Continuum cardinals

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April 2, 2014

These notes form a survey with proofs of the known results concerning combinatorial characteristics of the continuum, omitting any mention of consistency results and characteristics closely related to analysis. Essentially the notes are a condensed version of Blass [∞], which should be browsed to understand some of the background and omitted facts. At the end of the notes are a diagram indicating the relationships between the numbers, and indices of terminology and symbols.

We define $f \leq g$ iff $f, g \in {}^\omega\omega$ and $f(m) \leq g(m)$ for all $m \in \omega$.

We define $f \leq^* g$ iff $f, g \in {}^\omega\omega$ and $\exists m \forall n \geq m[f(n) \leq g(n)]$.

A family $\mathscr{D} \subseteq {}^\omega\omega$ is *almost dominating* iff $\forall f \in {}^\omega\omega \exists g \in \mathscr{D}[f \leq^* g]$. Let $\mathfrak{d}$ be the smallest size of a almost dominating family; this is the *dominating* number.

A family $\mathscr{B} \subseteq {}^\omega\omega$ is *almost unbounded* iff there is no $f \in {}^\omega\omega$ such that $\forall g \in \mathscr{B}[g \leq^* f]$. Let $\mathfrak{b}$ be the smallest size of an almost unbounded family.

Theorem 1. (Blass Theorem 2.4) $\aleph_1 \leq \mathrm{cf}(\mathfrak{b}) = \mathfrak{b} \leq \mathrm{cf}(\mathfrak{d}) \leq \mathfrak{d} \leq 2^\omega$.

Proof. If $\{f_n : n \in \omega\} \subseteq {}^\omega\omega$, define $g \in {}^\omega\omega$ by setting $g(n) = \sup\{f_m(n) : m \leq n\}$ for all $n \in \omega$. Thus if $m \leq n$, then $f_m(n) \leq g(n)$, so $f_m \leq^* g$. Hence g is a $\leq^*$ bound for $\{f_n : n \in \omega\} \subseteq {}^\omega\omega$. This argument shows that $\aleph_1 \leq \mathfrak{b}$. Suppose that $\mathrm{cf}(\mathfrak{b}) < \mathfrak{b}$. Let X be almost unbounded with $|X| = \mathfrak{b}$. Then we can write $X = \bigcup_{\alpha<\mathrm{cf}(\mathfrak{b})} Y_\alpha$ with $|Y_\alpha| < \mathfrak{b}$ for all $\alpha < \mathrm{cf}(\mathfrak{b})$. Choose a bound g^α for Y_α for each $\alpha < \mathrm{cf}(\mathfrak{b})$, and then by the above argument choose a bound h for $\{g^\alpha : \alpha < \mathrm{cf}(\mathfrak{b})\}$. Then h is a bound for X, contradiction. Thus $\mathrm{cf}(\mathfrak{b}) = \mathfrak{b}$.

To prove that $\mathfrak{b} \leq \mathrm{cf}(\mathfrak{d})$, let D be a almost dominating family of size $\mathfrak{d}$, and write $D = \bigcup_{\alpha<\mathrm{cf}(\mathfrak{d})} E_\alpha$, with each E_α of size less than $\mathfrak{d}$. Since then E_α is not almost dominating, there is an $f^\alpha \in {}^\omega\omega$ such that for all $g \in E_\alpha$ we have $f^\alpha \not\leq^* g$. Suppose that $\mathrm{cf}(\mathfrak{d}) < \mathfrak{b}$, and accordingly let $h \in {}^\omega\omega$ be such that $f^\alpha \leq^* h$ for all $\alpha < \mathrm{cf}(\mathfrak{d})$. Choose $k \in D$ such that $h \leq^* k$. Say $k \in E_\alpha$. But $f^\alpha \leq^* h \leq^* k$, contradiction.

Finally, obviously ${}^\omega\omega$ is almost dominating, so $\mathfrak{d} \leq 2^\omega$. □

A *scale* is a almost dominating family which is well-ordered by $\leq^*$.

Theorem 2. (Blass Theorem 2.6) $\mathfrak{b} = \mathfrak{d}$ *iff there is a scale.*

Proof. ⇒: let D be a almost dominating family of size $\mathfrak{b}$; write $D = \{f_\alpha : \alpha < \mathfrak{b}\}$. We define $\langle g_\alpha : \alpha < \mathfrak{b}\rangle$ by recursion. Suppose that g_α has been defined for all $\alpha < \beta$, where $\beta < \mathfrak{b}$. Then $\{f_\alpha :< \alpha < \beta\} \cup \{g_\alpha : \alpha < \beta\}$ has size less than $\mathfrak{b}$, and so there is a function h which $\leq^*$-dominates all of these functions. Let $g_\beta(m) = h(m) + 1$ for all $m \in \omega$. Then clearly $g_\alpha \neq g_\beta$ for all $\alpha < \beta$. This completes the construction. Clearly $\{g_\alpha : \alpha < \mathfrak{b}\}$ is a scale.

⇐: let D be a scale. Let B be almost unbounded with $|B| = \mathfrak{b}$. For each $f \in B$ choose $g_f \in D$ such that $f \leq^* g_f$. Now if $h \in D$, then h is not a bound for B, and so there is an

$f \in B$ such that $f \not\leq^* h$. Hence also $g_f \not\leq^* h$, so $h \leq^* g_f$. Thus $\{g_f : f \in B\}$ is cofinal in D and hence clearly is itself a almost dominating family. Hence $\mathfrak{d} \leq \mathfrak{b}$, as desired. □

We consider ${}^\omega\omega$ as a topological space: the ω-power of the discrete space ω. Recall the definition of a base for this topology. For F a finite subset of ω and $G \in {}^F\mathscr{P}(\omega)$, define

$$\mathscr{O}_{F,G} = \{f \in {}^\omega\omega : \exists g \in G \forall n \in F[f(n) \in g(n)]\}.$$

Then $\{\mathscr{O}_{F,G} : F \in [\omega]^\omega, G \in {}^F\mathscr{P}(\omega)\}$ is a base for the topology on ${}^\omega\omega$. Of course ${}^\omega\omega$ is not compact. We need a characterization of its compact subsets:

Proposition 3. (Blass, remark in the proof of 2.8) *Let $C \subseteq {}^\omega\omega$. Then the following are equivalent:*

(i) C is compact.

(ii) C is closed, and there is an $F \in \prod_{m\in\omega}[\omega]^{<\omega}$ such that $C \subseteq \prod_{m\in\omega} F_m$.

(iii) C is closed, and there is a $g \in {}^\omega\omega$ such that $C \subseteq \{f \in {}^\omega\omega : f \leq g\}$.

Proof. (i)⇒(ii): C is closed, as a compact subset of a Hausdorff space. Now for each $m \in \omega$ let $F_m = \{f(m) : f \in C\}$. Thus $C \subseteq \prod_{m\in\omega} F_m$. Suppose that there is an $m \in \omega$ with F_m infinite. Then

$$C \subseteq \bigcup_{n\in F_m} \mathscr{O}_{\{m\},\{(m,\{n\})\}}$$

with no finite subcover, contradiction.

(ii)⇒(iii): With F as in (ii), choose $g(m) > F_m$ for all $m \in \omega$.

(iii)⇒(i): Suppose that C is covered by a family $\mathscr{F}$ of open sets. Then

$$\prod_{m\in\omega}[0, g(m)] \subseteq ({}^\omega\omega\backslash C) \cup \bigcup \mathscr{F},$$

so there is a finite subset $\mathscr{F}' \subseteq \mathscr{F}$ such that

$$\prod_{m\in\omega}[0, g(m)] \subseteq ({}^\omega\omega\backslash C) \cup \bigcup \mathscr{F}'.$$

Hence $C \subseteq \bigcup \mathscr{F}'$, as desired. □

Let I be a proper ideal on a set X, containing all sets $\{x\}$ for $x \in X$. Then we define (Blass, 2.7)

$$\begin{aligned}
\text{add}(I) &= \min\{|Y|, Y \subseteq I \text{ and } \bigcup Y \notin I\};\\
\text{cov}(I) &= \min\{|Y|, Y \subseteq I \text{ and } \bigcup Y = X\};\\
\text{non}(I) &= \min\{|Y|, Y \subseteq X \text{ and } Y \notin I\};\\
\text{cof}(I) &= \min\{|Y|, Y \subseteq I \text{ and } \forall x \in I \exists y \in Y (x \subseteq y)\}.
\end{aligned}$$

These are respectively the *additivity*, *covering number*, *uniformity*, and *cofinality* of I. We also define $\mathscr{K}_\sigma$ to be the least σ-algebra of subsets of ${}^\omega\omega$ containing all the compact sets (Blass, before 2.8).

Proposition 4. (Blass, remark after 2.7) *Suppose that I is a proper ideal on a set X, containing all sets $\{x\}$ for $x \in X$. Then* $\mathrm{add}(I) \le \mathrm{cf}(\mathrm{non}(I))$.

Proof. Let $Y \subseteq X$ with $Y \notin I$ and $|Y| = \mathrm{non}(I)$.

Case 1. $|Y|$ is regular. Let $Z = \{\{y\} : y \in Y\}$. Then $Z \subseteq I$ and $\bigcup Z = Y \notin I$. So $\mathrm{add}(I) \le \mathrm{cf}(\mathrm{non}(I))$.

Case 2. $|Y|$ is singular. Write $Y = \bigcup_{\alpha<\mathrm{cf}(|Y|)} W_\alpha$ with each $|W_\alpha| < |Y|$. Then each $W_\alpha \in I$, but $Y \notin I$, so $\mathrm{add}(I) \le \mathrm{cf}(\mathrm{non}(I))$. □

Proposition 5. (Blass, remark after 2.7) *Suppose that I is a proper ideal on a set X, containing all sets $\{x\}$ for $x \in X$. Then* $\mathrm{add}(I) \le \mathrm{cov}(I)$. □

Proposition 6. (Blass, remark after 2.7) *Suppose that I is a proper ideal on a set X, containing all sets $\{x\}$ for $x \in X$. Then* $\mathrm{cov}(I) \le \mathrm{cof}(I)$.

Proof. Let $Y \subseteq X$ with $|Y| = \mathrm{cof}(I)$ and $\forall x \in I \exists y \in Y[x \subseteq y]$. Then $\bigcup Y = X$, since if $x \in X$ then $\{x\} \in I$; choosing $y \in Y$ such that $\{x\} \subseteq y$, we get $x \in y$ and so $x \in \bigcup Y$. □

Proposition 7. (Blass, remark after 2.7) *Suppose that I is a proper ideal on a set X, containing all sets $\{x\}$ for $x \in X$. Then* $\mathrm{add}(I) \le \mathrm{cf}(\mathrm{cof}(I))$.

Proof. If $\mathrm{cof}(I)$ is regular, this follows from Propositions 5,6. Suppose it is singular. Let $Y \stackrel{\text{def}}{=} \bigcup_{\alpha<\mathrm{cf}(\mathrm{cof}(I))} Z_\alpha$ be such that $\forall x \in I \exists y \in Y[x \subseteq y]$, $|Y| = \mathrm{cof}(I)$, and $|Z_\alpha| < |Y|$ for all $\alpha < \mathrm{cf}(\mathrm{cof}(I))$. For each $\alpha < \mathrm{cf}(\mathrm{cof}(I))$ choose $x_\alpha \in I$ such that $\forall y \in Z_\alpha[x_\alpha \not\subseteq y]$. Then $\bigcup_{\alpha<\mathrm{cf}(\mathrm{cof}(I))} x_\alpha \notin I$, as otherwise there is a $y \in Y$ such that $\bigcup_{\alpha<\mathrm{cf}(\mathrm{cof}(I))} x_\alpha \subseteq y$; say $y \in Z_\alpha$; then $x_\alpha \subseteq y$, contradiction. □

Proposition 8. (Blass, remark after 2.7) *Suppose that I is a proper ideal on a set X, containing all sets $\{x\}$ for $x \in X$. Then* $\mathrm{non}(I) \le \mathrm{cf}(\mathrm{cof}(I))$.

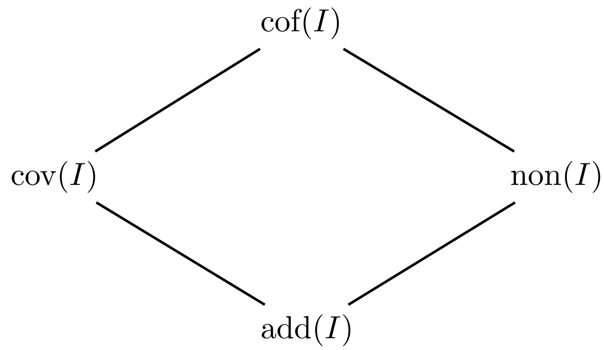
Proof. Let $Y \subseteq I$ be such that $|Y| = \mathrm{cof}(I)$ and $\forall x \in I \exists y \in Y[x \subseteq y]$. Since I is proper, we have $y \neq X$ for all $y \in Y$; choose $x_y \in X \setminus Y$. We claim that $\{x_y : y \in Y\} \notin I$. Otherwise, choose $z \in Y$ such that $\{x_y : y \in Y\} \subseteq z$. Then $x_z \in z$, contradiction. □

Propositions 4–8 are summarized in the diagram on the next page.

Lemma 9. *For any $g \in {}^\omega\omega$ we have $\{f \in {}^\omega\omega : f \le^* g\} \in \mathscr{K}_\sigma$.*

Proof. Let $I = \{(m, s) : m \in \omega$ and $s \in {}^m\omega\}$. So I is countable. Now

$$\{f \in {}^\omega\omega : f \le^* g\} \subseteq \bigcup_{(m,s)\in I} \{f \in {}^\omega\omega : f \le s \cup (g \restriction (\omega \setminus m)\}.$$ □



Proposition 10. (Blass, remark before 2.8 and part of proof of 2.8) *For any* $F \subseteq {}^\omega\omega$ *the following are equivalent:*

(i) $F \in \mathscr{K}_\sigma$.

(ii) F *is covered by a countable union of compact subsets.*

(iii) There is a $g \in {}^\omega\omega$ *such that* $F \subseteq \{f \in {}^\omega\omega : f \leq^* g\}$.

Proof. (i)⇒(ii): Let $I = \{X \subseteq {}^\omega\omega : X$ is covered by a countable union of compact subsets of ${}^\omega\omega\}$. Clearly I is a σ-ideal containing the compact sets. So $\mathscr{K}_\sigma \subseteq I$.

(ii)⇒(iii): Assume (ii). By Proposition 3 we can write $F \subseteq \bigcup_{g\in M}\{f : f \leq g\}$ for some countable $M \subseteq {}^\omega\omega$. Now M is bounded under $\leq^*$ by Theorem 1, so there is an $h \in {}^\omega\omega$ such that $g \leq^* h$ for all $g \in M$. Thus $F \subseteq \{f \in {}^\omega\omega : f \leq^* h\}$.

(iii)⇒(i): by Lemma 3. □

Corollary 11. $\mathscr{K}_\sigma$ *is a proper ideal over* ${}^\omega\omega$ *containing all singletons.*

Proof. Obviously it contains all singletons. By Proposition 10(iii), it is proper. □

Theorem 12. (Blass 2.8) $\mathrm{add}(\mathscr{K}_\sigma) = \mathrm{non}(\mathscr{K}_\sigma) = \mathfrak{b}$ *and* $\mathrm{cov}(\mathscr{K}_\sigma) = \mathrm{cof}(\mathscr{K}_\sigma) = \mathfrak{d}$.

Proof. Easy by Proposition 10(iii). □

(Blass 2.9) An *interval partition* is a partition P of ω whose members are finite intervals. For such a partition, we introduce the following notation:

$$P = \{[i^P_n, i^P_{n+1}) : n \in \omega\},$$

where $0 = i^P_0 < i^P_1 < \cdots$. Given two interval partitions P, Q, we say that P *almost dominates* Q iff $\exists m \forall n \geq m \exists k[[i^Q_k, i^Q_{k+1}) \subseteq [i^P_n, i^P_{n+1})]$.

(Blass, in the proof of 2.10) Now with each interval partition P we associate a function $\mathrm{func}_P \in {}^\omega\omega$ as follows. For each $x \in \omega$, choose n such that $x \in [i^P_n, i^P_{n+1})$, and let $\mathrm{func}_P(x) = i^P_{n+2} - 1$. Conversely, with each $g \in {}^\omega\omega$ we associate an interval partition $\mathrm{part}_g = Q$ as follows. We define $\langle i^Q_k : k \in \omega\rangle$ by recursion. Let $i^Q_0 = 0$. If i^Q_k has been defined, let i^Q_{k+1} be minimum such that $i^Q_{k+1} > i^Q_k$ and for any $x \leq i^Q_k$ we have $g(x) < i^Q_{k+1}$.

Theorem 13. (Blass 2.10) *(i) If P is an interval partition and $g \in {}^\omega\omega$, and if* $\text{func}_P \leq^* g$, *then* part_g *almost dominates P.*

(ii) If P is an interval partition and $g \in {}^\omega\omega$, and if P almost dominates part_g, *then* $g \leq^* \text{func}_P$.

(iii) $\mathfrak{d}$ is the smallest size of a family $\mathscr{P}$ of interval partitions such that every interval partition is almost dominated by some member of $\mathscr{P}$.

(iv) $\mathfrak{b}$ is the smallest size of a family $\mathscr{P}$ of interval partitions such that there is no interval partition which almost dominates each member of $\mathscr{P}$.

Proof. For both (i) and (ii) let, for brevity, $Q = \text{part}_g$.

(i) Choose p such that $\text{func}_P(n) \leq g(n)$ for all $n \geq p$. Take any $n \geq p$. Choose k such that $i^Q_n \in [i^P_k, i^P_{k+1})$. Take any $x \in [i^P_{k+1}, i^P_{k+2})$. Then $p \leq n \leq i^Q_n$, so

$$i^Q_n < i^P_{k+1} \leq x \leq i^P_{k+2} - 1 = f(i^Q_n) \leq g(i^Q_n) < i^Q_{n+1}.$$

Thus $[i^P_{k+1}, i^P_{k+2}) \subseteq [i^Q_n, i^Q_{n+1})$, as desired.

(ii) By definition, choose m so that for all $n \geq m$ there is a k such that $[i^Q_k, i^Q_{k+1}) \subseteq [i^P_n, i^P_{n+1})$. Take any $x \geq i^P_m$; we claim that $g(x) \leq \text{func}_P(x)$. For take n such that $x \in [i^P_n, i^P_{n+1})$. Then $n+1 \geq m$, so we can choose $k \in \omega$ such that $[i^Q_k, i^Q_{k+1}) \subseteq [i^P_{n+1}, i^P_{n+2}))$ Now $x < i^P_{n+1} \leq i^Q_k$, so $g(x) \leq i^Q_{k+1} - 1 \leq i^P_{n+2} - 1 = \text{func}_P(x)$, as desired.

(iii) and (iv) follow immediately from (i) and (ii). □

(Blass 3.1) Given $X, Y \in [\omega]^\omega$, we say that X *splits* Y iff $Y \cap X$ and $Y \backslash X$ are infinite. A *splitting family* is a subset $S \subseteq [\omega]^\omega$ such that every $Y \in [\omega]^\omega$ is split by some member of S. The *splitting number* $\mathfrak{s}$ is the smallest cardinality of a splitting family.

Lemma 14. (Blass, before 3.3) $\omega < \mathfrak{s}$.

Proof. Suppose that $\{Y_i : i < \omega\}$ is a splitting family. It is clear how to construct by recursion an $\varepsilon \in {}^\omega 2$ such that $\bigcap_{j<i} Y_j^{\varepsilon(j)}$ is infinite for every $i < \omega$. Now construct $\langle m_i : i < \omega$ by letting $m_i \in \bigcap_{j<i} Y_j^{\varepsilon(j)} \backslash \{m_j : j < i\}$ for every $i < \omega$. Clearly $Z \stackrel{\text{def}}{=} \{m_i : i < \omega\}$ is not split by any Y_i. □

If P is an interval partition, let $\varphi(P) = \bigcup_{n \in \omega} [i^P_{2n}, i^P_{2n+1})$. If $X \in [\omega]^\omega$ define an interval partition $\psi(X) = Q$ as follows:

$$\begin{aligned} i^Q_0 &= 0; \\ i^Q_{n+1} &= \text{ least } j > i^Q_n \text{ such that } [i^Q_n, j) \cap X \neq \emptyset. \end{aligned}$$

Lemma 15. (Blass, proof of 3.3) *If P almost dominates $\psi(X)$, then $\varphi(P)$ splits X.*

Proof. Let $\Psi(X) = Q$, as above. Choose m such that $\forall n \geq m \exists k [[i^Q_k, i^Q_{k+1}) \subseteq [i^P_n, i^P_{n+1})]$; hence $\forall n [X \cap [i^P_n, i^P_{n+1}) \neq \emptyset]$. So the lemma follows. □

Theorem 16. (Blass 3.3) $\mathfrak{s} \leq \mathfrak{d}$.

Proof. Let D be an almost dominating family of interval partitions, with $|D| = \mathfrak{d}$. Then $\{\varphi(P) : P \in D\}$ is a splitting family by Lemma 14. $\square$

(Blass 3.4) Let $n \in \omega\backslash 1$, $k \in \omega\backslash 2$, $f : [\omega]^n \to k$, and $H \subseteq \omega$. Then H is *homogeneous for f* iff $f \upharpoonright [H]^n$ is constant; it is *almost homogeneous for f* iff there is a finite set $F \subseteq H$ such that $H\backslash F$ is homogeneous for f.

Proposition 17. *If $n \in \omega\backslash 1$ and $k \in \omega\backslash 2$, then there is no $H \in [\omega]^\omega$ such that H is almost homogeneous for all $f \in {}^{[\omega]^n}k$.*

Proof. Suppose that such an H exists. Let $H = \{m_0, m_1, \ldots\}$ with $m_0 < m_1 < \cdots$. Define $f : [\omega]^n \to k$ by setting, for each $Y \in [\omega]^n$,

$$f(Y) = \begin{cases} 0 & \text{if } Y \notin [H]^n, \\ 0 & \text{if } Y \in [H]^n \text{ and } \min(Y) \text{ has the form } m_{2i} \text{ for some } i, \\ 1 & \text{otherwise.} \end{cases}$$

Clearly H is not almost homogeneous for f, contradiction. $\square$

(Blass 3.4) We now define, for every positive integer n,

$$\text{par}_n = \min\{|F| : F \subseteq {}^{[\omega]^n}2 : \neg\exists X \in [\omega]^\omega \forall f \in F[X \text{ is almost homogeneous for } f\}.$$

Proposition 18. (Blass, after 3.4) $\mathfrak{s} = \text{par}_1$.

Proof. First suppose that F satisfies the condition in the definiton of par_1, with $|F| = \text{par}_1$. For each $f \in F$ let $P_f = \{m \in \omega : f(m) = 1\}$, and let $M = \{P_f : f \in F\}$. We claim that M is a splitting family; this will prove $\mathfrak{s} \le \text{par}_1$. So, suppose that $Y \in [\omega]^\omega$. Choose $f \in F$ such that Y is not almost homogeneous for f. Then $Y \cap P_f$ is infinite, as otherwise, since f has the constant value 0 on $Y\backslash P_f$, $Y\backslash P_f$ would be homogeneous for f. Similarly $Y\backslash P_f$ is infinite.

Second, suppose that S is a splitting family. Let F be the collection of all characteristic functions of members of S. So if we show that F satisfies the conditions in the definition of par_n, this will prove that $\mathfrak{s} \ge \text{par}_1$. Suppose that $Y \in [\omega]^\omega$, and choose $M \in S$ which splits Y. Let f be the characteristic function of M. If N is any finite subset of Y, then $(Y\backslash N) \cap S$ and $(Y\backslash N)\backslash S$ are both infinite, and so f is not constant on $Y\backslash N$. $\square$

Proposition 19. (Blass, after 3.4) *Suppose that $2 \le k \in \omega$ and n is a positive integer. Then*

$$\text{par}_n = \min\{|F| : F \subseteq {}^{[\omega]^n}k : \neg\exists X \in [\omega]^\omega \forall f \in F[X \text{ is almost homogeneous for } f]\}$$

Proof. If F is as in the definition of par_n, clearly F works as in the right side. So $\ge$ holds. Now suppose that F is as in the right side. For each $f \in F$ and $i < k$ define $g_{fi} : [\omega]^n \to 2$ by setting, for any $x \in [\omega]^n$,

$$g_{fi}(x) = \begin{cases} 0 & \text{if } f(x) = i, \\ 1 & \text{otherwise.} \end{cases}$$

Thus $G \overset{\text{def}}{=} \{g_{fi} : f \in F, i < k\}$ has the same size as X, so it suffices, in order to prove $\leq$, to show that G satisfies the condition in the definition of par_n. So suppose that $X \in [\omega]^\omega$ and X is almost homogeneous for each g_{fi}. We claim that X is almost homogeneous for each $f \in F$ (contradiction). For, take any $f \in F$. For each $i < k$ let M_i be a finite subset of X such that g_{fi} is constant on $[X\backslash M_i]^n$. We claim that f is constant on $\left[X\backslash \bigcup_{i<k} M_i\right]$, as desired. For, take any two $x, y \in X\backslash \bigcup_{i<k} M_i$. Say $f(x) = i$. Then since $x, y \in X\backslash M_i$, we get $g_{fi}(y) = g_{fi}(x) = 0$, and hence $f(y) = i$, as desired. □

Example 20. (Blass, after 3.4) *If n is a positive integer and $k \in \omega\backslash 2$, then there is a countable $F \subseteq {}^{[\omega]^n}k$ such that there is no $M \in [\omega]^\omega$ such that M is homogeneous for each $f \in F$.*

Proof. Let $[\omega]^n = \{a_\alpha : \alpha < \omega\}$, with $a_\alpha \neq a_\beta$ if $\alpha \neq \beta$. For each $\alpha < \omega$ we define $g_\alpha : [\omega]^n \to k$ by setting, for each $x \in [\omega]^n$,

$$g_\alpha(x) = \begin{cases} 1 & \text{if } x = a_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Let $F = \{g_\alpha : \alpha < \omega\}$. Suppose that $M \in [\omega]^\omega$. Choose $\alpha < \omega$ so that $a_\alpha \in [M]^n$, and choose $x \in [M]^n$ with $x \neq a_\alpha$. Then $g_\alpha(a_\alpha) = 1$ and $g_\alpha(x) = 0$, so M is not homogeneous for g_α. □

Lemma 21. (Blass, in proof of 3.5) *If $m \leq n$, then* $\mathrm{par}_n \leq \mathrm{par}_m$. □

Corollary 22. (Blass, in proof of 3.5) $\mathrm{par}_n \leq \mathfrak{s}$ *for every positive integer n.* □

Theorem 23. (Blass 3.5) *For every integer $n \geq 2$,* $\mathrm{par}_n = \min(\mathfrak{b}, \mathfrak{s})$.

