Foundations of Mathematics

Mathematics is based upon set theory, which itself is based upon logic. The purpose of these notes is to describe the logical foundations upon which set theory and mathematics are built. There are two main goals of the notes. First is to describe precisely the languages upon which mathematics is founded, giving the completeness theorem which shows that these languages are satisfactory. The second goal is to describe the limitations of this formal development, giving Gödel's famous incompleteness theorem.

1. Sentential logic

We go into the mathematical theory of the simplest logical notions: the meaning of "and", "or", "implies", "if and only if" and related notions. The basic idea here is to describe a formal language for these notions, and say precisely what it means for statements in this language to be true. The first step is to describe the language, without saying anything mathematical about meanings. We need very little background to carry out this development. ω is the set of all natural numbers $0, 1, 2, \ldots$. Let ω^+ be the set of all positive integers. For each positive integer m let $m' = \{0, \ldots, m-1\}$. A *finite sequence* is a function whose domain is m' for some positive integer m; the values of the function can be arbitary.

To keep the treatment strictly mathematical, we will define the basic "symbols" of the language to just be certain positive integers, as follows:

Negation symbol: the integer 1.
Implication symbol: the integer 2.
Sentential variables: all integers ≥ 3.

Let Expr be the collection of all finite sequences of positive integers; we think of these sequences as *expressions*. Thus an expression is a function mapping m' into ω^+, for some positive integer m. Such sequences are frequently indicated by $\langle \varphi_0, \ldots, \varphi_{m-1} \rangle$. The case $m = 1$ is important; here the notation is $\langle \varphi \rangle$.

The one-place function $\neg$ mapping Expr into Expr is defined by $\neg\varphi = \langle 1 \rangle^\frown \varphi$ for any expression φ. Here in general $\varphi^\frown \psi$ is the sequence φ followed by the sequence ψ.

The two-place function $\rightarrow$ mapping Expr $\times$ Expr into Expr is defined by $\varphi \rightarrow \psi = \langle 2 \rangle^\frown \varphi^\frown \psi$ for any expressions φ, ψ. (For any sets A, B, $A \times B$ is the set of all ordered pairs (a, b) with $a \in A$ and $b \in B$. So Expr $\times$ Expr is the set of all ordered pairs (φ, ψ) with φ, ψ expressions.)

For any natural number n, let $S_n = \langle n + 3 \rangle$.

Now we can write down some things which look like logical statements, but are actually just certain finite sequences of positive integers:

S_{10},
$S_0 \rightarrow \neg S_1$,
$S_0 \rightarrow \neg\neg S_0$,
$\neg\neg S_0 \rightarrow S_0$,
$(S_0 \rightarrow S_1) \rightarrow (\neg S_1 \rightarrow \neg S_0)$,

$$[S_0 \to (S_1 \to S_2)] \to [(S_0 \to S_1) \to (S_0 \to S_2)]$$

As actual finite sequences, these are

$S_{10} = \langle 13 \rangle$.
$S_0 \to \neg S_1 = \langle 2, 3, 1, 4 \rangle$.
$S_0 \to \neg\neg S_0 = \langle 2, 3, 1, 1, 3 \rangle$.
$\neg\neg S_0 \to S_0 = \langle 2, 1, 1, 3, 3 \rangle$.
$(S_0 \to S_1) \to (\neg S_1 \to \neg S_0) = \langle 2, 2, 3, 4, 2, 1, 4, 1, 3 \rangle$.
$[S_0 \to (S_1 \to S_2)] \to [(S_0 \to S_1) \to (S_0 \to S_2)] = \langle 2, 2, 3, 2, 4, 5, 2, 2, 3, 4, 2, 3, 5 \rangle$.

These equalities are actually little theorems; for illustration,

$$\begin{aligned}\neg\neg S_0 \to S_0 &= \langle 2 \rangle ^\frown \neg\neg S_0 ^\frown S_0 \\ &= \langle 2 \rangle ^\frown \langle 1 \rangle ^\frown \neg S_0 ^\frown S_0 \\ &= \langle 2, 1 \rangle ^\frown \langle 1 \rangle ^\frown S_0 ^\frown S_0 \\ &= \langle 2, 1, 1, 3, 3 \rangle.\end{aligned}$$

There are many expressions which are meaningless. For example, the expression

$\langle 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \rangle$ can be expressed as
$\neg(S_0 \to S_1 S_2 S_3 S_4 S_5 S_6 S_7)$ or as
$\neg(S_0 S_1 \to S_2 S_3 S_4 S_5 S_6 S_7)$,

neither of which make sense.

Now we define the notion of a sentential formula—an expression which, suitably interpreted, makes sense. We do this definition by defining a *sentential formula construction*, which by definition is a sequence $\langle \varphi_0, \ldots, \varphi_{m-1} \rangle$ with the following property: for each $i < m$, one of the following holds:

$\varphi_i = S_j$ for some natural number j.

There is a $k < i$ such that $\varphi_i = \neg\varphi_k$.

There exist $k, l < i$ such that $\varphi_i = (\varphi_k \to \varphi_l)$.

Then a *sentential formula* is an expression which appears in some sentential formula construction. Here are three examples of sentential formula constructions:

$\langle S_3 \rangle$.

$\langle S_2, \neg S_2, S_2 \to S_2, S_2 \to \neg S_2 \rangle$.

$\langle S_4, \neg S_4, \neg\neg S_4, \neg\neg\neg S_4 \rangle$.

The first one is a function f with domain $1'$ such that $f(0) = S_3$. The second is a function g with domain $4'$ such that $g(0) = S_2$, $g(1) = \neg S_2$, $g(2) = S_2 \to S_2$, and $g(3) = S_2 \to \neg S_2$. The third is a function h with domain $4'$ such that $h(0) = S_4$, $h(1) = \neg S_4$, $h(2) = \neg\neg S_4$ and $h(3) = \neg\neg\neg S_4$. It is easy to check that these functions do satisfy the conditions for a sentential formula construction. For example, $g(0) = S_2$, $g(1) = \neg g(0)$, $g(2) = g(0) \to g(0)$, and $g(3) = g(0) \to g(1)$.

These constructions prove that the following are sentential formulas: S_3, S_2, $\neg S_2$, $S_2 \to S_2$, $S_2 \to \neg S_2$, S_4, $\neg S_4$, $\neg\neg S_4$, $\neg\neg\neg S_4$.

