4. The completeness theorem

The completeness theorem, in its simplest form, says that for any sentence φ, $\vdash \varphi$ iff $\models \varphi$. We already know the direction $\Rightarrow$, in Theorem 3.2.

A more general form of the completeness theorem is that $\Gamma \vdash \varphi$ iff $\Gamma \models \varphi$, for any set $\Gamma \cup \{\varphi\}$ of formulas. Again the direction $\Rightarrow$ is given in Theorem 3.2.

Basic for the proof of the completeness theorem is the notion of consistency. A set Γ of formulas is *consistent* iff there is a formula φ such that $\Gamma \not\vdash \varphi$.

Lemma 4.1. *For any set* Γ *of formulas the following conditions are equivalent:*

(i) Γ *is inconsistent.*

(ii) There is a formula φ *such that* $\Gamma \vdash \varphi$ *and* $\Gamma \vdash \neg\varphi$.

(iii) $\Gamma \vdash \neg(v_0 = v_0)$.

Proof. (i)$\Rightarrow$(ii): Assume (i). Since $\Gamma \vdash \psi$ for every formula ψ, (ii) is clear.

(ii)$\Rightarrow$(iii): Assume (ii). Then the following is a Γ-proof:

A Γ-proof of φ.
A Γ-proof of $\neg\varphi$.
A $\emptyset$-proof of $\varphi \to (\neg\varphi \to \neg(v_0 = v_0)$. (This is a tautology; see Lemma 3.3.)
$\neg\varphi \to \neg(v_0 = v_0)$.
$\neg(v_0 = v_0)$.

(iii)$\Rightarrow$(i): By (iii) we have $\Gamma \vdash \neg(v_0 = v_0)$, while by Proposition 3.4 we have $\Gamma \vdash v_0 = v_0$. $\square$

A *sentence* is a formula which has no variable occurring free in it. A set Γ of sentences *has a model* iff there is a structure $\overline{A}$ for the language in question such that $\overline{A} \models \varphi[a]$ for every $\varphi \in \Gamma$ and every $a : \omega \to A$.

The following first-order version of the deduction theorem, Theorem 1.12, will be useful.

Theorem 4.2. (First-order deduction theorem) *If* $\Gamma \cup \{\psi\}$ *is a set of formulas,* φ *is a sentence, and* $\Gamma \cup \{\varphi\} \vdash \psi$, *then* $\Gamma \vdash \varphi \to \psi$.

Proof. Let $\langle \chi_0, \ldots, \chi_{m-1} \rangle$ be a $(\Gamma \cup \{\varphi\})$-proof with $\chi_i = \psi$ for some $i < m$. We modify this proof, replacing each χ_j by one or more formulas, converting the proof to a Γ-proof, in such a way that $\varphi \to \chi_j$ is in the new proof for every $j < m$. If χ_j is a logical axiom or a member of Γ, we replace it by the three formulas

$$\chi_j \to (\varphi \to \chi_j)$$
$$\chi_j$$
$$\varphi \to \chi_j.$$

If χ_j is φ, we replace it by the five formulas giving a little proof of $\varphi \to \varphi$; see Lemma 1.11. If there exist $k, l < j$ such that χ_k is $\chi_l \to \chi_j$, we replace χ_j by the formulas

$$\begin{aligned}
&(\varphi \to \chi_k) \to [(\varphi \to \chi_l) \to (\varphi \to \chi_j)] \\
&(\varphi \to \chi_l) \to (\varphi \to \chi_j) \\
&\varphi \to \chi_j.
\end{aligned}$$

If there exist $k < j$ and $l \in \omega$ such that χ_j is $\forall v_l \chi_k$, we replace χ_j by the formulas

$$\begin{aligned}
&\forall v_l(\varphi \to \chi_k) \\
&\text{a proof of } \forall v_l(\varphi \to \chi_k) \to (\varphi \to \forall v_l \chi_k) \qquad \text{see Proposition 3.38} \\
&\varphi \to \chi_j.
\end{aligned}$$

□

Theorem 4.3. *Suppose that every consistent set of sentences has a model. Then* $\Gamma \vdash \varphi$ *iff* $\Gamma \models \varphi$, *for every set* $\Gamma \cup \{\varphi\}$ *of formulas.*

Proof. Assume that every consistent set of sentences has a model. Note again that $\Gamma \vdash \varphi$ implies that $\Gamma \models \varphi$, by Theorem 3.2. We prove the converse by proving its contrapositive. Thus suppose that $\Gamma \cup \{\varphi\}$ is a set of formulas such that $\Gamma \nvdash \varphi$. We want to show that $\Gamma \not\models \varphi$, i.e., there is a model of Γ which is not a model of φ. For any formula ψ, let $[\![\psi]\!]$ be the *closure* of ψ, i.e., the sentence

$$\forall v_{i(0)} \ldots \forall v_{i(m-1)} \psi,$$

where $i(0) < \cdots < i(m-1)$ lists all the integers j such that v_j occurs free in ψ. Let $\Gamma' = \{[\![\psi]\!] : \psi \in \Gamma\}$. We claim that $\Gamma' \cup \{\neg[\![\varphi]\!]\}$ is consistent. Suppose not. Then $\Gamma' \cup \{\neg[\![\varphi]\!]\} \vdash \neg(v_0 = v_0)$. Hence by the deduction theorem, $\Gamma' \vdash \neg[\![\varphi]\!] \to \neg(v_0 = v_0)$, so $\Gamma' \vdash v_0 = v_0 \to [\![\varphi]\!]$. Hence, using Proposition 3.4, $\Gamma' \vdash [\![\varphi]\!]$. Now in a Γ'-proof that has $[\![\varphi]\!]$ as a member, replace each formula

$$\forall v_{i(0)} \ldots \forall v_{i(m-1)} \psi,$$

with $\psi \in \Gamma$, by the sequence

$$\begin{aligned}
&\psi \\
&\forall v_{i(m-1)} \psi \\
&\cdots\cdots\cdots \\
&\forall v_{i(0)} \ldots \forall v_{i(m-1)} \psi.
\end{aligned}$$

This converts the proof into a Γ-proof one of whose members is $[\![\varphi]\!]$. Thus $\Gamma \vdash [\![\varphi]\!]$. Using Corollary 3.28, it follows that $\Gamma \vdash \varphi$, contradiction.

Hence $\Gamma' \cup \{\neg[\![\varphi]\!]\}$ is consistent. Since this is a set of sentences, by supposition it has a model $\overline{M}$. Clearly $\overline{M}$ is a model of Γ. Since $\overline{M}$ is a model of $\neg[\![\varphi]\!]$, clearly there is an $a \in {}^{\omega}M$ such that $\overline{M} \models \neg\varphi[a]$. Thus $\overline{M}$ is not a model of φ. This shows that $\Gamma \not\models \varphi$. □

To prove that every consistent set of sentences has a model, we need several lemmas, starting with some additional facts about structures and satisfaction.

Lemma 4.4. *Suppose that $\overline{A}$ is a structure, a and b map ω into A, φ is a formula, and $a_i = b_i$ for every i such that v_i occurs free in φ. Then $\overline{A} \models \varphi[a]$ iff $\overline{A} \models \varphi[b]$.*

Proof. Induction on φ. For φ an atomic equality formula $\sigma = \tau$, the hypothesis means that $a_i = b_i$ for all i such that v_i occurs in σ or τ. Hence, using Proposition 2.4,

$$\overline{A} \models \varphi[a] \text{ iff } \sigma^{\overline{A}}(a) = \tau^{\overline{A}}(a) \text{ iff } \sigma^{\overline{A}}(b) = \tau^{\overline{A}}(b) \text{ iff } \overline{A} \models \varphi[b].$$

For φ an atomic non-equality formula $\mathbf{R}\eta_0 \ldots \eta_{m-1}$, the hypothesis means that $a_i = b_i$ for all i such that v_i occurs in one of the terms η_j. Hence, again using Proposition 2.4,

$$\begin{aligned} \overline{A} \models \varphi[a] \quad &\text{iff} \quad \langle \eta_0^{\overline{A}}(a), \ldots, \eta_{m-1}^{\overline{A}}(a) \rangle \in \mathbf{R}^{\overline{A}} \\ &\text{iff} \quad \langle \eta_0^{\overline{A}}(b), \ldots, \eta_{m-1}^{\overline{A}}(b) \rangle \in \mathbf{R}^{\overline{A}} \\ &\text{iff} \quad \overline{A} \models \varphi[b]. \end{aligned}$$

Assume inductively that φ is $\neg\psi$. The hypothesis implies that $a_i = b_i$ for all i such that v_i occurs free in ψ. Hence

$$\begin{aligned} \overline{A} \models \varphi[a] \quad &\text{iff} \quad \text{not}(\overline{A} \models \psi[a]) \\ &\text{iff} \quad \text{not}(\overline{A} \models \psi[b]) \quad \text{(induction hypothesis)} \\ &\text{iff} \quad \overline{A} \models \varphi[b]. \end{aligned}$$

Assume inductively that φ is $\psi \to \chi$. The hypothesis implies that $a_i = b_i$ for all i such that v_i occurs free in ψ or in χ. Hence

$$\begin{aligned} \overline{A} \models \varphi[a] \quad &\text{iff} \quad \text{not}(\overline{A} \models \psi[a]) \text{ or } \overline{A} \models \chi[a] \\ &\text{iff} \quad \text{not}(\overline{A} \models \psi[b]) \text{ or } \overline{A} \models \chi[b] \quad \text{(induction hypothesis)} \\ &\text{iff} \quad \overline{A} \models \varphi[b]. \end{aligned}$$

Now assume inductively that φ is $\forall v_k \psi$. By symmetry it suffices to show that $\overline{A} \models \varphi[a]$ implies that $\overline{A} \models \varphi[b]$. So, assume that $\overline{A} \models \varphi[a]$. Take any $u \in A$. Then $\overline{A} \models \psi[a^k_u]$. We claim that $(a^k_u)_i = (b^k_u)_i$ for every i such that v_i occurs free in ψ. If $i \neq k$ this is true since v_i also occurs free in φ, so that $a_i = b_i$; and $(a^k_u)_i = a_i = b_i = (b^k_u)_i$. If $i = k$, then $(a^k_u)_i = u = (b^k_u)_i$. It follows now by the inductive hypothesis that $\overline{A} \models \psi[b^k_u]$. Since u is arbitrary, $\overline{A} \models \varphi[b]$. □

As in the case of terms (see Proposition 2.4 and the comments after it), Lemma 4.4 enables us to simplify the notation $\overline{A} \models \varphi[a]$. Instead of a full assignment $a : \omega \to A$, it suffices to take a function $a : \{0, \ldots, m\} \to A$ such that every variable v_i occurring free in φ is such that $i \leq m$. Then $\overline{A} \models \varphi[a]$ means that $\overline{A} \models \varphi[b]$ for any b (or some b) such that b extends a. If φ is a sentence, thus with no free variables, then $\overline{A} \models \varphi$ means that $\overline{A} \models \varphi[b]$ for any, or some, $b : \omega \to A$.

Lemma 4.5. *Suppose that τ, ρ, and ν are terms, and ρ is obtained from τ by replacing all occurrences of v_i in τ by ν. Then for any structure $\overline{A}$ and any assignment $a : \omega \to A$, $\rho^{\overline{A}}(a) = \tau^{\overline{A}}\left(a^i_{\nu^{\overline{A}}(a)}\right)$.*

Proof. By induction on τ. If τ is v_k with $k \neq i$, then ρ is the same as τ, and both sides of the above equation are equal to a_k. If τ is v_i, then ρ is ν, and $\rho^{\overline{A}}(a) = \nu^{\overline{A}}(a) = v_i^{\overline{A}}\left(a^i_{\nu^{\overline{A}}(a)}\right) = \tau^{\overline{A}}\left(a^i_{\nu^{\overline{A}}(a)}\right)$. If τ is an individual constant $\mathbf{k}$, then ρ is equal to τ, and both sides of the equation in the lemma are equal to $\mathbf{k}^{\overline{A}}$.