Proof. By Corollary 21, $\mathrm{par}_n \leq \mathfrak{s}$. Next we show that $\mathrm{par}_n \leq \mathfrak{b}$. By Lemma 15 it suffices to take the case $n = 2$. Let B be an almost unbounded subset of ${}^\omega\omega$ with $|B| = \mathfrak{b}$. We may assume that each member of B is strictly increasing. For each $g \in B$ define $f_g : {}^{[\omega]^2} \to 2$ by setting for any $\{x, y\} \in [\omega]^2$ with $x < y$,

$$f_g(\{x, y\}) = \begin{cases} 1 & \text{if } g(x) < y, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that there is no set $H \in [\omega]^\omega$ which is almost homogeneous for all f_g's; this will prove $\mathrm{par}_n \leq \mathfrak{b}$. Suppose that there is such an H.

(1) If $K \subseteq \omega$ and $f_g[[K]^2] \subseteq \{0\}$, then K is finite.

For, assume that $K \neq \emptyset$, and let x be its first element. If $z \in K\backslash\{x\}$, then $f_g(\{x, z\}) = 0$, and hence $z \leq g(x)$. So (1) holds.

Now we define $h, k : \omega \to \omega$ as follows. For any $x \in \omega$, $h(x)$ and $k(x)$ are the first and second elements of H which are greater than x. Now take any $g \in B$; we will show that $g <^* k$ (contradiction). Let F be a finite subset of H such that $f_g \restriction [H\backslash F]$ is constant. By (1), this constant value is 1. Thus if $x > F$, we have $h(x), k(x) \in H\backslash F$ and $h(x) < k(x)$, so

$f_g(\{h(x), k(x)\}) = 1$, and hence $g(h(x)) < k(x)$. So $g(x) < g(h(x)) < k(x)$. Thus $g <^* k$, as desired.

So we have shown $\leq$ in the theorem.

For $\geq$, we prove the following statement by induction on n:

(2) If n is a positive integer, $\langle f_\xi : \xi < \kappa \rangle$ is a system of members of ${}^{[\omega]^n}2$, and $\kappa < \min(\mathfrak{s}, \mathfrak{b})$, then there is a set homogeneous for all of the f_ξ's.

This holds for $n = 1$ by Proposition 18. Now suppose that $n > 2$ and we know the result for $n - 1$. Suppose that $\langle f_\xi : \xi < \kappa \rangle$ is a sequence of members of ${}^{[\omega]^n}2$ with $\kappa < \min(\mathfrak{b}, \mathfrak{s})$. We want to find a set almost homogeneous for all of them. Let $c : \omega \to [\omega]^{n-1}$ be a bijection. For each $\xi < \kappa$ and $p \in \omega$ let

$$f_{\xi,p}(m) = \begin{cases} f_\xi(c(p) \cup \{m\}) & \text{if } m \notin c(p), \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\{f_{\xi,p} : \xi < \kappa, p \in \omega\}$ is a family of less than $\mathfrak{s}$ functions mapping ω into 2 (using Lemma 14). Hence by Proposition 18 there is an infinite set A almost homogeneous for all of them. So for each $\xi < \kappa$ and $p \in \omega$ we can choose $g_\xi(p) \in \omega$ and $j_\xi(p) \in 2$ such that $f_{\xi,p}(x) = j_\xi(p)$ for all $x \in A$ such that $x \geq g_\xi(p)$. Write $A = \{m_i : i < \omega\}$, m strictly increasing. For each $a \in [\omega]^{n-1}$ let $c_\xi(a) = j_\xi(c^{-1}(\{m_i : i \in a\}))$. By the inductive hypothesis, let M be an infinite set almost homogeneous for each j_ξ. Choose b_ξ and k_ξ such that c_ξ takes on the constant value k_ξ on $[M \backslash b_\xi]^{n-1}$. Let $B = \{m_i : i \in M\}$.

(3) If $a \in [B \backslash m_{b_\xi}]^{n-1}$ then $j_\xi(c^{-1}(a)) = k_\xi$.

In fact, write $a = \{m_i : i \in s\}$. Then $s \subseteq M$, and $m_i \geq m_{b_\xi}$ and hence $i \geq b_\xi$, for each $i \in s$. So $s \in [M \backslash b_\xi]^{n-1}$, so $k_\xi = c_\xi(s) = j_\xi(c^{-1}(\{m_i : i \in s\})) = j_\xi(c^{-1}(a))$. So (3) holds.

Since $\kappa < \mathfrak{b}$, choose h such that $g_\xi \leq^* h$ for all $\xi < \kappa$. Choose a_ξ so that $g_\xi(p) \leq h(p)$ for all $p \geq a_\xi$.

Now we define $x_0 < x_1 < \cdots$ in B by recursion. Suppose that x_s has been defined for all $s < t$. Choose $x_t \in B$ so that $x_s < x_t$ for all $s < t$, and also $h(p) < x_t$ for all p such that $c(p) \in [\{x_0, \ldots, x_{t-1}\}]^{n-1}$. Let $H = \{x_i : i < \omega\}$. We claim that H is almost homogeneous for each f_ξ. Let $\xi < \kappa$. Choose t such that $t > c(p)$ for each $p < a_\xi$, and also $t \geq m_{b_\xi}$. Suppose that $a \in [H \backslash t]^n$. Let m be the largest element of a, and let $p = c^{-1}(a \backslash \{m\})$. Then $c(p)$ consists of members of H which are $\geq t$, so $a_\xi \leq p$. Thus $g_\xi(p) \leq h(p) < m$. Also note that $a \backslash \{m\} \in [B \backslash m_{b_\xi}]^{n-1}$. So

$$f_\xi(a) = f_{\xi,p}(m) = j_\xi(p) = j_\xi(c^{-1}(a \backslash \{m\})) = k_\xi. \qquad \square$$

(Blass 3.6) A family $X \subseteq [\omega]^\omega$ is *unsplittable* iff there is no $a \in [\omega]^\omega$ such that $\forall x \in X[x \cap a$ is infinite and $x \backslash a$ is infinite].

Proposition 24. $[\omega]^\omega$ *is unsplittable.*

Proof. Suppose, on the contrary that a splits $[\omega]^\omega$. Applying this to a itself gives a contradiction. $\square$

(Blass 3.6) We define the *reaping number* $\mathfrak{r}$ to be the smallest cardinality of an unsplittable family.

A family $X \subseteq [\omega]^\omega$ is σ-*unsplittable* iff there does not exist a countable $Y \subseteq [\omega]^\omega$ such that $\forall x \in X \exists y \in Y[x \cap y$ is infinite and $x \backslash y$ is infinite$]$.

Proposition 25. *$[\omega]^\omega$ is σ-unsplittable.*

Proof. Suppose on the contrary that $M \in [[\omega]^\omega]^{\leq\sigma}$ and M splits $[\omega]^\omega$. Write $M = \{a_i : i < \omega\}$. Define recursively $\varepsilon \in {}^\omega 2$ such that $\bigcap_{i<m} a_i^{\varepsilon(i)}$ is infinite for all $m < \omega$. Define $s_m \in \bigcap_{i<m} a_i^{\varepsilon(i)} \backslash \{s_n : n < m\}$ for all $m \in \omega$. Let $a = \{s_m : m \in \omega\}$. Then for all $i < \omega$, $a \cap a_i$ is finite or $a \backslash a_i$ is finite, contradiction. □

(Blass 3.6) We define the σ- *reaping number* $\mathfrak{r}_\sigma$ to be the smallest cardinality of a σ-unsplittable family.

Proposition 26. (Blass, after 3.6) $\mathfrak{r} \leq \mathfrak{r}_\sigma$.

Proposition 27. (Blass 3.8) $\mathfrak{b} \leq \mathfrak{r}$.

Proof. Let $R \subseteq [\omega]^\omega$ be unsplittable with $|R| = \mathfrak{r}$. We claim that no interval partition P almost dominates each member of $\{\psi(X) : X \in R\}$. In fact, otherwise by Lemma 15, $\varphi(P)$ splits R, contradiction. So the proposition follows by Theorem 13(iv). □

Proposition 28. *For every positive integer n and every $f : [\omega]^n \to 2$ there is an $X \in [\omega]^\omega$ which is homogeneous for f.*

Proof. Ramsey's theorem. □

(Blass 3.9) For every positive integer, $\mathfrak{hom}_n$ is the smallest size of a family $H \subseteq [\omega]^\omega$ such that for every $f : [\omega]^n \to 2$ there is an $X \in H$ which is almost homogeneous for f.

Proposition 29. *For every $f : \omega \to \omega$ there is an $X \in [\omega]^\omega$ such that $f \restriction X$ is one-one or $f \restriction X$ is a constant function.*

Proof. Define $i \equiv j$ iff $f(i) = f(j)$. If some equivalence class X is infinite, then $f \restriction X$ is constant. Otherwise there is an infinite set X with pairwise inequivalent elements, and then $f \restriction X$ is one-one. □

(Blass 3.9) $\mathfrak{hom}_{1,c}$ is the smallest size of a family $H \subseteq [\omega]^\omega$ such that for every $f : \omega \to \omega$ there is an $X \in H$ such that $f \restriction X$ is almost one-one or $f \restriction X$ is almost constant. Here *almost one-one* means that $f \restriction (X \backslash F)$ is one-one for some finite $F \subseteq X$, and *almost constant* means that $f \restriction (X \backslash F)$ is constant for some finite $F \subseteq X$.

Proposition 30. (Blass, after 3.9) *Suppose that n is a positive integer. Then*

$$\begin{aligned}\mathfrak{hom}_n = \min\{H : H \subseteq [\omega]^\omega \text{ and } \forall f : [\omega]^n \to 2 \exists X \in H \\ [X \text{ is homogeneous for } f]\}.\end{aligned}$$

Proof. Since homogeneous implies almost homogeneous, $\leq$ holds. Now suppose that H is as in the definition of almost homogeneous, with $|H| = \mathfrak{hom}_n$. Let $H' = \{X \backslash F : X \in H \text{ and } F \in [X]^{<\omega}\}$. Clearly for every $f : [\omega]^n \to 2$ there is a $K \in H'$ such that K is homogeneous for f. This proves $\geq$. □

Proposition 31. (Blass, after 3.9)

$$\mathfrak{hom}_{1,c} = \min\{|H| : H \subseteq [\omega]^\omega \text{ and } \forall f : \omega \to \omega \exists X \in H$$
$$[f \restriction X \text{ is one-one or} f \restriction X \text{ is constant}]\}.$$

Proof. Like the proof of Proposition 30. □

Proposition 32. (Blass, after 3.9) $\mathfrak{hom}_1 = \mathfrak{r}$.

Proof. First let H be as in the definition of $\mathfrak{hom}_1$, with $|H| = \mathfrak{hom}_1$. We claim that H is unsplittable. For, suppose that $a \in [\omega]^\omega$ and $\forall X \in H(X \cap a$ is infinite and $X \backslash a$ is infinite). Let f be the characteristic function of a. Clearly no member of H is almost homogeneous for f. This is a contradiction. Hence $\geq$ holds.

Conversely, suppose that H is an unsplittable family, with $|H| = \mathfrak{r}$. We claim that H is as in the definition of $\mathfrak{hom}_1$. For, suppose that $f : \omega \to 2$. Say wlog that $a \stackrel{\text{def}}{=} \{i \in \omega : f(i) = 0\}$ is infinite. Now a does not split H, so there is an $X \in H$ such that $X \cap a$ is finite or $X \backslash a$ is finite. If $X \cap a$ is finite, then $X \backslash a$ is homogeneous for f, so X is almost homogeneous for f. If $X \backslash a$ is finite, then $X \cap a = X \backslash (X \backslash a)$ is homogeneous for f, so X is almost homogeneous for f. So $\leq$ holds. □

Proposition 33. (Blass, after 3.9) *If $m \leq n$, then $\mathfrak{hom}_m \leq \mathfrak{hom}_n$.*

Proof. Suppose that $H \subseteq [\omega]^\omega$, $|H| = \mathfrak{hom}_n$, and $\forall f : [\omega]^n \to 2 \exists a \in H[a$ is almost homogeneous for $f]$. We want to show that $\forall f : [\omega]^m \to 2 \exists a \in H[a$ is almost homogeneous for $f]$. So let $f : [\omega]^m \to 2$. For each $x \in [\omega]^n$ let x^- be the set consisting of its first m members. Define $g : [\omega]^n \to 2$ by setting $g(x) = f(x^-)$ for any $x \in [\omega]^n$. Choose $a \in H$ such that a is almost homogeneous for g. Say that F is a finite subset of H and $g \restriction [H \backslash F]^n$ takes the constant value $\varepsilon \in 2$. For any $x \in [H \backslash F]^m$, let x^+ be the superset of x obtained by adjoining $n - m$ elements of $H \backslash F$ all greater than the members of x. Then $\varepsilon = g(x^+) = f(x)$, as desired. □

Theorem 34. (Blass 3.10) $\max(\mathfrak{d}, \mathfrak{r}) \leq \mathfrak{hom}_{1,c}$.

Proof. Suppose that H is as in the definition of $\mathfrak{hom}_{1,c}$, with $|H| = \mathfrak{hom}_{1,c}$. If a splits H, then $\forall x \in H[x \cap a$ is infinite and $x \backslash a$ is infinite], so the characteristic function on a is neither one-one nor constant on any member of H; cf. Proposition 31. This is a contradiction. So H is unsplittable, and so $\mathfrak{r} \leq \mathfrak{hom}_{1,c}$.

With each $x \in H$ associate an interval partition π_x such that each interval contains at least three members of x. By Theorem 8(iii) it suffices to show that every interval partition P is almost dominated by some π_x. Define $f : \omega \to \omega$ by setting $f(m) = k$ iff $m \in [i^P_k, i^P_{k+1})$ for all $m \in \omega$. Choose $x \in H$ such that $f \restriction x$ is one-one or constant, using Proposition 26 again. Since obviously f is not constant on x, it follows that f is one-one on x. It follows that x has at most one element in common with each interval $[i^P_k, i^P_{k+1})$. Take any $n \in \omega$. Then there are at least three integers $a, b, c \in x$ such that $i^{\pi_x}_n \leq a < b < c < i^{\pi_x}_{n+1}$. Let $f(a) = u$, $f(b) = v$, and $f(c) = w$. Then

$$i^{\pi_x}_n \leq a < i^P_{u+1} \leq i^P_v \leq b < i^P_{v+1} \leq i^P_w \leq c < i^{\pi_x}_{n+1},$$

and hence $[i^P_v, i^P_{v+1}) \subseteq [i^{\pi_x}_n, i^{\pi_x}_{n+1})$, as desired. This proves that $\mathfrak{d} \leq \mathfrak{hom}_{1,c}$. □

Theorem 35. (Blass 3.10) $\mathfrak{hom}_n \leq \max(\mathfrak{r}_\sigma, \mathfrak{d})$.

Proof. For $n = 1$ this is true by Propositions 26 and 32. Now suppose that $n > 1$ and it is true for $n-1$. Let S be a family satisfying the conditions for $\mathfrak{hom}_{n-1}$, with $|S| \leq \max(\mathfrak{r}_\sigma, \mathfrak{d})$. Let R be a σ-unsplittable family with $|R| = \mathfrak{r}_\sigma$. For each $A \in R$ let e_A be the strictly increasing enumeration of A. Let D be an almost dominating family such that $|D| = \mathfrak{d}$. Let $c : \omega \to [\omega]^{n-1}$ be a bijection.

(1) If $A \in R$, $B \in S$, and $h \in D$, then there is an infinite subset C of B with the property that for all $p \in \omega$, if $e_A^{-1}[c(p)] \in [C]^{n-1}$, $y \in C$, and $e_A^{-1}[c(p)] < y$, then $h(p) < y$.

In fact, we define $\langle x_m : m < \omega \rangle$ by recursion. If x_q has been defined for all $q < m$, note that the set $[\{x_q : q < m\}]^{n-1}$ is finite; choose

$$x_m \in B \backslash (\{x_q : q < m\} \cup \{h(p) + 1 : e_A^{-1}[c(p)] \in [\{x_q : q < m\}]^{n-1}\}).$$

Now let $C = \{x_m : m \in \omega\}$. Clearly (1) holds.

For each $A \in R$, $B \in S$, and $h \in D$ we choose C as in (1) and let $H(A, B, h) = e_A[C]$. Let E be the collection of all these sets $H(A, B, h)$. We claim that E satisfies the conditions for $\mathfrak{hom}_n$ (as desired).

To prove this, let $f : [\omega]^n \to 2$ be given. For $p, m \in \omega$ define

$$f_p(m) = \begin{cases} f(c(p) \cup \{m\}) & \text{if } m \notin c(p), \\ 0 & \text{otherwise.} \end{cases}$$

Now by the choice of R, there is an $A \in R$ such that for all $p \in \omega$, either $A \cap f_p^{-1}[\{1\}]$ or $A \backslash f_p^{-1}[\{1\}]$ is finite. Thus there is a $j(p) \in 2$ and a $g(p) \in \omega$ such that $f_p(m) = j(p)$ for all $m \geq g(p)$.

For every $a \in [\omega]^{n-1}$ define $k(a) = j(c^{-1}(e_A[a]))$. By the inductive hypothesis there is a $B \in S$ which is almost homogeneous for k. Choose b and i so that for all $a \in [B \backslash s]^{n-1}$ we have $k(a) = i$. Also choose $h \in D$ which almost dominates g, and choose e so that $h(p) \geq g(p)$ whenever $p \geq e$.

Choose t so that the following hold:

(2) For all $p < e$ we have $c(p) \subseteq t$.

(3) For all $u < s$ we have $e_A(u) < t$.

We claim that $H(A, B, h) \backslash t$ is homogenous for f (as desired). To prove this, note the following consequences of (2) and (3) respectively:

(2') If $c(p) \subseteq H(A,B,h)\backslash t$, then $e \leq p$.

(3') If $e_A[v] \in [H(A,B,h)\backslash t]^{n-1}$, then $s \leq v$.

Now take any $a \in [H(A,B,h)\backslash t]^n$. Let m be the greatest element of a, and choose p so that $c(p) = a\backslash\{m\}$. Thus $f(a) = f_p(m)$.

Now by (2') we have $e \leq p$, and so $g(p) \leq h(p)$. Also, $e_A^{-1}[c(p)] \in [C]^{n-1}$, $e_A^{-1}(m) \in C$, and $e_A^{-1}[c(p)] < e_A^{-1}(m)$. Hence by (1) we get $h(p) < e_A^{-1}(m)$. Now $m \leq e_A(m)$ since e_A is strictly increasing, so $e_A^{-1}(m) \leq m$. So altogether we get $m \geq g(p)$, and hence $f_p(m) = j(p)$.

Next, $e_A^{-1}[a\backslash\{m\}] \geq s$ by (3'), so

$$j(p) = j(c^{-1}(a\backslash\{m\}) = j(c^{-1}(e_A[e_A^{-1}[a\backslash\{m\}]])) = k(e_A^{-1}[a\backslash\{m\}]) = i.$$

Together with the above, this gives $f(a) = i$, finishing the proof. □

Proposition 36. (Blass, remark on page 14) $\mathfrak{hom}_{1,c} \leq \mathfrak{hom}_2$.

Proof. Let H satisfy the conditions for $\mathfrak{hom}_2$ as given in Proposition 30. Now suppose that $f : \omega \to \omega$. Define $g : [\omega]^2 \to 2$ by setting, for $i, j \in \omega$, $i \neq j$,

$$g(\{i,j\}) = \begin{cases} 1 & \text{if } f(i) = f(j), \\ 0 & \text{otherwise.} \end{cases}$$

Choose $X \in H$ such that X is homogeneous for g. Then $f \restriction X$ is one-one or constant. □

Proposition 37. (Blass, page 14) $\mathfrak{r}_\sigma \leq \mathfrak{hom}_2$.

Proof. Let H be as in Proposition 30, with $|H| = \mathfrak{hom}_2$. We want to show that H is σ-unsplittable To this end, suppose that $Y \in [[\omega]^\omega]^{\leq\omega}$. We want to show that there is an $X \in H$ such that $\forall y \in Y[X \cap y$ is finite or $X\backslash y$ is finite]. Write $Y = \{y_n : n \in \omega\}$, and for each $n \in \omega$ let f_n be the characteristic function of y_n. Thus we want to show that some $X \in H$ is almost homogeneous for each function f_n. For each $x \in \omega$ define $g(x) \in {}^\omega 2$ by setting $(g(x))(n) = f_n(x)$. Define $h : [\omega]^2 \to 2$ by setting, for natural numbers $x < y$, $h(\{x,y\}) = 0$ iff $g(x) <_{\rm lex} g(y)$. Choose $X \in H$ homogeneous for g.

Case 1. $g \restriction [X]^2$ has constant value 0. Then

(1) $\forall i \in \omega \exists w_i \in \omega \forall x \geq w_i[(g(x))(i) = (g(w_i))(i)]$.

We prove this by induction on i. Assume that it is true for all $j < i$. Let $p = \max_{j<i} w_j$ ($= 0$ if $i = 0$). If $p \leq x < v$ and $j < i$, then $(g(x))(j) = (g(v))(i)$. Since $g(x) <_{\rm lex} g(v)$, it follows that $(g(x))(i) = 0 = (g(v))(i)$, or $(g(x))(i) = 0$ and $(g(v))(i) = 1$, or $(g(x))(i) = 1 = (g(v))(i)$. So there is a $w_i \geq p$ such that $\forall x \geq w_i[(g(x))(i) = (g(w_i))(i)$. so (1) holds.

By (1), for any i and any $x \geq w_i$ we have $f_i(x) = f_i(w_i)$, as desired.

Case 2. $g \restriction [X]^2$ has constant value 1. Similarly. □

Theorem 38. (Blass 3.10) $\mathfrak{hom}_n = \max\{\mathfrak{d}, \mathfrak{r}_\sigma\}$ *and* $\max\{\mathfrak{d}, \mathfrak{r}\} \leq \mathfrak{hom}_{1,c} \leq \max\{\mathfrak{d}, \mathfrak{r}_\sigma\}$.

Proof. Using successively Theorem 34, Proposition 36, Proposition 33, and Theorem 35, we have

$$\max\{\mathfrak{d},\mathfrak{r}\}\le \mathfrak{hom}_{1,c}\le \mathfrak{hom}_2\le \mathfrak{hom}_n\le \max\{\mathfrak{d},\mathfrak{r}_\sigma\}.$$

This gives the second statement of the Theorem, and also $\le$ in the first statement. Then Proposition 37 yields the first statement. □

A *chopped real* is a pair (x,P) such that $x\in {}^\omega 2$ and P is an interval partition. A real $y\in {}^\omega 2$ *matches* a chopped real (x,P) iff $x\restriction I=y\restriction I$ for infinitely many $I\in P$.

The topology on ${}^\omega 2$ is determined by the basis consisting of all sets $U_z=\{y\in {}^\omega 2: z\subseteq y\}$ for $z\subseteq \omega\times 2$ a finite function. A subset $X\subseteq {}^\omega 2$ is *nowhere dense* iff ${}^\omega 2\backslash\overline{X}$ is dense.

Lemma 39. (Blass, in proof of 5.2) *X is nowhere dense iff for every finite function $z\subseteq\omega\times 2$ there is a finite function $w\subseteq\omega\times 2$ such that $z\subseteq w$ and $U_w\cap X=\emptyset$.*

Proof. Suppose that X is nowhere dense and $z\subseteq\omega\times 2$ is a finite function. Then $U_z\backslash\overline{X}\neq\emptyset$. Take any $f\in U_z\backslash\overline{X}$. Then there is a finite function $t\subseteq\omega\times 2$ such that $f\in U_t$ and $U_t\cap X=\emptyset$. Let $w=z\cup t$. Then w is as desired. Conversely, suppose that for every finite function $z\subseteq\omega\times 2$ there is a finite function $w\subseteq\omega\times 2$ such that $z\subseteq w$ and $U_w\cap X=\emptyset$. Let $z\subseteq\omega\times 2$ be a finite function. Choose w as indicated. Let $f\in U_w$. Then $f\notin\overline{X}$, as desired. □

Clearly the union of two nowhere dense sets is nowhere dense.

A subset $X\subseteq {}^\omega 2$ is *meager* iff X is a countable union of nowhere dense sets.