The following proposition formulates the principle of *induction on sentential formulas.*

Proposition 1.1. *Suppose that M is a collection of sentential formulas, satisfying the following conditions.*

(i) S_i is in M, for every natural number i.

(ii) If φ is in M, then so is $\neg\varphi$.

(iii) If φ and ψ are in M, then so is $\varphi \to \psi$.

Then M consists of all sentential formulas.

Proof. Suppose that θ is a sentential formula; we want to show that $\theta \in M$. Let $\langle \tau_0, \ldots, \tau_m \rangle$ be a sentential formula construction with $\tau_t = \theta$, where $0 \le t \le m$. We prove by complete induction on i that for every $i \le m$, $\tau_i \in M$. Hence by applying this to $i = t$ we get $\theta \in M$.

So assume that for every $j < i$, the sentential formula τ_j is in M.

Case 1. τ_i is S_s for some s. By (i), $\tau_i \in M$.

Case 2. τ_i is $\neg\tau_j$ for some $j < i$. By the inductive hypothesis, $\tau_j \in M$, so $\tau_i \in M$ by (ii).

Case 3. τ_i is $\tau_j \to \tau_k$ for some $j, k < i$. By the inductive hypothesis, $\tau_j \in M$ and $\tau_k \in M$, so $\tau_i \in M$ by (iii). $\square$

Proposition 1.2. *(i) Any sentential formula is a nonempty sequence.*

(ii) For any sentential formula φ, exactly one of the following conditions holds:

(a) φ is S_i for some $i \in \omega$.

(b) φ begins with 1, and there is a sentential formula ψ such that $\varphi = \neg\psi$.

(c) φ begins with 2, and there are sentential formulas ψ, χ such that $\varphi = \psi \to \chi$.

(iii) No proper initial segment of a sentential formula is a sentential formula.

(iv) If φ and ψ are sentential formulas and $\neg\varphi = \neg\psi$, then $\varphi = \psi$.

(v) If $\varphi, \psi, \varphi', \psi'$ are sentential formulas and $\varphi \to \psi = \varphi' \to \psi'$, then $\varphi = \varphi'$ and $\psi = \psi'$.

Proof. (i): Clearly every entry in a sentential formula construction is nonempty, so (i) holds.

(ii): First we prove by induction that one of (a)–(c) holds. This is true of sentential variables—in this case, (a) holds. If it is true of a sentential formula φ, it is obviously true of $\neg\varphi$; so (b) holds. Similarly for $\to$, where (c) holds.

Second, the first entry of a formula differs in cases (a),(b),(c), so exactly one of them holds.

(iii): We prove this by complete induction on the length of the formula. So, suppose that φ is a sentential formula and we know for any formula ψ shorter than φ that no proper initial segment of ψ is a formula. We consider cases according to (ii).

Case 1. φ is S_i for some i. Only the empty sequence is a proper initial segment of φ in this case, and the empty sequence is not a sentential formula, by (i).

Case 2. φ is $\neg\psi$ for some formula ψ. If χ is a proper initial segment of φ and it is a formula, then χ begins with 1 and so by (ii), χ has the form $\neg\theta$ for some formula θ. But then θ is a proper initial segment of ψ and ψ is shorter than φ, so the inductive hypothesis is contradicted.

Case 3. φ is $\psi \to \chi$ for some formulas ψ and χ. That is, φ is $\langle 2\rangle^\frown\psi^\frown\chi$. If θ is a proper initial segment of φ which is a formula, then by (ii), θ has the form $\langle 2\rangle^\frown\xi^\frown\eta$ for some formulas ξ, η. Now $\psi^\frown\chi = \xi^\frown\eta$, so ψ is an initial segment of ξ or ξ is an initial segment of ψ. Since ψ and ξ are both shorter than φ, it follows from the inductive hypothesis that $\psi = \xi$. Hence $\chi = \eta$, and $\varphi = \theta$, contradiction.

(iv) is rather obvious; if $\neg\varphi = \neg\psi$, then φ and ψ are both the sequence obtained by deleting the first entry.

(v): Assume the hypothesis. Then $\varphi \to \psi$ is the sequence $\langle 2\rangle^\frown\varphi^\frown\psi$, and $\varphi' \to \psi'$ is the sequence $\langle 2\rangle^\frown\varphi'^\frown\psi'$. Since these are equal, φ and φ' start at the same place in the sequence. By (iii) it follows that $\varphi = \varphi'$. Deleting the initial segment $\langle 2\rangle^\frown\varphi$ from the sequence, we then get $\psi = \psi'$. □

Parts (iv) and (v) of this proposition enable us to define values of sentential formulas, which supplies a mathematical meaning for the truth of formulas. A *sentential assignment* is a function mapping the set $\{0, 1, \ldots\}$ of natural numbers into the set $\{0, 1\}$. Intuitively we think of 0 as “false” and 1 as “true”. The definition of values of sentential formulas is a special case of definition by recursion. It is given in the following proposition.

Proposition 1.3. *For any sentential assignment f there is a function F mapping the set of sentential formulas into $\{0, 1\}$ such that the following conditions hold:*

(i) $F(S_n) = f(n)$ for every natural number n.

(ii) $F(\neg\varphi) = 1 - F(\varphi)$ for any sentential formula φ.

(iii) $F(\varphi \to \psi) = 0$ iff $F(\varphi) = 1$ and $F(\psi) = 0$.

A proof of this proposition is sketched in an exercise.

With f a sentential assignment, and with F as in this proposition, for any sentential formula φ we let $\varphi[f] = F(\varphi)$. Thus:

$$\begin{aligned} S_i[f] &= f(i); \\ (\neg\varphi)[f] &= 1 - \varphi[f]; \\ (\varphi \to \psi)[f] &= \begin{cases} 0 & \text{if } \varphi[f] = 1 \text{ and } \psi[f] = 0, \\ 1 & \text{otherwise.} \end{cases} \end{aligned}$$

The definition can be recalled by using *truth tables*:

φ	$\neg\varphi$
1	0
0	1

φ	ψ	$\varphi \to \psi$
1	1	1
1	0	0
0	1	1
0	0	1

Other logical notions can be defined in terms of $\neg$ and $\to$. We define

$\varphi \wedge \psi = \neg(\varphi \to \neg\psi)$.
$\varphi \vee \psi = \neg\varphi \to \psi$.
$\varphi \leftrightarrow \psi = (\varphi \to \psi) \wedge (\psi \to \varphi)$.