Now suppose inductively that τ is $\mathbf{F}\eta_0 \ldots \eta_{m-1}$. Let μ_i be obtained from η_i by replacing all occurrences of v_i by ν. Then

$$\begin{aligned}
\rho^{\overline{A}}(a) &= (\mathbf{F}\mu_0 \ldots \mu_{m-1})^{\overline{A}}(a) \\
&= \mathbf{F}^{\overline{A}}(\mu_0^{\overline{A}}(a), \ldots, \mu_{m-1}^{\overline{A}}(a)) \\
&= \mathbf{F}^{\overline{A}}\left(\eta_0\left(a^i_{\nu^{\overline{A}}(a)}\right), \ldots, \eta_{m-1}\left(a^i_{\nu^{\overline{A}}(a)}\right)\right) \\
&= (\mathbf{F}\eta_0 \ldots \eta_{m-1})\left[a^i_{\nu^{\overline{A}}(a)}\right] \\
&= \tau^{\overline{A}}\left(a^i_{\nu^{\overline{A}}(a)}\right).
\end{aligned}$$

□

Lemma 4.6. *Suppose that φ is a formula, ν is a term, no free occurrence of v_i in φ is within a subformula of the form $\forall v_k \mu$ with v_k a variable occurring in ν, and $\overline{A}$ is a structure. Then $\overline{A} \models \mathrm{Subf}^{v_i}_{\nu}\varphi[a]$ iff $\overline{A} \models \varphi\left[a^i_{\nu^{\overline{A}}(a)}\right]$.*

Proof. By induction on φ. For φ a formula $\sigma = \tau$, let ρ and η be obtained from σ and τ by replacing all occurrences of v_i by ν. Then by Lemma 4.5,

$$\begin{aligned}
\overline{A} \models \mathrm{Subf}^{v_i}_{\nu}\varphi[a] \quad &\text{iff} \quad \overline{A} \models (\rho = \eta)[a] \\
&\text{iff} \quad \rho^{\overline{A}}(a) = \eta^{\overline{A}}(a) \\
&\text{iff} \quad \sigma^{\overline{A}}\left(a^i_{\nu^{\overline{A}}(a)}\right) = \tau^{\overline{A}}\left(a^i_{\nu^{\overline{A}}(a)}\right) \\
&\text{iff} \quad \overline{A} \models (\sigma = \tau)\left(a^i_{\nu^{\overline{A}}(a)}\right) \\
&\text{iff} \quad \overline{A} \models \varphi\left(a^i_{\nu^{\overline{A}}(a)}\right).
\end{aligned}$$

For φ a formula $\mathbf{R}\sigma_0 \ldots \sigma_{m-1}$, let η_i be obtained from σ_i by replacing all occurrences of v_i by ν. Then

$$\begin{aligned}
\overline{A} \models \mathrm{Subf}^{v_i}_{\nu}\varphi[a] \quad &\text{iff} \quad \overline{A} \models (\mathbf{R}\eta_0 \ldots \eta_{m-1})[a] \\
&\text{iff} \quad \langle \eta_0^{\overline{A}}(a), \ldots, \eta_{m-1}^{\overline{A}}(a) \in \mathbf{R}^{\overline{A}} \\
&\text{iff} \quad \left\langle \sigma_0^{\overline{A}}\left(a^i_{\nu^{\overline{A}}(a)}\right), \ldots \sigma_{m-1}^{\overline{A}}\left(a^i_{\nu^{\overline{A}}(a)}\right)\right\rangle \in \mathbf{R}^{\overline{A}} \\
&\text{iff} \quad \overline{A} \models (\mathbf{R}\sigma_0 \ldots \sigma_{m-1})\left[a^i_{\nu^{\overline{A}}(a)}\right] \\
&\text{iff} \quad \overline{A} \models \varphi\left[a^i_{\nu^{\overline{A}}(a)}\right].
\end{aligned}$$

Now suppose inductively that φ is $\neg\psi$. Then

$$\begin{aligned}
\overline{A} \models \mathrm{Subf}^{v_1}_{\nu}\varphi[a] \quad &\text{iff} \quad \overline{A} \models (\neg \mathrm{Subf}^{v_1}_{\nu}\psi)\,[a] \\
&\text{iff} \quad \text{not}\left(\overline{A} \models (\mathrm{Subf}^{v_1}_{\nu}\psi)\right)[a] \\
&\text{iff} \quad \text{not}\left(\overline{A} \models \psi\left[a^i_{\nu^{\overline{A}}(a)}\right]\right) \\
&\text{iff} \quad \overline{A} \models \varphi\left[a^i_{\nu^{\overline{A}}(a)}\right].
\end{aligned}$$

Suppose inductively that φ is $\psi \to \chi$. Then

$$\begin{aligned}
\overline{A} \models \mathrm{Subf}^{v_1}_{\nu}\varphi[a] \quad &\text{iff} \quad \text{not}\left(\overline{A} \models \mathrm{Subf}^{v_1}_{\nu}\psi[a]\right) \text{ or } \overline{A} \models \mathrm{Subf}^{v_1}_{\nu}\chi[a] \\
&\text{iff} \quad \text{not}\left(\overline{A} \models \psi\left[a^i_{\nu^{\overline{A}}(a)}\right]\right) \text{ or } \overline{A} \models \chi\left[a^i_{\nu^{\overline{A}}(a)}\right] \\
&\text{iff} \quad \overline{A} \models \varphi\left[a^i_{\nu^{\overline{A}}(a)}\right].
\end{aligned}$$

Finally, suppose inductively that φ is $\forall v_k\psi$. Now if v_i does not occur free in φ, then $\mathrm{Subf}^{v_i}_{\nu}\varphi$ is just φ itself, and $\overline{A} \models \varphi[a]$ iff $\overline{A} \models \varphi[a^i_{\nu^{\overline{A}}(a)}]$ by Lemma 4.4. Hence we may assume that v_i occurs free in φ.

If $k = i$, then $\mathrm{Subf}^{v_i}_{\nu}\varphi$ is φ, and by Lemma 4.4, $\overline{A} \models \varphi\left[a^i_{\nu^{\overline{A}}(a)}\right]$ iff $\overline{A} \models \varphi[a]$; so the theorem holds in this case. Now suppose that $k \neq i$. Then $\mathrm{Subf}^{v_i}_{\nu}\varphi$ is $\forall v_k \mathrm{Subf}^{v_i}_{\nu}\psi$. Suppose that $\overline{A} \models \mathrm{Subf}^{v_i}_{\nu}\varphi[a]$. Take any $u \in A$. Then $\overline{A} \models \mathrm{Subf}^{v_i}_{\nu}\psi[a^k_u]$. Now no free occurrence of v_i in ψ is within a subformula of the form $\forall v_s\mu$ with v_s occurring in ν. Hence by the inductive hypothesis $\overline{A} \models \psi\left[(a^k_u)^i_{\nu^{\overline{A}}(a^k_u)}\right]$. Now since φ is $\forall v_k\psi$ and v_i occurs free in φ, the assumption of the lemma says that v_k does not occur in ν. Hence $\nu^{\overline{A}}(a) = \nu^{\overline{A}}(a^k_u)$ by Proposition 2.4. Hence $\overline{A} \models \psi\left[(a^k_u)^i_{\nu^{\overline{A}}(a)}\right]$. Since $\left(a^k_u\right)^i_{\nu^{\overline{A}}(a)} = \left(a^i_{\nu^{\overline{A}}(a)}\right)^k_u$, it follows that $\overline{A} \models \varphi\left[a^i_{\nu^{\overline{A}}(a)}\right]$.

Conversely, suppose that $\overline{A} \models \varphi\left[a^i_{\nu^{\overline{A}}(a)}\right]$. Take any $u \in A$. Then $\overline{A} \models \psi\left[\left(a^i_{\nu^{\overline{A}}(a)}\right)^k_u\right]$. Since $\left(a^i_{\nu^{\overline{A}}(a)}\right)^k_u = \left(a^k_u\right)^i_{\nu^{\overline{A}}(a)}$, and $\nu^{\overline{A}}(a) = \nu^{\overline{A}}(a^k_u)$ (see above), by the inductive hypothesis we get $\overline{A} \models \mathrm{Subf}^{v_i}_{\nu}\psi[a^k_u]$. It follows that $\overline{A} \models \mathrm{Subf}^{v_i}_{\nu}\varphi[a]$. $\square$

A set Γ of sentences is *complete* iff for every sentence φ, $\Gamma \vdash \varphi$ or $\Gamma \vdash \neg\varphi$. Γ is *rich* iff for every sentence of the form $\exists v_i\varphi$ there is an individual constant $\mathbf{c}$ such that $\Gamma \vdash \exists v_i\varphi \to \mathrm{Subf}^{v_i}_{\mathbf{c}}(\varphi)$.

The main lemma for the completeness proof is as follows.

Lemma 4.7. *If Γ is a complete, rich, consistent set of sentences, then Γ has a model.*

Proof. Let $B = \{\sigma : \sigma \text{ is a term in which no variable occurs}\}$. We define $\equiv$ to be the set

$$\{(\sigma, \tau) : \sigma, \tau \in B \text{ and } \Gamma \vdash \sigma = \tau\}.$$

By Propositions 3.4–3.6, $\equiv$ is an equivalence relation on B. Let π be the function which assigns to each $\sigma \in B$ the equivalence class $[\sigma]_\equiv$, and let A be the set of all equivalence classes.

We recall some basic facts about equivalence relations. An *equivalence relation* on a set M is a set R of ordered pairs (a,b) with $a,b \in M$ satisfying the following conditions:

(reflexivity) $(a,a) \in R$ for all $a \in M$.

(symmetry) For all $(a,b) \in R$ we have $(b,a) \in R$.

(transitivity) For all a,b,c, if $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$.

Given an equivalence relation R on a set M, for each $a \in M$ we let $[a]_R = \{b \in M : (a,b) \in R\}$; this is the *equivalence class* of a. Some basic facts:

(a) For any $a,b \in M$, $(a,b) \in R$ iff $[a]_R = [b]_R$.

Proof. $\Rightarrow$: suppose that $(a,b) \in R$. Suppose also that $x \in [a]_R$. Thus $(a,x) \in R$. Since R is symmetric, $(b,a) \in R$. Since R is transitive, $(b,x) \in R$. Hence $x \in [b]_R$. This proves that $[a]_R \subseteq [b]_R$. Suppose that $x \in [b]_R$. Thus $(b,x) \in R$. Since also $(a,b) \in R$, by transitivity we get $(a,x) \in R$. So $x \in [a]_R$. This proves that $[b]_R \subseteq [a]_R$, and completes the proof that $[a]_R = [b]_R$.

$\Leftarrow$: Assume that $[a]_R = [b]_R$. Since R is reflexive on M, we have $(b,b) \in R$, and hence $b \in [b]_R$. Now $[a]_R = [b]_R$, so $b \in [a]_R$. Hence $(a,b) \in R$. $\square$

(b) For any $a,b \in M$, $[a]_R = [b]_R$ or $[a]_R \cap [b]_R = \emptyset$.

Proof. Suppose that $[a]_R \cap [b]_R \neq \emptyset$; say $x \in [a]_R \cap [b]_R$. Thus $(a,x) \in R$ and $(b,x) \in R$. By symmetry, $(x,b) \in R$. By transitivity, $(a,b) \in R$. By (a), $[a]_R = [b]_R$. $\square$

We are now going to define a structure with universe A. If $\mathbf{k}$ is an individual constant, let $\mathbf{k}^{\overline{A}} = [\mathbf{k}]_\equiv$.

(1) If $\mathbf{F}$ is an m-ary function symbol and $\sigma_0, \ldots, \sigma_{m-1}, \tau_0, \ldots, \tau_{m-1}$ are members of B such that $\sigma_i \equiv \tau_i$ for all $i < m$, then $\mathbf{F}\sigma_0 \ldots \sigma_{m-1} \equiv \mathbf{F}\tau_0 \ldots \tau_{m-1}$.

In fact, the hypothesis implies that $\Gamma \vdash \sigma_i = \tau_i$ for all $i < m$. By Proposition 3.7,

$$\vdash \bigwedge_{i<m} (\sigma_i = \tau_i) \rightarrow \mathbf{F}\sigma_0 \ldots \sigma_{m-1} = \mathbf{F}\tau_0 \ldots \tau_{m-1};$$

it follows that $\Gamma \vdash \mathbf{F}\sigma_0 \ldots \sigma_{m-1} = \mathbf{F}\tau_0 \ldots \tau_{m-1}$, so that $\mathbf{F}\sigma_0 \ldots \sigma_{m-1} \equiv \mathbf{F}\tau_0 \ldots \tau_{m-1}$.

(2) If $\mathbf{F}$ is an m-ary function symbol, then there is a function $\mathbf{F}^{\overline{A}}$ mapping m-tuples of members of A into A, such that for any $\sigma_0, \ldots, \sigma_{m-1} \in B$, $\mathbf{F}^{\overline{A}}([\sigma_0]_\equiv, \ldots, [\sigma_{m-1}]_\equiv) = [\mathbf{F}\sigma_0 \ldots \sigma_{m-1}]_\equiv$.

In fact, we can define $\mathbf{F}^{\overline{A}}$ as a set of ordered pairs:

$$\begin{aligned}\mathbf{F}^{\overline{A}} = \{(x,y) : &\text{there are } \sigma_0, \ldots, \sigma_{m-1} \in B \text{ such that} \\ &x = \langle [\sigma_0]_\equiv, \ldots, [\sigma_{m-1}]_\equiv \rangle \text{ and } y = [\mathbf{F}\sigma_0 \ldots \sigma_{m-1}]_\equiv\}\end{aligned}$$

Then $\mathbf{F}^{\overline{A}}$ is a function. For, suppose that $(x, y), (x, z) \in \mathbf{F}^{\overline{A}}$. Accordingly choose elements $\sigma_0, \ldots \sigma_{m-1} \in B$ and $\tau_0, \ldots \tau_{m-1} \in B$ such that $x = \langle [\sigma_0]_\equiv, \ldots, [\sigma_{m-1}]_\equiv \rangle = \langle [\tau_0]_\equiv, \ldots, [\tau_{m-1}]_\equiv$, $y = [\mathbf{F}\sigma_0 \ldots \sigma_{m-1}]_\equiv$, and $z = [\mathbf{F}\tau_0 \ldots \varphi_{m-1}]_\equiv$. Thus for any $i < m$ we have $[\sigma_i]_\equiv = [\tau_i]_\equiv$, hence $\sigma_i \equiv \tau_i$. From (1) it then follows that $\mathbf{F}\sigma_0 \ldots \sigma_{m-1} \equiv \mathbf{F}\tau_0 \ldots \varphi_{m-1}$, hence $y = z$. So $\mathbf{F}^{\overline{A}}$ is a function. Clearly then (2) holds.