Theorem 40. (Blass 5.2) *A subset $M\subseteq {}^\omega 2$ is meager iff there is a chopped real (x,P) such that no member of M matches (x,P).*

Proof. Let $(x,P)=(x,\{I_n:n\in\omega\})$ be a chopped real. Then

$$\{y\in {}^\omega 2: y \text{ matches } (x,P)\}=\bigcap_k\bigcup_{n\ge k}\{y\in {}^\omega 2: x\restriction I_n=y\restriction I_n\}. \tag{1}$$

Now each set $\{y\in {}^\omega 2: x\restriction I_n=y\restriction I_n\}$ is open, since it is equal to $U_{x\restriction I_n}$. Hence $\bigcup_{n\ge k}\{y\in {}^\omega 2: x\restriction I_n=y\restriction I_n\}$ is open. It is also dense; for suppose that $z\subseteq\omega\times 2$ is a finite function. Choose $n\ge k$ such that $I_n\cap\mathrm{dmn}(z)=\emptyset$, and let $y\in {}^\omega 2$ extend z and $x\restriction I_n$. Then $y\in U_z\cap\bigcup_{n\ge k}\{y\in {}^\omega 2: x\restriction I_n=y\restriction I_n\}$.

Hence (1) says that $\{y\in {}^\omega 2: y \text{ matches } (x,P)\}$ is a countable intersection of dense open sets. Hence its complement is a countable union of nowhere dense sets, i.e., it is meager. Hence $\Leftarrow$ in the theorem holds.

For the converse, suppose that M is meager; say $M=\bigcup_{n\in\omega}F_n$ with each F_n nowhere dense. Since the union of two nowhere dense sets is nowhere dense, we may assume that $F_0\subseteq F_1\subseteq\cdots$. We now define an interval partition $\langle I_n:n\in\omega\rangle$ and a sequence of functions $\langle z_n:n\in\omega\rangle$, each $z_n\in {}^{I_n}2$. Suppose that these have been defined for all indices $<n$, so that $\langle I_0,\ldots,I_{n-1}\rangle$ are consecutively contiguous. If $n=0$ let $m=0$; otherwise let m be the right endpoint of I_{n-1}. Let $\langle u_i:i<2^m\rangle$ enumerate ${}^m 2$. We now define consecutively contiguous intervals $\langle J_i : i < 2^m \rangle$ and functions $\langle w_i : i < 2^m \rangle$, each $w_i \in {}^{J_i}2$, with m the left endpoint of J_0. Suppose that these have been defined for all $j < i$. By Lemma 34 there is a finite function $t \subseteq \omega \times 2$ such that $u_i \cup \bigcup_{j<i} w_j \subseteq t$ and $U_t \cap F_n = \emptyset$. Increasing t if necessary, we may assume that there is an interval J_i starting with m if $i = 0$ and contiguous with J_{i-1} if $i > 0$ such that $t = u_i \cup \bigcup_{j \leq i} w_j$ for some $w_i \in {}^{J_i}2$. This completes the construction of $\langle J_i : i < 2^m \rangle$ and $\langle w_i : i < 2^m \rangle$. Let $I_n = \bigcup_{i<2^m} J_i$ and $z_n = \bigcup_{i<2^m} w_i$. This completes the construction of $\langle I_n : n \in \omega \rangle$ and $\langle z_n : n \in \omega \rangle$. Let $x = \bigcup_{n \in \omega} z_n$.

Let $P = \langle I_n : n \in \omega \rangle$. Then (x, P) is a chopped real. We claim that no member of M matches (x, P). For, let $y \in M$; say $y \in F_k$. We claim that $x \restriction I_n \neq y \restriction I_n$ for all $n > k$. For, let m be the right endpoint of I_{n-1}, and set $u_i = x \restriction m$. Then by construction $U_{x \restriction p} \cap F_n = \emptyset$, where p is the right endpoint of I_n. Since $y \in F_n$, it follows that $x \restriction I_n \neq y \restriction I_n$. □

Now we define $\text{Match}(x, P) = \{y \in {}^\omega 2 : y \text{ matches } (x, P)\}$.

Proposition 41. (Blass 5.3) $\text{Match}(x, P) \subseteq \text{Match}(x', P')$ *iff for all but finitely many* $I \in P$ *there is a* $J \in P'$ *such that* $J \subseteq I$ *and* $x' \restriction J = x \restriction J$.

Proof. $\Rightarrow$: Suppose that $\text{Match}(x, P) \subseteq \text{Match}(x', P')$ but there are infinitely many $I \in P$ such that for all $J \in P'$, $J \subseteq I$ implies that $x \restriction J \neq x' \restriction J$. Let K be an infinite pairwise non-contiguous set of such I. We define $y \restriction \bigcup K = x \restriction \bigcup K$ while $y(i) = 1 - x'(i)$ for all $i \in \omega \backslash \bigcup K$. Thus $y \in \text{Match}(x, P)$. Hence $y \in \text{Match}(x', P')$. So $L \stackrel{\text{def}}{=} \{J \in P' : y \restriction J = x' \restriction J\}$ is infinite. It follows that $J \subseteq \bigcup K$ for all $J \in L$, so for each $J \in L$ there is an $I \in K$ such that $J \subseteq I$. Then $x \restriction J \neq x' \restriction J$, contradicting $x' \restriction J = y \restriction J = x \restriction J$.

$\Leftarrow$: Suppose that $y \in \text{Match}(x, P) \backslash \text{Match}(x', P')$, while for all but finitely many $I \in P$ there is a $J \in P'$ such that $J \subseteq I$ and $x' \restriction J = x \restriction J$. Then $K \stackrel{\text{def}}{=} \{I \in P : y \restriction I = x \restriction I\}$ is infinite, so there are infinitely many $I \in K$ such that there is a $J \in P'$ such that $J \subseteq I$ and $x' \restriction J = x \restriction J$. This implies that $y \in \text{Match}(x', P')$, contradiction. □

$\mathscr{B}$ is the ideal of meager subsets of ${}^\omega 2$.

Proposition 42. $\text{add}(\mathscr{B}) \leq \mathfrak{b}$.

Proof. By Theorem 13(iv), let $\mathscr{P}$ be a set of interval partitions of size $\mathfrak{b}$ such that no interval partition almost dominates all members of $\mathscr{P}$. Fix $x \in {}^\omega 2$. Now by Theorem 40, a set is meager iff it is a subset of ${}^\omega 2 \backslash \text{Match}(y, P)$ for some chopped real (y, P). In particular,

$$Y \stackrel{\text{def}}{=} \{{}^\omega 2 \backslash \text{Match}(x, Q) : Q \in \mathscr{P}\}$$

is a subset of $\mathscr{B}$. Let $Z = \bigcup Y$. We claim that Z is not meager; the proposition follows from this. For, suppose that Z is meager. By Theorem 40 let (x', P) be a chopped real such that $Z \subseteq {}^\omega 2 \backslash \text{Match}(x', P)$. Thus for all $Q \in \mathscr{P}$ we have $\text{Match}(x', P) \subseteq \text{Match}(x, Q)$. By Theorem 41 it follows that P almost dominates each $Q \in \mathscr{P}$, contradiction. □

(Blass 6.1) If $\mathscr{F}$ is a family of sets, a *pseudo-intersection* of $\mathscr{F}$ is an infinite set A such that $A \subseteq^* B$ for all $B \in \mathscr{F}$.

(Blass 6.2) A *tower* is a sequence $\langle T_\xi : \xi < \alpha \rangle$ with the following properties:

(1) α is an ordinal, and each T_ξ is an infinite subset of ω.

(2) If $\xi < \eta < \alpha$, then $T_\eta \subseteq^* T_\xi$.

(3) $\{T_\xi : \xi < \alpha\}$ does not have a pseudo-intersection.

The *tower number* $\mathfrak{t}$ is the smallest ordinal α which is the length of a tower.

Proposition 34. (Blass 6.4) $\mathfrak{t}$ *is a regular uncountable cardinal.*

Proof. Obviously $\mathfrak{t}$ is a regular cardinal. Now suppose that $\langle T_n : n \in \omega \rangle$ is a sequence of infinite subsets of ω such that $T_n \subseteq^* T_m$ whenever $m < n$; we want to show that this sequence has a pseudo-intersection. For each $m \in \omega$ choose

$$i_m \in \bigcap_{n \leq m} T_n \backslash \{i_n : n < m\}.$$

This is possible because $T_m \backslash T_n$ is finite for each $n < m$, and hence

$$\bigcap_{n \leq m} T_n = T_m \backslash \bigcup_{n < m} (T_m \backslash T_n)$$

is infinite. Clearly $\{i_m : m \in \omega\}$ is a pseudo-intersection of $\{T_m : m \in \omega\}$. □

Clearly there is a tower $\langle T_\xi : \xi < \mathfrak{t} \rangle$ such that $T_\xi \backslash T_\eta$ is infinite whenever $\xi < \eta < \mathfrak{t}$.

(Blass 6.5) A set $\mathscr{D} \subseteq [\omega]^\omega$ is *open* iff $\forall X, Y \in [\omega]^\omega [X \subseteq^* Y \in \mathscr{D} \to X \in \mathscr{D}]$. $\mathscr{D}$ is *dense* iff $\forall Y \in [\omega]^\omega \exists X \in \mathscr{D} [X \subseteq Y]$. Obviously $[\omega]^\omega$ itself is dense open. We say that $\mathscr{D}$ is *weakly dense* iff $\forall Y \in [\omega]^\omega \exists X \in \mathscr{D} [X \subseteq^* Y]$.

Proposition 35. *If $\mathscr{D}$ is weakly dense, then there is a $\mathscr{D}'$ such that $\mathscr{D} \subseteq \mathscr{D}' \subseteq [\omega]^\omega$, $|\mathscr{D}| = |\mathscr{D}'|$, and $\mathscr{D}'$ is dense.*

Proof. Let $\mathscr{D}' = \{X \in [\omega]^\omega :$ there is a finite $F \subseteq \omega$ such that $X \cup F \in \mathscr{D}\}$. □

Proposition 36. *For every $X \in [\omega]^\omega$ there is a dense open family $\mathscr{D}$ such that $X \notin \mathscr{D}$.*

Proof. Let $X = Y \cup Z$ with $Y, Z \in [\omega]^\omega$ and $Y \cap Z = \emptyset$. Define

$$\mathscr{D} = \{W \in [\omega]^\omega : W \subseteq^* Y \text{ or } W \subseteq^* Z \text{ or } W \cap X \text{ is finite}\}.$$

Clearly $\mathscr{D}$ is as desired. □

This proposition justifies the following definition: the *distributivity number* $\mathfrak{h}$ is the smallest size of a collection $\mathscr{A}$ of dense open sets such that $\bigcap \mathscr{A} = \emptyset$.

Proposition 37. (Blass 6.6, modified)

$$\begin{aligned}\mathfrak{h} &= \min\{|\mathscr{A}| : \mathscr{A} \text{ is a family of open weakly dense sets and } \bigcap \mathscr{A} = \emptyset\} \\ &= \min\{\kappa : \mathscr{P}(\omega)/\text{fin is not } (\kappa, \infty)\text{-distributive}\}\end{aligned}$$

Proof. Let $\mathfrak{h}' = \min\{|\mathscr{A}| : \mathscr{A}$ is a family of open weakly dense sets and $\bigcap \mathscr{A} = \emptyset\}$ and $\mathfrak{h}'' = \min\{\kappa : \mathscr{P}(\omega)/\text{fin}$ is not (κ, ∞)-distributive$\}$.

We will use Proposition 14.9 of the BA handbook.

$\mathfrak{h} \le \mathfrak{h}''$: Suppose that $\mathscr{P}(\omega)/\text{fin}$ is not (κ, ∞)-distributive. So by Proposition 14.9 there is a family $\mathscr{P}$ of partitions of $\mathscr{P}(\omega)/\text{fin}$ such that $|\mathscr{P}| \le \kappa$ and $\mathscr{P}$ does not have a common refinement. Let Q be maximal subject to the following conditions: Q is a family of pairwise disjoint nonzero members of $\mathscr{P}(\omega)/\text{fin}$ and $\forall a \in Q \forall P \in \mathscr{P} \exists b \in P[a \le b]$. Then Q is not a partition, as otherwise it would refine $\mathscr{P}$. Let $X \in [\omega]^\omega$ be such that $a \cdot [X] = 0$ for all $a \in Q$. Let $f : \omega \to X$ be a bijection. For each $P \in \mathscr{P}$ let

$$\mathscr{D}_P = \{Y \in [\omega]^\omega : \exists Z \in [\omega]^\omega([Z] \in P \text{ and } [f[Y]] \le [Z])\}.$$

We claim that $\mathscr{D}_P$ is dense open. It is clearly open. Now suppose that $W \in [\omega]^\omega$. Since $[f[W]] \ne 0$, there is a $Z \in [\omega]^\omega$ such that $[Z] \in P$ and $[f[W]] \cdot [Z] \ne 0$. Let $Y = W \cap f^{-1}[Z]$. Then $Y \in [\omega]^\omega$ and $[f[Y]] \le [Z]$. So $Y \in \mathscr{D}_P$. So the claim is established.

Now suppose that $\bigcap_{P \in \mathscr{P}} \mathscr{D}_P \ne \emptyset$; we will get a contradiction, and this will prove $\mathfrak{h} \le \mathfrak{h}''$. Take $Y \in \bigcap_{P \in \mathscr{P}} \mathscr{D}_P$. For any $P \in \mathscr{P}$ we have $Y \in \mathscr{D}_P$, and so we can choose $Z \in [\omega]^\omega$ such that $[Z] \in P$ and $[f[Y]] \le [Z]$. Then $[f[Y]] \le [X]$ and $Q \cup \{[f[Y]]\}$ satisfies the conditions defining Q, contradiction.

$\mathfrak{h}' \le \mathfrak{h}$: obvious.

$\mathfrak{h}'' \le \mathfrak{h}'$: Suppose that $\mathscr{A}$ is a family of open weakly dense sets with empty intersection. Let $\kappa = |\mathscr{A}|$. We show that $\mathscr{P}(\omega)/\text{fin}$ is not (κ, ∞)-distributive, and this will prove $\mathfrak{h}'' \le \mathfrak{h}'$. If $\mathscr{D} \in \mathscr{A}$, let $P_{\mathscr{D}}$ be a maximal set of nonzero pairwise disjoint elements of $\mathscr{P}(\omega)/\text{fin}$ such that $\forall a \in P \exists X \in \mathscr{D}(a \le [X])$. Clearly $P_{\mathscr{D}}$ is a partition. Suppose that Q is a common refinement of $\{\mathscr{D}_P : P \in \mathscr{A}\}$; we will get a contradiction, which will finish the proof. Take any $[X] \in Q$. If $\mathscr{D} \in \mathscr{A}$, then there is a $[Y] \in P_{\mathscr{D}}$ such that $[X] \le [Y]$. By the definition of $P_{\mathscr{D}}$, there is a $Z \in \mathscr{D}$ such that $[Y] \le [Z]$. Then $X \in \mathscr{D}$ since $\mathscr{D}$ is open. So $X \in \bigcap \mathscr{A}$, contradiction. $\square$

Proposition 38. (Blass 6.7) *The intersection of fewer than* $\mathfrak{h}$ *dense open sets is dense open. Thus* $\mathfrak{h}$ *is regular.*

Proof. The second statement clearly follows from the first. Now let $\mathscr{A}$ be a family of dense open sets, with $|\mathscr{A}| < \mathfrak{h}$. Clearly $\bigcap \mathscr{A}$ is open. To show that it is dense, let $Y \in [\omega]^\omega$. Let $f : \omega \to Y$ be a bijection. For each $\mathscr{D} \in \mathscr{A}$ let $\mathscr{D}' = \{f^{-1}[Z] : Z \in \mathscr{D}$ and $Z \subseteq Y\}$. Clearly $\mathscr{D}'$ is dense open. Hence we can choose $X \in \bigcap \{\mathscr{D}' : \mathscr{D} \in \mathscr{A}\}$. Hence $f[X] \in \bigcap \mathscr{A}$ and $f[X] \subseteq Y$, as desired. $\square$

Proposition 39. (Blass 6.8) $\mathfrak{t} \le \mathfrak{h}$.

Proof. Suppose that $\mathscr{A}$ is a family of dense open sets with $|\mathscr{A}| < \mathfrak{t}$; we want to find a member of $\bigcap \mathscr{A}$. Write $\mathscr{A} = \{\mathscr{D}_\alpha : \alpha < \kappa\}$ with $\kappa < \mathfrak{t}$. We now define a sequence $\langle T_\alpha : \alpha \le \kappa\}$ by recursion. Let $T_0 = \omega$. If $T_\alpha \in [\omega]^\omega$ has been chosen, let $T_{\alpha+1} \in \mathscr{D}_\alpha$ be a subset of T_α; this is possible because $\mathscr{D}_\alpha$ is dense. For $\alpha \le \kappa$ limit, let T_α be a pseudo-intersection of $\{T_\beta : \beta < \alpha\}$; this is possible because $\alpha < \mathfrak{t}$. This finishes the construction.

By openness we have $T_\kappa \in \mathscr{A}$. $\square$

Proposition 40. (Blass 6.9) $\mathfrak{h} \le \mathfrak{b}, \mathfrak{s}$.

Proof. By Theorem 18 it suffices to show that $\mathfrak{h} \le \mathrm{par}_2$. So, suppose that $F \subseteq {}^{[\omega]^2}2$ and $|F| < \mathfrak{h}$; we want to find $X \in [\omega]^\omega$ which is almost homogeneous for all $f \in F$. For each $f \in F$ let $\mathscr{D}_f = \{X \in [\omega]^\omega : X$ is almost homogeneous for $f\}$. We claim that each $\mathscr{D}_f$ is dense open. For openness, suppose that $X \subseteq^* Y \in \mathscr{D}_f$. Choose G, H finite such that $f \restriction [Y \backslash G]^2$ is constant and $X \backslash Y = H$. Then $f \restriction [X \backslash (G \cup H)]^2 \subseteq f \restriction [Y \backslash G]^2$ is constant; so $X \in \mathscr{G}_f$. For denseness, take any $Y \in [\omega]^\omega$. Then $f \restriction [Y]^2 : [Y]^2 \to 2$, so by Ramsey's theorem there is an infinite $X \subseteq Y$ such that $f \restriction [X]^2$ is constant; so $X \in \mathscr{D}_f$, as desired.

Take $H \in \bigcap_{f \in F} \mathscr{D}_f$. Clearly H is almost homogeneous for all $f \in F$, as desired. $\square$

Proposition 41. (Blass 6.14) *If $\omega \le \kappa < \mathfrak{t}$, then $2^\kappa = 2^\omega$.*

Proof. By recursion we define $t_\eta \in [\omega]^\omega$ for every $\eta \in {}^{<\mathfrak{t}}2$. Let $t_\emptyset = \omega$. If t_η has been defined, let $t_{\eta 0}$ and $t_{\eta 1}$ be disjoint infinite subsets of t_η. If η has limit length $\alpha < \mathfrak{t}$ and $t_{\eta \restriction \beta}$ has been defined for all $\beta < \alpha$, in such a way that $\beta < \gamma < \alpha$ implies that $t_{\eta \restriction \gamma} \subseteq^* t_{\eta \restriction \beta}$, let t_η be an infinite set $\subseteq^*$ each $t_{\eta \restriction \beta}$ for $\beta < \alpha$; this is possible because $\alpha < \mathfrak{t}$. Now $\langle t_\eta : \eta$ has length $\kappa \rangle$ is a system of infinite almost disjoint subsets of ω, and the desired conclusion follows. $\square$

Corollary 42. (Blass 6.15) $\mathfrak{t} \le \mathrm{cf}(2^\omega)$.

Proof. For each $\kappa < \mathfrak{t}$ we have $\kappa < \mathrm{cf}(2^\kappa) = \mathrm{cf}(2^\omega)$ by Proposition 41. $\square$

(Blass 6.16, 6.17) A family $\mathscr{F} \subseteq [\omega]^\omega$ is *almost disjoint* iff $A \cap B$ is finite for any two distinct members A and B of $\mathscr{F}$. We use MAD to abbreviate "maximal almost disjoint". Note that every finite partition of ω is MAD.

Proposition 43. (part of Blass 6.18) *If $\mathscr{A}$ is a MAD family, then*

$$\mathscr{A}{\downarrow} \stackrel{\text{def}}{=} \{X \in [\omega]^\omega : \exists A \in \mathscr{A}[X \subseteq^* A]\}$$

is dense open.

Proof. Clearly $\mathscr{A}{\downarrow}$ is open. Now suppose that $Y \in [\omega]^\omega$. Then there is an $X \in \mathscr{A}$ such that $Y \cap X$ is infinite. Thus $Y \cap X \in \mathscr{A}{\downarrow}$ and $Y \cap X \subseteq Y$, so $\mathscr{A}{\downarrow}$ is dense. $\square$

Proposition 44. (part of Blass 6.18) *If $\mathscr{D}$ is dense open, then there is a MAD family $\mathscr{A}$ such that $\mathscr{A}{\downarrow} \subseteq \mathscr{D}$.*

Proof. Fix $X \in \mathscr{D}$, and let $\mathscr{A}_0$ be a partition of X into infinitely many infinite subsets. Since $\mathscr{D}$ is open, we have $\mathscr{A}_0 \subseteq \mathscr{D}$. Now by Zorn's lemma let $\mathscr{A}$ be maximal subject to these conditions: (1) $\mathscr{A}_0 \subseteq \mathscr{A} \subseteq [\omega]^\omega$; (2) any two distinct members of $\mathscr{A}$ are almost disjoint; (3) $\mathscr{A} \subseteq \mathscr{D}$. By (1), $\mathscr{A}$ is infinite. By (3) and again using the openness of $\mathscr{D}$, we have $\mathscr{A}{\downarrow} \subseteq \mathscr{D}$. Hence it suffices to show that $\mathscr{A}$ is maximal subject only to (2).

Suppose that $X \in [\omega]^\omega$ and $X \cap Y$ is finite for all $Y \in \mathscr{A}$. Because $\mathscr{D}$ is dense, choose $Z \in \mathscr{D}$ such that $Z \subseteq X$. Now $\mathscr{A} \cup \{Z\}$ satisfies (1)–(3) and $Z \notin \mathscr{A}$, contradiction. □

Proposition 45. *Suppose that $\mathscr{A}$ is a MAD family and $X \in [\omega]^\omega$. Then the following conditions are equivalent:*

(i) $X \notin \mathscr{A}\downarrow$.

(ii) There exist distinct $a, b \in \mathscr{A}$ such that $|a \cap X| = |b \cap X| = \omega$. □

Proposition 46. (Blass 6.19) $\mathfrak{h}$ *is the smallest size of a collection $\mathscr{B}$ of MAD families such that for all $X \in [\omega]^\omega$ there is an $\mathscr{A} \in \mathscr{B}$ such that $X \notin \mathscr{A}\downarrow$.*

Proof. First suppose that $\mathscr{E}$ is a collection of dense open sets such that $\bigcap \mathscr{E} = \emptyset$ and $|\mathscr{E}| = \mathfrak{h}$. For each $\mathscr{D} \in \mathscr{E}$, by Proposition 44 let $\mathscr{A}_\mathscr{D}$ be a MAD family such that $\mathscr{A}_\mathscr{D}\downarrow \subseteq \mathscr{D}$. Thus also $\bigcap_{\mathscr{D}\in\mathscr{E}} \mathscr{A}_\mathscr{D}\downarrow = \emptyset$. Hence for any $X \in [\omega]^\omega$ there is a $\mathscr{D} \in \mathscr{E}$ such that $X \notin \mathscr{A}_\mathscr{D}$. This proves $\geq$.

Conversely, suppose that $\mathscr{B}$ is as in the proposition, of smallest size. Then $\{\mathscr{A}\downarrow : \mathscr{A} \in \mathscr{B}\}$ is a collection of dense open sets, by Proposition 43. Its intersection is empty, and this proves $\leq$. □

(Blass, before 6.20) A *base matrix tree for* $[\omega]^\omega$ is a family $\mathcal{T} \subseteq [\omega]^\omega$ satisfying the following conditions:

(1) $\mathcal{T}$ is a tree under reverse almost inclusion, with root ω.

(2) Each level of $\mathcal{T}$ above the root is a MAD family.

(3) Every $X \in [\omega]^\omega$ has a subset in $\mathcal{T}$.

Theorem 47. (Blass 6.20) *There is a base matrix tree of height* $\mathfrak{h}$.

Proof. By the definition of $\mathfrak{h}$, let $\langle \mathscr{D}_\alpha : \alpha < \mathfrak{h}\rangle$ be a system of dense open sets with empty intersection. By Proposition 38 we may assume that $\mathscr{D}_\alpha \subseteq^* \mathscr{D}_\beta$ if $\beta < \alpha < \mathfrak{h}$. We define the levels $\mathcal{T}_\alpha$ of the desired tree $\mathcal{T}$ by induction, as follows. Of course, let $\mathcal{T}_0 = \{\omega\}$. Now suppose that $\lambda < \mathfrak{h}$ is limit, and $\mathcal{T}_\alpha$ has been defined for all $\alpha < \lambda$ so that (2) holds. Then by Propositions 38 and 43, the set $E \stackrel{\text{def}}{=} \bigcap_{\alpha<\lambda} T_\alpha\downarrow$ is dense open. By Proposition 44, let T_λ be a MAD family such that $T_\lambda\downarrow \subseteq E$. Note that if $\alpha < \lambda$ and $A \in T_\lambda$, then $A \subseteq^* B$ for some $B \in T_\alpha$. In fact, $A \in T_\lambda\downarrow$, hence $A \in T_\alpha\downarrow$, and the existence of B follows.

