2. Fundamentals of first-order logic

We now introduce the main object of study in these notes: first-order logic. This is the most important and most widely studied of mathematical treatments of logic. The basic idea is to have a formal way of studying common mathematical structures. The structures we will consider are those implicit in most mathematical subjects. They consist of a *universe*, which is a set within which all action takes place, and various operations and relations on that universe. The universes we consider are ω, the set of all natural numbers $0, 1, \ldots$, the set $\mathbb{Q}$ of rational numbers, the set $\mathbb{R}$ of real numbers, and an arbitrary set. One can imagine other structures, and first-order logic is frequently applied to structures such as groups, rings, fields, etc., with which we do not assume familiarity. For us, typical structures are the following:

(ω, S), where $S(n) = n + 1$ for all $n \in \omega$.
$(\omega, +)$.
$(\omega, S, 0, +, \cdot)$. This is a basic structure for number theory, and will play an important role in our discussion of Gödel's incompleteness theorem.
$(\omega, <)$.
$(\mathbb{Q}, +, \cdot)$.
$(\mathbb{R}, +, \cdot, 0, 1, <)$. This structure is implicit in calculus.
(A, f), where A is any nonempty set and f maps A into A.
A, a nonempty set.

Different first-order languages are needed for each of these structures. All first-order languages have the following *symbols* in common. Again, as for sentential logic, we take these to be certain natural numbers.

1 (negation)
2 (implication)
3 (the equality symbol)
4 (the universal quantifier)
$5m$ for each positive integer m (variables ranging over elements, but not subsets, of a given structure) We denote $5m$ by v_{m-1}. Thus v_0 is 5, v_1 is 10, and in general v_i is $5(i + 1)$.

Special first-order languages have additional symbols for the functions and relations and special elements involved. These will always be taken to be some positive integers not among the above; thus they are positive integers greater than 4 but not divisible by 5. So we have in addition to the above logical symbols some non-logical symbols:

Relation symbols, each of a certain positive rank.
Function symbols, also each having a specified positive rank.
Individual constants.

For example, for the structures given above we need the following non-logical symbols:

For (ω, S): a one-place function symbol $\mathbf{S}$, taken to be 6.

For $(\omega, +)$: a two-place function symbol $\mathbf{+}$, taken to be 7.

For $(\omega, S, 0, +, \cdot)$: the symbols 6 and 7 above, and also an individual constant 0, taken to be 8, and a two-place function symbol $\bullet$, taken to be 9.

For $(\omega, <)$: a two-place relation symbol $\mathbf{R}$, taken to be 11.

For $(\mathbb{Q}, +, \cdot)$: symbols 7 and 9 as above

For $(\mathbb{R}, +, \cdot, 0, 1, <)$: symbols 7, 9, 8, 11 as above, and also an individual constant 1, taken to be 12.

For (A, f): a one-place function symbol $\mathbf{f}$, taken to be 13.

For A alone, there are no individual constants, function symbols, or relation symbols.

In summary, we have the following non-logical symbols:

Symbol	taken to be	type
$\mathbf{S}$	6	one-place function symbol
$+$	7	two-place function symbol
0	8	individual constant
$\bullet$	9	two-place function symbol
$<$	11	two-place relation symbol
1	12	individual constant
$\mathbf{f}$	13	one-place function symbol

Thus altogether the symbols are $1, 2, \ldots, 13, 15, 20, 25, \ldots$. Additional symbols, taken from $14, 16, 17, \ldots,$ will be needed in the proof of the completeness theorem. Of course the non-logical symbols which we have chosen are rather arbitrary. We could have chosen different ones with no essential differences in what follows, and we could consider many other structures. For brevity let $M = \{m \in \omega^+ : m \geq 4 \text{ and } m \text{ is not divisible by } 5\}$.

Formally, a first-order language is a quadruple (Rel, Fcn, Cn, rnk) such that Rel, Fcn, Cn are pairwise disjoint subsets of M (the sets of relation symbols, function symbols, and individual constants), and rnk is a function mapping $Rel \cup Fcn$indexfunction symbols into the positive integers; $rnk(\mathbf{S})$ gives the *rank* of the symbol $\mathbf{S}$. For example, the first-order language for the structure $(\mathbb{R}, +, \cdot, 0, 1, <)$ is the quadruple $(\{11\}, \{7, 9\}, \{8, 12\}, rnk)$, where rnk is the function with domain $\{7, 9, 11\}$ such that $rnk(7) = rnk(9) = rnk(11) = 2$.

Since the symbols of a first-order language are just certain natural numbers, the language is countable. It is also possible to consider uncountable languages, but we do not want to assume a knowledge of infinite sets.

Now we will define the notions of terms and formulas, which give a precise formulation of meaningful expressions. Terms are certain finite sequences of symbols. A *term construction sequence* is a sequence $\langle \tau_0, \ldots, \tau_{m-1} \rangle$, $m > 0$, with the following properties: for each $i < m$ one of the following holds:

τ_i is $\langle v_j \rangle$ for some natural number j.

τ_i is $\langle \mathbf{c} \rangle$ for some individual constant $\mathbf{c}$.

τ_i is $\langle \mathbf{F} \rangle \frown \sigma_0 \frown \sigma_1 \frown \cdots \frown \sigma_{n-1}$ for some n-place function symbol $\mathbf{F}$, with each σ_j equal to τ_k for some $k < i$, depending upon j.

A *term* is a sequence appearing in some term construction sequence. Note the similarity of this definition with that of sentential formula given in Chapter 1.

Frequently we will slightly simplify the notation for terms. Thus we might write simply v_j, or $\mathbf{c}$, or $\mathbf{F}\sigma_0 \ldots \sigma_{n-1}$ for the above.

For the languages associated with the structures above, terms look like this:

For (ω, S): v_i, $\mathbf{SS}v_3$, $\mathbf{S}v_{109}$. As sequences these are $\langle 5(i+1) \rangle$, $\langle 6, 6, 20 \rangle$, $\langle 6, 550 \rangle$. Here are term construction sequences showing that these are terms:

(1) $\langle \langle v_i \rangle \rangle$

(2) $\langle \langle v_3 \rangle, \langle \mathbf{S}v_3 \rangle, \langle \mathbf{SS}v_3 \rangle \rangle$.

