, since $\mathrm{sg}(x) = \overline{\mathrm{sg}}(\overline{\mathrm{sg}}(x))$ for any $x \in \omega$; thus $\mathrm{sg} = \mathbf{C}_1^1(\overline{\mathrm{sg}}, \overline{\mathrm{sg}})$. □

For any $m, n \in \omega$, $\mathrm{rm}(m, n)$ is the remainder upon dividing m by n if $n \neq 0$, and is 0 if $n = 0$. Thus for $n \neq 0$ we write $m = nq + r$ with $r < n$, and $\mathrm{rm}(m, n) = r$.

Proposition 7.10. rm *is recursive.*

Proof. Let f be the binary operation on ω defined as follows, by recursion: for any $x, y \in \omega$,

$$
\begin{aligned}
f(x, 0) &= 0, \\
f(x, y+1) &= \mathrm{sg}(x) \cdot (f(x, y) + 1) \cdot \mathrm{sg}(|x - f(x, y) - 1|).
\end{aligned}
$$

We check that $\mathrm{rm}(y, x) = f(x, y)$ for all $x, y \in \omega$. If $x = 0$, then $\mathrm{rm}(y, x) = 0$ by definition, and $\mathrm{sg}(0) = 0$, so $f(0, y) = 0$ for all y. Now suppose that $x \neq 0$. Then we prove that $\mathrm{rm}(y, x) = f(x, y)$ for all y by induction on y. For $y = 0$, note that $0 = x \cdot 0 + 0$, so $\mathrm{rm}(0, x) = 0$; and also $f(x, 0) = 0$. Now assume that $\mathrm{rm}(y, x) = f(x, y)$. Thus there is a natural number q such that $y = x \cdot q + f(x, y)$, with $f(x, y) < x$.

Case 1. $f(x, y) + 1 < x$. Then $|x - f(x, y) - 1| > 0$, hence $\mathrm{sg}(|x - f(x, y) - 1|) = 1$ and also $\mathrm{sg}(x) = 1$ since $x \neq 0$, so $f(x, y+1) = f(x, y) + 1 = \mathrm{rm}(y+1, x)$.

Case 2. $f(x, y) + 1 = x$. Then $|x - f(x, y) - 1| = 0$, hence $\mathrm{sg}(|x - f(x, y) - 1|) = 0$ and so $f(x, y+1) = 0 = \mathrm{rm}(y+1, x)$.

So we have shown that $\mathrm{rm}(y, x) = f(x, y)$ for all x, y. Hence $\mathrm{rm} = C_2^2(f, \mathbf{I}_1^2, \mathbf{I}_0^2)$. So we will be finished with rm once we show that f is recursive. That is done in these steps:

let $t_0(x, y, z) = z + 1$ for all x, y, z; recursive since $t_0 = \mathbf{C}_3^1(\mathbf{s}, \mathbf{I}_2^3)$;
let $t_1(x, y, z) = |x - z - 1|$ for all x, y, z; recursive since $t_1 = \mathbf{C}_3^2(|\ -\ |, \mathbf{I}_0^2, t_0)$;
let $t_2(x, y, z) = \mathrm{sg}(|x - z - 1|)$ for all x, y, z; recursive since $t_2 = \mathbf{C}_3^1(\mathrm{sg}, t_1)$;
let $t_3(x, y, z) = (z + 1) \cdot \mathrm{sg}(|x - z - 1|)$ for all x, y, z;
recursive since $t_3 = \mathbf{C}_3^2(\cdot, t_0, t_2)$;
let $t_4(x, y, z) = \mathrm{sg}(x)$ for all x; recursive since $t_4 = \mathbf{C}_3^1(\mathrm{sg}, \mathbf{I}_0^3)$;
let $t_5(x, y, z) = \mathrm{sg}(x) \cdot (z + 1) \cdot \mathrm{sg}(|x - z - 1|)$ for all x, y, z;
recursive since $t_5 = \mathbf{C}_3^2(\cdot, t_4, t_3)$.

Now we claim that $f = \mathbf{Q}_1(\mathbf{k}_0, t_5)$. We prove that $f(x, y) = \mathbf{Q}_1(\mathbf{k}_0, t_5)(x, y)$ for all x, y by induction on y, with x fixed. First, $f(x, 0) = 0$ and $\mathbf{Q}_1(\mathbf{k}_0, t_5)(x, 0) = \mathbf{k}_0(x) = 0$. Now suppose that $f(x, y) = \mathbf{Q}_1(\mathbf{k}_0, t_5)(x, y)$. Then

$$
\begin{aligned}
\mathbf{Q}_1(\mathbf{k}_0, t_5)(x, y+1) &= t_5(x, y, \mathbf{Q}_1(\mathbf{k}_0, t_5)(x, y)) \\
&= t_5(x, y, f(x, y)) \\
&= \mathrm{sg}(x) \cdot (f(x, y) + 1) \cdot \mathrm{sg}(|x - f(x, y) - 1|) \\
&= f(x, y+1).
\end{aligned}
$$

□

The two-place function $\lfloor\ /\ \rfloor$ is defined like this: for any $m, n \in \omega$,

$$\lfloor m/n \rfloor = \begin{cases} \text{largest natural number} \leq m/n & \text{if } n \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 7.11. *The function* $\lfloor\ /\ \rfloor$ *is recursive.*

Proof. In fact, define

$$\begin{aligned} g(x, 0) &= 0, \\ g(x, y+1) &= g(x, y) + \overline{\mathrm{sg}}(|x - \mathrm{rm}(y, x) - 1|). \end{aligned}$$

Then $\lfloor y/x \rfloor = g(x, y)$ for all $x, y \in \omega$. In fact, if $x = 0$, then $\lfloor y/0 \rfloor = 0$ by definition; and we show that $g(0, y) = 0$ for all y by induction: $g(0, 0) = 0$. Assume that $g(0, y) = 0$. Note that $\mathrm{rm}(y, 0) = 0$; hence $|0 - \mathrm{rm}(y, 0) - 1| = 1$ and $g(0, y+1) = g(0, y) + \overline{\mathrm{sg}}(|0 - \mathrm{rm}(y, 0) - 1|) = 0$. Thus $\lfloor y/0 \rfloor = g(0, y)$ for all y.