Working out the truth tables for these new notions shows that they mean approximately what you would expect:

φ	ψ	$\neg\psi$	$\varphi \to \neg\psi$	$\varphi \wedge \psi$	$\neg\varphi$	$\varphi \vee \psi$	$\varphi \to \psi$	$\psi \to \varphi$	$\varphi \leftrightarrow \psi$
1	1	0	0	1	0	1	1	1	1
1	0	1	1	0	0	1	0	1	0
0	1	0	1	0	1	1	1	0	0
0	0	1	1	0	1	0	1	1	1

(Note that $\vee$ corresponds to *non-exclusive or*: φ or ψ, or both.)

The following simple proposition is frequently useful.

Proposition 1.4. *If f and g map $\{0, 1, \ldots\}$ into $\{0, 1\}$ and $f(m) = g(m)$ for every m such that S_m occurs in φ, then $\varphi[f] = \varphi[g]$.*

Proof. Induction on φ. If φ is S_i for some i, then the hypothesis says that $f(i) = g(i)$; hence $S_i[f] = f(i) = g(i) = S_i[g]$. Assume that it is true for φ. Now S_m occurs in φ iff it occurs in $\neg\varphi$. Hence if we assume that $f(m) = g(m)$ for every m such that S_m occurs in $\neg\varphi$, then also $f(m) = g(m)$ for every m such that S_m occurs in φ, so $(\neg\varphi)[f] = 1 - \varphi[f] = 1 - \varphi[g] = (\neg\varphi)[g]$. Assume that it is true for both φ and ψ, and $f(m) = g(m)$ for every m such that S_m occurs in $\varphi \to \psi$. Now if S_m occurs in φ, then it also occurs in $\varphi \to \psi$, and hence $f(m) = g(m)$. Similarly for ψ. It follows that

$$(\varphi \to \psi)[f] = 0 \text{ iff } (\varphi[f] = 1 \text{ and } \psi[f] = 0) \text{ iff } (\varphi[g] = 1 \text{ and } \psi[g] = 0) \text{ iff } (\varphi \to \psi)[g] = 0.$$

□

This proposition justifies writing $\varphi[f]$ for a finite sequence f, provided that f is long enough so that m is in its domain for every m for which S_m occurs in φ.

A sentential formula φ is a *tautology* iff it is true under every assignment, i.e., $\varphi[f] = 1$ for every assignment f. We give some tautologies, with proofs that they are:

1. $\varphi \to \varphi$:

φ	$\varphi \to \varphi$
1	1
0	1

2. $\varphi \to \neg\neg\varphi$:

φ	$\neg\varphi$	$\neg\neg\varphi$	$\varphi \to \neg\neg\varphi$
1	0	1	1
0	1	0	1

3. $(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$: for brevity, let χ abbreviate this formula:

φ	ψ	$\varphi \to \psi$	$\neg\varphi$	$\neg\psi$	$\neg\psi \to \neg\varphi$	χ
1	1	1	0	0	1	1
1	0	0	0	1	0	1
0	1	1	1	0	1	1
0	0	1	1	1	1	1

4. $[\varphi \to (\psi \to \chi)] \to [(\varphi \to \psi) \to (\varphi \to \chi)]$. Let ρ denote this formula:

φ	ψ	χ	$\psi \to \chi$	$\varphi \to (\psi \to \chi)$	$\varphi \to \psi$	$\varphi \to \chi$	$(\varphi \to \psi) \to (\varphi \to \chi)$	ρ
1	1	1	1	1	1	1	1	1
1	1	0	0	0	1	0	0	1
1	0	1	1	1	0	1	1	1
1	0	0	1	1	0	0	1	1
0	1	1	1	1	1	1	1	1
0	1	0	0	1	1	1	1	1
0	0	1	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1

Of course one can use truth tables to show that certain formulas are not tautologies. We give some examples.

$(\varphi \to \psi) \to (\psi \to \varphi)$ is not a tautology:

φ	ψ	$\varphi \to \psi$	$\psi \to \varphi$	$(\varphi \to \psi) \to (\psi \to \varphi)$
1	1	1	1	1
1	0	0	1	1
0	1	1	0	0
0	0	1	1	1

Thus the formula is not a tautology, because of row 3. We could have stopped at that row if we only wanted to know whether or not the formula was a tautology.

$(\neg\varphi \to \varphi) \to \neg\varphi$ is not a tautology:

φ	$\neg\varphi$	$\neg\varphi \to \varphi$	$(\neg\varphi \to \varphi) \to \neg\varphi$
1	0	1	0

Notice that the truth tables get much longer as the number of variables involved increases; for example, with 6 variables there will be 64 rows. This motivates the idea of arguing in an ordinary mathematical way to determine whether something is a tautology. In the case of an implication, one can assume that some assignment gives the value 0 and then see what that implies; either one reaches an assignment showing that the statement is indeed not a tautology, or one gets a contradiction which proves that it is a tautology. We give some illustrations of this method.

$S_0 \to (S_1 \to (S_0 \to S_1))$ is a tautology. We indicate the provisional assignment below the symbols, and mark the steps in proving this by numbers above the symbols:

2	1	3	2	4	3	4
S_0	$\to$	$(S_1$	$\to$	$(S_0$	$\to$	$S_1))$
1	0	1	0	1	0	0

We have tentatively assigned two values to S_1, giving a contradiction.

$S_0 \to (S_1 \to (\neg S_2 \to (S_3 \to S_2)))$ is not a tautology:

2	1	3	2	4	5	3	6	4	6
S_0	$\to$	$(S_1$	$\to ($	$\neg$	S_2	$\to ($	S_3	$\to$	$S_2)))$
1	0	1	0	1	0	0	1	0	0

This time there is no contradiction, and we have produced an assignment where the sentence is false.

So, this method sometimes saves work. The next example illustrates a case where it is not so simple.

$(S_0 \vee S_1) \to \neg S_1$ is not a tautology:

	2		1	2	3
$(S_0$	$\vee$	$S_1)$	$\to$	$\neg$	S_2

1			0	0	1

Now there are 3 possibilities to make $S_0 \vee S_1$ true, which in principle we should try one after the other. Fortunately, the first one gives a valid assignment:

4	2	4	1	2	3
$(S_0$	$\vee$	$S_1)$	$\to$	$\neg$	S_2
1	1	1	0	0	1

So the indicated sentence is indeed not a tautology.