For $\mathbf{R}$ and m-ary relation symbol we define

$$\mathbf{R}^{\overline{A}} = \{x : \exists \sigma_0, \ldots \sigma_{m-1} \in B[x = \langle [\sigma_0]_\equiv, \ldots, [\sigma_{m-1}]_\equiv \rangle \text{ and } \Gamma \vdash \mathbf{R}\sigma_0 \ldots \sigma_{m-1}]\}.$$

(3) If $\mathbf{R}$ is an m-ary relation symbol and $\sigma_0, \ldots, \sigma_{m-1} \in B$, then $\langle [\sigma_0]_\equiv, \ldots, [\sigma_{m-1}]_\equiv \rangle \in \mathbf{R}^{\overline{A}}$ iff $\Gamma \vdash \mathbf{R}\sigma_0 \ldots \sigma_{m-1}$.

In fact, $\Leftarrow$ follows from the definition. Now suppose that $\langle [\sigma_0]_\equiv, \ldots, [\sigma_{m-1}]_\equiv \rangle \in \mathbf{R}^{\overline{A}}$. Then by definition there exist $\tau_0, \ldots, \tau_{m-1} \in B$ such that

$$\langle [\sigma_0]_\equiv, \ldots, [\sigma_{m-1}]_\equiv \rangle = \langle [\tau_0]_\equiv, \ldots, [\tau_{m-1}]_\equiv \rangle \text{ and } \Gamma \vdash \mathbf{R}\tau_0 \ldots \tau_{m-1}.$$

Thus $[\sigma_i]_\equiv = [\tau_i]_\equiv$, hence $\sigma_i \equiv \tau_i$, hence $\Gamma \vdash \sigma_i = \tau_i$, for each $i < m$. Now by Proposition 3.8, $\vdash \bigwedge_{i<m}(\sigma_i = \tau_i) \to (\mathbf{R}\sigma_0 \ldots \sigma_{m-1} \leftrightarrow \mathbf{R}\tau_0 \ldots \tau_{m-1})$. It follows that $\Gamma \vdash \mathbf{R}\sigma_0 \ldots \sigma_{m-1}$, as desired; so (3) holds.

(4) For any $\sigma \in B$ we have $\sigma^{\overline{A}} = [\sigma]_\equiv$.

We prove (4) by induction on σ. If σ is an individual constant $\mathbf{k}$, then by definition $\mathbf{k}^{\overline{A}} = [\mathbf{k}]_\equiv$. Now suppose that (4) is true for $\tau_0, \ldots, \tau_{m-1} \in B$ and σ is $\mathbf{F}\tau_0 \ldots \tau_{m-1}$. Then

$$\sigma^{\overline{A}} = \mathbf{F}^{\overline{A}}([\tau_0]_\equiv, \ldots, [\tau_{m-1}]_\equiv) = [\mathbf{F}\tau_0 \ldots \tau_{m-1}]_\equiv = [\sigma]_\equiv,$$

proving (4).

The following claim is the heart of the proof.

(5) For any sentence φ, $\Gamma \vdash \varphi$ iff $\overline{A} \models \varphi$.

We prove (5) by induction on the number m of the symbols $=$, relation symbols, $\neg$, $\to$, and $\forall$ in φ. For $m = 1$, φ is atomic, and we have

$$\begin{aligned}
\Gamma \vdash \sigma = \tau \quad & \text{iff} \quad \sigma \equiv \tau \\
& \text{iff} \quad [\sigma]_\equiv = [\tau]_\equiv \\
& \text{iff} \quad \sigma^{\overline{A}} = \tau^{\overline{A}} \quad \text{by (4)} \\
& \text{iff} \quad \overline{A} \models \sigma = \tau; \\
\Gamma \vdash \mathbf{R}\sigma_0 \ldots \sigma_{m-1} \quad & \text{iff} \quad \langle [\sigma_0]_\equiv, \ldots, [\sigma_{m-1}]_\equiv \rangle \in \mathbf{R}^{\overline{A}} \quad \text{by (3)} \\
& \text{iff} \quad \langle \sigma_0^{\overline{A}}, \ldots, \sigma_{m-1}^{\overline{A}} \rangle \in \mathbf{R}^{\overline{A}} \quad \text{by (4)} \\
& \text{iff} \quad \overline{A} \models \mathbf{R}\sigma_0 \ldots \sigma_{m-1}.
\end{aligned}$$

Now we take the inductive steps.

$$\begin{aligned}
\Gamma \vdash \neg\psi \quad &\text{iff} \quad \text{not}(\Gamma \vdash \psi) \\
&\text{iff} \quad \text{not}(\overline{A} \models \psi) \\
&\text{iff} \quad \overline{A} \models \neg\psi; \\
\Gamma \vdash \psi \to \chi \quad &\text{iff} \quad \text{not}(\Gamma \vdash \psi) \text{ or } \Gamma \vdash \chi \\
&\text{iff} \quad \text{not}(\overline{A} \models \psi) \text{ or } \overline{A} \models \chi \\
&\text{iff} \quad \overline{A} \models \psi \to \chi.
\end{aligned}$$

Finally, suppose that φ is $\forall v_i\psi$. First suppose that $\Gamma \vdash \varphi$. We want to show that $\overline{A} \models \varphi$, so take any $\sigma \in B$ and let $u = [\sigma]_\equiv$; we want to show that $\overline{A} \models \psi[a^i_u]$, where $a : \omega \to A$. Let χ be the sentence $\text{Subf}^{v_i}_\sigma\psi$. Then by Theorem 3.27 we have $\Gamma \vdash \chi$, and hence by the inductive assumption $\overline{A} \models \chi$. By (4) we have $\sigma^{\overline{A}} = [\sigma]_\equiv$. Hence by Lemma 4.6 we get $\overline{A} \models \psi[a^i_u]$.

Second suppose that $\Gamma \not\vdash \varphi$. Then by completeness $\Gamma \vdash \neg\varphi$, and hence $\Gamma \vdash \exists v_i\neg\psi$. Hence by richness there is an individual constant $\mathbf{c}$ such that $\Gamma \vdash \exists v_i\neg\psi \to \text{Subf}^{v_i}_{\mathbf{c}}(\neg\psi)$, hence $\Gamma \vdash \neg\text{Subf}^{v_i}_{\mathbf{c}}\psi$, and so $\Gamma \not\vdash \text{Subf}^{v_i}_{\mathbf{c}}\psi$. By the inductive assumption, $\overline{A} \not\models \text{Subf}^{v_i}_{\mathbf{c}}\psi$, and so by (4) and Lemma 4.6, $\overline{A} \not\models \psi[a^i_u]$, where $a : \omega \to A$ and $u = [\mathbf{c}]_\equiv$. So $\overline{A} \not\models \varphi$.

This finishes the proof of (5). Applying (5) to members φ of Γ we see that $\overline{A}$ is a model of Γ. $\square$

The following rather technical lemma will be used in a few places below.

Lemma 4.8. *Suppose that Γ is a set of formulas in $\mathscr{L}$, and $\langle\psi_0, \ldots, \psi_{m-1}\rangle$ is a Γ-proof in $\mathscr{L}$. Suppose that C is a set of individual constants such that no member of C occurs in any member of Γ. Let v_j be a variable not occurring in any formula ψ_k, and for each k let ψ'_k be obtained from ψ_k by replacing each member of C by v_j. Similarly, for each term σ let σ' be obtained from σ by replacing each member of C by v_j. Then $\langle\psi'_0, \psi'_1, \ldots, \psi'_{m-1}\rangle$ is a Γ-proof in $\mathscr{L}$.*

Proof. Assume the hypotheses. We need to show that if ψ_k is a logical axiom, then so is ψ'_k. We consider the possibilities one by one:

$(\varphi \to (\psi \to \varphi))'$ is $\varphi' \to (\psi' \to \varphi')$;
$((\varphi \to (\psi \to \chi) \to ((\varphi \to \psi) \to (\varphi \to \chi)))'$ is
$\quad(\varphi' \to (\psi' \to \chi') \to ((\varphi' \to \psi') \to (\varphi' \to \chi'))$;
$((\neg\varphi \to \neg\psi) \to (\psi \to \varphi))'$ is $(\neg\varphi' \to \neg\psi') \to (\psi' \to \varphi')$;
$(\forall v_k(\varphi \to \psi) \to (\forall v_k\varphi \to \forall v_k\psi))'$ is $\forall v_k(\varphi' \to \psi') \to (\forall v_k\varphi' \to \forall v_k\psi')$;
$(\varphi \to \forall v_k\varphi)'$ is $\varphi' \to \forall v_k\varphi'$ if v_k does not occur in φ;
$(\exists v_k(v_k = \sigma))'$ is $\exists v_k(v_k = \sigma')$ if v_k does not occur in σ;
$(\sigma = \tau \to (\sigma = \rho \to \tau = \rho))'$ is $(\sigma' = \tau' \to (\sigma' = \rho' \to \tau' = \rho')$;
$(\sigma = \tau \to (\rho = \sigma \to \rho = \tau))'$ is $(\sigma' = \tau' \to (\rho' = \sigma' \to \rho' = \tau')$;

$$(\sigma = \tau \to \mathbf{F}\xi_0 \ldots \xi_{i-1}\sigma\xi_{i+1} \ldots \xi_{m-1} = \mathbf{F}\xi_0 \ldots \xi_{i-1}\tau\xi_{i+1} \ldots \xi_{m-1})' \text{ is}$$
$$\sigma' = \tau' \to \mathbf{F}\xi'_0 \ldots \xi'_{i-1}\sigma'\xi'_{i+1} \ldots \xi'_{m-1} = \mathbf{F}\xi'_0 \ldots \xi'_{i-1}\tau'\xi'_{i+1} \ldots \xi'_{m-1};$$
$$(\sigma = \tau \to (\mathbf{R}\xi_0 \ldots \xi_{i-1}\sigma\xi_{i+1} \ldots \xi_{m-1} \to \mathbf{R}\xi_0 \ldots \xi_{i-1}\tau\xi_{i+1} \ldots \xi_{m-1}))' \text{ is}$$
$$\sigma' = \tau' \to (\mathbf{R}\xi'_0 \ldots \xi'_{i-1}\sigma'\xi'_{i+1} \ldots \xi'_{m-1} \to \mathbf{R}\xi'_0 \ldots \xi'_{i-1}\tau'\xi'_{i+1} \ldots \xi'_{m-1}).$$

Now back to our claim that $\langle \psi'_0, \ldots, \psi'_{m-1}\rangle$ is a Γ-proof. If ψ_k is a logical axiom, then by the above, ψ'_k is a logical axiom. If $\psi_k \in \Gamma$, then no member of C occurs in ψ_k, and hence $\psi'_k = \psi_k$. Suppose that $s, t < k$ and ψ_s is $\psi_t \to \psi_k$. Then ψ'_s is $\psi'_t \to \psi'_k$. If $s < k$ and $t \in \omega$, and ψ_k is $\forall v_t \psi_s$, then ψ'_k is $\forall v_t \psi'_s$. Thus our claim holds. □

Lemma 4.9. *Suppose that* $\mathbf{c}$ *is an individual constant not occurring in any formula in* $\Gamma \cup \{\varphi\}$. *Suppose that* $\Gamma \vdash \mathrm{Subf}^{v_i}_{\mathbf{c}}\varphi$. *Then* $\Gamma \vdash \varphi$.

Proof. Let $\langle \psi_0, \ldots, \psi_{m-1}\rangle$ be a Γ-proof with $\psi_j = \mathrm{Subf}^{v_i}_{\mathbf{c}}\varphi$. Let v_j and the sequence $\langle \psi'_0, \ldots, \psi'_{m-1}\rangle$ be as in Lemma 4.8, with $C = \{\mathbf{c}\}$. Then by Lemma 4.8, $\langle \psi'_0, \ldots, \psi'_{m-1}\rangle$ is a Γ-proof. Note that ψ'_j is $\mathrm{Subf}^{v_i}_{v_j}\varphi$. Thus $\Gamma \vdash \mathrm{Subf}^{v_i}_{v_j}\varphi$. Hence $\Gamma \vdash \forall v_j \mathrm{Subf}^{v_i}_{v_j}\varphi$, and so by Theorem 3.27, $\Gamma \vdash \varphi$. □

A first-order language $\mathscr{L}$ is *finite* iff $\mathscr{L}$ has only finitely many non-logical symbols. Note that in a finite language there are infinitely many integers which are not symbols of the language. We prove the main completeness theorem only for finite languages. This is not an essential restriction. With an expanded notion of first-order language the present proof still goes through.

Lemma 4.10. *Let* $\mathscr{L}$ *be a finite first-order language. Let* $\mathscr{L}'$ *extend* $\mathscr{L}$ *by adding individual constants* $\mathbf{c}_0, \mathbf{c}_1, \ldots$. *Suppose that* Γ *is a consistent set of formulas in* $\mathscr{L}$. *Then it is also consistent as a set of formulas in* $\mathscr{L}'$.