Now assume that $x \neq 0$. We prove that $\lfloor y/x \rfloor = g(x, y)$ for all y by induction on y, with x fixed. First, $g(x, 0) = 0 = \lfloor 0/x \rfloor$. Now assume that $\lfloor y/x \rfloor = g(x, y)$. By the division algorithm write $y = x \cdot q + r$ with $r < x$. Then $\frac{y}{x} = q + \frac{r}{x}$, hence $q = \lfloor y/x \rfloor$.

Case 1. $r + 1 < x$. Then

$$\frac{y+1}{x} = \frac{y}{x} + \frac{1}{x} = q + \frac{r}{x} + \frac{1}{x} = q + \frac{r+1}{x},$$

so that $\lfloor (y+1)/x \rfloor = q$. Also, $\mathrm{rm}(y+1, x) = r+1$, hence $x - \mathrm{rm}(y+1, x) - 1 = x - r \neq 0$, so $\overline{\mathrm{sg}}(|x - \mathrm{rm}(y+1, x) - 1|) = 0$ and $g(x, y+1) = g(x, y) = q$.

Case 2. $r+1 = x$. Then $x - \mathrm{rm}(y+1, x) - 1 = 0$, so $g(x, y+1) = g(x, y) + 1 = \lfloor (y+1)/x \rfloor$.

This finishes the proof that $\lfloor y/x \rfloor = g(x, y)$ for all $x, y \in \omega$. Hence $\lfloor\ /\ \rfloor = \mathbf{C}^2_2(g, \mathbf{I}^2_1, \mathbf{I}^2_0)$. So It remains to prove that g is recursive. We do this in steps:

let $t_0(x, y, z) = \mathrm{rm}(y, x)$; recursive since $t_0 = \mathbf{C}^2_3(\mathbf{rm}, \mathbf{I}^3_1, \mathbf{I}^3_0)$;
let $t_1(x, y, z) = \mathrm{rm}(y, x) + 1$; recursive since $t_1 = \mathbf{C}^1_3(\mathbf{s}, t_0)$;
let $t_2(x, y, z) = |x - \mathrm{rm}(y, x) - 1|$; recursive since $t_2 = C^2_3(|\ -\ |, \mathbf{I}^3_0, t_1)$;
let $t_3(x, y, z) = \overline{\mathrm{sg}}(|x - \mathrm{rm}(y, x) - 1|)$; recursive since $t_3 = \mathbf{C}^1_3(\overline{\mathrm{sg}}, t_2)$;
let $t_4(x, y, z) = z + \overline{\mathrm{sg}}(|x - \mathrm{rm}(y, x) - 1|)$; recursive since $t_4 = \mathbf{C}^2_3(+, \mathbf{I}^3_2, t_3)$.

Now $g = \mathbf{Q}_1(\mathbf{k}_0, t_4)$. We prove that $g(x, y) = \mathbf{Q}_1(\mathbf{k}_0, t_4)(x, y)$ by induction on y, with x fixed. $\mathbf{Q}_1(\mathbf{k}_0, t_4)(x, 0) = \mathbf{k}_0(x) = 0 = g(x, 0)$. Assume that $\mathbf{Q}_1(\mathbf{k}_0, t_4)(x, y) = g(x, y)$. Then

$$\begin{aligned} \mathbf{Q}_1(\mathbf{k}_0, t_4)(x, y+1) &= t_4(x, y, \mathbf{Q}_1(\mathbf{k}_0, t_4)(x, y)) \\ &= \mathbf{Q}_1(\mathbf{k}_0, t_4)(x, y) + \overline{\mathrm{sg}}(|x - \mathrm{rm}(y, x) - 1|) \\ &= g(x, y) + \overline{\mathrm{sg}}(|x - \mathrm{rm}(y, x) - 1|) \\ &= g(x, y+1). \end{aligned}$$

□

Proposition 7.12. *Suppose that f is an m-ary recursive function, m a positive integer. Define another m-ary function g, like this. For any $a_0, \ldots, a_{m-1} \in \omega$,*

$$g(a_0, \ldots, a_{m-1}) = \sum_{x < a_{m-1}} f(a_0, \ldots, a_{m-2}, x).$$

Then also g is recursive. (A sum over the empty set, i.e., with $a_{m-1} = 0$, is taken to be 0.)

Proof. We just reformulate the definition of g so that it is pretty obvious that g is recursive:

$$\begin{aligned} g(a_0, \ldots, a_{m-2}, 0) &= 0; \\ g(a_0, \ldots, a_{m-2}, y+1) &= \sum_{x < y+1} f(a_0, \ldots, a_{m-2}, x) \\ &= \sum_{x<y} f(a_0, \ldots, a_{m-2}, x) + f(a_0, \ldots, a_{m-2}, y) \\ &= g(a_0, \ldots, a_{m-2}, y) + f(a_0, \ldots, a_{m-2}, y). \end{aligned}$$

To really show that g is recursive, let $h(a_0, \ldots, a_m) = a_m + f(a_0, \ldots, a_{m-1})$ for all $a_0, \ldots, a_{m-1} \in \omega$. Then h is recursive, since

$$h = \mathbf{C}^2_{m+1}(+, I^{m+1}_m, \mathbf{C}^m_{m+1}(f, \mathbf{I}^{m+1}_0, \ldots, \mathbf{I}^{m+1}_{m-1}))).$$

Now we take two cases.

Case 1. $m = 1$. Then $g = \mathbf{Q}_0(0, h)$. We prove that $g(b) = \mathbf{Q}_0(0, h)(b)$ for all $b \in \omega$ by induction. $\mathbf{Q}_0(0, h)(0) = 0 = g(0)$. Assume that $\mathbf{Q}_0(0, h)(b) = g(b)$. Then $\mathbf{Q}_0(0, h)(b+1) = h(b, \mathbf{Q}_0(0, h)(b)) = h(b, g(b)) = g(b) + f(b) = g(b+1)$.