Here is a list of common tautologies:

(T1) $\varphi \to \varphi$.
(T2) $\varphi \leftrightarrow \neg\neg\varphi$.
(T3) $(\varphi \to \neg\varphi) \to \neg\varphi$.
(T4) $(\varphi \to \neg\psi) \to (\psi \to \neg\varphi)$.
(T5) $\varphi \to (\neg\varphi \to \psi)$.
(T6) $(\varphi \to \psi) \to [(\psi \to \chi) \to (\varphi \to \chi)]$.
(T7) $[\varphi \to (\psi \to \chi)] \to [(\varphi \to \psi) \to (\varphi \to \chi)]$.
(T8) $(\varphi \wedge \psi) \to (\psi \wedge \varphi)$.
(T9) $(\varphi \wedge \psi) \to \varphi$.
(T10) $(\varphi \wedge \psi) \to \psi$.
(T11) $\varphi \to [\psi \to (\varphi \wedge \psi)]$.
(T12) $\varphi \to (\varphi \vee \psi)$.
(T13) $\psi \to (\varphi \vee \psi)$.
(T14) $(\varphi \to \chi) \to [(\psi \to \chi) \to ((\varphi \vee \psi) \to \chi)]$.
(T15) $\neg(\varphi \wedge \psi) \leftrightarrow (\neg\varphi \vee \neg\psi)$.
(T16) $\neg(\varphi \vee \psi) \leftrightarrow (\neg\varphi \wedge \neg\psi)$.
(T17) $[\varphi \vee (\psi \vee \chi)] \leftrightarrow [(\varphi \vee \psi) \vee \chi]$.
(T18) $[\varphi \wedge (\psi \wedge \chi)] \leftrightarrow [(\varphi \wedge \psi) \wedge \chi]$.
(T19) $[\varphi \wedge (\psi \vee \chi)] \leftrightarrow [(\varphi \wedge \psi) \vee (\varphi \wedge \chi)]$.
(T20) $[\varphi \vee (\psi \wedge \chi)] \leftrightarrow [(\varphi \vee \psi) \wedge (\varphi \vee \chi)]$.
(T21) $(\varphi \to \psi) \leftrightarrow (\neg\varphi \vee \psi)$.
(T22) $\varphi \wedge \psi \leftrightarrow \neg(\neg\varphi \vee \neg\psi)$.
(T23) $\varphi \vee \psi \leftrightarrow \neg(\neg\varphi \wedge \neg\psi)$.

Note that (T17) and (T18) are sort of associative laws. We say "sort of" because $\varphi\vee(\psi\vee\chi)$ and $(\varphi \vee \psi) \vee \chi$ are different formulas, although they are logically equivalent in the sense that (T17) is a tautology. To see that they are different formulas, for simplicity take the special case in which φ is S_0, ψ is S_1, and χ is S_2. Then

$$\begin{aligned}
S_0 \vee (S_1 \vee S_2) &= S_0 \vee (\neg S_1 \to S_2) \\
&= \neg S_0 \to (\neg S_1 \to S_2) \\
&= \neg S_0 \to (\langle 1,4\rangle \to S_2) \\
&= \neg S_0 \to \langle 2,1,4,5\rangle \\
&= \langle 1,3\rangle \to \langle 2,1,4,5\rangle \\
&= \langle 2,1,3,2,1,4,5\rangle,
\end{aligned}$$

while

$$
\begin{aligned}
(S_0 \vee S_1) \vee S_2 &= (\neg S_0 \to S_1) \vee S_2 \\
&= (\langle 1,3\rangle \to S_1) \vee S_2 \\
&= \langle 2,1,3,4\rangle \vee S_2 \\
&= \neg\langle 2,1,3,4\rangle \to S_2 \\
&= \langle 1,2,1,3,4\rangle \to S_2 \\
&= \langle 2,1,2,1,3,4,5\rangle.
\end{aligned}
$$

The apparatus so far developed can be used to decide whether certain common language arguments are logically correct. We give two illustrations of this, taken from Suppes, "Introduction to Logic".

(1) If prices are high, then wages are high. Prices are high, or there are price controls. Also, if there are price controls, then there is no inflation. But there is inflation. Therefore, wages are high.

To analyze this, we let S_0 be "prices are high", S_1 be "wages are high", S_2 be "there are price controls", and S_3 be "there is inflation". Then the argument is

$$[(S_0 \to S_1) \wedge (S_0 \vee S_2) \wedge (S_2 \to \neg S_3) \wedge S_3] \to S_1$$

So, we wonder whether this is a tautology. We try to get a truth assignment making it false, using the above notation:

$$
\begin{array}{llllllllllllllll}
3 & 1 & 2 & & 4 & 1 & 5 & & 6 & 1 & 7 & 8 & & 1 & & 1 \\
[(S_0 & \to & S_1) & \wedge & (S_0 & \vee & S_2) & \wedge & (S_2 & \to & \neg & S_3) & \wedge & S_3] & \to & S_1 \\
0 & 1 & 0 & & 0 & 1 & 1 & & 1 & 1 & 1 & 0 & & 1 & 0 & 0
\end{array}
$$

Two different values have been assigned to S_3; this means that the argument is correct.

(2) Either logic is difficult, or not many students like it. If mathematics is easy, then logic is not difficult. Therefore, if many students like logic, then mathematics is not easy.

Let S_0 be "logic is difficult", S_1 be "students like logic", S_2 be "mathematics is easy". Thus the argument is formalized as follows:

$$(S_0 \to \neg S_1) \wedge (S_2 \to \neg S_0) \to (S_1 \to \neg S_2).$$

Again we try to get an assignment making this false:

$$
\begin{array}{llllllllllllll}
8 & 2 & 9 & 5 & & 5 & 2 & 6 & 7 & 1 & 3 & 2 & 3 & 4 \\
(S_0 & \to & \neg & S_1) & \wedge & (S_2 & \to & \neg & S_0) & \to & (S_1 & \to & \neg & S_2) \\
0 & 1 & 0 & 1 & & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1
\end{array}
$$

No contradiction has been reached, so we have an assignment showing that the argument is invalid. Namely, if logic is not difficult, students like logic, and mathematics is easy, then the argument does not work.

We have chosen negation and implication as the basic logical connectives, and then we have defined other common connectives in terms of them. We now want to show that every logical connective whatsoever can be defined in terms of them. The proof depends on a detailed description of values of rather complicated formulas. We need to define general disjunctions and conjunctions; this is done by recursion.

Proposition 1.5. *There is a function F whose domain is is the set of positive integers with the following properties:*

(i) F_1 is the function with domain the set of all sentential formulas, and $F_1(\varphi) = \varphi$ for every sentential formula φ.