Suppose not. Let $\langle \psi_0, \ldots, \psi_{m-1}\rangle$ be a Γ-proof in the $\mathscr{L}'$ sense with ψ_i the formula $\neg(v_0 = v_0)$. Let C be the set of all constants $\mathbf{c}_i$ which appear in some formula ψ_k. Let v_j and $\langle \psi'_0, \psi'_1, \ldots, \psi'_{m-1}\rangle$ be as in Lemma 4.8. Then by Lemma 4.8, $\langle \psi'_0, \psi'_1, \ldots, \psi'_{m-1}\rangle$ is a Γ-proof. Clearly each ψ'_k is a $\mathscr{L}$ formula. Note that $\psi'_i = \psi_i = \neg(v_0 = v_0)$. So Γ is inconsistent in $\mathscr{L}$, contradiction. □

Lemma 4.11. *Let* $\mathscr{L}$ *be a finite first-order language. Let* $\mathscr{L}'$ *extend* $\mathscr{L}$ *by adding individual constants* $\mathbf{c}_0, \mathbf{c}_1, \ldots$.

Then there is an enumeration $\langle \varphi_0, \varphi_1, \ldots\rangle$ *of all of the sentences of* $\mathscr{L}'$, *and also an enumeration* $\langle \psi_0, \psi_1, \ldots\rangle$ *of all the sentences of* $\mathscr{L}'$ *of the form* $\exists v_i \chi$.

Proof. Recall that a formula is a certain finite sequence of positive integers. First we describe how to list all finite sequences of positive integers. Given positive integers m and n, we can list all sequences of members of $\overline{m}$ of length n by just listing them in dictionary order. For example, with $m = 3$ and $n = 2$ our list is

$$\langle 1, 1\rangle$$
$$\langle 1, 2\rangle$$

$$\langle 1,3\rangle$$
$$\langle 2,1\rangle$$
$$\langle 2,2\rangle$$
$$\langle 2,3\rangle$$
$$\langle 3,1\rangle$$
$$\langle 3,2\rangle$$
$$\langle 3,3\rangle$$

To list all finite sequences, we then do the following:

(1) List all sequences of members of $\overline{1}$ of length 1. (There is only one such, namely $\langle 1\rangle$.)

(2) List all sequences of members of $\overline{2}$ of length 1 or 2. Here they are:

$$\langle 1\rangle$$
$$\langle 2\rangle$$
$$\langle 1,1\rangle$$
$$\langle 1,2\rangle$$
$$\langle 2,1\rangle$$
$$\langle 2,2\rangle$$

(3) List all sequences of members of $\overline{3}$ of length 1,2, or 3.

(4) General step: list all members of $\overline{m}$ of length $1, 2, \ldots m$.

Let $\langle \psi_0, \psi_1, \ldots\rangle$ be the listing described. Now we go through this list and select the ones which are sentences of $\mathscr{L}'$, giving the desired list $\langle \varphi_0, \varphi_1, \ldots\rangle$. Similarly for the list $\langle \psi_0, \psi_1, \ldots\rangle$ of all sentences of the form $\exists v_i\chi$. $\square$

Lemma 4.12. *Let $\mathscr{L}$ be a finite first-order language. Let $\mathscr{L}'$ extend $\mathscr{L}$ by adding individual constants $\mathbf{c}_0, \mathbf{c}_1, \ldots$.*

Suppose that Γ is a consistent set of sentences of $\mathscr{L}'$. Then there is a rich consistent set Δ of sentences with $\Gamma \subseteq \Delta$.

Proof. By Lemma 4.11, let $\langle \psi_0, \psi_1, \ldots\rangle$ enumerate all the sentences of $\mathscr{L}'$ of the form $\exists v_i\chi$; say that ψ_k is $\exists v_{t(k)}\psi'_k$ for all $k \in \omega$. Now we define an increasing sequence $\langle j(k) : k \in \omega\rangle$ by recursion. Suppose that $j(k)$ has been defined for all $k < l$. Let $j(l)$ be the smallest natural number not in the set

$$\{j(k) : k < l\} \cup \{s : \mathbf{c}_s \text{ occurs in some formula } \psi_k \text{ with } k \leq l\}.$$

For each $l \in \omega$ let

$$\Theta_l = \Gamma \cup \{\exists v_{t(k)}\psi'_k \rightarrow \mathrm{Subf}^{v_{t(k)}}_{\mathbf{c}_{j(k)}}\psi'_k : k < l\}.$$

We claim that each set Θ_l is consistent. We prove this by induction on l. Note that $\Theta_0 = \Gamma$, which is given as consistent. Now suppose that we have shown that Θ_l is consistent.

Now $\Theta_{l+1} = \Theta_l \cup \{\exists v_{t(l)}\psi'_l \to \mathrm{Subf}^{v_{t(l)}}_{\mathbf{c}_{j(l)}}\psi'_l\}$. Assume that Θ_{l+1} is inconsistent. Then $\Theta_{l+1} \vdash \neg(v_0 = v_0)$. By the deduction theorem 4.2, it follows that

$$\Theta_l \vdash (\exists v_{t(l)}\psi'_l \to \mathrm{Subf}^{v_{t(l)}}_{\mathbf{c}_{j(l)}}\psi'_l) \to \neg(v_0 = v_0),$$

hence easily

$$\Theta_l \vdash \neg(\exists v_{t(l)}\psi'_l \to \mathrm{Subf}^{v_{t(l)}}_{\mathbf{c}_{j(l)}}\psi'_l),$$

so that using tautologies

$$\Theta_l \vdash \exists v_{t(l)}\psi'_l \quad \text{and}$$
$$\Theta_l \vdash \neg\mathrm{Subf}^{v_{t(l)}}_{\mathbf{c}_{j(l)}}\psi'_l.$$

Now by the definition of the sequence $\langle j(k) : k \in \omega\rangle$, it follows that $\mathbf{c}_{j(l)}$ does not occur in any formula in $\Theta_l \cup \{\psi'_l\}$. Hence by Lemma 4.9 we get $\Theta_l \vdash \neg\psi'_l$, and so $\Theta_l \vdash \forall v_{t(l)}\neg\psi'_l$. But we also have $\Theta_l \vdash \exists v_{t(l)}\psi'_l$, so that Θ_l is inconsistent, contradiction.

Now let $\Delta = \bigcup_{l\in\omega}\Theta_l$. We claim that Δ is consistent. Suppose not. Then $\Delta \vdash \neg(v_0 = v_0)$. Let $\langle\varphi_0, \ldots \varphi_{m-1}\rangle$ be a Δ-proof with $\varphi_i = \neg(v_0 = v_0)$. For each $k < m$ such that $\varphi_k \in \Delta$, choose $s(k) \in \omega$ such that $\varphi_k \in \Theta_{s(k)}$. Let l be such that $s(l)$ is largest among all $k < m$ such that $\varphi_k \in \Theta_{s(k)}$. Then $\langle\varphi_0, \ldots \varphi_{m-1}\rangle$ is a $\Theta_{s(l)}$-proof, and hence $\Theta_{s(l)}$ is inconsistent, contradiction.

Now clearly $\Gamma \subseteq \Delta$, since $\Theta_0 = \Gamma$. We claim that Δ is rich. For, let $\exists v_l\chi$ be a sentence. Say $\exists v_l\chi$ is ψ_m. Then $\exists v_l\chi$ is $\exists v_{t(m)}\psi'_m$, so that $l = t(m)$ and $c = \psi'_m$. Now the formula

$$\exists v_{t(m)}\psi'_m \to \mathrm{Subf}^{v_{t(m)}}_{c_{j(m)}}\psi'_m$$

is a member of Θ_{m+1}, and hence is a member of Δ. This formula is $\exists v_l\chi \to \mathrm{Subf}^{v_l}_{c_{j(m)}}\chi$. Hence Δ is rich. □

Lemma 4.13. *Let $\mathscr{L}$ be a finite first-order language. Let $\mathscr{L}'$ extend $\mathscr{L}$ by adding individual constants $\mathbf{c}_0, \mathbf{c}_1, \ldots$.*

Suppose that Γ is a consistent set of sentences of $\mathscr{L}'$. Then there is a consistent complete set Δ of sentences with $\Gamma \subseteq \Delta$.

Proof. By Lemma 4.11, let $\langle\varphi_0, \varphi_1, \ldots\rangle$ be an enumeration of all the sentences of $\mathscr{L}'$. We now define by recursion sets Θ_i of sentences. Let $\Theta_0 = \Gamma$. Suppose that Θ_i has been defined so that it is consistent. If $\Theta_i \cup \{\varphi_i\}$ is consistent, let $\Theta_{i+1} = \Theta_i \cup \{\varphi_i\}$. Otherwise let $\Theta_{i+1} = \Theta_i \cup \{\neg\varphi_i\}$. We claim that in this otherwise case, still Θ_{i+1} is consistent. Suppose not. Then $\Theta_{i+1} \vdash \neg(v_0 = v_0)$, i.e., $\Theta_i \cup \{\neg\varphi_i\} \vdash \neg(v_0 = v_0)$. By the deduction theorem, $\Theta_i \vdash \neg\varphi_i \to \neg(v_0 = v_0)$, and then by Proposition 3.4 and a tautology $\Theta_i \vdash \varphi_i$. It follows that $\Theta_i \cup \{\varphi_i\}$ is consistent; otherwise $\Theta_i \cup \{\varphi_i\} \vdash \neg(v_0 = v_0)$, hence by the deduction theorem $\Theta_i \vdash \varphi_i \to \neg(v_0 = v_0)$, so by Proposition 4.3 and a tautology $\Theta_i \vdash \neg\varphi_i$. Together with $\Theta_i \vdash \varphi_i$, this shows that Θ_i is inconsistent, contradiction. So, $\Theta_i \cup \{\varphi_i\}$ is consistent. But this contradicts our "otherwise" condition. So, Θ_{i+1} is consistent.

This finishes the construction. Let $\Delta = \bigcup_{i\in\omega}\Theta_i$. Then Δ is consistent. In fact, suppose not. Then $\Delta \vdash \neg(v_0 = v_0)$. Let $\langle\psi_0, \ldots, \psi_{m-1}\rangle$ be a Δ-proof with $\psi_i = \neg(v_0 =$

v_0). Let $\langle \chi_0, \ldots, \chi_{n-1} \rangle$ enumerate all of the members of Δ which are in the proof. Say $\chi_j \in \Theta_{s(j)}$ for each $j < n$. Let t be maximum among all the $s(j)$ for $j < n$. Then each χ_k is in Θ_t, so that $\langle \psi_0, \ldots, \psi_{m-1} \rangle$ is a Θ_t-proof. It follows that Θ_t is inconsistent, contradiction.

So Δ is consistent. Since $\Theta_0 = \Gamma$, we have $\Gamma \subseteq \Delta$. Finally, Δ is complete, since every sentence is equal to some φ_i, and our construction assures that $\varphi_i \in \Delta$ or $\neg\varphi_i \in \Delta$. □

Lemma 4.14. *Let $\mathscr{L}$ be a first-order language. Let $\mathscr{L}'$ extend $\mathscr{L}$ by adding new non-logical symbols Suppose that $\overline{M}$ is an $\mathscr{L}'$-structure, and $\overline{N}$ is the $\mathscr{L}$-structure obtained from $\overline{M}$ by removing the denotations of the new non-logical symbols. Suppose that φ is a formula of $\mathscr{L}$, and $a : \omega \to M$. Then $\overline{M} \models \varphi[a]$ iff $\overline{N} \models \varphi[a]$.*

Proof. First we prove the following similar statement for terms:

(1) If σ is a term of $\mathscr{L}$, then $\sigma^{\overline{M}}(a) = \sigma^{\overline{N}}(a)$.

We prove this by induction on σ:

$$\begin{aligned}
v_i^{\overline{M}}(a) &= a_i = v_i^{\overline{N}}(a); \\
\mathbf{k}^{\overline{M}}(a) &= \mathbf{k}^{\overline{M}} = \mathbf{k}^{\overline{N}} = \mathbf{k}^{\overline{N}}(a) \quad \text{for } \mathbf{k} \text{ an individual constant of } \mathscr{L} \\
(\mathbf{F}\sigma_0 \ldots \sigma_{m-1})^{\overline{M}}(a) &= \mathbf{F}^{\overline{M}}(\sigma_0^{\overline{M}}(a), \ldots \sigma_{m-1}^{\overline{M}}(a)) \\
&= \mathbf{F}^{\overline{N}}(\sigma_0^{\overline{N}}(a), \ldots \sigma_{m-1}^{\overline{N}}(a)) \\
&= (\mathbf{F}\sigma_0 \ldots \sigma_{m-1})^{\overline{N}}(a).
\end{aligned}$$

Here $\mathbf{F}$ is a function symbol of $\mathscr{L}$. Thus (1) holds.