Case 2. $m > 1$. We claim that $g = \mathbf{Q}_{m-1}(\mathbf{k}^{m-1}_0, h)$. We prove that

$$g(a_0, \ldots, a_{m-2}, b) = \mathbf{Q}_{m-1}(\mathbf{k}^{m-1}_0, h)(a_0, \ldots, a_{m-2}, b)$$

for all $a_0, \ldots, a_{m-2}, b \in \omega$ by induction on b, with $a_0, \ldots, a_{m-2}$ fixed.

$$\mathbf{Q}_{m-1}(\mathbf{k}^{m-1}_0, h)(a_0, \ldots, a_{m-2}, 0) = \mathbf{k}^{m-1}_0(a_0, \ldots, a_{m-2}) = 0 = g(a_0, \ldots, a_{m-2}, 0).$$

Assume that $\mathbf{Q}_{m-1}(\mathbf{k}^{m-1}_0, h)(a_0, \ldots, a_{m-2}, b) = g(a_0, \ldots, a_{m-2}, b)$. Then

$$\begin{aligned} \mathbf{Q}_{m-1}(\mathbf{k}^{m-1}_0, h)(a_0, \ldots, a_{m-2}, b+1) &= h(a_0, \ldots, a_{m-2}, b, \mathbf{Q}_{m-1}(\mathbf{k}^{m-1}_0, h)(a_0, \ldots, a_{m-2}, b)) \\ &= h(a_0, \ldots, a_{m-2}, b, g(a_0, \ldots, a_{m-2}, b)) \\ &= g(a_0, \ldots, a_{m-2}, b) + f(a_0, \ldots, a_{m-2}, b) \\ &= g(a_0, \ldots, a_{m-2}, b+1). \end{aligned}$$

□

Proposition 7.13. *Suppose that f is an m-ary recursive function, m a positive integer. Define another m-ary function g, like this. For any $a_0, \ldots, a_{m-1} \in \omega$,*

$$g(a_0, \ldots, a_{m-1}) = \prod_{x < a_{m-1}} f(a_0, \ldots, a_{m-2}, x).$$

Then also g is recursive. (A product over the empty set, i.e., with $a_{m-1} = 0$, is taken to be 1.)

Proof. We just reformulate the definition of g so that it is pretty obvious that g is recursive:

$$\begin{aligned} g(a_0, \ldots, a_{m-2}, 0) &= 1; \\ g(a_0, \ldots, a_{m-2}, y+1) &= \prod_{x<y+1} f(a_0, \ldots, a_{m-2}, x) \\ &= \prod_{x<y} f(a_0, \ldots, a_{m-2}, x) \cdot f(a_0, \ldots, a_{m-2}, y) \\ &= g(a_0, \ldots, a_{m-2}, y) \cdot f(a_0, \ldots, a_{m-2}, y). \end{aligned}$$

To show that g is recursive, let $h(a_0, \ldots, a_m) = a_m \cdot f(a_0, \ldots, a_{m-1})$ for all $a_0, \ldots, a_{m-1} \in \omega$. Then h is recursive, since

$$h = \mathbf{C}^2_{m+1}(\cdot, I^{m+1}_m, \mathbf{C}^m_{m+1}(f, \mathbf{I}^{m+1}_0, \ldots, \mathbf{I}^{m+1}_{m-1}))).$$

Now we take two cases.

Case 1. $m = 1$. Then $g = \mathbf{Q}_0(1, h)$. We prove that $g(b) = \mathbf{Q}_0(1, h)(b)$ for all $b \in \omega$ by induction. $\mathbf{Q}_0(1, h)(0) = 1 = g(0)$. Assume that $\mathbf{Q}_0(1, h)(b) = g(b)$. Then $\mathbf{Q}_0(1, h)(b+1) = h(b, \mathbf{Q}_0(1, h)(b)) = h(b, g(b)) = g(b) \cdot f(b) = g(b+1)$.

Case 2. $m > 1$. We claim that $g = \mathbf{Q}_{m-1}(\mathbf{k}^{m-1}_1, h)$. We prove that

$$g(a_0, \ldots, a_{m-2}, b) = \mathbf{Q}_{m-1}(\mathbf{k}^{m-1}_1, h)(a_0, \ldots, a_{m-2}, b)$$

for all $a_0, \ldots, a_{m-2}, b \in \omega$ by induction on b, with $a_0, \ldots, a_{m-2}$ fixed.

$$\mathbf{Q}_{m-1}(\mathbf{k}^{m-1}_1, h)(a_0, \ldots, a_{m-2}, 0) = \mathbf{k}^{m-1}_1(a_0, \ldots, a_{m-2}) = 1 = g(a_0, \ldots, a_{m-2}, 0).$$

Assume that $\mathbf{Q}_{m-1}(\mathbf{k}^{m-1}_1, h)(a_0, \ldots, a_{m-2}, b) = g(a_0, \ldots, a_{m-2}, b)$. Then

$$\begin{aligned} \mathbf{Q}_{m-1}(\mathbf{k}^{m-1}_1, h)(a_0, \ldots, a_{m-2}, b+1) &= h(a_0, \ldots, a_{m-2}, b, \mathbf{Q}_{m-1}(\mathbf{k}^{m-1}_1, h)(a_0, \ldots, a_{m-2}, b)) \\ &= h(a_0, \ldots, a_{m-2}, b, g(a_0, \ldots, a_{m-2}, b)) \\ &= g(a_0, \ldots, a_{m-2}, b) \cdot f(a_0, \ldots, a_{m-2}, b) \\ &= g(a_0, \ldots, a_{m-2}, b+1). \end{aligned}$$

□

Proposition 7.14. *The factorial function is recursive.*

Proof. $x! = \prod_{y<x} \mathbf{s}(y)$. Thus ! is recursive by Propositon 7.13 applied to $\mathbf{s}$. □

Proposition 7.15. *$\emptyset$ and ω are recursive, and $\{x\}$ is recursive for every $x \in \omega$.*

Proof. $\chi_\emptyset = \mathbf{k}_0$ and $\chi_\omega = \mathbf{k}_1$. Now suppose that $x \in \omega$. Then for any $y \in \omega$ we have $\chi_{\{x\}}(y) = \overline{\mathrm{sg}}(|x - y|)$. Let $h(y) = |x - y|$ for any $y \in \omega$. Then h is recursive, since $h = \mathbf{C}^2_1(| \ - \ |, \mathbf{k}_x, \mathbf{I}^1_0)$. Now $\chi_{\{x\}} = \mathbf{C}^1_1(\overline{\mathrm{sg}}, h)$, so $\chi_{\{x\}}$ is recursive. □

For any positive integer m we denote by ${}^m\omega$ the set of all functions mapping m' into ω; this is the set of all m-termed sequences of natural numbers. Here $m' = \{0, \ldots, m-1\}$ for any natural number m. For any sets A, B, we denote by $A \backslash B$ the set of all elements of A which are not in B.