(3) $\langle \langle v_{109} \rangle, \langle \mathbf{S}v_{109} \rangle \rangle$.

For $(\omega, +)$: officially we should write things like $++v_0v_1v_0$. But it is easier to understand and read if we adhere to common mathematical usage, and write, for this term, $(v_0 + v_1) + v_0$. Note that this is different from the term $v_0 + (v_1 + v_0)$, although the two "mean" the same. (We will see shortly what "mean" means). These two terms are $\langle 7, 7, 5, 10, 5 \rangle$ and $\langle 7, 5, 7, 10, 5 \rangle$ respectively. Here are term construction sequences for these two terms:

(1) $\langle \langle v_0 \rangle, \langle v_1 \rangle, \langle +v_0v_1 \rangle, \langle ++v_0v_1v_0 \rangle \rangle$.

(2) $\langle \langle v_0 \rangle, \langle v_1 \rangle, \langle +v_1v_0 \rangle, \langle +v_0+v_1v_0 \rangle \rangle$.

$(\omega, S, 0, +, \cdot)$. A typical term is $+v_2 \bullet v_0\mathbf{S}0$. Using normal mathematical notation, this is $v_2 + (v_0 \bullet \mathbf{S}0)$. As a sequence it is $\langle 7, 15, 9, 5, 6, 8 \rangle$.

$(\omega, <)$: the only terms are the variables v_i.

$(\mathbb{Q}, +, \cdot)$: again we use a more conventional notation. For example, we could write $v_0 \bullet (v_1 + v_2)$, which formally should be $\bullet v_0 + v_1v_2$ or, as a sequence, $\langle 9, 5, 7, 10, 15 \rangle$.

$(\mathbb{R}, +, \cdot, 0, 1, <)$: here the terms are polynomials with non-negative integer coefficents, in the usual sense. Thus the polynomial x^2+1 can be considered to be the term $(v_0 \bullet v_0) + 1$ or, formally, $+\bullet v_0v_01$; as a sequence, $\langle 7, 9, 5, 5, 12 \rangle$.

(A, f): here the terms are just like those for (ω, S), except for using $\mathbf{f}$ rather than $\mathbf{S}$.

A: the only terms are the variables.

The following two propositons are very similar, in statement and proof, to Propositions 1.1 and 1.2. The first one is the principle of *induction on terms.*

Proposition 2.1. *Let T be a collection of terms satisfying the following conditions:*

(i) Each variable is in T.

(ii) Each individual constant is in T.

(iii) If $\mathbf{F}$ is a function symbol of rank m and $\tau_0, \ldots, \tau_{m-1} \in T$, then also $\mathbf{F}\tau_0 \ldots \tau_{m-1} \in T$.

Then T consists of all terms.

Proof. Let τ be a term. Say that $\langle \sigma_0, \ldots, \sigma_{m-1} \rangle$ is a term construction sequence and $\sigma_i = \tau$. We prove by complete induction on j that $\sigma_j \in T$ for all $j < m$; hence $\tau \in T$. Suppose that $j < m$ and $\sigma_k \in T$ for all $k < j$. If $\sigma_j = \langle v_s \rangle$ for some s, then $\sigma_j \in T$. If $\sigma_j = \langle \mathbf{c} \rangle$ for some individual constant $\mathbf{c}$, then $s_j \in T$. Finally, suppose that σ_j is $\mathbf{F}\sigma_{k_0} \ldots \sigma_{k_{n-1}}$ with each $k_t < j$. Then $\sigma_{k_t} \in T$ for each $t < n$ by the inductive hypothesis, and it follows that $\sigma_j \in T$. This completes the inductive proof. $\square$

Proposition 2.2. *(i) Every term is a nonempty sequence.*

(ii) If τ is a term, then exactly one of the following conditions holds:

(a) τ is an individual constant.

(b) τ is a variable.

(c) There exist a function symbol $\mathbf{F}$, say of rank m, and terms $\sigma_0, \ldots, \sigma_{m-1}$ such that τ is $\mathbf{F}\sigma_0 \ldots \sigma_{m-1}$.

(iii) No proper initial segment of a term is a term.

(iv) If $\mathbf{F}$ and $\mathbf{G}$ are function symbols, say of ranks m and n respectively, and if $\sigma_0, \ldots, \sigma_{m-1}, \tau_0, \ldots, \tau_{n-1}$ are terms, and if $\mathbf{F}\sigma_0 \ldots \sigma_{m-1}$ is equal to $\mathbf{G}\tau_0, \ldots \tau_{n-1}$, then $\mathbf{F} = \mathbf{G}$, $m = n$, and $\sigma_i = \tau_i$ for all $i < m$.

Proof. (i): This is clear since any entry in a term construction sequence is nonempty.

(ii): Also clear.

(iii): We prove this by complete induction on the length of a term. So suppose that τ is a term, and for any term σ shorter than τ, no proper initial segment of σ is a term. We consider cases according to (ii).

Case 1. τ is an individual constant. Then τ has length 1, and any proper initial segment of τ is empty; by (i) the empty sequence is not a term.

Case 2. τ is a variable. Similarly.

Case 3. There exist an m-ary function symbol $\mathbf{F}$ and terms $\sigma_0, \ldots, \sigma_{m-1}$ such that τ is $\mathbf{F}\sigma_0 \ldots \sigma_{m-1}$. Suppose that ρ is a term which is a proper initial segment of τ. By (i), ρ is nonempty, and the first entry of ρ is $\mathbf{F}$. By (ii), ρ has the form $\mathbf{F}\xi_0 \ldots \xi_{m-1}$ for certain terms $\xi_0, \ldots, \xi_{m-1}$. Since both σ_0 and ξ_0 are shorter terms than τ, and one of them is an initial segment of the other, the induction hypothesis gives $\sigma_0 = \xi_0$. Let $i < m$ be maximum such that $\sigma_i = \xi_i$. Since ρ is a proper initial segment of τ, we must have $i < m - 1$. But σ_{i+1} and ξ_{i+1} are shorter terms than τ and one is a segment of the other, so by the inductive hypthesis $\sigma_{i+1} = \xi_{i+1}$, contradicting the choice of i.