Now we prove the lemma itself by induction on φ:

$$\begin{aligned}
\overline{M} \models (\sigma = \tau)[a] \quad &\text{iff} \quad \sigma^{\overline{M}}(a) = \tau^{\overline{M}}(a) \\
&\text{iff} \quad \sigma^{\overline{N}}(a) = \tau^{\overline{N}}(a) \\
&\text{iff} \quad \overline{N} \models (\sigma = \tau)[a]; \\
\overline{M} \models (\mathbf{R}\sigma_0 \ldots \sigma_{m-1})[a] \quad &\text{iff} \quad \langle \sigma_0^{\overline{M}}(a), \ldots, \sigma_{m-1}^{\overline{M}}(a) \rangle \in \mathbf{R}^{\overline{M}} \\
&\text{iff} \quad \langle \sigma_0^{\overline{N}}(a), \ldots, \sigma_{m-1}^{\overline{N}}(a) \rangle \in \mathbf{R}^{\overline{N}} \\
&\text{iff} \quad \overline{N} \models (\mathbf{R}\sigma_0 \ldots \sigma_{m-1})[a]; \\
\overline{M} \models (\neg\varphi)[a] \quad &\text{iff} \quad \text{not}(\overline{M} \models \varphi[a]) \\
&\text{iff} \quad \text{not}(\overline{N} \models \varphi[a]) \\
&\text{iff} \quad \overline{N} \models (\neg\varphi)[a]; \\
\overline{M} \models (\varphi \to \psi)[a] \quad &\text{iff} \quad \text{not}(\overline{M} \models \varphi[a]) \text{ or } \overline{M} \models \psi[a] \\
&\text{iff} \quad \text{not}(\overline{N} \models \varphi[a]) \text{ or } \overline{N} \models \psi[a] \\
&\text{iff} \quad \overline{N} \models (\varphi \to \psi)[a]; \\
\overline{M} \models (\forall v_i\varphi)[a] \quad &\text{iff} \quad \text{for all } u \in M(\overline{M} \models \varphi[a^i_u]) \\
&\text{iff} \quad \text{for all } u \in N(\overline{N} \models \varphi[a^i_u]) \\
&\text{iff} \quad \overline{N} \models (\forall v_i\varphi)[a].
\end{aligned}$$

□

Theorem 4.15. (Completeness Theorem 1) *Every consistent set of sentences in a finite language has a model.*

Proof. Let Γ be a consistent set of sentences in the finite language $\mathscr{L}$. Let $\mathscr{L}'$ be obtained from $\mathscr{L}$ by adjoining individual constants $\mathbf{c}_i$ for each $i \in \omega$. By Lemmas 4.12 and 4.13 let Δ be a consistent rich complete set of sentences in $\mathscr{L}'$ such that $\Gamma \subseteq \Delta$. By Lemma 4.7, let $\overline{M}$ be a model of Δ. Let $\overline{N}$ be the $\mathscr{L}$-structure obtained from $\overline{M}$ by removing the denotations of the constants $\mathbf{c}_i$ for $i \in \omega$. By Lemma 4.14, $\overline{N}$ is a model of Γ. □

Theorem 4.16. (Completeness Theorem 2) *Let $\Gamma \cup \{\varphi\}$ be a set of formulas in a finite language. Then $\Gamma \vdash \varphi$ iff $\Gamma \models \varphi$.*

Proof. By Theorems 4.3 and 4.15. □

Theorem 4.17. (Completeness Theorem 3) *For any formula φ, $\vdash \varphi$ iff $\models \varphi$.*

Proof. Note that the implicit language $\mathscr{L}$ here is arbitrary, not necessarily finite. $\Rightarrow$ holds by Theorem 4.3. Now suppose that $\models \varphi$ in the sense of $\mathscr{L}$: for every $\mathscr{L}$-structure $\overline{M}$ and every $a : \omega \to M$ we have $\overline{M} \models \varphi[a]$. Let $\mathscr{L}'$ be the language whose non-logical symbols are those occurring in φ. There are finitely many such symbols, so $\mathscr{L}'$ is a finite language. By Lemma 4.14 we have $\models \varphi$ in the sense of $\mathscr{L}'$. Hence by Theorem 4.16, $\vdash \varphi$ in the sense of $\mathscr{L}'$. But every $\mathscr{L}'$-proof is also an $\mathscr{L}$-proof; so $\vdash \varphi$ in the sense of $\mathscr{L}$. □

One of the most important consequences of Completeness Theorem 1 is the following result, which is at the beginning of real model theory.

Theorem 4.18. (The Compactness Theorem) *If Γ is a set of sentences in a finite language and every finite subset of Γ has a model, then Γ itself has a model.*

Proof. Suppose to the contrary that Γ does not have a model. Then by Theorem 4.15, Γ is inconsistent. So $\Gamma \vdash \neg(v_0 = v_0)$. In a Γ-proof with $\neg(v_0 = v_0)$ as a member, let Δ be the set of all $\varphi \in \Gamma$ that appear as entries in the proof. So Δ is a finite subset of Γ, and so has a model $\overline{M}$. But the proof shows that $\Delta \vdash \neg(v_0 = v_0)$. So by Theorem 4.2, $\Delta \models \neg(v_0 = v_0)$. Since $\overline{M}$ is a model of Δ, it follows that $\overline{M}$ is a model of $\neg(v_0 = v_0)$, contradiction. □

We give one consequence of the compactness theorem, and formulate more in the exercises.

Theorem 4.19. *If $\mathscr{L}$ is a finite language, then there is no set Γ of sentences of $\mathscr{L}$ such that an $\mathscr{L}$-structure is finite iff it is a model of Γ.*

Proof. Suppose there is such a set Γ. Adjoin to Γ all of the sentences

$$\exists v_0 \ldots \exists v_n \bigwedge_{i<j\leq n} \neg(v_i = v_j)$$

for n a positive integer. Note that such a sentence holds in a model iff the model has at least $n+1$ elements. Let Γ' be the result of adjoining all of these sentences. Now every finite subset Δ of Γ' has a model. In fact, if n is largest such that the above sentence is in Δ, take an $\mathscr{L}$-structure $\overline{M}$ with $n+1$ elements. By hypothesis, $\overline{M}$ is a model of Γ, and it is also a model of all of the new sentences which are in Δ, so it is a model of Δ. By the compactness theorem it follows that Γ' has a model $\overline{N}$. But since all of the new sentences are in Γ', $\overline{N}$ must be infinite. It is a model of Γ, contradiction. □

As the final topic of this chapter we consider the role of definitions. By a *theory* we mean a pair $(\mathscr{L}, \Gamma)$ such that $\mathscr{L}$ is a first-order language and Γ is a set of formulas in $\mathscr{L}$. A theory $(\mathscr{L}', \Gamma')$ is a *simple definitional expansion* of a theory $(\mathscr{L}, \Gamma)$ provided that the following conditions hold:

(1) $\mathscr{L}'$ is obtained from $\mathscr{L}$ by adding one new non-logical symbol.

(2) If the new symbol of $\mathscr{L}'$ is an m-ary relation symbol $\mathbf{R}$, then there is a formula φ of $\mathscr{L}$ with free variables among $v_0, \ldots, v_{m-1}$ such that

$$\Gamma' = \Gamma \cup \{\mathbf{R}v_0 \ldots v_{m-1} \leftrightarrow \varphi\}.$$

(3) If the new symbol of $\mathscr{L}'$ is an individual constant $\mathbf{c}$, then there is a formula φ of $\mathscr{L}$ with free variables among v_0 such that $\Gamma \vdash \exists! v_0 \varphi$ and

$$\Gamma' = \Gamma \cup \{\mathbf{c} = v_0 \leftrightarrow \varphi\}.$$

Here $\exists! v_0 \varphi$ is the formula $\exists v_0[\varphi \wedge \forall v_i[\mathrm{Subf}^{v_0}_{v_i}\varphi \rightarrow v_0 = v_i]$, where i is minimum such that v_i does not occur in φ.

(4) If the new symbol of $\mathscr{L}'$ is an m-ary function symbol $\mathbf{F}$, then there is a formula φ of $\mathscr{L}$ with free variables among $v_0, \ldots, v_m$ such that $\Gamma \vdash \forall v_0 \ldots \forall v_{m-1} \exists! v_m \varphi$ and

$$\Gamma' = \Gamma \cup \{\mathbf{F}v_0 \ldots v_{m-1} = v_m \leftrightarrow \varphi\}.$$

The basic facts about definitions are that the defined terms can always be eliminated, and adding a definition does not change what is is provable in the original language. In order to prove these two facts, we first show that any formula can be put in a certain normal form, which is interesting in its own right. This normal form will be defined shortly.

Lemma 4.20. *If* $\mathbf{c}$ *is an individual constant and* $i \neq 0$*, then* $\vdash \mathbf{c} = v_i \leftrightarrow \exists v_0(v_0 = v_i \wedge \mathbf{c} = v_0)$.

Proof. We argue model-theoretically. Suppose that $\overline{A}$ is a structure and $a : \omega \to A$. If $\overline{A} \models (\mathbf{c} = v_i)[a]$, then $\mathbf{c}^{\overline{A}} = a_i$. Then $v_0^{\overline{A}}(a^0_{a_i}) = a_i$ and $v_i^{\overline{A}}(a^0_{a_i}) = a_i$. Hence $\overline{A} \models (v_0 = v_i \wedge \mathbf{c} = v_0)[a^0_{a_i}]$, and so $\overline{A} \models \exists v_0(v_0 = v_i \wedge \mathbf{c} = v_0)[a]$. Thus $\overline{A} \models (\vdash \mathbf{c} = v_i)[a]$ implies that $\overline{A} \models \exists v_0(v_0 = v_i \wedge \mathbf{c} = v_0)[a]$.

Conversely, suppose that $\overline{A} \models \exists v_0(v_0 = v_i \wedge \mathbf{c} = v_0)[a]$. Choose $x \in A$ such that $\overline{A} \models (v_0 = v_i \wedge \mathbf{c} = v_0)[a^0_x]$. Then $x = v_0^{\overline{A}}(a^0_x) = v_i^{\overline{A}}(a^0_x) = a_i$ and $\mathbf{c}^{\overline{A}} = v_0^{\overline{A}}(a^0_x) = a_i = v_i^{\overline{A}}(a)$. Hence $\overline{A} \models (\vdash \mathbf{c} = v_i)[a]$.

So we have shown that $\overline{A} \models (\vdash \mathbf{c} = v_i)[a]$ iff $\overline{A} \models \exists v_0(v_0 = v_i \wedge \mathbf{c} = v_0)[a]$. It follows that $\models \mathbf{c} = v_i \leftrightarrow \exists v_0(v_0 = v_i \wedge \mathbf{c} = v_0)$. Hence by the completeness theorem, $\vdash \mathbf{c} = v_i \leftrightarrow \exists v_0(v_0 = v_i \wedge \mathbf{c} = v_0)$. $\square$

Lemma 4.21. *Suppose that* $\mathbf{R}$ *is an* m*-ary relation symbol and* $\langle i(0), \ldots, i(m-1)\rangle$ *is a sequence of natural numbers such that* $m \leq i(j)$ *for all* $j < m$. *Then*

$$\vdash \mathbf{R}v_{i(0)} \ldots v_{i(m-1)} \leftrightarrow \exists v_0 \ldots \exists v_{m-1}\left[\bigwedge_{j<m}(v_j = v_{i(j)}) \wedge \mathbf{R}v_0 \ldots, v_{m-1}\right].$$

Proof. Again we argue model-theoretically. Suppose that $\overline{A}$ is a structure and $a : \omega \to A$. First suppose that $\overline{A} \models \mathbf{R}v_{i(0)} \ldots v_{i(m-1)}[a]$. Thus $\langle a_{i(0)}, \ldots, a_{i(m-1)}\rangle \in \mathbf{R}^{\overline{A}}$. Let

$$b = (\cdots(a^0_{i(0)})^1_{i(1)} \cdots)^{m-1}_{i(m-1)}.$$

Then for any $j < m$ we have $v_j^{\overline{A}}(b) = b_j = a_{i(j)} = b_{i(j)} = v_{i(j)}^{\overline{A}}(b)$. It follows that $\overline{A} \models \bigwedge_{j<m}(v_j = v_{i(j)})[b]$. Also, $\langle b_0, \ldots, b_{m-1}\rangle = \langle a_{i(0)}, \ldots, a_{i(m-1)}\rangle \in \mathbf{R}^{\overline{A}}$. Hence $\overline{A} \models \mathbf{R}v_0 \ldots v_{m-1}[b]$. Thus

$$\overline{A} \models \left[\bigwedge_{j<m}(v_j = v_{i(j)}) \wedge \mathbf{R}v_0 \ldots, v_{m-1}\right][b]$$

and hence

$$\overline{A} \models \exists v_0 \ldots \exists v_{m-1}\left[\bigwedge_{j<m}(v_j = v_{i(j)}) \wedge \mathbf{R}v_0 \ldots, v_{m-1}\right][a] \tag{1}$$

Hence we have shown that $\overline{A} \models \mathbf{R}v_{i(0)} \ldots v_{i(m-1)}[a]$ implies (1).

Now suppose conversely that (1) holds. Choose $x(0), \ldots, x(m-1) \in A$ such that

$$\overline{A} \models \left[\bigwedge_{j<m}(v_j = v_{i(j)}) \wedge \mathbf{R}v_0 \ldots, v_{m-1}\right][b],$$

where $b = (\cdots(a^0_{x(0)})^1_{x(1)}) \cdots)^{m-1}_{x(m-1)}$. For any $j < m$ we have $b_j = x(j) = v_j^{\overline{A}}(b) = v_{i(j)}^{\overline{A}}(b) = v_{i(j)}^{\overline{A}}(a) = a_{i(j)}$. We also have $\langle b_0, \ldots, b_{m-1}\rangle \in \mathbf{R}^{\overline{A}}$. Hence $\langle a_{i(0)}, \ldots, a_{i(m-1)}\rangle \in \mathbf{R}^{\overline{A}}$, and it follows that $\overline{A} \models \mathbf{R}v_{i(0)} \ldots v_{i(m-1)}[a]$.