Proposition 7.16. *Let m be a positive integer, and let A and B be m-ary recursive relations on ω. Then $A \cup B$, $A \cap B$, and ${}^m\omega \backslash A$ are recursive.*

Proof. For any $a_0, \ldots, a_{m-1} \in \omega$ we have

$$\begin{aligned}\chi_{A\cap B}(a_0, \ldots, a_{m-1}) &= \chi_A(a_0, \ldots, a_{m-1}) \cdot \chi_B(a_0, \ldots, a_{m-1}),\\ \chi_{{}^m\omega\backslash A}(a_0, \ldots, a_{m-1}) &= \overline{\mathrm{sg}}\left(\chi_A(a_0, \ldots, a_{m-1})\right),\end{aligned}$$

so $A \cap B$ and ${}^m\omega \backslash A$ are recursive: $\chi_{A\cap B} = \mathbf{C}^2_m(\cdot, \chi_A, \chi_B)$ and ${}^m\omega \backslash A = \mathbf{C}^1_m(\overline{\mathrm{sg}}, \chi_A)$. Now $A \cup B$ is recursive by elementary set theory: $A \cup B = {}^m\omega \backslash ({}^m\omega \backslash A \cap {}^m\omega \backslash B)$. □

A subset X of a set A is *cofinite* iff $A \backslash X$ is finite.

Corollary 7.17. *All finite and cofinite subsets of ω are recursive.*

Proof. $\emptyset$ is recursive by Proposition 7.15. If F is finite and nonempty, write $F = \{a_0, \ldots, a_m\}$. Then $F = \{a_0\} \cup \ldots \cup \{a_m\}$, and so F is recursive by Propositions 7.15 and 7.16. If F is cofinite, then $\omega \backslash F$ is finite, hence recursive by what was just shown. Then $F = \omega \backslash (\omega \backslash F)$ is recursive by Proposition 7.16. □

Proposition 7.18. *The following binary relations on ω are recursive: $=$, $\neq$, $<$, $\leq$, $>$, and $\geq$.*

Proof. For any $x, y \in \omega$ we have $\chi_=(x, y) = \overline{\mathrm{sg}}(|x - y|)$, hence $\chi_= = \mathbf{C}^1_2(\overline{\mathrm{sg}}, \mathbf{C}^2_2(| - |, \mathbf{I}^2_0, \mathbf{I}^2_1))$, so $\chi_=$ is recursive, hence $=$ is recursive.
$\neq = (\omega \backslash =)$ so $\neq$ is recursive by what was just shown and Proposition 7.16.
$\chi_<(x, y) = \overline{\mathrm{sg}}(\lfloor (x+1)/(y+1) \rfloor)$, so $<$ is recursive, as follows:

let $t_1(x, y) = x + 1$; recursive since $t_1 = \mathbf{C}^1_2(\mathbf{s}, \mathbf{I}^2_0)$;
let $t_2(x, y) = y + 1$; recursive since $t_2 = \mathbf{C}^1_2(\mathbf{s}, \mathbf{I}^2_1)$;
let $t_3(x, y) = \lfloor (x+1)/(y+1) \rfloor$; recursive since $t_3 = \mathbf{C}^2_2(\lfloor\ /\ \rfloor, t_1, t_2)$;
$\chi_< = \mathbf{C}^1_2(\overline{\mathrm{sg}}, t_3)$.

$\leq = (= \cup <)$, so $\leq$ is recursive by the above and Proposition 7.16.
$> = ({}^2\omega \backslash \leq)$, so $>$ is recursive by the above and Proposition 7.16.
$\geq = ({}^2\omega \backslash <)$, so $\geq$ is recursive by the above and Proposition 7.16. □

Proposition 7.19. *Let m be a positive integer, and let R be an m-ary recursive relation. Define*

$$S = \{(a_0, \ldots, a_{m-1}) : \textit{ there is an } x < a_{m-1} \textit{ such that } (a_0, \ldots, a_{m-2}, x) \in R\}.$$

Then S is recursive.

Proof. Let $a_0, \ldots, a_{m-1} \in \omega$.

$$\chi_S(a_0, \ldots, a_{m-1}) = \mathrm{sg}\left(\sum_{x<a_{m-1}} \chi_R(a_0, \ldots, a_{m-2}, x)\right)$$

Thus if we let $t(a_0, \ldots, a_{m-1}) = \sum_{x<a_{m-1}} \chi_R(a_0, \ldots, a_{m-2}, x)$, then t is recursive by Proposition 7.12, and $\chi_S = \mathbf{C}^1_m(\mathrm{sg}, t)$, so χ_S is recursive. □

Proposition 7.20. *Let $m > 1$ and let R be an m-ary recursive relation. Define*

$$T = \{(a_0, \ldots, a_{m-2}) : \text{ there is an } x < a_0 \text{ such that } (a_0, \ldots, a_{m-2}, x) \in R\}.$$

Then T is recursive.

Proof. Let S be as in Proposition 7.19. Then $\chi_T = \mathbf{C}^m_{m-1}(\chi_S, \mathbf{I}^{m-1}_0, \ldots, \mathbf{I}^{m-1}_{m-2}, \mathbf{I}^{m-1}_0)$. □

Proposition 7.21. *Let m be a positive integer, and let R be an m-ary recursive relation. Define*

$$S = \{(a_0, \ldots, a_{m-1}) : \text{ there is an } x \leq a_{m-1} \text{ such that } (a_0, \ldots, a_{m-2}, x) \in R\}.$$

Then S is recursive.

Proof. Let S' be as in Proposition 7.19:

$$S' = \{(a_0, \ldots, a_{m-1}) : \text{ there is an } x < a_{m-1} \text{ such that } (a_0, \ldots, a_{m-2}, x) \in R\}.$$

So by Proposition 7.19, S' is recursive. Now

$$S = \{\langle a_0, \ldots, a_{m-1}\rangle : \langle a_0, \ldots, a_{m-2}, \mathbf{s}(a_{m-1})\rangle \in S'\};$$
$$\text{hence } S = \mathbf{C}^m_m(S', \mathbf{I}^m_0, \ldots, \mathbf{I}^m_{m-2}, \mathbf{C}^1_m(\mathbf{s}, \mathbf{I}^m_{m-1})),$$

showing that S is recursive. □

Corollary 7.22. *Let m be a positive integer, and let R be an m-ary recursive relation. Define*

$$S = \{(a_0, \ldots, a_{m-1}) : \text{ for all } x < a_{m-1} \text{ we have } (a_0, \ldots, a_{m-2}, x) \in R\}.$$

Then S is recursive.