At a successor stage $2\alpha + 1$, let $T_{2\alpha+1}$ be MAD such that $T_{2\alpha+1}\downarrow \subseteq \mathscr{D}_\alpha \cap (T_{2\alpha}\downarrow)$; this is possible by Propositions 38 and 44.

Now we take a successor stage $2\alpha + 2$. Let

$$\mathscr{A}_\alpha = \{X \in [\omega]^\omega : |\{Y \in T_{2\alpha+1} : |X \cap Y| = \omega\}| = 2^\omega\}.$$

We claim:

(1) There is a one-one function $\psi_\alpha : \mathscr{A}_\alpha \to T_{2\alpha+1}$ such that $|X \cap \psi_\alpha(X)| = \omega$ for all $X \in \mathscr{A}_\alpha$.

In fact, let $\langle X_\xi : \xi < \gamma\rangle$ be a one-one enumeration of $\mathscr{A}_\alpha$, with $\gamma \le 2^\omega$. We define $\psi_\alpha(X_\xi)$ by induction. Suppose that $\psi_\alpha(X_\eta)$ has been defined for all $\eta < \xi$. Then we choose

$$\psi_\alpha(X_\xi) \in \{Y \in T_{2\alpha+1} : |X_\xi \cap Y| = \omega\}\backslash\{\psi_\alpha(X_\eta) : \eta < \xi\};$$

this is possible since $|\{Y \in T_{2\alpha+1} : |X_\xi \cap Y| = \omega\}| = 2^\omega > |\beta|$. Clearly this gives (1).

Now partition each $Y \in T_{2\alpha+1}$ into two infinite sets Y' and Y'', in such a way that if $Y \in \mathrm{rng}(\psi)$, then $Y' \subseteq \psi_\alpha^{-1}(Y)$. Now we define

$$T_{2\alpha+2} = \{Y', Y'' : Y \in T_{2\alpha+1}\}.$$

This finishes the definition of T.

Clearly the first two conditions in the definition of base matrix tree hold. For the third condition it suffices to take any $X \in [\omega]^\omega$ and show that $X \in \mathscr{A}_\alpha$ for some $\alpha < \mathfrak{h}$. We build a subset T' of T which is itself a tree under ${}^*\supseteq$, as follows. The root of T' is ω. Now suppose that the n-th level T'_n of T' has been defined so that:

(a) $n \in \omega$.

(b) All members of T'_n are on the same level α_n of T.

(c) There are exactly 2^n members of T'_n.

(d) $X \cap Y$ is infinite for each $Y \in T'_n$.

Now we claim:

(2) For each $Z \in T'_n$ there is a $\beta_Z < \mathfrak{h}$ such that for every $\gamma \in [\beta_Z, \mathfrak{h})$ we have $|\{Y \in T_\gamma : X \cap Z \cap Y \text{ is infinite}\}| \ge 2$.

In fact, $Z \cap X$ is infinite by (d), and so by the choice of $\langle \mathscr{D}_\alpha : \alpha < \mathfrak{h}\rangle$, there is an $\eta < \mathfrak{h}$ such that $Z \cap X \notin \mathscr{D}\eta$. Hence because the $\mathscr{D}_\xi$'s are decreasing, also $Z \cap X \notin \mathscr{D}_\xi$ for any $\xi \ge \eta$. Take any such ξ. Since $T_{2\xi+1} \subseteq \mathscr{D}_\xi$, we also have $Z \cap X \notin T_{2\xi+1}$. Hence because $T_{2\xi+1}$ is MAD, there is a $Y \in T_{2\xi+1}$ such that $Z \cap X \cap Y$ is infinite. But also $Z \cap X \backslash Y$ is infinite, as otherwise $Z \cap X \subseteq^* Y$ and hence $Z \cap X \in T_{2\xi+1}{\downarrow} \subseteq \mathscr{D}_\xi$, contradiction. This gives another member Y' of $T_{2\xi+1}$ such that $Z \cap X \cap Y'$ is infinite, again because $T_{2\xi+1}$ is MAD. Thus the conclusion of (2) holds.

Now let α_{n+1} be the supremum of all β_Z for $Z \in T'_n$. Now let T'_{n+1} consist of 2^{n+1} elements of T at level $T_{\alpha_{n+1}}$, two successors of each $Z \in T'_n$, each having infinite intersection with $X \cap Z$; thus each of the two successors is $\subseteq^* Z$. This finishes the definition of T'.

Now by Propositions 34, 38 and 39, $\mathfrak{h}$ is regular and uncountable. Hence the ordinal $\gamma \stackrel{\text{def}}{=} \sup_{n\in\omega} \alpha_n$ is less than $\mathfrak{h}$, so we have a level $\gamma + 1$ of T. Now let $\langle Z_n : n \in \omega\rangle$ be a branch through T'; thus Z_n is at level α_n in T, and $Z_{n+1} \subseteq^* Z_n$ for all n. Hence $\langle Z_n \cap X : n \in \omega\rangle$ is a $\subseteq^*$-decreasing sequence of infinite subsets of ω. By Proposition 34, there is some infinite $W \subseteq^* Z_n \cap X$ for all $n \in \omega$. Then there is a $Y \in T_{\gamma+1}$ such that $W \cap Y$ is infinite, since $T_{\gamma+1}$ is MAD. Because T is a tree, we have $Y \subseteq^* Z_n$ for each n. Moreover, $Y \cap W \subseteq^* Y \cap Z_n \cap X$, so $Y \cap X$ is infinite. Thus we each branch through T' we have found a Y below (in the $\subseteq^*$-sense) each member of the branch, with $Y \cap X$ infinite. Clearly then $X \in \mathscr{A}_\gamma$. □

(Blass 6.22) A family $\mathscr{F} \subseteq [\omega]^\omega$ has the *strong finite intersection property*, SFIP, iff the intersection of any finite subset of $\mathscr{F}$ is infinite. We define

$$\mathfrak{p} = \min\{|\mathscr{F}| : \mathscr{F} \subseteq [\omega]^\omega,\ \mathscr{F} \text{ has SFIP, but has no pseudo-intersection}\}.$$

This is well defined, since a tower is such a family $\mathscr{F}$.

Proposition 48. (Blass 6.23) $\aleph_1 \le \mathfrak{p} \le \mathfrak{t}$.

Proof. Obviously $\mathfrak{p} \le \mathfrak{t}$. Now suppose that $\mathscr{F} = \{A_m : m \in \omega\}$ has SFIP; we show that it has a pseudo-intersection, thus proving the first inequality in the proposition. Let $B_m = A_0 \cap \ldots \cap A_m$ for all $m \in \omega$. Each B_m is infinite. By the argument for the proof of Proposition 34, $\{B_m : m \in \omega\}$ has a pseudo-intersection. Hence so does $\mathscr{F}$. □

Proposition 49. (Blass 6.24) *Suppose that $\langle C_n : n \in \omega\rangle$ is a sequence of infinite subsets of ω such that $C_n \subseteq^* C_m$ if $m < n$. Suppose that $\mathscr{A}$ is a family of size less than $\mathfrak{d}$ of infinite subsets of ω, each of which has infinite intersection with each C_n. Then $\{C_n : n \in \omega\}$ has a pseudo-intersection B that has infinite intersection with each member of $\mathscr{A}$.*

Proof. Let $C'_n = \bigcap_{m \le n} C_m$ for all $n \in \omega$. If $A \in \mathscr{A}$, then

$$A \cap C'_n = (A \cap C_n)\backslash \bigcup_{m<n} (C_n \backslash C_m),$$

so $A \cap C'_n$ is still infinite. So it suffices to work with the C'_n's rather than the C_n's.

For each $h \in {}^\omega\omega$ let $B_h = \bigcup_{n\in\omega}(C'_n \cap h(n))$. Then $B_h\backslash C'_n \subseteq \bigcup_{m<n} h(m)$, so that $B_h \subseteq^* C'_n$. Hence it suffices to find $h \in {}^\omega\omega$ so that B_h has infinite intersection with each member of $\mathscr{A}$.

For each $A \in \mathscr{A}$ and each $n \in \omega$, let $f_A(n)$ be the n-th element of the infinite set $A \cap C'_n$ (starting the numbering at 0). Since $|\mathscr{A}| < \mathfrak{d}$, the set $\{f_A : A \in \mathscr{A}\}$ is not almost dominating, and so we can choose $h \in {}^\omega\omega$ such that $h \not\le^* f_A$ for all $A \in \mathscr{A}$. Thus for each $A \in \mathscr{A}$, the set $\{n \in \omega : h(n) > f_A(n)\}$ is infinite, so that $h(n) \cap A \cap C'_n$ has at least n elements for infinitely many n, and so $B_h \cap A$ is infinite, as desired. □

Theorem 50. (Blass 6.25) *If $\mathfrak{p} = \aleph_1$, then $\mathfrak{t} = \aleph_1$.*

Proof. Since $\aleph_1 \le \mathfrak{t} \le \mathfrak{d}$, the result follows if $\mathfrak{d} = \aleph_1$. So, assume that $\mathfrak{d} > \aleph_1$.

Let $\mathscr{A} = \{A_\alpha : \alpha < \omega_1\}$ be a family of infinite subsets of ω with SFIP but with no pseudo-intersection. We may assume that $\mathscr{A}$ is closed under finite intersections. We now define $\langle T_\alpha : \alpha < \omega_1\rangle$ by recursion. Let $T_0 = \omega$. Now suppose that we have constructed T_β for all $\beta < \alpha$ in such a way that:

(1) $T_{\beta+1} \subseteq A_\beta$ for $\beta + 1 < \alpha$,

(2) T_β has infinite intersection with each A_γ, for each $\beta < \alpha$,

(3) $T_\gamma \subseteq^* T_\beta$ if $\beta < \gamma < \alpha$.

If α is a successor ordinal $\beta + 1$, let $T_\alpha = T_\beta \cap A_\beta$. Clearly (1)–(3) continue to hold. Now suppose that α is a limit ordinal. Apply the lemma with $\langle C_n : n \in \omega\rangle$ a cofinal subsequence of $\langle T_\beta : \beta < \alpha\rangle$ to get T_α which continues to satisfy (1)–(3). This finishes the construction.

The T_α's have no pseudo-intersection because of (1). □

(Blass 6.26) A family $\mathscr{G} \subseteq [\omega]^\omega$ is *groupwise dense* iff it is open and for every interval partition P, some union of intervals of P belongs to $\mathscr{G}$. Obviously $[\omega]^\omega$ itself is groupwise dense.

Proposition 51. *The intersection of all groupwise dense families is empty.*

Proof. Let $X \in [\omega]^\omega$; we find a groupwise dense family $\mathscr{G}$ such that $X \notin \mathscr{G}$. Let

$$\mathscr{G} = \{Y \in [\omega]^\omega : X \not\subseteq^* Y\}.$$

Clearly $X \notin \mathscr{G}$, and $\mathscr{G}$ is open. Now suppose that P is an interval partition. Let $Z = \{n \in \omega : [i^P_n, i^P_{n+1}) \cap X \neq \emptyset\}$. Then Z is infinite, since X is infinite. Write $Z = U \cup V$ with U, V disjoint and infinite. Let $Y = \bigcup_{n\in U}[i^P_n, i^P_{n+1})$. Clearly $Y \in \mathscr{G}$. □

(Blass 6.26) By Proposition 51 we can define the *groupwise density number* $\mathfrak{g}$ as follows:

$$\mathfrak{g} = \min\{|\mathscr{H}| : \mathscr{H} \text{ is a system of groupwise dense families and } \bigcap \mathscr{H} = \emptyset\}.$$

Proposition 52. (Blass 6.27.2) *The intersection of fewer than $\mathfrak{g}$ groupwise dense families is groupwise dense.*

Proof. Let $\mathscr{A}$ be a collection of groupwise dense families with $|\mathscr{A}| < \mathfrak{g}$. Clearly $\bigcap \mathscr{A}$ is open. To check the other condition, let P be an arbitrary interval partition. For each $\mathscr{G} \in \mathscr{A}$ let

$$H_{\mathscr{G}} = \{X \in [\omega]^\omega : \bigcup_{n\in X}[i^P_n, i^P_{n+1}) \in \mathscr{G}\}.$$

We want to show that $\bigcap_{\mathscr{G}\in\mathscr{A}} H_{\mathscr{G}}$ is nonempty, and to do this it suffices to show that each $H_{\mathscr{G}}$ is groupwise dense, since $|\mathscr{A}| < \mathfrak{g}$.

First suppose that $Y \subseteq^* X \in H_{\mathscr{G}}$. Then

$$\left(\bigcup_{n\in Y}[i^P_n, i^P_{n+1})\right) \setminus \left(\bigcup_{n\in X}[i^P_n, i^P_{n+1})\right) \subseteq \bigcup_{n\in Y\setminus X}[i^P_n, i^P_{n+1}),$$

and the set on the right is finite. Hence $\bigcup_{n\in Y}[i^P_n, i^P_{n+1}) \in \mathscr{G}$, and so $Y \in H_{\mathscr{G}}$.

Second, suppose that Q is another interval partition; we want to show that some union of members of Q is in $H_{\mathscr{G}}$. Let $j_p = i^P_{i^Q_p}$ for all $p \in \omega$. Thus $j = i^P_{i^Q_0} = i^P_0 = 0$, and $j_p = i^P_{i^Q_p} < i^P_{i^Q_{p+1}} = j_{p+1}$. Hence $R \stackrel{\text{def}}{=} \{[j_p, j_{p+1}) : p \in \omega\}$ is an interval partition. Hence there is a $Z \in [\omega]^\omega$ such that $\bigcup_{n\in Z}[j_n, j_{n+1}) \in \mathscr{G}$. Let $X = \{p : \exists n \in Z(i^Q_n \leq p < i^Q_{n+1})\}$. Then

$$\begin{aligned}\bigcup_{p\in X}[i^P_p, i^P_{p+1}) &= \bigcup_{n\in Z}\bigcup\{[i^P_p, i^P_{p+1}) : i^Q_n \leqq p < i^Q_{n+1}\}\\ &= \bigcup_{n\in Z}[i^P_{i^Q_n}, i^P_{i^Q_{n+1}})\\ &= \bigcup_{n\in Z}[j_n, j_{n+1}) \in \mathscr{G},\end{aligned}$$

as desired. $\square$

Corollary 53. (Blass 6.27.3) $\mathfrak{g}$ *is regular.* $\square$

Proposition 54. (Blass 6.27.4) *Every groupwise dense family is dense, and so* $\mathfrak{h} \le \mathfrak{g}$.

Proof. Let $\mathscr{G}$ be groupwise dense, and let $Y \in [\omega]^\omega$. Let P be an interval partition such that each interval of P intersects Y. Then $\mathscr{G}$ contains the union Z of infinitely many members of P, and so $Z \cap Y$ is infinite. But $\mathscr{G}$ is open, so $Z \cap Y \in \mathscr{G}$, as desired. $\square$

Proposition 55. (Blass 6.27.4) $\mathfrak{g} \le \mathfrak{d}$.

Proof. Let $\mathscr{D}$ be a dominating family of size $\mathfrak{d}$. For each $f \in \mathscr{D}$ let

$$\mathscr{G}_f = \{X \in [\omega]^\omega : \{n : X \cap [n, f(n)) = \emptyset\} \text{ is infinite}\}.$$

We claim that $\mathscr{G}_f$ is groupwise dense. To show that it is open, suppose that $Y \subseteq^* X \in \mathscr{G}_f$. If $X \cap [n, f(n)) = \emptyset$ and $(Y \backslash X) \cap [n, f(n)) = \emptyset$, then

$$Y \cap [n, f(n)) = [(Y \cap X) \cap [n, f(n))] \cup [(Y \backslash X) \cap [n, f(n))] = \emptyset.$$

It follows that

$$\{n : X \cap [n, f(n)) = \emptyset\} \subseteq \{n : Y \cap [n, f(n)) = \emptyset\} \cup \{n : (Y \subseteq X) \cap [n, f(n)) \ne \emptyset\},$$

and $\{n : (Y \subseteq X) \cap [n, f(n)) \ne \emptyset\}$ is finite, so $\{n : Y \cap [n, f(n)) = \emptyset\}$ is infinite, and so $Y \in \mathscr{G}_f$.

For the other property of groupwise density, suppose that P is an interval partition. Define $\langle n(j) : j \in \omega\rangle$ by recursion as follows. Let $n(0) = 0$. If $n(j)$ has been defined, choose $n(j+1) > n(j)$ so that $f(i^P_{n(j)+1}) \le i^P_{n(j+1)}$. Let $X = \bigcup_{j\in\omega}[i^P_{n(j)}, i^P_{n(j)+1}$. Thus X is an infinite subset of ω. For every $j \in \omega$, $X \cap [i^P_{n(j)+1}, f(i^P_{n(j)+1})) = \emptyset$. So $X \in \mathscr{G}_f$, as desired.

To finish the proof it suffices to show that $\bigcap_{f\in\mathscr{D}} \mathscr{G}_f = \emptyset$. Suppose to the contrary that $X \in \bigcap_{f\in\mathscr{D}} \mathscr{G}_f$. Define $g : \omega \to \omega$ by letting $g(n)$ be the least member of X greater than n, for every $n \in \omega$. Choose $f \in \mathscr{D}$ such that $g \le^* f$, and choose $m \in \omega$ such that $\forall n > m(g(n) < f(n))$. Now $X \in \mathscr{G}_f$. Choose $n > m$ such that $X \cap [n, f(n)) = \emptyset$. But $n < g(n) < f(n)$ and $g(n) \in X$, contradiction. $\square$

Proposition 56. (Blass 7.14) *Suppose that* $\mathscr{A}, \mathscr{C} \subseteq [\omega]^\omega$, $|\mathscr{A}| + |\mathscr{C}| < \mathfrak{p}$, *and* $\forall F \in [\mathscr{C}]^{<\omega}$ *and all* $Y \in \mathscr{A}$, *the set* $\bigcap F \cap Y$ *is infinite. Then* $\mathscr{C}$ *has a pseudo-intersection* X *such that* $X \cap Y$ *is infinite for all* $Y \in \mathscr{A}$.

Proof. Define

$$\mathscr{H}_1 = \{[C]^{<\omega} : C \in \mathscr{C}\},$$
$$\mathscr{H}_2 = \{\{F \in [\omega\backslash n]^{<\omega} : F \cap A \ne \emptyset\} : A \in \mathscr{A},\ n \in \omega\}.$$

(1) $\mathscr{H}_1 \cup \mathscr{H}_2$ has the strong finite intersection property.

To prove this, suppose that $\mathscr{F}$ is a finite subset of $\mathscr{C}$ and $\mathscr{G}$ is a finite set of pairs (A, n) such that $A \in \mathscr{A}$ and $n \in \omega$; we want to show that

$$(*) \qquad \bigcap_{C \in \mathscr{F}} [C]^{<\omega} \cap \bigcap_{(A,n) \in \mathscr{G}} \{F \in [\omega \backslash n]^{<\omega} : F \cap A \neq \emptyset\}$$

is infinite. Choose $m < n$ for each n such that $(A, n) \in \mathscr{G}$ for some A, and for each A such that $(A, n) \in \mathscr{G}$ for some n, choose $p_A \in (\bigcap \mathscr{F} \cap A) \backslash n$. Then if F is any finite subset of $[\omega \backslash m]$ such that $p_A \in F$ for all A for which $(A, n) \in G$ for some n, it follows that F is in the set $(*)$.

Thus (1) holds. Note that $|\mathscr{H}_1 + |\mathscr{H}_2| < \mathfrak{p}$, since $\omega < \mathfrak{p}$. Now $|[\omega]^{<\omega}| = \omega$, so we can apply the definition of to $\mathscr{P}([\omega]^{<\omega})/\text{fin}$ rather than $\mathscr{P}(\omega)/\text{fin}$. It follows that there is an $\mathscr{I} \subseteq [\omega]^{<\omega}$ which is a pseudo-intersection of $\mathscr{H}_1 \cup \mathscr{H}_2$.

Since $\mathscr{I}$ is infinite, clearly also $\bigcup \mathscr{I}$ is infinite.

(2) $\bigcup \mathscr{I}$ is a pseudo-intersection of $\mathscr{C}$.

For, suppose that $C \in \mathscr{C}$. Now $\mathscr{I} \subseteq^* [C]^{<\omega}$. Choose $\mathscr{F} \subseteq [\omega]{<}\,\omega$ such that $\mathscr{I} \backslash \mathscr{F} \subseteq [C]^{<\omega}$. Let $n \in \omega$ be greater than each member of $\bigcup \mathscr{F}$. We claim that $\bigcup \mathscr{I} \backslash n \subseteq C$. For, suppose that $m \in \bigcup \mathscr{I} \backslash n$. Choose $I \in \mathscr{I}$ such that $m \in I$. Since $m \geq n$, we have $I \notin \mathscr{F}$. So $I \in [C]^{<\omega}$, and hence $m \in C$, as desired. This proves (2).

Finally, suppose that $A \in \mathscr{A}$. Take any $n < \omega$; we show that $\bigcup \mathscr{I} \cap A$ has a member $\geq n$; this will finish the proof. Now $\mathscr{I} \subseteq^* \{F \in [\omega \backslash n]^{<\omega} : F \cap A \neq \emptyset\}$, so we can find a finite $\mathscr{G} \subseteq [\omega]^{<\omega}$ such that $\mathscr{I} \backslash \mathscr{G} \subseteq \{F \in [\omega \backslash n]^{<\omega} : F \cap A \neq \emptyset\}$. Take any $F \in \mathscr{I} \backslash \mathscr{G}$. Then $F \in [\omega \backslash n]^{<\omega}$ and $F \cap A \neq \emptyset$, so $\bigcup \mathscr{I} \cap A$ has a member $\geq n$. □

Proposition 57. (Blass 7.15) $\mathfrak{p}$ *is regular.*

Proof. Suppose not. Let $\mathscr{A} \subseteq [\omega]^\omega$ with $|\mathscr{A}| = \mathfrak{p}$, $\mathscr{A}$ closed under finite intersections, and write $\mathscr{A} = \bigcup_{\alpha < \text{cf}(\mathfrak{p})} \mathscr{B}_\alpha$, where $|\mathscr{B}_0| = \omega$, $\mathscr{B}_\alpha \subseteq \mathscr{B}_\beta$ for $\alpha < \beta < \text{cf}(\mathfrak{p})$, and $|\mathscr{B}_\alpha| < \mathfrak{p}$ for each $\alpha < \text{cf}(\mathfrak{p})$. Moreover, let each $\mathscr{B}_\alpha$ be closed under finite intersection.

(1) Suppose that $\mathscr{C} \subseteq [\omega]^\omega$, $|\mathscr{C}| < \mathfrak{p}$, and $\forall F \in [\mathscr{C}]^{<\omega}$ and all $Y \in \mathscr{A}$, the set $\bigcap F \cap Y$ is infinite. Then $\mathscr{C}$ has a pseudo-intersection X such that $X \cap Y$ is infinite for all $Y \in \mathscr{A}$.

[Note that this does not follow directly from Proposition 56, since $\mathscr{A}$ has size $\mathfrak{p}$.] To prove (1), note that for any $\alpha < \text{cf}(\mathfrak{p})$, the set $\mathscr{C} \cup \mathscr{B}_\alpha$ has the SFIP and size less than $\mathfrak{p}$, so it has a pseudo-intersection Z_α. Now we apply Proposition 56 to $\mathscr{C}$ and $\{Z_\alpha : \alpha < \text{cf}(\mathfrak{p})\}$: we get a pseudo-intersection X of $\mathscr{C}$ such that $X \cap Z_\alpha$ is infinite for all $\alpha < \text{cf}(\mathfrak{p})$. For any $Y \in \mathscr{A}$, choose $\alpha < \text{cf}(\mathfrak{p})$ with $Y \in \mathscr{B}_\alpha$. Then $Z_\alpha \backslash Y$ is finite, hence $X \cap Z_\alpha \backslash Y$ is finite, hence $X \cap Y$ is infinite, as desired.

Now we are going to define by recursion $\langle C_\alpha : \alpha \leq \text{cf}(\mathfrak{p})\rangle$ so that the following conditions hold:

(2) $C_\alpha \in [\omega]^\omega$.

(3) C_α is a pseudo-intersection of $\mathscr{A}_\alpha$.

(4) If $\beta < \alpha$, then $C_\alpha \backslash C_\beta$ is finite.

(5) $\forall Y \in \mathscr{A}(C_\alpha \cap Y$ is infinite).

As soon as we have done this, a contradiction is reached because $C_{\mathrm{cf}(\mathfrak{p})}$ is a pseudo-intersection of $\mathscr{A}$ by (3) and (4).

So, suppose that C_α has been defined for all $\alpha < \gamma$ so that (2)–(5) hold, where $\gamma \le \mathrm{cf}(\mathfrak{p})$. We want to apply (1) with $\mathscr{C}$ replaced by $\{C_\alpha : \alpha < \gamma\} \cup \mathscr{A}_\gamma$. To check the hypotheses, suppose that $F \in [\gamma]^{<\omega}$, $G \in [\mathscr{A}_\gamma]^{<\omega}$, and $Y \in \mathscr{A}$; we want to show that $\bigcap_{\alpha \in F} C_\alpha \cap \bigcap G \cap Y$ is infinite. Wlog $F \neq \emptyset \neq G$. Let β be the largest element of F. Since $\mathscr{A}$ is closed under $\cap$, we have $\bigcap G \cap Y \in \mathscr{A}$. By (5), $C_\beta \cap \bigcap G \cap Y$ is infinite. Now $\bigcup_{\alpha \in F\backslash\{\beta\}}(C_\beta \backslash C_\alpha) = C_\beta \backslash \bigcap_{\alpha \in F \backslash \{\beta\}} C_\alpha$ is finite, so $\bigcap_{\alpha \in F} C_\alpha \cap \bigcap G \cap Y$ is infinite, as desired.