(ii) For any integer $m > 1$, F_m is a function with domain the set of all m-tuples of sentential formulas, such that for any sequence $\langle \varphi_0, \ldots, \varphi_{m-1} \rangle$ of sentential formulas we have

$$F_m(\varphi_0, \ldots, \varphi_{m-1}) = F_{m-1}(\varphi_0, \ldots, \varphi_{m-2}) \vee \varphi_{m-1}.$$

Again a proof is sketched in an exercise.

Let F be as in Proposition 1.5. If $\langle \varphi_0, \ldots, \varphi_{m-1} \rangle$ is a sequence of sentential formulas, we denote $F_m(\varphi_0, \ldots, \varphi_{m-1})$ by $\bigvee_{i \le m-1} \varphi_i$. Thus we have

$$\bigvee_{i \le 0} \varphi_i = \varphi_0;$$

$$\bigvee_{i \le n+1} \varphi_i = \left(\bigvee_{i \le n} \varphi_i \right) \vee \varphi_{n+1}.$$

Similarly we define

$$\bigwedge_{i \le 0} \varphi_i = \varphi_0;$$

$$\bigwedge_{i \le n+1} \varphi_i = \left(\bigwedge_{i \le n} \varphi_i \right) \wedge \varphi_{n+1}.$$

Frequently we write $\varphi_0 \vee \ldots \vee \varphi_m$ in place of $\bigvee_{i \le m} \varphi_i$; and similarly for $\wedge$.

For any sentential formula φ we define $\varphi^1 = \varphi$ and $\varphi^0 = \neg\varphi$. The following lemma will be useful.

Lemma 1.6. *If φ is a formula, $\varepsilon \in \{0, 1\}$, and $f : \omega \to \{0, 1\}$, then for any $i \in \omega^+$,*

$$S_i^\varepsilon[f] = \begin{cases} 1 & \text{if } \varepsilon = f(i), \\ 0 & \text{if } \varepsilon \neq f(i). \end{cases}$$

Proof. The proof is clear from the following table.

ε	$f(i)$	$S_i^\varepsilon[f]$
1	1	$S_i[f] = f(i) = 1$
1	0	$S_i[f] = f(i) = 0$
0	1	$\neg S_i[f] = 1 - f(i) = 0$
0	0	$\neg S_i[f] = 1 - f(i) = 1$

□

Now the mathematical meaning of $\neg$ is completely described by the function f giving its truth table. Thus f is a one place function mapping $\{0,1\}$ into $\{0,1\}$, with $f(0) = 1$ and $f(1) = 0$. Similarly, the mathematical meaning of $\rightarrow$ is described by the two-place function g mapping $\{0,1\} \times \{0,1\}$, with $g(1,1) = 1$, $g(1,0) = 0$, $g(0,1) = 1$, and $g(0,0) = 1$. This motivates considering the set $\mathscr{F}$ of all functions of any number of variables defined on $\{0,1\}$ with values in $\{0,1\}$ as embodying all possible sentential connectives.

Theorem 1.7. *For any positive integer k and any function F mapping k-tuples of members of $\{0,1\}$ into $\{0,1\}$ there is a sentential formula φ involving only at most $S_0, \ldots, S_{k-1}$ such that for every function f mapping $\{0,1,\ldots\}$ into $\{0,1\}$ we have*

$$\varphi[f] = F(f(0), \ldots, f(k-1)).$$

Proof. If F takes on only the value 0, we can let φ be $S_0 \wedge \neg S_0$. Now suppose that F takes on the value 1 at least once, and let $\langle f_0, \ldots, f_{m-1}\rangle$ enumerate all of the k-tuples of members of $\{0,1\}$ on which F takes the value 1, where $m \geq 1$. Then we can let φ be

$$\bigvee_{i<m} \bigwedge_{j<k} S_j^{f_i(j)}.$$

In fact, if $\varphi[g] = 1$, then there is an $i < m$ such that $\left(\bigwedge_{j<k} S_j^{f_i(j)}\right)[g] = 1$. Hence for every $j < k$ we have $S_j^{f_i(j)}[g] = 1$, and hence by the lemma, $g(j) = f_i(j)$. Thus $f_i = \langle g(0), \ldots, g(k-1)\rangle$, and so $F(g(0), \ldots, g(k-1)) = F(f_i) = 1$.

On the other hand, if $\varphi[g] = 0$, then $\left(\bigvee_{i<m} \bigwedge_{j<k} S_j^{f_i(j)}\right)[g] = 0$, and this means that $\left(\bigwedge_{j<k} S_j^{f_i(j)}\right)[g] = 0$ for every $i < m$, and it follows that for each $i < m$ there is a $j < k$ such that $(S_j^{f_i(j)})[g] = 0$; by the lemma, $f_i(j) \neq g(j)$. Thus $\langle g(0), \ldots, g(k-1)\rangle \neq f_i$ for all $i < m$, and hence $F(g(0), \ldots, g(k-1)) = 0$. □

We have taken $\neg$ and $\rightarrow$ as primitive. By (T21)–(T23) each of the following could be taken as primitive:

$\neg$ and $\vee$.

$\neg$ and $\wedge$.

It is of some interest that there is a single connective which could be taken as primitive, all others being obtained from it; see an exercise.

Theorem 1.8. (Disjunctive normal form) *If φ is a sentential formula which is true under some sentential assignment, and if every sentential variable S_i occurring in φ has $i < m$, then there is a nonempty set M of m-termed sequences of 0's and 1's such that the following formula is a tautology:*

$$\varphi \leftrightarrow \bigvee_{\varepsilon \in M} \bigwedge_{i<m} S_i^{\varepsilon(i)}.$$

Proof. Let

$$M = \{\varepsilon : \varepsilon \text{ is an } m\text{-termed sequence of 0's and 1's and } \varphi[f] = 1 \text{ for some } f \supseteq \varepsilon\}.$$

Note that M is nonempty, since φ is true under some assignment. Now take any sentential assignment f.

Suppose that $\varphi[f] = 1$. Then $\varepsilon \stackrel{\text{def}}{=} \langle f(0), \ldots, f(m-1)\rangle \in M$. For each $i < m$ we have $S_i^{\varepsilon(i)}[f] = 1$ by the lemma, and hence $\left(\bigwedge_{i<m} S_i^{\varepsilon(i)}\right)[f] = 1$. It follows that $\left(\bigvee_{\delta\in M} \bigwedge_{i<m} S_i^{\delta(i)}\right)[f] = 1$.