(iv): $\mathbf{F}$ is the first entry of $\mathbf{F}\sigma_0 \ldots \sigma_{m-1}$ and $\mathbf{G}$ is the first entry of $\mathbf{G}\tau_0, \ldots \tau_{n-1}$, so $\mathbf{F} = \mathbf{G}$. Then by (ii) we get $m = n$. By induction using (iii), each $\sigma_i = \tau_i$. $\square$

We introduced at the beginning of this chapter some typical structures for first-order logic. Now we want to give the general notion of a structure. For a given first-order language $\mathscr{L} = (Rel, Fcn, Cn, rnk)$, an *$\mathscr{L}$-structure* is a quadruple $\overline{A} = (A, Rel', Fcn', Cn')$ such that A is a nonempty set (the *universe* of the structure), Rel' is a function assigning to each relation symbol $\mathbf{R}$ a $rnk(\mathbf{R})$-ary relation on A, i.e., a collection of $rnk(\mathbf{R})$-tuples of elements of A, Fcn' is a function assigning to each function symbol $\mathbf{F}$ a $rnk(\mathbf{F})$-ary opeation on A, i.e., a function assigning a value in A to each $rnk(\mathbf{F})$-tuple of elements of A, and Cn' is a function assigning to each individual constant $\mathbf{c}$ an element of A. Usually instead of $Rel'(\mathbf{R})$, $Fcn'(\mathbf{F})$ and $Cn'(\mathbf{c})$ we write $\mathbf{R}^{\overline{A}}$, $\mathbf{F}^{\overline{A}}$, and $\mathbf{c}^{\overline{A}}$.

The typical structures introduced at the beginning of this chapter can easily be put into this general framework. For example, the structure $(\mathbb{Q}, +, \cdot)$ can be considered to be the structure $(\mathbb{Q}, Rel', Fcn', Cn')$ with $Rel' = Cn' = \emptyset$ and Fcn' the function with domain $\{7, 9\}$ such that $Fcn'(7)$ is $+$ and $Fcn'(9)$ is $\cdot$.

Now we define the "meaning" of terms. This is a recursive definition, similar to the definition of the values of sentential formulas under assignments.

Proposition 2.3. *Let $\overline{A}$ be a structure, and a a function mapping ω into A. (A is the universe of $\overline{A}$.) Then there is a function F mapping the set of terms into A with the following properties:*

(i) $F(v_i) = a_i$ for each $i \in \omega$.

(ii) $F(\mathbf{c}) = \mathbf{c}^{\overline{A}}$ for each individual constant $\mathbf{c}$.

(iii) $F(\mathbf{F}\sigma_0 \ldots \sigma_{m-1}) = \mathbf{F}^{\overline{A}}(F(\sigma_0), \ldots, F(\sigma_{m-1}))$ for every m-ary function symbol $\mathbf{F}$ and all terms $\sigma_0, \ldots, \sigma_{m-1}$

With F as in Proposition 2.3, we denote $F(\sigma)$ by $\sigma^{\overline{A}}(a)$. Thus

$$\begin{aligned} v_i^{\overline{A}}(a) &= a_i; \\ \mathbf{c}^{\overline{A}}(a) &= \mathbf{c}^{\overline{A}}; \\ (\mathbf{F}\tau_0 \ldots \tau_{m-1})^{\overline{A}}(a) &= \mathbf{F}^{\overline{A}}(\tau_0^{\overline{A}}(a), \ldots, \tau_{m-1}^{\overline{A}}(a)). \end{aligned}$$

Here v_i is any variable, $\mathbf{c}$ any individual constant, and $\mathbf{F}$ any function symbol (of some rank, say m).

What $\sigma^{\overline{A}}(a)$ means intuitively is: replace the individual constants and function symbols by the actual members of A and functions on A given by the structure $\overline{A}$, and replace the variables v_i by coresponding elements a_i of A; calculate the result, giving an element of A. We illustrate this with the structures given at the beginning of this chapter.

$\overline{A} = (\omega, S)$. Let $a = \langle 7, 2, 0, 0, 0, \ldots \rangle$. Then
$(\mathbf{SSS}v_1)^{\overline{A}}(a) = SSS2 = 5$;
$(\mathbf{SSSSSSS}v_0)^{A}(a) = 14$.

$\overline{A} = (\omega, +)$. Let $a = \langle 3, 4, 5, \ldots \rangle$. Then $((v_3 + v_0) + v_9)^{\overline{A}}(a) = 6 + 3 + 12 = 21$.

$\overline{A} = (\omega, S, 0, +, \cdot)$. Let $a = \langle 0, 1, 2, \ldots \rangle$. Then

$$((\mathbf{S}0 + v_2) \cdot v_3)(a) = (S0 + 2) \cdot 3 = 9.$$

$\overline{A} = (\omega, <)$. Here the terms are only the variables. For example, with $a = \langle 2^m : m \in \omega \rangle$ we have $v_3^{\overline{A}}(a) = 8$.

$\overline{A} = (\mathbb{Q}, +, \cdot)$. Here the terms are essentially polynomials with positive integer coefficients and with 0 constant term. For example, the polynomial $2x^2 + x$ corresponds to the term $(v_0 \cdot v_0 + v_0 \cdot v_0) + v_0$, whose value at $\langle 3, 4, 5, \ldots\rangle$ is 21.

$\overline{A} = (\mathbb{R}, +, \cdot, 0, 1, <)$. Now the terms are all polynomials with positive integer coefficients. For example, x^2+3 corresponds to the term $v_0 \cdot v_0 + (1+(1+1))$, whose value at $\langle \sqrt{2}, \sqrt{2}, \ldots\rangle$ is 5.

$\overline{A} = (A, f)$, with A any set and $f : A \to A$. For example, take $A = \{0, 3, 4\}$ and $f : A \to A$ with $f(0) = 3$, $f(3) = 4$, $f(4) = 0$. Consider the term $\mathbf{f}\,\mathbf{f}\,\mathbf{f}\,\mathbf{f}\,\mathbf{f}v_2$ and the sequence $a = \langle 0, 3, 4, 4, 4, \ldots\rangle$. Then the value of this term at a is

$$f(f(f(f(f(4))))) = f(f(f(f(0)))) = f(f(f(3))) = f(f(4)) = f(0) = 3.$$

$\overline{A} = A$. The terms are just the variables. For example, $v_0(a) = x$ when $a = \langle x, x, \ldots\rangle$ with x a particular element of A.