So we have shown that $\overline{A} \models \mathbf{R}v_{i(0)} \ldots v_{i(m-1)}[a]$ iff (1). Therefore

$$\models \mathbf{R}v_{i(0)} \ldots v_{i(m-1)} \leftrightarrow \exists v_0 \ldots \exists v_{m-1}\left[\bigwedge_{j<m}(v_j = v_{i(j)}) \wedge \mathbf{R}v_0 \ldots, v_{m-1}\right].$$

and it follows by the completeness theorem that

$$\vdash \mathbf{R}v_{i(0)} \ldots v_{i(m-1)} \leftrightarrow \exists v_0 \ldots \exists v_{m-1} \left[\bigwedge_{j<m} (v_j = v_{i(j)}) \wedge \mathbf{R}v_0 \ldots, v_{m-1} \right]. \qquad \square$$

The proof of the following lemma is very similar to the proof of Lemma 4.21.

Lemma 4.22. *Suppose that* $\mathbf{F}$ *is an* m*-ary function symbol and* $\langle i(0), \ldots, i(m) \rangle$ *is a sequence of natural numbers such that* $m+1 \leq i(j)$ *for all* $j \leq m$*. Then*

$$\vdash \mathbf{F}v_{i(0)} \ldots v_{i(m-1)} = v_{i(m)} \leftrightarrow \exists v_0 \ldots \exists v_m \left[\bigwedge_{j \leq m} (v_j = v_{i(j)}) \wedge \mathbf{F}v_0 \ldots, v_{m-1} = v_m \right].$$

Proof. Again we argue model-theoretically. Suppose that $\overline{A}$ is a structure and $a : \omega \to A$. First suppose that $\overline{A} \models \mathbf{F}v_{i(0)} \ldots v_{i(m-1)} = v_{i(m)}[a]$. Thus $\mathbf{F}^{\overline{A}}(a_{i(0)}, \ldots, a_{i(m-1)}) = a_{i(m)}$. Let

$$b = (\cdots (a^0_{i(0)})^1_{i(1)} \cdots)^m_{i(m)}.$$

Then for any $j \leq m$ we have $v_j^{\overline{A}}(b) = b_j = a_{i(j)} = b_{i(j)} = v_{i(j)}^{\overline{A}}(b)$. It follows that $\overline{A} \models \bigwedge_{j \leq m}(v_j = v_{i(j)})[b]$. Also,

$$\begin{aligned}
\mathbf{F}(b_0, \ldots, b_{m-1}) &= \mathbf{F}(a_{i(0)}, \ldots, a_{i(m-1)}) \\
&= a_{i(m)} \\
&= b_m.
\end{aligned}$$

Hence $\overline{A} \models (\mathbf{F}v_0 \ldots v_{m-1} = v_m)[b]$. Thus

$$\overline{A} \models \left[\bigwedge_{j \leq m} (v_j = v_{i(j)}) \wedge \mathbf{F}v_0 \ldots, v_{m-1} = v_m \right] [b]$$

and hence

$$\overline{A} \models \exists v_0 \ldots \exists v_m \left[\bigwedge_{j \leq m} (v_j = v_{i(j)}) \wedge \mathbf{F}v_0 \ldots, v_{m-1} = v_m \right] [a] \tag{1}$$

Hence we have shown that $\overline{A} \models \mathbf{R}v_{i(0)} \ldots v_{i(m-1)}[a]$ implies (1).

Now suppose conversely that (1) holds. Choose $x(0), \ldots, x(m) \in A$ such that

$$\overline{A} \models \left[\bigwedge_{j \leq m} (v_j = v_{i(j)}) \wedge \mathbf{F}v_0 \ldots, v_{m-1} = v_m \right] [b],$$

where $b = (\cdots(a^0_{x(0)})^1_{x(1)})\cdots)^m_{x(m)}$. For any $j \le m$ we have $b_j = x(j) = v_j^{\overline{A}}(b) = v_{i(j)}^{\overline{A}}(b) = v_{i(j)}^{\overline{A}}(a) = a_{i(j)}$. We also have $\langle \mathbf{F}^{\overline{A}}(b_0, \ldots, b_{m-1}) = b_m$. Hence $\mathbf{F}^{\overline{A}}(a_{i(0)}, \ldots, a_{i(m-1)}) = a_{i(m)}$, and it follows that $\overline{A} \models (\mathbf{F}v_{i(0)} \ldots v_{i(m-1)}) = v_{i(m)})[a]$.

So we have shown that $\overline{A} \models (\mathbf{F}v_{i(0)} \ldots v_{i(m-1)}) = v_{i(m)})[a]$ iff (1). Therefore

$$\models \mathbf{F}v_{i(0)} \ldots v_{i(m-1)} = v_{i(m)} \leftrightarrow \exists v_0 \ldots \exists v_{m-1} \left[\bigwedge_{j \le m} (v_j = v_{i(j)}) \wedge \mathbf{F}v_0 \ldots, v_{m-1} = v_m \right].$$

and it follows by the completeness theorem that

$$\vdash \mathbf{F}v_{i(0)} \ldots v_{i(m-1)} = v_{i(m)} \leftrightarrow \exists v_0 \ldots \exists v_{m-1} \left[\bigwedge_{j \le m} (v_j = v_{i(j)}) \wedge \mathbf{F}v_0 \ldots, v_{m-1} = v_m \right]. \quad \square$$

A formula φ is *standard* provided that every atomic subformula of φ has one of the following forms:

$v_i = v_j$ for some $i, j \in \omega$.
$\mathbf{c} = v_0$ for some individual constant $\mathbf{c}$.
$\mathbf{R}v_0 \ldots v_{m-1}$ for some m-ary relation symbol $\mathbf{R}$.
$\mathbf{F}v_0 \ldots v_{m-1} = v_m$ for some m-ary function symbol $\mathbf{F}$.

Lemma 4.23. *If σ is a term and $i \in \omega$, then there is a standard formula with the same free variables as $\sigma = v_i$ such that $\vdash \sigma = v_i \leftrightarrow \varphi$.*

Proof. We proceed by induction on σ. If σ is a variable v_j, then $\sigma = v_i$ is $v_j = v_i$, which is already standard. If σ is an individual constant $\mathbf{c}$, then the desired conclusion follows from Lemma 4.20. Now suppose that σ is $\mathbf{F}\tau_0 \ldots \varphi_{m-1}$ for some m-ary function symbol $\mathbf{F}$ and some terms $\tau_0, \ldots, \tau_{m-1}$, where we know the result for each τ_i. Let $n \in \omega$ be greater than each j such that j occurs in σ, and also greater than i and m. Then we claim

(*) $\vdash \sigma = v_i \leftrightarrow \exists v_n \ldots \exists v_{n+m} \left[\bigwedge_{j<m} (\tau_j = v_{n+j}) \wedge v_i = v_{n+m} \wedge \mathbf{F}v_n \ldots v_{n+m-1} = v_{n+m} \right]$.

To prove this claim, suppose that $\overline{A}$ is a structure and $a : \omega \to A$. First suppose that $\overline{A} \models (\sigma = v_i)[a]$. Thus $\mathbf{F}^{\overline{A}}(\tau_0^{\overline{A}}(a), \ldots \tau_{m-1}^{\overline{A}}(a)) = a_i$. Let

$$b = (\cdots(a^n_{\tau_0^{\overline{A}}(a)})^{n+1}_{\tau_1^{\overline{A}}(a)}) \cdots)^{n+m-1}_{\tau_{m-1}^{\overline{A}}(a)})^{n+m}_{a_i}.$$

Then for each $j < m$ we have $\tau_j^{\overline{A}}(a) = \tau_j^{\overline{A}}(b)$ since n is greater than each k such that v_k occurs in τ_j, using Proposition 2.4. Also, $b_{n+j} = \tau_j^{\overline{A}}(a) = \tau_j^{\overline{A}}(b)$. So

$$(1) \qquad \overline{A} \models \bigwedge_{j<m} (\tau_j = v_{n+j})[b].$$

Moreover, $b_{n+m} = a_i = b_i$, so

(2) $$\overline{A} \models (v_{n+m} = v_i)[b].$$

Next, for each $j < m$ we have $\tau_j^{\overline{A}}(a) = b_{n+j}$, $b_{n+m} = a_i$, and $\mathbf{F}^{\overline{A}}(\tau_0^{\overline{A}}(a), \ldots \tau_{m-1}^{\overline{A}}(a)) = a_i$, so

(3) $$\overline{A} \models (\mathbf{F}v_n \ldots v_{n+m-1} = v_{n+m})[b].$$

Putting (1)–(3) together, we get

$$\overline{A} \models \left[\bigwedge_{j<m} (\tau_j = v_{n+j}) \wedge v_i = v_{n+m} \wedge \mathbf{F}v_n \ldots v_{n+m-1} = v_{n+m}\right][b]$$

and hence

(4) $$\overline{A} \models \exists v_n \ldots \exists v_{n+m} \left[\bigwedge_{j<m} (\tau_j = v_{n+j}) \wedge v_i = v_{n+m} \wedge \mathbf{F}v_n \ldots v_{n+m-1} = v_{n+m}\right][a]$$

Thus we have shown that $\overline{A} \models [\sigma = v_i][a]$ implies (4).

Conversely, suppose that (4) holds. Choose $x(0), \ldots x(m) \in A$ such that

$$\overline{A} \models \left[\bigwedge_{j<m} (\tau_j = v_{n+j}) \wedge v_i = v_{n+m} \wedge \mathbf{F}v_n \ldots v_{n+m-1} = v_{n+m}\right][b],$$

where $b = (\cdots(a^n_{x(0)})^{n=1}_{x(1)}) \cdots)^{n+m}_{x(m)}$. Then for any $j < m$ we have $b_{n+j} = \tau_j^{\overline{A}}(b) = \tau_j^{\overline{A}}(a)$ since n is greater than each k such that v_k occurs in τ_j. Also, $b_{n+m} = b_i = a_i$ and $\mathbf{F}^{\overline{A}}(b_n, \ldots, b_{n+m-1}) = b_{n+m}$. It follows that $\mathbf{F}^{\overline{A}}(\tau_0^{\overline{A}}(a), \ldots, \tau_{m-1}^{\overline{A}}(a)) = a_i$. Thus $\overline{A} \models [\sigma = v_i][a]$. So we have shown that $\overline{A} \models [\sigma = v_i][a]$ iff (4). By the completeness theorem, this proves the claim $(*)$.

Now by the inductive hypothesis, for each $j < m$ let ψ_j be a standard formula such that $\vdash \tau_j = v_{n+j} \leftrightarrow \psi_j$. By Lemma 4.22 let χ be a standard formula such that $\vdash \mathbf{F}v_n \ldots v_{n+m-1} = v_{n+m} \leftrightarrow \chi$. Then by $(*)$ and Lemma 3.20 there is a standard formula φ such that $\vdash \sigma = v_i \leftrightarrow \varphi$. The condition on free variables is clear. □

Theorem 4.24. *For any formula φ there is a standard formula ψ with the same free variables as φ such that $\vdash \varphi \leftrightarrow \psi$.*

Proof. We proceed by induction on φ. First suppose that φ is an atomic equality formula $\sigma = \tau$. Let $i \in \omega$ be such that v_i does not occur in φ. Then

(1) $\vdash \varphi \leftrightarrow \exists v_i(\sigma = v_i \wedge \tau = v_i)$.

In fact, let $\overline{A}$ be a structure and $a : \omega \to A$. First suppose that $\overline{A} \models \sigma = \tau[a]$. Say $\sigma^{\overline{A}}(a) = \tau^{\overline{A}}(a) = x$. Then $\overline{A} \models [\sigma = v_i \wedge \tau = v_i][a^i_x]$, and hence $\overline{A} \models \exists v_i(\sigma = v_i \wedge \tau = v_i)[a]$.

Conversely, suppose that $\overline{A} \models \exists v_i(\sigma = v_i \wedge \tau = v_i)[a]$. Choose $x \in A$ such that $\overline{A} \models [\sigma = v_i \wedge \tau = v_i][a^i_x]$. Now v_i does not occur in φ, so using Proposition 2.4 we get $\sigma^{\overline{A}}(a) = \sigma^{\overline{A}}[a^i_x] = \tau^{\overline{A}}[a^i_x] = \tau^{\overline{A}}[a]$.

Now (1) follows by the completeness theorem.

By (1), Lemma 4.23, and Theorem 3.20 it follows that there is a standard formula ψ with the same free variables as φ such that $\vdash \varphi \leftrightarrow \psi$.

Second, suppose that φ is $\mathbf{R}\sigma_0 \ldots \sigma_{m-1}$ for some m-ary relation symbol and some terms $\sigma_0, \ldots \sigma_{m-1}$. Let n be greater than m and all k such that v_k occurs in φ. Then

(2) $\vdash \varphi \leftrightarrow \exists v_n \ldots \exists v_{n+m-1} \left[\bigwedge_{j<m}(\sigma_j = v_{n+j}) \wedge \mathbf{R}v_n \ldots v_{n+m-1}\right]$.

We leave the proof of (2) to an exercise.

Now (2), Lemma 4.23, and Theorem 3.20 again give the desired formula ψ.

The inductive steps involving $\neg$, $\to$, and $\forall$ follow using Theorem 3.20. □

The following theorem expresses that defined notions can be eliminated.

Theorem 4.25. *Let $(\mathscr{L}', \Gamma')$ be a simple definitional expansion of $(\mathscr{L}, \Gamma)$, and let φ be a formula of $\mathscr{L}'$. Then there is a formula ψ of $\mathscr{L}$ with the same free variables as φ such that $\Gamma' \vdash \varphi \leftrightarrow \psi$.*

(Note here that $\vdash$ is in the sense of $\mathscr{L}'$.)