So we apply (1) and get C_γ such that C_γ is a pseudo-intersection of $\{C_\alpha : \alpha < \gamma\} \cup \mathscr{A}_\gamma$ and $C_\gamma \cap Y$ is infinite for all $Y \in \mathscr{A}$. So (2)–(5) hold, and the construction is complete. □

Now we return to the discussion of MAD families; see Proposition 43 and the following material.

Proposition 58. (Blass 8.1) *There is a MAD family of size* 2^ω.

Proof. Since $|\bigcup_{m\in\omega} {}^m\omega| = \omega$, it suffices to construct an almost disjoint family $\mathscr{F} \subseteq [\bigcup_{m\in\omega} {}^m\omega]^\omega$ of size 2^ω. Let

$$\mathscr{F} = \{\{f \restriction m : m \in \omega\} : f \in {}^\omega 2\}.$$ □

The *almost disjointness number* $\mathfrak{a}$ is the smallest size of an infinite MAD family. (Blass 8.3)

Proposition 59. (Blass 8.4) $\mathfrak{b} \le \mathfrak{a}$.

Proof. Suppose that $\mathscr{A}$ is an infinite MAD family. Let $\langle C_n : n \in \omega\rangle$ be a one-one enumeration of some of the members of $\mathscr{A}$. We define

$$D_0 = C_0 \cup \left(\omega \backslash \bigcup_{n\in\omega} C_n\right);$$

$$D_{n+1} = C_{n+1} \backslash \bigcup_{m\le n} C_m.$$

Clearly $\langle D_n : n \in \omega\rangle$ is a partition of ω into infinite subsets. For each $n \in \omega$, let $f_n : D_n \to \omega$ be a bijection. Let $\mathscr{A}' = \mathscr{A} \backslash \{C_n : n \in \omega\}$. For each $A \in \mathscr{A}$ define $g_A : \omega \to \omega$ by letting $g_A(n)$ be the least natural number such that $\forall m \in A \cap D_n(f_n(m) < g_A(n))$, for any $n \in \omega$. We claim that $\{g_A : A \in \mathscr{A}'\}$ is unbounded, as desired.

To see this, suppose that $g_A \le^* h$ for all $A \in \mathscr{A}'$. Define $X = \{f_n^{-1}(h(n)) : n \in \omega\}$. Thus $|D_n \cap X| = 1$ for all $n \in \omega$. Hence X is infinite. Now take any $A \in \mathscr{A}'$; we show that $X \cap A$ is finite. Choose $p \in \omega$ so that $g_A(n) \le h(n)$ for all $n \ge p$. Then, we claim,

$$(*) \qquad A \cap X \subseteq \bigcup_{n<p} (A \cap D_n).$$

(Hence $A \cap X$ is finite, as desired.) To prove $(*)$, suppose that $m \in A \cap X$. Choose $n \in \omega$ so that $m = f_n^{-1}(h(n))$. Then $m \in A \cap D_n$, so $f_n(m) < g_A(n)$. But $f_n(m) = h(n)$, so it follows that $n < p$. □

Proposition 60. (Blass 8.6) *If $\mathscr{A} \subseteq [\omega]^\omega$ and $|\mathscr{A}| < 2^\omega$, then $\mathscr{A} \cap \mathscr{B} = \emptyset$ for some groupwise dense $\mathscr{B}$.*

Proof. Assume the hypothesis. Let $\mathscr{B} = \{Y \in [\omega]^\omega : \forall X \in \mathscr{A}(X \not\subseteq^* Y)\}$. Clearly $\mathscr{B}$ is open. For the main groupwise dense property, let P be an interval partition. Let $\mathscr{C}$ be a family of 2^ω infinite almost disjoint subsets of ω. For each $C \in \mathscr{C}$, let $C' = \bigcup_{n\in\omega}[i_n^P, i_{n+1}^P)$. Then $\{C' : C \in \mathscr{C}\}$ is still almost disjoint and of size 2^ω. For each $C \in \mathscr{C}$, let $f(C)$ be a member $X \in \mathscr{A}$ such that $X \subseteq^* C'$ if there is such a member, and let it be $\emptyset$ otherwise. Then if C, D are distinct members of $\mathscr{C}$ and $\emptyset \neq f(C), f(D)$, then clearly $f(C) \neq f(D)$. Since f maps the set $\mathscr{C}$ of size 2^ω into the set $\mathscr{A} \cup \{\emptyset\}$ of size less than 2^ω, it follows that there is a $C \in \mathscr{C}$ such that $f(C) = \emptyset$. Thus $C' \in \mathscr{B}$, as desired. □

Proposition 61. (Blass 8.7) $\mathfrak{g} \leq \mathrm{cf}(2^\omega)$.

Proof. Write $[\omega]^\omega = \bigcup_{\alpha<\mathrm{cf}(2^\omega)} \mathscr{A}_\alpha$, with $|\mathscr{A}_\alpha| < 2^\omega$ for each $\alpha < \mathrm{cf}(2^\omega)$. By Proposition 60, for each $\alpha < \mathrm{cf}(a^\omega)$ let $\mathscr{D}_\alpha$ be groupwise dense with $\mathscr{A}_\alpha \cap \mathscr{D}_\alpha = \emptyset$. Then

$$\bigcap_{\alpha<\mathrm{cf}(2^\omega)} \mathscr{D}_\alpha = [\omega]^\omega \cap \bigcap_{\alpha<\mathrm{cf}(2^\omega)} \mathscr{D}_\alpha = \left(\bigcup_{\alpha>\mathrm{cf}(2^\omega)} \mathscr{A}_\alpha\right) \cap \bigcap_{\alpha<\mathrm{cf}(2^\omega)} \mathscr{D}_\alpha = \emptyset.$$

Hence $\mathfrak{g} \leq \mathrm{cf}(2^\omega)$ by the definition of $\mathfrak{g}$. □

(Blass 8.8) A family $\mathscr{I} \subseteq [\omega]^\omega$ is *independent* iff for any finite disjoint $F, G \subseteq \mathscr{I}$ the set $(\bigcap_{X\in F} X) \cap (\bigcap_{X\in G}(\omega\backslash X))$ is infinite.

Proposition 62. (Blass, following 8.8) *If $\mathscr{I} \subseteq [\omega]^\omega$ is infinite, then it is independent iff for any finite disjoint $F, G \subseteq \mathscr{I}$ the set $(\bigcap_{X\in F} X) \cap (\bigcap_{X\in G}(\omega\backslash X))$ is nonempty.*

Proof. $\Rightarrow$: trivial. $\Leftarrow$: Assume the new condition, and suppose that $F, G \subseteq \mathscr{I}$ are finite and disjoint. Then for any finite disjoint $H, K \subseteq \mathscr{I}\backslash(F \cup G)$ we have

$$\left(\bigcap_{X\in F} X\right) \cap \left(\bigcap_{X\in G} (\omega\backslash X)\right) \cap \left(\bigcap_{X\in H} X\right) \cap \left(\bigcap_{X\in K} (\omega\backslash X)\right) \neq \emptyset,$$

and hence $(\bigcap_{X\in F} X) \cap (\bigcap_{X\in G}(\omega\backslash X))$ is infinite. □

Proposition 63. (Blass 8.9) *There is a family of 2^ω independent members of $[\omega]^\omega$.*

Proof. Let $C = \{(a, B) : a \in [\omega]^{<\omega}$ and $B \subseteq \mathscr{P}(a)\}$. Since $|C| = \omega$, it suffices to find 2^ω independent subsets of C. For each $X \subseteq \omega$ let $c_X = \{(a, B) \in C : a \cap X \in B\}$. Suppose that F, G are finite disjoint subsets of $\mathscr{P}(\omega)$. Let $a \in [\omega]^{<\omega}$ be such that $X \cap a \neq Y \cap a$ for all $X \in F$ and $Y \in G$. Take any $a' \in [\omega]^{<\omega}$ such that $a \subseteq a'$. Let $B = \{a' \cap X : X \in F\}$.

Clearly $(a', B) \in c_X$ for all $X \in F$. If $Y \in G$ and $(a', B) \in c_Y$, then $a' \cap Y \in B$, so there is an $X \in F$ such that $a' \cap Y = a' \cap X$. Hence also $a \cap Y = a \cap X$, contradiction. $\square$

(Blass 8.11) We define the *independence number*

$$\mathfrak{i} = \min\{|\mathscr{I} : \mathscr{I} \subseteq [\omega]^\omega \text{ is maximal independent}\}.$$

Proposition 64. (Blass 8.12) $\mathfrak{r} \leq \mathfrak{i}$.

Proof. Let $\mathscr{I} \subseteq [\omega]^\omega$ be maximal independent, with size $\mathfrak{i}$. Let R be the set of all monomials over $\mathscr{I}$. By maximality, R satisfies the conditions defining $\mathfrak{r}$. $\square$

Proposition 65. (Blass 8.13) $\mathfrak{d} \leq \mathfrak{i}$.

Proof. Suppose that $\mathscr{I} \subseteq [\omega]^\omega$ is independent and $|\mathscr{I}| < \mathfrak{d}$; we show that it is not maximal.

Let $\langle D_n : n \in \omega\rangle$ be a one-one enumeration of some of the elements of $\mathscr{I}$, and let $\mathscr{I}' = \mathscr{I}\backslash\{D_n : n \in \omega\}$. For each $\varepsilon \in {}^\omega 2$ and each $n \in \omega$ define

$$C_n^\varepsilon = \bigcap_{k<n} D_k^{\varepsilon(k)}.$$

Let

$$\mathscr{A} = \left\{ \bigcap_{X\in F} X \cap \bigcap_{X\in G} (\omega\backslash X) : F, G \text{ are finite disjoint subsets of } \mathscr{I} \right\}.$$

We apply Proposition 49 to $\langle C_n^\varepsilon : n \in \omega\rangle$ and $\mathscr{A}$ to get a pseudo-intersection B^ε of $\{C_n^\varepsilon : n \in \omega\}$ which has infinite intersection with each element of $\mathscr{A}$. Thus

(1) $B^\varepsilon \subseteq^* \bigcap_{k<n} D_k^{\varepsilon(k)}$ for all $n \in \omega$.

(2) B^ε has infinite intersection with each element of $\mathscr{A}$.

(3) $B^\varepsilon \cap B^\delta$ is finite for distinct $\varepsilon, \delta \in {}^\omega 2$.

This is clear from (1).

(4) There are countable disjoint $Q, Q' \subseteq {}^\omega 2$ such that for every $p \in {}^{<\omega}2$ there are $f \in Q$ and $g \in Q'$ such that $p \subseteq f$ and $p \subseteq g$.

In fact, enumerate ${}^{<\omega}2$ as $\langle p_n : n \in \omega\}$. Now we define functions $f_n, g_n \in {}^\omega 2$ by induction as follows: they are distinct elements of the set

$$\{h \in {}^\omega 2 : p_n \subseteq h\}\backslash\{f_m, g_m : m < n\}.$$

Then we let $Q = \{f_n : n \in \omega\}$ and $Q' = \{g_n : n \in \omega\}$. Clearly (4) holds.

(5) There exists $\langle E^\varepsilon : \varepsilon \in Q \cup Q'\rangle$ such that the E^ε's are pairwise disjoint, $E^\varepsilon \subseteq B^\varepsilon$, and $B^\varepsilon\backslash E^\varepsilon$ is finite.

To prove this, enumerate $Q \cup Q'$ as $\langle \varepsilon_n : n \in \omega \rangle$ without repetitions, and let $E^{\varepsilon_n} = B^{\varepsilon_n} \backslash \bigcup_{m<n} B^{\varepsilon_m}$ for all m; clearly (5) then holds.

Now we define

$$Z = \bigcup_{\varepsilon \in Q} E^{\varepsilon}, \quad \text{and} \quad Z' = \bigcup_{\varepsilon \in Q'} E^{\varepsilon}.$$

(6) Z has infinite intersection with each set $\left(\bigcap_{X \in F} X\right) \cap \left(\bigcap_{X \in G} (\omega \backslash X)\right)$ with F, G finite disjoint subsets of $\mathscr{I}$.

In fact, take such F, G. Let $F' = F \cap \mathscr{I}'$ and $G' = G \cap \mathscr{I}'$. Choose $n \in \omega$ such that for all $k \in \omega$, if $D_k \in F \cup G$ then $k < n$. Define $p \in {}^n 2$ by setting, for each $k < n$,

$$p(k) = \begin{cases} 1 & \text{if } D_k \in F, \\ 0 & \text{otherwise.} \end{cases}$$

Choose $\varepsilon \in Q$ such that $p \subseteq \varepsilon$. Then

$$\begin{aligned} \left(\bigcap_{X \in F} X\right) \cap \left(\bigcap_{X \in G} (\omega \backslash X)\right) &= \left(\bigcap_{X \in F'} X\right) \cap \left(\bigcap_{X \in G'} (\omega \backslash X)\right) \cap \left(\bigcap_{D_k \in F \cup G} D_k^{\varepsilon(k)}\right) \\ &\supseteq \left(\bigcap_{X \in F'} X\right) \cap \left(\bigcap_{X \in G'} (\omega \backslash X)\right) \cap \left(\bigcap_{k<n} D_k^{\varepsilon(k)}\right) \\ &\supseteq^* \left(\bigcap_{X \in F'} X\right) \cap \left(\bigcap_{X \in G'} (\omega \backslash X)\right) \cap B^{\varepsilon}. \\ &\supseteq^* \left(\bigcap_{X \in F'} X\right) \cap \left(\bigcap_{X \in G'} (\omega \backslash X)\right) \cap E^{\varepsilon}. \end{aligned}$$

The last intersection is infinite, and is a subset of Z since $\varepsilon \in Q$, as desired; (6) holds.

Similarly,

(7) Z' has infinite intersection with each set $\bigcap_{X \in F} X \cap \bigcap_{X \in G} (\omega \backslash X)$, with F, G finite disjoint subsets of $\mathscr{I}$.

Since $\omega \backslash Z \supseteq Z'$, this finishes the proof. □

For the remainder of these notes we discuss filters and ultrafilters. Unlike Blass, we allow improper and principal filters in the definition, but in the results we usually require that filters are proper and contain all cofinite sets. We denote by Cofin the collection of all cofinite sets.

Proposition 66. (Blass 9.1, 9.2) *If $\mathscr{F} \subseteq [\omega]^{\omega}$ has the strong finite intersection property, then*

$$\emptyset \neq \langle \mathrm{Cofin} \cup \mathscr{F} \rangle^{\mathrm{fi}} = \left\{ X \subseteq \omega : \bigcap M \subseteq^* X \text{ for some } M \in [\mathscr{F}]^{<\omega} \right\}.$$

Proof. If X is in the left side, then there is a finite $F \subseteq \omega$ and a finite $G \subseteq \mathscr{F}$ such that $(\omega \backslash F) \cap \bigcap G \subseteq X$. Hence $\bigcap G \backslash X \subseteq F$, showing that X is in the right side. The other inclusion is proved similarly. □

(Blass 9.2) If $\mathscr{F} \subseteq \mathscr{P}(\omega)$ and $f : \omega \to \omega$, then we define

$$f((\mathscr{F})) = \{X \subseteq \omega : f^{-1}[X] \in \mathscr{F}\}.$$

Proposition 67. (Blass, after 9.2) *Let $f : \omega \to \omega$.*
(i) If $\emptyset \notin \mathscr{F}$, then $\emptyset \notin f((\mathscr{F}))$.
(ii) If $\mathscr{F}$ is a filter, then so is $f((\mathscr{F}))$.
(iii) If $\mathscr{F}$ is an ultrafilter, then so is $f((\mathscr{F}))$. □

A function $f : A \to B$ is *finite-to-one* iff $\forall b \in B(f^{-1}[\{b\}]$ is finite).

Proposition 68. (Blass, after 9.2) *If $f : \omega \to \omega$ is finite-to-one and $\mathscr{F}$ is a nonprincipal ultrafilter, then so is $f((\mathscr{F}))$.*

Similarly, if a filter $\mathscr{F}$ contains all the cofinite sets, then so does $f((\mathscr{F}))$.

Proof. For any $m \in \omega$, $f^{-1}[\{m\}]$ is finite; so $f^{-1}[\omega\backslash\{m\}] = (\omega\backslash f^{-1}[\{m\}]) \in \mathscr{F}$, and hence $\omega\backslash\{m\} \in f((\mathscr{F}))$. □

Proposition 69. (Blass, in proof of 9.22) *If f is finite-to-one and $\mathscr{F}$ is groupwise dense, then so is $f((\mathscr{F}))$.*

Proof. For openness, suppose that $X \subseteq^* Y \in f((\mathscr{F}))$. Thus $f^{-1}[Y] \in \mathscr{F}$. Now $f^{-1}[X]\backslash f^{-1}[Y] = f^{-1}[X\backslash Y]$ is finite, so $f^{-1}[X] \subseteq^* f^{-1}[Y]$, and so $f^{-1}[X] \in \mathscr{F}$ since $\mathscr{F}$ is open. So $X \in f((\mathscr{F}))$, as desired.

Now suppose that P is an interval partition. Define $\langle j_m : m < \omega\rangle$ and $\langle k(m) : m < \omega\rangle$ by recursion, as follows $j_m = 0 = k(m)$. Suppose that j_m and $k(m)$ have been defined so that $f^{-1}[[0, i^P_{k(m)})] \subseteq [0, j_m)$. (This holds trivially for $m = 0$.) Now $f^{-1}[[0, i^P_{k(m)})]$ is finite, so we can choose $k(m+1) > k(m)$ so that $\emptyset \neq f^{-1}[[i^P_{k(m)}, i^P_{k(m+1)})]$. Now choose $j_{m+1} > j_m$ so that $f^{-1}[[i^P_{k(m)}, i^P_{k(m+1)})] \subseteq [j_m, j_{m+1})$. This finishes the construction.

Let M be an infinite subset of ω such that $\bigcup_{m\in M}[j_m, j_{m+1}) \in \mathscr{F}$. Now by construction, $f^{-1}[\bigcup_{m\in M}[i^P_{k(m)} i^P_{k(m+1)})] \subseteq \bigcup_{m\in M}[j_m, j_{m+1})$, so by the openness of $\mathscr{F}$ we get $f^{-1}[\bigcup_{m\in M}[i^P_{k(m)} i^P_{k(m+1)})] \in \mathscr{F}$, and so $\bigcup_{m\in M}[i^P_{k(m)} i^P_{k(m+1)}) \in f((\mathscr{F}))$, as desired. □

(Blass 9.3) A filter $\mathscr{F}$ on ω is *feeble* iff $\mathscr{F}$ contains all the cofinite sets, is proper, and there is a finite-to-one $f : \omega \to \omega$ such that $f((\mathscr{F})) = \{X \subseteq \omega : \omega\backslash X$ is finite$\}$.

With regard to the following proposition, note that if $\mathscr{F}$ is a proper filter containing all of the cofinite sets, then $\mathscr{F}$ is closed under almost supersets. In fact, if $X \in \mathscr{F}$ and $X \subseteq^* Y$, then $X\backslash Y$ is finite, and so $(\omega\backslash X) \cup Y \in \mathscr{F}$. Intersecting with X, we get $X \cap Y \in \mathscr{F}$, hence $Y \in \mathscr{F}$.

Proposition 70. (Blass 9.4) *Let $\mathscr{F} \subseteq [\omega]^\omega$ be closed under almost supersets. Then the following are equivalent:*

(i) There is a finite-to-one f such that $f((\mathscr{F})) = \{X \subseteq \omega : X$ is infinite and $\omega\backslash X$ is finite$\}$.

(ii) There is a partition P of ω into finite sets such that $\forall X \in \mathscr{F}(\{Y \in P : X \cap Y = \emptyset\}$ is finite).

(iii) There is an interval partition P such that $\forall X \in \mathscr{F}(\{Y \in P : X \cap Y = \emptyset\}$ is finite).

(iv) $\{X \subseteq \omega : X$ is infinite and $\omega \backslash X \in \mathscr{F}\}$ is not groupwise dense.

Proof. (i)⇒(ii): Assume (i), given by a finite-to-one function f. Let $P = \{f^{-1}[\{m\}] : m \in \omega\backslash\{\emptyset\}$. Clearly P is a partition of ω into finite sets. Suppose that $X \in \mathscr{F}$. Let $Z = \{m \in \omega : X \cap f^{-1}[\{m\}] = \emptyset\}$. Then $X \subseteq \omega\backslash f^{-1}[Z]$, since if $n \in X \cap f^{-1}[Z]$ then $f(n) \in Z$, and so $n \in X \cap f^{-1}[f(n)]$, contradiction. Thus $X \subseteq f^{-1}[\omega\backslash Z]$, so $\omega\backslash Z \in f((\mathscr{F}))$. So $\omega\backslash Z$ is cofinite, and hence Z is finite. Now if $Y \in P$ and $X \cap Y = \emptyset$, choose $m \in \omega$ such that $Y = f^{-1}[\{m\}]$; so $m \in Z$. This proves (ii).

(ii)⇒(iii): Let P be a partition as in (ii). Let Q be an interval partition such that for all $n \in \omega$ there is an $a \in P$ such that $a \subseteq [i_n^Q, i_{n+1}^Q)$. So for each $Y \in Q$ we get an $a(Y) \in P$ such that $a(Y) \subseteq Y$. This function a is one-one, and for any $X \in \mathscr{F}$ it maps $\{Y \in Q : X \cap Y = \emptyset\}$ into $\{Z \in P : X \cap Z = \emptyset\}$. By (ii) the latter set is finite, and so the former is also, as desired.

(iii)⇒(iv): Assume (iii), and let P be as indicated. Suppose that Y is a union of intervals of P, Y is infinite, and $\omega\backslash Y \in \mathscr{F}$. Then $\{a \in P : (\omega\backslash Y) \cap a = \emptyset\} = \{a \in P : a \subseteq Y\}$ is infinite, contradiction. So (iv) holds.

(iv)⇒(i): Assume (iv). Let $\mathscr{D}$ be the set in (iv). Then $\mathscr{D}$ is open. For, suppose that Z is infinite and $Z \subseteq^* Y \in \mathscr{D}$. Then $\omega\backslash Y \subseteq^* \omega\backslash Z$, so $\omega\backslash Z \in \mathscr{F}$ by an assumption of the proposition. Thus $Z \in \mathscr{D}$. This shows that $\mathscr{D}$ is open.

So because (iv) holds, it follows that there is an interval partition P such that no union of intervals of P is in $\mathscr{D}$. Let $f : \omega \to \omega$ be defined by $f[[i_n^P, i_{n+1}^P)] = \{n\}$ for all $n \in \omega$. Now if $X \subseteq \omega$ is cofinite, then $\omega\backslash f^{-1}[X] = f^{-1}[\omega\backslash X]$ is finite, and so $f^{-1}[X] \in \mathscr{F}$. Thus $X \in f((\mathscr{F}))$.

Now suppose that $X \in f((\mathscr{F}))$. So $f^{-1}[X] \in \mathscr{F}$. Now $\omega\backslash f^{-1}[X]$ is a union of intervals of P, and its complement is in $\mathscr{F}$, so by the definition of $\mathscr{D}$ it follows that $f^{-1}[\omega\backslash X] = \omega\backslash f^{-1}[X]$ is finite. Hence $\omega\backslash X$ is finite, and so X is cofinite. □

Proposition 71. (Blass 9.5) *There is a nonprincipal ultrafilter on ω every generating set of which has size 2^ω.*

Proof. Let $\mathscr{I}$ be an independent set of size 2^ω. We claim that the set

$$(*) \qquad \mathscr{I} \cup \left\{ \bigcup_{X \in J} (\omega\backslash X) : J \in [\mathscr{I}]^\omega \right\}$$

has the strong finite intersection property. To prove this, suppose that $F \in [\mathscr{I}]^{<\omega}$ and $G \in [[\mathscr{I}]^\omega]^{<\omega}$. For every $J \in G$ choose $K(J) \in J\backslash F$. Then

$$\bigcap_{X \in F} X \cap \bigcap_{J \in G} \bigcup_{Y \in J} (\omega\backslash Y) \supseteq \bigcap_{X \in F} X \cap \bigcap_{J \in G} (\omega\backslash K(J)),$$

and the set on the right is infinite because $\mathscr{I}$ is independent. Thus, as claimed, $(*)$ has the strong finite intersection property.

Hence the set $(*)$, together with the cofinite sets, generates a proper filter, and we extend it to an ultrafilter U. So U is nonprincipal. Suppose that M generates U and $|M| < 2^\omega$. We may assume that M is closed under $\cap$. For every $X \in \mathscr{I}$ we have $X \in U$, and hence there is a $Y(X) \in M$ such that $Y(X) \subseteq X$. Thus Y is a function from $\mathscr{I}$, which has size 2^ω, into M, which has size $< 2^\omega$. So there is a $Z \in M$ such that $\{X \in \mathscr{I} : Y(X) = Z\}$ is infinite. Let N be a countable subset of $\{X \in \mathscr{I} : Y(X) = Z\}$. Then $Z \subseteq \bigcap_{W \in N} W$, but also $\bigcup_{W \in N}(\omega \backslash W) \in U$, so $\emptyset = Z \cap \bigcup_{W \in N}(\omega \backslash W) \in U$, contradiction. □

(Blass 9.6) The *ultrafilter number* is $\mathfrak{u} = \min\{|X| : X$ filter-generates a non-principal ultrafilter$\}$.