Proposition 2.4. *Suppose that τ is a term, $\overline{A}$ is a structure, a, b assignments, and $a(i) = b(i)$ for all i such that v_i occurs in τ. Then $\tau^{\overline{A}}(a) = \tau^{\overline{A}}(b)$.*

Proof. By induction on τ:

$$\begin{aligned}
\mathbf{c}^{\overline{A}}(a) &= \mathbf{c}^{\overline{A}} = \mathbf{c}^{\overline{A}}(b);\\
v_i^{\overline{A}}(a) &= a(i) = b(i) = v_i^{\overline{A}}(b);\\
(\mathbf{F}\sigma_0 \ldots \sigma_{m-1})^{\overline{A}}(a) &= \mathbf{F}^{\overline{A}}(\sigma_0^{\overline{A}}(a), \ldots, \sigma_{m-1}^{\overline{A}}(a))\\
&= \mathbf{F}^{\overline{A}}(\sigma_0^{\overline{A}}(b), \ldots, \sigma_{m-1}^{\overline{A}}(b))\\
&= (\mathbf{F}\sigma_0 \ldots \sigma_{m-1})^{\overline{A}}(b).
\end{aligned}$$

The last step here is the induction step (many of them, one for each function symbol and associated terms). The inductive assumption is that $a(i) = b(i)$ for all i for which v_i occurs in $\mathbf{F}\sigma_0 \ldots \sigma_{m-1}$; hence also for each $j < m$, $a(i) = b(i)$ for all i for which v_i occurs in σ_j, so that the inductive hypothesis can be applied. □

This proposition enables us to simplify our notation a little bit. If n is such that each variable occurring in τ has index less than n, then in the notation $\varphi^{\overline{A}}(a)$ we can just use the first n entries of a rather than the entire infinite sequence. For example, for the above illustrations we can simplify things like this:

$\overline{A} = (\omega, S)$:
$(\mathbf{SSS}v_1)^{\overline{A}}(7, 2) = SSS2 = 5$;
$(\mathbf{SSSSSSS}v_0)^{A}(7) = 14$.

$\overline{A} = (\omega, +)$: $((v_3 + v_0) + v_9)^{\overline{A}}(3, 4, 5, 6, 7, 8, 9, 10, 11, 12) = 6 + 3 + 12 = 21$.

$\overline{A} = (\omega, S, 0, +, \cdot)$: $((\mathbf{S}0 + v_2) \cdot v_3)(0, 1, 2, 3) = 9$.

$\overline{A} = (\omega, <)$: $v_3^{\overline{A}}(1, 2, 4, 8) = 8$.

$\overline{A} = (\mathbb{Q}, +, \cdot)$: $((v_0 \cdot v_0 + v_0 \cdot v_0) + v_0)^{\overline{A}}(3) = 21$.

$\overline{A} = (\mathbb{R}, +, \cdot, 0, 1, <)$: $(v_0 \cdot v_0 + (1 + (1 + 1)))^{\overline{A}}(\sqrt{2}) = 5$.

$\overline{A} = (A, f)$, with A any set and $f : A \to A$. For example, take $A = \{0, 3, 4\}$ and $f : A \to A$ with $f(0) = 3$, $f(3) = 4$, $f(4) = 0$. Then $(\mathbf{f}\,\mathbf{f}\,\mathbf{f}\,\mathbf{f}\,\mathbf{f} v_2)^{\overline{A}}(0, 3, 4) = 3$.

$\overline{A} = A$: $v_0(x) = x$ for any $x \in A$.

We turn to the definition of formulas. For any terms σ, τ we define $\sigma = \tau$ to be the sequence $\langle 3\rangle^\frown\sigma^\frown\tau$. Such a sequence is called an *atomic equality formula*. An *atomic non-equality formula* is a sequence of the form $\langle \mathbf{R}\rangle^\frown\sigma_0^\frown \cdots {}^\frown \sigma_{m-1}$ where $\mathbf{R}$ is an m-ary relation symbol and $\sigma_0, \ldots \sigma_{m-1}$ are terms. An *atomic formula* is either an atomic equality formula or an atomic non-equality formula.

Examples in the language for $(\mathbf{R}, +, \cdot, 0, 1, <)$ are: $x + y = 0$, $x^2 = 1$ (atomic equality formulas) and $x \bullet y < 2$ (an atomic non-equality formula). More formally, these are $v_0 + v_1 = 0$, $v_0 \bullet v_0 = 1$, and $v_0 \bullet v_1 < 1 + 1$ or, as sequences, $\langle 3, 7, 5, 10, 8\rangle$, $\langle 3, 9, 5, 5, 12\rangle$, $\langle 11, 9, 5, 10, 7, 12, 12\rangle$. We define $\neg$, a function assigning to each sequence φ of symbols of a first-order language the sequence $\neg\varphi \stackrel{\text{def}}{=} \langle 1\rangle^\frown\varphi$. $\to$ is the function assigning to each pair (φ, ψ) of sequences of symbols the sequence $\varphi \to \psi \stackrel{\text{def}}{=} \langle 2\rangle^\frown\varphi^\frown\psi$. $\forall$ is the function assigning to each pair (i, φ) with $i \in \omega$ and φ a sequence of symbols the sequence $\forall v_i\varphi \stackrel{\text{def}}{=} \langle 4, 5i + 5\rangle^\frown\varphi$. A *formula construction sequence* is a sequence $\langle \varphi_0, \ldots, \varphi_{m-1}\rangle$ such that for each $i < m$ one of the following holds:

(1) φ_i is an atomic formula.

(2) There is a $j < i$ such that φ_i is $\neg\varphi_j$

(3) There are $j, k < i$ such that φ_i is $\varphi_j \to \varphi_k$.

(4) There exist $j < i$ and $k \in \omega$ such that φ_i is $\forall v_k\varphi_j$.