On the other hand, suppose that $\left(\bigvee_{\delta\in M} \bigwedge_{i<m} S_i^{\delta(i)}\right)[f] = 1$. Choose $\delta \in M$ such that $\left(\bigwedge_{i<m} S_i^{\delta(i)}\right)[f] = 1$. Then $S_i^{\delta(i)}[f] = 1$ for all $i < m$, and so by the lemma, $f(i) = \delta(i)$ for all $i < m$. By the definition of M there is a $g : \omega \to \{0, 1\}$ such that $\delta \subseteq g$ and $\varphi[g] = 1$. By Proposition 1.4, $\varphi[f] = 1$. $\square$

Now we describe a proof system for sentential logic. Formulas of the following form are *sentential axioms*; φ, ψ, χ are arbitrary sentential formulas.

(1) $\varphi \to (\psi \to \varphi)$.

(2) $[\varphi \to (\psi \to \chi)] \to [(\varphi \to \psi) \to (\varphi \to \chi)]$.

(3) $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$.

Proposition 1.9. *Every sentential axiom is a tautology.*

Proof. For (1):

φ	ψ	$\psi \to \varphi$	$\varphi \to (\psi \to \varphi)$
1	1	1	1
1	0	1	1
0	1	0	1
0	0	1	1

(2) was treated above, as an example of a tautology.

For (3):

φ	ψ	$\neg\varphi$	$\neg\psi$	$\neg\varphi \to \neg\psi$	$\psi \to \varphi$	(3)
1	1	0	0	1	1	1
1	0	0	1	1	1	1
0	1	1	0	0	0	1
0	0	1	1	1	1	1

□

If Γ is a collection of sentential formulas, then a Γ-*proof* is a finite sequence $\langle \psi_0, \ldots, \psi_m \rangle$ such that for each $i \le m$ one of the following conditions holds:

(a) ψ_i is a sentential axiom.

(b) $\psi_i \in \Gamma$.

(c) There exist $j, k < i$ such that ψ_k is $\psi_j \to \psi_i$. (Rule of *modus ponens*, abbreviated MP).

We write $\Gamma \vdash \varphi$ if there is a Γ-proof with last entry φ. We also write $\vdash \varphi$ in place of $\emptyset \vdash \varphi$.

Proposition 1.10. *(i) If $\Gamma \vdash \varphi$, f is a sentential assignment, and $\psi[f] = 1$ for all $\psi \in \Gamma$, then $\varphi[f] = 1$.*

(ii) If $\vdash \varphi$, then φ is a tautology.

Proof. For (i), let $\langle \psi_0, \ldots, \psi_m \rangle$ be a Γ-proof. Suppose that f is a sentential assignment and $\chi[f] = 1$ for all $\chi \in \Gamma$. We show by complete induction that $\psi_i[f] = 1$ for all $i \le m$. Suppose that this is true for all $j < i$.

Case 1. ψ_i is a sentential axiom. Then $\psi_i[f] = 1$ by Proposition 1.9.

Case 2. $\psi_i \in \Gamma$. Then $\psi_i[f] = 1$ by assumption.

Case 3. There exist $j, k < i$ such that ψ_k is $\psi_j \to \psi_i$. By the inductive assumption, $\psi_k[f] = \psi_j[f] = 1$. Hence $\psi_i[f] = 1$.

(ii) clearly follows from (i), □

Now we are going to show that, conversely, if φ is a tautology then $\vdash \varphi$. This is a kind of completeness theorem, and the proof is a highly simplified version of the proof of the completeness theorem for first-order logic which will be given later.

Lemma 1.11. $\vdash \varphi \to \varphi$.

Proof.

(a)	$[\varphi \to [(\varphi \to \varphi) \to \varphi]] \to [[\varphi \to (\varphi \to \varphi)] \to (\varphi \to \varphi)]$	(2)
(b)	$\varphi \to [(\varphi \to \varphi) \to \varphi]$	(1)
(c)	$[\varphi \to (\varphi \to \varphi)] \to (\varphi \to \varphi)$	(a), (b), MP
(d)	$\varphi \to (\varphi \to \varphi)$	(1)

(e) $\qquad \varphi \to \varphi \qquad$ (c), (d), MP

□

Theorem 1.12. (The deduction theorem) *If* $\Gamma \cup \{\varphi\} \vdash \psi$*, then* $\Gamma \vdash \varphi \to \psi$*.*

Proof. Let $\langle \chi_0, \ldots, \chi_m \rangle$ be a $(\Gamma \cup \{\varphi\})$-proof with last entry ψ. We replace each χ_i by several formulas so that the result is a Γ-proof with last entry $\varphi \to \psi$.

If χ_i is a logical axiom or a member of Γ, we replace it by the two formulas $\chi_i \to (\varphi \to \chi_i)$, $\varphi \to \chi_i$.

If χ_i is φ, we replace it by the five formulas in the proof of Lemma 1.11; the last entry is $\varphi \to \varphi$.

If χ_i is obtained from χ_j and χ_k by modus ponens, so that χ_k is $\chi_j \to \chi_i$, we replace χ_i by the formulas

$$[\varphi \to (\chi_j \to \chi_i)] \to [(\varphi \to \chi_j) \to (\varphi \to \chi_i)]$$
$$(\varphi \to \chi_j) \to (\varphi \to \chi_i)$$
$$\varphi \to \chi_i$$

Clearly this is as desired. □

Lemma 1.13. $\vdash \psi \to (\neg\psi \to \varphi)$*.*

Proof. By axiom (1) we have $\{\psi, \neg\psi\} \vdash \neg\varphi \to \neg\psi$. Hence axiom (3) gives $\{\psi, \neg\psi\} \vdash \psi \to \varphi$, and hence $\{\psi, \neg\psi\} \vdash \varphi$. Now two applications of Theorem 1.12 give the desired result. □

Lemma 1.14. $\vdash (\varphi \to \psi) \to [(\psi \to \chi) \to (\varphi \to \chi)]$*.*

Proof. Clearly $\{\varphi \to \psi, \psi \to \chi, \varphi\} \vdash \chi$, so three applications of Theorem 1.12 give the desired result. □

Lemma 1.15. $\vdash (\neg\varphi \to \varphi) \to \varphi$*.*

Proof. Clearly $\{\neg\varphi \to \varphi, \neg\varphi\} \vdash \varphi$ and also $\{\neg\varphi \to \varphi, \neg\varphi\} \vdash \neg\varphi$, so by Lemma 1.13, $\{(\neg\varphi \to \varphi, \neg\varphi\} \vdash \neg(\varphi \to \varphi)$. Then Theorem 1.12 gives $\{\neg\varphi \to \varphi\} \vdash \neg\varphi \to \neg(\varphi \to \varphi)$, and so using axiom (3), $\{\neg\varphi \to \varphi\} \vdash (\varphi \to \varphi) \to \varphi$. Hence by Lemma 1.8, $\{\neg\varphi \to \varphi\} \vdash \varphi$, and so Theorem 1.9 gives the desired result. □

Lemma 1.16. $\vdash (\varphi \to \psi) \to [(\neg\varphi \to \psi) \to \psi]$*.*

Proof.