Proof. By Proposition 7.16, ${}^m\omega\backslash R$ is recursive. Define

$$T = \{(a_0, \ldots, a_{m-1}) : \text{there is an } x < a_{m-1} \text{ such that } (a_0, \ldots, a_{m-2}, x) \in ({}^m\omega\backslash R)\}.$$

Then T is recursive by Proposition 7.19. We claim that $S = ({}^m\omega\backslash T)$, so that S is recursive by Proposition 7.16. In fact,

$$\begin{aligned}
(a_0,\ldots,a_{m-1}) \in ({}^m\omega\backslash T) \quad & \text{iff} \quad (a_0,\ldots,a_{m-1}) \notin T \\
& \text{iff} \quad \text{there does not exist an } x < a_{m-1} \text{ such that} \\
& \qquad (a_0,\ldots,a_{m-1},x) \in {}^m\omega\backslash R \\
& \text{iff} \quad \text{there does not exist an } x < a_{m-1} \text{ such that} \\
& \qquad (a_0,\ldots,a_{m-1},x) \notin R \\
& \text{iff} \quad \text{for all } x < a_{m-1} \text{ we have } (a_0,\ldots,a_{m-2},x) \in R \\
& \text{iff} \quad (a_0,\ldots,a_{m-1}) \in S.
\end{aligned}$$

□

Proposition 7.23. *Let $m,n \in \omega$ with $m,n > 0$. Suppose that $g_0,\ldots,g_{m-1}$ are n-ary recursive functions, and $R_0,\ldots,R_{m-1}$ are pairwise disjoint n-ary recursive relations with union ${}^n\omega$. Let f be the n-ary function defined as follows. For any $a_0,\ldots,a_{n-1} \in \omega$,*

$$f(a_0,\ldots,a_{n-1}) = \begin{cases} g_0(a_0,\ldots,a_{n-1}) & \textit{if } \langle a_0,\ldots,a_{n-1}\rangle \in R_0, \\ g_1(a_0,\ldots,a_{n-1}) & \textit{if } \langle a_0,\ldots,a_{n-1}\rangle \in R_1, \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots \\ g_{m-1}(a_0,\ldots,a_{n-1}) & \textit{if } \langle a_0,\ldots,a_{n-1}\rangle \in R_{m-1}. \end{cases}$$

Then f is recursive.

Proof. Induction on m. For $m = 1$, by hypothesis $R_0 = {}^n\omega$ and $f = g_0$; so f is recursive. Now assume the result for $m-1$ with $m > 1$, and assume the hypotheses of the proposition. Let $R'_i = R_i$ for all $i < m-2$, and $R'_{m-2} = R_{m-2} \cup R_{m-1}$. Then by Proposition 7.16, R'_{m-2} is recursive. So all of $R'_0,\ldots,R'_{m-2}$ are recursive. For $i,j < m-2$ with $i \neq j$ we have $R'_i \cap R'_j = R_i \cap R_j = \emptyset$, and for $i < m-2$ we have

$$R'_i \cap R'_{m-2} = R_i \cap (R_{m-2} \cup R_{m-1}) = (R_i \cap R_{m-2}) \cup (R_i \cap R_{m-1}) = \emptyset \cup \emptyset = \emptyset.$$

So the R'_i are pairwise disjoint for $i < m-2$. Furthermore,

$$R'_0 \cup \ldots \cup R'_{m-2} = R_0 \cup \ldots \cup R_{m-3} \cup R_{m-2} \cup R_{m-1} = {}^n\omega.$$

Now define

$$g'_{m-2}(a_0,\ldots,a_{n-1}) = \begin{cases} g_{m-2}(a_0,\ldots,a_{n-1}) & \text{if } \langle a_0,\ldots,a_{n-1} \in R_{m-2}, \\ g_{m-1}(a_0,\ldots,a_{n-1}) & \text{if } \langle a_0,\ldots,a_{n-1} \in R_{m-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Then g'_{m-2} is recursive, since

$$\begin{aligned} g'_{m-2}(a_0,\ldots,a_{n-1}) = {} & \chi_{R_{m-2}}(a_0,\ldots,a_{n-1}) \cdot g_{m-2}(a_0,\ldots,a_{n-1}) + \\ & \chi_{R_{m-1}}(a_0,\ldots,a_{n-1}) \cdot g_{m-1}(a_0,\ldots,a_{n-1}), \end{aligned}$$

and so $g'_{m-2} = \mathbf{C}^2_n(+, \mathbf{C}^2_n(\cdot, \chi_{R_{m-2}}, g_{m-2}), \mathbf{C}^2_n(\cdot, \chi_{R_{m-1}}, g_{m-1}))$, showing that g'_{m-2} is recursive. For $i < m-2$ let $g'_i = g_i$. Define

$$f'(a_0,\ldots,a_{n-1}) = \begin{cases} g'_0(a_0,\ldots,a_{n-1}) & \text{if } \langle a_0,\ldots,a_{n-1}\rangle \in R'_0, \\ g'_1(a_0,\ldots,a_{n-1}) & \text{if } \langle a_0,\ldots,a_{n-1}\rangle \in R'_1, \\ \dots\dots\dots\dots\dots\dots & \dots\dots\dots\dots\dots\dots\dots\dots \\ g'_{m-2}(a_0,\ldots,a_{n-1}) & \text{if } \langle a_0,\ldots,a_{n-1}\rangle \in R'_{m-2}. \end{cases}$$

Then f' is recursive by the induction hypothesis. We claim that actually $f' = f$. For, take any $a_0, \ldots, a_{n-1} \in \omega$.

If $\langle a_0,\ldots,a_{n-1}\rangle \in R_i$ with $i < m-2$, then
$$f'(a_0,\ldots,a_{n-1}) = g'_i(a_0,\ldots,a_{n-1}) = g_i(a_0,\ldots,a_{n-1});$$
if $\langle a_0,\ldots,a_{n-1}\rangle \in R_{m-2}$, then
$$f'(a_0,\ldots,a_{n-1}) = g'_{m-2}(a_0,\ldots,a_{n-1}) = g_{m-2}(a_0,\ldots,a_{n-1});$$
if $\langle a_0,\ldots,a_{n-1}\rangle \in R_{m-1}$, then
$$f'(a_0,\ldots,a_{n-1}) = g'_{m-2}(a_0,\ldots,a_{n-1}) = g_{m-1}(a_0,\ldots,a_{n-1}).$$
□

Proposition 7.24. *Let m be a positive integer, and let R be an m-ary recursive relation on ω. We define an m-ary operation f on ω as follows. For any $a_0, \ldots, a_{m-1} \in \omega$,*

$$f(a_0,\ldots,a_{m-1}) = \begin{cases} \text{the least } x < a_{m-1} \text{ such that} & \\ \qquad \langle a_0,\ldots,a_{m-2},x\rangle \in R & \text{if there is such an } x, \\ a_{m-1} & \text{otherwise.} \end{cases}$$

Then f is recursive.