A *formula* is an expression which appears as an entry in some formula construction sequence.

The following is the principle of *induction on formulas*.

Proposition 2.5. *Suppose that Γ is a set of formulas satisfying the following conditions:*
(i) Every atomic formula is in Γ.
(ii) If $\varphi \in \Gamma$, then $\neg\varphi \in \Gamma$.
(iii) If $\varphi, \psi \in \Gamma$, then $(\varphi \to \psi) \in \Gamma$.
(iv) If $\varphi \in \Gamma$ and $i \in \omega$, then $\forall v_i\varphi \in \Gamma$.

Then Γ is the set of all formulas.

Proof. It suffices to take any formula construction sequence $\langle \varphi_0, \ldots, \varphi_{m-1}\rangle$ and show by complete induction on i that $\varphi_i \in \Gamma$ for all $i \in \omega$. We leave this as an exercise. □

Proposition 2.6. *(i) Every formula is a nonempty sequence.*
(ii) If φ is a formula, then exactly one of the following conditions holds:

(a) φ is an atomic equality formula, and there are terms σ, τ such that φ is $\sigma = \tau$.

(b) φ is an atomic non-equality formula, and there exist a positive integer m, a relation symbol $\mathbf{R}$ of rank m, and terms $\sigma_0, \ldots, \sigma_{m-1}$, such that φ is $\mathbf{R}\sigma_0 \ldots \sigma_{m-1}$.

(c) There is a formula ψ such that φ is $\neg\psi$.

(d) There are formulas ψ, χ such that φ is $\psi \to \chi$.

(e) There exist a formula ψ and a natural number i such that φ is $\forall v_i \psi$.

(iii) No proper initial segment of a formula is a formula.

(iv) (a) If φ is an atomic equality formula, then there are unique terms σ, τ such that φ is $\sigma = \tau$.

(b) If φ is an atomic non-equality formula, then there exist a unique positive integer m, a unique relation symbol $\mathbf{R}$ of rank m, and unique terms $\sigma_0, \ldots, \sigma_{m-1}$, such that φ is $\mathbf{R}\sigma_0 \ldots \sigma_{m-1}$.

(c) If φ is a formula and the first symbol of φ is 1, then there is a unique formula ψ such that φ is $\neg\psi$.

(d) If φ is a formula and the first symbol of φ is 2, then there are unique formulas ψ, χ such that φ is $\psi \to \chi$.

(e) If φ is a formula and the first symbol of φ is 4, then there exist a unique natural number i and a unique formula ψ such that φ is $\forall v_i \psi$.

Proof. (i): First note that this is true of atomic formulas, since an atomic formula must have at least a first symbol 3 or some relation symbol. Knowing this about atomic formulas, any entry in a formula construction sequence is nonempty, since the entry is either an atomic formula or else begins with 1,2, or 4.

(ii): This is true on looking at any entry in a formula construction sequence: either the entry begins with 3 or a relation symbol and hence (a) or (b) holds, or it begins with 1, 2, or 4, giving (c), (d) or (e). Only one of (a)–(e) holds because of the first symbol in the entry.

(iii): We prove this by complete induction on the length of the formula. Thus suppose that φ is a formula of length m, and for any formula ψ of length less than m, no proper initial segment of ψ is a formula. Suppose that χ is a proper initial segment of φ and χ is a formula; we want to get a contradiction. By (ii) we have several cases.

Case 1. φ is an atomic equality formula $\sigma = \tau$ for certain terms σ, τ. Thus φ is $\langle 3\rangle^\frown\sigma^\frown\tau$. Since χ is a formula which begins with 3 (since χ is an initial segment of φ and is nonempty by (i)), (ii) yields that χ is $\langle 3\rangle^\frown\rho^\frown\xi$ for some terms ρ, ξ. Hence $\sigma^\frown\psi = \rho^\frown\xi$. Thus σ is an initial segment of ρ or ρ is an initial segment of σ. By Proposition 2.2(iii) it follows that $\sigma = \rho$. Then also $\tau = \xi$, so $\varphi = \chi$, contradiction.

Case 2. φ is an atomic non-equality formula $\mathbf{R}\sigma_0 \ldots \sigma_{m-1}$ for some m-ary relation symbol $\mathbf{R}$ and some terms $\sigma_0, \ldots, \sigma_{m-1}$. Then χ is a formula which begins with $\mathbf{R}$, and so there exist terms $\tau_0, \ldots, \tau_{m-1}$ such that χ is $\mathbf{R}\tau_0 \ldots \tau_{m-1}$. By induction using Proposition 2.2(iii), $\sigma_i = \tau_i$ for all $i < m$, so $\varphi = \chi$, contradiction.

Case 3. φ is $\neg\psi$ for some formula ψ. Then 1 is the first entry of χ, so by (ii) χ has the form $\neg\rho$ for some formula ρ. Thus ρ is a proper initial segment of ψ, contradicting the inductive hypothesis, since ψ is shorter than φ.

Case 4. φ is $\psi \to \theta$ for some formulas ψ, θ, i.e., it is $\langle 2\rangle^\frown\psi^\frown\theta$. Then χ starts with 2, so by (ii) χ has the form $\langle 2\rangle^\frown\sigma^\frown\tau$ for some formulas σ, τ. Now both ψ and σ are shorter than φ, and one is an initial segment of the other. So $\psi = \sigma$ by the inductive assumption. Then τ is a proper initial segment of θ, contradicting the inductive assumption.

Case 5. φ is $\langle 4, 5(i+1)\rangle^\frown\psi$ for some $i \in \omega$ and some formula ψ. Then by (ii), χ is $\langle 4, 5(i+1)\rangle^\frown\theta$ for some formula θ. So θ is a proper initial segment of ψ, contradiction.

(iv): These conditions follow from Proposition 2.2(iii) and (iii). □

Now we come to a fundamental definition connecting language with structures. Again this is a definition by recursion; it is given in the following proposition. First a bit of notation. If $a : \omega \to A$, $i \in \omega$, and $s \in A$, then by a^i_s we mean the sequence which is just like a except that $a^i_s(i) = s$.