Proposition 72. (Blass 9.7) $\mathfrak{r} \leq \mathfrak{u}$.

Proof. Let X filter-generate a nonprincipal ultrafilter U. We may assume that X is closed under $\cap$. For any $a \subseteq \omega$, either $a \in U$ or $(\omega \backslash a) \in U$; so there is a $b \in X$ such that $b \subseteq a$ or $b \subseteq (\omega \backslash a)$. □

(Blass 9.8) A *π-base* for a filter U is a set $X \subseteq [\omega]^\omega$ such that for any $b \in U$ there is an $a \in X$ such that $a \subseteq b$. Note that it is not required that $X \subseteq U$.

Also, a *weak π-base* for a filter U is a set $X \subseteq [\omega]^\omega$ such that for any $b \in U$ there is an $a \in X$ such that $a \subseteq^* b$. Clearly every π-base for U is a weak π-base for U.

Proposition 73. *Let X be a weak π-base for a nonprincipal ultrafilter U. Then X is infinite, and the set*

$$X' = \{b \backslash x : b \in X,\ x \in [\omega]^{<\omega}\}$$

is a π-base for U with $|X| = |X'|$.

Proof. Suppose that X is finite. Let

$$c = \left(\bigcap(X \cap U)\right) \cap \left(\bigcap\{b \in U : \omega \backslash b \in X\}\right).$$

Thus $c \in U$. Since U is nonprincipal, there is a $d \in U$ such that $d \subseteq c$ and $c \backslash d$ is infinite. Choose $x \in X$ such that $x \subseteq^* d$. If $x \in U$, then $x \subseteq^* d \subseteq c \subseteq x$, contradiction. If $\omega \backslash x \in U$, then $x \subseteq^* d \subseteq c \subseteq \omega \backslash x$, contradicting x infinite. Thus X is infinite.

Clearly $X' \subseteq [\omega]^\omega$. Suppose that $b \in U$. Choose $x \in X$ such that $x \subseteq^* b$. Then $x \backslash (x \backslash b) \subseteq b$ and $x \backslash (x \backslash b) \in X'$. Thus X' is a π-base for U. In particular, it is a weak π-base, and so it is infinite by the first part of this proof. Since $|X'| \leq |X| \cup \omega$ and X is infinite, we have $|X'| \leq |X|$. Now $X = \bigcup_{c \in X'}\{b \in X : b \backslash x = c$ for some $x \in [\omega]^{<\omega}\}$, and each set $\{b \in X : b \backslash x = c$ for some $x \in [\omega]^{<\omega}\}$ is countable, so $|X| \leq |X'| \cdot \omega = |X'|$. So $|X| = |X'|$. □

Proposition 74. (Blass, in proof of 9.9) *Suppose that $X \subseteq [\omega]^\omega$. Then the following conditions are equivalent:*

(i) X is a weak π-base for some nonprincipal ultrafilter.

(ii) There is a nonprincipal ultrafilter U such that $U \cap \{b \subseteq \omega : \forall a \subseteq^* b(a \notin X)\} = \emptyset$.

(iii) There does not exist an $F \in [\{b \subseteq \omega : \forall a \subseteq^* b(a \notin X)\}]^{<\omega}$ *such that* $\omega \backslash \bigcup F$ *is finite.*

(iv) If P *is a finite partition of* ω*, then there exist an* $a \in P$ *and a* $b \in X$ *such that* $b \subseteq^* a$*.*

Proof. (i)⇒(ii): obvious.

(ii)⇒(iii): Assume (ii), but suppose that F exists as indicated. Since U is nonprincipal, it follows that $\bigcup F \in U$, and hence $b \in U$ for some $b \in F$, contradicting (ii).

(iii)⇒(iv): Assume (iii), and suppose that P is a finite partition of ω. Then by (iii) we have $P \not\subseteq \{b \subseteq \omega : \forall a \subseteq^* b(a \notin X)\}$, so there is a $b \in P$ such that for some $a \subseteq^* b$ we have $a \in X$, as desired.

(iv)⇒(iii): Assume (iv), but suppose that F exists as in (iii). Let $F = \{b_0, \ldots, b_{m-1}\}$. Define $a_i = b_i \backslash \bigcup_{j<i} b_j$ for $i < m$, and let $a_m = \omega \backslash \bigcup F$. Then $P \stackrel{\text{def}}{=} \{a_i : i \leq m\} \backslash \{\emptyset\}$ is a finite partition of ω. So choose $a_i \in P$ and $c \in X$ such that $c \subseteq^* a_i$. So a_i is infinite, and hence $i < m$. So $c \subseteq^* b_i$, contradiction.

(iii)⇒(i): Assume (iii). Let $Y = \{b \subseteq \omega : \forall a \subseteq^* (\omega \backslash b)(a \notin X)\}$. We claim that Y has fip. For, suppose that G is a finite subset of Y. Let $F = \{b \subseteq \omega : (\omega \backslash b) \in G\}$. Thus $F \subseteq \{b \subseteq \omega : \forall a \subseteq^* b(a \notin X)\}$ Hence by (iii), $\bigcap G = \omega \backslash \bigcup F$ is infinite, as desired.

Note that, trivially, every cofinite set is in Y. So Y is contained in a nonprincipal ultrafilter U. We claim that X is a weak π-base for U. To prove this, suppose that $b \in U$, but $\forall x \in X(x \not\subseteq^* b)$. Thus $\omega \backslash b \in Y$, so $\omega \backslash b \in U$, contradiction. □

Proposition 75. (Blass 9.9) $\mathfrak{r} = \min\{|X| :$ *X is a π-base for some nonprincipal ultrafilter*$\}$.

Proof. $\leq$ holds by the proof of Proposition 72. Now suppose that $X \subseteq [\omega]^\omega$ is unsplittable, with $|X| = \mathfrak{r}$. Thus

$$\forall a \in [\omega]^\omega \exists x \in X(x \subseteq^* a \text{ or } x \subseteq^* (\omega \backslash a)).$$

For every $a \in [\omega]^\omega$, let i_a be a bijection from ω onto a. Define

$$\begin{aligned} Y_0 &= X, \\ Y_{n+1} &= Y_n \cup \{i_b[a] : b \in Y_n, a \in X\}, \\ Z &= \bigcup_{n\in\omega} Y_n. \end{aligned}$$

Thus $|Z| = |X|$. We claim that Z is a weak π-base for some nonprincipal ultrafilter; by Proposition 73, this will complete the proof. To prove this, we apply Proposition 74(iv). We prove by induction on $|P|$ that if P is a finite partition of ω, then there exist $a \in P$ and $b \in Z$ such that $b \subseteq^* a$. For $|P| = 1$ we have $P = \{\omega\}$ and the conclusion is trivial. Assume inductively that $|P| = n+1$. Let $a, b \in P$, $a \neq b$. Let $Q = (P \backslash \{a, b\}) \cup \{a \cup b\}$. So Q is a finite partition of ω and $|Q| = n$. Hence choose $c \in Q$ and $d \in Z$ such that $d \subseteq^* c$. If $c \in P \backslash \{a, b\}$ we are through, so suppose that $c = a \cup b$. Note that $d = (d \cap a) \cup (d \backslash c) \cup (d \cap b)$, and these three sets are pairwise disjoint. Now clearly each member of Z is infinite. Since $d \in Z$ and $d \subseteq^* c$, it follows that c is infinite. Also, $d \backslash c$ is finite. Wlog $d \cap a$ is infinite.

Say $d \in Y_m$. Choose $e \in X$ such that $e \subseteq^* i_d^{-1}[d \cap a]$ or $e \subseteq^* (\omega \backslash i_d^{-1}[d \cap a])$. Then $i_d[e] \in Y_{m+1} \subseteq Z$ and $i_d[e] \subseteq^* d \cap a$ or $i_d[e] \subseteq^* (d \backslash (d \cap a))$. Thus $i_d[e] \subseteq^* a$ or $i_d[e] \subseteq^* b$, as desired. □

Proposition 76. (Blass 9.10, 9.11) *If $\mathscr{F}$ is a filter containing the cofinite sets and with a π-base having fewer than $\mathfrak{b}$ elements, then $\mathscr{F}$ is feeble; in particular, if $\mathscr{F}$ is generated by fewer than $\mathfrak{b}$ elements, then $\mathscr{F}$ is feeble.*

Proof. Let $\mathscr{F}$ be as indicated, with a π-basis X of size less than $\mathfrak{b}$. We may assume that the cofinite sets are in X. For each $A \in X$, let $P(A)$ be an interval partition such that each interval of $P(A)$ contains a point of A. By Theorem 8(iv), there is an interval partition Q such that for all $A \in X$, Q almost dominates $P(A)$. We claim now that (iii) in Proposition 70 holds, completing the proof. To see this, suppose that $Z \in \mathscr{F}$, and choose $A \in X$ such that $A \subseteq Z$. Since Q dominates $P(A)$, choose m such that for all $n \geq m$ there is a k such that $[i_k^{P(A)}, i_{k+1}^{P(A)}) \subseteq [i_n^Q, i_{n+1}^Q)$. Thus if $n \geq m$, then for some k,

$$Z \cap [i_n^Q, i_{n+1}^Q) \supseteq A \cap [i_k^{P(A)}, i_{k+1}^{P(A)}) \neq \emptyset,$$

and thus $\{U \in Q : U \cap Z = \emptyset\}$ is finite, as desired. □

Proposition 77. (Blass 9.10) *There is a non-feeble filter containing all cofinite sets and generated by $\mathfrak{b}$ sets.*

Proof. *Case 1.* $\mathfrak{b} = \mathfrak{d}$. By Theorem 8(iii), let $\langle P_\alpha : \alpha < \mathfrak{b} \rangle$ be a family of interval partitions such that every interval partition is almost dominated by some P_α. We now define subsets X_α of $[\omega]^\omega$ for $\alpha < \mathfrak{b}$ by recursion. Let X_0 be all of the cofinite sets. Suppose that X_β has been defined for all $\beta < \alpha$ so that $|X_\beta| \leq \max(\omega, |\beta|)$ and X_β has the strong finite intersection property.

Subcase 1.1. There is a $Y \in \langle \bigcup_{\beta<\alpha} X_\beta \rangle^{\text{fi}}$ such that $Y \cap [i_n^{P_\alpha}, i_{n+1}^{P_\alpha}) = \emptyset$ for infinitely many n. Then let $X_\alpha = \bigcup_{\beta<\alpha} X_\beta$. Clearly X_α has the strong finite intersection property, and $|X_\alpha| \leq \max(\omega, |\alpha|)$.

Subcase 1.2. A Y as in Subcase 1.1 does not exist. Let

$$X_\alpha = \left(\bigcup_{\beta<\alpha} X_\beta \right) \cup \left\{ \bigcup_{n\in\omega} [i_{2n}^{P_\alpha}, i_{2n+1}^{P_\alpha}) \right\}.$$

Clearly $|X_\alpha| \leq \max(\omega, |\alpha|)$. To show that X_α has the strong finite intersection property, suppose that $F \in [\bigcup_{\beta<\alpha} X_\beta]^{<\omega}$; we want to show that $(\bigcap F) \cap \bigcup_{n\in\omega} [i_{2n}^{P_\alpha}, i_{2n+1}^{P_\alpha})$ is infinite. Clearly $\bigcap F \in \langle \bigcup_{\beta<\alpha} X_\beta \rangle^{\text{fi}}$. By the subcase we are in, $\bigcap F \cap [i_{2n}^{P_\alpha}, i_{2n+1}^{P_\alpha}) \neq \emptyset$ for infinitely many n, so $(\bigcap F) \cap \bigcup_{n\in\omega} [i_{2n}^{P_\alpha}, i_{2n+1}^{P_\alpha})$ is infinite. This finishes the construction.

Let $Z = \bigcup_{\alpha<\mathfrak{b}} X_\alpha$. By the construction, for all $\alpha < \mathfrak{b}$ there is a $U \in Z$ such that $U \cap [i_n^{P_\alpha}, i_{n+1}^{P_\alpha}) = \emptyset$ for infinitely many n. Let $\mathscr{F} = \langle Z \rangle^{\text{fi}}$. Suppose that $\mathscr{F}$ is feeble; we shall obtain a contradiction.

By Proposition 70(iii), let Q be an interval partition such that for all $V \in \mathscr{F}$, the set $\{k : V \cap [i_k^Q, i_{k+1}^Q) = \emptyset\}$ is finite. Choose $\alpha < \mathfrak{b}$ such that P_α almost dominates Q.

Thus there is an m such that $\forall n \geq m \exists k([i_k^Q, i_{k+1}^Q) \subseteq [i_n^{P_\alpha}, i_{n+1}^{P_\alpha})$. Now choose $V \in Z$ such that $V \cap [i_n^{P_\alpha}, i_{n+1}^{P_\alpha}) = \emptyset$ for infinitely many n. Since $V \in \mathscr{F}$, the set $N \stackrel{\text{def}}{=} \{k : V \cap [i_k^Q, i_{k+1}^Q) = \emptyset\}$ is finite. Choose $m' \geq m$ so that $i_{k+1}^Q \leq i_{m'}^{P_\alpha}$ for all $k \in N$. Then for every $n \geq m'$ there is a $k(n)$ such that $[i_{k(n)}^Q, i_{k(n)+1}^Q) \subseteq [i_n^{P_\alpha}, i_{n+1}^{P_\alpha})$. But if $n \geq m'$, then $V \cap [i_{k(n)}^Q, i_{k(n)+1}^Q) \subseteq V \cap [i_n^{P_\alpha}, i_{n+1}^{P_\alpha})$, and the latter set is empty for infinitely many n, so N is infinite, contradiction. This finishes Case 1.

Case 2. $\mathfrak{b} < \mathfrak{d}$. Let $\mathscr{B} \subseteq {}^\omega\omega$ be almost unbounded, with $|\mathscr{B}| = \mathfrak{b}$. For each $g \in B$ we can define g' by $g'(m) = \max\{g(n) : n \leq m\}$, and we obtain $g \leq g'$ and $g'(m) \leq g'(n)$ if $m < n$. Hence we may assume:

(1) If $g \in \mathscr{B}$ and $m < n$, then $g(m) \leq g(n)$.

Given $f, g \in {}^\omega\omega$, define $f \vee g$ by $(f \vee g)(m) = \max\{f(m), g(m)\}$. Clearly we may assume:

(2) If $f, g \in \mathscr{B}$, then $f \vee g \in \mathscr{B}$.

Now $|\mathscr{B}| = \mathfrak{b} < \mathfrak{d}$, so there is a $f \in {}^\omega\omega$ not almost dominated by any member of $\mathscr{B}$. Clearly we may assume:

(3) If $m < n$, then $f(m) \leq f(n)$.

Now for each $g \in \mathscr{B}$ let

$$X_g = \{n \in \omega : g(n) < f(n)\}.$$

Since $f \not\leq^* g$, each such set is infinite. If $g_0, \ldots, g_{m-1} \in \mathscr{B}$, then

$$X_{g_0} \cap \ldots \cap X_{g_{m-1}} = X_h,$$

where $h = g_0 \vee \ldots \vee g_{m-1}$. Hence the set $\{X_g : g \in \mathscr{B}\}$ is closed under $\cap$. Since each element of this set is infinite, it follows that $\{X_g : g \in \mathscr{B}\} \cup \{X \subseteq \omega : X \text{ is cofinite}\}$ has fip, and so generates a filter $\mathscr{F}$ containing all cofinite sets. We finish the proof by assuming that $\mathscr{F}$ is feeble and getting a contradiction.

By Proposition 70(iii), let P be an interval partition such that

(4) $\forall X \in \mathscr{F}(\{Y \in P : X \cap Y = \emptyset\}$ is finite).

Now we define $f' \in {}^\omega\omega$ by setting, for each $k \in \omega$, $f'(k) = f(i_{n+2}^P)$, where n is such that $k \in [i_n^P, i_{n+1}^P)$. We claim that for any $g \in \mathscr{B}$, $g <^* f'$ (contradiction). To prove this, note that $X_g \in \mathscr{F}$, so by (4) we can choose m so that for all $n \geq m$, $X_g \cap [i_n^P, i_{n+1}^P) \neq \emptyset$. Now take any $k \geq i_m^P$. Choose n so that $k \in [i_n^P, i_{n+1}^P)$. Then $n \geq m$, so choose $c \in [i_{n+1}^P, i_{n+2}^P) \cap X_g$. Then

$$g(k) \leq g(c) < f(c) \leq f(i_{n+2}^P) = f'(k). \qquad \square$$

Proposition 78. (Blass, following 9.11) *Every filter which contains the cofinite sets is the intersection of at most 2^ω nonprincipal ultrafilters.*

Proof. Let $\mathscr{F}$ be a filter which contains the cofinite sets. For each $A \in \mathscr{P}(\omega) \backslash \mathscr{F}$ the set $\mathscr{F} \cup \{\omega \backslash A\}$ has the strong finite intersection property, and hence is contained in an ultrafilter $\mathscr{G}_A$. Clearly

$$\mathscr{F} = \bigcap_{A \in \mathscr{P}(\omega) \backslash \mathscr{F}} \mathscr{G}_A. \qquad \square$$

Proposition 79. (Blass 9.12) *The intersection of fewer than 2^ω nonprincipal ultrafilters is not feeble.*

Proof. Suppose to the contrary that $\mathscr{U}$ is a family of $< 2^\omega$ nonprincial ultrafilters such that $\mathscr{F} \stackrel{\text{def}}{=} \bigcap \mathscr{U}$ is feeble. Let $f : \omega \to \omega$ be finite-to-one such that $f((\mathscr{F})) = \{X \subseteq \omega : X$ is cofinite$\}$. Let $\mathscr{A}$ be a family of 2^ω almost disjoint infinite subsets of ω. Then for every $A \in \mathscr{A}$, $\omega \backslash A \notin f((\mathscr{F}))$, since A is infinite. Hence $\omega \backslash f^{-1}[A] = f^{-1}[\omega \backslash A] \notin \mathscr{F}$, so there is a $\mathscr{G}_A \in \mathscr{U}$ such that $\omega \backslash f^{-1}[A] \notin \mathscr{G}_A$, and hence $f^{-1}[A] \in \mathscr{G}_A$. Now the mapping $A \mapsto \mathscr{G}_A$ maps $\mathscr{A}$, which has size 2^ω, into $\mathscr{U}$, which has size less than 2^ω. Hence we can choose distinct $A, B \in \mathscr{A}$ such that $\mathscr{G}_A = \mathscr{G}_B$. Hence $f^{-1}[A \cap B] = f^{-1}[A] \cap f^{-1}[B] \in \mathscr{G}_A$. But $A \cap B$ is finite, and hence so is $f^{-1}[A \cap B]$, since f is finite-to-one. This contradicts $\mathscr{G}_A$ being nonprincipal. □

Proposition 80. (Blass 9.13) *There is a set $\mathscr{U}$ of nonprincipal ultrafilters such that $|\mathscr{U}| = \mathfrak{d}$ and there is no finite-to-one function f such that $f((\bigcap \mathscr{U}))$ is an ultrafilter.*

Proof. By Proposition 8(iii) let $\mathscr{P}$ be a family of interval partitions such that $|\mathscr{P}| = \mathfrak{d}$ and every interval partition is almost dominated by some member of $\mathscr{P}$. For each $P \in \mathscr{P}$ let A_P and B_P be nonprincipal ultrafilters such that

$$\bigcup_{n\in\omega} [i^P_{8n}, i^P_{8n+1}) \in A_P \text{ and } \bigcup_{n\in\omega} [i^P_{8n+4}, i^P_{8n+5}) \in B_P.$$

Let $\mathscr{U} = \{A_P : P \in \mathscr{P}\} \cup \{B_P : P \in \mathscr{P}\}$. Suppose that $f : \omega \to \omega$ is a finite-to-one function such that $f((\bigcap \mathscr{U}))$ is an ultrafilter; we want to get a contradiction. If $n \in \omega$ and $f^{-1}[\{n\}] \neq \emptyset$, let $l(n)$ be the least member of $f^{-1}[\{n\}]$ and $r(n)$ the largest member. Note that l is a one-one function. Now we define $\langle i_m : m \in \omega\rangle$ by recursion. Let $i_0 = 0$ and $i_1 = 1$. If i_n has been defined, where $n \geq 1$, let i_{n+1} be the largest of the following integers:

$i_n + 1$;
$\max\{r(m) : m \in \omega, l(m) \in [i_{n-1}, i_n)\} + 1$ if such m's exist.

Note here that, since l is one-one, there are only finitely many m's such that $l(m) \in [i_{n-1}, i_n)$. Let Q be the partition with classes $\{[i_n, i_{n+1}) : n \in \omega\}$. We claim

(1) For each $m \in \omega$ there is an $n \in \omega$ such that $f^{-1}[\{m\}] \subseteq [i^Q_n, i^Q_{n+2})$.

In fact, if $f^{-1}[\{m\}] = \emptyset$, this is trivially true (take $n = 0$), while if $f^{-1}[\{m\}] \neq \emptyset$, choose n so that $l(m) \in [i^Q_n, i^Q_{n+1})$. Then the conclusion of (1) holds by construction.

Now choose $P \in \mathscr{P}$ such that P almost dominates Q. Thus we can choose m such that for all $n \geq m$ there is a k such that $[i^Q_k, i^Q_{k+1}) \subseteq [i^P_n, i^P_{n+1})$. Choose k such that $[i^Q_k, i^Q_{k+1}) \subseteq [i^P_m, i^P_{m+1})$. Then we claim

(2) For every $l \geq k$ there is an n such that $[i^Q_l, i^Q_{l+1}) \subseteq [i^P_n, i^P_{n+2})$.

For, choose n such that $i^Q_l \in [i^P_n, i^P_{n+1})$. By the choice of k we must have $n \geq m$. Hence we can choose s such that $[i^Q_s, i^Q_{s+1}) \subseteq [i^P_{n+1}, i^P_{n+2})$. Hence $l < s$ and so $[i^Q_l, i^Q_{l+1}) \subseteq [i^P_n, i^P_{n+2})$, as desired.

Now choose p greater than all m such that there is an $l < k$ such that $\emptyset \neq f^{-1}[\{m\}] \subseteq [i^Q_l, i^Q_{l+2})$.

(3) For all $m \geq p$ there is an n such that $f^{-1}[\{m\}] \subseteq [i^P_n, i^P_{n+4})$.

We may assume that $f^{-1}[\{m\}] \neq \emptyset$. Now by (1) choose l such that $f^{-1}[\{m\}] \subseteq [i^Q_l, i^Q_{l+2})$. Then $l \geq k$ by the choice of p, and the desired result follows by (2).

By (3) we have:

$$(4) \qquad \forall m \geq p \left(f^{-1}[\{m\}] \cap \bigcup_{n\in\omega} [i^P_{8n}, i^P_{8n+1}) = \emptyset \text{ or } f^{-1}[\{m\}] \cap \bigcup_{n\in\omega} [i^P_{8n+4}, i^P_{8n+5}) = \emptyset \right)$$

$$(5) \qquad f\left[\bigcup_{n\in\omega} [i^P_{8n}, i^P_{8n+1})\right] \cap f\left[\bigcup_{n\in\omega} [i^P_{8n+4}, i^P_{8n+5})\right] \text{ is finite.}$$

In fact, suppose that $m \geq p$ and $m \in f\left[\bigcup_{n\in\omega}[i^P_{8n}, i^P_{8n+1})\right]$. Choose $k \in \bigcup_{n\in\omega}[i^P_{8n}, i^P_{8n+1})$ such that $m = f(p)$. Thus $p \in f^{-1}[\{m\}] \cap \bigcup_{n\in\omega}[i^P_{8n}, i^P_{8n+1})$, and so by (4) we have $m \notin f\left[\bigcup_{n\in\omega}[i^P_{8n+4}, i^P_{8n+5})\right]$. So (5) holds.

By (5), one of the sets there is not in $f((\mathscr{U}))$. Say that $C = \bigcup_{n\in\omega}[i^P_{8n}, i^P_{8n+1})$ and $f[C] \notin f((\mathscr{U}))$. Hence $\omega\backslash f[C] \in f((\mathscr{U}))$, so that $f^{-1}[\omega\backslash f[C]] \in \bigcap \mathscr{U} \subseteq A_P$. Hence $C \cap f^{-1}[\omega\backslash f[C]] \in A_P$. But if $m \in C \cap f^{-1}[\omega\backslash f[C]]$, then $m \in C$ and $f(m) \in \omega\backslash f[C]$, contradiction. $\square$

(Blass 9.14) For $\mathscr{F} \subseteq [\omega]^\omega$ we define

$$\begin{aligned} \sim\mathscr{F} &= [\omega]^\omega \backslash \mathscr{F}, \\ \mathscr{F}\sim &= \{X \in [\omega]^\omega : \omega\backslash X \in \mathscr{F}\}, \\ \widetilde{\mathscr{F}} &= \{X \in [\omega]^\omega : \omega\backslash X \notin \mathscr{F}\}. \end{aligned}$$

Proposition 81. (Blass, following 9.14, proof of 9.22) *Suppose that $\mathscr{F} \subseteq [\omega]^\omega$. Then*

(i) $\sim(\mathscr{F}\sim) = (\sim\mathscr{F})\sim = \widetilde{\mathscr{F}}$.

(ii) If f is finite-to-one, then $f((\sim\mathscr{F})) = \sim f((\mathscr{F}))$.

(iii) If f is finite-to-one, then $f((\mathscr{F}\sim)) = f((\mathscr{F}))\sim$.

(iv) If $\mathscr{F}$ is closed under supersets, then $\widetilde{\mathscr{F}} = \{X \in [\omega]^\omega : \forall Y \in \mathscr{F}(X \cap Y \neq \emptyset\})$.