Proof. For any $a_0, \ldots, a_{m-1}, x \in \omega$ let

$$g(a_0,\ldots,a_{m-1},x) = \begin{cases} \overline{\mathrm{sg}}\,(\chi_R(a_0,\ldots,a_{m-2},x)) & \text{if } x < a_{m-1} \\ 0 & \text{otherwise.} \end{cases}$$

Now g is special, since $g(a_0,\ldots,a_{m-1},a_{m-1}) = 0$ for all $a_0,\ldots,a_{m-1} \in \omega$. To see that g is recursive, we use these steps:

let $t_1(a_0,\ldots,a_{m-1},x) = \chi_<(x, a_{m-1})$; recursive, since $t_1 = \mathbf{C}^2_{m+1}(\chi_<, \mathbf{I}^{m+1}_m, \mathbf{I}^{m+1}_{m-1})$;
let $t_2(a_0,\ldots,a_{m-1},x) = \overline{\mathrm{sg}}(\chi_R(a_0,\ldots,a_{m-1},x))$; recursive, since $t_2 = \mathbf{C}^1_{m+1}(\overline{\mathrm{sg}}, \chi_R)$;

so g is recursive, since $g = \mathbf{C}^2_{m+1}(\cdot, t_1, t_2)$. For any $a_0,\ldots,a_{m-1}$, let $h(a_0,\ldots,a_{m-1})$ be the least x such that $g(a_0,\ldots,a_{m-1},x) = 0$. Thus h is recursive, since $h = \mathbf{M}_m(g)$. Note that for any $a_0,\ldots,a_{m-2},x \in \omega$, $\langle a_0,\ldots,a_{m-2},x\rangle \in R$ iff $\overline{\mathrm{sg}}(\chi_R(a_0,\ldots,a_{m-2},x)) = 0$. Hence for any $a_0,\ldots,a_{m-1} \in \omega$, if there is an $x < a_{m-1}$ such that $\langle a_0,\ldots,a_{m-2},x\rangle \in R$, then the least such x is also the least $x < a_{m-1}$ such that $\overline{\mathrm{sg}}(\chi_R(a_0, \ldots, a_{m-2}, x)) = 0$. So for any $a_0, \ldots, a_{m-1} \in \omega$, if there is an $x < a_{m-1}$ such that $\langle a_0, \ldots, a_{m-2}, x\rangle \in R$, then

$$\begin{aligned} h(a_0, \ldots, a_{m-1}) &= \text{ the least } x < a_{m-1} \text{ such that } g(a_0, \ldots, a_{m-2}, x) = 0 \\ &= \text{ the least } x < a_{m-1} \text{ such that } \overline{\mathrm{sg}}(\chi_R(a_0, \ldots, a_{m-2}, x)) = 0 \\ &= \text{ the least } x < a_{m-1} \text{ such that } \langle a_0, \ldots, a_{m-2}, x\rangle \in R \\ &= f(a_0, \ldots, a_{m-1}). \end{aligned}$$

On the other hand, if $a_0, \ldots, a_{m-1} \in \omega$ and $\langle a_0, \ldots, a_{m-1}, x\rangle \notin R$ for all $x < a_{m-1}$, then

$$h(a_0, \ldots, a_{m-1}) = a_{m-1} = f(a_0, \ldots, a_{m-1}) \qquad \square$$

Now we introduce some more elementary functions. If $m \in \omega$ and $m > 1$, then $\min_m$ and $\max_m$ are the following m-ary functions. For any $a_0, \ldots, a_{m-1} \in \omega$,

$$\begin{aligned} \min_m(a_0, \ldots, a_{m-1}) &= \text{ minimum of } a_0, \ldots, a_{m-1}; \\ \max_m(a_0, \ldots, a_{m-1}) &= \text{ maximum of } a_0, \ldots, a_{m-1}. \end{aligned}$$

For $m = 2$ we drop the subscript, writing only min and max.

Proposition 7.25. *The following functions and relations are recursive:* min, max, $\min_m$ *and* $\max_m$ *for each integer* $m > 1$.

Proof. For any $a, b \in \omega$ we have

$$\min(a, b) = \begin{cases} a & \text{if } a \le b \\ b & \text{otherwise,} \end{cases}$$

We can reformulate this as follows:

$$\min(a, b) = \begin{cases} \mathbf{I}_0^2(a, b) & \text{if } (a, b) \in (\le) \\ \mathbf{I}_1^2(a, b) & \text{if } (a, b) \in (>). \end{cases}$$

Moreover, $(\le) \cap (>) = \emptyset$ and $(\le) \cup (<) = {}^2\omega$. Hence min is recursive by Theorem 7.23.

Similarly,

$$\max(a, b) = \begin{cases} \mathbf{I}_1^2(a, b) & \text{if } (a, b) \in (\le) \\ \mathbf{I}_0^2(a, b) & \text{if } (a, b) \in (>). \end{cases}$$

Moreover, $(\le) \cap (>) = \emptyset$ and $(\le) \cup (<) = {}^2\omega$. Hence max is recursive by Theorem 7.23.

We show that $\min_m$ is recursive for all $m > 1$ by induction on m. Since $\min_2 = \min$, the case $m = 2$ holds. Assume that $\min_m$ is recursive. Now

$$\min_{m+1}(a_0, \ldots, a_m) = \min(\min_m(a_0, \ldots, a_{m-1}), a_m);$$

hence

$$\min_{m+1} = \mathbf{C}^2_{m+1}(\min, \mathbf{C}^m_{m+1}(\min_m, \mathbf{I}_0^{m+1}, \ldots, \mathbf{I}_{m-1}^{m+1}), \mathbf{I}_m^{m+1}),$$

proving that $\min_{m+1}$ is recursive. So by induction, $\min_m$ is recursive for all $m \geq 2$.

The case of $\max_m$ is very similar. $\square$

We write $a|b$ iff a divides b, i.e., there is a natural number c such that $b = a \cdot c$. Note that by this definition, every natural number divides 0: take $c = 0$.