Proposition 2.7. *Suppose that $\overline{A}$ is an $\mathscr{L}$-structure. Then there is a function G assigning to each formula φ and each sequence $a : \omega \to A$ a value $G(\varphi, a) \in \{0, 1\}$, such that*

(i) For any terms σ, τ, $G(\sigma = \tau, a) = 1$ iff $\sigma^{\overline{A}}(a) = \tau^{\overline{A}}(a)$.

(ii) For each m-ary relation symbol $\mathbf{R}$ and terms $\sigma_0, \ldots, \sigma_{m-1}$, $G(\mathbf{R}\sigma_0 \ldots \sigma_{m-1}, a) = 1$ iff $\langle \sigma_0^{\overline{A}}(a), \ldots, \sigma_{m-1}^{\overline{A}}(a)\rangle \in \mathbf{R}^{\overline{A}}$.

(iii) For every formula φ, $G(\neg\varphi, a) = 1 - G(\varphi, a)$.

(iv) For all formulas φ, ψ, $G(\varphi \to \psi, a) = 0$ iff $G(\varphi, a) = 1$ and $G(\psi, a) = 0$.

(v) For all formulas φ and any $i \in \omega$, $G(\forall v_i\varphi, a) = 1$ iff for every $s \in A$, $G(\varphi, a^i_s) = 1$.

With G as in Proposition 2.7, we write $\overline{A} \models \varphi[a]$ iff $G(\varphi, a) = 1$. $\overline{A} \models \varphi[a]$ is read: "$\overline{A}$ is a model of φ under a" or "$\overline{A}$ models φ under a" or "φ is satisfied by a in $\overline{A}$" or "φ holds in $\overline{A}$ under the assignment a". In summary:

$\overline{A} \models (\sigma = \tau)[a]$ iff $\sigma^{\overline{A}}(a) = \tau^{\overline{A}}(b)$. Here σ and τ are terms.

$\overline{A} \models (\mathbf{R}\sigma_0 \ldots \sigma_{m-1})[a]$ iff the m-tuple $\langle \sigma_0^{\overline{A}}, \ldots, \sigma_{m-1}^{\overline{A}}\rangle$ is in the relation $R^{\overline{A}}$. Here $\mathbf{R}$ is an m-ary relation symbol, and $\sigma_0, \ldots, \sigma_{m-1}$ are terms.

$\overline{A} \models (\neg\varphi)[a]$ iff it is not the case that $\overline{A} \models \varphi[a]$.

$\overline{A} \models (\varphi \to \psi)[a]$ iff either it is not true that $\overline{A} \models \varphi[a]$, or it is true that $\overline{A} \models \psi[a]$. (Equivalently, iff ($\overline{A} \models \varphi[a]$ implies that $\overline{A} \models \psi[a]$).

$\overline{A} \models (\forall v_i\varphi)[a]$ iff $\overline{A} \models \varphi[a^i_s]$ for every $s \in A$.

Before giving examples of this notion, we define some additional logical notions:

$\varphi \vee \psi$ is the formula $\neg\varphi \to \psi$; $\varphi \vee \psi$ is called the *disjunction* of φ and ψ.

$\varphi \wedge \psi$ is the formula $\neg(\varphi \to \neg\psi)$; $\varphi \wedge \psi$ is called the *conjunction* of φ and ψ.

$\varphi \leftrightarrow \psi$ is the formula $(\varphi \to \psi) \wedge (\psi \to \varphi)$; $\varphi \leftrightarrow \psi$ is called the *equivalence* between φ and ψ.

$\exists v_i\varphi$ is the formula $\neg\forall v_i\neg\varphi$; $\exists$ is the *existential quantifier*.

These notions mean the following.

Proposition 2.8. *Let $\overline{A}$ be a structure and $a : \omega \to A$.*

(i) $\overline{A} \models (\varphi \vee \psi)[a]$ iff $\overline{A} \models \varphi[a]$ or $\overline{A} \models \psi[a]$ (or both).

(ii) $\overline{A} \models (\varphi \wedge \psi)[a]$ *iff* $\overline{A} \models \varphi[a]$ *and* $\overline{A} \models \psi[a]$.
(iii) $\overline{A} \models (\varphi \leftrightarrow \psi)[a]$ *iff* ($\overline{A} \models \varphi[a]$ *iff* $\overline{A} \models \psi[a]$).
(iv) $\overline{A} \models \exists v_i \varphi[a]$ *iff there is a* $b \in A$ *such that* $\overline{A} \models \varphi[a^i_b]$.

Proof. The proof consists in reducing the statements to ordinary mathematical usage. (i):

$$\begin{aligned}
\overline{A} \models (\varphi \vee \psi)[a] \quad & \text{iff} \quad \overline{A} \models (\neg\varphi \to \psi)[a] \\
& \text{iff} \quad \text{either it is not true that } \overline{A} \models (\neg\varphi)[a] \text{ or it is true that } \overline{A} \models \psi[a] \\
& \text{iff} \quad \text{not}(\text{not}(\overline{A} \models \varphi[a])) \text{ or } \overline{A} \models \psi[a] \\
& \text{iff} \quad \overline{A} \models \varphi[a] \text{ or } \overline{A} \models \psi[a].
\end{aligned}$$

(ii):

$$\begin{aligned}
\overline{A} \models (\varphi \wedge \psi)[a] \quad & \text{iff} \quad \text{not}(\overline{A} \models (\varphi \to \neg\psi)[a]) \\
& \text{iff} \quad \text{not}(\text{not}(\overline{A} \models \varphi[a]) \text{ or } \overline{A} \models \neg\psi[a]) \\
& \text{iff} \quad \text{not}(\text{not}(\overline{A} \models \varphi[a]) \text{ or not}(\overline{A} \models \psi[a])) \\
& \text{iff} \quad \overline{A} \models \varphi[a] \text{ and } \overline{A} \models \psi[a].
\end{aligned}$$