Proof. Let χ be a standard formula (of $\mathscr{L}'$) such that $\vdash \varphi \leftrightarrow \chi$, such that χ has the same free variables as φ. Now we consider cases depending on what the new symbol s of $\mathscr{L}'$ is. Let θ be as in the definition of simple definitional expansion.

Case 1. s is an individual constant $\mathbf{c}$. Then we let ψ be obtained from χ by replacing every subformula $\mathbf{c} = v_0$ of χ by θ.

Case 2. s is an m-ary relation symbol $\mathbf{R}$. Then we let ψ be obtained from χ by replacing every subformula $\mathbf{R}v_0 \ldots v_{m-1}$ of χ by θ.

Case 3. s is an m-ary function symbol $\mathbf{F}$. Then we let ψ be obtained from χ by replacing every subformula $\mathbf{F}v_0 \ldots v_{m-1} = v_m$ of χ by θ. □

The following theorem expresses that a simple definitional expansion does not increase the set of old formulas which are provable.

Theorem 4.26. *Let $(\mathscr{L}', \Gamma')$ be a simple definitional expansion of $(\mathscr{L}, \Gamma)$ with $\mathscr{L}$ finite, and let φ be a formula of $\mathscr{L}$. Suppose that $\Gamma' \vdash \varphi$. Then $\Gamma \vdash \varphi$.*

Proof. By the completeness theorem we have $\Gamma' \models \varphi$, and it suffices to show that $\Gamma \models \varphi$. So, suppose that $\overline{A} \models \psi$ for each $\psi \in \Gamma$. In order to show that $\overline{A} \models \varphi$, suppose that $a : \omega \to A$; we want to show that $\overline{A} \models \varphi[a]$. We define an $\mathscr{L}'$-structure $\overline{A}'$ by defining the denotation of the new symbol s of $\mathscr{L}'$. The three cases are treated similarly, but we give full details for each of them.

Case 1. s is $\mathbf{c}$, an individual constant. By the definition of simple definitional expansion, there is a formula χ of $\mathscr{L}$ with free variables among v_0 such that $\Gamma \vdash \exists! v_0 \chi$, and

$\Gamma' = \Gamma \cup \{\mathbf{c} = v_0 \leftrightarrow \chi\}$. Then $\Gamma \models \exists! v_0\chi$. Since $\overline{A} \models \Gamma$, it follows that $\overline{A} \models \chi[a^0_x]$ for a unique $x \in A$. Let $\mathbf{c}^{\overline{A}'} = x$. We claim that $\overline{A}' \models (\mathbf{c} = v_0 \leftrightarrow \chi)$. In fact, suppose that $b : \omega \to A$. If $\overline{A}' \models (\mathbf{c} = v_0)[b]$, then $b_0 = \mathbf{c}^{\overline{A}'} = x$. Then a^0_x and b agree at 0, so by Lemma 4.4, since the free variables of χ are among v_0, we have $\overline{A} \models \chi[b]$. By Lemma 4.14, $\overline{A}' \models \chi[b]$. Conversely, suppose that $\overline{A}' \models \chi[b]$. Then b and $a^0_{b(0)}$ agree on 0, so $\overline{A}' \models \chi[a^0_{b(0)}]$. Hence $\overline{A} \models \chi[a^0_{b(0)}]$ by Lemma 4.14. Since also $\overline{A} \models \chi[a^0_x]$ and $\overline{A} \models \exists! v_0\chi$, it follows that $b(0) = x$. Hence $\overline{A}' \models (\mathbf{c} = v_0)[b]$. This proves the claim.

By the claim, $\overline{A}'$ is a model of Γ'. Hence it is a model of φ. By Lemma 4.14, $\overline{A}$ is a model of φ, as desired.

Case 2. s is $\mathbf{F}$, an m-ary function symbol. By the definition of simple definitional expansion, there is a formula χ of $\mathscr{L}$ with free variables among $v_0, \ldots, v_m$ such that $\Gamma \vdash \forall v_0 \ldots \forall v_{m-1} \exists! v_m \chi$, and $\Gamma' = \Gamma \cup \{\mathbf{F}v_0 \ldots v_{m-1} = v_m \leftrightarrow \chi\}$. Then $\Gamma \models \forall v_0 \ldots \forall v_{m-1} \exists! v_m \chi$. Let $x(0), \ldots, x(m-1) \in A$. Since $\overline{A} \models \Gamma$, it follows that $\overline{A} \models \chi[(\cdots(a^0_{x(0)})^1_{x(1)})\cdots)^{m-1}_{x(m-1)})^m_y]$ for a unique $y \in A$. Let $\mathbf{F}^{\overline{A}'}(x(0), \ldots, x((m-1)) = y$. We claim that $\overline{A}' \models (\mathbf{F}v_0 \ldots v_{m-1} = v_m \leftrightarrow \chi)$. In fact, suppose that $b : \omega \to A$. If $\overline{A}' \models (\mathbf{F}v_0 \ldots v_{m-1} = v_m)[b]$, then $\mathbf{F}^{\overline{A}'}(b_0, \ldots, b_{m-1}) = b_m$. Now b and $(\cdots(a^0_{b_0})^1_{b_1})\cdots)^m_{b_m}$ and b agree on $\{0, \ldots, m\}$, so by the definition of $\mathbf{F}^{\overline{A}'}$ we get $\overline{A} \models \chi[(\cdots(a^0_{b_0})^1_{b_1})\cdots)^m_{b_m}]$, and hence also $\overline{A} \models \chi[b]$, and by Lemma 4.14 $\overline{A}' \models \chi[b]$.

Conversely, suppose that $\overline{A}' \models \chi[b]$. Then $\overline{A} \models [(\cdots(a^0_{b_0})^1_{b_1})\cdots)^m_{b_m}]$, and therefore $\mathbf{F}^{\overline{A}'}(b_0, \ldots, b_{m-1}) = b_m$. This proves the claim.

By the claim, $\overline{A}'$ is a model of Γ'. Hence it is a model of φ. By Lemma 4.14, $\overline{A}$ is a model of φ, as desired.

Case 3. s is $\mathbf{R}$, an m-ary relation symbol. By the definition of simple definitional expansion, there is a formula χ of $\mathscr{L}$ with free variables among $v_0, \ldots, v_{m-1}$ such that $\Gamma' = \Gamma \cup \{\mathbf{R}v_0 \ldots v_{m-1} \leftrightarrow \chi\}$. Let

$$\begin{aligned}\mathbf{R}^{\overline{A}'} = \{\langle a_0, \ldots, a_{m-1}\rangle : \overline{A} \models \chi[a] \\ \text{for some } a : \omega \to A \text{ which extends } \langle a_0, \ldots, a_{m-1}\rangle\}.\end{aligned}$$

We claim that $\overline{A}' \models (\mathbf{R}v_0 \ldots v_{m-1} \leftrightarrow \chi)$. In fact, suppose that $b : \omega \to A$. If $\overline{A}' \models (\mathbf{R}v_0 \ldots v_{m-1}[b]$, then $\langle b_0, \ldots, b_{m-1}\rangle \in \mathbf{R}^{\overline{A}'}$, and so there is an extension $a : \omega \to A$ of $\langle b_0, \ldots, b_{m-1}\rangle$ such that $\overline{A} \models \chi[a]$. Since a and b agree on all k such that v_k occurs in χ, it follows that $\overline{A} \models \chi[b]$, and hence $\overline{A}' \models \chi[b]$.

Conversely, suppose that $\overline{A}' \models \chi[b]$. Then $\overline{A} \models \chi[b]$ by Lemma 4.14, and it follows that $\langle b_0, \ldots, b_{m-1}\rangle \in \mathbf{R}^{\overline{A}'}$. This proves the claim.

By the claim, $\overline{A}'$ is a model of Γ'. Hence it is a model of φ. By Lemma 4.14, $\overline{A}$ is a model of φ, as desired. □

Theorem 4.27. *Let m be an integer ≥ 2, and suppose that $(\mathscr{L}_{i+1}, \Gamma_{i+1})$ is a simple definitional expansion of $(\mathscr{L}_i, \Gamma_i)$ for each $i < m$. Suppose that φ is an $\mathscr{L}_m$ formula. Then there is an $\mathscr{L}_0$ formula ψ with the same free variables as φ such that $\Gamma_m \vdash \varphi \leftrightarrow \psi$.*

Proof. By induction on m. If $m = 2$, the conclusion follows from Theorem 4.25. now assume the result for m and suppose that $(\mathscr{L}_{i+1}, \Gamma_{i+1})$ is a simple definitional expansion of $(\mathscr{L}_i, \Gamma_i)$ for each $i \leq m$. Let φ be a formula of $\mathscr{L}_{m+1}$. Then by Theorem 4.25 there is a formula ψ of $\mathscr{L}$ with the same free variables as φ such that $\Gamma_{m+1} \vdash \varphi \leftrightarrow \psi$. By the inductive hypothesis, there is a formula χ with the same free variables as ψ such that $\Gamma_m \vdash \psi \leftrightarrow \chi$. Then $\Gamma_{m+1} \vdash \varphi \leftrightarrow \chi$. $\square$

Theorem 4.28. *Let m be an integer ≥ 2, and suppose that $(\mathscr{L}_{i+1}, \Gamma_{i+1})$ is a simple definitional expansion of $(\mathscr{L}_i, \Gamma_i)$ for each $i < m$. Also assume that $\mathscr{L}_0$ is finite. Suppose that φ is an $\mathscr{L}_0$ formula and $\Gamma_m \vdash \varphi$. Then $\Gamma_0 \vdash \varphi$.*

Proof. By induction on m. If $m = 2$, the conclusion follows from Theorem 4.26. now assume the result for m and suppose that $(\mathscr{L}_{i+1}, \Gamma_{i+1})$ is a simple definitional expansion of $(\mathscr{L}_i, \Gamma_i)$ for each $i \leq m$. Suppose that φ is an $\mathscr{L}_0$ formula and $\Gamma_{m+1} \vdash \varphi$. Then by Theorem 4.26, $\Gamma_m \vdash \varphi$, and so by the inductive assumption, $\Gamma_0 \vdash \varphi$. $\square$

The above facts about definitions help to clarify the foundations of mathematics. As already mentioned, almost any mathematical theorem can be put in the form ZFC $\vdash \varphi$ for some formula φ. The language $\mathscr{L}$ here has only one non-logical symbol, the binary relation symbol $\in$ for membership. Many other symbols occur in a development of set theory, including those in various fields of mathematics. These can all be considered as being introduced by definitions, as above. To make these comments more definite, we consider the symbols $\subseteq$, $\emptyset$, and $\cap$. Let $\mathscr{L}_0$ be the language of set theory, so that it has only one non-logical constant, the binary relation symbol $\in$ for membership. Then we have claimed that the foundations of mathematics is embodied in the theory $(\mathscr{L}_0, \mathrm{ZFC})$.

Let $(\mathscr{L}_1, \mathrm{ZFC}_1)$ be the simple definitional expansion of $(\mathscr{L}_0, \mathrm{ZFC})$ obtained by adjoining to $\mathscr{L}_0$ a new binary relation symbol $\subseteq$ and adjoining to ZFC the formula

$$v_0 \subseteq v_1 \leftrightarrow \forall v_2(v_2 \in v_0 \to v_2 \in v_1).$$

Here we write $v_0 \subseteq v_1$ instead of $\subseteq v_0 v_1$.

Proposition 4.29. ZFC $\vdash \exists! v_0 \forall v_1[\neg(v_1 \in v_0)]$.