Proposition 7.26. $|$ *is recursive.*

Proof. Note that for any $a, b \in \omega$,

$$a \,|\, b \text{ iff there is a } c \leq b \text{ such that } b = ac.$$

To see that this implies that $|$ is recursive we do the following steps:

let $t_1(b, a, c) = a \cdot c$; recursive since $t_1 = \mathbf{C}^2_3(\cdot, \mathbf{I}^3_1, \mathbf{I}^3_2)$;
let $t_2(b, a, c) = |b - a \cdot c|$; recursive since $t_2 = \mathbf{C}^2_3(|\ -\ |, \mathbf{I}^3_0, t_1)$;
let $R = \{\langle b, a, c\rangle : b = a \cdot c\}$; recursive since $\chi_R = \mathbf{C}^1_3(\overline{\mathrm{sg}}, t_2)$;
let $S = \{(b, a, s)$ there is a $c \leq s$ such that $b = a \cdot c$; recursive by Prop. 7.21;
$\chi_| = \{\langle a, b\rangle : \langle b, a, b\rangle \in S\}$; recursive since $\chi_| = \mathbf{C}^3_2(\chi_S, \mathbf{I}^2_0, \mathbf{I}^2_1, \mathbf{I}^2_0)$. $\square$

Pr is the set of all prime numbers.

Proposition 7.27. Pr *is recursive.*

Proof. For any $a \in \omega$ we have: a is a prime iff $a > 1$ and for every $b < a$, $b|a$ implies that $b = 1$. Pr is recursive, by these steps:

let $S = \{(b, a) : \mathrm{not}(b|a)\}$; recursive by Props. 7.16 and 7.26;
let $t(b, a) = |b - 1|$; recusive since $t = \mathbf{C}^2_2(|\ -\ |, \mathbf{I}^2_0, \mathbf{C}^1_2(\mathbf{k}_1, \mathbf{I}^2_0))$;
let $T = \{(b, a) : b = 1\}$; recursive since $\chi_T = \mathbf{C}^1_2(\overline{\mathrm{sg}}, t)$;
let $U = S \cup T$; recursive by the above and Prop. 7.16
let $U' = \{(a, b) : (b, a) \in U\}$; recursive since $\chi_{U'} = \mathbf{C}^2_2(\chi_U, \mathbf{I}^2_1, \mathbf{I}^2_0)$;
let $V = \{(c, a) : \text{for all } b < a[(c, b) \in U']\}$; recursive by Cor. 7.22
$\chi_{\mathrm{Pr}} = \mathbf{C}^2_1(\chi_V, \mathbf{I}^1_0, \mathbf{I}^1_0)$. $\square$

Recall that $\mathbf{p}$ enumerates all the prime numbers: $\mathbf{p}_0 = 2$, $\mathbf{p}_1 = 3$, $\mathbf{p}_2 = 5$, etc.

Proposition 7.28. $\mathbf{p}$ *is recursive.*

Proof. Note that

$$\mathbf{p}_0 = 2;$$
$$\mathbf{p}_{k+1} = \text{ least } x(\mathbf{p}_k < x \text{ and } x \in \mathrm{Pr}).$$

To see that this works, we do these steps:

let $S = \{\langle m,n,x\rangle : n < x\}$; recursive since $\chi_S = \mathbf{C}^2_3(\chi_<, \mathbf{I}^3_1, \mathbf{I}^3_2)$;
let $T = \{(m,n,x) : x \in \mathrm{Pr}\}$; recursive since $\chi_T = \mathbf{C}^1_3(\chi_{\mathrm{Pr}}, \mathbf{I}^3_2)$;
$S \cap T$ is recursive by Proposition 7.16

$$\text{let } h(m,n,x) = \begin{cases} 0 & \text{if } \langle m,n,x\rangle \in S \cap T \\ 1 & \text{if } \langle m,n,x\rangle \in {}^3\omega\backslash(S \cap T) \end{cases}$$

recursive by Proposition 7.23

let $f(m,n) =$ the least x such that $h(m,n,x) = 0$; recursive since $f = \mathbf{M}_2(h)$;
$\mathbf{p} = \mathbf{Q}_0(2, f)$.

The last equality works like this; we prove by induction on m that $\mathbf{Q}_0(2,f)(m) = \mathbf{p}_m$ for all $m \in \omega$. First, $\mathbf{Q}_0(2,f)(0) = 2 = \mathbf{p}_0$. Now assume that $\mathbf{Q}_0(2,f)(m) = \mathbf{p}_m$. Then

$$\begin{aligned} \mathbf{Q}_0(2,f)(m+1) &= f(m, \mathbf{Q}_0(2,f)(m)) \\ &= f(m, \mathbf{p}_m) \\ &= \text{the least } x \text{ such that } h(m, \mathbf{p}_m, x) = 0 \\ &= \text{the least } x \text{ such that } \langle m, \mathbf{p}_m, x\rangle \in S \cap T \\ &= \text{the least } x \text{ such that } \mathbf{p}_m < x \text{ and } x \in \mathrm{Pr} \\ &= \mathbf{p}_{m+1}. \end{aligned}$$

□

Now for any natural numbers a, i we define $(a)_i$ to be 0 if $a = 0$, and otherwise $(a)_i$ is the exponent of $\mathbf{p}_i$ in the prime decomposition of a. Thus $(12)_0 = (2^2 \cdot 3^1) = 2$, $(12)_1 = 1$, $(12)_i = 0$ for all $i > 1$.

Proposition 7.29. ( ) *is recursive.*

Proof. We make the following steps

let $t_0(i,b,x) = \mathbf{p}^x_i$; recursive since $t_0 = \mathbf{C}^2_3(\exp, \mathbf{C}^1_3(\mathbf{p}, \mathbf{I}^3_0), \mathbf{I}^3_2)$;
let $R = \{(i,b,x) : \mathbf{p}^x_i | b\}$; recursive since $\chi_R = \mathbf{C}^2_3(|, t_0, \mathbf{I}^3_1)$;
let $S = \{(i,b,x) : \mathrm{not}(p^x_i | b)\}$; recursive since $S = {}^3\omega\backslash R$;

$$\text{let } t_1(i,b,a) = \begin{cases} \text{the least } x < a \text{ such that } (i,b,x) \in S & \text{if there is such an } x \\ a & \text{otherwise;} \end{cases}$$

recursive by Proposition 7.24

let $t_2(a,i) = t_1(i,a,a+1)$; recursive since $t_2 = \mathbf{C}^3_2(t_1, \mathbf{I}^2_0, \mathbf{C}^1_2(\mathbf{s}, \mathbf{I}^2_1))$;
let $t_3(a,i) = \mathbb{P}(t_2(a,i)$; recursive since $t_3 = \mathbf{C}^1_2(\mathbb{P}, t_2)$.