$\{\varphi \to \psi, \neg\varphi \to \psi, \neg\psi\} \vdash \neg\varphi \to \neg\psi$ using axiom (1)

$\{\varphi \to \psi, \neg\varphi \to \psi, \neg\psi\} \vdash \psi \to \varphi$ using axiom (3)

$\{\varphi \to \psi, \neg\varphi \to \psi, \neg\psi\} \vdash \neg\varphi \to \varphi$ using Lemma 1.14

$\{\varphi \to \psi, \neg\varphi \to \psi, \neg\psi\} \vdash \varphi$ by Lemma 1.15

$\{\varphi \to \psi, \neg\varphi \to \psi, \neg\psi\} \vdash \psi$

$\{\varphi \to \psi, \neg\varphi \to \psi\} \vdash \neg\psi \to \psi$ by Theorem 1.12

$\{\varphi \to \psi, \neg\varphi \to \psi\} \vdash \psi$ by Lemma 1.15

Now two applications of Theorem 1.12 give the desired result. $\square$

Theorem 1.17. *There is a sequence $\langle\varphi_0, \varphi_1, \ldots\rangle$ containing all sentential formulas.*

Proof. One can obtain such a sequence by the following procedure.

(1) Start with S_0.

(2) List all sentential formulas of length at most two which involve only S_0 or S_1; they are S_0, S_1, $\neg S_0$, and $\neg S_1$.

(3) List all sentential formulas of length at most three which involve only S_0, S_1, or S_2; they are S_0, S_1, S_2, $\neg S_0$, $\neg S_1$, $\neg S_2$, $\neg\neg S_0$, $\neg\neg S_1$, $\neg\neg S_2$, $S_0 \to S_0$, $S_0 \to S_1$, $S_0 \to S_2$, $S_1 \to S_0$, $S_1 \to S_1$, $S_1 \to S_2$, $S_2 \to S_0$, $S_2 \to S_1$, $S_2 \to S_2$.

(4) Etc. $\square$

Theorem 1.18. *If not($\Gamma \vdash \varphi$), then there is a sentential assignment f such that $\psi[f] = 1$ for all $\psi \in \Gamma$, while $\varphi[f] = 0$.*

Proof. Let $\langle\chi_0, \chi_1, \ldots\rangle$ list all the sentential formulas. We now define $\Delta_0, \Delta_1, \ldots$ by recursion. Let $\Delta_0 = \Gamma$. Suppose that Δ_i has been defined so that not($\Delta_i \vdash \varphi$). If not($\Delta_i \cup \{\chi_i\} \vdash \varphi$) then we set $\Delta_{i+1} = \Delta_i \cup \{\chi_i\}$. Otherwise we set $\Delta_{i+1} = \Delta_i$. Let $\Theta = \bigcup_{i\in\omega} \Delta_i$. By induction we have not($\Delta_i \vdash \varphi$) for each $i \in \omega$. In fact, we have $\Delta_0 = \Gamma$, so not($\Delta_0 \vdash \varphi$) by assumption. If not($\Delta_i \vdash \varphi$), then not($\Delta_{i+1} \vdash \varphi$) by construction.

Hence also not($\Theta \vdash \varphi$), since $\Theta \vdash \varphi$ means that there is a Θ-proof with last entry φ, and any Θ-proof involves only finitely many formulas χ_i, and they all appear in some Δ_j, giving $\Delta_j \vdash \varphi$, contradiction.

$(*)$ For any formula χ_i, either $\chi_i \in \Theta$ or $\neg\chi_i \in \Theta$.

In fact, suppose that $\chi_i \notin \Theta$ and $\neg\chi_i \notin \Theta$. Say $\neg\chi_i = \chi_j$. Then by construction, $\Delta_i \cup \{\chi_i\} \vdash \varphi$ and $\Delta_j \cup \{\neg\chi_i\} \vdash \varphi$. So $\Theta \cup \{\chi_i\} \vdash \varphi$ and $\Theta \cup \{\neg\chi_i\} \vdash \varphi$. Hence by Theorem 1.12, $\Theta \vdash \chi_i \to \varphi$ and $\Theta \vdash \neg\chi_i \to \varphi$. So by Lemma 1.16 we get $\Theta \vdash \varphi$, contradiction.

$(**)$ If $\Theta \vdash \psi$, then $\psi \in \Theta$.

In fact, clearly not($\Theta \cup \{\psi\} \vdash \varphi$) by Theorem 1.12, so $(**)$ follows.

Now let f be the sentential assignment such that $f(i) = 1$ iff $S_i \in \Theta$. Now we claim

$(***)$ For every sentential formula ψ, $\psi[f] = 1$ iff $\psi \in \Theta$.

We prove this by induction on ψ. It is true for $\psi = S_i$ by definition. Now suppose that it holds for ψ. Suppose that $(\neg\psi)[f] = 1$. Thus $\psi[f] = 0$, so by the inductive assumption, $\psi \notin \Theta$, and hence by $(*)$, $\neg\psi \in \Theta$. Conversely, suppose that $\neg\psi \in \Theta$. If $(\neg\psi)[f] = 0$, then $\psi[f] = 1$, hence $\psi \in \Theta$ by the inductive hypothesis. Hence by Lemma 1.13, $\Theta \vdash \varphi$, contradiction. So $(\neg\psi)[f] = 1$.