(iii):

$$\begin{aligned}
\overline{A} \models (\varphi \leftrightarrow \psi)[a] \quad & \text{iff} \quad \overline{A} \models ((\varphi \to \psi) \wedge (\psi \to \psi))[a] \\
& \text{iff} \quad \overline{A} \models ((\varphi \to \psi)[a] \text{ and } \overline{A} \models (\psi \to \psi))[a] \\
& \text{iff} \quad (\overline{A} \models \varphi[a] \text{ implies that } \overline{A} \models \psi[a]) \text{ and} \\
& \qquad\ (\overline{A} \models \psi[a] \text{ implies that } \overline{A} \models \varphi[a]) \\
& \text{iff} \quad (\overline{A} \models \varphi[a] \text{ iff } \overline{A} \models \psi[a]).
\end{aligned}$$

(iv):

$$\begin{aligned}
\overline{A} \models \exists v_i \varphi[a] \quad & \text{iff} \quad \overline{A} \models \neg\forall v_i \neg\varphi[a] \\
& \text{iff} \quad \text{not}(\text{for all } b \in A(\overline{A} \models \neg\varphi[a^i_b])) \\
& \text{iff} \quad \text{not}(\text{for all } b \in A(\text{not}(\overline{A} \models \varphi[a^i_b]))) \\
& \text{iff} \quad \text{there is a } b \in A \text{ such that } \overline{A} \models \varphi[a^i_b].
\end{aligned}$$ □

Now we want to give several examples of translating ordinary mathematical statements about structures into first-order logic.

(1) One of the Peano postulates for the natural numbers says that if m and n are natural numbers and $m + 1 = n + 1$, then $m = n$. A translation into our language for (ω, S) is

$$\forall v_0 \forall v_1 [\mathbf{S}v_0 = \mathbf{S}v_1 \to v_0 = v_1].$$

If φ is this formula, then $(\omega, S) \models \varphi[a]$ for any $a : \omega \to \omega$.

(2) The commutative law for addition of natural numbers is expressed by the formula

$$\forall v_0 \forall v_1 [v_0 + v_1 = v_1 + v_0]$$

in the language for $(\omega, +)$; and $(\omega, +) \models \varphi[a]$ for this formula φ, for any $a : \omega \to \omega$.

(3) In the language for $(\omega, <)$, to say that $a_0 + 1 < a_2$ one can use the formula

$$\exists v_2 [\mathbf{R} v_0 v_2 \wedge \mathbf{R} v_2 v_1];$$

if we call this formula ψ, then $(\omega, <) \models \psi[a]$ iff $a_0 + 1 < a_1$.

(4) In the language for $(\mathbf{Q}, +, \cdot)$, the formula

$$\forall v_1 [v_0 + v_1 = v_1]$$

defines 0, in the sense that $(\mathbf{Q}, +, \cdot) \models \forall v_1 [v_0 + v_1 = v_1][a]$ iff $a_0 = 0$.

(5) In the language for $(\mathbb{R}, +, \cdot, 0, 1, <)$, the formula

$$\forall v_0 [0 < v_0 \to \exists v_1 [v_1 \bullet v_1 = v_0]]$$

expresses that every positive real number has a square root.

(6) In the language for (A, f), the following formula expresses that f is a one-one function:

$$\forall v_0 \forall v_1 [\mathbf{f} v_0 = \mathbf{f} v_1 \to v_0 = v_1].$$

(7) The following formula holds in $(\mathbb{R}, +, \cdot, 0, 1, <)$ under the assignment a iff $|a_0 - a_1| < 1$:

$$v_0 < v_1 + 1 \wedge v_1 < v_0 + 1.$$

(8) For the structure $(\mathbb{R}, +, \cdot, 0, 1, <, f)$, where f is a function mapping $\mathbb{R}$ into $\mathbb{R}$, the following formula expresses that f is continuous at the argument a_0, given an assignment a. Here we use $\mathbf{f}$ as the symbol corresponding to f.

$$\begin{aligned} &\forall v_1 [0 < v_1 \to \exists v_2 [0 < v_2 \wedge \forall v_3 [v_3 < v_0 + v_2 \wedge v_0 < v_3 + v_2 \\ &\to \mathbf{f}(v_0) < \mathbf{f}(v_3) + v_1 \wedge \mathbf{f}(v_3) < \mathbf{f}(v_0) + v_1]]]. \end{aligned}$$

Explanation: this would normally be written like this:

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x [|x - a_0| < \delta \to |f(x) - f(a_0)| < \varepsilon].$$

We are using v_1 in place of ε, v_2 in place of δ, v_3 in place of x, and the absolute value is expressed as in (7).

(9) The formula $\forall v_0 \forall v_1 (v_0 = v_1)$ holds in a structure iff the structure has only one element. The formula $\exists v_0 \exists v_1 (\neg (v_0 = v_1) \wedge \forall v_2 (v_0 = v_2 \vee v_1 = v_2))$ holds in a structure iff the structure has exactly two elements.

We say that $\overline{A}$ is a *model* of φ iff $\overline{A} \models \varphi[a]$ for every $a : \omega \to A$. If Γ is a set of formulas, we write $\Gamma \models \varphi$ iff every structure which models each member of Γ also models φ. $\models \varphi$ means that every structure models φ. φ is then called *universally valid.*

A different meaning for $\Gamma \vdash \varphi$ is sometimes found in books or articles. Define $\Gamma \vdash' \varphi$ iff for every structure $\overline{A}$ in the implicit language and every $a : \omega \to A$, if $\overline{A} \models \psi[a]$ for each $\psi \in \Gamma$, then $\overline{A} \models \varphi[a]$. The two notions $\Gamma \vdash \varphi$ and $\Gamma \vdash' \varphi$ are different. For example, $\{v_0 = v_1\} \vdash v_0 = v_2$, since a model of $\{v_0 = v_1\}$ has only one element, and hence is a model of $v_0 = v_2$. However, if A has two distinct elements a, b, then $\overline{A} \models (v_0 = v_1)[\langle a, a, b, b, \ldots\rangle]$ but $\overline{A} \not\models (v_0 = v_2)[\langle a, a, b, b, \ldots\rangle]$; hence $\{v_0 = v_1\} \not\models' v_0 = v_2$.

In the important special case in which $\Gamma \cup \{\varphi\}$ is a set of sentences, the two notions coincide. (The notion of a sentence is defined in the next chapter.)