Proof. The formula $\vdash v_0 = v_0 \to ([v_0 \in v_1 \leftrightarrow v_0 \in v_2 \wedge \neg(v_0 = v_0)] \to \neg(v_0 \in v_1))$ is a tautology, and hence by Proposition 3.4 we get

$$\vdash [v_0 \in v_1 \leftrightarrow v_0 \in v_2 \wedge \neg(v_0 = v_0)] \to \neg(v_0 \in v_1).$$

Hence by generalization and (L2) we get

$$\vdash \forall v_0([v_0 \in v_1 \leftrightarrow v_0 \in v_2 \wedge \neg(v_0 = v_0)]) \to \forall v_0(\neg(v_0 \in v_1)).$$

Now using generalization, (L2), and a tautology we obtain

$$\vdash \exists v_1 \forall v_0([v_0 \in v_1 \leftrightarrow v_0 \in v_2 \wedge \neg(v_0 = v_0)]) \to \exists v_1 \forall v_0(\neg(v_0 \in v_1)).$$

The hypothesis of the implication here is an instance of the comprehension axiom. Hence

$$\text{ZFC} \vdash \exists v_1 \forall v_0 (\neg(v_0 \in v_1)).$$

Using the change of bound variable theorem 3.25 we get in succession

$$\begin{aligned}
&\text{ZFC} \vdash \exists v_2 \forall v_0 (\neg(v_0 \in v_2)) \\
&\text{ZFC} \vdash \exists v_2 \forall v_1 (\neg(v_1 \in v_2)) \\
(1) \qquad &\text{ZFC} \vdash \exists v_0 \forall v_1 (\neg(v_1 \in v_0))
\end{aligned}$$

By the extensionality axiom and change of bound variable theorem 3.25 we get in succession

$$\begin{aligned}
&\text{ZFC} \vdash \forall v_0 \forall v_1 [\forall v_2 (v_2 \in v_0 \leftrightarrow v_2 \in v_1) \to v_0 = v_1]; \\
&\text{ZFC} \vdash \forall v_0 \forall v_1 [\forall v_3 (v_3 \in v_0 \leftrightarrow v_3 \in v_1) \to v_0 = v_1]; \\
&\text{ZFC} \vdash \forall v_0 \forall v_2 [\forall v_3 (v_3 \in v_0 \leftrightarrow v_3 \in v_2) \to v_0 = v_2]; \\
(2) \qquad &\text{ZFC} \vdash \forall v_0 \forall v_2 [\forall v_1 (v_1 \in v_0 \leftrightarrow v_1 \in v_2) \to v_0 = v_2].
\end{aligned}$$

Now the following formula is a tautology: $\neg(v_1 \in v_0) \to [\neg(v_1 \in v_2) \to (v_1 \in v_0 \leftrightarrow v_1 \in v_2)]$. Hence using generalization and (L2) we get

$$\vdash \forall v_1 (\neg(v_1 \in v_0)) \to [\forall v_1 (\neg(v_1 \in v_2)) \to \forall v_1 (v_1 \in v_0 \leftrightarrow v_1 \in v_2)].$$

From (2) we then easily get

$$\text{ZFC} \vdash \forall v_1 (\neg(v_1 \in v_0)) \to [\forall v_1 (\neg(v_1 \in v_2)) \to v_0 = v_2],$$

and hence by generalization and Proposition 3.38 we have

$$\text{ZFC} \vdash \forall v_1 (\neg(v_1 \in v_0)) \to \forall v_2 [\forall v_1 (\neg(v_1 \in v_2)) \to v_0 = v_2],$$

Then by a tautology,

$$\text{ZFC} \vdash \forall v_1 (\neg(v_1 \in v_0)) \to [\forall v_1 (\neg(v_1 \in v_0)) \wedge \forall v_2 [\forall v_1 (\neg(v_1 \in v_2)) \to v_0 = v_2]],$$

Now using generalization and (L2) we get

$$\text{ZFC} \vdash \exists v_0 \forall v_1 (\neg(v_1 \in v_0)) \to \exists v_0 [\forall v_1 (\neg(v_1 \in v_0)) \wedge \forall v_2 [\forall v_1 (\neg(v_1 \in v_2)) \to v_0 = v_2]],$$

From (1) it then follows that

$$\text{ZFC} \vdash \exists v_0 [\forall v_1 (\neg(v_1 \in v_0)) \wedge \forall v_2 [\forall v_1 (\neg(v_1 \in v_2)) \to v_0 = v_2]],$$

which is the desired conclusion. $\square$

We now let $\mathscr{L}_2$ be the extension of $\mathscr{L}_1$ by adding an individual constant $\emptyset$, and $\mathrm{ZFC}_2 = \mathrm{ZFC}_1 \cup \{\emptyset = v_0 \leftrightarrow \forall v_1(\neg(v_1 \in v_0))\}$. Thus $(\mathscr{L}_2, \mathrm{ZFC}_2)$ is a simple definitional expansion of $(\mathscr{L}_1, \mathrm{ZFC}_1)$.

Proposition 4.30. $\mathrm{ZFC} \vdash \forall v_0 \forall v_1 \exists! v_2 \forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1)$.

Proof. An instance of the comprehension axioms gives

$$\mathrm{ZFC} \vdash \exists v_1 \forall v_0(v_0 \in v_1 \leftrightarrow v_0 \in v_2 \wedge v_0 \in v_3).$$

Hence by generalization we get

$$\mathrm{ZFC} \vdash \forall v_2 \forall v_3 \exists v_1 \forall v_0(v_0 \in v_1 \leftrightarrow v_0 \in v_2 \wedge v_0 \in v_3).$$

Now the change of bound variables theorem 3.25 gives successively

$$\begin{aligned}
&\mathrm{ZFC} \vdash \forall v_4 \forall v_3 \exists v_1 \forall v_0(v_0 \in v_1 \leftrightarrow v_0 \in v_4 \wedge v_0 \in v_3);\\
&\mathrm{ZFC} \vdash \forall v_4 \forall v_5 \exists v_1 \forall v_0(v_0 \in v_1 \leftrightarrow v_0 \in v_4 \wedge v_0 \in v_5);\\
&\mathrm{ZFC} \vdash \forall v_4 \forall v_5 \exists v_2 \forall v_0(v_0 \in v_2 \leftrightarrow v_0 \in v_4 \wedge v_0 \in v_5);\\
&\mathrm{ZFC} \vdash \forall v_4 \forall v_5 \exists v_2 \forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_4 \wedge v_3 \in v_5);\\
&\mathrm{ZFC} \vdash \forall v_0 \forall v_5 \exists v_2 \forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_5);\\
&\mathrm{ZFC} \vdash \forall v_0 \forall v_1 \exists v_2 \forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1).
\end{aligned}$$

Now two applications of Corollary 3.28 gives

$$\mathrm{ZFC} \vdash \exists v_2 \forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1). \tag{1}$$

By the extensionality axiom and successive applications of the change of bound variables theorem 3.25 we have

$$\begin{aligned}
&\mathrm{ZFC} \vdash \forall v_0 \forall v_1[\forall v_2(v_2 \in v_0 \leftrightarrow v_2 \in v_1) \rightarrow v_0 = v_1];\\
&\mathrm{ZFC} \vdash \forall v_0 \forall v_4[\forall v_2(v_2 \in v_0 \leftrightarrow v_2 \in v_4) \rightarrow v_0 = v_4];\\
&\mathrm{ZFC} \vdash \forall v_0 \forall v_4[\forall v_3(v_3 \in v_0 \leftrightarrow v_3 \in v_4) \rightarrow v_0 = v_4].
\end{aligned}$$

Then by Corollary 3.28 twice we get

$$\mathrm{ZFC} \vdash \forall v_3(v_3 \in v_0 \leftrightarrow v_3 \in v_4) \rightarrow v_0 = v_4]. \tag{2}$$

Now the following formula is a tautology:

$$(v_3 \in v_2 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \rightarrow [(v_3 \in v_4 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \rightarrow (v_3 \in v_0 \leftrightarrow v_3 \in v_4)].$$

Hence by generalization and (L2) we get

$$\begin{aligned}
\vdash &\forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \rightarrow\\
&[\forall v_3(v_3 \in v_4 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \rightarrow \forall v_3(v_3 \in v_0 \leftrightarrow v_3 \in v_4)].
\end{aligned}$$

Using (2) it follows that

$$\begin{aligned}\text{ZFC} \vdash &\forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \rightarrow \\ &[\forall v_3(v_3 \in v_4 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \rightarrow v_0 = v_4].\end{aligned}$$

By generalization and Proposition 3.38 we then obtain

$$\begin{aligned}\text{ZFC} \vdash &\forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \rightarrow \\ &\forall v_4[\forall v_3(v_3 \in v_4 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \rightarrow v_0 = v_4].\end{aligned}$$

By a tautology we then have

$$\begin{aligned}\text{ZFC} \vdash &\forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \rightarrow \forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \wedge \\ &\forall v_4[\forall v_3(v_3 \in v_4 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \rightarrow v_0 = v_4].\end{aligned}$$

Generalization and (L2) yield

$$\begin{aligned}\text{ZFC} \vdash &\exists v_2 \forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \rightarrow \exists v_2[\forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \wedge \\ &\forall v_4[\forall v_3(v_3 \in v_4 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \rightarrow v_0 = v_4]].\end{aligned}$$

Hence by (1) we have

$$\begin{aligned}\text{ZFC} \vdash &\exists v_2[\forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \wedge \\ &\forall v_4[\forall v_3(v_3 \in v_4 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \rightarrow v_0 = v_4]].\end{aligned}$$

This is the desired result. □

Now we let $\mathscr{L}_3$ be the extension of $\mathscr{L}_2$ by adding a binary function symbol $\cap$, and

$$\text{ZFC}_3 = \text{ZFC}_2 \cup \{v_0 \cap v_1 = v_2 \leftrightarrow \forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1)\}.$$

Then $(\mathscr{L}_3, \text{ZFC}_3)$ is a simple definitional expansion of $(\mathscr{L}_2, \text{ZFC}_2)$.

EXERCISES

E4.1. Suppose that $\Gamma \vdash \varphi \rightarrow \psi$, $\Gamma \vdash \varphi \rightarrow \neg\psi$, and $\Gamma \vdash \neg\varphi \rightarrow \varphi$. Prove that Γ is inconsistent.

E4.2. Let $\mathscr{L}$ be a language with just one non-logical constant, a binary relation symbol $\mathbf{R}$. Let Γ consist of all sentences of the form $\exists v_1 \forall v_0[\mathbf{R}v_0v_1 \leftrightarrow \varphi]$ with φ a formula with only v_0 free. Show that Γ is inconsistent. Hint: take φ to be $\neg\mathbf{R}v_0v_0$.

E4.3. Show that the first-order deduction theorem fails if the condition that φ is a sentence is omitted. Hint: take $\Gamma = \emptyset$, let φ be the formula $v_0 = v_1$, and let ψ be the formula $v_0 = v_2$.

E4.4. In the language for $\overline{A} \stackrel{\text{def}}{=} (\omega, S, 0, +, \cdot)$, let τ be the term $v_0 + v_1 \cdot v_2$ and ν the term $v_0 + v_2$. Let a be the sequence $\langle 0, 1, 2, \ldots \rangle$. Let ρ be obtained from τ by replacing the occurrence of v_1 by ν.

(a) Describe ρ as a sequence of integers.
(b) What is $\rho^{\overline{A}}(a)$?
(c) What is $\nu^{\overline{A}}(a)$?
(d) Describe the sequence $a^1_{\nu^{\overline{A}}(a)}$ as a sequence of integers.
(e) Verify that $\rho^{\overline{A}}(a) = \tau^{\overline{A}}(a^1_{\nu^{\overline{A}}(a)})$ (cf. Lemma 4.4.)

E4.5. In the language for $\overline{A} \stackrel{\text{def}}{=} (\omega, S, 0, +, \cdot)$, let φ be the formula $\forall v_0(v_0 \cdot v_1 = v_1)$, let ν be the formula $v_1 + v_1$, and let $a = \langle 1, 0, 1, 0, \ldots \rangle$.
(a) Describe $\mathrm{Subf}^{v_1}_{\nu}\varphi$ as a sequence of integers
(b) What is $\nu^{\overline{A}}(a)$?
(c) Describe $a^1_{\nu^{\overline{A}}(a)}$ as a sequence of integers.
(d) Determine whether $\overline{A} \models \mathrm{Subf}^{v_1}_{\nu}\varphi[a]$ or not.
(e) Determine whether $\overline{A} \models \varphi[a^1_{\nu^{\overline{A}}(a)}]$ or not.

E4.6. Show that the condition in Lemma 4.6 that

no free occurrence of v_i in φ is within a subformula of the form $\forall v_k \mu$ with v_k a variable occurring in ν

is necessary for the conclusion of the lemma.

E4.7. Let $\overline{A}$ be an $\mathscr{L}$-structure, with $\mathscr{L}$ arbitrary. Define $\Gamma = \{\varphi : \varphi$ is a sentence and $\overline{A} \models \varphi[a]$ for any $a : \omega \to A\}$. Prove that Γ is complete and consistent.

E4.8. Call a set Γ *strongly complete* iff for every formula φ, $\Gamma \vdash \varphi$ or $\Gamma \vdash \neg\varphi$. Prove that if Γ is strongly complete, then $\Gamma \vdash \forall v_0 \forall v_1 (v_0 = v_1)$.

E4.9. Prove that if Γ is rich, then for every term σ with no variables occurring in σ there is an individual constant $\mathbf{c}$ such that $\Gamma \vdash \sigma = \mathbf{c}$.

E4.10. Prove that if Γ is rich, then for every sentence φ there is a sentence ψ with no quantifiers in it such that $\Gamma \vdash \varphi \leftrightarrow \psi$.

E4.11. Describe sentences in a language for ordering which say that $<$ is a linear ordering and there are infinitely many elements. Prove that the resulting set Γ of sentences is not complete.

E4.12. Prove that if a sentence φ holds in every infinite model of a set Γ of sentences, then there is an $m \in \omega$ such that it holds in every model of Γ with at least m elements.

E4.13. Let $\mathscr{L}$ be the language of ordering. Prove that there is no set Γ of sentences whose models are exactly the well-ordering structures.

E4.14. Suppose that Γ is a set of sentences, and φ is a sentence. Prove that if $\Gamma \models \varphi$, then $\Delta \models \varphi$ for some finite $\Delta \subseteq \Gamma$.

E4.15. Suppose that f is a function mapping a set M into a set N. Let $R = \{(a, b) : a, b \in M$ and $f(a) = f(b)\}$. Prove that R is an equivalence relation on M.

E4.16. Suppose that R is an equivalence relation on a set M. Prove that there is a function f mapping M into some set N such that $R = \{(a, b) : a, b \in M \text{ and } f(a) = f(b)\}$.

E4.17. Let Γ be a set of sentences in a first-order language, and let Δ be the collection of all sentences holding in every model of Γ. Prove that $\Delta = \{\varphi : \varphi \text{ is a sentence and } \Gamma \vdash \varphi\}$.

E4.18. Prove (2) in the proof of Theorem 4.24.