Next suppose that $(***)$ holds for ψ and χ; we show that it holds for $\psi \to \chi$. Suppose that $(\psi \to \chi)[f] = 1$. If $\chi[f] = 1$, then $(\psi \to \chi)[f] = 1$, and also $\chi \in \Theta$ by the inductive hypothesis. By axiom (1), $\Theta \vdash \psi \to \chi$. Hence by $(**)$, $(\psi \to \chi) \in \Theta$. Suppose that

$\chi[f] = 0$. Then $\psi[f] = 0$ also, since $(\psi \to \chi)[f] = 1$. By the inductive hypothesis and $(*)$ we have $\neg\psi \in \Theta$. Hence $\Theta \vdash \neg\chi \to \neg\psi$ by axiom (1), so $\Theta \vdash \psi \to \chi$ by axiom (3). So $(\psi \to \chi) \in \Theta$ by $(**)$.

Conversely, suppose that $(\psi \to \chi) \in \Theta$. Working for a contradiction, suppose that $(\psi \to \chi)[f] = 0$. Thus $\psi[f] = 1$ and $\chi[f] = 0$. So $\psi \in \Theta$ and $\neg\chi \in \Theta$ by the inductive hypothesis and $(*)$. Since $(\psi \to \chi) \in \Theta$ and $\psi \in \Theta$, we get $\Theta \vdash \chi$. Since also $\neg\chi \in \Theta$, we get $\Theta \vdash \varphi$ by Lemma 1.10, contradiction.

This finishes the proof of $(***)$.

Since $\Gamma \subseteq \Theta$, $(***)$ implies that $\psi[f] = 1$ for all $\psi \in \Gamma$. Also $\varphi[f] = 0$ since $\varphi \notin \Theta$. □

Corollary 1.19. *If $\varphi[f] = 1$ whenever $\psi[f] = 1$ for all $\psi \in \Gamma$, then $\Gamma \vdash \varphi$.*

Proof. This is the contrapositive of Theorem 1.18. □

Theorem 1.20. *$\vdash \varphi$ iff φ is a tautology.*

Proof. $\Rightarrow$ is given by Proposition 1.10(ii). $\Leftarrow$ follows from Corollary 1.19 by taking $\Gamma = \emptyset$. □

EXERCISES

E1.1. Verify that

$$S_0 \to \neg S_1 = \langle 2, 3, 1, 4 \rangle$$

and

$$(S_0 \to S_1) \to (\neg S_1 \to \neg S_0) = \langle 2, 2, 3, 4, 2, 1, 4, 1, 3 \rangle.$$

E1.2. Show that the function h defined after the definition of sentential formula construction satisfies the conditions for a sentential formula construction.

E1.3. Prove that there is a sentential formula of each positive integer length.

E1.4. Prove that m is the length of a sentential formula not involving $\neg$ iff m is odd.

E1.5. Prove Proposition 1.3 as follows. Let f be a sentential assignment. For each positive integer m, let A_m be the set of all sentential formulas of length at most m. An m-approximation is a function G assigning to each member of A_m a value 0 or 1 so that the following conditions hold:

(1) If $S_i \in A_m$, then $G(S_i) = f(i)$.
(2) If $\neg\varphi \in A_m$, then $G(\neg\varphi) = 1 - G(\varphi)$.
(3) If $\varphi \to \psi$ is in A_m, then $G(\varphi \to \psi) = 0$ iff $G(\varphi) = 1$ and $G(\psi) = 0$.

Prove:

(4) If G and G' are m-approximations, then $G = G'$.

(5) For each positive integer m there is an m-approximation.

Then one can define the desired function F by setting $F(\varphi) = G(\varphi)$ where G is an m-approximation with φ of length m.

E1.6. Prove that a truth table for a sentential formula involving n basic formulas has 2^n rows.

E1.7. Use the truth table method to show that the formula

$$(\varphi \rightarrow \psi) \leftrightarrow (\neg\varphi \vee \psi)$$

is a tautology.

E1.8. Use the truth table method to show that the formula

$$[\varphi \vee (\psi \wedge \chi)] \leftrightarrow [(\varphi \vee \psi) \wedge (\varphi \vee \chi)]$$

is a tautology.

E1.9. Use the truth table method to show that the formula

$$(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \neg\psi)$$

is not a tautology. It is not necessary to work out the full truth table.

E1.10. Use the informal method described in the notes to determine whether or not the following is a tautology:

$$S_0 \rightarrow (S_1 \rightarrow (S_2 \rightarrow (S_3 \rightarrow S_1))).$$

E1.11. Use the informal method described in the notes to determine whether or not the following is a tautology:

$$(\{[(\varphi \rightarrow \psi) \rightarrow (\neg\chi \rightarrow \neg\theta)] \rightarrow \chi\} \rightarrow \tau) \rightarrow [(\tau \rightarrow \varphi) \rightarrow (\theta \rightarrow \varphi)].$$

E1.12. Determine whether the following statements are logically consistent. If the contract is valid, then Horatio is liable. If Horatio is liable, he will go bankrupt. Either Horatio will go bankrupt or the bank will lend him money. However, the bank will definitely not lend him money.

E1.13. Prove Proposition 1.5. Hint: For m a positive integer, let G_m be the set of all functions f with the following properties.

(1) The domain of f is m'.

(2) For each $i \in m'$, f_i is itself a function whose domain is the set of all i-tuples of sentential formulas.

(3) $f_1(\varphi) = \varphi$ for every sentential formula φ.

(4) If $1 < i \leq m$ and $\langle\psi_1, \ldots, \psi_i\rangle$ is a sequence of sentential formulas, then

$$f_i(\psi_1, \ldots, \psi_i) = f_{i-1}(\psi_1, \ldots, \psi_{i-1}) \vee \psi_i.$$

Prove:

(5) If $0 < n < m$ and $f \in G_m$, then f restricted to n' is in G_n.

(6) If m is a positive integer and $f, g \in G_m$, then $f = g$.

(7) For each positive integer m the set G_m is nonempty

Then one can define, for each positive integer m, $F_m = f_m$ for f the unique member of G_m.

E1.14. Let $\varphi \mid \psi$ be defined by the following truth table:

φ	ψ	$\varphi \mid \psi$
1	1	0
1	0	0
0	1	0
0	0	1

Prove that for any k, any function mapping k-tuples of members of $\{0,1\}$ into $\{0,1\}$ can be obtained from $\mid$.

E1.15. Give a formula in disjunctive normal form equivalent to the following formula:

$$(S_0 \to (S_1 \to S_2)) \to (S_1 \to S_0).$$

E1.16. Write out an actual proof for $\{\psi\} \vdash \neg\psi \to \varphi$. This can be done by following the proof of Lemma 1.13, expanding it using the proof of the deduction theorem.