Some examples of universally valid formulas are:

$v_0 = v_0$.
$v_0 = v_1 \to (\mathbf{R}v_0v_2 \to \mathbf{R}v_1v_2)$, where $\mathbf{R}$ is a binary relation symbol.
$\forall v_0\varphi \to \varphi$.
$\varphi \to \exists v_0\varphi$.
$v_0 = v_1 \to \forall v_2(v_0 = v_1)$.

Now we want to apply the material of Chapter 1 concerning sentential logic. By definition, a tautology in a first-order language is a formula ψ such that there exist formulas $\varphi_0, \varphi_1, \ldots$ and a sentential tautology χ such that ψ is obtained from χ by replacing each symbol S_i occurring in χ by φ_i, for each $i < \omega$.

Theorem 2.9. *If ψ is a tautology in a first-order language, then ψ holds in every structure for that language.*

Proof. Let $\overline{A}$ be any structure, and $b : \omega \to A$ any assignment. We want to show that $\overline{A} \models \psi[b]$. Let formulas $\varphi_0, \varphi_1, \ldots, \chi$ be given as in the above definition. For each sentential formula θ, let θ' be the first-order formula obtained from θ by replacing each sentential variable S_i by φ_i. Thus χ' is ψ. We define a sentential assignment f by setting, for each $i \in \omega$,

$$f(i) = \begin{cases} 1 & \text{if } \overline{A} \models \varphi_i[b], \\ 0 & \text{otherwise.} \end{cases}$$

Then we claim:

(*) For any sentential formula θ, $\overline{A} \models \theta'[b]$ iff $\theta[f] = 1$.

We prove this by induction on θ:

If θ is S_i, then θ' is φ_i, and our condition holds by definition. If inductively θ is $\neg\tau$, then θ' is $\neg\tau'$, and

$$\begin{aligned} \overline{A} \models \theta'[b] \quad &\text{iff} \quad \text{not}(\overline{A} \models \tau'[b]) \\ &\text{iff} \quad \text{not}(\tau[f] = 1) \\ &\text{iff} \quad \tau[f] = 0 \\ &\text{iff} \quad \theta[f] = 1. \end{aligned}$$

Finally if inductively θ is $\tau \rightarrow \xi$, then θ' is $\tau' \rightarrow \xi'$, and

$$\begin{aligned} \overline{A} \models \theta'[b] \quad & \text{iff} \quad (\overline{A} \models \tau'[b] \text{ implies that } \overline{A} \models \xi'[b] \\ & \text{iff} \quad \tau[f] = 1 \text{ implies that } \xi[f] = 1 \\ & \text{iff} \quad \theta[f] = 1. \end{aligned}$$

This finishes the proof of (*).

Applying (*) to χ, we get $\overline{A} \models \chi'[b]$, i.e., $\overline{A} \models \psi[b]$. □

Theorem 2.10. (Disjunctive normal form) *Suppose that $\varphi_0, \varphi_1, \ldots$ is a sequence of first-order formulas, ψ is a first-order formula which has a model, and ψ is obtained from a sentential formula χ by replacing each symbol S_i in χ by φ_i, for all $i < \omega$. Suppose that every S_i occurring in χ has $i < m$. Then there is a nonempty set M of m-termed sequences of 0's and 1's such that*

$$\emptyset \models \psi \leftrightarrow \bigvee_{\varepsilon \in M} \bigwedge_{i<m} \varphi_i^{\varepsilon(i)},$$

where ρ^1 is ρ and ρ^0 is $\neg\rho$, for any formula ρ.

Recall here the definitions of $\bigvee$ and $\bigwedge$ given in Chapter 1 for sentential logic; we take the same definitions for first-order logic.

Proof. By the proof of Theorem 2.9, χ is true under some sentential assignment. Hence our theorem follows from Theorem 1.8 and Theorem 2.9. □

EXERCISES

E2.1. Give the exact definition of the language for the structure $(\omega, <)$.

E2.2. Give the exact definition of the language for the set A (no individual constants, function symbols, or relation symbols).

E2.3. Describe a term construction sequence which shows that $+ \bullet v_0 v_0 v_1$ is a term in the language for $(\mathbb{R}, +, \cdot, 0, 1, <)$.

E2.4. In any first-order language, show that the sequence $\langle v_0, v_0 \rangle$ is not a term. Hint: use Proposition 2.2.

E2.5. In the language for $(\omega, S, 0, +, \cdot)$, show that the sequence $\langle +, v_0, v_1, v_2 \rangle$ is not a term. Hint: use Proposition 2.2.

E2.6. Show how the structure (A, f) introduced at the beginning of the chapter can be put in the general framework of structures.

E2.7. Show how the structure $(\omega, S, 0, +, \cdot)$ introduced at the beginning ot the chapter can be put in the general framework of structures.

E2.8. Prove that in the language for the structure $(\omega, +)$, a term has length m iff m is odd.

E2.9. Complete the proof of Proposition 2.5.

E2.10. Give a formula φ in the language for $(\mathbb{Q}, +, \cdot)$ such that for any $a : \omega \to \mathbb{Q}$, $(\mathbb{Q}, +, \cdot) \models \varphi[a]$ iff $a_0 = 1$.

E2.11. Give a formula φ which holds in a structure, under any assignment, iff the structure has at least 3 elements.

E2.12. Give a formula φ which holds in a structure, under any assignment, iff the structure has exactly 4 elements.

E2.13. Write the formula given in (1) at the end of this chapter as a sequence of integers.

E2.14. Write a formula φ in the language for $(\omega, <)$ such that for any assignment a, $(\omega, <) \models \varphi[a]$ iff $a_0 < a_1$ and there are exactly two integers between a_0 and a_1.

E2.15. Prove that the formula

$$v_0 = v_1 \to (\mathbf{R}v_0v_2 \to \mathbf{R}v_1v_2)$$

is universally valid, where $\mathbf{R}$ is a binary relation symbol.

E2.16. Give an example showing that the formula

$$v_0 = v_1 \to \forall v_0(v_0 = v_1)$$

is not universally valid.

E2.17. Prove that $\exists v_0 \forall v_1 \varphi \to \forall v_1 \exists v_0 \varphi$ is universally valid.