We claim that $t_3(a,i) = (a)_i$ for all $a, i \in \omega$. For $a = 0$ we have $p^0_i|0$, hence $(i,0,0) \in R$, so $(i,0,0) \notin S$, so there is no $x < 1$ such that $(i,a,x) \in S$, hence $t_1(i,0,1) = 1$, $t_2(0,i) = t_1(i,0,1) = 1$, and $t_3(0,i) = 0 = (0)_i$. Now suppose that $a \neq 0$, and let $(a)_i = y$.

Then $\mathbf{p}_i^{y+1}$ does not divide a, so $(i,a,y+1) \in S$, while $(i,a,z) \notin S$ for all $z \leq y$. So $t_1(i,a,a+1) = y+1$, and $t_2(a,i) = y+1$ and $t_3(a,i) = y$. □

For any positive integer m we let $\mathrm{len}(m)$ be the least i such that $\mathbf{p}_i$ does not divide m. len stands for "length", and the motivation is that if m is the Gödel number of a sequence of length n, then i is equal to n. We also let $\mathrm{len}(0) = 0$.

Proposition 7.30. len *is recursive.*

Proof. We use these steps:

let $S = \{(a,x) : \mathbf{p}_x | a\}$; recursive since $\chi_S = \mathbf{C}^2_2(|, \mathbf{C}^1_2(\mathbf{p}, \mathbf{I}^2_1).\mathbf{I}^2_0)$;

let $R = \{(a,x) : \mathrm{not}(\mathbf{p}_x | a)\}$; recursive by Proposition 7.16

let $f(a,b) = \begin{cases} \text{least } x < b \text{ such that not}(p_x|a) & \text{if there is such an } x, \\ b & \text{otherwise;} \end{cases}$

recursive by Proposition 7.24

let $g(a) = f(a,a)$; recursive since $g = \mathbf{C}^2_1(f, \mathbf{I}^2_0, \mathbf{I}^2_1)$.

We claim now that $g = \mathrm{len}$. First take 0. We have $g(0) = f(0,0) = 0$. Now suppose that $m \neq 0$. Then $g(m) = f(m,m) =$ least $x < m$ such that $\mathrm{not}(p_x|m)$, as desired. □

EXERCISES

The definition of recursive functions includes the rather complicated minimalization operator. The class of *primitive recursive* functions is defined just like the set of recursive functions, but without using the minimalization operator. Most of the functions that one encounters are primitive recursive. So the question arises, are they the same as the recursive functions? The purpose of the following exercises is to show that the two classes are different. In fact, the well-known Ackermann function is recursive but not primitive recursive.

The Ackermann function is the function $f : \omega \times \omega \to \omega$ such that for any $m, n \in \omega$ we have

$$\begin{aligned} f(0,n) &= n+1, \\ f(m+1,0) &= f(m,1), \\ f(m+1,n+1) &= f(m, f(m+1,n)). \end{aligned}$$

To show that this function exists, and is recursive, we define the following set M.

$$\begin{aligned} M = \Big\{ z \in \omega : & z > 1 \wedge \forall i < \mathrm{len}(z)[\mathrm{len}((z)_i) \leq 3 \wedge \\ & [((z)_i)_0 = 0 \wedge ((z)_i)_2 = ((z)_i)_1 + 1) \vee \\ & (((z)_i)_0 \neq 0 \wedge ((z)_i)_1 = 0 \wedge \exists j < i[((z)_j)_0 = \mathbb{P}((z)_i)_0) \\ & \quad \wedge ((z)_j)_1 = 1 \wedge ((z)_j)_2 = ((z)_i)_2]) \vee \\ & (((z)_i)_0 \neq 0 \wedge ((z)_i)_1 \neq 0 \wedge \exists j < i \exists k < i[\\ & \quad ((z)_j)_0 = ((z)_i)_0 \wedge ((z)_j)_1 = \mathbb{P}((z)_i)_1) \wedge ((z)_k)_0 = \mathbb{P}(((z)_i)_0) \wedge \\ & \quad ((z)_k)_1 = ((z)_j)_2 \wedge ((z)_k)_2 = ((z)_i)_2]] \Big\} \end{aligned}$$

E7.1. Prove that

$$\forall a \forall b \forall z \in M \forall i < \text{len}(z) \forall c \forall w \in M \forall j < \text{len}(w) \forall d$$
$$[(z)_i = 2^a \cdot 3^b \cdot 5^c \wedge (w)_j = 2^a \cdot 3^b \cdot 5^d \rightarrow c = d]$$

E7.2. Prove that

$$\forall a \forall b \exists c \exists z \in M \exists i < \text{len}(z)[(z)_i = 2^a \cdot 3^b \cdot 5^c].$$

Now we define $f(a, b)$ to be the c given by exercise E7.2; it is unique by exercise E7.1.

E7.3. Show that f satisfies the conditions of the Ackermann function.

E7.4 Show that M is recursive.

E7.5. Prove that the Ackermann function is recursive.

E7.6. Show that $n < f(m, n)$ for all $m, n \in \omega$. Hint: prove $\forall m \forall n[n < f(m, n)]$ by induction on m; in the inductive step, use induction on n.

E7.7. Show that $f(m, n) < f(m, n + 1)$ for all $m, n \in \omega$.

E7.8. Show that $f(m, n) < f(m, p)$ for all $m, n, p \in \omega$ such that $n < p$.

E7.9. Show that $f(m, n + 1) \leq f(m + 1, n)$ for all $m, n \in \omega$.

E7.10. Show that $f(m, n) < f(m + 1, n)$ for all $m, n \in \omega$.

E7.11. Show that $f(m, n) < f(p, n)$ for all $m, n, p \in \omega$ such that $m < p$.

E7.12. Show that $f(1, y) = y + 2$ for all $y \in \omega$.

E7.13. Show that $f(2, y) = 2y + 3$ for all $y \in \omega$.

E7.14. Show that for every primitive recursive function g, say that g is n-ary, there is a $c \in \omega$ such that for all $x_1, \ldots, x_n \in \omega$ we have $g(x_1, \ldots, x_n) < f(c, \max(x_1, \ldots, x_n)$.

E7.15. Show that f is not primitive recursive.