(v) If $\mathscr{F}$ is a proper filter, then $\mathscr{F} \subseteq \widetilde{\mathscr{F}}$.

(vi) If $\mathscr{F}$ is a proper filter, then $\mathscr{F}$ is an ultrafilter iff $\mathscr{F} = \widetilde{\mathscr{F}}$.

Proof. (i): Let $a \in [\omega]^\omega$. Then $a \in \sim(\mathscr{F}\sim)$ iff $a \notin (\mathscr{F}\sim)$ iff $\omega\backslash a \notin \mathscr{F}$ iff $a \in \widetilde{\mathscr{F}}$. And $a \in (\sim\mathscr{F})\sim$ iff $\omega\backslash a \in (\sim\mathscr{F})$ iff $\omega\backslash a \notin \mathscr{F}$ iff $a \in \widetilde{\mathscr{F}}$.

(ii): For any a, $a \in f((\sim\mathscr{F}))$ iff $f^{-1}[a] \in \sim\mathscr{F}$ iff $f^{-1}[a] \notin \mathscr{F}$ iff not$(f^{-1}[a] \in \mathscr{F})$ iff not$(a \in f((\mathscr{F})))$ iff $a \in \sim f((\mathscr{F}))$.

(iii): For any a, $a \in f((\mathscr{F}\sim))$ iff $f^{-1}[a] \in (\mathscr{F}\sim)$ iff $\omega\backslash f^{-1}[a] \in \mathscr{F}$ iff $f^{-1}[\omega\backslash a] \in \mathscr{F}$ iff $(\omega\backslash a) \in f((\mathscr{F}))$ iff $a \in (f((\mathscr{F}))\sim)$.

(iv): Suppose that $X \in \widetilde{\mathscr{F}}$. If $Y \in \mathscr{F}$ and $X \cap Y = \emptyset$, then $Y \subseteq (\omega\backslash X)$, hence $(\omega\backslash X) \in \mathscr{F}$, contradicting $X \in \widetilde{\mathscr{F}}$. Now suppose that $\forall Y \in \mathscr{F}(X \cap Y \neq \emptyset)$. Then $\omega\backslash X \notin \mathscr{F}$, so $X \in \widetilde{\mathscr{F}}$.

(v): obvious.

(vi): Assume that $\mathscr{F}$ is a filter. If it is an ultrafilter and $X \in \widetilde{\mathscr{F}}$, then we cannot have $(\omega\backslash X) \in \mathscr{F}$, and hence $X \in \mathscr{F}$. Conversely, suppose that $\mathscr{F} = \widetilde{\mathscr{F}}$. Let $X \subseteq \omega$, and suppose that $X \notin \mathscr{F}$. So $X \notin \widetilde{\mathscr{F}}$, and hence $(\omega\backslash X) \in \mathscr{F}$. So $\mathscr{F}$ is an ultrafilter. □

Proposition 82. (Blass 9.15) *Suppose that $\mathscr{X}, \mathscr{Y} \subseteq [\omega]^\omega$, $|\mathscr{X}| < \mathfrak{g}$, and $\mathscr{Y}\sim$ is groupwise dense.*

Then there is a finite-to-one $f : \omega \to \omega$ such that $\forall X \in \mathscr{X}\, \exists Y \in \mathscr{Y}(f[Y] \subseteq f[X])$.

Proof. First we note:

(1) For all $Y \in \mathscr{Y}$ and all finite F, the sets $Y\backslash F$ and $Y \cup F$ are in $\mathscr{Y}$.

For, $Y \subseteq^* (Y\backslash F)$, hence $(\omega\backslash(Y\backslash F)) \subseteq^* (\omega\backslash Y) \in (\mathscr{Y}\sim)$, and $\mathscr{Y}\sim$ is groupwise dense, so $(\omega\backslash(Y\backslash F)) \in (\mathscr{Y}\sim)$, and hence $(Y\backslash F) \in \mathscr{Y}$. Similarly, $(Y \cup F) \in \mathscr{Y}$.

Now for each $X \in \mathscr{X}$, let

$$\mathscr{G}_X = \{Z \in [\omega]^\omega : \exists Y \in \mathscr{Y} \forall a, b \in Z([a, b) \cap Y \neq \emptyset \to [a, b) \cap X \neq \emptyset)\}.$$

We claim that $\mathscr{G}_X$ is groupwise dense. First we show that it is open. So, suppose that $W \subseteq^* Z \in \mathscr{G}_X$. Choose $Y \in \mathscr{Y}$ according to $Z \in \mathscr{G}_X$. For each integer m, let m' be the least member of X which is greater than m. Now let

$$Y' = Y\backslash \left(\bigcup_{b \in W\backslash Z} [0, b) \cup \bigcup_{a \in W\backslash Z} [a, a' + 1) \right).$$

Thus $Y' \in \mathscr{Y}$ by (1). Now suppose that $a, b \in W$ and $[a, b) \cap Y' \neq \emptyset$.

Case 1. $a, b \in Z$. Then $[a, b) \cap X \neq \emptyset$ by the choice of Y.

Case 2. $b \in W\backslash Z$. Then $[a, b) \cap Y' = \emptyset$, contradiction.

Case 3. $b \in Z$, but $a \in W\backslash Z$. Choose $c \in [a, b) \cap Y'$. Then $a' < c$ by the definition of Y', so $a' \in [a, b) \cap X$, as desired.

Thus we have shown that $\mathscr{G}_X$ is open.

Now we turn to the "real" groupwise density property. Thus suppose that P is an interval partition; we want to show that some union of intervals of P is in $\mathscr{G}_X$. We define $\langle j_n : n < \omega \rangle$ by recursion. Let $j_0 = 0 = i^P_0$. Having defined j_n to be some i^P_k, choose i^P_m so that $[j_n, i^P_m) \cap X \neq \emptyset$. Then let $j_{n+1} = i^P_m$. Let $Q = \{[j_n, j_{n+1}) : n \in \omega\}$. Thus each interval of Q intersects X, and is a union of intervals of P. So it suffices to show that some union of intervals of Q is in $\mathscr{G}_X$.

Since $\mathscr{Y}\sim$ is groupwise dense, there is some infinite $M \subseteq \omega$ such that the set $Z \stackrel{\text{def}}{=} \bigcup_{n \in M}[j_n, j_{n+1})$ is in $\mathscr{Y}\sim$. Let $Y = \omega\backslash Z$; so $Y \in \mathscr{Y}$. We claim that $Z \in \mathscr{G}_X$, as witnessed by Y. For, suppose that $a, b \in Z$ and $[a, b) \cap Y \neq \emptyset$. Thus $a < b$ and there is a $c \in \omega \backslash Z$ such that $a < c < b$. So there is an n such that $[j_n, j_{n+1}) \subseteq [a, b)$. Hence $[a, b) \cap X \supseteq [j_n, j_{n+1}) \cap X \neq \emptyset$. This finishes the proof that $\mathscr{G}_X$ is groupwise dense.

Now since $|\mathscr{X}| < \mathfrak{g}$, we can choose $Z \in \bigcap_{X \in \mathscr{X}} \mathscr{G}_X$. Since each $\mathscr{G}_X$ is open and $Z \cup \{0\} \subseteq^* Z$, we may assume that $0 \in Z$. Define $f : \omega \to \omega$ by setting $f(n) = |\{m \in Z : m \leq n\}|$. Since $0 \in Z$, we always have $f(n) > 0$. Write $Z = \{i_n : n \in \omega\}$ in increasing order. Then for any positive integer n, $f^{-1}[\{n\}] = [i_{n-1}, i_n)$. Now suppose that $a \in Y$. Let $f(a) = n$. Then $a \in f^{-1}[\{n\}] = [i_{n-1}, i_n)$. Thus $i_{n-1}, i_n \in Z$ and $[i_{n-1}, i_n) \cap Y \neq \emptyset$. Hence $[i_{n-1}, i_n) \cap X \neq \emptyset$. Choose $d \in [i_{n-1}, i_n) \cap X$. Then $f(d) = n$. This shows that $f[Y] \subseteq f[X]$. □

Proposition 83. (Blass 9.17) *For any proper filter $\mathscr{F}$ containing all cofinite sets, $\mathscr{F}$ is unsplittable iff $\mathscr{F}$ is an ultrafilter.*

Proof. ⇒: Assume that $\mathscr{F}$ is unsplittable. Thus

$$(*) \qquad \forall a \in [\omega]^\omega \exists x \in \mathscr{F}[x \subseteq^* a \text{ or } x \subseteq^* (\omega \backslash a)].$$

Now let $a \subseteq \omega$. If a is finite, then $(\omega \backslash a) \in \mathscr{F}$. Suppose that a is infinite. Choose x by $(*)$.

If $x \subseteq^* a$, then $x \backslash a$ is finite, hence $a \cup (\omega \backslash x) \in \mathscr{F}$, so $a \in \mathscr{F}$.

If $x \subseteq^* (\omega \backslash a)$, then $x \cap a$ is finite, hence $(\omega \backslash x) \cup (\omega \backslash a) \in \mathscr{F}$, so $(\omega \backslash a) \in \mathscr{F}$.

Thus $\mathscr{F}$ is an ultrafilter.

⇐: obvious. □

Proposition 84. (Blass 9.16, 9.17) *Assume that $\mathfrak{r} < \mathfrak{g}$. Then for any proper filter $\mathscr{F}$ containing all cofinite sets, either $\mathscr{F}$ is feeble or there is a finite-to-one $f : \omega \to \omega$ such that $f((\mathscr{F}))$ is a nonprincipal ultrafilter.*

Proof. Assume that $\mathscr{F}$ is not feeble. Let $\mathscr{X} \subseteq [\omega]^\omega$ be unsplittable, with $|\mathscr{X}| = \mathfrak{r}$. By Proposition 70(iv), $\mathscr{F}\sim$ is groupwise dense. Hence we can apply Proposition 82 to $\mathscr{X}$ and $\mathscr{F}$ to obtain a finite-to-one f such that $\forall X \in \mathscr{X} \exists Y \in \mathscr{F}(f[Y] \subseteq f[X])$.

Now it suffices by Proposition 83 to show that $f((\mathscr{F}))$ is unsplittable. Let $a \in [\omega]^\omega$ be given. Choose $x \in \mathscr{X}$ such that $x \subseteq^* f^{-1}[a]$ or $x \subseteq (\omega \backslash f^{-1}[a])$. (If $f^{-1}[a]$ is finite, any element of $\mathscr{X}$ will work.) Then choose $y \in \mathscr{F}$ such that $f[y] \subseteq f[x]$. Then $y \subseteq f^{-1}[f[x]]$, so $f^{-1}[f[x]] \in \mathscr{F}$, so that $f[x] \in f((\mathscr{F}))$. Since $f[x] \subseteq^* a$ or $f[x] \subseteq^* (\omega \backslash a)$, the proof is complete. □

(Blass 9.17) The *principle of filter dichotomy* is the statement that for every proper filter $\mathscr{F}$ containing all cofinite sets, either $\mathscr{F}$ is feeble or there is a finite-to-one $f : \omega \to \omega$ such that $f((\mathscr{F}))$ is an ultrafilter. Thus Proposition 84 says that $\mathfrak{r} < \mathfrak{g}$ implies filter dichotomy.

Proposition 85. (Blass 9.18, 9.19) *Assume filter dichotomy. Then for any family $\mathscr{A}$ of fewer than 2^ω nonprincipal ultrafilters there is a finite-to-one f such that $f((\mathscr{U})) = f((\mathscr{V}))$ for any $\mathscr{U}, \mathscr{V} \in \mathscr{A}$.*

Proof. By Proposition 79, $\bigcap \mathscr{A}$ is not feeble, so there is a finite-to-one f such that $f((\bigcap \mathscr{A}))$ is an ultrafilter. Suppose that $\mathscr{U}, \mathscr{V} \in \mathscr{A}$ and $X \in f((\mathscr{U}))$. So $f^{-1}[X] \in \mathscr{U}$.

If $(\omega\backslash X) \in f((\bigcap\mathscr{A}))$, then $\omega\backslash f^{-1}[X] = f^{-1}[\omega\backslash X] \in \bigcap\mathscr{A} \subseteq \mathscr{U}$, contradiction. So $X \in f((\bigcap\mathscr{A}))$, hence $f^{-1}[X] \in \bigcap\mathscr{A} \subseteq \mathscr{V}$, so $X \in f((\mathscr{V}))$. □

(Blass 9.19) *Near coherence of filters*, NCF for short, is the statment that for any two nonprincipal ultrafilters $\mathscr{U}, \mathscr{V}$ there is a finite-to-one f such that $f((\mathscr{U})) = f((\mathscr{V}))$. Thus a special case of Proposition 84 is that filter dichotomy implies NCF.

Proposition 86. (Blass 9.19) *The following conditions are equivalent:*

(i) NCF.

(ii) For any two proper filters $\mathscr{F}, \mathscr{G}$ containing the cofinite sets, there is a finite-to-one function f such that $f((\mathscr{F})) \cup f((\mathscr{G}))$ has fip.

Proof. (i)⇒(ii): Assume (i) and the hypothesis of (ii). Extend $\mathscr{F}, \mathscr{G}$ to ultrafilters $\mathscr{F}', \mathscr{G}'$ respectively. By NCF, let f be finite-to-one such that $f((\mathscr{F}')) = f((\mathscr{G}'))$. Since $f((\mathscr{F})) \cup f((\mathscr{G})) \subseteq f((\mathscr{F}')) \cup f((\mathscr{G}'))f((\mathscr{F}'))$, the conclusion of (ii) follows.

(ii)⇒(i): Assume (ii) and suppose that $\mathscr{F}$ and $\mathscr{G}$ are ultrafilters. Applying (ii), we get a finite-to-one f such that $f((\mathscr{F})) \cup f((\mathscr{G}))$ has fip. Since $f((\mathscr{F}))$ and $f((\mathscr{G}))$ are ultrafilters, they must be identical. □

Proposition 87. (Blass 9.20) *Filter dichotomy implies that $\mathfrak{b} = \mathfrak{u}$ and $\mathfrak{d} = 2^\omega$.*

Proof. By Proposition 80 let $\mathscr{A}$ be a collection of nonprincipal ultrafilters such that $|\mathscr{A}| = \mathfrak{d}$ and there is no finite-to-one function f such that $f((\bigcap\mathscr{A}))$ is an ultrafilter. By filter dichotomy, the filter $\bigcap\mathscr{A}$ is feeble. By Proposition 79, $\mathfrak{d} = 2^\omega$.

By Proposition 77 let $\mathscr{F}$ be a non-feeble filter containing all cofinite sets which is generated by a set X such that $|X| = \mathfrak{b}$. We may assume that X is closed under $\cap$. By filter dichotomy, there is a finite-to-one function f such that $f((\mathscr{F}))$ is an ultrafilter. Let $Y = \{f[a] : a \in X\}$. Thus $|Y| \le \mathfrak{b}$. Moreover, $Y \subseteq f((\mathfrak{F}))$. For, if $a \in X$, then $a \subseteq f^{-1}[f[a]]$ and so $f^{-1}[f[a]] \in \mathscr{F}$, and so $f[a] \in f((\mathscr{F}))$. Now Y generates $f((\mathscr{F}))$. For suppose that $a \in f((\mathscr{F}))$. Thus $f^{-1}[a] \in \mathscr{F}$, so there is an $x \in X$ such that $x \subseteq f^{-1}[a]$. Hence $f[x] \subseteq a$, as desired. It follows that $\mathfrak{u} \le \mathfrak{b}$. The other inclusion holds by Propositions 22 and 72. □

Proposition 88. (Blass 9.22) *Assume that $\mathfrak{u} < \mathfrak{g}$. Suppose that $\mathscr{Y} \subseteq [\omega]^\omega$ is closed under almost supersets. Then there is a finite-to-one f such that one of the following conditions holds:*

(i) $f((\mathscr{Y})) = \{X \in [\omega]^\omega : X$ is cofinite$\}$.

(ii) $f((\mathscr{Y})) = [\omega]^\omega$.

(iii) $f((\mathscr{Y}))$ is a nonprincipal ultrafilter.

Proof. Let $\mathscr{Y}$ be as indicated.

Suppose that $\mathscr{Y}\sim$ is not groupwise dense. Then by Proposition 70, (i) holds.

Suppose that $\sim\mathscr{Y}$ is not groupwise dense. Now $\sim\mathscr{Y} = \sim\mathscr{Y}\sim\sim$, so by Proposition 70 we see that $f((\sim\mathscr{Y}\sim))$ is the collection of all cofinite sets. Suppose that $a \in [\omega]^\omega\backslash f((\mathscr{Y}))$. Then $f^{-1}[a] \notin \mathscr{Y}$, so $f^{-1}[a] \in \sim\mathscr{Y}$, hence $f^{-1}[\omega\backslash a] = \omega\backslash f^{-1}[a] \in \sim\mathscr{Y}\sim$, so $\omega\backslash a \in f((\sim\mathscr{Y}\sim))$, hence $\omega\backslash a$ is cofinite, contradiction. So (ii) holds.

Thus we may assume:

(1) $\mathscr{Y}\sim$ is groupwise dense.

(2) $\sim\mathscr{Y}$ is groupwise dense.

By (1) and Proposition 82, there is a finite-to-one function g such that $\forall X \in \mathscr{X}\exists Y \in \mathscr{Y}(g[Y] \subseteq g[X])$. Hence

(3) $g((\mathscr{U})) \subseteq g((\mathscr{Y}))$.

In fact, suppose that $a \in g((\mathscr{U}))$. Then $g^{-1}[a] \in U$, so we can choose $X \in \mathscr{X}$ such that $X \subseteq g^{-1}[a]$. Then we can choose $Y \in \mathscr{Y}$ such that $g[Y] \subseteq g[X]$. Then $Y \subseteq g^{-1}[g[X]] \subseteq g^{-1}[a]$, so $g^{-1}[a] \in \mathscr{Y}$ since $\mathscr{Y}$ is closed under almost supersets. So $a \in g((\mathscr{Y}))$, as desired for (3).

(4) $\{g[X] : X \in \mathscr{X}\}$ is a base for $g((\mathscr{U}))$.

In fact, let $a \in g((\mathscr{U}))$. Thus $g^{-1}[a] \in \mathscr{U}$, so there is an $X \in \mathscr{X}$ such that $X \subseteq g^{-1}[a]$. Hence $g[X] \subseteq a$, as desired.

Now by (2) and Propositions 69 and 81, $g((\sim\mathscr{Y})) = g((\sim\mathscr{Y}\sim\sim)) = g((\sim\mathscr{Y}\sim))\sim$ is groupwise dense. Hence we can apply Proposition 82 with $\mathscr{X}$ and $\mathscr{Y}$ replaced by $\{g[X] : X \in \mathscr{X}\}$ and $g((\sim\mathscr{Y}\sim))$ to obtain a finite-to-one function f such that $\forall X \in \mathscr{X}\exists Y \in g((\sim\mathscr{Y}\sim))(f[Y] \subseteq f[g[X]])$. Hence

(5) $f((g((\mathscr{U})))) \subseteq f((g((\sim\mathscr{Y}\sim))))$.

In fact, let $a \in f((g((\mathscr{U}))))$. Thus $f^{-1}[a] \in g((\mathscr{U}))$, so by (4), choose $X \in \mathscr{X}$ such that $g[X] \subseteq f^{-1}[a]$. Then choose $Y \in g((\sim\mathscr{Y}\sim))$ such that $f[Y] \subseteq f[g[X]])$. Thus $f[Y] \subseteq a$, so $Y \subseteq f^{-1}[a]$, and it follows that $f^{-1}[a] \in g((\sim\mathscr{Y}\sim))$. Hence $a \in f((g((\sim\mathscr{Y}\sim))))$. This proves (5).

It follows from (5) using Proposition 81 that

$$\begin{aligned} f((g((\mathscr{U})))) &\subseteq \sim f((g((\mathscr{Y}))))\sim; \quad \text{hence} \\ f((g((\mathscr{Y}))))\sim &\subseteq \sim f((g((\mathscr{U})))); \quad \text{hence} \\ f((g((\mathscr{Y})))) &\subseteq \sim f((g((\mathscr{U}))))\sim. \end{aligned}$$

By Proposition 81 we have $\sim f((g((\mathscr{U}))))\sim = f((g((\mathscr{U}))))$. So we have $f((g((\mathscr{Y})))) \subseteq f((g((\mathscr{U}))))$.

The other inclusion follows from (3), so (iii) holds. □

(Blass 9.23) An ultrafilter $\mathscr{U}$ on ω is *selective* iff $\mathscr{U}$ is nonprincipal and for every function $f : \omega \to \omega$ there is an $a \in U$ such that $f \restriction a$ is one-one or constant.

An ultrafilter $\mathscr{U}$ on ω is a *P-point* iff $\mathscr{U}$ is nonprincipal and for every function $f : \omega \to \omega$ there is an $a \in U$ such that $f \restriction a$ is finite-to-one or constant.

An ultrafilter $\mathscr{U}$ on ω is a *Q-point* iff $\mathscr{U}$ is nonprincipal and for every finite-to-one function $f : \omega \to \omega$ there is an $a \in U$ such that $f \restriction a$ is one-one.

Proposition 89. (Blass 9.24) *An ultrafilter is selective iff it is both a P-point and a Q-point.* □

Proposition 90. (Blass 9.24) *For any nonprincipal ultrafilter $\mathscr{U}$ the following conditions are equivalent:*

(i) $\mathscr{U}$ is selective.

(ii) If P is a partition of ω and $a \notin \mathscr{U}$ for all $a \in P$, then there is a $d \in \mathscr{U}$ such that $|a \cap d| = 1$ for all $a \in P$.

Proof. (i)⇒(ii): Assume (i) and the hypothesis of (ii). Then P is clearly infinite. Write $P = \{a_n : n \in \omega\}$ with a one-one. For each $m \in \omega$, let $f(m)$ be the unique n such that $n \in a_m$. Now we apply (i). If f is constant on $d \in \mathscr{U}$, say with value n, then $d \subseteq a_n$ and so $a_n \in \mathscr{U}$, contradiction. So f is one-one on some $d \in \mathscr{U}$. Thus $|d \cap a_n| \leq 1$ for all $n \in \omega$. Hence there is an $e \supseteq d$ such that $|e \cap a_n| = 1$ for all $n \in \omega$, and $e \in \mathscr{U}$.

(ii)⇒(i): Assume (ii), and suppose that $f : \omega \to \omega$. Then $\{f^{-1}[\{n\}] : n \in \omega\} \backslash \{\emptyset\}$ is a partition of ω. If $f^{-1}[\{n\}] \in \mathscr{U}$ for some n, then f is constant on this member of $\mathscr{U}$. So, suppose that $f^{-1}[\{n\}] \notin \mathscr{U}$ for all $n \in \omega$. Then by (ii), there is a $d \in \mathscr{U}$ such that $|d \cap f^{-1}[\{n\}]| = 1$ for all $n \in \omega$ for which $f^{-1}[\{n\}] \neq \emptyset$. Thus f is one-one on d. □

Proposition 91. (Blass 9.24) *For any nonprincipal ultrafilter $\mathscr{U}$ the following conditions are equivalent:*

(i) $\mathscr{U}$ is a P-point.

(ii) For every sequence $\langle a_n : n \in \omega \rangle$ of members of $\mathscr{U}$ such that $a_m \subseteq a_n$ if $n < m$, the family $\{a_n : n \in \omega\}$ has a pseudo-intersection which is in $\mathscr{U}$.

(iii) For every sequence $\langle a_n : n \in \omega \rangle$ of members of $\mathscr{U}$ such that $a_m \subseteq^* a_n$ if $n < m$, the family $\{a_n : n \in \omega\}$ has a pseudo-intersection which is in $\mathscr{U}$.

Proof. (i)⇒(ii): Assume (i) and the hypothesis of (ii). Define $c = \bigcap_{n\in\omega} a_n$. If $c \in \mathscr{U}$, we are through, so suppose that $\omega \backslash c \in \mathscr{U}$. Now let $b_0 = \omega$, and $b_{n+1} = a_n \backslash c$. Then $\langle b_n : n \in \omega \rangle$ is such that $b_0 = \omega$, $b_{i+1} \subseteq b_i$ for all i, each $b_i \in \mathscr{U}$, and $\bigcap_{i\in\omega} b_i = \emptyset$. If there is an i such that $b_j = b_i$ for all $j > i$, again we are through, so suppose that for every i there is a $j > i$ such that $b_j \neq b_i$. Then there is a strictly increasing sequence $\langle m(j) : j < \omega \rangle$ such that $m(0) = 0$ and $b_{m(j+1)} \subset b_{m(j)}$ for all j. Note that for each $n \in \omega$ there is a largest j such that $n \in b_{m(j)}$. Now define $f : \omega \to \omega$ by setting $f(n) = j$ where j is largest such that $n \in b_{m(j)}$. Thus $f[b_{m(j)} \backslash b_{m(j+1)}] = \{j\}$. Choose $d \in \mathscr{U}$ such that f is finite-to-one or constant on d. If f takes the constant value j on d, then $d \subseteq b_{m(j)} \backslash b_{m(j+1)}$, and hence $\omega \backslash b_{m(j+1)} \in \mathscr{U}$, contradiction. So f is finite-to-one on d. Now for any $i < \omega$ we have $f(n) < j$ for all $n \in d \backslash b_{m(j)}$, so $d \backslash b_{m(j)} \subseteq f^{-1}[j]$, so that $d \backslash b_{m(j)}$ is finite. Thus d is the required pseudo-intersection.

(ii)⇒(iii): Assume (ii) and the hypothesis of (iii). Let $b_m = \bigcap_{n\leq m} a_n$ for each $m \in \omega$. Then by (ii) let d be a pseudo-intersection of $\langle b_m : m \in \omega \rangle$. If $m \in \omega$, then $b_m \subseteq a_m$, and hence $d \backslash a_m \subseteq d \backslash b_m$, so that $d \backslash a_m$ is finite. Thus d is a pseudo-intersection of $\langle a_m : m \in \omega \rangle$.

(iii)⇒(i): Assume (iii), and suppose that $f : \omega \to \omega$. For each $n \in \omega$ let $a_n = \{m \in \omega : f(m) \geq n\}$. Thus $a_m \subseteq a_n$ if $n < m$.

Case 1. There is an $n \in \omega$ such that $a_n \notin \mathscr{U}$. So $\omega \backslash a_n \in \mathscr{U}$. Now

$$\omega \backslash a_n = \{m \in \omega : f(m) < n\} = \bigcup_{i<n} \{m \in \omega : f(m) = i\},$$

so there is an $i < n$ such that $\{m \in \omega : f(m) = i\} \in \mathscr{U}$. This means that f is constant on a member of $\mathscr{U}$.

Case 2. $a_n \in \mathscr{U}$ for all $n \in \omega$. Then by (iii), let d be a pseudo-intersection of $\langle a_n : n \in \omega\rangle$. Then for any $n \in \omega$, $\{m \in d : f(m) = n\} \subseteq d\backslash a_{n+1}$, and the latter set is finite. So f is finite-to-one on d. □

For the next proposition we consider the Boolean algebra $\mathscr{P}(\omega)/\text{fin}$. Elements of this BA are denoted by $[a]$ with $a \subseteq \omega$. If $\mathscr{U}$ is a nonprincipal ultrafilter on ω, then $\{[a] : a \in \mathscr{U}\}$ is an ultrafilter on $\mathscr{P}(\omega)/\text{fin}$, as is easily checked. This ultrafilter will be denoted by $\mathscr{U}'$. Note that for any subset a of ω, $a \in \mathscr{U}$ iff $[a] \in \mathscr{U}'$. In fact, $\Rightarrow$ is clear, so suppose that $[a] \in \mathscr{U}'$. Then there is a $b \in \mathscr{U}$ such that $[a] = [b]$. Hence $b\backslash a$ is finite, so $a \cup (\omega\backslash b) \in \mathscr{U}$. Since $b \in \mathscr{U}$, it follows by taking intersections that $a \in \mathscr{U}$. Every ultrafilter on $\mathscr{P}(\omega)/\text{fin}$ has the form $\mathscr{U}'$ for some nonprincipal ultrafilter $\mathscr{U}$ on ω, as is easily seen.

Proposition 92. (Blass 9.24) *For any nonprincipal ultrafilter $\mathscr{U}$ the following conditions are equivalent:*

(i) $\mathscr{U}$ is a P-point.

(ii) Let $\mathscr{X} = \text{Ult}(\mathscr{P}(\omega)/\text{fin})$. If $\mathscr{O}$ is a countable collection of open neighborhoods of $\mathscr{U}'$ in $\mathscr{X}$, then there is an open neighborhood V of $\mathscr{U}'$ such that $V \subseteq \bigcap \mathscr{O}$.

Proof. (i)⇒(ii): Assume (i) and the hypothesis of (ii). For each $U \in \mathscr{O}$ choose $a_U \in \mathscr{U}$ such that $\mathcal{S}([a_U]) \subseteq U$. Write $\mathscr{O} = \{U(m) : m \in \omega\}$. Let $b_m = \prod_{n \leq m} a_{U(m)}$. By (i), choose $c \in \mathscr{U}$ such that c is a pseudo-intersection of $\langle b_m : m \in \omega\rangle$. Thus $\mathscr{U}' \in \mathcal{S}([c])$. We claim that $\mathcal{S}([c]) \subseteq \bigcap \mathscr{O}$. (As desired.) To prove this, suppose that $U \in \mathscr{U}$ and $\mathscr{V}$ is a nonprincipal ultrafilter on ω such that $\mathscr{V}' \in \mathcal{S}([c])$. Thus $[c] \in \mathscr{V}'$, so $c \in \mathscr{V}$. Now $c \subseteq^* a_U$, so $c\backslash a_U$ is finite and hence $(\omega\backslash c) \cup a_U \in \mathscr{V}$ and hence $a_U \in \mathscr{V}$. Hence $\mathscr{V}' \in \mathcal{S}([a_U])$, and so $\mathscr{V}' \in U$, as desired.

(ii)⇒(i): Assume (ii), and, in order to apply Proposition 91, assume that $\langle a_m : m \in \omega\rangle$ is a sequence of members of $\mathscr{U}$ such that $a_m \subseteq a_n$ if $n < m$. Then

$$\mathscr{O} \stackrel{\text{def}}{=} \{\mathcal{S}([a_m]) : m \in \omega\}$$

is a countable collection of open neighborhoods of $\mathscr{U}'$. Hence by (ii), there is an open neighborhood V of $\mathscr{U}'$ such that $V \subseteq \bigcap \mathscr{O}$. Let d be a subset of ω such that $\mathscr{U}' \in \mathcal{S}([d]) \subseteq V$. Then we have $[d] \in \mathscr{U}'$, hence $d \in \mathscr{U}$. Now if $m \in \omega$, then $\mathcal{S}([a_m]) \in \mathscr{O}$, and so $\mathcal{S}([d]) \subseteq V \subseteq \mathcal{S}([a_m])$. Hence $[d] \leq [a_m]$, so that $d\backslash a_m$ is finite, as desired. □

Proposition 93. (Blass 9.24) *There is a nonprincipal ultrafilter on ω which is neither a P-point nor a Q-point.*

Proof. Let $f : \omega \to \omega \times \omega$ be a bijection. Let $\mathscr{U}$ be a nonprincipal ultrafilter on ω. Then we define

$$\mathscr{V} = \{X \subseteq \omega : \{a \in \omega : \{b \in \omega : (a, b) \in f[X]\} \in \mathscr{U}\} \in \mathscr{U}\}$$

We claim that $\mathscr{V}$ is the desired nonprincipal ultrafilter. The proof of this takes several easy steps. For brevity let $Z^X_a = \{b \in \omega : (a, b) \in f[X]\}$ for any $a \in \omega$. Thus the definition of $\mathscr{V}$ can be abbreviated as follows:

$$\mathscr{V} = \{X \subseteq \omega : \{a \in \omega : Z^X_a \in \mathscr{U}\} \in \mathscr{U}\}.$$

1) $\emptyset \notin \mathscr{V}$. For, $f[\emptyset] = \emptyset$, and for any $a \in \omega$, $Z_a^\emptyset = \{b \in \omega : (a,b) \in \emptyset\} = \emptyset$. So $\{a \in \omega : Z_a^\emptyset \in \mathscr{U}\} = \{a \in \omega : \emptyset \in \mathscr{U}\} = \emptyset \notin \mathscr{U}$. Thus $\emptyset \notin \mathscr{V}$.

2) If $X \subseteq Y$ and $X \in \mathscr{V}$, then $Y \in \mathscr{V}$. For, $f[X] \subseteq f[Y]$, so for any $a \in \omega$, $Z_a^X \subseteq Z_a^Y$. Now $\{a \in \omega : Z_a^X \in \mathscr{U}\} \in \mathscr{U}$. Since for any a, $Z_a^X \in \mathscr{U}$ implies that $Z_a^Y \in \mathscr{U}$, we have $\{a \in \omega : Z_a^X \in \mathscr{U}\} \subseteq \{a \in \omega : Z_a^Y \in \mathscr{U}\}$, and it follows that $\{a \in \omega : Z_a^Y \in \mathscr{U}\} \in \mathscr{U}$. So $Y \in \mathscr{V}$.

3) If $X, Y \in \mathscr{V}$, then $X \cap Y \in \mathscr{V}$. Note that for any $a \in \omega$,

$$\begin{aligned} Z_a^X \cap Z_a^Y &= \{b \in \omega : (a,b) \in f[X]\} \cap \{b \in \omega : (a,b) \in f[Y]\} \\ &= \{b \in \omega : (a,b) \in f[X] \cap f[Y]\} \\ &= \{b \in \omega : (a,b) \in f[X \cap Y]\} \\ &= Z_a^{X \cap Y}. \end{aligned}$$

Hence for any a, $Z_a^X \in \mathscr{U}$ and $Z_a^Y \in \mathscr{U}$ imply that $Z_a^{X \cap Y} \in \mathscr{U}$. So

$$\{a \in \omega : Z_a^X \in \mathscr{U}\} \cap \{a \in \omega : Z_a^Y \in \mathscr{U}\} \subseteq \{a \in \omega : Z_a^{X \cap Y} \in \mathscr{U}\}.$$

Since the two sets on the left here are in $\mathscr{U}$, so is the set on the right, and this implies that $X \cap Y \in \mathscr{V}$.

4) If $X \subseteq \omega$, then $X \in \mathscr{V}$ or $(\omega \backslash X) \in \mathscr{V}$. For, suppose that $X \notin \mathscr{V}$. Thus $\{a \in \omega : Z_a^X \in \mathscr{U}\} \notin \mathscr{U}$, so $\{a \in \omega : Z_a^X \notin \mathscr{U}\} \in \mathscr{U}$, hence $\{a \in \omega : (\omega \backslash Z_a^X) \in \mathscr{U}\} \in \mathscr{U}$. Now

$$\begin{aligned} \omega \backslash Z_a^X &= \{b \in \omega : b \notin Z_a^X\} \\ &= \{b \in \omega : (a,b) \notin f[X]\} \\ &= \{b \in \omega : (a,b) \in f[\omega \backslash X]\} \\ &= Z_a^{\omega \backslash X}. \end{aligned}$$

It follows that $\{a \in \omega : Z_a^{\omega \backslash X} \in \mathscr{U}\} \in \mathscr{U}$, and so $(\omega \backslash X) \in \mathscr{V}$.

Note that so far we have shown that $\mathscr{V}$ is an ultrafilter on ω.

5) $\mathscr{V}$ is nonprincipal. For, if $m \in \omega$, let $f(m) = (c, d)$. Then for any $a \in \omega$,

$$\begin{aligned} Z_a^{\omega \backslash \{m\}} &= \{b \in \omega : (a,b) \in f[\omega \backslash \{m\}] \\ &= \begin{cases} \omega & \text{if } a \neq c, \\ \omega \backslash \{d\} & \text{if } a = c. \end{cases} \end{aligned}$$

At any rate, $Z_a^{\omega \backslash \{m\}} \in \mathscr{U}$. So $\{a \in \omega : Z_a^{\omega \backslash \{m\}} \in \mathscr{U}\} = \omega \in \mathscr{U}$, and so $(\omega \backslash \{m\}) \in \mathscr{V}$.

6) $\mathscr{V}$ is not a P-point. To prove this, define $g : \omega \to \omega$ by setting $g(a) = 1^{\text{st}}(f(a))$ for any $a \in \omega$. Here for any ordered pair (u, v), $1^{\text{st}}(u, v) = u$ and $2^{\text{nd}}(u, v) = v$. Suppose that $X \in \mathscr{V}$ is such that $g \restriction X$ is finite-to-one. Now $\emptyset \notin \mathscr{U}$, so we can choose $a \in \omega$ such that $Z_a^X \in \mathscr{U}$. The set Z_a^X is infinite, and for any $b \in Z_a^X$ we have $(a, b) \in f[X]$ and so

$g(f^{-1}(a,b)) = a$, and $f^{-1}(a,b) \in X$, contradicting that g is finite-to-one on X. Suppose that g is constant on X. For distinct members a_0, a_1 of $\{a \in \omega : Z^X_a \in \mathscr{U}$ and members $b_0 \in Z^X_{a_0}, b_1 \in Z^X_{a_1}$, we have $(a_0, b_0), (a_1, b_1) \in X$ and

$$g(f^{-1}(a_0, b_0)) = a_0 \neq a_1 = g(f^{-1}(a_1, b_1)),$$

contradiction.

Thus $\mathscr{V}$ is not a P-point.

7) $\mathscr{V}$ is not a Q-point. This time, define $h : \omega \to \omega$ by setting, for any $x \in \omega$,

$$h(x) = \begin{cases} 2^{\text{nd}}(f(x)) & \text{if } 1^{\text{st}}(f(x)) < 2^{\text{nd}}(f(x)), \\ x & \text{otherwise.} \end{cases}$$

Then h is finite-to-one, since for any $m \in \omega$ we have $h^{-1}[\{m\}] \subseteq \{m\} \cup \{f^{-1}(a, m) : a < m\}$.

Let $X = \{f^{-1}(a, b) : a, b \in \omega \text{ and } a < b\}$. For any $a \in \omega$ we have $Z^X_a = \{b \in \omega : (a, b) \in f[X]\} = \{b \in \omega : a, b \in \omega \text{ and } a < b\}$. This is a cofinite set, so $Z^X_a \in \mathscr{U}$. Hence $\{a \in \omega : Z^X_a \in \mathscr{U}\} = \omega \in \mathscr{U}$. This shows that $X \in \mathscr{V}$.

Now suppose that h is one-one on $Y \in \mathscr{V}$. Thus h is also one-one on $X \cap Y$. Now since $\mathscr{U}$ is nonprincipal, choose distinct members $a_0, a_1 \in \{a \in \omega : Z^X_a \in \mathscr{U}\}$. Now $\omega \backslash (\max(a_0, a_1) + 1)$ is cofinite, so $Z^X_{a_0} \cap Z^X_{a_1} \cap \omega \backslash (\max(a_0, a_1) + 1) \in \mathscr{U}$; choose b in this set. Then $b \in Z^X_{a_0} \cap Z^X_{a_1}$ and $a_0, a_1 < b$, so $h(f^{-a}(a_0, b)) = b = h(f^{-1}(a_1, b))$. Since $f^{-1}(a_0, b)$ and $f^{-1}(a_1, b)$ are distinct members of X, this is a contradiction. $\square$

Proposition 94. (Blass 9.25.1) *If $2^\omega = \mathfrak{d}$, then every proper filter containing all cofinite sets and generated by $< 2^\omega$ sets is contained in a P-point.*

Proof. Assume the hypothesis, with $\mathscr{F}$ a filter containing all cofinite sets and generated by $< 2^\omega$ sets. Let $\langle S^\alpha : \alpha < 2^\omega \rangle$ enumerate all sequences $\langle T_m : m \in \omega \rangle$ of members of $[\omega]^\omega$ such that $T_m \subseteq T_n$ for $n < m$. We define $\langle \mathscr{F}^\alpha : \alpha \leq 2^\omega \rangle$ by recursion so that each $\mathscr{F}^\alpha$ is a proper filter. Let $\mathscr{F}^0 = \mathscr{F}$. If α is a limit ordinal $\leq 2^\omega$, let $\mathscr{F}^\alpha = \bigcup_{\beta<\alpha} \mathscr{F}^\beta$. Now assume that $\mathscr{F}^\alpha$ has been defined, so that it is a proper filter generated by less than 2^ω elements, with $\alpha < 2^\omega$.

If there is an n such that $S^\alpha_n \notin \mathscr{F}^\alpha$, let $\mathscr{F}^{\alpha+1}$ be generated by $\mathscr{F}^\alpha \cup \{\omega \backslash S^\alpha_n\}$. Clearly $\mathscr{F}^{\alpha+1}$ is then a proper filter generated by less than 2^ω elements.

Suppose that there is no such n. Then we apply Proposition 49 to $\mathscr{F}^\alpha$ and $\langle S^\alpha_n : n \in \omega \rangle$ to get a set $U \in [\omega]^\omega$ which has infinite intersection with each member of $\mathscr{F}^\alpha$ and is a pseudo-intersection of $\{S^\alpha_n : n \in \omega\}$. Let $\mathscr{F}^{\alpha+1}$ be generated by $\mathscr{F}^\alpha \cup \{U\}$. Clearly $\mathscr{F}^{\alpha+1}$ is then a proper filter generated by less than 2^ω elements.

Now we claim that if $\mathscr{U}$ is an ultrafilter containing $\mathscr{F}^{\alpha+1}$, then $\mathscr{U}$ is a P-point. To prove this, we apply Proposition 9.1. Suppose that $\langle a_n : n \in \omega \rangle$ is a sequence of members of $\mathscr{U}$ such that $a_m \subseteq a_n$ if $n < m$. Choose $\alpha < 2^\omega$ such that $S^\alpha = \langle a_n : n \in \omega \rangle$. If $S^\alpha_n \notin \mathscr{F}^\alpha$, then $\omega \backslash a_n = \omega \backslash S^\alpha_n \in \mathscr{F}^{\alpha+1} \subseteq \mathscr{U}$, contradiction. Hence there is no such n, and so by construction there is a pseudo-intersection U of $\{a_n : n \in \omega\}$ such that $U \in \mathscr{U}$. $\square$

Proposition 95. (Blass 9.25.2) *There is a proper filter containing the cofinite sets and generated by $\mathfrak{d}$ sets which cannot be extended to a P-point.*

Proof. Let $f : \omega \to \omega \times \omega$ be a bijection. Let D be an almost dominating family of size $\mathfrak{d}$. Let $\mathscr{F}$ be the filter generated by the following set:

$$\begin{aligned}&\{X : \omega\backslash X \text{ is finite}\}\\ &\cup \{\{m \in \omega : 1^{\text{st}}(f(m)) \geq n\} : n \in \omega\}\\ &\cup \{\{m \in \omega : 2^{\text{nd}}(f(m)) > g(1^{\text{st}}(f(m)))\} : g \in D\}.\end{aligned}$$

We claim that $\mathscr{F}$ is proper. Otherwise there are finite $F \subseteq \omega$ and $G \subseteq D$ such that

$$(*) \qquad \bigcap_{n\in F} \{m \in \omega : 1^{\text{st}}(f(m)) \geq n\} \cap \bigcap_{g\in G} 2^{\text{nd}}(f(m)) > g(1^{\text{st}}(f(m)))\} = \emptyset.$$

But if $p = \max(F)$ and $q > g(p)$ for each $g \in G$, then $f^{-1}(p, q)$ is a member of $(*)$, contradiction.

Now suppose that $\mathscr{U}$ is an ultrafilter containing $\mathscr{F}$ and it is a P-point. Let $h(x) = 1^{\text{st}}(f(x))$ for all $x \in \omega$. Choose $a \in \mathscr{U}$ such that h is constant or finite-to-one on a. Suppose that it is constant on a, say with value n. Choose $p \in a \cap \{m : 1^{\text{st}}(f(m)) \geq n + 1\}$. Then $h(p) \geq n + 1$, contradiction.

Hence h is finite-to-one on a. For any $x \in a$, if $h(m) = x$ for some m, let $h(x) = \max\{2^{\text{nd}}(f(m)) : h(m) = x\}$; this is legal because h is finite-to-one on a. If $x \notin a$, or if $h^{-1}[\{x\}] = \emptyset$, let $h(x) = 0$. Choose $g \in D$ such that $k \leq^* g$. Choose m so that $k(p) \leq g(p)$ for all $p \geq m$. Take

$$p \in a \cap \{x : 2^{\text{n}}(f(x)) > g(1^{\text{st}}(f(x))\} \cap \{x : 1^{\text{st}}(f(x)) \geq m\}.$$

Thus $h(p) = 1^{\text{st}}(f(p)) \geq m$, so

$$2^{\text{nd}}(f(p)) \leq k(h(p)) \leq g(h(p)) < 2^{\text{nd}}(f(p)),$$

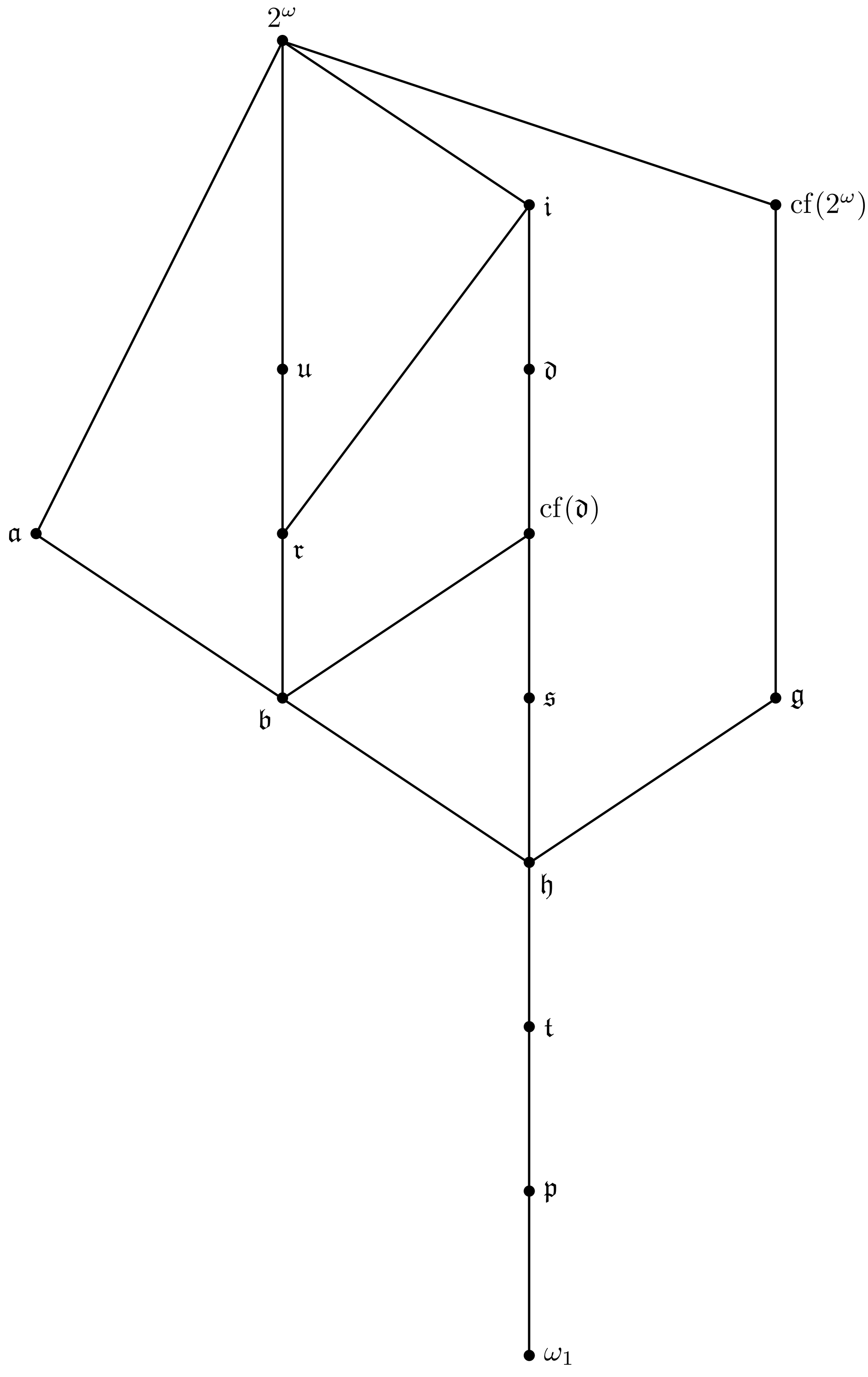
contradiction. □

Proposition 96. (Blass 9.25.3) *Every nonprincipal ultrafilter generated by fewer than $\mathfrak{d}$ sets is a P-point.*

Proof. Let $\mathscr{U}$ be a nonprincipal ultrafilter generated by D, with $|D| < \mathfrak{d}$. We may assume that D is closed under $\cap$. To apply Proposition 91, suppose that $\langle S_n : n \in \omega\rangle$ is a system of members of $\mathscr{U}$, with $S_m \subseteq S_n$ if $n < m$. By Proposition 49 let V be a pseudo-intersection of $\{S_m : m \in \omega\}$ which has infinite intersection with each member of D. Then $V \in \mathscr{U}$ (as desired). For, otherwise $(\omega\backslash V) \in \mathscr{U}$, and so there is an $a \in D$ such that $a\backslash\omega\backslash V$, so that $a \cap V = \emptyset$, contradiction. □

Main theorems

$\aleph_1 \leq \mathrm{cf}(\mathfrak{b}) \leq \mathrm{cf}(\mathfrak{d}) \leq \mathfrak{d}$. Theorem 1.
$\omega_1 \leq \mathfrak{s}$. Lemma 9.
$\mathfrak{s} \leq \mathfrak{d}$. Theorem 11.
$\mathfrak{b} \leq \mathfrak{r}$. Proposition 22.
$\mathfrak{t}$ is regular and uncountable. Proposition 34.
$\mathfrak{h}$ is regular. Proposition 38.
$\mathfrak{t} \leq \mathfrak{h}$. Proposition 39.
$\mathfrak{h} \leq \mathfrak{b}, \mathfrak{s}$. Proposition 40.
$\mathfrak{t} \leq \mathrm{cf}(2^\omega)$. Corollary 42.
$\aleph_1 \leq \mathfrak{p} \leq \mathfrak{t}$. Proposition 48.
$\mathfrak{p}$ is regular. Proposition 57.
$\mathfrak{b} \leq \mathfrak{a}$. Proposition 59.
$\mathfrak{r} \leq \mathfrak{i}$. Proposition 64.
$\mathfrak{d} \leq \mathfrak{i}$. Proposition 65.
$\mathfrak{r} \leq \mathfrak{u}$. Proposition 72.



Note: the inequality $\mathfrak{s} < \mathrm{cf}(\mathfrak{d})$ is not proved in these notes or in Blass [∞]; see Mildenberger [01].

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REFERENCES

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